HILBERT'S LEMNISCATE THEOREM FOR RATIONAL MAPS

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Abstract. In 1897 Hilbert proved that any Jordan curve in the complex plane can be approximated in a strong sense by a polynomial lemniscate (a level curve of $|p|$ for some polynomial p). We extend this by showing that any finite collection of pairwise disjoint Jordan curves can be similarly approximated by a rational lemniscate.

1. Introduction

A *rational lemniscate* is a set of the form

$$L_r := \{z \in \widehat{\mathbb{C}} : |r(z)| = 1\},$$

where r is a rational function and $\widehat{\mathbb{C}}$ is the Riemann sphere. If r is a polynomial, L_r is called a *polynomial lemniscate*. A *topological annulus* is a subset of $\widehat{\mathbb{C}}$ homeomorphic to a Euclidean annulus $\{z : 1 \leq |z| \leq r\}$ for some (and hence every) $1 < r \leq \infty$. A *Jordan curve* $\gamma \subset \widehat{\mathbb{C}}$ is a homeomorphic image of $\mathbb{T} := \{z : |z| = 1\}$, and γ is said to *separate* the two boundary components of a topological annulus A if each component of $\widehat{\mathbb{C}} \setminus \gamma$ contains exactly one of the two components of ∂A. In 1897, David Hilbert [Hil97] proved the following result, known as Hilbert's Lemniscate Theorem.

Theorem 1.1. *For every topological annulus $A \subset \mathbb{C}$, there exists a polynomial p whose lemniscate is an analytic Jordan curve separating the two boundary components of A.*

In this paper, we extend Hilbert's theorem to the rational setting. First we state the following preliminary version.

Theorem 1.2. *For any finite collection of pairwise disjoint topological annuli $(A_j)_{j=1}^N \subset \widehat{\mathbb{C}}$, there is a rational function r whose lemniscate L_r has N components, each of which is an analytic Jordan curve, and so that the two boundary components of any $(A_j)_{j=1}^N$ are separated by exactly one component of L_r.*

When $N = 1$, Theorem 1.2 as stated is weaker than Theorem 1.1 since the poles of r are not specified. In order to state a stronger version of Theorem 1.2 which contains Theorem 1.1 as a special case, it will be convenient to first introduce some more terminology.

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Notation 1.3. Given pairwise disjoint topological annuli $(A_j)_{j=1}^N \subset \widehat{\mathbb{C}}$, a rational function r satisfying the conclusions of Theorem 1.2 will be said to *separate* the $(A_j)_{j=1}^N$. The components of $\widehat{\mathbb{C}} \setminus \cup_j A_j$ have a natural bipartite tree structure (see Figure 1) defined by associating a vertex to each component of $\widehat{\mathbb{C}} \setminus \cup_j A_j$, and connecting two vertices by an edge if and only if the boundaries of the corresponding components both have non-empty intersection with a common A_j. A 2-coloring of $\widehat{\mathbb{C}} \setminus \cup_j A_j$ is an assignment of one of two colors (black or white) to each vertex so that any two adjacent vertices have different colors. Since the tree is bipartite, such a coloring always exists, and there are exactly two 2-colorings. We denote the Hausdorff distance between two sets E, F, by $d_H(E, F)$.

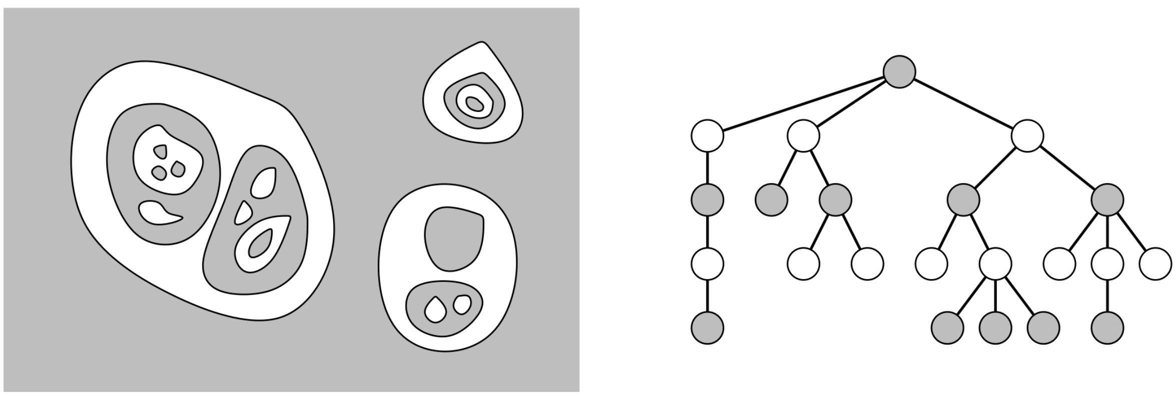

FIGURE 1. A finite family of disjoint closed curves or annuli (shown on the left) divides the Riemann sphere into regions that may be interpreted as the vertices of a bipartite tree (shown on the right).

The following theorem contains Theorems 1.1 and 1.2 as special cases, and is the main result of this paper. It roughly says that there exists a rational function r satisfying the conclusions of Theorem 1.2 with the minimum possible number of poles, and moreover up to three poles of r may be specified exactly with the remaining poles specified approximately.

Theorem 1.4. *Let $(A_j)_{j=1}^N \subset \widehat{\mathbb{C}}$ be a 2-colored collection of pairwise disjoint annuli, $\varepsilon > 0$, and suppose $P \subset \widehat{\mathbb{C}}$ consists of exactly one point from each black region. Let p_1, p_2, $p_3 \in P$. Then there exists a rational function r separating the $(A_j)_{j=1}^N$ with p_1, p_2, $p_3 \in r^{-1}(\infty)$, $|r^{-1}(\infty)| = |P|$, and $d_H(r^{-1}(\infty), P) < \varepsilon$.*

The maximum principle prevents connected components of a polynomial lemniscate from being nested, so rational functions are needed in Theorem 1.4 whenever one of the annuli surrounds another one. However, if the annuli are mutually exterior (meaning each A_j is bounded, and the bounded component of $\widehat{\mathbb{C}} \setminus A_j$ has empty intersection with A_k for all $k \neq j$), then one can 2-color the $(A_j)_{j=1}^N$ by letting the unique unbounded component of $\widehat{\mathbb{C}} \setminus \cup_j A_j$

be black, and all other components be white. Setting $P = \{\infty\}$ and $p_1 = p_2 = p_3 = \infty$, Theorem 1.4 says that there exists a polynomial lemniscate separating the $(A_j)_{j=1}^N$. This special case of Theorem 1.4 is due to Walsh and Russell (see Lemma III of [WR34]).

Polynomial and rational lemniscates have been extensively studied. Hilbert's 16th problem asks for the maximal possible number of connected components of a lemniscate on $\mathbb{R}^n$ and their possible arrangement in space. For polynomials and rational functions in one complex variable this is known: the maximal number of components is the degree and any combinatorial arrangement is possible. Thus for planar lemniscates attention has focused on their particular geometric properties. Polynomial lemniscates were used by Sharon and Mumford [SM06] for algorithmic shape recognition. This led to the study of the conformal properties of lemniscates by Frolova, Khavinson and Vasil'ev [FKV18], by Ebenfelt, Khavinson, and Shapiro [EKS11], by Fortier Bourque and Younsi [FBY15], and by Younsi [You16]. A question of Erdős, Herzog, and Piranian [EHP58] asks what is the maximum length of a lemniscate of a monic polynomial: $z^n - 1$ is conjectured to be the extreme case. This problem is still open, but related results are given by Borwein in [Bor95] and by Eremenko and Hayman in [EH99]. The expected length of random rational lemniscates is considered by Lerario and Lundberg in [LL15], [LL16]. Random polynomial lemniscates are considered by Lundberg and Ramachandran [LR17] and by Epstein, Hanin and Lundberg [EHL20]. Hilbert's theorem has been generalized to higher complex dimensions by Bloom, Levenberg and Lyubarskii [BLL08], and by Nivoche [Niv09]. Nagy and Totik consider placing a lemniscate between tangent curves in [NT05]. The rate of convergence in Hilbert's theorem has been studied by Andrievskii [And00], [And18], and Kosukhin [Kos05]. Results on generalized lemniscates for resolvants of operators are surveyed by Putinar in [Put05]. The geometry of higher dimensional, real polynomial lemniscates is a very active field, e.g., the "polynomial ham sandwich theorem" of Stone and Tukey [ST42] has recently given rise to the "polynomial method" in discrete geometry and combinatorics, as described by Guth in [Gut13]. Recently, several investigators have used real variable polynomial lemniscates to separate data points in the context of machine learning, e.g., [KLL$^+$], [MP19]. Polynomial lemniscates are also closely related to approximation of planar continua by Julia sets as described in [LY19] (see also [Lin15], [BP15], [BT21] and [MRW22]), and we describe a relevant application of Theorem 1.4 to this setting in Section 7 (see Corollary 7.1).

We discuss several other consequences and questions related to Theorem 1.4 in Section 7, but for now we will conclude the introduction by describing the proof of Theorem 1.4. Since the annuli $(A_j)_{j=1}^N$ are conformally equivalent to Euclidean annuli, there exist analytic Jordan curves $\Gamma_j \subset A_j$ that separate the boundary components of A_j. Let $\Gamma := \cup_{j=1}^N \Gamma_j$. Then $\Omega := \widehat{\mathbb{C}} \setminus \Gamma$ consists of $N+1$ finitely connected regions $(\Omega_j)_{j=0}^N$ with analytic boundaries. For each j there is a proper holomorphic map $\mathscr{A}_j : \Omega_j \to \mathbb{D}$ that maps each connected component of $\partial\Omega$ conformally onto $\mathbb{T}$. This is called the *Ahlfors map* for Ω_j; its existence was proven by Ahlfors in [Ahl50] in the context of finite, bordered Riemann surfaces; we are using his result in the special case of finitely connected planar domains. See also Royden's paper [Roy52], Theorem 13.1 of Bell's book [Bel92], and Theorem 5.58 of [Ran95] by Ransford.

