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Constructing Continuous Functions Holomorphic off a Curve

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We give new proofs of a theorem of A. Browder and J. Wermer and a theorem of A. Davie which characterize the plane sets K for which A_K is a Dirichlet algebra. The use of functional analysis in the original proofs is replaced by a construction involving bounded solutions of the $\bar{\partial}$ equation. In particular, this gives an explicit construction of nonconstant functions in these spaces. © 1989 Academic Press, Inc.

1. INTRODUCTION

If Ω is an open subset of the Riemann sphere, $\bar{\mathbf{C}}$, we let $H^\infty(\Omega)$ denote the space of bounded holomorphic functions on Ω and let $A(\Omega)$ denote the subspace of functions in $H^\infty(\Omega)$ which extend continuously to $\bar{\Omega}$, the closure of Ω. If $K \subset \bar{C}$ is compact we let $A_K \equiv A(\bar{C}\backslash K)$. In this paper we are concerned with constructing nonconstant functions in A_K. We will be interested in the special case when $K = \Gamma$ is a closed Jordan curve, so we introduce some notation for this case. Suppose Γ is a closed Jordan curve on the Riemann sphere and let Ω_1 and Ω_2 denote the two components of $\Omega = \bar{C}\backslash\Gamma$. Choose a point $z_1 \in \Omega_1$ and let $\omega_1(E) = \omega(z_1, E, \Omega_1)$ denote the harmonic measure on Γ with respect to z_1. Since Ω_1 is simply connected, the Riemann mapping theorem says there is a conformal map Φ_1 from the unit disk $D = \{|z| < 1\}$ to Ω_1 with $\Phi_1(0) = z_1$. Ω_1 is a Jordan domain, so by Carathéodory's theorem Φ_1 extends to a homeomorphism of $\mathbf{T} = \{|z| = 1\}$ to Γ. It is well known that ω_1 is the image of normalized Lebesgue measure on $\mathbf{T}$ under this correspondence. We choose $z_2 \in \Omega_2$ and define ω_2 and Φ_2 similarly.

Of course, A_K need not contain any nonconstant functions. For example, if K is a straight line, then Morera's theorem implies that any continuous

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function holomorphic off K is actually entire, and thus is constant by Liouville's theorem. It is an observation of Riemann that the same is true for any smooth curve Γ, and Painlevé showed $A_\Gamma = \mathbf{C}$ whenever Γ is rectifiable.

On the other hand, if K has interior then clearly $A(K)$ is nontrivial. For a less obvious example, suppose $K \subset D(0, 1)$ is a compact, totally disconnected subset of C with positive area. Then we can easily check that

$$F(z) = \frac{1}{z} * \chi_K(z) = \int_K \frac{1}{w - z}\, dx\, dy$$

defines a nonconstant element of A_K. This construction was given in a 1909 paper of Denjoy [12] and is based on an example of Pompéiu. It was a surprising result because previously it had been thought that any continuous function holomorphic off a totally disconnected set must be entire. Moreover, this construction used the (then) recent work of Cantor on the existence of perfect, totally disconnected sets and Lebesgue's work on integration (note that χ_K is not Riemann integrable). In a commentary on Denjoy's paper, Painlevé says [25],

> Il convient de signaler le rôle joué, dans ce résultat, par l'extension due à M. Lebesgue, de l'intégrale définie. Grace à cette opération, que nombre de géomètres jugeaient artificielle et trop abstraite, une question naturelle, une question fondamentale qui restait indécise à l'entrée de la théorie des fonctions uniformes, est aujourd'hui tranchée, et tranchée précisément dans le sens qui semblait le moins vraisemblable à la plupart des analystes.... L'intégration de *M*. Lebesgue pourra contribuer là encore à la formation d'examples décisifs.

Another interesting observation about A_K is due to John Wermer, and essentially says that any nonconstant function in A_K is necessarily badly behaved on K. More precisely, if K has no interior then for any $f \in A_K$, $f(\bar{\mathbf{C}}) \subset f(K)$, so that if f is nonconstant $f(K)$ must cover a disk. This is a simple consequence of the argument principle; for a proof see [5, Lemma 3.5.4].

It would be nice to be able to characterize the sets K for which A_K is nontrivial, but this problem seems to be very difficult. It is possible, however, to characterize the sets for which A_K is "large." To see what we might mean by "large," note that $A_K \subset C(K)$, but since elements of A_K are holomorphic off K the imaginary parts are determined (up to a constant) by the real parts. Moreover, Wermer's observation above shows that not every continuous real valued function on K can be the real part of an element of A_K. Thus the most we can hope for is that such functions can be approximated by the real parts of functions in A_K. This motivates the following definition.

We say that the function algebra A_K on K is a *Dirichlet algebra* on K if for every continuous, real valued function g on K and every $\varepsilon > 0$ there is an $f \in A_K$ such that

$$\| g - \operatorname{Re}(f) \|_K < \varepsilon,$$

where the norm is the "sup" norm on K. In this case we say K is a *Dirichlet set*.

In the special case of $K = \Gamma$ a curve, A. Browder and J. Wermer proved in 1963 the following:

THEOREM 1.1 (Browder and Wermer [6]). *A_Γ is a Dirichlet algebra on Γ iff $\omega_1 \perp \omega_2$.*

Here "$\omega_1 \perp \omega_2$" means the measures are mutually singular, i.e., there is a subset $E \subset \Gamma$ such that $\omega_1(E) = \omega_2(\Gamma \setminus E) = 0$. This result was later generalized by Davie to characterize all Dirichlet sets. One can check that if K is Dirichlet then it must be connected. In particular, each of the complementary components, Ω_j, of K is simply connected. Thus we can choose a conformal mapping Φ_j from D to Ω_j. It is well known that these maps extend non-tangentially at almost every point of $\mathbf{T}$ and we let $\{\Phi_j\}$ also denote these extensions. Then following Glicksberg in [19] we say Ω_j is *nicely connected* if there is a set of full measure $E_j \subset \mathbf{T}$ such that Φ_j is injective on E_j (this is independent of the particular choice of Riemann mapping). The following is Theorem 4.3.1 in [9].

THEOREM 1.2 (Davie). *A_K is a Dirichlet algebra iff each complementary component Ω_j of K is nicely connected and harmonic measures for different components are mutually singular.*

In [1, 3] geometric characterizations of these conditions are given. To state them we need a few definitions. If $K \subset \mathbf{C}$ is compact and $x \in K$ we say x satisfies a *double cone condition* with respect to K if there is a $\theta_0 \in [0, 2\pi)$, $0 < \varepsilon < \pi/2$, and $\delta > 0$ such that

$$\left\{ x + re^{i\theta} : 0 < |r| < \delta,\ |\theta - \theta_0| < \frac{\pi}{2} - \varepsilon \right\} \cap K = \varnothing,$$

i.e., if there are two symmetric cones with vertex x which do not hit K. We say $x \in K$ is a *tangent point* of K if there is a $\theta_0 \in [0, \pi/2)$ so that for any $\varepsilon > 0$ there exists a $\delta > 0$ with

$$\left\{ x + e^{i\theta} : 0 < |r| < \delta,\ |\theta - \theta_0| < \frac{\pi}{2} - \varepsilon \right\} \cap K = \varnothing.$$

Finally, a set E has *zero linear measure* if we can cover it by disks whose radii sum up to be as small as we wish. One can show that the two definitions are equivalent up to sets of zero linear measure.

