MODELS FOR THE EREMENKO-LYUBICH CLASS

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Abstract. If f is in the Eremenko-Lyubich class $\mathcal{B}$ (transcendental entire functions with bounded singular set) then $\Omega = \{z : |f(z)| > R\}$ and $f|_\Omega$ must satisfy certain simple topological conditions when R is sufficiently large. A model (Ω, F) is an open set Ω and a holomorphic function F on Ω that satisfy these same conditions. We show that any model can be approximated by an Eremenko-Lyubich function in a precise sense. In many cases, this allows the construction of functions in $\mathcal{B}$ with a desired property to be reduced to the construction of a model with that property, and this is often much easier to do.

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1. Introduction

The singular set of a entire function f is the closure of its critical values and finite asymptotic values and is denoted $S(f)$. The Eremenko-Lyubich class $\mathcal{B}$ consists of functions such that $S(f)$ is a bounded set (such functions are also called bounded type). The Speiser class $\mathcal{S} \subset \mathcal{B}$ (also called finite type) are those functions for which $S(f)$ is a finite set.

In [9] Eremenko and Lyubich showed that if $S(f) \subset \mathbb{D}_R = \{z : |z| < R\}$, then $\Omega = \{z : |f(z)| > R\}$ is a disjoint union of analytic, unbounded, Jordan domains, and that f acts a covering map $f : \Omega_j \to \{x : |z| > R\}$ on each component Ω_j of Ω. Building examples where Ω has a certain geometry is important for applications to dynamics. We would like to start with a model, i.e., a choice of Ω and a covering map $f : \Omega \to \{|z| > 1\}$ and ask if f can be approximated by an entire function F in $\mathcal{B}$ or $\mathcal{S}$. In this paper, we deal with approximation by functions in $\mathcal{B}$. It turns out that if Ω satisfies some obviously necessary topological conditions, the approximation by Eremenko-Lyubich functions is always possible in a sense strong enough to imply that the functions f and F have the same dynamical behavior on their Julia sets. This allows us to build entire functions in $\mathcal{B}$ with certain behaviors by simply exhibiting a model with that behavior (this is often much easier to do). In [5] we deal with the analogous question for the Speiser class; again the approximation is always possible, but in a slightly weaker sense (dynamically, given any model we can build a Speiser class functions that has the model's dynamics on some subset of its Julia set). In the next few paragraphs we introduce some notation to make these remarks more precise.

Suppose $\Omega = \cup_j \Omega_j$ is a disjoint union of unbounded simply connected domains such that

(1) sequences of components of Ω accumulate only at infinity,
(2) $\partial\Omega_j$ is connected for each j (as a subset of $\mathbb{C}$).

Such an Ω will be called a model domain. If $\overline{\Omega} \cap \{|z| \leq 1\} = \emptyset$, we say the model domain is disjoint type. The connected components $\{\Omega_j\}$ of Ω are called tracts. Given a model domain, suppose $\tau : \Omega \to \mathbb{H}_r = \{x + iy : x > 0\}$ is holomorphic so that

(1) The restriction of τ to each Ω_j is a conformal map $\tau_j : \Omega_j \to \mathbb{H}_r$,

(2) If $\{z_n\} \subset \Omega$ and $\tau(z_n) \to \infty$ then $z_n \to \infty$.

Given such a $\tau : \Omega \to \mathbb{H}_r$, we call $F(z) = \exp(\tau(z))$ a model function.

The second condition on τ is a careful way of saying that the conformal map on each component sends ∞ to ∞. Even after making this condition, we still have a (real) 2-dimensional family of conformal maps from each component of Ω to $\mathbb{H}_r$ determined by choosing where one base point in each component will map in $\mathbb{H}_r$. A choice of both a model domain Ω and a model function F on Ω will be called a model.

Given a model (Ω, F) we let

$$\Omega(\rho) = \{z \in \Omega : |F(z)| > e^\rho\} = \tau^{-1}(\{x + iy : x > \rho\}),$$

and

$$\Omega(\delta, \rho) = \{z \in \Omega : e^\delta < |F(z)| < e^\rho\} = \tau^{-1}(\{x + iy : \delta < x < \rho\}).$$

If Ω has connected components $\{\Omega_j\}$ we let $\Omega_j(\rho) = \Omega(\rho) \cap \Omega_j$ and similarly for $\Omega_j(\delta, \rho)$.

Moreover, a model has dynamics: we can iterate F as long as the iterates keep landing in Ω, and we define the Julia set of a model

$$\mathcal{J}(F) = \bigcap_{n \geq 0} \{z \in \Omega : F^n(z) \in \Omega\}.$$

Each function f in the Eremenko-Lyubich class that satisfies $S(f) \subset \mathbb{D}$ gives rise to a model by taking $\Omega = \{z : |f(z)| > 1\}$ and $\tau(z) = \log f(z)$. The log is well defined since each component of Ω is simply connected and f is non-vanishing on Ω. Eremenko and Lyubich proved in [9] that τ defined in this way is a conformal map from each component of Ω to $\mathbb{H}_r$. We call a model arising in this way an Eremenko-Lyubich model. If f is in the Speiser class, we call it a Speiser model.

An entire function f is called hyperbolic if $f \in \mathcal{B}$ and if there is a compact set K so that $f(K) \subset \mathrm{int(K)}$ and $f : f^{-1}(\mathbb{C} \setminus K) \to \mathbb{C} \setminus K$ is a covering map. This is equivalent to saying that the singular set is bounded and every point of $S(f)$ iterates to an attracting periodic cycle in the Fatou set. If we can take K to be connected, then f is called disjoint type. This implies the Fatou set of f (i.e., the set where the iterates of f form a normal family) is connected (e.g., see [15]). The assumption that $S(f) \subset \mathbb{D}$ and $\Omega = \{z : |f(z)| > 1\}$ implies that f is disjoint type. In this case, the usual Julia set of f (defined as the complement of the Fatou set) is the same set as

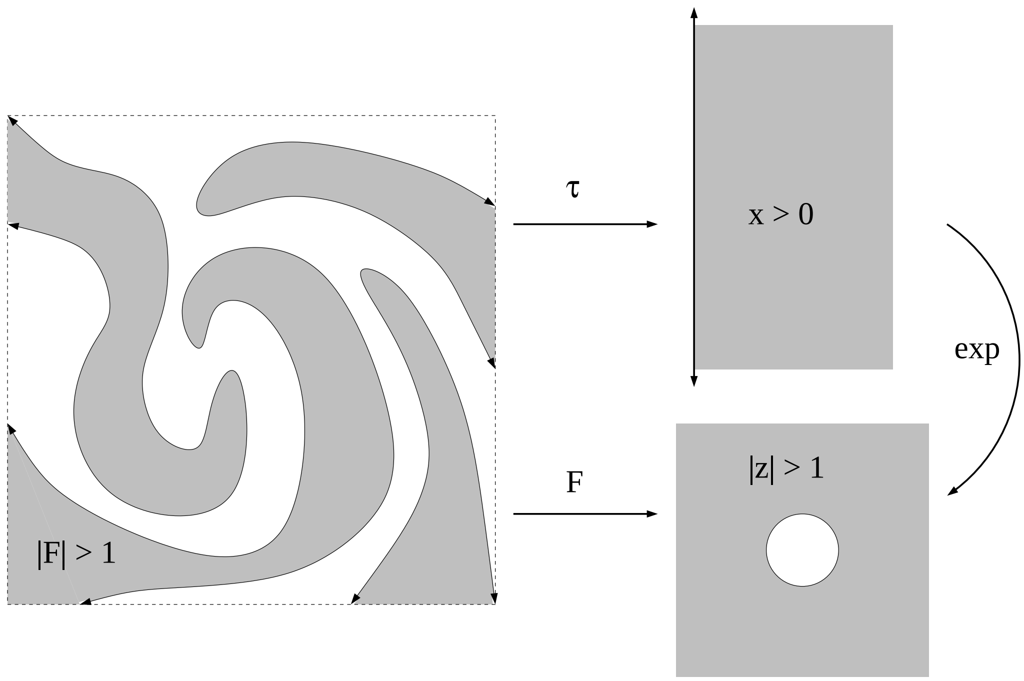

FIGURE 1. A model consists of an open set Ω which may have a number of connected components called tracts. Each tract is mapped conformally by τ to the right half-plane and then by the exponential function to the exterior of the unit disk. The composition of these two maps is the model function F. In this paper, we are interested in knowing if a holomorphic model function on Ω can be approximated by holomorphic function on the entire plane.