The $(\Omega_j)_{j=0}^N$ come with a 2-coloring, and we let $B \subset \{0, \ldots, N\}$ denote the indices of black regions, and $W = \{0, \ldots, N\} \setminus B$ the indices of white regions. For each $n \in \mathbb{N}$, consider the function g_n on Ω defined by

$$g_n(z) = \begin{cases} \mathscr{A}_j^{-n}(z), & \text{if } z \in \Omega_j, j \in B \\ \mathscr{A}_j^{n}(z), & \text{if } z \in \Omega_j, j \in W. \end{cases}$$

This defines a holomorphic map $g_n : \Omega \to \widehat{\mathbb{C}} \setminus \mathbb{T}$ which restricts to a proper map $g_n : \Omega_j \to g_n(\Omega_j)$ for each j, where $g_n(\Omega_j) = \mathbb{D}$ if $j \in W$ and $g_n(\Omega_j) = \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ if $j \in B$. If g_n were continuous across Γ, then g_n would be a rational map whose lemniscate coincides with Γ, however this will not be the case except in some exceptional circumstances. However, for every n we can construct a $m = m(n) \in \mathbb{N}$ and a K-quasiregular function f_n on the Riemann sphere that coincides with $\mathscr{A}_j^n(z)$ in Ω_j for each $j \in W$, and with $\mathscr{A}_j^{-m}(z)$ outside a small neighborhood of Γ in Ω_j for each $j \in B$. The Measurable Riemann Mapping Theorem (or MRMT for short) then implies there is a quasiconformal map ϕ_n of the sphere to itself so that $r_n := f_n \circ \phi_n^{-1}$ is rational. Moreover, we will prove the dilatation bound K for f_n is independent of n, and the region where f_n is not holomorphic has area which $\to 0$ as $n \to \infty$. This implies that we can choose ϕ_n tending uniformly to the identity as $n \to \infty$. Thus $L_{r_n} = \phi(L_{f_n}) \approx L_{f_n} = \Gamma$ for large n. The components of the lemniscate L_{r_n} will not be Jordan curves, but it will be deduced that there is a $R \approx 1$ so that the lemniscate of $R \cdot r_n$ has the desired properties. This describes a proof of Theorem 1.2. The above will be adapted to give a proof of Theorem 1.4 by replacing the maps $\mathcal{A}_j$ for $j \in B$ by proper maps $\mathcal{B}_j : \Omega'_j \to \mathbb{D}$ with a unique specified zero, where Ω'_j is a perturbation of Ω_j. We prove the existence of such a $\mathcal{B}_j$ in Section 5.

Thus the bulk of the proof of Theorem 1.4 is to construct the quasiregular map f_n with the necessary bounds on the dilatation of f_n and the area of $\{z : \mathrm{supp}((f_n)_{\overline{z}}) \neq 0\}$. The idea is similar to the quasiconformal folding method from [Bis15], but we will give a self-contained construction that is simpler. Here we only require the dilatation bound K to be independent of n, but it is allowed to depend on Γ. If we used the full strength of the methods in [Bis15], then K could also be taken independent of the geometry of Γ (as long as the components of Γ are analytic). The proof of Theorem 1.4 does not require this extra control, although it might be useful in future work, for instance in bounding the degree of the rational map r in terms of the geometry of the annuli $(A_j)_{j=1}^n$.

2. Linear interpolation across a strip

In this section, we record several simple results about constructing a quasiconformal map from an infinite strip to itself that interpolates between the identity on one boundary line and a biLipschitz homeomorphism on the other. First, some notation and definitions.

In this paper, i will always denote the imaginary unit in $\mathbb{C}$, never an integer index. A mapping η is said to be λ-*periodic* if $\eta(z+\lambda)=\eta(z)$ for all z in the domain of η. For $c\geq 0$ and $k,n\in\mathbb{Z}$, we define (see Figure 2):

(1) $H_c:=\{z:0<\mathrm{Re}(z)<c\}$ (a vertical strip of width c),
(2) $\ell_c^{k,n}:=\{c\}\times[k/n,(k+1)/n]$ (short segments on the right side of H_c).

We denote the imaginary axis as $i\mathbb{R}$ and sometimes refer to this as the "left side of H_c". Similarly we call $i\mathbb{R}+c=\{z:\mathrm{Re}(z)=c\}$ the "right side of H_c". For brevity, we call the intervals $\ell_0^{k,n}$ "n-intervals" or "n-edges", and we call their endpoints the "n-vertices".

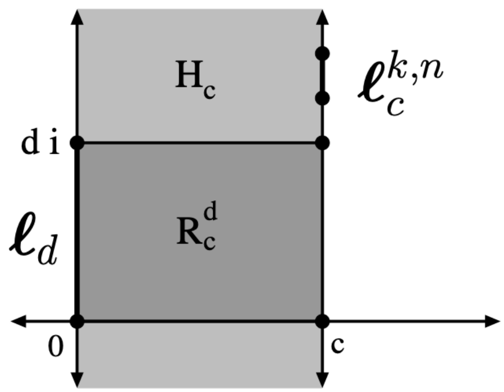

FIGURE 2. An illustration of various definitions given in the text.

A $\mathbb{C}$-valued map f is called K-biLipschitz if

$$\frac{1}{K}\leq\frac{|f(x)-f(y)|}{|x-y|}\leq K,$$

for all x,y in the domain of f.

Lemma 2.1. *Let $f:i\mathbb{R}\to i\mathbb{R}$, and suppose $C<\infty$ and $n\in\mathbb{N}$ are so that f is C-biLipschitz and $\sup_{z\in i\mathbb{R}}|z-f(z)|\leq C/n$. Then, for all $c>0$, there is a $K<\infty$ depending on C and c (but not on f or n) so that the map*

$$F(x+iy):=x+\frac{nx}{c}iy+\left(1-\frac{nx}{c}\right)f(iy):H_{c/n}\to H_{c/n} \tag{2.1}$$

is K-biLipschitz.

Remark 2.2. The map (2.1) is the linear interpolation between the map $z\mapsto f(z)$ on the left side of $H_{c/n}$ and the identity on the right side of $H_{c/n}$.

Proof. It will be convenient to switch from $\mathbb{C}$-notation to $\mathbb{R}^2$-notation for the proof, in other words we use (x,y) in place of $x+iy$. Let (x,y), $(s,t)\in H_{c/n}$. We want to show

$$\frac{1}{K}\leq\frac{|F(x,y)-F(s,t)|}{|(x,y)-(s,t)|}\leq K,$$

where K depends only on C and c. We will consider two cases: when $|x-s| \geq |y-t|$ and when $|x-s| \leq |y-t|$. First suppose $|x-s| \geq |y-t|$ (see the left side of Figure 3). Then

$$\frac{1}{2}|(x,y)-(s,t)| \leq |x-s| \leq |F(x,y)-F(s,t)|,$$

and

$$|F(x,y)-F(s,t)| \leq |x-s| + C|y-t| \leq (C+1)|x-s| \leq (C+1)|(x,y)-(s,t)|.$$

This concludes the case that $|x-s| \geq |y-t|$.

Next suppose $|x-s| \leq |y-t|$ (see the right side of Figure 3). Then under F, the two horizontal segments in $H_{c/n}$ containing the points (x,y) and (s,t) each map to (possibly non-horizontal) line segments crossing $H_{c/n}$ with absolute slope $\leq C/c$ and at least vertical distance $|y-t|/C$ apart at their endpoints. Some simple trigonometry (see the far right side of Figure 3) then implies that these segments are at least distance $c|y-t|/(C\sqrt{c^2+C^2})$ apart inside $H_{c/n}$. Hence

$$\frac{c}{2C\sqrt{c^2+C^2}}|(x,y)-(s,t)| \leq \frac{c}{C\sqrt{c^2+C^2}}|y-t| \leq |F(x,y)-F(s,t)|$$

and

$$|F(x,y)-F(s,t)| \leq |x-s| + C|y-t| \leq (C+1)|y-t| \leq (C+1)|(x,y)-(s,t)|.$$

This concludes the case $|x-s| \leq |y-t|$, and hence we have proven that $F : H_{c/n} \to H_{c/n}$ is K-biLipschitz with K depending only on c and C. □

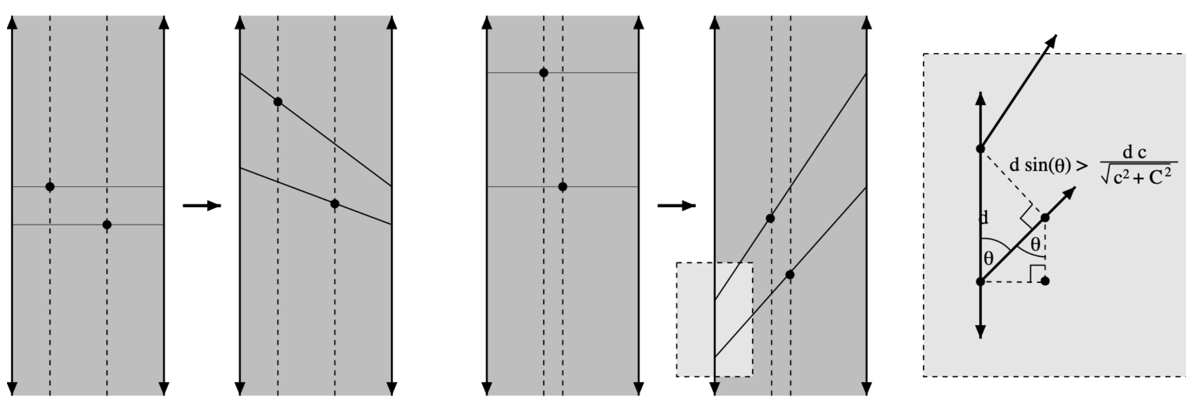

FIGURE 3. The proof of Lemma 2.1. The left side shows the argument when $|x-s| \geq |y-t|$; the right side shows the other case. On the far right, note that if the image segments have slope $< C/c$, then $\sin(\theta) > c/\sqrt{c^2+C^2}$, so the minimal distance between the segments is $> dc/\sqrt{c^2+C^2}$ where d is the distance between their nearer endpoints.

Recall that for a function of $z = x + iy$, we set $f_z := (f_x - if_y)/2$ and $f_{\overline{z}} := (f_x + if_y)/2$. The dilatation of f is $\mu_f := f_{\overline{z}}/f_z$. If $|\mu_f(z)| = k < 1$ at a point, then the tangent map of f at z maps a circle to an ellipse of eccentricity $K = (k+1)/(k-1)$. A homeomorphism $f : \Omega \to \Omega'$ between planar domains is called K-*quasiconformal* (or K-QC for brevity) if it is locally in the Sobolev space $W^{1,2}$ and its partial derivatives satisfy $|f_z| \leq k|f_{\overline{z}}|$ almost everywhere, where $k = (K-1)/(K+1)$. A map is called quasiconformal if it is K-QC for some $K < \infty$. See [Ahl06] for details and basic properties of quasiconformal mappings. A K-biLipschitz between planar domains is automatically K^2-quasiconformal between those domains (this follows from the "geometric" definition of quasiconformality: see for instance Theorem IV.4.1 of [LV73]). Thus the map produced by Lemma 2.1 is a quasiconformal map from $H_{c/n}$ to itself.

