Anti-Self-Dual 4-Manifolds, Quasi-Fuchsian Groups, and Almost-Kähler Geometry

Christopher J. Bishop* and Claude LeBrun†
Department of Mathematics, Stony Brook University

For Karen Uhlenbeck,
in celebration of her
seventy-fifth birthday.

Abstract

It is known that the almost-Kähler anti-self-dual metrics on a given 4-manifold sweep out an open subset in the moduli space of anti-self-dual metrics. The purpose of this paper is to show by example that this subset is not however generally closed, and so need not sweep out entire connected components in the moduli space. Our construction hinges on an unexpected link between harmonic functions on certain hyperbolic 3-manifolds and self-dual harmonic 2-forms on associated 4-manifolds.

1 Introduction

An oriented Riemannian 4-manifold (M^4, g) is said to be *anti-self-dual* if it satisfies $W_+ = 0$, where the self-dual Weyl curvature W_+ is by definition the orthogonal projection of the Riemann curvature tensor $\mathcal{R} \in \odot^2\Lambda^2$ into the trace-free symmetric square $\odot^2_0\Lambda^+$ of the bundle of self-dual 2-forms. This is a conformally invariant condition, and so is best understood as a condition on the conformal class $[g] := \{u^2 g \mid u : M \to \mathbb{R}^+\}$ rather than on the representative metric g. For example, any oriented *locally-conformally-flat*

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4-manifold is anti-self-dual; indeed, these are precisely the 4-manifolds that are anti-self-dual with respect to *both* orientations. One compelling reason for the study of anti-self-dual 4-manifolds is that $W_+ = 0$ is exactly the integrability condition needed to make the total space of the 2-sphere bundle $S(\Lambda^+) \to M$ into a complex 3-fold Z in a natural way [2], and the *Penrose correspondence* [19] then allows one to completely reconstruct $(M^4, [g])$ from the complex geometry of the so-called twistor space Z.

A rather different link between anti-self-dual metrics and complex geometry is provided by the observation that a Kähler manifold (M^4, g, J) of complex dimension two is anti-self-dual if and only if its scalar curvature vanishes [17]. This makes a special class of extremal Kähler manifolds susceptible to study via twistor theory, and led, in the early 1990s, to various results on scalar-flat scalar surfaces that anticipated more recent theorems regarding more general extremal Kähler manifolds.

Of course, the Kähler condition is far from conformally invariant. Rather, in the present context, it should be thought of as providing a preferred conformal gauge for those special conformal classes that can be represented by Kähler metrics. But even among anti-self-dual conformal classes, those that can be represented by Kähler metrics tend to be highly non-generic. The following example nicely illustrates this phenomenon.

Example Let Σ be a compact Riemann surface of genus $\mathfrak{g} \geq 2$, and let M denote the compact oriented 4-manifold $\Sigma \times \mathbb{CP}_1$. If we equip Σ with its hyperbolic metric of Gauss curvature $K = -1$ and equip $S^2 = \mathbb{CP}_1$ with its usual "round" metric of Gauss curvature $K = +1$, the product metric on $M = \Sigma \times S^2$ is scalar-flat Kähler, and hence anti-self-dual. Moreover, since this metric admits orientation-reversing isometries, it is actually locally conformally flat. This last fact illustrates a useful consequence [2] of the 4-dimensional signature formula: if M is a compact oriented 4-manifold (without boundary) which has signature $\tau(M) = 0$, then every anti-self-dual metric on M is locally conformally flat.

Now the scalar-flat Kähler metric we have just described on $M = \Sigma \times S^2$ has universal cover $\mathcal{H}^2 \times S^2$, where $\mathcal{H}^2$ denotes the hyperbolic plane. In fact, one can show [6, 13] that *every* scalar-flat Kähler metric on $\Sigma \times S^2$ has universal cover homothetically isometric to this fixed model. Thus, starting with the representation $\pi_1(M) = \pi_1(\Sigma) \hookrightarrow PSL(2, \mathbb{R}) = SO_+(2, 1)$ that uniformizes Σ, any deformation of our example through scalar-flat Kähler metrics of fixed total volume arises from a deformation though representations $\pi_1(\Sigma) \hookrightarrow SO_+(2,1) \times SO(3)$. On the other hand, $\mathcal{H}^2 \times S^2$ is conformally isometric to the complement $S^4 - S^1$ of an equatorial circle in the 4-sphere, so the general deformation of the locally conformally flat structure on $M = \Sigma \times S^2$ instead just corresponds to a family of homomorphism $\pi_1(\Sigma) \hookrightarrow SO_+(5,1)$ taking values in the group of Möbius transformations of the 4-sphere. Remembering to only count these representations up to conjugation, we see that the moduli space of anti-self-dual conformal classes on M has dimension $30(\mathsf{g}-1)$, but that only a subspace of dimension $12(\mathsf{g}-1)$ arises from conformal structures that contain scalar-flat Kähler metrics. $\diamondsuit$

Because of this, we can hardly expect scalar-flat-Kähler metrics to provide a reliable model for general anti-self-dual metrics, even on 4-manifolds that arise as compact complex surfaces. However, the larger class of *almost-Kähler* anti-self-dual metrics shares many of the remarkable properties of the scalar-flat Kähler metrics, and yet sweeps out an open region in the moduli space of anti-self-dual conformal structures.

Recall that an oriented Riemannian manifold (M,g) equipped with a closed 2-form ω is said to be *almost-Kähler* if there is an orientation-compatible almost-complex structure $J : TM \to TM$, $J^2 = -\mathbf{1}$, such that $g = \omega(\cdot, J\cdot)$; in this language, a Kähler manifold just becomes an almost-Kähler manifold for which the almost-complex structure J happens to be integrable. But because J is algebraically determined by g and ω, there is an equivalent reformulation that avoids mentioning J explicitly. Indeed, an oriented Riemannian $2m$-manifold (M,g) is almost-Kähler with respect to the closed 2-form ω iff $\star\omega = \omega^{m-1}/(m-1)!$ and $|\omega| = \sqrt{m}$, where the Hodge star and pointwise norm on 2-forms are those determined by g and the fixed orientation. In particular, we see that ω is a harmonic 2-form on (M^{2m}, g), and that ω is an orientation-compatible symplectic form on M.

But this also makes it clear that the 4-dimensional case is transparently simple and natural. Indeed, a compact oriented Riemannian 4-manifold (M,g) is almost-Kähler with respect to ω iff ω is a self-dual harmonic 2-form on (M^4, g) of constant length $\sqrt{2}$. However, since the Hodge star operator is conformally invariant on middle-dimensional forms, a 2-form on a 4-manifold is harmonic with respect to g iff it is harmonic with respect to every other metric in the conformal class $[g]$. Since a conformal change of metric $g \rightsquigarrow u^2 g$ changes the point-wise norm of a 2-form by $|\omega| \rightsquigarrow u^{-2}|\omega|$, this means that any harmonic self-dual form ω that is simply everywhere non-zero determines a unique $\hat{g} = u^2 g \in [g]$ such that $(M, \hat{g})$ is almost-Kähler with respect to

ω. Moreover, the dimension $b_+(M) = [b_2(M) + \tau(M)]/2$ of the space of self-dual harmonic 2-forms is a topological invariant of M, and the collection of these forms depends continuously on the space of Riemannian metrics (with respect, for example, to the $C^{1,\alpha}$ topology). Thus, if there is a nowhere-zero self-dual harmonic 2-form ω with respect to g, the same is true for every metric $\tilde{g}$ that is sufficiently close to g in the C^2 topology. It therefore follows that the almost-Kähler metrics sweep out an open subset of the space of conformal classes on any smooth compact 4-manifold M^4. We emphasize that this is a strictly 4-dimensional phenomenon; it certainly does not persist in higher dimensions.

Because of this, almost-Kähler anti-self-dual metrics can provide an interesting window into the world of general anti-self-dual metrics, at least on 4-manifolds that happen to admit symplectic structures. Since the anti-self-duality condition $W_+ = 0$ largely compensates for the extra freedom in the curvature tensor that would otherwise result from relaxing the Kähler condition, many features familiar from the scalar-flat Kähler case turn out to persist in this broader context. One particularly intriguing consequence is that it is not difficult to find obstructions to the existence of almost-Kähler anti-self-dual metrics, even though the known obstructions to the existence of general anti-self-dual metrics are few and far between. For example, while we know [12, 20] that there are scalar-flat Kähler metrics (and hence anti-self-dual metrics) on $\mathbb{CP}_2 \# k \overline{\mathbb{CP}}_2$ for $k \geq 10$, we also know [10, 18] that almost-Kähler anti-self-dual metrics definitely do not exist on such blow-ups of the complex projective plane at $k \leq 9$ points. Is the latter indicative of a deeper non-existence theorem for general anti-self-dual metrics? Or is this merely the sort of false hope that arises from staring too long at a mirage?

This paper will try to shed some light on these matters by investigating a related question. If a smooth compact 4-manifold admits an almost-Kähler anti-self-dual metric, is every anti-self-dual metric in the same component of the moduli space also almost-Kähler? Since we have seen that the almost-Kähler condition is open on the level of conformal classes, this question amounts asking whether it is also closed in the anti-self-dual context. Our results will show that the answer is no. Indeed, we will display a large family of counter-examples that arise from the theory of quasi-Fuchsian groups.

Theorem A *There is an integer N such that, whenever Σ is a compact oriented surface of even genus $\mathfrak{g} \geq N$, the 4-manifold $M = \Sigma \times S^2$ admits locally-conformally-flat conformal classes $[g]$ that cannot be represented* *by almost-Kähler metrics. Moreover, certain such $[g]$ arise from quasi-Fuchsian groups $\pi_1(\Sigma) \hookrightarrow SO_+(3,1) \subset SO_+(5,1)$, and so can be exhibited as locally-conformally-flat deformations of the conformal structures represented by scalar-flat Kähler product metrics on $\Sigma \times S^2$.*

While the examples described by this result are all locally conformally flat, and thus live on 4-manifolds of signature zero, a variant of the same construction produces many explicit examples that live on 4-manifolds with $\tau < 0$, and so are certainly not locally conformally flat. These arise in connection with the second author's explicit construction [15] of scalar-flat Kähler metrics on blown-up ruled surfaces. The main idea is to deform the hyperbolic 3-manifolds that played a central role in the earlier construction, by replacing Fuchsian with quasi-Fuchsian subgroups of $PSL(2,\mathbb{C})$.