LEMMA 1.3. *If Ω_1 and Ω_2 are disjoint and simply connected, with harmonic measures ω_1 and ω_2, then $\omega_1 \perp \omega_2$ iff the set of points in $\mathbf{C}\backslash(\Omega_1 \cup \Omega_2)$ which satisfy a double cone condition (with one cone in each of Ω_1 and Ω_2) has zero linear measure.*

LEMMA 1.4. *If Ω is simply connected, then Ω is nicely connected iff the set of points in $\mathbf{C}\backslash\Omega$ satisfying a double cone condition (with respect to $\mathbf{C}\backslash\Omega$) has zero linear measure.*

Thus we obtain:

COROLLARY 1.5. *K is a Dirichlet set iff it is connected and the set of tangent points of K has zero linear measure. (Equivalently, iff the set of points satisfying a double cone condition has zero linear measure.)*

A_K being a Dirichlet algebra is also related to other types of approximation. Let $\Omega = \bar{\mathbf{C}}\backslash K$. We say A_K is *pointwise boundedly dense* (p.b.d.) in $H^\infty(\Omega)$ if there is a $C>0$ such that for any $f \in H^\infty(\Omega)$ there is a sequence $\{f_n\} \subset A_K$ such that $\|f_n\| \leqslant C\,\|f\|$ and $\{f_n\}$ converges pointwise to f on Ω. A_K is called *strongly pointwise boundedly dense* in $H^\infty(\Omega)$ if we can take $C=1$. Then the following is known:

THEOREM 1.6. *If K is connected then the following are equivalent:*

(1) *A_K is a Dirichlet algebra on K.*

(2) *A_K is pointwise boundedly dense in $H^\infty(\Omega)$.*

(3) *A_K is strongly pointwise boundely dense in $H^\infty(\Omega)$.*

The implication (1) $\Rightarrow$ (2) is due to Hoffman (see [13, 28]), (2) $\Rightarrow$ (3) to Davie in [10], and (3) $\Rightarrow$ (1) to Gamelin and Garnett in [13].

In this paper we shall give new proofs of the three theorems, replacing the use of the Hahn–Banach theorem in the original proofs by explicit constructions involving a $\bar{\partial}$ problem. Our proofs are not simpler than the original ones, but they are more geometrical and they illustrate the connection between these results and more recent developments on H^∞ and BMO. We will first consider the special (and easier) case $K=\Gamma$ a closed Jordan curve. We will then sketch the necessary changes for the general case. After that we will show how the construction also recovers some related results.

2. Proof of Necessity

We start by showing that the mutual singularity of the harmonic measures is a necessary condition for A_Γ to be either a Dirichlet algebra or pointwise boundedly dense in $H^\infty(\Omega)$.

Suppose ω_1 and ω_2 are not mutually singular. Then the results of [1, 3] say there is a neighborhood of a point on Γ in which Γ looks like a Lipschitz graph. More precisely, by Lemma I.9.1 of [1] we may assume (after rescaling) that we are in the situation pictured in Fig. 1: a subarc of Γ is trapped between two Lipschitz graphs, Γ_2 and Γ_3, of length $2R \gg 1$ and agreeing except for sets of length less than ε (R and ε to be fixed later). Choose points z_1 and z_2 at distance 1 from Γ on either side, and let Γ_1 and Γ_4 be circular arcs of radius R connecting the ends of Γ_2 and Γ_3, respectively.

Now suppose $f \in A_\Gamma$. We will estimate $|f(z_1) - f(z_2)|$. By the Cauchy integral formula,

$$2\pi i f(z_1) = \int_{\Gamma_1} \frac{f(w)\,dw}{z_1 - w} + \int_{\Gamma_2} \frac{f(w)\,dw}{z_1 - w}$$

and by our assumptions,

$$\left| \int_{\Gamma_2} \frac{f(w)\,dw}{z_1 - w} - \int_{\Gamma_3} \frac{f(w)\,dw}{z_1 - w} \right| \leqslant \|f\|\, \varepsilon.$$

By the Cauchy integral formula again,

$$\left| 2\pi i f(z_1) - \int_{\Gamma_1 + \Gamma_4} \frac{f(w)\,dw}{z_1 - w} \right| \leqslant \|f\|\, \varepsilon.$$

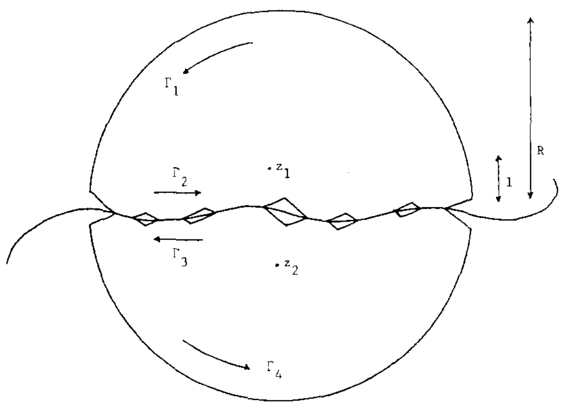

Fig. 1. The curves Γ_i.

Now subtract the corresponding inequality for z_2 and obtain

$$2\pi\,|f(z_1)-f(z_2)| \leqslant 2\,\|f\|\,\varepsilon + \left|\int_{\Gamma_1+\Gamma_4}\left(\frac{1}{z_1-w}-\frac{1}{z_2-w}\right)f(w)\,dw\right|$$
$$\leqslant \|f\|\left(2\varepsilon+\frac{C}{R}\right)$$
$$\leqslant \frac{1}{4}$$

if ε is small enough and R is large enough (depending on $\|f\|$). But this means we cannot pointwise approximate the holomorphic function χ_{Ω_1} by any bounded sequence in A_Γ. Thus A_Γ is not pointwise boundedly dense in $H^\infty(\Omega)$.

To show A_Γ cannot be a Dirichlet algebra on Γ we continue to consider Fig. 1, but now we estimate $|f'(z_1)|$. Using the Cauchy integral formula for derivatives and the preceding argument we get

$$2\pi\,|f'(z_1)| \leqslant \left|\int_{\Gamma_1+\Gamma_4}\frac{f(w)\,dw}{(z_1-w)^2}\right| + \|f\|\,\varepsilon$$
$$\leqslant \|f\|\left(\frac{C}{R}+\varepsilon\right)$$

so that $|f'(z_1)|$ is small if ε is small and R is large (depending on $\|f\|$). But if A_Γ were a Dirichlet algebra, we could take a continuous function g which was 0 on the "left" half of Fig. 1 and 1 on the "right" half, and approximate it to within $\frac{1}{10}$ by $\mathrm{Re}(f)$, $f\in A_\Gamma$. Set $F=e^f$. Then simple estimates and the mean value theorem imply there is a point z_1 satisfying the estimates in the argument above and such that

$$|F(z_1)| \geqslant |\nabla\,\mathrm{Re}(f)(z_1)| \geqslant \frac{1}{100}.$$

Since $F\in A_\Gamma$ and $\|F\|\leqslant e^2$ this is a contradiction (for appropriate choices of ε and R). Thus A_Γ cannot be a Dirichlet algebra on Γ.