A K-*quasiregular* map is a continuous function that is locally in $W^{1,2}$ and with dilatation bounded by $k = (K-1)/(K+1) < 1$. Thus it is simply the non-injective analog of a quasiconformal map. The famous Measurable Riemann Mapping Theorem (abbreviated henceforth by MRMT) implies that given any K-quasiregular map $g : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, there is a K-quasiconformal homeomorphism $\varphi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ so that $f = g \circ \varphi^{-1}$ is holomorphic, and φ is unique up to composition with a Möbius transformation. If we assume that g is K-quasiregular and μ_g is supported on a set of spherical area ε, then φ can be chosen to approach the identity as $\varepsilon \to 0$ (and K is kept fixed). We will use these facts in Section 6 to obtain a rational map r from a quasiregular map g, whose lemniscate is constructed to have the desired shape.

Lemma 2.3. *Let $\eta : i\mathbb{R} \to i\mathbb{R}$ be a periodic analytic homeomorphism, $c > 0$, and $n \in \mathbb{N}$. Then there exists a periodic K-quasiconformal homeomorphism $\beta : H_{c/n} \to H_{c/n}$ so that:*

(1) *$\beta \circ \eta^{-1}(\ell_0^{k,n}) = \eta^{-1}(\ell_0^{k,n})$ for all k,*
(2) *$\beta'(z) = n \cdot \mathrm{length}(\eta^{-1}(\ell_0^{k,n})) \cdot \eta'(z)$ for all k and $z \in \eta^{-1}(\ell_0^{k,n})$, and*
(3) *$\beta(z) = z$ if $\mathrm{Re}(z) = c/n$.*

Moreover, K depends only on η and c, and not on n.

Proof. Let $\phi : i\mathbb{R} \to i\mathbb{R}$ be the piecewise linear homeomorphism so that $\phi(ki/n) = \eta^{-1}(ki/n)$ for every integer k. Then $\beta := \phi \circ \eta : i\mathbb{R} \to i\mathbb{R}$ is a homeomorphism satisfying (1) and (2). Note that β satisfies the hypotheses of Lemma 2.1 with

$$C := \max\left\{\sup_{z \in i\mathbb{R}} |\eta'(z)|^2, 1/\inf_{z \in i\mathbb{R}} |\eta'(z)|^2\right\}.$$

Since C depends only on η (and not on n), it follows that the interpolation of β with the identity on the right side of H_c is K-quasiconformal with K depending only on η and c, and not on n. □

Lemma 2.4. *Let $\eta : i\mathbb{R} \to i\mathbb{R}$ be a periodic analytic homeomorphism, $c > 0$, and $n \in \mathbb{N}$. Then, for all $m > n \cdot \sup_{z \in i\mathbb{R}} |\eta'(z)|$, there exists a periodic K-quasiconformal homeomorphism $\varphi : H_{c/n} \to H_{c/n}$ so that:*

(1) *φ is the identity on the right side of $H_{c/n}$,*
(2) *$\varphi \circ \eta^{-1}$ maps n-vertices of $i\mathbb{R}$ to m-vertices,*
(3) *φ is affine on $\eta^{-1}(J)$ for each n-edge $J \subset i\mathbb{R}$.*

Moreover, K depends only on η and c, and not on n or m.

Proof. We will define φ first on the η^{-1}-images of the n-vertices on $i\mathbb{R}$. For each n-vertex v, set $\varphi(\eta^{-1}(v))$ to be the m-vertex closest to $\eta^{-1}(v)$. If w is an n-vertex different from v, then by our assumptions,

$$|\eta^{-1}(w) - \eta^{-1}(v)| \geq \inf |(\eta^{-1})'| \cdot |v - w| \geq 1/(n \sup |\eta'|) > 1/m.$$

Thus if $v \neq w$, then the closest m-vertex to $\eta^{-1}(w)$ is not also the closest to $\eta^{-1}(v)$, so φ is injective from n-vertices to m-vertices. Hence we may extend φ to a periodic, piecewise linear, C-biLipschitz homeomorphism of $i\mathbb{R}$ to itself satisfying conditions (2) and (3), where C depends only on η and not on n or m. Thus, by Lemma 2.1, the linear interpolation of φ with the identity on the right side of $H_{c/n}$ satisfies the conclusions of the lemma. $\square$

3. Simple folding and triangular interpolation

In this section we introduce a "folding" map we will need in order to prove our main result. First we define a special sequence of trees as follows.

Definition 3.1. See Figure 4. First, we define T_0 to consist of the three vertices 0, $i/2$, i and the two edges $\{0\} \times [0, 1/2]$, $\{0\} \times [1/2, 1]$. If $j \geq 1$, we define T_j by adjoining to T_0 the collection of j vertices $V_j := \{k/(2j) + i/2\}_{k=1}^{j}$ and j edges $E_j := \{[k/(2j), (k+1)/(2j)] \times \{1/2\}\}_{k=0}^{j-1}$.

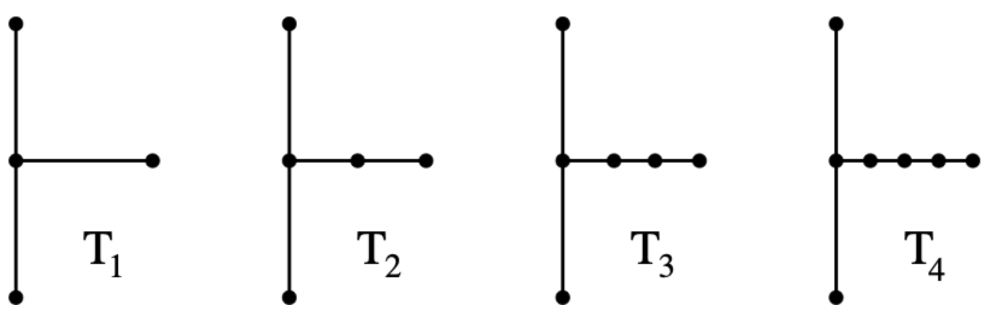

Figure 4. Some trees $\{T_j\}$ from Definition 3.1.

Definition 3.2. We define $\ell_d := \{0\} \times [0, d]$ (a vertical segment on $i\mathbb{R}$ of height d), and $R_c^d := [0, c] \times [0, d]$ (a rectangle inside H_c with width c and height d)

Lemma 3.3. *For all $j \geq 0$, there exists a QC map $\psi_j : (0,1)^2 \setminus T_j \to (0,1)^2$ so that:*

(1) *$\psi_j(z) = z$ for $z \in \partial([0,1]^2) \setminus \ell_1$.*
(2) *ψ_j is affine on $\ell_1 \cup T_j$.*
(3) *$\psi_j(V_j) = \{ki/(2j+2)\}_{k=0}^{2j+2}$.*

Remark 3.4. In Lemma 3.3, ψ_j is defined in $(0,1)^2 \setminus T_j$, but we are also using the symbol ψ_j to denote the extension of ψ_j to the boundary $\partial((0,1)^2 \setminus T_j) = \partial([0,1]^2) \cup ([0,1/2] \times \{1/2\})$. Thus, conclusions (2) and (3) require some explanation since the boundary extension of ψ_j to the segment $[0,1/2) \times \{1/2\}$ is 2-valued. By (2), we mean that the boundary extension from either side of $[0,1/2) \times \{1/2\}$ is affine, and by (3) we mean that the union of both boundary extensions map V_j onto $\{ki/(2j+2)\}_{k=0}^{2j+2}$ (this is illustrated in Figure 5).

Proof. Figure 5 shows two triangulations of $R_1^1 = [0,1]^2$ and the map ψ_j is given by affine maps on each triangle, chosen to be the identity on the top, right and bottom edges of R_1 and to agree along any common edge or vertex between two triangles (this determines the map uniquely). The figure illustrates the case $j = 2$, but a similar construction is possible for all $j \geq 0$. □

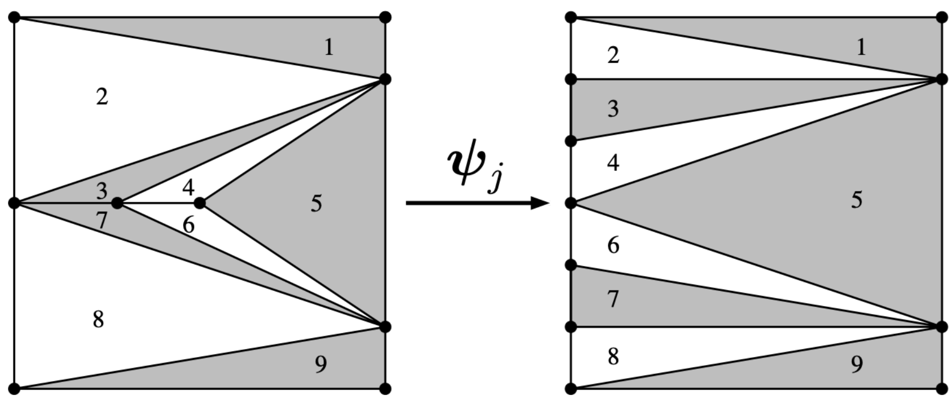

FIGURE 5. The map ψ_j is defined as piecewise affine between these triangulations. Shown is the case $j = 2$.

Remark 3.5. By examining the triangle labeled "2" in Figure 5, one can check that the quasiconformal constant for ψ_j tends to infinity as j tends to ∞. The quasiconformal folding construction in [Bis15] uses a much more intricate construction to achieve uniform bounds on the quasiconformal dilatation of the corresponding ψ_j map. As noted in the introduction, this more difficult result is not needed for the proofs given in this paper.

Definition 3.6. Let m, $d \in \mathbb{N}$ and consider a subset $I \subset \{ik/m\}_{k=0}^{dm} \subset \ell_d$. We say I is *admissible* if 0, $di \in I$. Let $\mathbb{N}_I := \{k \in \mathbb{N} : ik/m \in I \setminus \{di\}\}$. Given $k \in \mathbb{N}_I$, we let

$k^+ := \min\{n \in \mathbb{N}_I : n > k\}$. Finally, we set

$$\min(I) := \min\{\text{length}(\mathcal{I}) : \mathcal{I} \text{ is a component of } \ell_d \setminus I\},$$

and similarly for $\max(I)$.

