CORRECTIONS FOR QUASICONFORMAL FOLDING

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Introduction

David Martí-Pete and Mitsuhiro Shishikura found some errors in my paper "Constucting entire functions by quasiconformal folding". Fortunately, all these points are fairly easy to correct and this note is a summary of several email exchanges between the three of us in which we have resolved the difficulties. Some of these mistakes have already been adressed in later papers on folding, e.g., [3], [4], [2], but it seems worthwhile to gather the questions and repsonses together in one place.

I would be happy to add further details if any of the explanations seem incomplete. I would also be happy to hear about any other points that deserve a more careful explanation. I hope to eventually incorporate all such suggestions into a set of lecture notes on planar quasiconformal mappings,which will also give some applications to geometric function theory and dynamics, including a chapter on the folding theorem.

1. Backwards arrows

Question: *This is something that confused us in the beginning. It is only a typo and hence nothing needs to be done about it, but since it appears in the introduction it is more significant and it is good to be aware of it.*

In the construction it is used that $\eta = \psi^{-1} \circ \iota \circ \lambda \circ \tau$. *However, there are a few places in which* ψ^{-1} *is replaced by* ψ *in the definition of* η. *For example, towards the end of p. 5, there is a centered equation where the definition of* η *is wrong, and suggests that the image of* η *is* W, *while it should be* $\mathbb{H}_r$ *as* cosh *comes after* η. *Another place*

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where this happens is in p. 8., again in the centered equation that defines Ω'_j, the map ϕ_j should be the inverse, which is correct in the next line. This is also a bit confusing in the diagram in Figure 2.

Response: This is a careless typo on my part. I do not have a definite opinion on which direction should be the "forward" map and which the inverse, and most likely I changed my mind while writing the paper and did not make all the necessary changes to the manuscript.

2. Bounded Geomery

Question: *In the definition of bounded geometry, the condition (iv), namely that for non-consecutive edges we have diam(e)/dist(e,f) ensures that the graph cannot "double back" on itself. However, we are concerned about the fact that there is no condition to prevent such a thing for edges that share a vertex. Imagine that we have a sequence of pairs of edges $(e_n, f_n)_n$ so that in each pair, the two edges have a common vertex, and the other two endpoints become closer and closer. These pairs could form a circular shape in the limit. Would this be a valid tree for your result?*

Response: I agree that one needs a condition that prevents two edges that meet at a vertex from getting too close to each other away from the vertex. This was the picture I had in mind, and is what is used in the proof. I think a compact way to express the condition is to say that the union of edges that meet at a vertex for a uniformly bi-Lipschitz star, i.e., the M-bi-Lipschitz image of N equal length segments meeting at the origin and in the directions of the Nth roots of unity. Here M and N should be uniformly bounded over the whole tree. With this change the given proof is correct.

For example, this is the formulation given in my preprint with Simon Albrecht, [2]. In all applications that I know of, a even stronger condition than bounded geometry holds: uniformly analytic. This means that each edge is an analytic image of segment with uniform bounds. This condition is discussed in my paper [1]. I would be happy to receive suggestions for a more optimal formulation of the bounded geometry condition, perhaps one that was quasiconformally invariant.

3. Lemma 4.1

Question: *In the proof of Lemma 4.1, you say that it is enough to choose a point w such that dist$(e, w) \simeq \text{dist}(f, w) \simeq \text{dist}(w, \partial\Omega)$. But we thought that this is not enough in the case where e, f are the two sides of the same edge in the tree. Perhaps some statement about the hyperbolic distance should be used here instead of the Euclidean distance* dist*?*

Response: I claim that two adjacent intervals I, J on the real line have comparable lengths if and only if there is a point z in the upper half-plane from which these intervals have harmonic measures both bounded away from zero and the harmonic measures of the two rays that make up the real line minus these two intervals are also bounded away from zero.

One direction of this claim is easy: if I, J have comparable lengths, then choose z above their common endpoint and with height comparable to the shorter length. Then clearly all four harmonic measures with respect to z are comparable to 1. On the other hand, if I is much shorter than J, then any point z from which I has large harmonic measure must lie in neighborhood of size comparable to $|I|$ around I and hence the ray separated from I by J will have small harmonic measure.

To deduce Lemma 4.1 from this fact, we first suppose e and f are opposite sides of the same edge γ of T that map to adjacent intervals under the conformal map from a complementary component Ω of T to the right half-plane. Thus one endpoint, v, of γ must have degree 1 in T. By rescaling we may also assume γ has diameter 1. Fix $\epsilon > 0$. Since γ is a C^2 curve with uniform bounds, I claim we can choose a point $z \in \Omega$ distance 2ϵ from γ from which the both e and f have harmonic measure bounded away from zero. To see this, fix $\epsilon > 0$ and think of a round annulus centered at v with inner radius ϵ and outer radius 3ϵ.