Theorem B *Let $k \geq 2$ be an integer, and let N be the integer of Theorem A. Then if Σ is a compact oriented surface of even genus $\mathbf{g} \geq N$, the connected sum $M = (\Sigma \times S^2)\#k\overline{\mathbb{CP}}_2$ admits anti-self-dual conformal structures that cannot be represented by almost-Kähler metrics. Moreover, some such $[g]$ can be explicitly constructed from configurations of k points in quasi-Fuchsian hyperbolic 3-manifolds diffeomorphic to $\Sigma \times \mathbb{R}$, and so can be exhibited as anti-self-dual deformations of conformal structures that are represented by scalar-flat Kähler structures on blown-up ruled surfaces.*

Dedication and acknowledgments: This paper is dedicated to Karen Uhlenbeck, on the occasion of her 75^{th} birthday. While Karen is of course primarily known for her deep contributions to geometric analysis, her role in encouraging the work of younger mathematicians has also been a consistent feature of her long and fruitful career. The second author, whose taste and interests have been significantly shaped and influenced by Uhlenbeck's work, is therefore delighted to finally have an opportunity to thank Karen for the encouragement she offered him at the outset of this project. On the other hand, the first author, who has been presumably been off Prof. Uhlenbeck's radar for many years, would like to belatedly thank her for having passed him on his topology oral exam at the University of Chicago in 1984, while offering this paper as evidence that this decision might not have been a mistake after all. The authors would also like to thank many other colleagues, including Dennis Sullivan, Ian Agol, and Inyoung Kim, for helpful and stimulating conversations.

2 Fuchsian and Quasi-Fuchsian Groups

Let $\mathcal{H}^3$ denote hyperbolic 3-space, which we will visualize using either the Poincaré ball model or the upper-half-space model. Either way, we see a 2-sphere at infinity; in the Poincaré model, it is simply the boundary 2-sphere of the closed 3-ball D^3, while in the upper-half-space model it becomes the boundary plane plus an extra point, called ∞. In either picture, isometries of $\mathcal{H}^3$ extend to to the boundary sphere as conformal transformations, and every global conformal transformation of S^2 conversely arises this way from a unique isometry. We will henceforth choose to emphasize isometries of $\mathcal{H}^3$ and conformal maps of $S^2 = \mathbb{CP}_1$ that preserve orientation. The group of such isometries is then exactly the complex automorphism group $PSL(2, \mathbb{C})$ of $\mathbb{CP}_1$; the fact that this can be identified with the Lorentz group $SO_+(3, 1)$ (which most naturally acts on yet a third model for $\mathcal{H}^3$, the hyperboloid of unit future-pointing time-like vectors in Minkowski space) is one of those rare low-dimensional coincidences in Lie group theory that underlie many important phenomena in low-dimensional geometry and topology. Another relevant coincidence is that $SO_+(5, 1) = PGL(2, \mathbb{H})$, so that oriented conformal transformations of the 4-sphere can be understood as fractional linear transformation of the quaternionic projective line $\mathbb{HP}_1$; thus the natural exension $SO_+(3, 1) \hookrightarrow SO_+(5, 1)$ of conformal transformations from S^2 to S^4 can also be understood as arising from the the inclusion $PSL(2, \mathbb{C}) \hookrightarrow PGL(2, \mathbb{H})$ induced by including the complex numbers $\mathbb{C}$ into the quaternions $\mathbb{H}$.

A *Kleinian group* Γ is by definition a discrete subgroup of $PSL(2, \mathbb{C})$. Since discrete means that the identity element is isolated, this implies that the orbit of any point in $\mathcal{H}^3 \subset D^3$ can only accumulate on the boundary sphere. The set of accumulation points of any orbit is called the limit set, and denoted $\Lambda = \Lambda(\Gamma)$; this can easily be shown to be independent of the particular orbit we choose. A *Fuchsian group* is by definition a Kleinian group which sends some geometric disk $D^2 \subset S^2$ to itself; and since the boundary circle of any such disk is the image of $\mathbb{RP}^1 \subset \mathbb{CP}_1$ under a Möbius transformation, this is equivalent to saying that a Fuchsian group is a Kleinian group which is conjugate to a subgroup of $PSL(2, \mathbb{R})$. In this case, the limit set $\Lambda(\Gamma)$ must be a closed subset of the invariant circle. A Fuchsian group is said to be of the first type if $\Lambda(\Gamma)$ is the whole circle. A Kleinian group Γ is called *quasi-Fuchsian* if it is quasiconformally conjugate to a Fuchsian group Γ' of the first type, meaning that $\Gamma = \Phi^{-1} \circ \Gamma' \circ \Phi$ for some quasiconformal homeomorphism Φ of $\mathbb{CP}_1$. In particular, the limit set of such a group is a quasi-circle, meaning a Jordan curve that is the image of a geometric circle under some quasiconformal map. This implies [8] that the limit set has Hausdorff dimension < 2, and so in particular has Lebesgue area zero. We note in passing that there are other equivalent characterizations of such groups; for example, a finitely generated Kleinian group is quasi-Fuchsian if and only if its limit set is a closed Jordan curve.

The special class of quasi-Fuchsian groups Γ that will concern us here consists of those Γ which are group-isomorphic to the fundamental group $\pi_1(\Sigma)$ of some compact oriented surface Σ of genus $\mathtt{g} \geq 2$. We will call these[1] quasi-Fuchsian groups of *Bers type*, in honor of Lipman Bers' pioneering contribution to the subject. Given two orientation-compatible complex structures j and j' on Σ, Bers [3] showed that there is a quasi-Fuchsian group $\pi_1(\Sigma) \hookrightarrow PSL(2, \mathbb{C})$ with limit set a quasi-circle Λ, such that $(\mathbb{CP}_1 - \Lambda)/\pi_1(\Sigma)$ is biholomorphic to the disjoint union $(\Sigma, j) \sqcup (\Sigma, -j')$. The hyperbolic manifold $X = \mathcal{H}^3/\pi_1(\Sigma)$ is then diffeomorphic to $\Sigma \times (-1, 1)$, and the 3-manifold-with-boundary $\overline{X} := (D^3 - \Lambda)/\pi_1(\Sigma)$, which is diffeomorphic to $\Sigma \times [-1, 1]$, carries a conformal structure which extends the conformal class of the hyperbolic metric on $X \subset \bar{X}$ and induces the two specified conformal structures on the two boundary components $\Sigma \times \{\pm 1\}$. The quasi-Fuchsian group that accomplishes this is moreover unique up to conjugation in $PSL(2, \mathbb{C})$, so that these Bers groups are classified by pairs of points in Teichmüller space.

3 Constructing Anti-Self-Dual 4-Manifolds

We will now construct a menagerie of explicit anti-self-dual 4-manifolds, starting from any quasi-Fuchsian group $\Gamma \subset PSL(2, \mathbb{C})$ of Bers type. Recall that our definition requires that Γ to have limit set $\Lambda(\Gamma) \subset \mathbb{CP}_1$ equal to a quasi-circle, and to be group-isomorphic to $\pi_1(\Sigma)$ for some compact oriented surface Σ of genus $\mathtt{g} \geq 2$. Let $X = \mathcal{H}^3/\Gamma$ denote the associated hyperbolic 3-manifold, and let $\overline{X} := (D^3 - \Lambda)/\Gamma$ be its canonical compactification as a 3-manifold-with-boundary. We will use h to denote the hyperbolic metric on X, and will let $[\bar{h}]$ denote the conformal structure on $\overline{X}$ induced by the Euclidean conformal structure on D^3, all the while remembering that the restriction of $[\bar{h}]$ to the interior X of $\overline{X}$ is just the conformal class $[h]$ of the

[1] Since the standard terminology would instead describe such Γ as finitely-generated convex-co-compact quasi-Fuchsian groups without elliptic elements, the introduction of a shorter name seems both necessary and appropriate!

hyperbolic metric h.

The simplest version of our construction proceeds by defining P to be the 4-manifold $X \times S^1$, and then compactifying this as $M = (\overline{X} \times S^1)/\sim$, where the equivalence relation $\sim$ collapses $(\partial\overline{X}) \times S^1$ to $\partial\overline{X}$ by contracting each circle factor to a point. As a set, M is therefore just the disjoint union of P and $\partial\overline{X}$. The point of interest, though, is that the topological space M can be made into a smooth 4-manifold in a way that simultaneously endows it with a locally-conformally-flat conformal structure. To see this, let us first recall that the Riemannian product $\mathcal{H}^3 \times S^1$ is conformally flat, since

$$z^2\left(\frac{dx^2+dy^2+dz^2}{z^2}+dt^2\right) = (dx^2+dy^2)+(dz^2+z^2dt^2) \tag{3.1}$$

shows that a certain conformal rescaling of its metric is just the Euclidean metric on $\mathbb{R}^4 - \mathbb{R}^2$, written in cylindrical coordinates. We can generalize this by letting

$$u : \overline{X} \to [0,\infty)$$

be a smooth defining function for $\partial\overline{X}$, in the sense that $\partial\overline{X} = u^{-1}(\{0\})$ and that $du \neq 0$ along $\partial\overline{X}$. Inspection of (3.1) then demonstrates that

$$g = u^2(h+dt^2) \tag{3.2}$$

defines a smooth locally-conformally-flat metric on M, since near any boundary point of $\overline{X}$ we can choose local coordinates (x, y, z) that express h in the upper-half-space model, and we then automatically have $u = wz$ for some smooth positive function w on the corresponding coordinate domain. Of course, the metric g defined by (3.2) depends on the defining function u, but its conformal class $[g]$ does not. This gives $M = (\overline{X} \times S^1)/\sim$ the structure of a smooth oriented locally-conformally-flat 4-manifold in a very natural manner. It is moreover easy to see that M is diffeomorphic to $\Sigma \times S^2$, since $\overline{X}$ is diffeomorphic to $\Sigma \times [-1, 1]$, and collapsing each boundary circle of $[-1, 1] \times S^1$ exactly produces a 2-sphere S^2.

Of course, none of this should come as a surprise, since the inclusion

$$\begin{array}{ccc} PSL(2,\mathbb{C}) & \hookrightarrow & PGL(2,\mathbb{H}) \\ \| & & \| \\ SO_+(3,1) & \hookrightarrow & SO_+(5,1) \end{array}$$

exactly allows Γ to act on $S^4 = \mathbb{HP}_1$ in a manner that is free and properly discontinuous outside the limit set $\Lambda = \Lambda(\Gamma) \subset S^2 \subset S^4$. The smooth compact 4-manifold M we constructed above is exactly $(S^4 - \Lambda)/\mathit{\Gamma}$, and the locally-conformally-flat conformal structure $[g]$ with which we endowed it is simply the push-forward of the standard conformal structure on S^4. However, the approach we have just detailed has certain specific virtues; not only does it nicely generalize to yield a construction of more general anti-self-dual 4-manifolds, but it will also lead, in §4 below, to a concrete picture of harmonic 2-forms on $(M, [g])$ in terms of harmonic functions on (X, h).