Definition 3.7. Let m, $d \in \mathbb{N}$. For any admissible $I \subset \{ik/m\}_{k=0}^{dm}$, we define a tree $\mathcal{T}_I = (V_I, E_I)$ as follows. For each $k \in \mathbb{N}_I$, consider the square S_k in the right-half plane with left side $\{0\} \times [k/m, k^+/m]$. Let $\phi_k(z) := a_k z + b_k$ be the Euclidean similarity where $a_k > 0$, $b_k \in \mathbb{R}$ are so that $\phi_k(S_k) = [0,1]^2$, and let $j(k) := k^+ - k - 1$. We define

$$\mathcal{T}_I := \bigcup_{k \in \mathbb{N}_I} \phi_k^{-1}(T_{j(k)}). \tag{3.1}$$

For $1 \le C < \infty$ we let $L_C(x,y) := (x/C, y)$ and set $\mathcal{T}_I^C := L_C(\mathcal{T}_I)$, with $V_{\mathcal{I}}^C := L_C(V_I)$ and $E_{\mathcal{I}}^C := L_C(E_I)$.

We refer the reader to Figure 6 for an illustration of the map in the following lemma.

Lemma 3.8. *Let m, $d \in \mathbb{N}$, $\varepsilon > 0$, $1 \le C < \infty$ and $I \subset \{ik/m\}_{k=0}^{dm}$ be admissible so that $\max(I) < C\varepsilon$. Then there exists a K-quasiconformal homeomorphism $\psi : R_\varepsilon^d \setminus \mathcal{T}_I^C \to R_\varepsilon^d$ so that:*

1. *ψ is affine on each edge in $\mathcal{T}_I^C$,*
2. *$\psi(V_{\mathcal{I}}^C) = \{ik/2m\}_{k=0}^{2dm}$,*
3. *if e is an edge in $\mathcal{T}_I^C \setminus \ell_d$ and $x \in e$, then the two limits $\{\xi^\pm\} = \lim_{z \to x} \psi(z)$ satisfy $\exp(2\pi m \xi^+) = \overline{\exp(2\pi m \xi^-)}$, and*
4. *$\psi(z) = z$ if $\mathrm{Re}(z) = \varepsilon$.*

Moreover, K depends only on an upper bound for $\max\{C, m \cdot \max(I)\}$, and not on ε, d nor otherwise on I, C, or m.

Proof. First we consider the case $C = 1$. Then, with notation as in Definition 3.7, the assumption $\max(I) < C\varepsilon$ implies that $\mathcal{T}_I \subset R_\varepsilon^d$. The proof will be completed, roughly speaking, by using Lemma 3.3 to map each $S_k \setminus \phi_k^{-1}(T_{j(k)})$ on the left of Figure 6 to the corresponding S_k on the right of Figure 6. More precisely, we define

$$\psi := \phi_k^{-1} \circ \psi_{j(k)} \circ \phi_k \text{ in } S_k \setminus \mathcal{T}_I \text{ for } k \in \mathbb{N}_I, \tag{3.2}$$

and $\psi(z) := z$ for $z \notin \cup_k S_k$. The conclusions (1), (2), and (4) now follow from the definition of ψ and Proposition 3.3.

Let us verify (3). For any edge e in $\mathcal{T}_I^C \setminus \ell_d$, $\psi(e)$ consists of two disjoint $2m$-edges. The map $z \to \exp(2\pi m z)$ sends each $2m$-edge onto either the upper or lower half of the unit circle. Moreover, the two edges $\psi(e)$ map under $z \to \exp(2\pi m z)$ to opposite halves of the circle (since they are always separated by an even number of $2m$-edges). Because the map ψ is piecewise affine, if $x \in e$ then the two limits $\{\xi^\pm\} = \lim_{z \to x} \psi(z)$ map to conjugate points under $z \to \exp(2\pi m z)$, and hence (3) is verified.

Lastly we investigate the dependence of K on the parameters. Lemma 3.3 implies that the dilatation constant of (3.2) depends only on an upper bound for

$$\sup_k j(k) < \sup_k (k^+ - k) = m \cdot \max(I),$$

and so K depends only on an upper bound for $m \cdot \max(I)$. This concludes the proof of Lemma 3.8 in the case that $C = 1$, and if $C > 1$ the map $L_C \circ \psi \circ L_C^{-1}$ (for ψ as above) satisfies the conclusions of the Proposition. ☐

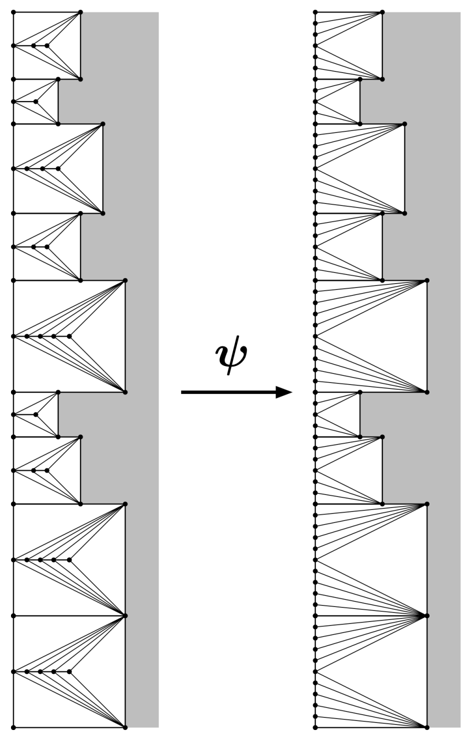

Figure 6. Illustrated is the quasiconformal map of Lemma 3.8. The map is the identity in the shaded region and on edges common to adjacent squares. It is piecewise affine between corresponding triangles, as described earlier in Figure 5.

4. The main interpolation result

We are now ready to prove our main interpolation result. Its proof will use a map σ defined as follows.

Definition 4.1. Let $\mu(z) := (z+1)/(z-1)$ be the conformal mapping of $\{z : |z| > 1\}$ onto the right-half plane, taking the triple $(-1, 1, \infty)$ to $(0, \infty, 1)$. Note that $\mu^{-1} = \mu$. Consider the 3-quasiconformal map $\nu : \{\mathrm{Re}(z) > 0\} \to \mathbb{C} \setminus (-\infty, 0]$ defined by

$$\nu(re^{i\theta}) = \begin{cases} re^{i\theta}, & |\theta| \leq \pi/4, \\ re^{i3(\theta-\pi/4)+i\pi/4}, & \pi/4 < \theta < \pi/2 \\ re^{i3(\theta+\pi/4)+i\pi/4}, & -\pi/4 > \theta > -\pi/2. \end{cases}$$

Thus $\sigma := \mu^{-1} \circ \nu \circ \mu = \mu \circ \nu \circ \mu$ is 3-quasiconformal from $\{|z| > 1\}$ onto $\mathbb{C} \setminus [-1, 1]$ (see Figure 7).

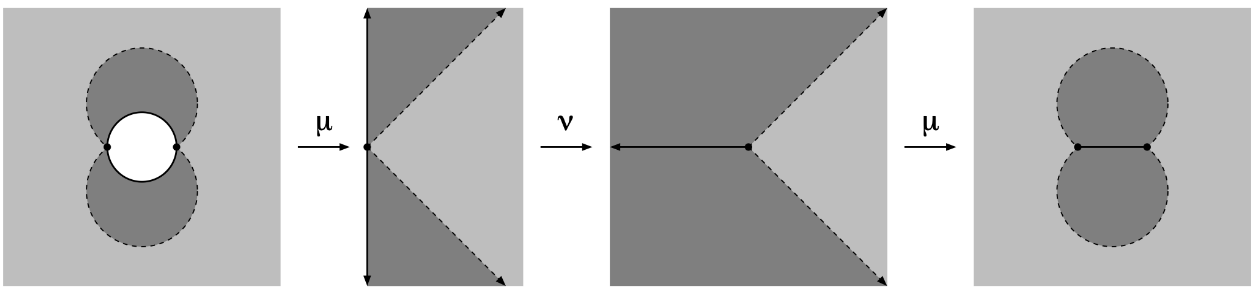

FIGURE 7. A quasiconformal map $\sigma : \{|z| > 1\} \to \mathbb{C} \setminus [-1, 1]$. The composition $\mu \circ \nu \circ \mu$ is the identity in the light gray area, and 3-QC in the dark gray.

Theorem 4.2. *Let $\eta : i\mathbb{R} \to i\mathbb{R}$ be a di-periodic analytic homeomorphism for $d \in \mathbb{N}$ with $\eta(0) = 0$, and let $c > 0$, $n \in \mathbb{N}$. Then for all $m \in \mathbb{N}$ with $m > n \cdot \sup_{z \in i\mathbb{R}} |\eta'(z)|$, there exists a di-periodic K-quasiregular mapping $h : H_{c/n} \to \exp(2\pi cm/n) \cdot \mathbb{D}$ satisfying:*

(1) $h(z) = \exp(2\pi mz)$ *on* $i\mathbb{R} + c/n$
(2) $h(z) = \exp(2\pi n\eta(z))$ *on* $i\mathbb{R}$.

Moreover, the constant K depends only on η and an upper bound for

$$\max\left\{\frac{1}{c}, \frac{m}{n \cdot \inf_{z \in i\mathbb{R}} |\eta'(z)|}\right\}, \tag{4.1}$$

and not otherwise on c, n, d, or m.

Proof. Fix η, d, c, n, m as in the statement. Apply Lemma 2.4 to η, c, n and m to obtain a map φ. Define a subset I of the vertical segment ℓ_d by

$$I := \varphi \circ \eta^{-1}(\{ik/n\}_{k=0}^{dn}) \subset \{ik/m\}_{k=0}^{dm}. \tag{4.2}$$

Note that $\{0, di\} \in I$ since $\eta(0) = 0$ by assumption and hence $\eta(di) = di$ by periodicity. Hence I is admissible. We set $C := 2/(c \cdot \inf_{z \in i\mathbb{R}} |\eta'(z)|)$ and $\varepsilon := c/n$, so that the maximum gap in I satisfies

$$\max(I) < \frac{\sup_{z\in i\mathbb{R}} |(\eta^{-1})'(z)|}{n} + \frac{1}{m} < \frac{2\sup_{z\in i\mathbb{R}} |(\eta^{-1})'(z)|}{n} = C\varepsilon.$$

Thus Lemma 3.8 applies to m, d, ε, C, and I to produce a map

$$\psi : R^d_{c/n} \setminus \mathcal{T}^C_I \to R^d_{c/n}.$$

Define a tree $\mathcal{T}$ by extending $\mathcal{T}^C_I$ by periodicity, and extend ψ by periodicity to yield a map

$$\psi : H_{c/n} \setminus \mathcal{T} \to H_{c/n}.$$

Finally, let β be the map obtained by applying Lemma 2.3 to η, c, and n, and set

$$\tilde{\mathcal{T}} := (\varphi \circ \beta)^{-1}(\mathcal{T}).$$

We will show that a modification of the mapping

$$\tilde{h}(z) := \exp(2\pi m \cdot \psi \circ \varphi \circ \beta(z)) : H_{c/n} \setminus \tilde{\mathcal{T}} \to \exp(2\pi cm/n) \cdot \mathbb{D} \tag{4.3}$$

satisfies the conclusion of Theorem 4.2.