3. Three $\bar\partial$ Lemmas

Before describing the construction in detail, we will review a few facts about the $\bar\partial$ problem that we will need later. For an arc $I\subset\mathbf{T}$ we let $Q(I)$ denote the "cube" over I,

$$Q(I)=\{r\xi\colon \xi\in I,\ 1-|I|<r<1\}.$$

Then we say a positive measure μ on D is a *Carleson measure* if

$$\sup_{I \subset \mathbf{T}} \frac{\mu(Q(I))}{|I|} \equiv \|\mu\|_C < \infty$$

and $\|\mu\|_C$ is the *Carleson norm* of μ. Such measures were introduced by Carleson in his solution of the corona problem on D [8]. They are related to the $\bar{\partial}$ problem as follows. If μ is a Carleson measure we can find a function F which solves

$$\bar{\partial} F = \mu$$

(in the sense of distributions) and satisfies

$$\|F\|_{L^\infty(T)} \leqslant C \|\mu\|_C$$

for some universal $C > 0$ (see [15, Chap. VIII] or [23]). Unfortunately, this estimate only holds on $\mathbf{T}$, not on all of D. For example, if μ is a point mass, then any solution must have a pole and so is unbounded on D. However, with addition assumptions we can get estimates on all of D. For example, if γ is a collection of arcs in D and μ is arclength measure on γ then we have:

LEMMA 3.1. *There is a $\delta > 0$ so that if μ is a Carleson measure and φ is a smooth function on D which satisfies*

$$\operatorname{supp}(\nabla\varphi) \subset \{z \in D: \operatorname{dist}(z, \gamma) < \delta(1 - |z|)\}$$
$$|\nabla\varphi(z)| \leqslant \varepsilon\delta^{-1}(1 - |z|)^{-1}$$

then for any bounded, continuous function b on D, there exists a $F \in L^\infty(D)$ such that $\bar{\partial} F = b\bar{\partial}\varphi$ and $\|F\|_D \leqslant C\varepsilon\|b\|_D$, *where C depends only on the Carleson norm of μ.*

This is essentially Lemma 3.3 of [18], but we include a proof for completeness. The basic idea is implicit in Carleson's paper [8] and was first used explicitly by Peter Jones in [22]. Suppose $\{z_n\}$ is a sequence of points on γ which satisfy

$$|z_n - z_m| \geqslant \frac{1}{10}(1 - |z_n|), \qquad n \neq m.$$

Then it is known [15, p. 341] that such a sequence is interpolating for $H^\infty(D)$ with constant depending only on $\|\mu\|_C$ (i.e., for any sequence of complex numbers $\{a_n\}$ of modulus less than 1, there is an $F \in H^\infty(D)$,

$\|F\|_\infty \leqslant C = C(\|\mu\|_C)$ such that $F(z_n) = a_n$). Thus by [15, Theorem VII.2.1] there exist holomorphic functions $\{h_n\}$ such that

$$h_n(z_n) = 1$$

$$\sum_n |h_n(z)| \leqslant C_1 \equiv C_1(\|\mu\|_C).$$

By Schwarz's lemma there is a $\delta = \delta(C_1) > 0$ such that

$$|h_n(w)| > \frac{1}{2}$$

if $|z_n - \omega| < \delta(1 - |z_n|)$. Let

$$\mathscr{D} = \left\{ z \in D\colon \operatorname{dist}(z, \gamma) \leqslant \frac{\delta}{2}(1 - |z|) \right\}$$

and choose a finite collection of sequences $\{z_n^j\}$ for $j = 1, 2, ..., N$ such that

$$|z_n^j - z_m^j| \geqslant \frac{1}{10}(1 - |z_n^j|) \qquad n \neq m$$

and so that for every $z \in \mathscr{D}$, there exists z_n^j with

$$|z - z_n^j| \leqslant \delta(1 - |z_n^j|).$$

For fixed j, let $\{h_n^j\}$ be the functions associated to $\{z_n^j\}$. Next write $\mathscr{D}$ as a disjoint union of sets

$$\mathscr{D}_n^j \subset \{z\colon |z_n^j - z| < \delta(1 - |z_n^j|)\}.$$

Now define

$$F(z) = \frac{1}{\pi} \sum_{n,j} \int_{\mathscr{D}_n^j} \frac{h_n(z)\, b(\omega)\, \bar{\partial}\varphi(w)}{z - w}\, dx\, dy.$$

Then we can check that $\bar{\partial} F = b \bar{\partial} \varphi$ formally, so we only have to check the convergence of the series. If we write $F = \sum H_n^j$, then

$$|H_n^j(z)| \leqslant \frac{2}{\pi} |h_n^j(z)| \int_{\mathscr{D}_n^j} \frac{\|b\|_D \varepsilon \delta^{-1}(1 - |w|)}{z - \omega}\, dx\, dy$$

$$\leqslant C \|b\|_D \frac{\varepsilon}{\delta} |h_n^j(z)|\, \delta.$$

Thus

$$\begin{aligned}|F(z)| &\leqslant C\|b\|_D \varepsilon \sum_{n,j} |h_n^j(z)| \\ &\leqslant C\|b\|_D \varepsilon C_1 N \\ &\leqslant C\varepsilon \|b\|_D.\end{aligned}$$

This completes the proof of Lemma 3.1.

We will now use this lemma to solve two special $\bar{\partial}$ problems. The first roughly corresponds to the fact that given a closed set $E \subset \mathbf{T}$ of measure zero, we can find a nonconstant holomorphic function which vanishes exactly on E. Given an interval I on $\mathbf{T}$ we let

$$T = \{r\xi : 1 > r > \operatorname{dist}(\xi, I^c),\ \xi \in I\}$$

denote the solid "tent" above I.

LEMMA 3.2. *Suppose $U \subset \mathbf{T}$ is open and let $U = \bigcup_j I_j$ be its decomposition into disjoint intervals. Then for any $\beta > 0$, $\delta > 0$ there is an $\varepsilon > 0$ such that if $|U| < \varepsilon$ then there is a $\varphi \in C^\infty(D)$ such that*:

(1) $\varphi = 1$ *on T_j for all j.*

(2) $\varphi = 0$ on $\{|z| < 1 - \beta\}$.