If ϵ is small enough (depending on the bounded geometry constants) this annulus only hits T in γ. A Brownian motion has a fixed positive probability of traveling around this annulus and hitting itself without hitting either boundary of the annulus; doing this in one direction, it must hit e and in the other direction it must hit f, and hence both sides have harmonic measure bounded away from zero. Next consider the set of points A that are between (Euclidean) distance ϵ and 3ϵ from γ. This contains an ϵ-neighborhood of the curve of points that are distance 2ϵ from γ, and this curve surrounds γ. Again by bounded geometry, a Brownian path started at z has a fixed (though very small) probability of making a loop around γ inside A in either direction, and such paths must hit T at points that correspond to the two rays described above. Thus all four desired harmonic measures are bounded away from zero, and hence the τ images of e and f have comparable lengths.

This argument covers the case when e, f are two sides of the same edge γ. Next we consider what happens when e, f are sides of adjacent edges of T that meet at a vertex v of T. I won't write out the argument in the same level of detail, but the general idea is the same. We consider the two sets of points that are Euclidean distance 2ϵ from e and f respectively. These curves surround two different edges of T that meet at a common endpoint, so they must intersect at a point z and both e and f have harmonic measure bounded away from zero at this point (depending only on the bounded geometry constants). Also similar to the above, z is contained in two topological annuli that surround e and f respectively and a Brownian motion started at z has a positive probability of "making a loop" (again depending on the bounded geometry constant) in each of these, and these numbers give a lower bound for the harmonic measure of the two rays that are the complement of the τ images of I and J. This completes the proof of Lemma 4.1.

4. Exp-Cosh interpolation

Question: *We believe that Lemma 7.1 about the exp-cosh interpolation is false as stated, and this requires a substantial modification of the construction. However, we think this can be amended, and we describe a possible solution below.*

The problem is that the integer multiples of πi are critical points for the map cosh *but they are regular points for* exp. *Hence, we cannot have a quasiconformal map of the right half-plane that one the boundary equals* cosh *on one side of such points and* exp *in the other side. This happens if we have at least one interval in the collection J_2. The problem is that if such quasiconformal map existed, then it would contradict the fact that the boundary map should be quasisymmetric, as in one side the map is of the form x^2 and in the other x. It can also be seen by counting the turns of the map around these points and seeing that it does not match.*

The solution we have found involves introducing a new type of vertices of the graph around D-components that will no longer be critical points, but instead they will have a critical point very nearby. These new type of vertices will be preimages of two points $\pm\alpha \in (-1, 1) \setminus \{0\}$. Then the folding can be modified so that the trees grow from critical points that are near each of these points (there is an edge of the tree that joins the new special vertices and these critical points). This ensures that the size of the neighbourhood of the tree decreases as we introduce more of these special points around the D-components.

Response: Although the statement of Lemma 7.1 may be incorrect, the proof is basically OK. In particular, the first sentence, "The proof is basically a picture; see Figures 23 and 24.", is correct. On the boundary we should take ν_j to be equal to exp followed by one of three maps: either the identity map or the Mobius transformation

$$m(z) = i(z - i)/(z + i),$$

or its inverse. We use the identity for type 2 intervals and m for type 1 intervals when $\exp(z)$ lies on the upper half-unit-circle and its inverse when $\exp(z)$ lies in the lower half-unit-circle. The map m is chosen to map the upper half-unit-circle to the segment $[-1, 1]$; it is an elliptic transformation that fixes $-1, +1$ and rotates by 90 degrees. Its inverse maps the lower half-unit-circle to $[-1, 1]$. As illustrated in Figure 24, the upper half-plane minus the unit disk can be 3-QC mapped to the whole upper half-plane with boundary values equal to the identity on $\mathbb{R} \setminus [-1, 1]$ and the map m on the upper half-unit-circle; the map is a stretch on an 45 degree crescent to a 135 degree crescent as illustrated and is the identity elsewhere.

This map ν_j is then clearly 3-quasiregular and maps segments on the boundary of the right half plane either to the top or bottom half of the unit circle or to the segment $[-1, 1]$. Moreover, it has the desired length preserving property that points on the same type of interval that are the same distance from $2\pi i\mathbb{Z}$ will map to the same points and allow us to glue across the tree T continuously.

In this case when there are only R-components, there are no type 2 intervals, so we will always use the m map, and after we solve the Beltrami equation, we will indeed get the cosh mapping from the half-plane to the complement of $[-1, 1]$, so the general construction does generalize the "all R components" case. However, I should not have used cosh in the statement of the lemma (at least not for the boundary values). This is a case where the picture is accurate, but the formula has a mistake.