Before proceeding further, we should notice some other key features of the metrics g defined by (3.2). The vector field $\xi = \partial/\partial t$ is a *Killing field* of g, and generates an isometric action of $S^1 = U(1)$ on (M, g). We can usefully restate this by observing that ξ is a *conformal Killing field* of $(M, [g])$, and that the special metrics $g \in [g]$ given by (3.2) are simply the metrics in the fixed conformal class that are invariant under the induced circle action. Now observe that $\overline{X} = M/S^1$, and that the inverse image P of $X \subset \overline{X}$ is a flat principle circle bundle — namely, the trivial one! We now mildly generalize the construction by instead considering arbitrary flat principal S^1-bundles (P, θ) over X. On such a principal bundle, there is still a vector field ξ that generates the free S^1-action on P, and saying that θ is a connection 1-form just means that it's an S^1-invariant 1-form on P such that $\theta(\xi) \equiv 1$; requiring that such a connection be flat then imposes the condition that $d\theta = 0$, and, since this then says that θ is locally exact, is obviously equivalent to saying that there is a system of local trivializations of P in which $\theta = dt$ and $\xi = \partial/\partial t$. Given any smooth defining function $u : \overline{X} \to [0, \infty)$ for $\partial\overline{X}$, the local arguments we used before now show that we can compactify $(P, u^2[h + \theta^2])$ as a smooth locally-conformally-flat 4-manifold (M, g) by adding a copy of $\partial\overline{X}$. On the other hand, the extra freedom of choosing a flat S^1-connection on X is completely encoded in a monodromy homomorphism $\pi_1(X) \to S^1 = SO(2)$, and since $\pi_1(X) = \pi_1(\Sigma) \cong \mathit{\Gamma}$, the graph of such a homomorphism is a subgroup $\widetilde{\mathit{\Gamma}} \subset SO_+(3, 1) \times SO(2)$ that projects bijectively to the quasi-Fuchsian group $\mathit{\Gamma} \subset SO_+(3, 1)$. Identifying $\widetilde{\mathit{\Gamma}}$ with its image under the natural inclusion $SO_+(3, 1) \times SO(2) \hookrightarrow SO_+(5, 1)$ then allows $\widetilde{\mathit{\Gamma}}$ to act conformally on S^4 with the same limit set $\Lambda \subset S^2 \subset S^4$ as $\mathit{\Gamma}$, and the additional locally-conformally-flat structures on $M \approx \Sigma \times S^2$ introduced above are exactly the ones that then arise as quotients $(S^4 - \Lambda)/\widetilde{\mathit{\Gamma}}$. For all of these conformal structure, $(M, [g])$ comes equipped with an S^1-action generated by a conformal Killing field ξ; moreover, they all have $\overline{X} = M/S^1$, with $\partial\overline{X}$ exactly given by the image of the zero locus of ξ.

We will now describe a generalization of the above ansatz that constructs anti-self-dual 4-manifolds that are not locally conformally flat. Let (X, h) once again be the hyperbolic 3-manifold associated with some quasi-Fuchsian group $\Gamma \subset PSL(2, \mathbb{C})$ of Bers type; and let us emphasize that we take $\mathcal{H}^3$ to have a standard orientation, so that $X = \mathcal{H}^3/\Gamma$ also comes equipped with a preferred orientation from the outset. For some positive integer k, now choose a configuration $\{p_1, p_2, \ldots, p_k\}$ of k distinct points in X. Our objective will be to construct a compact anti-self-dual 4-manifold $(M, [g])$ with a semi-free[2] conformally isometric S^1-action with k isolated fixed points $\{\hat{p}_1, \hat{p}_2 \ldots, \hat{p}_k\}$ and two fixed surfaces Σ_+ and Σ_-, such that $M/S^1 = \overline{X}$, with $\hat{p}_j$ mapping to p_j, and with the $\Sigma_+ \sqcup \Sigma_-$ mapping to $\partial\overline{X}$. The construction will actually require the configuration $\{p_1, p_2, \ldots, p_k\}$ to satisfy a mild constraint, but configurations with this property will turn out to exist for all $k \geq 2$.

The ingredients needed for our construction will include the Green's functions G_{p_j} of the chosen points. By definition, each $G_{p_j} : X - \{p_j\} \to \mathbb{R}^+$ is a positive harmonic function that tends to zero at $\partial\overline{X}$, and solves

$$\Delta G_{p_j} = 2\pi\delta_{p_j} \tag{3.3}$$

in the distributional sense, where $\Delta = -\text{div grad}$ is the (modern geometer's) Laplace-Beltrami operator of the hyperbolic metric h. This Green's function can be constructed explicitly by lifting the problem to $\mathcal{H}^3$, where the inverse image of p_j becomes the orbit Γq_j of an arbitrarily chosen point q_j in the preimage. Superimposing the hyperbolic Green's functions for the points in the orbit then leads one to express the solution as a Poincaré series

$$G_{p_j}(q) = \sum_{\phi \in \Gamma} \frac{1}{e^{2\,\text{dist}(\phi q_j, q)} - 1}, \tag{3.4}$$

where dist denotes the hyperbolic distance in $\mathcal{H}^3$. The fact that this expression converges away from the orbit Γq_j follows from Sullivan's theorem on critical exponents [22] for Poincaré series, because the limit set $\Lambda(\Gamma)$ has Hausdorff dimension < 2. It is then easy to show that the singular function defined by (3.4) solves (3.3) in the distributional sense, and elliptic regularity therefore shows that G_{p_j} is smooth on $X - \{p_j\}$. Moreover, Sullivan's theorem also implies that G_{p_j} extends continuously to the boundary of $\overline{X}$ by zero. Regularity theory for boundary-degenerate elliptic operators [9,

[2] An action is called semi-free if it is free on the complement of its fixed-point set.

Theorem 11.7] then implies that this extension of G_{p_j} is actually smooth on $\overline{X} - \{p_j\}$, and has vanishing normal derivative at $\partial\overline{X}$.

Let us next define a harmonic function $V : X - \{p_1, p_2, \ldots, p_k\} \to \mathbb{R}^+$ by

$$V = 1 + G_{p_1} + G_{p_2} + \cdots + G_{p_k}. \qquad (3.5)$$

Since V satisfies Laplace's equation on the complement of $\{p_1, \ldots, p_k\}$, we have $d \star dV = 0$ in this region, and the 2-form defined there by

$$F = \star dV$$

is therefore closed. Our construction now asks us to find a principal circle bundle $P \to X - \{p_1, p_2, \ldots, p_k\}$ equipped with a connection form θ whose curvature is exactly F. On any contractible region $U \subset X - \{p_1, p_2, \ldots, p_k\}$, this can always be done simply by taking any 1-form ϑ with $d\vartheta = F$, and then setting $\theta = dt + \vartheta$ on $U \times S^1$. However, there is a cohomological obstruction to gluing these local models together consistently; namely, we need $[\frac{1}{2\pi}F]$ to be an integer class in deRham cohomology, because it will ultimately represent the first Chern class $c_1(P) \in H^2(X - \{p_1, p_2, \ldots, p_k\}, \mathbb{Z})$. This motivates the following definition:

Definition 3.1 *If $X = \mathcal{H}^3/\Gamma$ is a quasi-Fuchsian hyperbolic 3-manifold of Bers type, and if $\{p_1, \ldots, p_k\}$ is a configurations of $k \geq 0$ distinct points in X, we will say that $\{p_1, \ldots, p_k\}$ is* quantizable *if $\frac{1}{2\pi} \star dV$ represents an element of $H^2(X - \{p_1, \ldots, p_k\}, \mathbb{Z}) \subset H^2(X - \{p_1, \ldots, p_k\}, \mathbb{R})$ in deRham cohomology.*

This "quantization condition" is equivalent to demanding that $\frac{1}{2\pi}\int_Y F$ be an integer for every smooth compact oriented surface $Y \subset X - \{p_1, p_2, \ldots, p_k\}$ without boundary. However, $H_2(X - \{p_1, p_2, \ldots, p_k\}, \mathbb{Z})$ is in fact generated by k small disjoint 2-spheres $S_1, S_2, \ldots, S_k$ around the k points of the configuration $\{p_1, p_2, \ldots, p_k\}$, together with a single copy of Σ that is homologous to a boundary component Σ_+ of $\partial\overline{X}$ in $\overline{X} - \{p_1, \ldots, p_k\}$. We can therefore check our quantization condition by just evaluating the integral of $F = \star dV$ on these $k + 1$ generators.

In order to evaluate the corresponding integrals, it will often be helpful to pass to the universal cover $\mathcal{H}^3$ of X, where (3.4) then tells us that

$$\star dG_{p_j} = -\frac{1}{2} \sum_{\phi \in \Gamma} \boldsymbol{\phi}^* \alpha;$$

here α denotes the pull-back of the standard area form on the unit 2-sphere S^2 in $T_{q_j}\mathcal{H}^3$ via the radial geodesic projection $(\mathcal{H}^3 - \{q_j\}) \to S^2$, and $\boldsymbol{\phi}^*\alpha$ is the pull-back of this singular form via the action of $\boldsymbol{\phi} \in \mathit{\Gamma}$ on $\mathcal{H}^3$. By representing the sphere S_j by a small 2-sphere around q_j that is contained in a fundamental domain for the action, we see that $\star dG_{p_j}$ restricts to S_j as $-\frac{1}{2}\alpha$ plus an exact form, and that $\star dG_{p_i}$ is exact on S_j for $i \neq j$. We thus have

$$\frac{1}{2\pi}\int_{S_j} F = \frac{1}{2\pi}\int_{S_j} \star dV = \frac{1}{2\pi}\int_{S^2}\left(-\frac{1}{2}\alpha\right) = -1 \in \mathbb{Z} \tag{3.6}$$

for every $j = 1, \ldots, k$, and our quantization condition is therefore automatically satisfied for the homology generators $[S_1], \ldots, [S_k]$.