(3) *For any bounded, continuous function b on D, the equation $\bar{\partial} B = b \bar{\partial} \varphi$ has a solution with $\|B\|_D \leqslant \delta \|b\|_D$.*

The proof is similar to that of [17, Lemma 2.2] where Garnett and Jones construct a BMO function which is large on U and vanishes away from U. The reader may also wish to compare it to Section 3 of [21]. Consider the Hardy–Littlewood maximal function of the characteristic function of U,

$$m(\xi) \equiv M(\chi_U)(\xi) = \sup_{\xi \in I} \frac{|U \cap I|}{|I|}.$$

Clearly m is equal to 1 on U and is lower semi-continuous. Fix an integer $N < |\log_2(\varepsilon)|$ and for $1 \leqslant n \leqslant N$ suppose $\{I_j^n\}$ is the interval decomposition of the open set $\{m(\xi) > 2^{-n}\}$. Note that

$$\sum_{I_j^n \subset I_k^{n+1}} |I_j^n| \leqslant \frac{1}{2} |I_k^{n+1}|$$

and if $I_j^n \subset I_k^{n+1}$ then

$$|I_j^n| \leqslant \frac{1}{2} \operatorname{dist}(I_j^n, (I_k^{n+1})^c).$$

These imply that arclength on the contour γ consisting of all the tents

$$\gamma_j^n = \{r\xi : r = \operatorname{dist}(\xi, (I_j^n)^c),\ \xi \in I_j^n\}$$

is a Carleson measure with norm bounded independent of ε and N. We define φ by

$$\varphi(z) = \sum_{n=1}^{N} \frac{1}{N} \chi_{T_j^n}(z),$$

where T_j^n is the solid tent over I_j^n. Observe that φ satisfies condition (1) of the lemma, and if ε is small enough then all the intervals $\{I_j^n\}$ have length less than $\beta/2$, so φ also satisfies (2). Finally, since $\bar{\partial}\varphi$ looks like $1/N$ times arclength measure on γ, we can smooth out φ (as in [15, p. 357]) so that it satisfies the hypotheses of Lemma 3.1. This proves the result.

The next fact we wish to recall involves solving a $\bar{\partial}$ problem, not on the unit disk, but on an arbitrary simply connected domain Ω. Fix an $\eta > 0$ and consider the grid of squares of the form

$$Q = \{z = x + iy : k\eta \leqslant x \leqslant (k+1)\eta,\ j\eta \leqslant y \leqslant (j+1)\eta\}$$

for integers k and j. Let $\mathscr{C}$ be the collection of such squares satisfying

$$Q \cap \Omega \neq \varnothing, \qquad \operatorname{dist}(Q, \partial\Omega) \leqslant 3\eta.$$

Set

$$\Omega_\eta = \bigcup_{\mathscr{C}} Q \cap \Omega$$

$$\Gamma_\eta = \bigcup_{\mathscr{C}} \partial Q \cap \Omega.$$

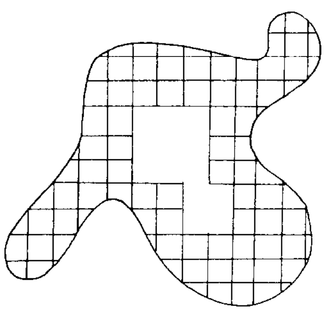

FIG. 2. Ω_η and Γ_η.

LEMMA 3.3. *Suppose Ω is simply connected and Γ_η is as above. Also suppose $\varphi \in C^\infty(\Omega)$ satisfies*

$$\operatorname{supp}(\nabla\varphi) \subset \{z \in \Omega: \operatorname{dist}(z, \Gamma_\eta) \leqslant \delta \operatorname{dist}(z, \partial\Omega)\}$$
$$|\nabla\varphi(z)| \leqslant \varepsilon(\delta \operatorname{dist}(z, \partial\Omega))^{-1}.$$

Then if δ is small enough there is a universal $C>0$ such that for any bounded, continuous function b on Ω there is a solution to $\bar\partial B = b\bar\partial\varphi$ which satisfies $\|B\|_\Omega \leqslant C\varepsilon \|b\|_\Omega$. (In particular, C does not depend on Ω or η.)

The proof is exactly the same as that of Lemma 3.1 once we know that any collection of points $\{z_n\}$ on $\Gamma_\eta \cap \Omega$ which satisfies

$$|z_n - z_m| \geqslant \frac{1}{10} \operatorname{dist}(z_n, \partial\Omega), \qquad n \neq m$$

must be an interpolating sequence for $H^\infty(\Omega)$ with uniformly bounded constant. This is essentially proven by Garnett, Gehring, and Jones in [16]. Actually, they only consider the case when the $\{z_n\}$'s all lie on a straight line, but the desired result is an easy consequence. Divide the squares in $\mathscr{C}$ into 64 subcollections $\mathscr{C}_j$, $1 \leqslant j \leqslant 64$ so that

$$Q_1, Q_2 \in \mathscr{C}_j, \qquad Q_1 \neq Q_2 \Rightarrow \operatorname{dist}(Q_1, Q_2) \geqslant 7\eta$$

and divide Γ_η into 128 subcontours $\{\Gamma_j\}$ each corresponding to taking one side from each square in a given $\mathscr{C}_k$. Corollary 4.3 of [16] says that the points which lie on a single line segment form an interpolating sequence. The proof that each sequence $\{z_n\} \cap \Gamma_j$ is interpolating is similar, except that we need to apply Lemma 3.4 of [16] with $\partial\Omega$ replaced by $\partial\Omega \cap \{|w - w_j| \leqslant 3 \operatorname{dist}(w_j, \partial\Omega)\}$. This version is an easy consequence of the lemma as stated. Thus $\{z_n\} \cap \Gamma_j$ is interpolating for each j and $\{z_n\}$ is hyperbolically separated (by [16, Lemma 3.1]) so $\{z_n\}$ is also interpolating [15, Exercise VII.2]. This proves Lemma 3.3.

We should also mention that this result is essentially equivalent to the theorem of Hayman and Wu which states that if $\Phi: D \to \Omega$ is conformal, then arclength on $\Phi^{-1}(\mathbf{R} \cap \Omega)$ is a Carleson measure with norm bounded independently of Ω and Φ (see [20] or [16]).

4. THE BROWDER–WERMER THEOREM

We now turn to the proof of sufficiency. So assume $\omega_1 \perp \omega_2$ and (without loss of generality) that Γ is bounded. First we will show that A_Γ is strongly pointwise boundedly dense in $H^\infty(\Omega)$ and then that A_Γ is a Dirichlet algebra on Γ.

First we introduce some notation concerning the continuity of a function f defined on $\mathbf{C}$. We set

$$J(f, \delta, x) = \text{diameter}(f(D(x, \delta)))$$
$$= \sup\{|f(z) - f(w)| : z, w \in D(x, \delta)\}$$

$$J(f, x) = \lim_{\delta \to 0} J(f, \delta, x)$$

$$J(f) = \sup_{x \in \mathbf{C}} J(f, x).$$

Clearly $J(f, x)$ is upper semicontinuous and f is continuous at x iff $J(f, x) = 0$.

Now take $f \in H^\infty(\Omega)$, $\|f\|_\Omega = 1$. It is enough to show f can be uniformly approximated on compact sets of Ω by holomorphic functions on Ω which extend continuously to $\bar{\Omega}$, so fix a compact set $\tilde{K}$ in Ω and an $\eta > 0$. We will construct a sequence $\{F_n\}$ in $H^\infty(\Omega)$ which satisfies

$$\|f - F_n\|_{\tilde{K}} \leqslant (1 - 2^{-n})\,\eta \tag{4.1}$$

$$\|F_n\|_\Omega \leqslant 1 - 2^{-n}\eta < 1 \tag{4.2}$$

$$\|F_n - F_{n+1}\|_\Omega \leqslant \frac{3}{4} J(F_n) \tag{4.3}$$

$$J(F_{n+1}) \leqslant \frac{3}{4} J(F_n). \tag{4.4}$$

Such a sequence obviously converges to the desired function F.