Let $X \subset \mathbb{C}$ be the set where the map σ of Definition 4.1 is not conformal (this is the dark shaded region in the leftmost picture of Figure 7). The set $\tilde{h}^{-1}(X)$ has dm components in R^d_ε, one neighboring each side of each edge of $\tilde{\mathcal{T}} \cap R^d_\varepsilon$. We let U denote the union of those components of $\tilde{h}^{-1}(X)$ which neighbor an edge of $\tilde{\mathcal{T}} \setminus i\mathbb{R}$. We define

$$h(z) := \begin{cases} \sigma \circ \tilde{h}(z) & \text{if } z \in U \\ \tilde{h}(z) & \text{if } z \in H_{c/n} \setminus U. \end{cases} \tag{4.4}$$

By Lemma 3.8(3), the map h extends continuously across $\tilde{\mathcal{T}}$, and since $\sigma(z) = z$ for $z \in \partial X$ we have that $\sigma \circ \tilde{h}(z) = \tilde{h}(z)$ for $z \in \partial U$. Thus,

$$h : H_{c/n} \to \exp(2\pi cm/n) \cdot \mathbb{D}$$

is a quasiregular mapping.

We claim that h satisfies the conclusions of Theorem 4.2. Since $\beta(z) = \varphi(z) = \psi(z) = z$ for $z \in i\mathbb{R} + c/n$ by Lemmas 2.3, 2.4 and 3.8, it follows that from (4.3) that

$$h(z) = \exp(2\pi mz) \text{ for } z \in i\mathbb{R} + c/n.$$

In order to prove the relation

$$h(z) = \exp(2\pi n\eta(z)) \text{ on } i\mathbb{R}, \tag{4.5}$$

we first note that if $\mathcal{I}$ is a component of $\mathcal{T} \cap i\mathbb{R}$, then:

$$\exp(2\pi m\psi \circ \varphi \circ \beta(\mathcal{I})) = \exp(2\pi n\eta(\mathcal{I})). \tag{4.6}$$

Moreover, by Lemmas 2.4 and 3.8, the map $z \mapsto 2\pi m \cdot \psi \circ \varphi(z)$ is affine on $\beta(\mathcal{I}) = \mathcal{I}$, and the map $z \mapsto 2\pi nz$ is affine on $\eta(\mathcal{I})$. Thus, by Part (2) of Lemma 2.3, the maps

$z \mapsto 2\pi m\psi \circ \varphi \circ \beta(z)$ and $z \mapsto 2\pi n\eta(\mathcal{I})$ have the same derivative (pointwise) on $\mathcal{I}$. This implies (together with (4.3) and (4.6)) the relation (4.5).

It remains to study the dependence of K on the parameters. We denote the quasiregularity constant of h by $K(h)$ (similarly for $K(\varphi)$, $K(\beta)$, $K(\psi)$, $K(\sigma)$). By Lemmas 2.3, 2.4 we have that $K(\varphi)$, $K(\beta)$ depend only on η and c (and not on n, d or m). By Lemma 3.8, $K(\psi)$ depends only on an upper bound for

$$\max\{C, m \cdot \max(I)\} \leq \max\left\{\frac{2}{c \cdot \inf_{z\in i\mathbb{R}} |\eta'(z)|}, \frac{2m}{n \cdot \inf_{z\in i\mathbb{R}} |\eta'(z)|}\right\},$$

which in turn depends only on (4.1) and η, (and not otherwise on c, n, d or m). Since $K(\sigma) = 3$, it follows from the definition of h that $K(h)$ depends only on η and (4.1), and not otherwise on c, n, d or m. □

Definition 4.3. We denote the set of critical values of an analytic function by $\mathrm{CV}(g)$, and we set $A(1, r) := \{z \in \mathbb{C} : 1 \leq |z| \leq r\}$. If $\gamma \subset \widehat{\mathbb{C}}$ is a Jordan curve, we say two covering maps f, $g : \gamma \to \mathbb{T}$ are *of the same orientation* if they are simultaneously orientation preserving or reversing for any given orientations of γ, $\mathbb{T}$.

We will use the following consequence of Theorem 4.2 in the proof of Theorem 1.4.

Corollary 4.4. *Let γ be an analytic Jordan curve, $d \in \mathbb{N}$, and suppose f, $g : \gamma \to \mathbb{T}$ are d-to-1 covering maps of the same orientation which extend holomorphically to a neighborhood of γ. Assume furthermore that $f^{-1}(1) \cap g^{-1}(1) \neq \emptyset$. For any $r > 1$ so that $A(1, r) \cap \mathrm{CV}(g) = \emptyset$, denote by A_r the annular component of $g^{-1}(A(1, r))$ satisfying $\gamma \subset \partial A_r$. Then for all such r, and for any n, $m \in \mathbb{N}$ with $m > n \cdot \sup_{z\in\gamma} |f'(z)|/\inf_{z\in\gamma} |g'(z)|$, there exists a K-quasiregular mapping $h : A_{\sqrt[n]{r}} \to r^{m/n}\mathbb{D}$ so that:*

(1) *$h(z) = f(z)^n$ for $z \in \gamma$.*
(2) *$h(z) = g(z)^m$ for $z \in \partial(A_{\sqrt[n]{r}}) \setminus \gamma$.*

Moreover, K depends on γ, f, g, r and an upper bound for m/n, but not on d or otherwise on n or m.

Proof. Fix γ, f, g, d, r and m, n as in the statement. Let $c := \log(r)/2\pi$. We may assume without loss of generality that 0 is in the interior of γ. Thus $g : A_r \to A(1, r)$ has a lift $\hat{g}$ under the exponential, in other words $\hat{g} : \frac{1}{2\pi} \log(A_r) \to H_c$ satisfies

$$\exp\left(2\pi\hat{g}\left(\frac{\log(z)}{2\pi}\right)\right) = g(z) \text{ for } z \in A_r.$$

Similarly, we denote by $\hat{f}$ the lift of $f : \gamma \to \mathbb{T}$ under the exponential. Since $f^{-1}(1) \cap g^{-1}(1) \neq \emptyset$, the lifts $\hat{f}$, $\hat{g}$ may be chosen so that $\hat{f}(0) = \hat{g}(0)$. Since f, g are of the same orientation, the map $\eta(z) := \hat{f} \circ \hat{g}^{-1} : i\mathbb{R} \to i\mathbb{R}$ is a di-periodic analytic homeomorphism satisfying $\eta(0) = 0$.

Hence Theorem 4.2 applies to η, d, c, n, m to produce a mapping which we denote by $\hat{h}$. We claim that the map

$$h := \hat{h} \circ \hat{g} \circ \left(z \mapsto \frac{\log z}{2\pi} \right) : A_{\sqrt[n]{r}} \to r^{m/n}\mathbb{D} \tag{4.7}$$

satisfies the conclusions of the corollary. Indeed, conclusion (1) (resp. conclusion (2)) follows directly from conclusion (1) (resp. conclusion (2)) of Theorem 4.2 together with the fact that $\hat{f}$ (resp. $\hat{g}$) is a lift of f (resp. g). Similarly, the the dependence of the dilatation of (4.7) on the parameters follows from the analagous conclusion for the dilatation of $\hat{h}$ in Theorem 4.2. □

5. Proper Mappings with Prescribed Zeros

In this section we prove the existence of a proper map $B : \Omega \to \mathbb{D}$ on a multiply connected domain Ω. The existence of such a map is due to Ahflors (as discussed in the introduction), but unlike in Ahlfors's theorem, we would like to be able to specify a unique zero for B so that we can specify the poles in Theorem 1.4. In order to do this we will need to consider a perturbation of Ω, as will be explained later in this section (see Theorem 5.5).

Definition 5.1. We say a domain $\Omega \subset \mathbb{C}$ is *analytic* if $\partial\Omega$ has finitely many components, each of which is an analytic Jordan curve. We call a domain $\Omega' \subset \mathbb{C}$ an *analytic ε-perturbation* of Ω if Ω' is analytic and there exists a homeomorphism $\phi : \overline{\Omega} \to \overline{\Omega'}$ satisfying $||\phi(z) - z||_{\overline{\Omega}} < \varepsilon$.

We recall a continuous mapping $B : \Omega \to \mathbb{D}$ is said to be *proper* if for every compact $K \subset \mathbb{D}$, the set $B^{-1}(K)$ is a compact subset of Ω.

Theorem 5.2. *Let Ω be an analytic domain, $p \in \Omega$, and $\varepsilon > 0$. Then there exists an analytic ε-perturbation Ω' of Ω and a proper mapping $B : \Omega' \to \mathbb{D}$ so that $B^{-1}(0) = \{p\}$.*

The idea of the proof of Theorem 5.2 is to consider the Green's function $z \mapsto G(z,p)$ on Ω with pole at p, and set

$$B(z) := \exp(-G(z,p) - i\tilde{G}(z,p)), \tag{5.1}$$

where $\tilde{G}$ is the harmonic conjugate of G. However, when Ω is multiply-connected (which is the main case of interest in this paper), the harmonic conjugate $\tilde{G}$ need not be single-valued, and hence (5.1) need not be single-valued. The first step in proving Theorem 5.2 is to find a function $h(z)$ harmonic in Ω so that replacing $G(z,p)$ with $G(z,p) + h(z)$ in (5.1) defines a single-valued function. The existence of h follows from a standard result about periods of harmonic functions (Theorem 5.4 below). To be as self-contained as possible, we give the details here.

Suppose an analytic domain Ω has m boundary components, denoted $(\gamma_j)_{j=1}^m$. For $j = 1, \dots, m$ let ω_j be the harmonic function on Ω that has boundary value 1 on γ_j and 0 on the other boundary components. Since the boundary components are analytic, each ω_j extends to be analytic across $\partial\Omega$, so the normal and tangential derivatives are well defined, and are themselves analytic, at every boundary point. The period of ω_j along γ_k is defined by

$$\mathrm{per}(\omega_j, \gamma_k) := \int_{\gamma_k} \frac{\partial \omega_j}{\partial n} ds,$$

where $\partial/\partial n$ denotes the outward pointing normal derivative, and ds denotes arclength measure.