the Julia set of the corresponding model

$$\mathcal{J}(f) = \bigcap_{n \geq 0} \{z : |f^n(z)| \geq 1\}.$$

Thus we can refer to $\mathcal{J}(f)$ where we think of f as either an entire function in $\mathcal{B}$ or as a model function on $\Omega = \{x : |f(z)| > 1\}$ without ambiguity. Basic facts about hyperbolic and disjoint type functions are discussed in [2].

The question now arises of whether or not the Eremenko-Lyubich models are only a very special subclass of general models. There are at least two ways to make such a comparison: geometric and dynamical. We start with our geometric result.

A homeomorphism of the plane is called quasiconformal if it is absolutely continuous on almost all vertical and horizontal lines and the partial derivatives $f_z = f_x - if_y$

and $f_{\bar z} = f_x + i f_y$ almost everywhere satisfy

$$|f_{\bar z}| \le k|f_z|,$$

where $0 \le k < 1$. Geometrically, the derivative of f exists almost everywhere and sends circle to ellipses of eccentricity at most $K = (1+k)/(1-k)$. This number K is called the quasiconstant of f. The ratio $\mu = f_{\bar z}/f_z$ is called the complex dilatation of f. The measurable Riemann mapping theorem (see e.g., [1], [14]) says that given any measurable μ with $|\mu| < k < 1$, there is a quasiconformal homeomorphism φ of the plane so that the complex dilatation of φ equal μ almost everywhere. We shall actually use the following consequence of this: if $\psi : \Omega \to \Omega'$ is a a quasiconformal map between planar domains, then there is a quasiconformal map $\varphi : \mathbb{C} \to \mathbb{C}$ so that $\psi \circ \varphi$ is conformal on $\varphi^{-1}(\Omega)$.

We can now state our main result:

Theorem 1.1 (All models occur). *Suppose (Ω, F) is a model and $0 < \rho \le 1$. Then there is $f \in \mathcal{B}$ and a quasiconformal $\varphi : \mathbb{C} \to \mathbb{C}$ so that $F = f \circ \varphi$ on $\Omega(2\rho)$. In addition,*

1. *$|f \circ \varphi| \le e^{2\rho}$ off $\Omega(2\rho)$ and $|f \circ \varphi| \le e^{\rho}$ off $\Omega(\rho)$. Thus the components of $\{z : |f(z)| > e^{\rho}\}$ are in 1-to-1 correspondence to the components of Ω via φ.*
2. *$S(f) \subset D(0, e^{\rho})$.*
3. *the quasiconstant of φ is $O(\rho^{-2})$ with a constant independent of F and Ω,*
4. *φ^{-1} is conformal except on the set $\Omega(\frac{\rho}{2}, 2\rho)$.*

Another useful way to state the result (for those familiar with the language), is that for any model F and any $\rho > 0$, F restricted to $\Omega(\rho)$ can be extended to a quasiregular function on $\mathbb{C}$ that is bounded off $\Omega(\rho)$ and has a quasiconstant bounded depending only on ρ. The extension is holomorphic off $\Omega(\rho/2)$. The precise definition and basic properties of quasiregular functions can be found in, e.g., [13], [14], [18], [20].

We say that two continuous maps $f : X \to X$ and $g : Y \to Y$ are conjugate if there is a homeomorphism $h : X \to Y$ so that

$$g = h \circ f \circ h^{-1}.$$

It is easy to see that if this holds then

$$g^n = h \circ f^n \circ h^{-1},$$

for all $n \geq 0$, so that the orbits of f correspond via h to the orbits of g. For our purposes this means the dynamics of f and g are "the same".

Lasse Rempe-Gillen has pointed out that Theorem 1.1 implies the following result:

Theorem 1.2. *If F is any disjoint type model, then there is a disjoint type Eremenko-Lyubich function f so that f and F are quasiconformally conjugate on a neighborhood of their Julia sets. More precisely, there is a quasiconformal $\varphi : \mathbb{C} \to \mathbb{C}$ so that*

$$f \circ \varphi = \varphi \circ F,$$

on an open set that contains both $\mathcal{J}(f)$ and $\mathcal{J}(F)$.

This means that any property of $\mathcal{J}(F)$ that is preserved by quasiconformal maps also holds for $\mathcal{J}(f)$, e.g., every component of $\mathcal{J}(f)$ is path connected or the Julia set has positive area. Since it is generally easier to build a model with a desired property than to build a entire function directly, this result is useful in constructing Eremenko-Lyubich functions with pathological behavior. For example, Rempe-Gillen uses this result in [17] to show there are functions in $\mathcal{B}$ so that the components of the Julia sets are pseudo-arcs, by building models that have this property.

Theorems 1.1 and 1.2 are inspired by results of Lasse Rempe-Gillen [15] that draw the same conclusions from a stronger hypothesis: he assumes that $F = e^\tau$ is defined on a model domain Ω with a single tract and restricts it to a slightly smaller domain than $\Omega(\rho)$; roughly, he omits a strip whose width grows logarithmically, i.e., $\tau^{-1}(\{x + iy : x > \max(1, \log |y|\})$. His version of Theorem 1.1 is proved by using a Cauchy integral construction to first approximate F uniformly and then show that uniform approximation implies quasiconformal approximation in the sense of Theorem 1.1. Rempe-Gillen then shows how to deduce Theorem 1.2 from Theorem 1.1 using an iterative construction. With his permission, we sketch his argument in Section 9 for the convenience of the reader (our application does not require the much more powerful results he also proved in [16]).

We sketch the proof of Theorem 1.1 quickly here to give the basic idea. Let $W = \mathbb{C} \setminus \overline{\Omega(\rho)}$. It is simply connected, non-empty and not the whole plane, so there is a conformal map $\Psi : W \to \mathbb{D}$. Since Ψ maps ∂W to the unit circle, if we knew that $F = f|_\Omega$ for some entire function f, then $B = e^{-\rho} \cdot F \circ \Psi^{-1}$ would be an inner function on $\mathbb{D}$ (i.e., a holomorphic function on $\mathbb{D}$ so that $|B| = 1$ almost everywhere on the boundary).