We start by setting $F_1 = (1 - \eta/2) f$. In general, given F_n we define F_{n+1} in two steps. First we will modify it to obtain a function g_n such that

$$g_n = F_n \qquad \text{on } \tilde{K}$$

$$\|g_n\|_\Omega < \|F_n\|_\Omega$$

$$\|g_n - F_n\|_\Omega < \frac{2}{3} J(F_n)$$

$$J(g_n) < \frac{2}{3} J(F_n).$$

This g_n is not continuous, but it does have a smaller "jump" than F so it is "closer" to being continuous. Also g_n will not be holomorphic, so we need to find an h_n such that

$$\bar{\partial} h_n = -\bar{\partial} g_n \qquad \text{on } \Omega$$

$$\|h_n\|_\Omega \leqslant \min\left(2^{-n-2}\eta, \frac{1}{100} J(F_n)\right).$$

Then $F_{n+1} = g_n + h_n \in H^\infty(\Omega)$ and satisfies (4.1)–(4.4), as required.

To ease notation we shall drop the "n" and simply write $F = F_n$, $g = g_n$, and $h = h_n$. Our first step in defining g is to construct a continuous function H_0 on a neighborhood of Γ which approximates F in the following sense:

$$|H_0(z) - F(z)| \leqslant \frac{3}{5} J(F). \tag{4.5}$$

To do this, we consider F as a pair of functions F_1, F_2 defined on Ω_1 and Ω_2, respectively. We may assume $F_1 \in A(\Omega_1)$ (otherwise pull back to the unit disk and dilate slightly; this approximates F_1 uniformly on $\tilde{K}$, decreases the sup norm, and increases $J(F)$ only a little). Similarly for F_2. On Γ let

$$H_0 = \frac{F_1 + F_2}{2}$$

and extend H_0 to be continuous on a neighborhood of Γ. Now suppose $\varepsilon = \varepsilon_1$ is small (to be fixed later). Since H_0 is uniformly continuous on a neighborhood of Γ we can choose $\delta = \delta_1$ so that

$$\delta < \frac{1}{10} \operatorname{dist}(\tilde{K}, \Gamma)$$

and

$$|z - w| < \delta \qquad \text{and} \qquad \operatorname{dist}(z, \Gamma) < \delta \Rightarrow |H_0(z) - H_0(w)| < \varepsilon.$$

For this δ consider the collection of squares $\mathscr{C} = \mathscr{C}_\delta$ defined in Section 3 and the associated Ω_δ and Γ_δ. Define an approximation to H_0 which is constant on each square of $\mathscr{C}$ (e.g., use the value of H_0 at the center of the square), fix $\delta = \delta_2 > 0$, and then "smooth out" this function along Γ_{δ_1} to obtain a function $H \in C^\infty(\Omega_\delta)$ satisfying

$$\|H\| \leqslant \|F\|_\Omega$$

$$\|H-F\| \leqslant \frac{1}{2} J(F) + \varepsilon \leqslant \frac{3}{5} J(F)$$

$$\operatorname{supp}(\nabla H) \subset \{z \in \Omega \colon \operatorname{dist}(z, \Gamma_{\delta_1}) \leqslant \delta_2 \operatorname{dist}(z, \partial\Omega)\}$$

$$|\nabla H(z)| \leqslant \varepsilon(\delta_2 \operatorname{dist}(z, \partial\Omega))^{-1}.$$

In particular, H satisfies the hypotheses of Lemma 3.3 on both components of Ω.

The function g will be written as a convex combination of F and H; i.e.,

$$g = (1-\varphi) F + \varphi H,$$

where $0 \leqslant \varphi \leqslant 1$. Consider Riemann mappings $\Phi_j \colon D \to \Omega_j$ for $j = 1, 2$. By compactness we can choose a $\beta > 0$ so that

$$\Phi_j^{-1}(\Omega \backslash \Omega_{\delta/4}) \subset \{|z| < 1 - \beta\}$$

for $j = 1, 2$. Fix $\varepsilon = \varepsilon_2$. Using Fatou's theorem on non-tangential limits of bounded holomorphic functions and our hypothesis we can find compact sets $E_j \subset \mathbf{T}$ which satisfy

(4.6) Φ_j and $F \circ \Phi_j$ have non-tangential limits at each point of E_j.

(4.7) Φ_j is injective on E_j.

(4.8) $\Phi_1(E_1)$ and $\Phi_2(E_2)$ are disjoint, compact subsets of Γ.

(4.9) $|\mathbf{T} \backslash E_j| < \varepsilon$.

We now apply Lemma 3.2 to each $U_j = \mathbf{T} \backslash E_j$ with β as above and

$$\delta < \min\left(2^{-n-2}\eta, \frac{1}{100} J(F)\right)$$

to get functions φ_j defined on D. We define φ on each Ω_j by $\varphi = \varphi_j \circ \Phi_j$.

With this φ, g clearly satisfies the first three conditions it's suppose to, so we need only estimate $J(g)$. Let $x \in \Gamma$ and $\delta > 0$ (to be fixed later) and suppose $z, w \in D(x, \delta) \cap \Omega$. If $\varphi(z) = 1$, then

$$\begin{aligned} |g(z) - g(w)| &= |H(z) - (1 - \varphi(w)) F(w) - \varphi(w) H(w)| \\ &\leqslant |H(z) - H(w)| + (1 - \varphi(w)) |F(w) - H(w)| \\ &\leqslant \varepsilon_1 + \frac{3}{5} J(F) \\ &\leqslant \frac{2}{3} J(F) \end{aligned}$$

if ε_1 is small enough. Similarly if $\varphi(w)=1$. Thus we may assume $\varphi(z), \varphi(w)<1$. By construction, $\operatorname{supp}(1-\varphi)$ lies in the union of two disjoint, compact sets, each corresponding to one of the components Ω_1, Ω_2. So if δ is small enough, z and w must lie in the same component, say Ω_1. By our construction F is continuous on the support of $(1-\varphi)$ so $|F(z)-F(w)|$ is small if δ is small. Thus

$$\begin{aligned} |g(z)-g(w)| &\leqslant |H(z)-H(w)|+|H(w)-F(w)|+|F(w)-F(z)| \\ &\leqslant \varepsilon_1+\frac{3}{5}J(F)+\varepsilon(\delta) \\ &\leqslant \frac{2}{3}J(F) \end{aligned}$$

if ε_1 and δ are small enough. Hence

$$J(g, \delta, x) \leqslant \frac{2}{3}J(F)$$

as required.