The harmonic conjugate $\widetilde{\omega}_j$ of ω_j is well defined locally, but when we analytically continue it around the boundary component γ_k, its value changes by

$$\int_{\gamma_k} \frac{\partial \widetilde{\omega}_j}{\partial t} ds = \int_{\gamma_k} \frac{\partial \omega_j}{\partial n} ds = \mathrm{per}(\omega_j, \gamma_k), \tag{5.2}$$

where $\partial/\partial t$ denotes the tangential derivative. Thus $\widetilde{\omega}_j$ is a multi-valued harmonic function whose value changes by $\mathrm{per}(\omega_j, \gamma_k)$ each time we travel around γ_k, hence the terminology "period". We will use the following result (Theorem 5.4 below) due to Heins [Hei50], and in greater generality to Khavinson [Kha84]. Its proof uses the following elementary fact from linear algebra (see [Tau49]):

Lemma 5.3. *Let $A = (a_{i,j})_{i,j=1}^n$ be a square matrix satisfying $\sum_{j\neq k} |a_{j,k}| < |a_{k,k}|$ for all $1 \leq k \leq n$. Then A is invertible.*

Proof. Let A be as in the statement of the lemma. It suffices to show A^T is invertible. Suppose by way of contradiction that there exists a non-zero vector $(v_1, ..., v_n) \in \mathbb{R}^n$ satisfying $A^T v = 0$. Let v_k be the component of v with the largest modulus. Then

$$|a_{k,k} v_k| = \left| \sum_{j\neq k} a_{j,k} v_j \right| \leq |v_k| \sum_{j \neq k} |a_{j,k}| < |a_{k,k}| \cdot |v_k|,$$

which is a contradiction. $\square$

For $x = (x_1, ..., x_n) \in \mathbb{R}^n$, we will denote by $||x|| := \sqrt{x_1^2 + ... + x_n^2}$ the L^2 norm of x. We remark that we are choosing some particular norm only to avoid ambiguity, and that any norm on $\mathbb{R}^n$ will work in what follows.

Theorem 5.4. *Suppose Ω is an analytic domain with boundary components $(\gamma_j)_{j=1}^m$, and $(\omega_j)_{j=1}^m$ are defined as above. Then, for any $(v_j)_{j=2}^m \subset \mathbb{R}$, there exist $(a_j)_{j=2}^m \subset \mathbb{R}$ so that*

$$\mathrm{per}\left(\sum_{j=2}^n a_j \omega_j, \gamma_k \right) = v_k \textit{ for } k = 2, ..., m. \tag{5.3}$$

Moreover, there exists a $C < \infty$ depending only on Ω so that $||(a_2, ..., a_m)|| \leq C||(v_2, ..., v_m)||$.

Proof. We set $\lambda_{j,k} := \mathrm{per}(\omega_j, \gamma_k)$. It suffices to show that the matrix $\Lambda := (\lambda_{j,k})_{j,k=2}^m$ is invertible, since (5.3) is equivalent to

$$\Lambda^T \begin{pmatrix} a_2 \\ \vdots \\ a_m \end{pmatrix} = \begin{pmatrix} v_2 \\ \vdots \\ v_m \end{pmatrix}.$$

Since $0 < \omega_j < 1$ in Ω and $\omega_j = 1$ on γ_j, the outward normal derivative $\partial\omega_j/\partial n$ on γ_j is nonnegative. In fact, we claim that $\partial\omega_j/\partial n$ is strictly positive on γ_j. Indeed, if we suppose by way of contradiction $\partial\omega_j/\partial n(z) = 0$ for $z \in \gamma_j$, then taking a harmonic conjugate $\tilde{\omega}_j$ defined in a small neighborhood U of z gives $(\omega_j + i\tilde{\omega}_j)'(z) = 0$, but this implies the image of $U \cap \Omega$ under $\exp(\omega_j + i\tilde{\omega}_j)$ intersects $\mathbb{C} \setminus \{z : |z| \leq e\}$, a contradiction to the property $\omega_j < 1$ in Ω. Thus $\partial\omega_j/\partial n$ is strictly positive on γ_j, and a similar argument proves that $\partial\omega_j/\partial n$ is strictly negative on γ_k for $k \neq j$.

The above shows that

$$\lambda_{j,j} > 0 \text{ and } \lambda_{j,k} < 0 \text{ for all } j \text{ and } k \neq j. \tag{5.4}$$

Moreover, since $\sum_{j=1}^m \omega_j$ is the constant function 1 on Ω, it follows that

$$\sum_{j=1}^m \lambda_{j,k} = 0 \text{ for every } k. \tag{5.5}$$

Thus (5.4) together with (5.5) implies that the $(m-1) \times (m-1)$ matrix Λ (obtained by deleting the first row and column from $(\lambda_{j,k})_{j,k=1}^m$) satisfies the hypothesis of Lemma 5.3, and hence Λ is invertible. Thus the theorem is proven, where we may take $C := ||(\Lambda^T)^{-1}||$ (which depends only on Ω). $\square$

Given an analytic domain Ω and a point $p \in \Omega$, as before we denote the Green's function for Ω with pole at p by $G(z,p)$. We recall that $z \mapsto G(z,p)$ is the unique harmonic function in $\Omega \setminus \{p\}$ satisfying

(1) $G(z,p) = 0$ for $z \in \partial\Omega$, and
(2) $G(z,p) + \log|z-p|$ is bounded in a neighborhood of p.

It will be useful to recall that the harmonic measure of a set $E \subset \partial\Omega$ satisfies

$$\omega(E) = \omega(E, p, \Omega) = -\frac{1}{2\pi}\int_E \frac{\partial G}{\partial n}(z,p)\,ds,$$

and so we have

$$\mathrm{per}(G(z,p), \gamma_j) = -2\pi \cdot \omega(\gamma_j, p, \Omega) \tag{5.6}$$

for any component γ_j of $\partial\Omega$. We will now prove Theorem 5.2, but first it will be useful to record the following theorem.

Theorem 5.5. *Let Ω be an analytic domain with boundary components $(\gamma_j)_{j=1}^m$, and $p \in \Omega$. Then there exists a proper mapping $B : \Omega \to \mathbb{D}$ satisfying $B^{-1}(0) = p$ if and only if*

$$\omega(\gamma_j, p, \Omega) \in \mathbb{Q} \text{ for each } 1 \leq j \leq m. \tag{5.7}$$

Proof. Assume (5.7) holds. Together with (5.6), the relation (5.7) implies that there exists $n \in \mathbb{N}$ so that

$$\mathrm{per}(n \cdot G(z,p), \gamma_j) \in 2\pi\mathbb{Z} \text{ for all } 1 \leq j \leq m. \tag{5.8}$$

The relation (5.8) together with (5.2) implies that the harmonic conjugate $n\tilde{G}(z,p)$ is such that

$$B(z) := \exp(-nG(z,p) - in\tilde{G}(z,p)) \tag{5.9}$$

is a single-valued, proper holomorphic mapping $B : \Omega \to \mathbb{D}$ with a unique zero at p.

Conversely, let us suppose the existence of B with the specified properties. Denote the order of the zero of B at p by $\mathrm{ord}(B,p)$. Then $\log|B| = 0$ on $\partial\Omega$, and

$$z \mapsto -\frac{\log|B(z)|}{\mathrm{ord}(B,p)} + \log|z-p|$$

is bounded in a neighborhood of p, and hence $-\log|B(z)|/\mathrm{ord}(B,p) = G(z,p)$. Since B is single-valued, we have

$$\mathrm{per}(\log|B|, \gamma_j) \in 2\pi\mathbb{Z}, \text{ and hence } \mathrm{per}(G(z,p), \gamma_j) \in 2\pi\mathbb{Q} \text{ for all } 1 \leq j \leq m. \tag{5.10}$$

The relation (5.7) now follows from (5.10) and (5.6). □

Proof of Theorem 5.2. For $x \in \mathbb{R}$, let $\lfloor x \rfloor$ denote (as usual) the greatest integer less than x, and $\mathrm{frac}(x) := x - \lfloor x \rfloor$ the fractional part of x. For each $n \in \mathbb{N}$, we set

$$v_j^n := -2\pi \cdot \mathrm{frac}\left(\frac{\mathrm{per}(-nG(z,p), \gamma_j)}{2\pi}\right) \text{ for } 2 \leq j \leq m. \tag{5.11}$$

Applying Theorem 5.4 to $(v_j^n)_{j=2}^m$ yields a function h_n harmonic in Ω satisfying

$$\mathrm{per}(-nG(z,p) + h_n, \gamma_j) \in 2\pi\mathbb{Z} \text{ for all } 2 \leq j \leq m, \text{ and} \tag{5.12}$$

$$||h_n||_\Omega \leq ||h_n||_{\partial\Omega} \leq C||(v_2^n, ..., v_m^n)|| \leq 2C\pi m \text{ for all } n, \tag{5.13}$$

where C is as in Theorem 5.4 (and in particular does not depend on n). Since

$$\sum_{j=1}^m \mathrm{per}(-nG(z,p) + h_n, \gamma_j) = 0,$$

we have that (5.12) holds for $j = 1$ as well. Thus, as in the proof of Theorem 5.5,

$$\mathcal{B}_n(z) := \exp(-nG(z,p) + h_n + i(-n\tilde{G}(z,p) + \tilde{h}_n)) \tag{5.14}$$

defines a holomorphic function in Ω satisfying $\mathcal{B}_n^{-1}(0) = p$. Because of the h_n term in (5.14), however, $\mathcal{B}_n$ need not be a proper mapping into the unit disc. We solve this as follows.

Note that $\partial G(z,p)/\partial n \neq 0$ for $z \in \partial\Omega$ (this follows from the same argument as in the proof of Theorem 5.4), and hence $\nabla G(z,p) \neq 0$ for all z in some neighborhood U of $\partial\Omega$. By (5.13) and the Cauchy estimates for harmonic functions, it follows that for large n we also have $\nabla(nG(z,p) + h_n) \neq 0$ for $z \in U$. Let

$$A_m := \left\{ z : \frac{1}{2\exp(2C\pi m)} \leq |z| \leq 2\exp(2C\pi m) \right\}. \tag{5.15}$$

Note that $\mathcal{B}_n^{-1}(A_m) \subset U$ for large n. Since $\mathcal{B}_n$ has no critical points in U (for large n), it follows that $\mathcal{B}_n^{-1}(A_m)$ consists of m components $(\mathcal{A}_j^n)_{j=1}^m$, where each $\mathcal{A}_j^n$ is a topological annulus with one boundary component $= \gamma_j$.