However, the integral of $\star dG_{p_j}$ on a surface $\Sigma \subset X - \{p_1, \ldots, p_k\}$ homologous to a boundary component Σ_+ of $\partial\overline{X}$ is a bit more complicated. The answer is best understood in terms of a special harmonic function on X that will come to play a starring role in this article:

Definition 3.2 *Let $\mathit{\Gamma} \cong \pi_1(\Sigma)$ be a quasi-Fuchsian group of Bers type, let $X = \mathcal{H}^3/\mathit{\Gamma}$ be the associated hyperbolic 3-manifold, and let $\overline{X} = [D^3 - \Lambda(\mathit{\Gamma})]/\mathit{\Gamma}$ be the associated 3-manifold-with-boundary, where $\partial X = [\mathbb{CP}_1 - \Lambda(\mathit{\Gamma})]/\mathit{\Gamma}$. Let Σ_+ be the component of ∂X on which the boundary orientation agrees with the given orientation of Σ, and let Σ_- be the other component. Then the* tunnel-vision function *of X is defined to be the unique continuous function $f : \overline{X} \to [0, 1]$ which is harmonic on (X, h), equal to 1 on Σ_+, and equal to 0 on Σ_-.*

The motivation for this terminology also motivates the proofs of several of our results. Think of X as a tunnel leading from Σ_- to Σ_+, and imagine that the tunnel mouth Σ_+ leads into bright daylight, while Σ_- leads into darkest night. How big does the bright tunnel opening appear from a point inside the tunnel? For an observer at $p \in X$, this amounts to asking what fraction of the geodesic rays emanating from p end up at Σ_+, where the measure used to determine this fraction is the usual one on the unit 2-sphere in T_pX. We can understand the answer by passing to the universal cover $\mathcal{H}^3$ of X, and letting $q \in \mathcal{H}^3$ be a preimage of p. The sphere at infinity can then be decomposed as a disjoint union $\Lambda \sqcup \Omega_+ \sqcup \Omega_-$, where $\Lambda = \Lambda(\mathit{\Gamma})$ has Lebesgue area zero, and where Ω_+ and Ω_- are the universal covers of Σ_+ and Σ_-, respectively. The question is now equivalent to asking for the fraction of geodesic rays emanating from q that end up at Ω_+. But this fraction is obviously just the average value of the characteristic function of Ω_+, computed with respect to the area measure on the sphere at infinity induced by identifying it with the unit sphere in $T_q\mathcal{H}^3$ via radial projection along geodesics. However, the Poisson integral formula [7, Chapter 5] tells us that, as q varies, this spherical average defines a harmonic function $f : \mathcal{H}^3 \to \mathbb{R}$ that tends to 1 on Ω_+ and 0 on Ω_-. Since f is manifestly Γ-invariant, it must moreover be the pull-back of a harmonic function on X, and since this harmonic function tends to 1 at Σ_+ and to 0 at Σ_-, it must therefore coincide with the tunnel-vision function f. This proves that the apparent area of the image of Σ_+, as seen from p, divided by the total area $4\pi^2$ of the unit 2-sphere, is exactly $f(p) = f(q)$. In other words, the value at p of our tunnel-vision function f is equal to what some analysts would call the harmonic measure $\omega(p, \Sigma_+, X)$ of Σ_+ in the space X in with respect to the reference point p.

This geometric description of the tunnel-vision function f has a flip-side that explains why we have chosen to introduce it at this particular juncture:

Lemma 3.3 *Let $p \in X$, and let $\Sigma \subset X$ be a surface which is homologous to Σ_+ in $\overline{X} - \{p\}$. Then*

$$\int_\Sigma \star dG_p = -2\pi f(p).$$

Proof. First notice that the 2-form $\star dG_p$ is smooth up to the boundary of $\overline{X}$. Indeed, if we use the upper-half-space model to represent the hyperbolic metric as $h = (dx^2 + dy^2 + dz^2)/z^2$ near some boundary point of $\overline{X}$, we then have $\star dG_p = z^{-1} \star\!\!\!\!\raisebox{0pt}{$\cdot$} dG_p$, where ☆ is the Hodge star with respect to the Euclidean metric $dx^2 + dy^2 + dz^2$. The fact that dG_p is smooth up to the boundary and vanishes there thus guarantees that $\star dG_p$ extends smoothly to all of $\overline{X} - \{p\}$.

Since $\star dG_p$ is consequently a smooth closed 2-form on $\overline{X} - \{p\}$, and because Σ is homologous to Σ_+ by hypothesis, Stokes' theorem now immediately tells us that $\int_\Sigma \star dG_p = \int_{\Sigma_+} \star dG_p$. To compute the latter integral, we now remember that the universal cover of Σ_+ is exactly Ω_+. Letting $\varPi \subset \Omega_+$ be a fundamental domain for the action of Γ on Ω_+, our expression (3.4) for the pull-back of G_p to $\mathcal{H}^3$ therefore tells us that

$$\begin{aligned}\int_{\Sigma_+} \star dG_p &= \int_{Д} \star d\left(\sum_{\phi\in\Gamma} \frac{1}{e^{2\,\mathrm{dist}(\phi q_j,q)}-1}\right)\\ &= \sum_{\phi\in\Gamma}\int_{Д} \star d\left(\frac{1}{e^{2\,\mathrm{dist}(\phi q_j,q)}-1}\right)\\ &= -\frac{1}{2}\sum_{\phi\in\Gamma}\int_{Д}\boldsymbol{\phi}^*\alpha\\ &= -\frac{1}{2}\sum_{\phi\in\Gamma}\int_{\phi(Д)}\alpha\\ &= -\frac{1}{2}\int_{\Omega_+}\alpha.\end{aligned}$$

where q is a preimage of p, and α is the pull-back of the area form on the unit sphere $S^2 \subset T_q$ to $\overline{\mathcal{H}^3} - \{q\}$ via geodesic radial projection. However, we have just observed that the Poisson integral formula tells us that $\frac{1}{4\pi}\int_{\Omega_+}\alpha$ is exactly the tunnel-vision function f evaluated at p. It thus follows that

$$\int_\Sigma \star dG_p = \int_{\Sigma_+}\star dG_p = -\frac{1}{2}\int_{\Omega_+}\alpha = -\frac{1}{2}[4\pi f(p)] = -2\pi f(p),$$

exactly as claimed. ∎

Adding up k such contributions now yields a useful corollary:

Lemma 3.4 *Let $\{p_1,\ldots,p_k\}$ be any configuration of k points in X, and let V be the positive potential defined by* (3.5). *If $\Sigma \subset X - \{p_1,\ldots,p_k\}$ is a surface homologous to the boundary component Σ_+ in $\overline{X} - \{p_1,\ldots,p_k\}$, then*

$$\frac{1}{2\pi}\int_\Sigma \star dV = -\sum_{j=1}^{k} f(p_j).$$

Since $f : X \to (0,1)$, it follows that our quantization condition can never be satisfied if $k = 1$. Fortunately, however, this problem does not reoccur for larger values of k:

Proposition 3.5 *Let (X, h) be any quasi-Fuchsian hyperbolic 3-manifold of Bers type. Then for every integer $k \geq 2$, there are quantizable configurations $\{p_1, \ldots, p_k\}$ of k distinct points in X.*

Proof. According to Definition 3.1, the claim just means that there are configurations $\{p_1, \ldots, p_k\}$ of distinct points in $X = \mathcal{H}^3/\Gamma$ for which $\frac{1}{2\pi} \star dV$ represents an element of $H^2(X - \{p_1, \ldots, p_k\}, \mathbb{Z}) \subset H^2(X - \{p_1, \ldots, p_k\}, \mathbb{R})$ in deRham cohomology. Since $H_2(X - \{p_1, \ldots, p_k\})$ is generated by Σ and small 2-spheres $S_1, \ldots, S_k$ about the points $p_1, \ldots, p_k$ of the configuration, we only need to arrange for the integrals of $\star dV$ on these generating surfaces to all be integers. However, since (3.6) shows that the integrals on $S_1, \ldots, S_k$ all equal -1, we only need to worry about the integral on Σ, which equals $-\sum_{j=1}^{k} f(p_j)$ by Lemma 3.4. But because $f : \overline{X} \to [0, 1]$ is continuous and achieves the values 0 and 1 exactly on its two boundary components, and because $\overline{X}$ is connected, every element of the interval $(0, 1)$ must occur as the value of f at some point of X. Moreover, since the restriction of f to (X, h) is harmonic, every such value is attained by uncountably many different points in X; indeed, the mean-value theorem guarantees that $f(p)$ also occurs as a value of f restricted to the sphere of radius ϱ about p, for every ϱ smaller than the injectivity radius of (X, h) at p. If m is any integer from 1 to $k - 1$, we can therefore pick distinct points $p_1, \ldots, p_k \in X$ with $f(p_1) = \cdots = f(p_k) = m/k$, which then ensures that $\frac{1}{2\pi} \int_\Sigma \star dV = -m$. Of course, the same reasoning also shows that this same goal can also be attained by specifying the $f(p_j)$ to be any other k elements in $(0, 1)$ that add up to m. ∎

In fact, the space of quantizable configurations is a real-analytic subvariety of the non-singular part of the k-fold symmetric product $X^{[k]}$, and is locally cut out by the vanishing of a single harmonic function. However, this space is disconnected if $k \geq 3$, since $m = \sum_{j=1}^{k} f(p_j)$ must be an integer for every quantizable configuration, and every integer from 1 to $k - 1$ arises in this manner.

With Proposition 3.5 in hand, we now proceed to construct anti-self-dual metrics on $(\Sigma \times S^2) \# k\overline{\mathbb{CP}}_2$ associated with each quasi-Fuchsian hyperbolic manifold (X, h) and each quantizable configuration of k distinct points $p_1, \ldots p_k$ in X. Indeed, given a quantizable configuration, let V be given by (3.5), set $F = \star dV$, and let $P \to X - \{p_1, \ldots, p_k\}$ be a principal circle bundle with first Chern class $c_1(P) = [\frac{1}{2\pi}F] \in H^2(X - \{p_1, \dots, p_k\}, \mathbb{Z})$. Let θ be a connection 1-form on P with curvature $d\theta = F$, let $u : \overline{X} \to [0, \infty)$ be a non-degenerate defining function for $\partial\overline{X}$, and then equip P with the Riemannian metric

$$g = u^2 \left(Vh + V^{-1}\theta^2 \right) .$$

Because h is hyperbolic, V is harmonic, and $d\theta = \star dV$, this "hyperbolic ansatz" metric is automatically anti-self-dual [14] with respect to a natural orientation of P. Moreover, its metric-space completion is actually a smooth anti-self-dual 4-manifold (M, g), obtained by adding one extra point $\hat{p}_j$ for each point p_j of the configuration, and a pair of surfaces $\Sigma_\pm$ conformal to the two components of $\overline{X}$; this can be proved [11, 14, 15, 16] by explicitly constructing the completion, using local models near $\Sigma_\pm$ and the $\hat{p}_j$. The resulting smooth compact anti-self-dual 4-manifolds (M, g) are then all diffeomorphic to $(\Sigma \times S^2)\#k\overline{\mathbb{CP}}_2$, and carry a conformally isometric semi-free S^1-action that is generated by a conformal Killing field ξ of period 2π. The invariant $m \in \{1, \dots, k-1\}$ of the configuration now becomes an invariant of the S^1-action, because the fixed surfaces Σ_+ and Σ_- of the action now have self-intersection numbers $-m$ and $-k + m$, respectively.

However, it is also worth noting that $(M, [g])$ is not uniquely determined by $(X, \{p_1, \dots, p_k\})$, because the principal-bundle-with-connection (P, θ) is not determined up to gauge-equivalence by its curvature F. Indeed, if Σ has genus $\mathbf{g} \geq 2$, there is a $2\mathbf{g}$-dimensional torus $H^1(\Sigma, \mathbb{R})/H^1(\Sigma, \mathbb{Z})$ of flat S^1-connections on $X \approx \Sigma \times \mathbb{R}$, and this torus then acts freely on S^1-connections over $X - \{p_1, \dots, p_k\}$ without changing their curvatures. This additional freedom in the construction supplements our freedom to choose $(X, h, \{p_1, \dots, p_k\})$, and generalizes the extra choice of a flat S^1-connection we previously encountered in the locally-conformally-flat case.