We now take care of the fact that g is not holomorphic. By our construction and Lemmas 3.2 and 3.3 each term of

$$-\bar{\partial}g = F\bar{\partial}\varphi + H\bar{\partial}\varphi + \varphi\bar{\partial}H$$

corresponds to a $\bar{\partial}$ problem on each Ω_j that we know how to solve with (small) uniform estimates. Thus we can find an h on Ω such that

$$\bar{\partial}h = -\bar{\partial}g$$

$$\|h\|_\Omega \leqslant \min\left(2^{-n-2}\eta, \frac{1}{100}J(F)\right)$$

as desired. Thus we have proven that $A(\Omega)=A_\Gamma$ is strongly pointwise boundedly dense in $H^\infty(\Omega)$.

Next we wish to show that A_Γ is a Dirichlet algebra on Γ. Fix $\eta>0$ and suppose g is a real valued continuous function on Γ and let u be its harmonic extension to Ω. Since each component of Ω is simply connected, u has a well defined harmonic conjugate u^* on Ω. We would like to apply the construction above to $f=u+iu^*$, but u^* may be unbounded, so first we must approximate it by a bounded holomorphic function; i.e., we need to find a $F\in H^\infty(\Omega)$ such that

$$\|\operatorname{Re}(f)-\operatorname{Re}(F)\|_\Omega \leqslant \eta.$$

However, this is easy by pulling back to the unit disk and dilating slightly.

We now apply the construction to F to obtain a sequence $\{F_n\}$ which in addition to (4.1)–(4.4) also satisfies

$$J(\mathrm{Re}(F-F_n)) \leqslant (1-2^{-n})\,\eta. \tag{4.10}$$

This can be done by taking $\delta=\delta_1$ in the definition of H so small that

$$J(\mathrm{Re}(F-F_n), 10\delta, x) \leqslant \left(1-2^{-n}+\frac{2^{-n-1}}{3}\right)\eta$$

for all x. From this we get

$$\|\mathrm{Re}(H)-\mathrm{Re}(F_n)\|_\Omega \leqslant \left(1-2^{-n}+\frac{2^{-n-1}}{3}\right)\eta.$$

Taking ε_1 and ε_2 small enough we get

$$\|h_n\|_\Omega \leqslant \frac{1}{3}\,2^{-n-1}\eta$$

$$\|H-H_0\|_\Omega \leqslant \frac{1}{3}\,2^{-n-1}\eta$$

which gives, since H_0 is continuous,

$$J(\mathrm{Re}(F)-\mathrm{Re}(F_n)) \leqslant (1-2^{-n-1})\,\eta$$

as required. This completes the proof that A_Γ is a Dirichlet algebra on Γ.

5. Davie's Theorem

In this section we will briefly describe the changes which need to be made in the preceding proof to give Davie's theorem. As before, we may assume K is bounded. The proof of necessity is essentially the same. To prove sufficiency we will construct a sequence $\{F_n\}$ which satisfies conditions (4.1)–(4.4), except that the constant 3/4 is replaced by 95/100. The proof goes exactly as before until we get to the definition of H_0. We can no longer just take the average of F's boundary values or even assume F has continuous boundary values on each Ω_j. However, we can construct an H_0 which satisfies

$$|H_0(z)-F(z)| \leqslant \frac{88}{100}\,J(F). \tag{5.1}$$

To do this, first choose a $\delta = \delta_1 > 0$ so small that

$$\delta < \frac{1}{10} \operatorname{dist}(\tilde{K}, K)$$

$$\sup_{x \in \mathbf{C}} J(F, 10\delta, x) < \frac{100}{99} J(F).$$

The key observation is that if E is a planar set and z, $w \in \bar{E}$ satisfy $|z - w| = \operatorname{diam}(E)$, then

$$E \subset \overline{D\left(\frac{z+w}{2}, \frac{\sqrt{3}}{2} \operatorname{diam}(E)\right)}$$

$$\subset D\left(\frac{z+w}{2}, \left(\frac{87}{100}\right) \operatorname{diam}(E)\right)$$

since $\sqrt{3}/2 = .86602 \cdots < 87/100 < 1$. To define H_0 first consider the points of the form $z = n\delta + im\delta$ for n and m integers and $\operatorname{dist}(z, \partial\Omega) < 10\delta$. For such a z choose a number $H_0(z)$ so that

$$F(B(z, 3\delta)) \subset D\left(H_0(z), \frac{88}{100} J(F)\right).$$

This will be possible because of the observation above applied to $E = E(D(z, 3\delta))$ which satisfies $(\sqrt{3}/2) \operatorname{diam}(E) \leqslant (88/100)\, J(F)$. Next, if z is any point in the square formed by four adjacent points $\{z_1, ..., z_4\}$ of the above type, then

$$F(D(z, \delta)) \subset F\left(\bigcap_{i=1}^{4} D(z_i, 3\delta)\right)$$

$$\subset \bigcap_{i=1}^{4} F(D(z_i, 3\delta))$$

$$\subset \bigcap_{i=1}^{4} D\left(H_0(z_i), \frac{88}{100} J(F)\right)$$

$$\subset D\left(w, \frac{88}{100} J(F)\right)$$

for any w in the convex hull of $\{F(z_1), ..., F(z_4)\}$. Therefore we can extend H_0 continuously to a neighborhood of $\partial\Omega$ so that it satisfies (5.1) and $\|H_0\| \leqslant \|F\|$.

Now suppose $\varepsilon = \varepsilon_1$ is small (to be fixed later). Since H_0 is uniformly continuous on a neighborhood of $\partial\Omega$ we can choose $\delta = \delta_2$ so that

$|z-w|<\delta$ and $\text{dist}(z, \partial\Omega)<\delta$ imply that $|H_0(z)-H_0(w)|<\varepsilon$. Let $\mathscr{C}=\mathscr{C}_\delta$, Ω_δ and Γ_δ be as before. Now define H approximating H_0 satisfying the same conditions as before except now

$$\|H-F\| \leqslant \frac{88}{100} J(F)+\varepsilon \leqslant \frac{89}{100} J(F).$$

Thus H will satisfy the hypotheses of Lemma 3.3 for each component Ω_j of Ω.

Now write

$$g=(1-\varphi) F+\varphi H,$$

where we define φ by defining it on each component Ω_j. First of all, note that all but finitely many of the Ω_j lie inside $\Omega_{\delta/4}$, because otherwise infinitely many would contain a disk of radius $\delta/4$, contradicting the boundedness of K. On these components we set $\varphi \equiv 1$. For the remaining components, say $\Omega_1, ..., \Omega_N$, we fix Riemann mappings $\Phi_j: D \to \Omega_j$. By compactness we can choose a $\beta>0$ so that

$$\Phi_j^{-1}(\Omega \backslash \Omega_{\delta/4}) \subset \{|z|<1-\beta\}$$

for $j=1,..., N$. Now let $\varepsilon_2>0$ (to be fixed later) and choose compact sets $E_j \subset \mathbf{T}$ which satisfy conditions (4.6) through (4.9). We now apply Lemma 3.2 to each $\mathbf{T} \backslash E_j$ with β as above and $\delta<\min(2^{-n-2}\eta, (1/100) J(F))$, to get functions φ_j. We complete the proof in exactly the same way, so we have now shown A_K is strongly pointwise boundedly dense in $H^\infty(\Omega)$.