For all $r > \exp(-2C\pi m)$, let

$$S_r : \left\{ z : \frac{1}{2\exp(2C\pi m)} \leq |z| \leq r \right\} \to \left\{ z : \frac{1}{2\exp(2C\pi m)} \leq |z| \leq 1 \right\} \tag{5.16}$$

denote a quasiconformal map satisfying $S_r(z) = z$ in a neighborhood of $|z| = \exp(-2C\pi m)$, and $S_r(z) = z/r$ in a neighborhood of $|z| = r$. The existence of S_r with dilatation depending only on an upper bound for $\max(1/r, r)$ is straightforward.

We define a function

$$\tilde{\mathcal{B}}_n(z) := \begin{cases} S_{|\mathcal{B}_n(\gamma_j)|} \circ \mathcal{B}_n(z) & \text{if } z \in \mathcal{A}_j^n \text{ for } 1 \leq j \leq m \\ \mathcal{B}_n(z) & \text{if } z \notin \cup_j \mathcal{A}_j^n. \end{cases} \tag{5.17}$$

The map $\tilde{\mathcal{B}}_n : \Omega \to \mathbb{D}$ is a proper, quasiregular mapping with dilatation bounded independently of n. Thus, by the MRMT there exists a quasiconformal mapping $\phi_n : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ (normalized to fix p as well two other arbitrary points in $\widehat{\mathbb{C}}$) so that

$$B_n := \tilde{\mathcal{B}}_n \circ \phi_n^{-1} : \phi_n(\Omega) \to \mathbb{D}$$

is a proper, holomorphic mapping which we claim satisfies the conclusions of the theorem for all sufficiently large n. Indeed, let $\Omega' := \phi_n(\Omega)$. Since S_r is holomorphic in a neighborhood of $|z| = r$, the map ϕ_n is conformal in a neighborhood of $\partial\Omega$. This implies that Ω' is analytic. Moreover, since the dilatation of ϕ_n is supported in $\cup_j \mathcal{A}_j^n$, and $\text{area}(\mathcal{A}_j^n) \to 0$ as $n \to \infty$, we have that $\phi_n : \overline{\Omega} \to \overline{\Omega'}$ satisfies $||\phi_n(z) - z||_{\overline{\Omega}} < \varepsilon$ for large n. Similarly, $B_n^{-1}(0) = \phi_n(p) = p$ by our choice of normalization for ϕ_n. This proves the theorem. $\square$

Remark 5.6. Theorem 5.2 will suffice for our purposes, but we mention that one can prove the slightly stronger conclusion that $\Omega' \subseteq \Omega$, although it is not possible in general to take $\Omega' = \Omega$ (by Theorem 5.5). The main adjustment needed for this improvement is to show that in Lemma 5.3, if the $(v_j)_{j=2}^m$ are positive then so are the $(a_j)_{j=2}^m$ (see Proposition 4.1.3 of [Gru78] for a short proof of this fact). We also mention that our proof of Theorem 5.2 shows the existence of a proper quasiregular map on the original domain Ω with a unique zero at p, such that the quasiregular map is holomorphic off of a region of arbitrarily small area near $\partial\Omega$.

6. Proof of Theorem 1.4: approximation by rational lemniscates

We fix a collection of N pairwise disjoint analytic Jordan curves $\{\Gamma_j\}_{j=1}^N$, and set $\Gamma := \cup_1^N \Gamma_j$. Denote the $N+1$ connected components of $\Omega := \widehat{\mathbb{C}} \setminus \Gamma$ by $\{\Omega_j\}_{j=0}^N$. As discussed in the introduction, the collection $\{\Omega_j\}_{j=0}^N$ has a natural tree structure. Since trees are bipartite, we can 2-color the regions $\{\Omega_j\}_{j=0}^N$ in one of two ways (a choice of color for any given Ω_j determines the colors for all other components). We fix such a 2-coloring and let B (black) and W (white) denote the indices of the two collections.

Let $P = (p_j)_{j\in B}$ be a collection of points satisfying $p_j \in \Omega_j$ for each $j \in B$. Applying Theorem 5.2 to each Ω_j (where $j \in B$) gives a perturbation Ω_j' of Ω_j and a proper map $\mathcal{B}_j : \Omega_j' \to \mathbb{D}$ satisfying $\mathcal{B}_j^{-1}(0) = p_j$. We set $\Gamma' := \cup_{j\in B} \partial\Omega_j'$ and $\Omega' := \mathbb{C} \setminus \Gamma'$. Let $(\Omega_j')_{j=0}^N$ (resp. $(\Gamma_j')_{j=1}^N$) denote the components of Ω' (resp. Γ'), with indexing chosen so that Ω_j' (resp. Γ_j') is a perturbation of Ω_j (resp. Γ_j) for each $0 \le j \le N$ (resp. $1 \le j \le N$).

For each $j \in W$, let $\mathcal{A}_j : \Omega_j' \to \mathbb{D}$ denote an Ahlfors map. For each $0 \le j \le N$, we define a map g_j on Ω_j' by $g_j := \mathcal{A}_j$ if $j \in W$ and $g_j := 1/\mathcal{B}_j$ if $j \in B$. For any $0 \le j \le N$, we let j_B, j_W denote the two indices satisfying $j_B \in B$, $j_W \in W$ and $\Gamma_j = \Omega_{j_B} \cap \Omega_{j_W}$.

Lemma 6.1. *There exists $d \in \mathbb{N}$ so that for any $0 \le j \le N$, the two maps $g_{j_W}^d : \Gamma_j' \to \mathbb{T}$ and $g_{j_B}^d : \Gamma_j' \to \mathbb{T}$ have the same degree.*

Proof. This follows from a simple recursive procedure: start with some region Ω_j and raise each of the maps g_k which are defined on some boundary component of Ω_j to the appropriate power d_k so that each of the maps $g_k^{d_k}$ have degree on $\partial\Omega_j \cap \partial\Omega_k$ equal to the product (over k) of the degrees of g_k on $\partial\Omega_j \cap \partial\Omega_k$. Next, do the same procedure on a neighboring component of Ω_j, making sure to also raise the maps defined in the previous step to the appropriate power in this next step. Because there are only finitely many Ω_j, this procedure terminates after finitely many steps and gives a d with the desired properties. $\square$

We set $G_j := g_j^d$ for $0 \le j \le N$. A recursive argument similar to the proof of Lemma 6.1 shows that, after dividing a collection of the maps $(G_j)_{j=0}^N$ by unimodular constants if necessary, we may assume that

$$(G_{j_W})^{-1}(1) \cap (G_{j_B})^{-1}(1) \neq \emptyset \text{ for all } 0 \le j \le N.$$

We henceforth fix a constant $r > 1$ so that the maps $(G_j)_{j\in B}$ have no critical values in $A(1, r)$. In particular, for every $n \in \mathbb{N}$ the set

$$\mathcal{A}^n := \bigcup_{j\in B} G_j^{-1}(A(1, \sqrt[n]{r})) \tag{6.1}$$

has N components, each of which is a topological annulus. We denote the components of $\mathcal{A}^n$ by $(\mathcal{A}^n_j)_{j=1}^N$, where the labelling is chosen so that $\Gamma'_j \subset \partial\mathcal{A}^n_j$.

Theorem 6.2. *For all $n \in \mathbb{N}$, there exists $m = m(n)$ and a K-quasiregular mapping $h_n : \mathcal{A}^n \to r^{m/n}\mathbb{D}$ so that for each $1 \leq j \leq N$, we have:*

1. $h_n(z) = G_{j_W}(z)^n$ *on* Γ'_j.
2. $h_n(z) = G_{j_B}(z)^m$ *on* $\partial(\mathcal{A}^n_j) \setminus \Gamma'_j$.

Moreover, K depends only on Γ and r, and not on n.

Proof. Define

$$m(n) := \min\left\{ k : k > n \frac{\sup_{1\leq j\leq N, z\in\Gamma'_j} |G'_{j_W}(z)|}{\inf_{1\leq j\leq N, z\in\Gamma'_j} |G'_{j_B}(z)|} \right\}. \tag{6.2}$$

Corollary 4.4 then applies to $\gamma := \Gamma'_j$, $f := G_{j_W}$, $g := G_{j_B}$ for each $1 \leq j \leq N$ to define maps

$$h_j : \mathcal{A}^n_j \to r^{m/n}\mathbb{D}.$$

We set

$$h_n := \bigsqcup_{1\leq j\leq N} h_j : \mathcal{A}^n \to r^{m/n}\mathbb{D}.$$

Conclusions (1) and (2) of Theorem 6.2 follow directly from conclusions (1) and (2) of Corollary 4.4. Since m/n is bounded independently of n by (6.2), Corollary 4.4 also implies that the dilatation of h_n is bounded independently of n. □

Definition 6.3. We define a sequence of $\widehat{\mathbb{C}}$-valued functions f_n by setting

$$f_n := G_j^n \text{ in } \Omega'_j \text{ if } j \in W,$$

$$f_n := G_j^m \text{ in } \Omega'_j \setminus \mathcal{A}^n \text{ if } j \in B, \text{ and}$$

$$f_n := h_n \text{ in } \mathcal{A}^n.$$

Lemma 6.4. *For each $n \in \mathbb{N}$, the mapping f_n of Definition 6.3 extends to a K-quasiregular function $f_n : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ with K independent of n, and*

$$\text{area}(\text{supp}((f_n)_{\overline{z}})) \xrightarrow{n\to\infty} 0. \tag{6.3}$$

Proof. The map f_n is defined on $\widehat{\mathbb{C}} \setminus \partial\mathcal{A}^n$, where it is K-quasiregular by Theorem 6.2, with K independent of n. Thus to show f_n extends to a K-quasiregular function $f_n : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, it suffices (by a standard removability result for quasiregular mappings) to show that f_n extends continuously across $\partial\mathcal{A}^n$. This follows from the definition of f_n and (1) and (2) of Theorem 6.2. Since f_n is holomorphic in $\widehat{\mathbb{C}} \setminus \mathcal{A}^n$, we have that $\text{supp}((f_n)_{\overline{z}}) \subset \overline{\mathcal{A}^n}$. Hence (6.3) follows from (6.1) since $\sqrt[n]{r} \to 1$ as $n \to \infty$. □

Lemma 6.5. *For all n, we have*

$$f_n^{-1}(r^{m/n}\mathbb{T}) = \bigsqcup_{1\leq j\leq N} \partial(\mathcal{A}_j^n) \setminus \Gamma_j'.$$