When Γ is a Fuchsian group, the anti-self-dual conformal class $[g]$ on M can actually be represented [15] by a scalar-flat Kähler metric, obtained by using the specific defining function $u = \sqrt{f(1-f)}$ for $\partial\overline{X}$ as our conformal factor. However, this does not happen for other quasi-Fuchsian groups:

Proposition 3.6 *Let $(M, [g])$ be an anti-self-dual 4-manifold arising from a quasi-Fuchsian group Γ of Bers type and a (possibly empty) quantizable configuration of points in $X = \mathcal{H}^3/\Gamma$ via the hyperbolic-ansatz construction. Then $[g]$ is represented by a global scalar-flat Kähler metric $g \in [g]$ if and only if Γ is a Fuchsian group.*

Proof. When Γ is Fuchsian, it was shown in [15] that the hyperbolic-ansatz construction and an appropriate choice of conformal factor yield a scalar-flat Kähler metric. In particular, because $[g]$ is represented by a global scalar-flat metric in the Fuchsian case, the constructed conformal class has Yamabe constant $Y_{[g]} = 0$. By contrast, however, one can show that $Y_{[g]} < 0$ if Γ is quasi-Fuchsian but not Fuchsian. Indeed, Jongsu Kim [11], generalizing a result of Schoen and Yau [21], showed that the Yamabe constant of an anti-self-dual manifold $(M^4, [g])$ arising from the hyperbolic ansatz is negative whenever the limit set $\Lambda(\Gamma)$ of the corresponding Kleinian group Γ has Hausdorff dimension > 1. The claim therefore follows from a result of Bowen [5] that $\dim_H \Lambda(\Gamma) > 1$ for any a quasi-Fuchsian group of Bers type that is not Fuchsian; cf. [4, 22]. ∎

4 Harmonic Forms and Harmonic Functions

Given a quasi-Fuchsian hyperbolic 3-manifold $X \approx \Sigma \times \mathbb{R}$ of Bers type, we have now seen how to construct anti-self-dual conformal classes $[g]$ on $(\Sigma \times S^2)\#k\overline{\mathbb{CP}}_2$, $k \neq 1$, from quantizable configurations of k points in X. Because these 4-manifolds M all have $b_+ = 1$, each such $(M, [g])$ carries exactly a 1-dimensional space of self-dual harmonic 2-forms; that is, there is a non-trivial self-dual harmonic 2-form ω on any such (M, g), and this form is unique up to multiplication by a non-zero real constant. Our next goal is to translate the question of whether $\omega \neq 0$ everywhere into a question about the quasi-Fuchsian hyperbolic manifold (X, h).

We begin with a local study of the problem. Recall that an open dense set P of M was constructed as a circle bundle $P \to X - \{p_1, \ldots, p_k\}$, equipped with a connection 1-form θ whose curvature $F = d\theta$ is given by $\star dV$, where V is the positive harmonic function on $(X - \{p_1, \ldots, p_k\}, h)$ given by (3.5). We then equipped P with a conformal class that is represented on P by

$$g_0 = Vh + V^{-1}\theta^{\otimes 2} \tag{4.1}$$

although we have until now generally tended to focus on conformally rescalings $g = u^2 g_0$ that were chosen so as to extend to the compact manifold M. We emphasize that (P, g_0) carries an isometric S^1-action that is generated by a Killing field ξ that satisfies $\theta(\xi) \equiv 1$.

Now suppose that ω is a self-dual 2-form on P which is invariant under the fixed isometric S^1-action on P. Here, we fix our orientation conventions so that, if e^1, e^2, e^3 is an oriented orthonormal co-frame on (X, h), then $V^{1/2}e^1, V^{1/2}e^2, V^{1/2}e^3, V^{-1/2}\theta$ is an oriented orthonormal co-frame with respect to g_0. It follows that

$$e^1 \wedge \theta + Ve^2 \wedge e^3$$

is a self-dual 2-form on (P, g_0) of point-wise norm $\sqrt{2}$, and, since $SO(3) \subset SO(4)$ acts transitively on the unit sphere in Λ^+, this implies that *any* ξ-invariant self-dual 2-form field of norm $\sqrt{2}$ can locally be expressed in this way by choosing an appropriate oriented orthonormal frame on X. It therefore follows that any ξ-invariant self-dual 2-form on P can be uniquely written as

$$\omega = \psi \wedge \theta + V \star \psi \tag{4.2}$$

for a unique 1-form ψ on $X - \{p_1, \ldots, p_k\}$.

Let us now also suppose that the self-dual 2-form ω is also *closed*; of course, since $\omega = *\omega$, where $*$ denotes the 4-dimensional Hodge star, this then implies that ω is co-closed, and hence harmonic. Since ω is invariant under the flow of ξ, Cartan's magic formula for the Lie derivative of a differential form therefore tells us that

$$0 = \mathcal{L}_\xi \omega = \xi \lrcorner\, d\omega + d(\xi \lrcorner\, \omega) = -d\psi$$

so that ψ must be a closed 1-form on $X - \{p_1, \ldots, p_k\}$. However, if we let μ_h denote the volume 3-form of (X, h), we also have

$$\begin{aligned} 0 &= d\omega \\ &= -\psi \wedge d\theta + d(V \star \psi) \\ &= -\psi \wedge \star dV + dV \wedge \star\psi + V(d \star \psi) \\ &= -\langle \psi, dV \rangle \mu_h + \langle dV, \psi \rangle \mu_h - V(d \star \psi) \\ &= -V(d \star \psi) \end{aligned}$$

and we therefore conclude that the 1-form ψ is *strongly harmonic*, in the sense that

$$d\psi = 0, \qquad d \star \psi = 0.$$

Conversely, if ψ is any strongly harmonic 1-form on $X - \{p_1, \ldots, p_k\}$, the 2-form ω defined on (P, g) by (4.2) is closed and self-dual, and hence harmonic.

We note in passing that this argument does not depend on the fact that h has constant curvature or that (P, g_0) is anti-self-dual; one just needs g_0 to be expressed as (4.1) for some metric h, some positive harmonic function V, and some connection 1-form θ on P with curvature $F = \star dV$.

Notice that the relationship between ω and ψ codified by (4.2) entails a simple relationship between the point-wise norms of these forms. Indeed, notice that

$$\omega \wedge \omega = 2\psi \wedge \theta \wedge V \star \psi = 2V(\psi \wedge \star\psi) \wedge \theta = 2V|\psi|_h^2 \ \mu_h \wedge \theta \ .$$

On the other hand, since g_0 has volume form

$$\mu_{g_0} = V^{1/2}e^1 \wedge V^{1/2}e^2 \wedge V^{1/2}e^3 \wedge V^{-1/2}\theta = V\mu_h \wedge \theta$$

the self-duality of ω thus implies that

$$|\omega|_{g_0}^2 \mu_{g_0} = \omega \wedge \omega = 2|\psi|_h^2 \mu_{g_0}$$

and hence that

$$|\omega|_g = \sqrt{2}|\psi|_h. \tag{4.3}$$

This will allow us to invoke the following removable singularities result:

Lemma 4.1 *Let p be a point of a smooth, oriented Riemannian n-manifold $(Y, \mathbf{g})$, $n \geq 2$, and let φ be a differential ℓ-form on $Y - \{p\}$ that is* strongly harmonic, *in the sense that $d\varphi = 0$ and $d \star \varphi = 0$. Also suppose that φ is bounded near p, in the sense that there is a neighborhood U of p and a positive constant C such that the point-wise norm $|\varphi|$ satisfies $|\varphi| < C$ on $U - \{p\}$. Then φ extends uniquely to Y as a smooth strongly harmonic ℓ-form.*

Proof. Let α be any smooth, compactly supported $(n - \ell - 1)$-form on Y. Letting B_ϵ denote the ϵ-ball around p for any small ϵ, and setting $S_\epsilon = \partial B_\epsilon$, we then have

$$\int_Y \varphi \wedge d\alpha = \lim_{\epsilon \to 0} \int_{Y - B_\epsilon} \varphi \wedge d\alpha = \pm \lim_{\epsilon \to 0} \int_{Y - B_\epsilon} d(\varphi \wedge \alpha) = \mp \lim_{\epsilon \to 0} \int_{S_\epsilon} \varphi \wedge \alpha = 0,$$

because $\varphi \wedge \alpha$ is bounded, and the area of S_ϵ tends to zero as $\epsilon \to 0$. Hence the L^∞ form φ satisfies $d\varphi = 0$ in the sense of currents. Similarly, if β is any smooth, compactly supported $(\ell - 1)$-form on Y, then

$$\int_Y \varphi \wedge \star d\beta = \lim_{\epsilon \to 0} \int_{Y - B_\epsilon} (d\beta) \wedge \star\varphi = \lim_{\epsilon \to 0} \int_{Y - B_\epsilon} d(\beta \wedge \star\varphi) = -\lim_{\epsilon \to 0} \int_{S_\epsilon} \beta \wedge \star\varphi = 0,$$

so that $d(\star\varphi) = 0$ in the sense of currents, too. Thus $\Delta\varphi = 0$ in the distributional sense, and elliptic regularity then guarantees that φ is a smooth ℓ-form on Y. Since φ is both closed and co-closed on the open dense subset $Y - \{p\}$, it therefore follows by continuity that its extension is also closed and co-closed on all of Y. ∎

We now restrict our attention to the problem at hand. Let $(M, [g])$ be a smooth compact 4-manifold produced from a quasi-Fuschsian hyperbolic 3-manifold (X, h) and a (possibly empty) quantizable configuration of points $\{p_1, \ldots, p_k\}$ by the hyperbolic-ansatz construction. Thus, $(M, [g])$ comes equipped with a conformally isometric S^1-action such that $\overline{X} = M/S^1$, and such that $P \to X - \{p_1, \ldots, p_k\}$ is the union of the free S^1-orbits. We can thus represent $[g]$ by a smooth metric of the form $g = \mathfrak{u}^2 g_0$, where g_0 is given by (4.1), and where $\mathfrak{u} : \overline{X} \to [0, \infty)$ is a smooth non-degenerate defining function for $\partial\overline{X} = \Sigma_- \sqcup \Sigma_+$. Let ω be a non-trivial self-dual 2-form on $(M, [g])$. In particular, the restriction of ω to the dense subset P is also non-trivial, and our previous calculations then show that $\psi = -\xi \lrcorner \omega$ therefore defines a strongly harmonic 1-form on $(X - \{p_1, \ldots, p_k\}, h)$. However, $|\omega|_g$ is bounded on M by compactness, so $|\omega|_{g_0} = \mathfrak{u}^2|\omega|_g$ is uniformly bounded on P. Equation (4.3) therefore tells us that $|\psi|_h = |\omega|_{g_0}/\sqrt{2}$ is uniformly bounded on $X - \{p_1, \ldots p_k\}$, and ψ consequently extends to all of X as a strongly harmonic 1-form by Lemma 4.1.