The proof of Theorem 1.1 reverses this observation. Given the model and the corresponding domain W and conformal map Ψ we construct a Blaschke product B (a special type of inner function) on the disk so that $G = B \circ \Psi$ approximates $F = e^\tau$ on $\partial\Omega(\rho)$ (the precise nature of the approximation will be described later). This step is fairly straightforward using standard estimates of the Poisson kernel on the disk. We then "glue" G to F across ∂W to get a quasi-regular function g that agrees with F on $\Omega(2\rho)$ and agrees with G on W. This takes several (individually easy) steps to accomplish. We then use the measurable Riemann mapping theorem to define a quasiconformal mapping $\phi : \mathbb{C} \to \mathbb{C}$ so that $f = g \circ \phi$ is holomorphic on the whole plane. The only critical points of g correspond to critical points of B, and critical points introduced into $\Omega(\rho, 2\rho)$ by the gluing process. We will show that both types of critical values have absolute value $\leq e^\rho$. A different argument shows that any finite asymptotic value of f must correspond to a limit of B along a curve in $\mathbb{D}$, so all finite asymptotic values of f are also bounded by e^ρ. Thus $f \in \mathcal{B}$. Since g is only non-holomorphic in $\Omega(\rho, 2\rho)$, we will also get that ϕ^{-1} is conformal everywhere except in $\Omega(\rho, 2\rho)$.

Given Theorem 1.1 for the Eremenko-Lyubich class $\mathcal{B}$, it is natural to ask the analogous question for the more restrictive Speiser class: can every model be approximated by a Speiser model? This question is addressed in [5], where an analog of Theorem 1.1 is proven for the Speiser class. In that paper we show that given a model (Ω, F) and any $\rho > 0$, there is a $f \in \mathcal{S}$ and a quasiconformal map $\phi : \mathbb{C} \to \mathbb{C}$ so that $f \circ \phi = e^\tau$ on $\Omega(2\rho)$. We may take ϕ to be conformal on $\Omega(\rho)$, and f may be taken with the two critical values $\pm e^\rho$ and no finite asymptotic values.

Note that this result omits the conclusion "$f \circ \phi$ is bounded off $\Omega(2\rho)$". In fact, the Speiser functions constructed in [5] will usually be unbounded off Ω; f can have "extra" tracts that do not correspond to tracts of the original model function F. It is shown in [5] that f has at most twice as many tracts as F and sometimes this many are needed. The Speiser version of Theorem 1.2 says that if (Ω, F) is any model, there is a Speiser class function f, a closed set $A \subset \mathcal{J}(f)$, an open neighborhood U of A and a quasiconformal map $\varphi : \mathbb{C} \to \mathbb{C}$ that conjugates f to F on U. Thus the dynamics of any model can be found "inside" the Julia set of a Speiser class function. See [5] for the precise statement.

Finally, the construction in this paper uses a construction called "simple folding". A more complicated version of this is used in [4] to construct functions in the Speiser class with prescribed geometry. The paper [5] on Speiser models uses the main result of [4] to prove the results described in the preceding paragraphs. Thus this paper can be thought of as a gentle introduction to [4], whereas [5] is a sequel to [4]. The results of both this paper and [5] originally appeared in a single preprint titled "The geometry of bounded type entire functions", but I have split this into two papers in an attempt to improve the exposition and to separate the simpler, self-contained arguments for the Eremenko-Lyubich class $\mathcal{B}$ from the more intricate arguments relying on [4] needed for the Speiser class $\mathcal{S}$.

Using quasiconformal methods to construct holomorphic functions with desired geometry has a long history and has been a crucial tool in several areas such as value distribution theory and, more recently, holomorphic dynamics. See [8] and [12] for surveys of applications to the first area and [6] for a recent survey of the second.

Many thanks to Adam Epstein, Alex Eremenko and Lasse Rempe-Gillen for very helpful discussions about the contents of this paper and about the folding constructions and their applications. The introduction of the paper and the formulation of the main result in terms of approximation of models was inspired by a lecture of Lasse Rempe-Gillen at an ICMS conference on transcendental dynamics in Edinburgh, May 2013. Also thanks to the anonymous referee whose comments prompted the revision of the entire manuscript and improved its clarity and correctness. Malik Younsi also read the revised manuscript and gave me numerous corrections and suggestions that I greatly appreciate.

2. Reduction of Theorem 1.1 to the case $\rho = 1$

We start the proof of Theorem 1.1 with the observation that it suffices to prove the result for $\rho = 1$.

To do this we define two quasiconformal maps, ψ_ρ and φ_ρ. Define

$$L(x) = \begin{cases} x, & 0 < x < \rho/2, \\ (\frac{2-\rho}{\rho})(x - \rho/2) + \rho/2 & \rho/2 \le x \le \rho, \\ x/\rho & \rho \le x \le 2\rho. \end{cases}$$

This is a piecewise linear map that sends $[\rho/2, \rho]$ to $[\rho/2, 1]$ and sends $[\rho, 2\rho]$ to $[1, 2]$. The slope on both intervals is less than $2/\rho$. For $z = x + iy \in \mathbb{H}_r$, define

$$\sigma_\rho(z) = \begin{cases} L(x) + iy & 0 < x \le 2\rho, \\ z + 2 - 2\rho & x > 2\rho. \end{cases}$$

This is quasiconformal $\mathbb{H}_r \to \mathbb{H}_r$ with quasiconstant $K \le 2/\rho$. Then set

$$\psi_\rho(z) = \begin{cases} z, & z \notin \Omega \\ \tau_j^{-1} \circ \sigma_\rho \circ \tau_j(z), & z \in \Omega_j. \end{cases}$$

Note that ψ_ρ is the identity near $\partial\Omega$, so ψ_ρ is quasiconformal on the whole plane by the Royden gluing lemma, e.g., Lemma 2 of [3], Lemma I.2 of [7] on page 303, or [19]. (Actually, since ψ_ρ is the identity off $\Omega(\rho/2)$ which has a smooth boundary, one can use a weaker version of the gluing lemma.)

Next, define

$$\varphi_\rho(z) = \begin{cases} z, & |z| < e^{\rho/2} \\ \exp(\sigma_\rho(\log(z))), & |z| \ge e^{\rho/2} \end{cases}.$$

Note that even though $\log(z)$ is multi-valued, the function σ_ρ does not change the imaginary part of its argument, so the exponential of $\sigma_\rho(\log(z))$ is well defined. This is clearly a quasiconformal map of the plane with quasiconstant $2/\rho$. Note also that these functions were chosen so that if $F = \exp \circ \tau$ is the model function associated to Ω and τ, then on Ω_j

$$\begin{aligned} F \circ \psi_\rho &= \exp \circ \tau_j \circ \tau_j^{-1} \circ \sigma_\rho \circ \tau_j \\ &= \exp \circ \sigma_\rho \circ \log \circ \exp \circ \tau_j \\ &= \varphi_\rho \circ F. \end{aligned} \tag{2.1}$$

Now apply Theorem 1.1 to the model (Ω, F) with $\rho = 1$ to get a $f \in \mathcal{B}$ and a quasiconformal map $\Phi : \mathbb{C} \to \mathbb{C}$ so that $f \circ \Phi = F$ on $\Omega(2)$ and $S(f) \subset D(0, e^1)$. Let $g_\rho = \varphi_\rho^{-1} \circ f \circ \Phi \circ \psi_\rho$. This is an entire function pre and post-composed with quasiconformal maps of the plane, so it is quasiregular. By the measurable Riemann mapping theorem, there is a quasiconformal $\Phi_\rho : \mathbb{C} \to \mathbb{C}$ so that $f_\rho = g_\rho \circ \Phi_\rho^{-1}$ is entire and clearly

$$S(f_\rho) = S(g_\rho) \subset \varphi_\rho^{-1}(S(f)) \subset \varphi_\rho^{-1}(D(0,e)) = D(0,e^\rho).$$