Proof. First, note that if $j \in W$, then $f_n(\Omega_j) \subset \mathbb{D}$, so $f_n^{-1}(r^{m/n}\mathbb{T})$ does not intersect any of the white regions $(\Omega_j)_{j\in W}$. If $j \in B$, then $f_n(\mathcal{A}_j^n) \subset r^{m/n}\mathbb{D}$. Hence, it follows from the definition of f_n that

$$f_n^{-1}(r^{m/n}\mathbb{T}) = \bigcup_{j\in B}(G_j^m)^{-1}(r^{m/n}\mathbb{T}) = \bigcup_{j\in B}(G_j)^{-1}(\sqrt[n]{r}\mathbb{T}) \tag{6.4}$$

Moreover, r was chosen so that $\sqcup_{j\in B} G_j : \mathcal{A}^n \to A(1, \sqrt[n]{r})$ is a covering map, and $\mathcal{A}^n$ has N components, each of which is a topological annulus. Thus (6.4) has N components, each of which is a Jordan curve coinciding with one of the $\partial(\mathcal{A}_j^n) \setminus \Gamma_j'$. □

Thus each $\sigma_n := f_n^{-1}(r^{m/n}\mathbb{T})$ is a collection of N closed Jordan curves that approximate Γ from inside the black regions. By Lemma 6.4 and the MRMT, for each n there exists a K-quasiconformal mapping $\phi_n : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ so that

$$r_n := f_n \circ \phi_n^{-1} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$$

is holomorphic (and hence a rational function). We normalize each ϕ_n so as to fix any three given points p_1, p_2, $p_3 \in P$ (if the p_1, p_2, p_3 are not distinct, then we specify that ϕ_n also fixes another 1 or 2 arbitrary points in $\widehat{\mathbb{C}}$).

Lemma 6.6. *The mappings ϕ_n converge to the identity uniformly on compact subsets of $\widehat{\mathbb{C}}$.*

Proof. This follows from (6.3) since the mappings ϕ_n are K-quasiconformal, all normalized to fix the same three points, and with K independent of n (by Lemma 6.5). □

Proof of Theorem 1.4. We are given N disjoint annuli $(A_j)_{j=1}^N$ and, as described in the introduction, we choose N analytic Jordan curves $(\Gamma_j)_{j=1}^N$, one in each annulus and that separate the boundary components of that annulus. We then construct curves $(\Gamma_j')_{j=1}^N$ as above which also lie inside the disjoint annuli, and rational mappings $(r_n)_{n=1}^\infty$. By Lemma 6.5, the set

$$r_n^{-1}(r^{m/n}\mathbb{T}) = \phi_n(f_n^{-1}(r^{m/n}\mathbb{T})) \tag{6.5}$$

consists of N components, each of which is a Jordan curve coinciding with

$$\sigma_j^n := \partial(\phi_n(\mathcal{A}_j^n)) \setminus \phi_n(\Gamma_j')$$

for some j. Because the σ_j^n are Jordan curves, they are automatically analytic (they contain no critical points). Since $\sqrt[n]{r}\cdot\mathbb{T} \to 1$ as $n \to \infty$, the annulus $\mathcal{A}_j^n$ lies inside any neighborhood of Γ_j' for large enough n, and hence the same is true of $\phi_n(\mathcal{A}_j^n)$ by Lemma 6.6. Thus the σ_j^n separate the boundary components of the $(A_j)_{j=1}^N$ for large n. By the normalization of ϕ_n

we have p_1, p_2, $p_3 \in r_n^{-1}(\infty)$ for all n, and $|r_n^{-1}(\infty)| = |\phi_n(P)| = |P|$ for all n since ϕ_n is a homeomorphism. Moreover $d_H(r_n^{-1}(\infty), P) \xrightarrow{n\to\infty} 0$ by Lemma 6.6. Therefore the rational map $r^{-m/n} \cdot r_n$ satisfies Theorem 1.4 if n is sufficiently large. $\square$

7. Consequences and Questions

We remark that approximation by polynomial or rational lemniscates is easier if one is only interested in Hausdorff approximation, but not the topological properties of the lemniscate or the number of their components. For instance, by Hilbert's theorem it is possible to approximate any finite collection of pairwise disjoint Jordan curves by polynomial lemniscates (see the center of Figure 8), and it is easy to approximate any compact set by disconnected lemniscates (again see Figure 8).

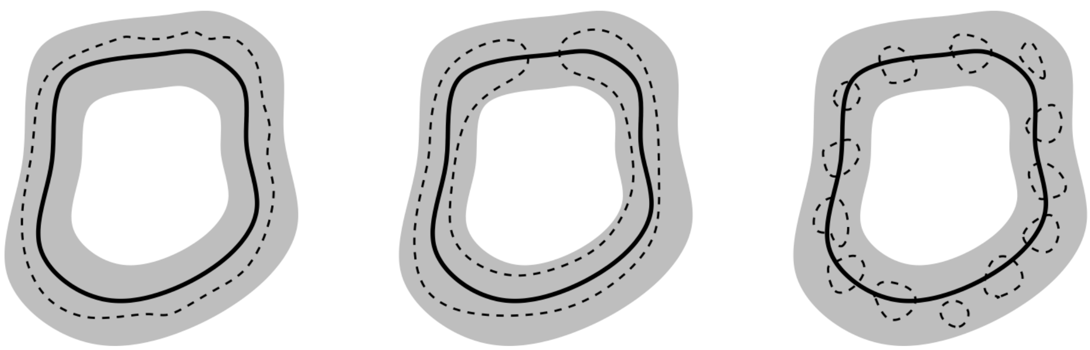

Figure 8. The left picture shows approximation of a curve (solid) by a lemniscate (dashed). In the center, the same solid curve is approximated in the Hausdorff metric by the dashed Jordan curve, but the dashed curve does not separate the boundary components of the annulus. The right side shows approximation by a lemniscate with several components: note that any compact set K can be approximated in the Hausdorff metric by a disconnected lemniscate of the polynomial $C\prod_1^n(z-x_j)$, if the points $\{x_j\}$ are ε-dense in K and C is sufficiently large.

Given a rational map r on $\widehat{\mathbb{C}}$ with an attracting fixed point at ∞, let $\mathcal{J}(r)$ denote its Julia set and $\mathcal{K}(r)$ its filled Julia set (the complement of the attracting basin of ∞). Theorem 1.4 implies the following.

Corollary 7.1. *Let $K \subset \mathbb{C}$ compact, suppose P contains exactly one point from each component of $\widehat{\mathbb{C}} \setminus K$, and let p_1, p_2, $p_3 \in P$. Then for all $\varepsilon > 0$, there exists $P' \subset P$ and a rational mapping $r : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ satisfying p_1, p_2, $p_3 \in r^{-1}(\infty)$, $d_H(r^{-1}(\infty), P') < \varepsilon$ and*

$$d_H(\mathcal{K}(r), K) < \varepsilon \text{ and } d_H(\mathcal{J}(r), \partial K) < \varepsilon. \tag{7.1}$$

When $\widehat{\mathbb{C}} \setminus K$ is connected and $P = \{\infty\}$, this is Theorem 1.2 of [LY19]. They give two proofs of their result. The second uses Hilbert's polynomial lemniscate theorem and replacing this by Theorem 1.4 in their proof gives the corollary.

It seems reasonable to ask if all of the poles in Theorem 1.4 can be specified exactly (not just three exactly and the remaining approximately).

Question 7.2. Can the conclusion "$p_1, p_2, p_3 \in r^{-1}(\infty)$, $|r^{-1}(\infty)| = |P|$, and $d_H(r^{-1}(\infty), P) < \varepsilon$" in Theorem 1.4 be improved to "$r^{-1}(\infty) = P$"?

One plausible approach to such an improvement would be to consider families of maps $(\mathcal{B}_j)_{j\in B}$ with unique zeros $(p'_j)_{j\in B}$ varying in a neighborhood of $(p_j)_{j\in B}$. If the quasiconformal map ϕ_n coming from the MRMT depended continuously on $(p'_j)_{j\in B}$, one could conclude by a fixpoint argument that one could choose the poles of r_n to coincide with the $(p_j)_{j\in B}$. However, in Lemma 2.4 the process of "choosing a closest m-point" is discontinuous and we have found no way yet to avoid this. A similar difficulty is encountered in trying to answer the following question.

Question 7.3. Let $(A_j)_{j=1}^N$ be a collection of pairwise disjoint topological annuli on a compact Riemann surface S, such that the connected components of $S \setminus \cup_j A_j$ are 2-colorable. Can the $(A_j)_{j=1}^N$ be separated by a meromorphic lemniscate on S with N components (each of which is an analytic Jordan curve) with specified poles in each black region?

Much of the proof in the current paper goes though on Riemann surfaces. Ahlfors [Ahl50] proved that Ahlfors functions exist on any finite bordered Riemann surface (a compact surface with a finite number of disjoint Jordan regions removed). The maps have finite degree, but this degree may be larger than the number of boundary components since it also depends on the genus of the surface. Thus we can create the required quasiregular map on S, even with specified poles. However, when we solve the Beltrami equation, we end up with a meromorphic function f on a different Riemann surface S'. We may take S' as close to S as we wish in the Teichmüller metric, but it is not yet clear if we can take $S' = S$. Perhaps a fixed point argument on Teichmüller space will work here. It is also not obvious whether a similar lemniscate theorem will hold on arbitrary Riemann surfaces, or under what conditions an Ahlfors function will exist on a (non-compact) bordered Riemann surface.

Question 7.4. Given n disjoint, finite sets $\{X_k\}_1^n$, each containing at most k points, can we compute the minimal degree of a rational map r so that each connected component of $r^{-1}(\widehat{\mathbb{C}} \setminus \mathbb{T})$ contains at one of the sets? Can this degree be bounded in terms of n and k alone?

Given two finite sets of size n and k it is easy to separate them using a polynomial of degree $\min(n, k)$ by placing zeros at the points of the smaller set. If we can separate the points with a linear polynomial, it means the sets can be separated by a circle. The lowest degree polynomial that separates two finite sets is thus some measure of how "overlapping" these sets are, and is likely to be connected to the pattern recognition and machine learning applications mentioned in the introduction. How can we compute the lowest degree polynomial whose lemniscate separates two finite point sets? The lowest degree polynomial with a connected lemniscate that separates them?

It is known that rational functions are much more efficient at approximating certain functions than polynomials. For instance, on $[-1, 1]$, the best polynomial approximation of degree n for $|x|$ has error $\simeq 1/n$, but the best rational approximation of degree n has error $\simeq \exp(-\pi\sqrt{n}) \ll 1/n$. See Chapter 25 of [Tre13]. Do rational lemniscates give more efficient separation of sets than polynomial lemniscates?

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