To prove A_K is Dirichlet fix $\eta>0$ and $g \in C(K)$. Again we extend g harmonically and take harmonic conjugates to obtain $f=u+iu^*$. However, it is not so clear that we can approximate u by the real part of something in $H^\infty(\Omega)$. What we shall need is:

Lemma 5.1. *Suppose Ω is simply connected and u is harmonic on Ω, with a continuous extension to $\bar{\Omega}$. Then for any $\eta>0$ there is a $F \in H^\infty(\Omega)$ with*

$$\|u-\operatorname{Re}(F)\|_\Omega \leqslant \eta.$$

To prove this, let $\Phi: D \to \Omega$ be conformal and set $\tilde{u}=u \circ \Phi$. We claim that $\tilde{u} \in \mathrm{VMO}$ (see [15, p. 250]). For if $z \in D$, $w=\Phi(z)$, $r=\text{dist}(w, \partial\Omega)$, and P_z is the Poisson kernel, then the harmonic measure estimates in Beurling's solution of the Carleman–Milloux problem give (see [24]),

$$\int_T |\tilde{u}(\xi)-\tilde{u}(z)| P_z(\xi)\, d\xi = \int_{\partial\Omega} |u(\zeta)-u(w)|\, d\omega_w(\zeta)$$

$$\leqslant \sum_{n=0}^{\infty} J(u, 2^n r, w)\, \omega_w(D(w, 2^{n-1}r))$$

$$\leqslant C \sum_{n=0}^{\infty} J(u, 2^n r, w)\, 2^{-n/2}.$$

Since u is continuous, this tends to zero uniformly as r does and so $\tilde{u}$ is in VMO. In particular, it proves that $\tilde{u}$ can be approximated in the BMO norm by contnuous functions, so for any $\varepsilon > 0$ we can write

$$\tilde{u} = v + w,$$

where v is harmonic and continuous on $\bar{D}$ and $\|w\|_{\mathrm{BMO}} < \varepsilon$ (of course, we are considering the boundary values of w on $\mathbf{T}$). By a result of Varopoulos [26, 27] we can find a $\tilde{w}$ on D with the same boundary values as w and such that $|\nabla \tilde{w}|\, dx\, dy$ is a Carleson measure with norm less than $C\varepsilon$. Thus we can solve the $\bar{\partial}$ problem

$$\bar{\partial} b = -\bar{\partial} \tilde{w}$$

with $\|b\|_{L^\infty(\mathbf{T})} \leqslant \eta/2$ if ε is small enough (see [15, Theorem VIII.1.1]). Now choose r so close to 1 that $|v(z) - v(rz))| < \eta/2$ for $z \in D$, and note that

$$F(z) = v(rz) + iv^*(rz) + \tilde{w}(z) + b(z)$$

defines a bounded holomorphic function on D. Using the maximum principle we see

$$|\tilde{u}(z) - \mathrm{Re}(F(z))| \leqslant \sup_{\mathbf{T}} (|v(z) + w(z) - v(rz) - \tilde{w}(z) - \mathrm{Re}(b(z))|)$$

$$\leqslant \sup_{\mathbf{T}} (|v(z) - v(rz)| + |w(z) - \tilde{w}(z)| + |b(z)|)$$

$$\leqslant \frac{\eta}{2} + 0 + \frac{\eta}{2}$$

as required. (I would like to thank Peter Jones for pointing out this argument to me.)

We now return to the problem of finding a bounded holomorphic function F approximating f. Let $\varepsilon > 0$ (to be fixed later). Since u is continuous on $\bar{\mathbf{C}}$ there is a $\delta > 0$ such that $|z - w| < \delta$ implies $|u(z) - u(w)| < \varepsilon$. For this δ consider Ω_δ and Γ_δ as before. If Ω_j is a component of Ω such that $\Omega_j \subset \Omega_\delta$, then we can approximate u by a function g satisfying the hypotheses of Lemma 3.2. Thus we can solve

$$\bar{\partial} h = -\bar{\partial} g$$

with

$$\|h\|_\Omega \leqslant C\varepsilon$$

and so $F = g + h$ is the desired function on Ω_j (if ε is small enough). On the finitely many remaining components we apply Lemma 5.1. Thus we get the desired F. Proceedings as in Section 4 proves A_K is Dirichlet.

6. Homeomorphisms of T

If $K = \Gamma$ is a curve, let $\Phi_i: D \to \Omega_i$ for $i = 1, 2$ denote fixed choices of the Riemann maps. By Carathéodory's theorem these maps extend to be homeomorphisms from $\mathbf{T}$ to Γ, so $\psi \equiv (\Phi_2)^{-1} \circ \Phi_1$ defines an orientation reversing homeomorphism of $\mathbf{T}$ to itself. Note that a function $f \in A_\Gamma$ corresponds to a pair of functions $f_1, f_2 \in A(D)$ which satisfy $f_1 = f_2 \circ \psi$ on $\mathbf{T}$. Our assumption that $\omega_1 \perp \omega_2$ is equivalent to ψ being singular; i.e., there is a set $E \subset \mathbf{T}$ such that $|\mathbf{T} \backslash E| = |\psi(E)| = 0$. One can easily apply the iterative procedure described above to a pair of holomorphic functions on D as follows. Define H_1 by approximating the continuous function $\frac{1}{2}(f_1 + f_2 \circ \psi)$ on $\mathbf{T}$ by a function on D satisfying the hypothesis of Lemma 3.3. We will write this as $H_1 \sim \frac{1}{2}(f_1 + f_2 \circ \psi)$. Similarly define $H_2 \sim \frac{1}{2}(f_2 + f_1 \circ \psi^{-1})$. Then choose E_1 and E_2 with

$$|E_1|, |E_2| > 1 - \varepsilon$$

$$\psi(E_1) \cap E_2 = \varnothing$$

and define the corresponding φ_1 and φ_2 as in Lemma 3.2. For $i = 1, 2$ set

$$g_i = f_i \varphi_i + H_i (1 - \varphi_i)$$

and solve for h_i as before. Then for $i = 1, 2$, $F_i = g_i + h_i$ is holomorphic and

$$\|F_1 - F_2 \circ \psi\| \leqslant \|h_1\| + \|h_2\| + \|g_1 - g_2 \circ \psi\|$$

$$\leqslant \frac{2}{3} \|f_1 - f_2 \circ \psi\|.$$

Thus iterating gives a pair of holomorphic functions with "matching" boundary values. This observation works for any singular homeomorphism of $\mathbf{T}$, regardless of whether it comes from a curve Γ or not. Thus we get:

COROLLARY 6.1 (Browder and Wermer [6]). *If* ψ *is a singular homeomorphism of* $\mathbf{T}$ *to itself then* $\{f \in A(D): f \circ \psi \in A(D)\}$ *is a Dirichlet algebra on* $\mathbf{T}$.