For $z \in \Omega(2\rho)$, $\psi_\rho(z) \in \Omega(2)$, so using this and (2.1)

$$\begin{aligned} f_\rho \circ \Phi_\rho(z) &= g_\rho(z) \\ &= \varphi_\rho^{-1}(f(\Phi((\psi_\rho(z)))) \\ &= \varphi_\rho^{-1}(F(\psi_\rho(z))) \\ &= F(z). \end{aligned}$$

Similarly, $|f_\rho \circ \Phi_\rho| = |g_\rho|$ is bounded by $e^{2\rho}$ off $\Omega(2\rho)$. The quasiconstant of Φ_ρ is, at worst, the product of the constants for Φ, ψ_ρ and φ_ρ, which is $K_1 \cdot 4\rho^{-2}$, where K_1 is the upper bound for the quasiconstant in Theorem 1.1 in the case $\rho = 1$.

Finally, our construction in the next section will show that Φ is conformal except on $\Omega(1,2)$ and that F has a quasiregular extension to the plane that is holomorphic except on $\Omega(1,2)$ and is bounded by e off $\Omega(1)$ and by e^2 off $\Omega(2)$. This implies that g_ρ is holomorphic except on $\Omega(\rho/2, 2\rho)$ (since ψ_ρ is holomorphic off $\Omega(\rho/2, 2\rho)$ and φ_ρ^{-1} is holomorphic off $\{e^{\rho/2} < |z| < e^2\}$.) This, in turn, implies that Φ_ρ is conformal except on $\Omega(\rho/2, 2\rho)$, as desired. Thus f_ρ satisfies Theorem 1.1 for the model (Ω, F) and the given $\rho > 0$.

3. The proof of Theorem 1.1

In this section we give the proof of Theorem 1.1 for $\rho = 1$, stating certain facts as lemmas to be proven in later sections.

Let $W = \mathbb{C} \setminus \overline{\Omega(1)}$. This is an open, connected, simply connected domain that is bounded by analytic arcs $\{\gamma_j\}$ that are each unbounded in both directions. See Figure 2. The same comments hold for the larger domain $W_2 = \mathbb{C} \setminus \overline{\Omega(2)}$.

Let $L_1 = \{x + iy : x = 1\}$ and $L_2 = \{x + iy : x = 2\}$. The vertical strip between these two lines will be denoted S. Note that L_1 is partitioned into intervals of length 2π by the points $1 + 2\pi i\mathbb{Z}$. This partition of L_1 will be denoted $\mathcal{J}$. Note that $\tau_j(\gamma_j) = L_1$, so each curve γ_j is partitioned by the image of $\mathcal{J}$ under τ_j^{-1}. We denote this partition of γ_j by $\mathcal{J}_j$. Because elements of $\mathcal{J}_j$ are all images of a fixed interval $J \in L_1 \subset \mathbb{H}_r$ under some conformal map of $\mathbb{H}_r$, the distortion theorem (e.g., Theorem

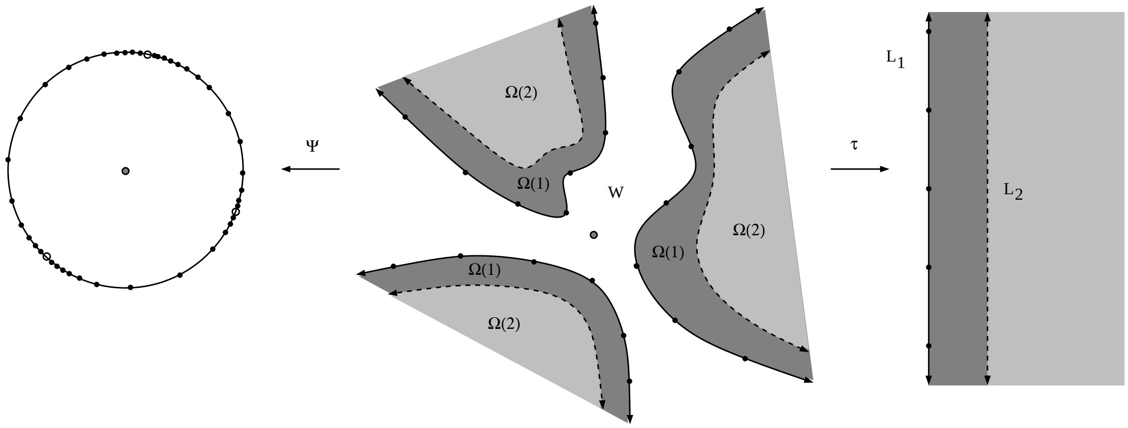

FIGURE 2. W is the complement of $\Omega(1)$; it is simply connected and bounded by smooth curves. We are given the holomorphic function $F = e^\tau$ on $\Omega(2)$ and we will define a holomorphic function on W using the Riemann map Ψ of W to the unit disk, and a specially chosen infinite Blaschke product B on the disk. We will then interpolate these functions in $\Omega(2) \setminus \Omega(1)$ by a quasiregular function. Each component of this set is mapped to a vertical strip by τ, and it is in these strips that we construct the interpolating functions. Note that the integer partition on the boundary of the half-plane pulls back under τ to a partition of each component of $\partial\Omega(1)$, and that Ψ maps these to a partition of the unit circle (minus the singular set of Ψ). The Blaschke product B will be constructed so that $B^{-1}(1)$ approximates this partition of the circle.

I.4.5 of [11]) implies they all lie in a compact family of smooth arcs and that adjacent elements of $\mathcal{J}_j$ have comparable lengths with a uniform constant, independent of j, Ω and F.

Let $\Psi : W \to \mathbb{D}$ be a conformal map given by the Riemann mapping theorem. We claim that Ψ can be analytically continued from W to W_2 across γ_j. Let R_1 denote reflection across L_1 and for $z \in \Omega_j \cap W = \tau_j^{-1}(\{x+iy : 0 < x < 1\})$ let $T = \tau_j^{-1} \circ R_1 \circ \tau_j$; this defines an anti-holomorphic 1-to-1 map from $\Omega_j(0,1)$ to $\Omega_j(1,2)$ that fixes each point of γ_j. We can then extend Ψ by the formula

$$\Psi(T(z)) = 1/\overline{\Psi(z)},$$

(where the right hand side denotes reflection of $\Psi(z)$ across the unit circle). The Schwarz reflection principle says this is an analytic continuation of Ψ to W_2.

Thus Ψ is a smooth map of each γ_j onto an arc I_j of the unit circle $\mathbb{T} = \partial\mathbb{D} = \{|z| = 1\}$. The complement of these arcs is a closed set $E \subset \mathbb{T}$. It is a standard fact of conformal mappings that since E is the set where a conformal map fails to have a finite limit, it has zero Lebesgue, indeed, zero logarithmic capacity. We will not need this fact, although we will use the easier fact that E can't contain an interval (i.e., a conformal map can't have infinite limits on an interval).

The partition $\mathcal{J}_j$ of γ_j transfers, via Ψ to a partition of $I_j \subset \mathbb{T}$ into infinitely many intervals $\{J^j_k\}, k \in \mathbb{Z}$. We will let $\mathcal{K} = \cup_{j,k} J^j_k$ denote the collection of all intervals that occur this way. Thus $\mathbb{T} = E \bigcup \cup_{K\in\mathcal{K}} K$.