However, even more is true. Notice that we can define a smooth Riemannian metric on $\overline{X}$ by $\bar{h} := \mathfrak{u}^2 h$, and we then have $|\psi|_{\bar{h}} = \mathfrak{u}^{-1}|\psi|_h = \mathfrak{u}^{-1}|\omega|_{g_0}/\sqrt{2} = \mathfrak{u}|\omega|_g/\sqrt{2}$. This shows that ψ has a continuous extension to the boundary of $\overline{X}$ by zero. Now notice that any loop in $X \approx \Sigma \times (0, 1)$ is freely homotopic to a loop that is arbitrarily close to $\partial\overline{X}$, and on which the integral of ψ is therefore as small as we like. But ψ is closed, and its integral on a loop is therefore invariant under free homotopy. This shows that the integral of ψ on any loop must vanish, and that $[\psi] \in H^1(X, \mathbb{R})$ therefore vanishes. Thus ψ is exact, and we therefore have $\psi = df$ for some smooth function on X. Moreover, since $d \star \psi = 0$, we have $\Delta f = - \star d \star df = 0$, so f is therefore a harmonic function on (X, h). Since we can, moreover, explicitly construct f from ψ by integration along paths from a base-point, and since $\psi \to 0$ at ∂X, this harmonic function on X tends to a constant on each boundary component, and, since $df = \psi \not\equiv 0$, the values on the two boundary components must moreover be different by the maximum principle.

By adding a constant if necessary, we can now arrange for f to tend to zero at Σ_-, and, at the price of perhaps replacing ω with a constant multiple, we can then also arrange for f to tend to 1 along Σ_+. This means that we have arranged for f to exactly be the tunnel-vision function f of Definition 3.2. This proves the following result:

Proposition 4.2 *Let $(M, [g])$ be an anti-self-dual 4-manifold arising via the hyperbolic ansatz from a quasi-Fuchsian hyperbolic 3-manifold (X, h) of Bers type and a (possibly empty) quantizable configuration $\{p_1, \ldots, p_k\}$ of points in X. Then any self-dual harmonic form on $(M, [g])$ restricts to the open dense subset $P \subset M$ as a constant multiple of*

$$\omega := df \wedge \theta + V \star df,$$

where f is the tunnel-vision function of the quasi-Fuchsian hyperbolic manifold (X, h), V is the potential assigned to the configuration $\{p_1, \ldots, p_k\}$ by (3.5)*, and θ is the connection 1-form with $d\theta = \star dV$ used to construct $[g]$ via* (3.2)*.*

To fully exploit this observation, we will need one other key fact about the tunnel-vision function:

Lemma 4.3 *Let (X, h) be a quasi-Fuchsian hyperbolic 3-manifold of Bers type, and let $f : \overline{X} \to [0, 1]$ be its tunnel-vision function. Then at every point of $\partial\overline{X}$, the first normal derivative of f is zero, but the second normal derivative of f is non-zero.*

Proof. Recall that $\partial\overline{X} = \Sigma_+ \sqcup \Sigma_-$. It will suffice show that near any point of Σ_- we can find local non-degenerate local defining functions u and $\tilde{u}$ for $\partial\overline{X}$ such that $u^2 \leq f \leq \tilde{u}^2$ near the given point, while near any point of Σ_+ we can similarly find local non-degenerate local defining functions u and $\tilde{u}$ such that $u^2 \leq 1 - f \leq \tilde{u}^2$ near the given point.

Let us first see what happens in the Fuchsian case. Here, the decomposition $\mathbb{CP}_1 = \Omega_+ \sqcup \Omega_- \sqcup \Lambda$ is just the decomposition of the sphere at infinity into a geometric circle and two geometric open disks. In the upper half-space model, we can thus take Ω_+ and Ω_- to be the halves of the xy-plane respectively given by $y > 0$ and $y < 0$. In this prototypical situation, the tunnel-vision function just pulls back to become

$$f_0 = \frac{1}{2}\left(\frac{y}{\sqrt{y^2 + z^2}} + 1\right)$$

which is harmonic on $z > 0$ with respect to $h = (dx^2 + dy^2 + dz^2)/z^2$, equals 1 when $y > 0$ and $z = 0$, and equals 0 when $y < 0$ and $z = 0$. Now notice that, when z is small, $f_0 = (z/2y)^2 + O((z/y)^3)$ when $y < 0$, while $1 - f_0 = (z/2y)^2 + O((z/y)^3)$ when $y > 0$. It is therefore easy to see that $\sqrt{f_0}$ and $\sqrt{1 - f_0}$ are themselves smooth non-degenerate defining functions for these boundary half-planes, and we are thus free to take $u = \tilde{u}$ to be these defining functions to emphasize that the claim is certainly true in this prototypical case.

Now, in the general quasi-Fuchsian case, we again have $\mathbb{CP}_1 = \Omega_+ \sqcup \Omega_- \sqcup \Lambda$, but Λ will just be a quasi-circle, and the open sets $\Omega_\pm$ could be dauntingly complicated. However, if p is any point of Σ_+, we can still represent it in the universal cover by some $q \in \Omega_+$, and, since Ω_+ is open in $\mathbb{CP}_1$, we may choose some closed geometric disk D_+ such that $y \in D_+ \subset \Omega_+$; moreover, since Ω_- is also open, we can also choose a second closed disk $D_- \subset \mathbb{CP}_1$ such that $\Omega_+ \subset D_-$ by taking $\mathbb{CP}_1 - D_-$ to be a small open disk around some $\tilde{q} \in \Omega_-$. Now let $f_\pm$ be the harmonic functions on $\mathcal{H}^3$ whose values at p are the average values of the characteristic functions of $D_\pm$ with respect to the visual measure at p. We then immediately have

$$f_+ \leq f \leq f_- \tag{4.4}$$

everywhere, because $D_+ \subset \Omega_+ \subset D_-$. On the other hand, our discussion of the Fuchsian case shows that $u = \sqrt{1 - f_-}$ and $\tilde{u} = \sqrt{1 - f_+}$ are non-degenerate defining functions for $S^2 = \partial\overline{\mathcal{H}^3}$ near q, and our last inequality then becomes

$$u^2 \leq 1 - f \leq \tilde{u}^2$$

as desired. On the other hand, if we instead take $\tilde{q} \in \Omega_-$ to represent a given point $\tilde{p} \in \Sigma_-$, our Fuchsian discussion shows that $u = \sqrt{f_+}$ and $\tilde{u} = \sqrt{f_-}$ are non-degenerate defining functions for $S^2 = \partial\overline{\mathcal{H}^3}$ near $\tilde{q}$, and we then have

$$u^2 \leq f \leq \tilde{u}^2,$$

exactly as required. This shows that f has vanishing first normal derivative but non-zero second normal derivative at every point of $\partial\overline{X}$, as claimed. ∎

This now allows us to prove the main result of this section.

Theorem 4.4 *Let $(M, [g])$ be an anti-self-dual 4-manifold arising via the hyperbolic ansatz from a quasi-Fuchsian 3-manifold (X, h) of Bers type and* *a (possibly empty) quantizable configuration of points in X. Then the anti-self-dual conformal class $[g]$ contains an almost-Kähler metric $g \in [g]$ if and only if the tunnel-vision function $f : X \to (0,1)$ has no critical points.*

Proof. The conformal class $[g]$ contains an almost-Kähler representative iff the non-trivial self-dual harmonic form ω satisfies $\omega \neq 0$ everywhere. We have just shown that, possibly after multiplying ω by a non-zero constant, we may assume that it is associated with the 1-form $\psi = df$ on $\overline{X}$. Since this means that $\sqrt{2}|df|_{\bar{h}} = u|\omega|_g$ with respect to any S^1-invariant metric in the conformal class $[g]$, a necessary condition for ω to be everywhere non-zero is that we must have $df \neq 0$ away from $\partial\overline{X}$. Conversely, if f has no critical points on X, the same calculation implies that ω must be non-zero away from the surfaces Σ_+ and Σ_-. On the other hand, regardless of the detailed behavior of f, Lemma 4.3 shows that $u^{-1}|df|_{\bar{h}}$ always has non-zero limit at every point of $\partial\overline{X}$, so we *always* have $\omega \neq 0$ at every point on the surfaces $\Sigma_\pm$, and it therefore follows that $\omega \neq 0$ on all of M unless the tunnel-vision function f has a critical point somewhere in $X \subset \overline{X}$. ∎

While this article will use Theorem 4.4 to construct anti-self-dual deformations of scalar-flat Kähler manifolds that are not conformally almost-Kähler, one could in principle use this same result to instead construct specific examples that *are* conformally almost-Kähler. For example, it would be interesting to thoroughly understand the tunnel-vision functions of the *nearly Fuchsian* 3-manifolds X first studied by Uhlenbeck [24]. By definition, these are the quasi-Fuchsian 3-manifolds of Bers type that contain a compact minimal surface $\Sigma \subset X$ of Gauss curvature > -2. Is the tunnel-vision function of a nearly Fuchsian manifold always free of critical points? If so, then Theorem 4.4 would imply that the anti-self-dual 4-manifolds arising from these special quasi-Fuchsian manifolds are always conformally almost-Kähler.

5 Tunnel-Vision Critical Points

Theorems A and B will now follow from Theorem 4.4 if we can produce an appropriate sequence of quasi-Fuchsian hyperbolic 3-manifolds (X,h) whose tunnel-vision functions $f : X \to (0,1)$ have critical points. The first step is to show that any Jordan curve can be approximated by the limit set of a suitable quasi-Fuchsian group $\mathit{\Gamma}$. Our proof is based on the measurable

Riemann mapping theorem in this section, even though a more elementary and constructive proof can be given using concrete reflection groups. This lemma is sometimes attributed to Sullivan and Thurston [23], who used the idea to construct 4-manifolds with unusual affine structures.

In what follows, we will work in the upper-half-space model of $\mathcal{H}^3$, so that $\mathbb{C} = \mathbb{R}^2$ will represent the complement of a point in the sphere at infinity, even though, as a matter of convention, we will find it convenient to endow it with its usual Euclidean metric. The latter will in particular allow us to speak of the the Hausdorff distance between two compact subsets, meaning by definition the infimum of all $\epsilon > 0$ so that each set is contained in the ϵ-neighborhood of the other.

Lemma 5.1 *For any piecewise smooth Jordan curve $\gamma \subset \mathbb{C}$ and any $\varepsilon > 0$, there is a positive integer N such that, for every compact oriented surface Σ of genus $\mathscr{g} \geq N$, there is quasi-Fuchsian group $\Gamma \cong \pi_1(\Sigma)$ of Bers type whose limit set $\Lambda(\Gamma) \subset \mathbb{C} \subset \mathbb{CP}_1$ is a quasi-circle whose Hausdorff distance from γ is less than ε. Moreover, if γ is invariant under $\zeta \mapsto -\zeta$, and if $\mathscr{g}$ is even, we can arrange for $\Lambda(\Gamma)$ to also be invariant under reflection through the origin.*

Proof. Let $\Psi : \mathbb{C} \to \mathbb{C}$ be a diffeomorphism that is holomorphic outside the unit disk and maps the unit circle $\mathbb{T} = \{|\zeta| = 1\}$ to γ. The diffeomorphism $\Phi : \mathbb{C} \to \mathbb{C}$ defined by $\Phi(\zeta) = \Psi(\zeta/[1-\epsilon])$ is then holomorphic outside the disk $D = \{|\zeta| < 1-\epsilon\}$ and maps the unit circle to an approximation $\gamma_\epsilon = \Phi(\mathbb{T})$ of γ whose Hausdoff distance from γ may be taken to be smaller than $\varepsilon/2$ by choosing ϵ to be sufficiently small.