Similarly, if $\{\psi_1, ..., \psi_n\}$ are all homeomorphisms of $\mathbf{T}$ to itself such that $\psi_{jk} = \psi_j \circ \psi_k^{-1}$ for $j \neq k$ are all singular, then we can find sets $E_1, ..., E_n$ such that

$$|E_j| > 1 - \varepsilon, \qquad \forall j$$

$$\psi_j(E_k) \text{ are disjoint for } k, j = 1, ..., n.$$

Using this we can mimic the proof above to show that

$$\{f: f \circ \psi_j \in A(D), j = 1, ..., n\}$$

is a Dirichlet algebra on $\mathbf{T}$, another result of Browder and Wermer.

7. DECOMPOSING CONTINUOUS FUNCTIONS

In their paper [7] Browder and Wermer show that if ψ is any orientation reversing homeomorphism of $\mathbf{T}$ to itself, then

$$A(D) + A_\psi(D) = A(D) + \{f \circ \psi: f \in A(D)\}$$

is a uniformly dense subspace of $C(\mathbf{T})$. A result of Browder implies $A(D) + A_\psi(D)$ is a closed subspace of $C(\mathbf{T})$ if ψ is singular, hence $C(\mathbf{T}) = A(D) + A_\psi(D)$ (see [5, Lemma 7.2.2 and p. 235]). In fact, he shows this for any singular homeomorphism of $\mathbf{T}$. We can recover this result from our construction; i.e.,

COROLLARY 7.1. *If* ψ *is any singular homeomorphism of* $\mathbf{T}$ *to itself and* $f \in C(\mathbf{T})$, *then we can find* $f_1, f_2 \in A(D)$ *such that* $\|f_1\|, \|f_2\| \leqslant \|f\|$ *and* $f = f_1 - f_2 \circ \psi$.

Of course, there is a corresponding result stated in terms of curves. We will not give a complete proof of this since it is so similar to what we have done already. We merely note that if $f_1, f_2 \in A(D)$, φ_1 and φ_2 are as above and $H_1 \sim (f + f_2 \circ \psi)$, $H_2 \sim (f_1 \circ \psi^{-1} - f \circ \psi^{-1})$, then we can set

$$g_1 = \varphi_1 f_1 + (1 - \varphi_1)(H_1)$$

$$g_2 = (1 - \varphi_2) f_2 + \varphi_2(H_2).$$

Then $\bar{\partial} g_i \, dx \, dy$, for $i = 1, 2$, are Carleson measures of the appropriate type and so we can find h_1 and h_2 such that $F_i = g_i + h_i$, for $i = 1, 2$, are holomorphic and satisfy

$$\|f - (F_1 - F_2 \circ \psi)\| \leq \|h_1\| + \|h_2\| + \|f - (g_1 - g_2 \circ \psi)\|$$
$$\leq \frac{2}{3} \|f - (f_1 - f_2 \circ \psi)\|.$$

Iterating gives Corollary 7.1.

8. P.B.D. and Distance Estimates

In [11] Davie, Gamelin, and Garnett show that A_K is pointwise boundedly dense in $H^\infty(\Omega)$ iff

$$\operatorname{dist}(h, A_K) = \operatorname{dist}(h, H^\infty(\Omega))$$

for every $h \in C(\bar{\mathbf{C}})$. The inequality "$\geqslant$" is trivial, of course. If K is connected, and satisfies the conditions of Davie's theorem our construction gives "$\leqslant$." This is because if $f \in H^\infty(\Omega)$ satisfies $\operatorname{dist}(h, f) = d$, then applying the construction to f gives a function $F \in A_K$ which at a given point is essentially a convex combination of f's values near that point plus a small error. Using the uniform continuity of h we can deduce $\operatorname{dist}(h, F) \leqslant d + \varepsilon$. On the other hand, one can use the methods of Section 2 to show equality must fail if ω_1 and ω_2 are not mutually singular.

9. A Capacity Characterization

Another "geometric" characterization of Dirichlet algebras is due to Gamelin and Garnett. In [13] they prove that A_K is a Dirichlet algebra on K iff for all $x \in K$ and $0 < \delta < \operatorname{diam}(K)$,

$$\alpha(D(x, \delta) \cap K) \geqslant \frac{\delta}{4}.$$

Here α is the *continuous analytic capacity*,

$$\alpha(E) \equiv \sup\{|f'(\infty)| : f \in A_E, \|f\| \leqslant 1\}.$$

By a Cauchy integral argument, similar to the one in Section 2, one can show the inequality fails if K does not satisfy the conditions of Davie's theorem (and as in [13] one may replace 1/4 by any positive constant). On the other hand, suppose it does, and let E be a connected subset of

$D(x, \delta) \cap \Gamma$ of diameter at least δ. It is a well known consequence on the Riemann mapping and Koebe $\frac{1}{4}$ theorems that

$$\begin{aligned}\gamma(E) &\equiv \sup\{|f'(\infty)| : f \in H^\infty(\bar{C} \backslash E), \|f\| \leqslant 1\} \\ &\geqslant \frac{1}{4} \operatorname{diam}(E) \\ &\geqslant \frac{\delta}{4}\end{aligned}$$

(e.g., [14, p. 9]). Now take $f \in H^\infty(\bar{\mathbf{C}} \backslash E)$ with $\|f\| \leqslant 1$ and $|f'(\infty)| > \delta/4 - \varepsilon$. By the construction we can approximate f uniformly on a neighborhood of $\{\infty\}$ and so we can find $F \in A_E$ with $\|F\| \leqslant 1$ and

$$|F'(\infty)| > |f'(\infty)| > \frac{\delta}{4} - 2\varepsilon.$$

This proves the desired inequality.

10. Remarks

When $K = \Gamma$ a closed curve, the results of this paper can also be deduced using the proof of the following result (see [2]):

THEOREM 10.1. *Suppose* $\psi: \mathbf{T} \to \mathbf{T}$ *is a singular homeomorphism and* $\varepsilon > 0$. *Then there exists a* $\varphi \in C(\mathbf{T})$, $0 \leqslant \varphi \leqslant 1$, *such that*

(1) $\|\varphi\|_{\mathrm{BMO}} \leqslant \varepsilon$.

(2) $\|\varphi \circ \psi\|_{\mathrm{BMO}} \leqslant \varepsilon$.

(3) $|\{\varphi = 1\}| \geqslant 1 - \varepsilon$.

(4) $|\{\varphi \circ \psi = 0\}| \geqslant 1 - \varepsilon$.

This is reminiscent of a result of Garnett and Jones concerning BMO functions which take the values zero and one on small, preassigned subsets of $\mathbf{T}$ (see [17, 21]). To recover our results, one extends the functions φ and $\varphi \circ \psi$ to the interior of the unit disk so that their gradients are Carleson measures of the appropriate type, takes the corresponding continuous function on $\bar{\mathbf{C}}$, and then uses this function as a partition of unity to "glue" together holomorphic functions on either side of Γ.

There are several open problems concerning the spaces A_K, but I will mention only the following: Does A_K always have a Schauder basis (when considered as a Banach space with the sup norm)? The answer is probably yes, but I don't know how to prove it. The corresponding question for the disk algebra was answered affirmatively by S. V. Bŏkarev in 1974 (see [4]).

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