Because Ψ conformally extends from W to W_2, $|\Psi'|$ has comparable minimum and maximum on each partition element of γ_j (with uniform constants). Thus the corresponding intervals $\{J^j_k\}$ have the property that adjacent intervals have comparable lengths (again with a uniform bound).

The hyperbolic distance between two points $z_1, z_2 \in \mathbb{D}$ is defined as

$$\rho(z_1, z_2) = \inf_\gamma \int_\gamma \frac{|dz|}{1 - |z|^2}.$$

See Chapter 1 of [11] for the basic properties of the hyperbolic metric. Here we will mostly need the facts that it is invariant under Möbius self-maps of the disk, that hyperbolic geodesics are circular arcs in $\mathbb{D}$ that are perpendicular to $\mathbb{T}$, and that points hyperbolic distance r from 0 are Euclidean distance

$$\frac{2}{\exp(2r) + 1} = O(\exp(-r)),$$

from the unit circle.

For any proper sub-interval $I \subset \mathbb{T}$, let γ_I be the hyperbolic geodesic with the same endpoints as I and let a_I be the point on γ_i that is closest to the origin (closest in either the Euclidean or hyperbolic metrics; it is the same point).

Since $\mathcal{K}$ are disjoint intervals on the circle,

$$\sum_{K\in\mathcal{K}} (1 - |a_K|) < \infty,$$

and so

$$B(z) = \prod_{\mathcal{K}} \frac{|a_K|}{a_K} \frac{a_K - z}{1 - \overline{a_K} z},$$

defines a convergent Blaschke product (see Theorem II.2.2 of [10]). Thus B is a bounded, non-constant, holomorphic function on $\mathbb{D}$ that vanishes exactly on the set $\{a_n\}$. Also, $|B|$ has radial limits 1 almost everywhere. Moreover, B extends meromorphically to $\mathbb{C} \setminus E$, where E is the accumulation set of its zeros on $\mathbb{T}$; this is the same set E as defined above using the map Ψ (the zeros accumulate at both endpoints of every component of $\mathbb{T} \setminus E$, and since these points are dense in E, the accumulation set of the zeros is the whole singular set E). The poles of the extension are precisely the points in the exterior of the unit disk that are the reflections across $\mathbb{T}$ of the zeros.

Any subset $\mathcal{M}$ of $\mathcal{K}$ also defines a convergent Blaschke product. Fix such a subset. The corresponding Blaschke product $B_{\mathcal{M}}$ induces a partition of each I_j with endpoints given by the set $\{e^{i\theta} : B_{\mathcal{M}}(e^{i\theta}) = 1\}$ and this induces a partition $\mathcal{H}_j$ of each γ_j via the map Ψ. This in turn, induces a partition $\mathcal{L}_j$ of L_1 via τ_j.

We would like to say that the partitions $\mathcal{L}_j$ and $\mathcal{J}$ are "almost the same". The first step to making this precise is a lemma that we will prove in Section 4:

Lemma 3.1. *There is a subset $\mathcal{M} \subset \mathcal{K}$ so that if B is the Blaschke product corresponding to $\mathcal{M}$ and $\mathcal{L}_j$ is the partition of L_1 corresponding to B via $\tau_j \circ \Psi^{-1}$, then each element of $\mathcal{J}$ hits at least 2 elements of $\mathcal{L}_j$ and at most M elements of $\mathcal{L}_j$, where M is uniform. In particular, no element of $\mathcal{J}$ can hit both endpoints of any element of $\mathcal{L}_j$ (elements of each partition are considered as closed intervals).*

In Section 5 we will prove

Lemma 3.2. *Suppose $K = [1 + ia, 1 + ib] \in \mathcal{L}_j$ and define*

$$\alpha(1 + iy) = \frac{1}{2\pi} \arg(B \circ \Psi \circ \tau_j^{-1}(1 + iy)),$$

where we choose a branch of α so $\alpha(1 + ia) = 0$ (recall that $B(\Psi(\tau_j^{-1}(1 + ia))) = 1 \in \mathbb{R}$). Set

$$\psi_1(z) = 1 + i(a(1 - \alpha(z)) + b\alpha(z)) = 1 + i(a + (b - a)\alpha(z)).$$

Then ψ_1 is a homeomorphism from K to itself so that $\alpha \circ \psi_1^{-1} : K \to [0, 1]$ is linear and ψ_1 can be extended to a quasiconformal homeomorphism of $R = K \times [1, 2]$ to itself that is the identity on the $\partial R \setminus K$ (i.e., it fixes points on the top, bottom and right side of R).

The main point of the proof is to show that $\arg(B \circ \Psi \circ \tau_j^{-1}) : K \to [0, 2\pi]$ is biLipschitz with uniform bounds.

Roughly, Lemma 3.1 says there are more elements of $\mathcal{J}$ than there are of $\mathcal{L}_j$. This is made a little more precise by the following:

Lemma 3.3. *There is a* 1*-to-*1*, order preserving map of* $\mathcal{L}_j$ *into (but not necessarily onto)* $\mathcal{J}$ *so that each interval* $K \in \mathcal{L}_j$ *is sent to an interval* J *with* $\operatorname{dist}(K, J) \leq 2\pi$*. Moreover, adjacent elements of* $\mathcal{L}_j$ *map to elements of* $\mathcal{J}$ *that are either adjacent or are separated by an even number of elements of* $\mathcal{J}$*.*

This will be proven in Section 6. Again, the proof is quite elementary.

Partition $\mathcal{J} = \mathcal{J}_1^j \cup \mathcal{J}_2^j$ according to whether the interval is associated to some element of $\mathcal{L}_j$ by Lemma 3.3 (i.e., $\mathcal{J}_1^j$ is the image of $\mathcal{L}_j$ under the map in the lemma). The maximal chains of adjacent elements of $\mathcal{J}_2^j$ will be called blocks. By the lemma, each block has an even number of elements. We will say that the block associated to an element $J \in \mathcal{J}_1^j$ is the block immediately above J.

Thus each interval K in $\mathcal{L}_j$ is associated to an interval J' that consists of the corresponding J given by Lemma 3.3 and its associated block. K and J' have comparable lengths and are close to each other, so the orientation preserving linear map from J' to K defines a piecewise linear map $\tilde{\psi}_2 : \mathbb{R} \to \mathbb{R}$ that is biLipschitz with a uniform constant. Using linear interpolation we can extend this to a biLipschitz map ψ_2 of the strip $S = \{x + iy : 1 < x < 2\}$ to itself that equals $\tilde{\psi}_2$ on L_1 (the left boundary) and is the identity on L_2 (the right side).

Each element $J \in \mathcal{J}_2^j$ is paired with a distinct element $J^* \in \mathcal{J}_2^j$ that belongs to the same block. The two outer-most elements of the block are paired, as are the pair adjacent to these, and so on. Similarly, each point z is paired with the other point z^* in the block that has the same distance to the boundary (the center of the block is an endpoint of $\mathcal{J}$ and is paired with itself).

For each $K \in \mathcal{L}_j$, let J_K be the corresponding element of $\mathcal{J}_1^j$ and let I_K be the union of J_K and its corresponding block. Let $R_K = [1, 2] \times I_K$. Let $U_K = R_K \setminus X_K$, where X_K is the closed segment connecting the upper left corner of R_K to the center of R_K. See Figure 3.