Given any $\delta \in (0, \epsilon)$, we now construct some $\mathbb{T}$-preserving Fuchsian groups with fundamental domains containing the disk $\{|\zeta| < 1-\delta\}$. To this end, endow the open unit disk in $\mathbb{C}$ with the hyperbolic metric $4|d\zeta|^2/(1-|\zeta|^2)^2$, and, for an arbitrary positive integer $\mathscr{g} \geq 2$, let $\mathscr{P}$ be the regular hyperbolic $4\mathscr{g}$-gon whose vertices are all equidistant from 0, and whose interior angles at these vertices are all equal to $\pi/2\mathscr{g}$. By drawing geodesic segments from 0 to the $4\mathscr{g}$ vertices and the $4\mathscr{g}$ midpoints of the sides of $\mathscr{P}$, we can then dissect $\mathscr{P}$ into $8\mathscr{g}$ hyperbolic isosceles right triangles, with interior angles $(\pi/2, \pi/4\mathscr{g}, \pi/4\mathscr{g})$. Now label the oriented edges of $\mathscr{P}$ as a_1, b_1, a_1^{-1}, b_1^{-1}, $\ldots$, $a_{\mathscr{g}}$, $b_{\mathscr{g}}$, ${a_{\mathscr{g}}}^{-1}$, ${b_{\mathscr{g}}}^{-1}$, starting at some reference point and proceeding counter-clockwise. We can then construct a genus-$\mathscr{g}$ hyperbolic surface Σ by identifying the edges of $\mathscr{P}$ in pairs according to this labeling scheme. The universal cover of Σ then becomes the open unit disk, and the fundamental group $\pi_1(\Sigma)$ is then represented as a $\mathbb{T}$-preserving Fuchsian group Γ_g with fundamental domain $\mathscr{P}$, where the relevant deck transformations form a finite-index subgroup of the $(2, 4g, 4g)$ triangle group that is generated by reflections through the sides of the isosceles right triangles into which we dissected $\mathscr{P}$. Now let я denote the hyperbolic distance from 0 to the midpoint of a side of $\mathscr{P}$, and let R denote the hyperbolic distance from 0 to a vertex of $\mathscr{P}$. Then $\mathscr{P}$ contains the disk of hyperbolic radius я about 0, and is contained in the disk of hyperbolic radius R with the same center. Moreover, since our isosceles right triangles have sides я, я, and R, we have $\mathrm{R} < 2\text{я}$ by the triangle inequality. Since the hyperbolic area of $\mathscr{P}$ is $4\pi(g-1)$, and since $\mathscr{P}$ is contained in a disk of hyperbolic radius R, with hyperbolic area $2\pi(\cosh \mathrm{R} - 1) < \pi e^{\mathrm{R}}$, it thus follows that $4(g-1) < e^{\mathrm{R}} < e^{2\text{я}}$. On the other hand, if a is the Euclidean radius of the disk of hyperbolic radius я, we have $\text{я} = \log(1+a)/(1-a)$. Thus

$$\frac{2}{1-a} > \frac{1+a}{1-a} > 2\sqrt{g-1}$$

and hence

$$a > 1 - \frac{1}{\sqrt{g-1}}.$$

This shows that if

$$g \geq N(\delta) := 1 + \lceil \frac{1}{\delta^2} \rceil,$$

our fundamental domain $\mathscr{P}$ for the Fuchsian group $\Gamma_g \cong \pi_1(\Sigma)$ will contain the disk of Euclidean radius $1-\delta$ about 0.

Now let $\mu = \Phi_{\bar{\zeta}}/\Phi_\zeta$ be the complex dilatation of Φ, and notice that μ is supported in the fundamental domain $\mathscr{P}$ of Γ_g provided that $g \geq N(\delta)$. With this proviso, we can then extend μ uniquely as a bounded measurable function μ_δ on $\mathbb{C}$ which is Γ_g-invariant, supported in the unit disk, and has L^∞ norm < 1. The Measurable Riemann Mapping Theorem [1] therefore guarantees the existence of a quasiconformal map Φ_δ with dilatation equal to μ_δ, and the Γ_g-invariance of μ_δ then guarantees that $\mathit{\Gamma}_\delta = \Phi_\delta \circ \Gamma_g \circ \Phi_\delta^{-1}$ is a quasi-Fuchsian group of Bers type. However, the region where $\mu \neq \mu_\delta$ is contained in the annulus $\{\zeta \,|\, 1-\delta < |\zeta| < 1\}$, so $\Phi_\delta \to \Phi$ on compact sets as $\delta \to 0$, and the limit set $\Lambda_\delta = \Phi_\delta(\mathbb{T})$ therefore converges to $\Phi(\mathbb{T}) = \gamma_\epsilon$ in the Hausdorff metric as we decrease δ. By choosing δ sufficiently small we thus can arrange for Λ_δ to be within Hausdorff distance $\varepsilon/2$ of γ_ϵ, and hence within Hausdorff distance ε of γ, as desired.

If γ is invariant under reflection through the origin, then Ψ and Φ can be chosen to share this symmetry, and the dilatation μ then consequently be reflection-invariant, too. Now the line joining the origin to our reference point separates the sides of $\mathcal{P}$ into two counter-clockwise lists of $2g$ sides. If g is even, the number of entries on each of these lists is divisible by 4, and since our listing of the sides as $\ldots, a_j, b_j, a_j^{-1}, b_j^{-1}, \ldots$ breaks them into quadruples, our rules for identifying the sides do not mix the two lists. Since reflection through the origin is the same as a $180°$ rotation, this involution compatibly intertwines with our rules for identifying the sides of $\mathcal{P}$ to obtain Σ, and so induces an orientation-preserving isometry of the hyperbolic surface Σ with two fix points — namely, the origin and the equivalence class consisting of all the vertices of $\mathcal{P}$. It follows that Γ_g and reflection through the origin generate a group extension

$$1 \to \Gamma_g \to \widehat{\Gamma}_g \to \mathbb{Z}_2 \to 0,$$

where $\widehat{\Gamma}_g$ is a larger Fuchsian group, but is no longer torsion-free. Notice each of the halves into which $\mathcal{P}$ is divided by our reference line is now a fundamental domain for the action of $\widehat{\Gamma}_g$ on the unit disk. The reflection symmetry of μ thus guarantees that its Γ_g-invariant extension $\tilde{\mu}$ of μ is actually $\widehat{\Gamma}_g$-invariant, so that $\widehat{\mathit{\Gamma}}_\delta = \Phi_\delta \circ \Gamma_g \circ \Phi_\delta^{-1}$ defines a larger quasi-Fuchsian group. However, Λ_δ is also the limit set of the group $\widehat{\mathit{\Gamma}}_\delta$. On the other hand, $\widehat{\mathit{\Gamma}}_\delta$ is exactly the group generated by $\mathit{\Gamma}_\delta$ and reflection through the origin. Thus, in this situation, all of the constructed limit sets Λ_δ approximating γ_ϵ are reflection-invariant, and all our claims have therefore been proven. $\blacksquare$

When $\mathit{\Gamma}$ is a quasi-Fuchsian group of Bers type, recall that we have defined the tunnel-vision function f of $X = \mathcal{H}^3/\mathit{\Gamma}$ to be the unique harmonic function on X which tends to 0 at one component Σ_- of $\partial\overline{X}$ and tends to 1 at the other component Σ_+ of $\partial\overline{X}$. The limit set $\Lambda = \Lambda(\mathit{\Gamma})$ then divides $\mathbb{CP}_1$ into two connected components Ω_- and Ω_+, which may be respectively identified with the universal covers of Σ_- and Σ_+ and the pull-back $\widetilde{f}$ of f to $\mathcal{H}^3$ then tends to 0 at Ω_- and tends to 1 at Ω_+. This means that we can reconstruct $\widetilde{f}$ from the open set Ω_+ using the *Poisson kernel* of $\mathcal{H}^3$. This 2-form on the sphere at infinity, depending on a point in hyperbolic 3-space, is explicitly given in the upper half-space model by $P_{(x,y,z)}d\xi \wedge d\eta$, where

$$P_{(x,y,z)}(\xi, \eta) = \frac{1}{\pi}\left[\frac{z}{(x-\xi)^2 + (y-\eta)^2 + z^2}\right]^2. \tag{5.1}$$

Note that the factor of $1/\pi$ is inserted here to make the form have total mass 1. Up to a constant factor, $P_{(x,y,z)} d\xi \wedge d\eta$ is just the "visual area form" obtained by identifying the unit sphere in $T_{(x,y,z)}\mathcal{H}^3$ with the sphere at infinity by following geodesic rays starting at (x, y, z). Given a bounded measurable function $F(\xi, \eta)$ on the sphere at infinity, we can uniquely extend it to $\mathcal{H}^3$ as a bounded harmonic function f via the hyperbolic-space version of the *Poisson integral formula*

$$f(x, y, z) = \int_{\mathbb{R}^2} F(\xi, \eta) P_{(x,y,z)}(\xi, \eta)\, d\xi \wedge d\eta; \tag{5.2}$$

for more details, see [7, Chapter 5]. For us, the importance of this formula stems from the fact that when $F = \chi_{\Omega_+}$ is the indicator function of Ω_+, the resulting f is exactly the pull-back $\widetilde{f}$ of the tunnel-vision function of $X = \mathcal{H}^3/\Gamma$.

In the proof that follows, we will never use the precise formula (5.1) for the Poisson kernel, but will instead just make use of the following qualitative properties of $P_{(x,y,z)}(\xi, \eta)$:

(I) For any $\lambda > 0$,

$$P_{(0,0,\lambda)}(\xi, \eta) = \lambda^{-2} P_{(0,0,1)}(\lambda^{-1}\xi, \lambda^{-1}\eta),$$

so that the Poisson kernel scales under dilations to preserve its mass;

(II) On any compact disk $D_\varrho = \{\xi^2 + \eta^2 \leq \varrho^2\}$, the Poisson kernel $P_{(0,0,1)}(\xi, \eta)$ is bigger than a positive constant, depending only on ϱ;

(III) $P_{(0,0,1)}(\xi, \eta) \leq 1$ on the whole plane; and

(IV) $\int_{\mathbb{R}^2 \setminus D_\varrho} P_{(0,0,1)}(\xi, \eta)\, d\xi \wedge d\eta \longrightarrow 0$ as $\varrho \to \infty$.

These facts can all be read off immediately from (5.1), although some readers might instead prefer to deduce them from the Poisson kernel's geometric interpretation in terms of visual measure.