5. Wandering domains

Question: *The argument that is given in the paper "Wandering domains for composition of entire functions" by Fagella, Godillon and Jarque (and also in "Several constructions in the Eremenko-Lyubich class" by Lazebnik), is based on the fact that the support of the dilatation of ϕ is an epsilon-thin set. In these papers it is argued that when we modify the values of d_n, then the dilatation of ϕ lies in an arbitrarily small annulus inside D_n. However, adding more vertices to D_n also means that in the vertical R-component that shares some part of the boundary with D_n, the map τ which is approximately $\lambda \sinh(z)$ (after composing with some affine map that sends this component to a horizontal half-strip) must increase the value of λ to ensure that the tau-size is bigger than π. Thus, in a neighbourhood of the boundary of the band S_+ we are changing the tree decorations and creating more folding. So these sets (comparable to a square with the edge as a side) must be in the support of ϕ which cannot become arbitrarily thin.*

Also, the boundaries of the vertical R-components that are in the common with S_+ have a fixed size, and the length of the vertices around D_n becomes very small as d_n increases. Since the length of consecutive vertices must be comparable for the tree to have bounded geometry, we need to be careful and add vertices in the segment that joins $a_n + i\pi/2$ and $a_n + i(\pi - 1)$. But we can achieve comparable lengths by keeping dividing by 2 the length of every edge until we reach a size that is comparable with the length of the edges around D_n.

We think that the problem of having a circular argument can be solved with using a different strategy that chooses the correct parameters at once using Rouché's theorem, rather than keep doing subsequential modifications to the construction. At the moment we are writing a paper about how to construct functions in the class B of finite order with oscillating wandering domains using quasiconformal interpolation instead of quasiconformal folding. And there we use this new approach, but it should be possible to use it in your construction as well.

Response: In the construction of the wandering example, there is a sequence of D-components $\{D_n\}$ that contain critical points of high order $\{d_n\}$ and critical values that chosen in the upper half-plane very near to $1/2$. The placement of the critical value is controlled by a quasiconformal map ρ_n of the disk sending the origin to the desired critical point, as described in the definition of D-components. The dilatation of ρ_n is supported in a fixed annulus in the unit disk, but the corresponding dilatation in the D-component D_n is supported on a very thin annulus (roughly $1/d_n$ thin).

In order to introduce a high degree critical point in a D-component, we have to place very many vertices on the boundary of the D-component and this requires the conformal maps in the neighboring R-components chosen so that the corresponding conformal partitions (i.e., the pullback of the integer partition on the boundary of the right half-plane) consists of arcs that are even smaller than the partitions of the D-component boundary. Since the R-components neighboring the D-component are also neighbors of R-component containing the positive real axis (i.e., the "central strip"), changing d_n may result in changing the folding between the neighboring vertical R-components and the horizontal central strip and this might involve changing the dilation on a fixed area, not area close to zero. Thus the change might not be close to the identity.

The difficulty is that I wrote the argument so that the integers $\{d_n\}$ are chosen inductively based on earlier steps of the construction. However, it is not hard to choose all the $\{d_n\}$ ahead of time, and only choose the maps $\{\rho_n\}$ for placing the critical values inductively. In this case, the dilatation only changes in the thin annulus, as described in the current proof, and the modified correction map will be close to the identity on any compact set bounded away from the D-components where we are making the change. Thus the main point is to see why we can "pre-select" the orders $\{d_n\}$. There are two conditions to meet.

First, when we make the correction by introducing the map ρ_n to move the critical point, we change the dilatation and need to incorporate a new QC correction map using the measurable Riemann mapping theorem. If the order d_n does not change, then the folding between different R-components or between a R-component and the D-component do not change; only the dilatation inside a thin annulus inside the D-component does.Thus we need to choose $\{d_n\}$ so large, that the corresponding annulus supporting the dilatation of the correction is so small that any quasiconformal map with dilation (with a known uniform bound) supported on this annulus inside D_n is close to the identity on a compact disk around the origin that contains the entore orbit before it enters D_n. We know the number of orbits points before we hit D_n (at least we can give an explicit upper bound since we get to choose the structure of the orbit), so we can choose each d_n large enough to satisfy this requirement without knowing the other d_k's.

The second condition is that the center half of D_n must be "shrunk" enough that its image "fits inside" the preimage of the next D-component to be visited; I will call this $\tilde{D}_{n+1}$. Again, we can choose d_n if we just know how small an image disk we need. In other words, we need a lower bound for the inradius of a preimage of $\tilde{D}_{n+1}$. For this we need to know an upper bound N for the number of iteration needed reach $\tilde{D}_{n+1}$ from a point near $1/2$ and a bound for the amount of expansion that a disk near $1/2$ will undergo along this finite orbit (since the maps are conformal on this finite orbit, we can use Koebe's theorem and we really just need any upper bound of the derivatives along the orbit of $1/2$). We don't know exactly what map we are iterating, since this changes slightly at each stage of the construction, but since every possible map that can arise has uniformly bounded dilatation that is supported in a fixed set that is positive distance from the ray $[1/2, \infty)$, we know that the possible maps (and hence their derivatives) lie in a compact family. Using the worst case in this family allows us to choose d_n so large that it will work for any of the maps that arise in our inductive construction.

Therefore we can choose $\{d_n\}$ growing so rapidly to infinity at the first step of the construction, that we never have to change these values; we only change the maps ρ_n the position the critical value corresponding to the nth D-component visited. The corresponding QC correction has dilatation supported in a thin annulus, as claimed in the paper, and now the argument stated in the paper works.

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