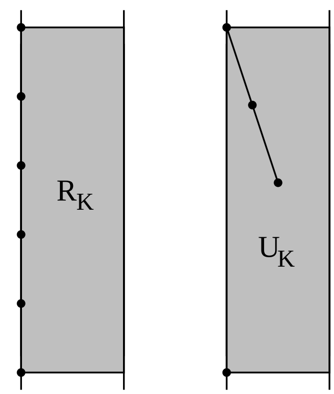

FIGURE 3. Definition of U_K

Lemma 3.4 (Simple folding)**.** *There is a quasiconformal map $\psi_3 : U_K \to R_K$ so that (ψ_3 depends on j and on K, but we drop these parameters from the notation)*

(1) *ψ_3 is the identity on $\partial R_K \setminus L_1$ (i.e., it is the identity on the the top, bottom and right side of R_K),*
(2) *ψ_3^{-1} extends continuously to the boundary and is linear on each element of $\mathcal{J}$ lying in I_K,*
(3) *ψ_3 maps I_K (linearly) to J_K,*
(4) *for each $z \in I_K$, $\psi_3^{-1}(z) = \psi_3^{-1}(z^*) \in X_k$ (i.e., ψ_3 maps opposite sides of X_k to paired points in I_k),*
(5) *the quasiconstant of ψ_3 depends only on $|I_K|/|J_K|$, i.e., on the number of elements in the block associated to K. It is independent of the original model and of the choice of j and K.*

We call this "simple folding" because it is a simple analog of a more complicated folding procedure given in [4]. In the lemma above, the image domain is a rectangle with a slit removed and the quasiconstant of ψ_3 is allowed to grow with n, the number of block elements. This growth is not important in this paper because here we only apply the folding construction in cases where this number n is uniformly bounded (this will occur in our application because of Lemma 3.1). In [4], the corresponding values may be arbitrarily large but the folding construction there must give a map with uniformly bounded quasiconstant regardless. The construction in [4] removes a collection of finite trees from R_k and does so in a way that keeps the quasiconstant of ψ_3 bounded independent of n (there are also complications involving how the construction on adjacent rectangles are merged).

We want to treat the boundary intervals in $\mathcal{J}_1$ and $\mathcal{J}_2$ slightly differently. The precise mechanism for doing this is:

Lemma 3.5 (exp-cosh interpolation). *There is a quasiregular map $\sigma_j : S \to D(0, e^2)$ so that*

$$\sigma_j(z) = \begin{cases} \exp(z), & z \in J \in \mathcal{J}_1^j, \\ e \cdot \cosh(z-1), & z \in J \in \mathcal{J}_2^j, \\ \exp(z), & z \in \mathbb{H}_r + 2. \end{cases}$$

The quasiconstant of ϕ_j is uniformly bounded, independent of all our choices.

This lemma will be proven in Section 8 and is completely elementary.

We now have all the individual pieces needed to construct the interpolation g_j between e^z on L_2 and $B \circ \Psi \circ \tau_j^{-1}$ on L_1. Let U_j be S minus all the segments X_K where $K \in \mathcal{L}_j$ as in Lemma 3.4. Define a quasiconformal map $\psi : U_j \to S$ by

$$\psi = \psi_1 \circ \psi_2 \circ \psi_3,$$

and let $g_j = \sigma_j \circ \psi$ map U_j into $D(0, e^2)$. By definition, each ψ_i, $i = 1, 2, 3$ is the identity on L_2, so $g_j(z) = e^z$ on L_2. For any $K \in \mathcal{L}_j$, the map ψ sends the boundary segments of ∂U_K that lie on some X_K linearly onto elements of $\mathcal{J}_2^j$, so boundary points on opposite sides of X_K get mapped to points that are equidistant from $2\pi i\mathbb{Z}$ and cosh agrees at any two such points. Thus g_j extends continuously across each slit X_K. Finally, the map ψ was designed so that g_j is continuous on S and agrees with $B \circ \Psi \circ \tau_j^{-1}$ on L_1. Thus $g_j \circ \tau_j$ continuously interpolates between $B \circ \Psi$ on W and F on $\Omega(2)$ and so defines a quasiregular g on the whole plane with a uniformly bounded constant. Thus by the measurable Riemann mapping theorem there is a quasiconformal $\varphi : \mathbb{C} \to \mathbb{C}$ so that $f = g \circ \varphi$ is entire.

The singular values of f are the same as for g. On $\Omega(2)$, $g = F = e^\tau$, so g has no critical points in this region. In U_j, $g = g_j$ is locally 1-to-1, so has no critical points there either. Thus the only critical points of g in $\Omega(1)$ are on the slits X_K, then these are mapped by g onto the circle of radius e around the origin. Thus every critical value of g (and hence f) must lie in $D(0, e)$.

If g has a finite asymptotic value outside $\overline{D(0, e)}$, then it must be the limit of g along some curve Γ contained in a single component of Ω. Then e^z has a finite limit along $\tau(\Gamma) \subset \mathbb{H}_r$; this is impossible, so f has no finite asymptotic values outside $\overline{D(0,e)}$. Thus $S(f) \subset \overline{D(0,e)}$, and so $f \in \mathcal{B}$.

This proves Theorem 1.1 except for the proof of the lemmas.

4. Blaschke partitions

In this section we prove Lemma 3.1. We start by recalling some basic properties of the Poisson kernel and harmonic measure in the unit disk $\mathbb{D}$.

The Poisson kernel on the unit circle with respect to the point $a \in \mathbb{D}$ is given by the formula

$$P_a(\theta) = \frac{1-|a|^2}{|e^{i\theta}-a^2|} = \frac{1-|a|^2}{1-2|a|\cos(\theta-\phi)+|a|^2},$$

where $a = |a|e^{i\phi}$. This is the same as $|\sigma'|$ where σ is any Möbius transformation of the disk to itself that sends a to zero. If $E \subset \mathbb{T}$, we write

$$\omega(E, a, \mathbb{D}) = \frac{1}{2\pi}\int_E P_a(e^{i\theta})d\theta,$$

and call this the harmonic measure of E with respect to a. This is the same as the (normalized) Lebesgue measure of $\sigma(E) \subset \mathbb{T}$ where $\sigma : \mathbb{D} \to \mathbb{D}$ is any Möbius transformation sending a to 0. It is also the same as the first hitting distribution on $\mathbb{T}$ of a Brownian motion started at a (although we will not use this characterization).

Suppose $I \subset \mathbb{T}$ is any proper arc, and, as before, let γ_I be the hyperbolic geodesic in $\mathbb{D}$ with the same endpoints as I; then γ_I is a circular arc in $\mathbb{D}$ that is perpendicular to $\mathbb{T}$ at its endpoints. Let a_I denote the point of γ_I that is closest to the origin.

Lemma 4.1. $\omega(I, a_I, \mathbb{D}) = \frac{1}{2}$.

Proof. Apply a Möbius transformation of $\mathbb{D}$ that sends a_I to the origin. Then γ_I must map to a diameter of the disk and I maps to a semi-circle. □

Given two disjoint arcs I, J in $\mathbb{T}$, let γ_I, γ_J be the two corresponding hyperbolic geodesics and let a_I^J be the point on γ_I that is closest to J and let a_J^I be the point on γ_J that is closest to I.