Given a quasi-Fuchsian group Γ of Bers type, we will now choose our upper-half-space model of $\mathcal{H}^3$ so that the the point at infinity belongs to $\Omega_- = \Omega_-(\Gamma)$. Thus, the limit set $\Lambda = \Lambda(\Gamma)$ will always be presented as a Jordan curve in $\mathbb{R}^2 = \mathbb{C}$, and $\Omega_+ = \Omega_+(\Gamma)$ will always be the bounded component of the complement of $\Lambda \subset \mathbb{C}$. Given a measurable set E subset the plane, we will let f_E denote the function f produced by (5.2) when $F = \chi_E$ is the indicator function of E. In particular, f_{Ω_+} is exactly the pull-back $\widetilde{f} : \mathcal{H}^3 \to (0,1)$ of the tunnel-vision function f of $X = \mathcal{H}^3/\Gamma$.

Theorem 5.2 *There is an integer N such that, for every compact oriented surface Σ of even genus $g \geq N$, there is a quasi-Fuchsian group $\Gamma \cong \pi_1(\Sigma)$ for which the corresponding tunnel-vision function f has at least two critical points.*

Proof. Given $\epsilon > 0$, set $\varepsilon = \epsilon^3$, and let Γ_ε be any quasi-Fuchsian group of Bers type whose limit set Λ_ε is invariant under $\zeta \mapsto -\zeta$ and lies within Hausdorff distance ε of the boundary of the "dogbone" domain

$$\Omega_\epsilon = \{z : |z-1| < \frac{1}{4}\} \cup \{z : |z+1| < \frac{1}{4}\} \cup \{z : |z| < 1, |\mathrm{Im}(z)| < \epsilon^3\}.$$

illustrated by Figure 1. By Lemma 5.1, such Γ_ε exist for all even genera $g \geq N$ for some N depending on ε, and hence on ϵ, but we will not choose a specific ϵ until much later, so our notation will help remind us of this point.

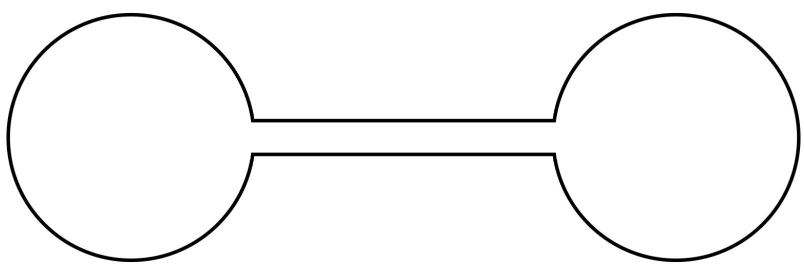

Figure 1: A "dogbone" domain. The harmonic measure of the interior defines a harmonic function in the upper half-space that has at least two critical points. This phenomenon persists when the boundary curve is approximated by quasi-Fuchsian limit sets, using Lemma 5.1.

As before, we work in the upper-half-space model $\mathbb{R}^2 \times \mathbb{R}^+$ of $\mathcal{H}^3$, with coordinates (x, y, z), and follow the convention is that $\Omega_{+,\varepsilon} := \Omega_+(\Gamma_\varepsilon)$ is always taken to be the region *inside* the Jordan curve $\Lambda_\varepsilon = \Lambda(\Gamma_\varepsilon)$. By Lemma 5.1, the limit set Λ_ε can always be chosen to be symmetric with respect to reflection though the origin in the $\zeta = \xi + i\eta$ plane, and this symmetry then implies that the corresponding harmonic function $f_\epsilon := f_{\Omega_{+,\varepsilon}}$

is then invariant under the isometry of $\mathcal{H}^3$ represented in our upper-half-space model by reflection $(x, y, z) \mapsto (-x, -y, z)$ through the z-axis. Since this means that the gradient $\nabla f_\epsilon = h^{-1}(df_\epsilon, \cdot)$ of f_ϵ with respect to the hyperbolic metic h is also invariant under this isometry, it follows that, along the hyperbolic geodesic represented by the positive z-axis, the gradient ∇f_ϵ must be everywhere tangent to the axis. Thus, to show that f_ϵ has a critical point on the positive z-axis, it suffices to show that its restriction to the z-axis is neither monotonically increasing nor decreasing. However, the maximum principle guarantees that $0 < f_\epsilon < 1$ on all of $\mathcal{H}^3$, and we also know that

$$\begin{aligned} \lim_{z \searrow 0} f_\epsilon(0,0,z) &= 1, \\ \lim_{z \nearrow \infty} f_\epsilon(0,0,z) &= 0, \end{aligned}$$

since by construction $0 \in \Omega_{+,\varepsilon}$ and $\infty \in \Omega_{-,\varepsilon}$. We thus just need to show that f_ϵ is not monotonically decreasing along the positive z-axis.

We will do this by showing that

$$f_\epsilon(0,0,\epsilon) < f_\epsilon(0,0,1) \tag{5.3}$$

whenever ϵ is sufficiently small. To see this, first observe that property (II) of the Poisson kernel implies that $f_\epsilon(0,0,1)$ is bigger than a positive constant independent of ϵ, since, for $\epsilon < 1/2$, the region $\Omega_{+,\varepsilon}$ is contained in the disk of radius $3/2$ about $\zeta = 0$, and contains a pair of disks of Euclidean radius $1/8$. On the other hand, if we dilate the upper half-space by a factor of $1/\epsilon$, thereby mapping $(0,0,\epsilon)$ to $(0,0,1)$, property (I) tells us that we can calculate $f_\epsilon(0,0,\epsilon)$ by instead calculating the Poisson integral at $(0,0,1)$ while replacing $\Omega_{+,\varepsilon}$ with its dilated image. However, the dilated copy of $\Omega_{+,\varepsilon}$ meets the disk of radius $1/\epsilon$ in a region of Euclidean area $< 4\epsilon$, so property (III) guarantees that the contribution of this large disk to the integral is $O(\epsilon)$; meanwhile, property (IV) guarantees that the the contribution of the exterior of this increasingly large disk tends to zero as $\epsilon \searrow 0$. This establishes that (5.3) holds for all small ϵ, since $f_\epsilon(0,0,1)$ is bounded away from zero. Moreover, this argument shows that this holds for a specific small ϵ that is independent of the detailed geometry of the approximating limit sets Λ_ε, just as long as they are within Hausdorff distance $\varepsilon = \epsilon^3$ of the "dogbone" curve specified by the parameter ϵ. Thus, there is some specific small ϵ such that the restriction of the pulled-back tunnel-vision function $\widetilde{f} = f_\epsilon$ associated with any allowed Λ_ε, $\varepsilon = \epsilon^3$, has the property that its restriction to the z-axis has at least two critical points – namely, a local minimum and a local maximum. Indeed, we can even arrange for $\widetilde{f}$ to have different values at these two critical points of the restriction to the axis by insisting that the local maximum be the *first* local maximum after the local minimum. By also insisting that the approximating limit sets be invariant under $\zeta \mapsto -\zeta$, we can thus guarantee that $\widetilde{f}$, and hence $f : X \to \mathbb{R}$ has a critical point for any such approximation. Fixing this ϵ, and applying Lemma 5.1 with $\varepsilon = \epsilon^3$, we thus deduce that there is some N such that, for every oriented surface Σ of even genus $\mathscr{g} \geq N$, there is a quasi-Fuchsian group $\Gamma \cong \pi_1(\Sigma)$ for which the the tunnel-vision function f of $X = \mathcal{H}^3/\Gamma$ has at least two critical values, and hence at least two critical points. ∎

Theorem A now follows immediately from Theorem 5.2, given the fact that any quasi-Fuchsian group is a deformation of a Fuchsian one. To prove Theorem B, we merely need add that we can continuously deform any quantizable configuration as we deform the relevant Fuchsian group into a given quasi-Fuchsian one.

Let us now conclude with a few additional comments in connection with Theorem 5.2. First of all, the restriction to very large genus $\mathscr{g}$ appears to be an inevitable limitation of our method of proof, because for any fixed genus the possible limit sets $\Lambda(\Gamma)$ only depend on a finite number of parameters, and so cannot provide arbitrarily good approximations of a freely specified curve. On the other hand, the requirement that the genus $\mathscr{g}$ be even seems relatively unimportant. For example, by replacing our dogbone regions with analogous domains that are invariant under $\mathbb{Z}_p$ for some prime p other than 2, and a similar argument then shows that the same phenomenon occurs for all genera $\mathscr{g}$ that are sufficiently large multiples of the given prime.

However, there is a more robust version of our strategy that should lead to a proof of the existence of critical points without imposing a symmetry condition. Given an Ω_+ that approximates a dogbone with an extremely narrow corridor between the two disks, we would like to understand the level sets S_t of the corresponding $\widetilde{f} : \mathcal{H}^3 \to (0, 1)$. When t is a regular value close to 1, this surface hugs the boundary of hyperbolic space and is a close approximation to the bounded region Ω_+. Now, if Ω_t were exactly a limiting dogbone consisting of two disks joined by a “corridor” of width zero, then we could write down the surfaces in closed form, and find that the surfaces

S_t consists of two "bubbles" over the disks when t is close to 1, but as t decreased these two bubbles eventually touch and then merge by adding a handle joining the two original components. We expect this change of topology to survive small perturbations of the regions in question, and that one should therefore be able to prove the existence of a critical point by a careful Morse-theoretic argument, just assuming that Ω_+ closely approximates a dogbone with a very narrow corridor between the disks. We believe that a careful implementation of this argument could be used to remove the even-genus restriction of Theorem 5.2, although we will not attempt to supply all the details in this paper.

Another issue we have not dealt with here is whether some of the critical points of the tunnel-vision function we have produced are typically non-degenerate in the sense of Morse. This is an intrinsically interesting question from the point of view of 4-manifolds, because it amounts to asking whether the corresponding harmonic 2-form is transverse to the zero section. When such transverse occurs, it is stable is under small perturbations of the metric, so that the failure of the metric to be conformally almost-Kähler then persists under small deformations of the conformal class. Numerical experiments have convinced us that our construction does consistently lead to critical points that are non-degenerate in the sense of Morse, and have suggested strategies for rigorously proving results to this effect. However, it seems best to leave a careful discussion of this work another time and forum.

Finally, while we have proved the existence of two critical points in certain specific situations, it seems natural to expect that there might be many more when the conformal structures on the two boundary components of $\overline{X}/\Gamma$ become widely separated in Teichmüller space. When this happens, the limit set becomes very chaotic, and a better understanding of the tunnel-vision function in this setting might eventually allow one to prove the existence of critical points even when the genus g is small. Conversely, it would be interesting to characterize those quasi-Fuchsian manifolds for which the tunnel-vision function has relatively few critical points; in particular, it would obviously be of great interest to have a precise characterization of those Γ for which the tunnel-vision function has no critical points at all.

In these pages, we seem to have just scratched the surface of a fascinating subject with deep connections with many natural questions in geometry and topology. We can only hope that some interested reader will take up the challenge, and address a few of the questions we have left unanswered here.

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