Lemma 4.2. $\omega(I, a_J^I, \mathbb{D}) = \omega(J, a_I^J, \mathbb{D})$

Proof. Everything is invariant under Möbius maps of the unit disk to itself, so use such a map to send I, J to antipodal arcs. Then the conclusion is obvious. □

Lemma 4.3. *If $z, w \in \mathbb{D}$ and $I \subset \mathbb{T}$, then*

$$\frac{\omega(I, z, \mathbb{D})}{\omega(I, w, \mathbb{D})} \leq C$$

where the constant C depends only on the hyperbolic distance between z and w.

Proof. Suppose $\sigma(z) = (z - w)/(1 - \overline{w}z)$ maps w to 0. Then $u(z) = \omega(I, \sigma(z), \mathbb{D})$ is a positive harmonic function on $\mathbb{D}$, so the lemma is just Harnack's inequality applied to u. □

Suppose $I, J, \subset \mathbb{T}$ are disjoint closed arcs and $\operatorname{dist}(I, J) \geq \epsilon \max(|I|, |J|)$. Then we call I and J ϵ-separated. This implies the hyperbolic geodesics γ_I, γ_J are separated in the hyperbolic metric (with a lower bounded depending only on ϵ), but the converse is not true.

Lemma 4.4. *If $I, J \subset \mathbb{T}$ are ϵ-separated, then the hyperbolic distance between a_I and a_I^J is bounded, depending only on ϵ.*

Proof. Assume I is the longer arc and consider hyperbolic geodesic S that connects a_I^J and a_J^I. Then S is perpendicular to γ_I at a_I^J, so if $1 - |a_I^J| \ll 1 - |a_I|$, S will hit the unit circle without hitting γ_j. See Figure 4. □

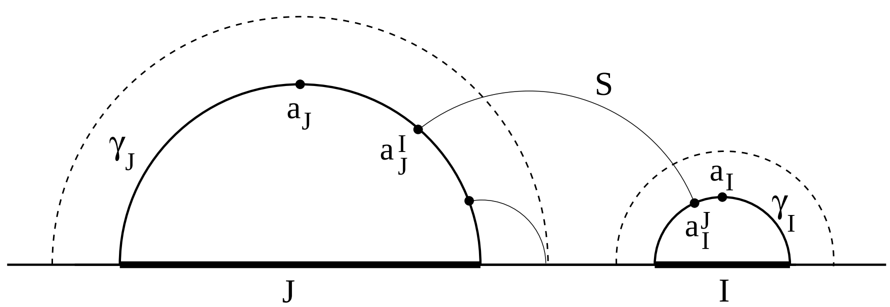

FIGURE 4. If the intervals I and J are ϵ-separated, then a shortest path between γ_I and γ_J must hit each geodesic near the "top" points. A perpendicular geodesic that starts too "low" on γ_J will hit the unit circle without hitting γ_I.

Lemma 4.5. *Suppose that I, J are ϵ-separated. Then*

$$\omega(I, a_J, \mathbb{D}) \simeq \omega(J, a_I, \mathbb{D}),$$

where the constant depends only on ϵ.

Proof. This follows immediately from our earlier results. □

Lemma 4.6. *Suppose that I and J are ϵ-separated and that a_J, a_I are at least distance R apart in the hyperbolic metric. Then*

$$\omega(J, a_I, \mathbb{D}) \leq C(\epsilon) e^{-R}.$$

Proof. Since the intervals are ϵ-separated, the hyperbolic distance between a_I and a_J is the same as the distance between a_I^J and a_J^I, up to a bounded additive factor. Thus if we apply a Möbius transformation of $\mathbb{D}$ so that $a_J = 0$, a_I is mapped to a point w with $1 - |w| = O(e^{-R})$, which implies $\omega(I, a_J, \mathbb{D}) = O(e^{-R})$. Since the intervals are ϵ-separated, the reverse inequality also holds by Lemma 4.5. □

Fix $M < \infty$ and suppose $\mathcal{K}$ is a collection of disjoint (except possibly for endpoints) closed intervals on $\mathbb{T}$ so that any two adjacent intervals have length ratio at most M. We say that two intervals I, J are S steps apart if there is a chain of $S+1$ adjacent intervals $J_0, \dots, J_S$ so that $I = J_0$ and $J = J_S$.

Note that if $I, J \in \mathcal{K}$ are adjacent, then a_I, a_J are at bounded hyperbolic distance T apart (and T depends only on M). Also, if $I, J \in \mathcal{K}$ are not adjacent, then they are ϵ-separated for some $\epsilon > 0$ that depends only on M.

Lemma 4.7. *For any $R > 0$ there is a collection $\mathcal{N} \subset \mathcal{K}$ so that*

(1) *for any $I \in \mathcal{K}$, there is a $J \in \mathcal{N}$ with $\rho(a_J, a_I) \leq R$*
(2) *for any $I, J \in \mathcal{N}$, $\rho(a_J, a_I) \geq R$.*

Proof. Just let $\mathcal{N}$ correspond to a maximal collection of the points $\{a_K\}$ with the property that any two of them are hyperbolic distance $\geq R$ apart. □

Fix a positive integer S. For each $J \in \mathcal{N}$ choose the shortest element of $\mathcal{K}$ that is at most S steps away from J. Let $\mathcal{M} \subset \mathcal{K}$ be the corresponding collection of intervals.

Lemma 4.8. *Suppose R, S, T are as above and $R \geq 4ST$. If $\mathcal{K}$ and $\mathcal{M}$ are as above, then for all $K \in \mathcal{K}$,*

$$\epsilon \leq \sum_{J \in \mathcal{M}} \omega(K, a_J, \mathbb{D}) \leq \mu,$$

where $\epsilon > 0$ depends only on R and $\mu \to 1/2$ as $S \to \infty$.

Proof. The left-hand inequality is easier and we do it first. Fix $K \in \mathcal{K}$. There is a $I \in \mathcal{N}$ with $\rho(a_I, a_K) \leq R$, and since adjacent elements of $\mathcal{K}$ have points that are only T apart in the hyperbolic metric, there is an element $J \in \mathcal{M}$ with $\rho(a_K, a_J) \leq R + ST \leq \frac{5}{4}R$. This implies $|J| \simeq |K| \simeq \mathrm{dist}(J, K)$ and these imply $\omega(K, a_J, \mathbb{D}) \geq \epsilon$ with ϵ depending only on ρ. Thus every element of $\mathcal{K}$ has harmonic measure bounded below with respect to some point corresponding to a single element of $\mathcal{M}$ and hence the sum of harmonic measures over all elements of $\mathcal{M}$ is also bounded away from zero uniformly.

Now we prove the right-hand inequality. By our choice of R, points a_J corresponding to distinct intervals in $\mathcal{M}$ are at least distance $R/2$ apart. Fix $K \in \mathcal{K}$. There is at most one point within hyperbolic distance $R/4$ of a_K and the harmonic measure it assigns K is at most $1/2$ since the point lies on or outside the geodesic γ_K.

All other points associated to elements of $\mathcal{M}$ are Euclidean distance $\geq \exp(R/8)|K|$ away from K or are within this distance of K, and are within Euclidean distance $\exp(-R/8)|K|$ of the unit circle (this is because of the Euclidean geometry of hyperbolic balls in the half-space). We call these two disjoint sets $\mathcal{M}_1$ and $\mathcal{M}_2$ respectively.

Using Lemma 4.5 we see that the

$$\sum_{J \in \mathcal{M}_1} \omega(K, a_J, \mathbb{D}) = O(\sum_{J \in \mathcal{M}_1} \omega(J, a_K, \mathbb{D})) = O(\exp(-R/8)).$$

To bound the sum over $\mathcal{M}_2$, we note that each interval in $\mathcal{M}_2$, is the endpoint of a chain of S adjacent intervals that are each at least as long as J. Since

$$|J| \leq \exp(-R/8)|K|,$$

and

$$\mathrm{dist}(J, K) \gtrsim |K|,$$

we can deduce

$$\omega(J, a_K, \mathbb{D}) \leq O(\frac{1}{S})\omega(a_K, J, \mathbb{D}),$$

so since the J's are all disjoint intervals,

$$\sum_{J \in \mathcal{M}_2} \omega(K, a_J, \mathbb{D}) = O(\frac{1}{S} \sum_{J \in \mathcal{M}_2} \omega(J, a_K, \mathbb{D})) = O(\frac{1}{S}).$$

Choosing first S large, and then R large (depending on S and separation constant of $\mathcal{K}$), both sums are as small as we wish, which proves the lemma. $\square$

Corollary 4.9. *Suppose B is as above and $K \in \mathcal{K}$. Then*

$$\epsilon \leq \frac{1}{|K|}\frac{\partial B}{\partial \theta} \leq C.$$

Proof. If I, J are ϵ-separated, then it is easy to verify that

$$\sup_{z\in J} P_{a_I}(z), \qquad \inf_{z\in J} P_{a_I}(z),$$

are comparable up to a bounded multiplicative factor that depends only on ϵ. The lemma then follows from our earlier estimates. □

We have now essentially proven Lemma 3.1; it just remains to reinterpret the terminology a little. For the reader's convenience we restate the lemma.

Lemma 4.10 (The Blaschke partition)**.** *There is a subset $\mathcal{M} \subset \mathcal{K}$ so that if B is the Blaschke product corresponding to $\mathcal{M}$ and $\mathcal{L}_j$ is the partition of L_1 corresponding to B via $\tau_j \circ \Psi^{-1}$, then each element of $\mathcal{J}$ hits at least 2 elements of $\mathcal{L}_j$ and at most M elements of $\mathcal{L}_j$, where M is uniform. In particular, no element of $\mathcal{J}$ can hit both endpoints of any element of $\mathcal{L}_j$ (elements of each partition are considered as closed intervals).*

Proof. A computation shows that for the Blaschke product

$$B(z) = \prod_n \frac{|a_n|}{a_n}\frac{z-a_n}{1-\bar{a}_n z},$$

the derivative satisfies

$$|\frac{\partial B}{\partial \theta}(e^{i\theta})| = \sum_n P_{a_n}(e^{i\theta}),$$

and the convergence is absolute and uniform on any compact set K disjoint from the singular set E of B (since B is a product of Möbius transformations, and the derivative of a Möbius transformation is a Poisson kernel, this formula is simply the limit of the n-term product formula for derivatives).

Lemma 4.8 now says we can choose $\mathcal{M}$ so that

$$2\pi\epsilon \leq \int_J |\frac{\partial}{\partial\theta} B| d\theta \leq \frac{3}{4}\cdot 2\pi = \frac{3\pi}{2}.$$

Since the integral over an element of $\mathcal{L}$ has integral exactly 2π, the lower bound means that an element of $\mathcal{L}$ can contain at most $1/\epsilon$ elements of $\mathcal{J}$ and hence can intersect at most $2 + \frac{1}{\epsilon}$ elements of $\mathcal{J}$. The upper bound says that each element K of

$\mathcal{L}$ must hit at least 2 elements of $\mathcal{J}$. Hence it is not contained in any single element of $\mathcal{J}$, and so no single element of $\mathcal{J}$ can hit both endpoints of K. $\square$

5. Straightening a biLipschitz map

Lemma 5.1. *Suppose $K = [1 + ia, 1 + ib] \in \mathcal{L}_j$ and define*

$$\alpha(1 + iy) = \frac{1}{2\pi} \arg(B \circ \Psi \circ \tau_j^{-1}(1 + iy)),$$

where we choose a branch of α so $\alpha(1 + ia) = 0$ (recall that $B(\Psi(\tau_j^{-1}(1 + ia))) = 1 \in \mathbb{R}$). Set

$$\psi_1(z) = 1 + i(a(1 - \alpha(z)) + b\alpha(z)) = 1 + i(a + (b - a)\alpha(z)).$$

Then ψ_1 is a homeomorphism from K to itself so that $\alpha \circ \psi_1^{-1} : K \to [0, 1]$ is linear and ψ_1 can be extended to a quasiconformal homeomorphism of $R = K \times [1, 2]$ to itself that is the identity on the $\partial R \setminus K$ (i.e., it fixes points on the top, bottom and right side of R).

Proof. The linearizing property of ψ_1 is clear from its definition, so we need only verify the quasiconformal extensions property.

Corollary 4.9 implies α' is bounded above and below by absolute constants. Let $R = K \times [1, 2]$ and define an extension of ψ_1 by

$$\psi_1(x + iy) = u(x, y) + iv(x, y) = x + i[(2 - x)\psi_1(1 + iy) + (x - 1)y)].$$

i.e., take the linear interpolation between ψ_1 on L_1 and the identity on L_2. We can easily compute

$$\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ y - \psi(y) & (2 - x)(b - a)\alpha'(y) + (x - 1) \end{pmatrix}.$$

Note that $|y - h(y)| \le |K|$ is absolutely bounded. Also, since $|b - a||\alpha'|$ is bounded above and away from 0, so is v_y. Thus the derivative matrix lies in a compact subset of the invertible 2×2 matrices and hence ψ_1 is quasiconformal (with only a little more work we could compute an explicit bound for the quasiconstant, and even prove that the extension is actually biLipschitz). $\square$

6. Aligning partitions

Now we prove Lemma 3.3, which we restate for convenience.

Lemma 6.1. *There is a* 1*-to-*1*, order preserving map of* $\mathcal{L}_j$ *into (but not necessarily onto)* $\mathcal{J}$ *so that each interval* $K \in \mathcal{L}_j$ *is sent to an interval* J *with* $\mathrm{dist}(K, J) \leq 2\pi$*. Moreover, adjacent elements of* $\mathcal{L}_j$ *map to elements of* $\mathcal{J}$ *that are either adjacent or are separated by an even number of elements of* $\mathcal{J}$*.*

Proof. For each $K \in \mathcal{K}$ choose $J \in \mathcal{J}$ so that J contains the lower endpoint of K (if two such intervals contain the endpoint, choose the upper one). No interval J is chosen twice, since Lemma 3.1 says that no J can hit both endpoints of any element of $\mathcal{L}$.

Fix an order preserving labeling of the chosen $\mathcal{J}$ by $\mathbb{Z}$ and denote it $\{J_n\}$. By the gap between J_n and J_{n+1} we mean the number of unselected elements of $\mathcal{J}$ that separate these two intervals. The position of J_0 is fixed. If the gap between J_0 and J_1 is even (including no gap), we leave J_1 where it is. If the gap is odd, there is a least one separating interval and we replace J_1 by the adjacent interval in $\mathcal{J}$ that is closer to J_0. If the gap between (the new) J_1 and J_2 is even, we leave J_2 alone; otherwise, we move it one interval closer to J_0. Continuing in this way, we can guarantee that for all $n \geq 0$, gaps are even and each J_n is either in its original position or adjacent to its original position. Thus its distance to the associated element of $\mathcal{K}$ is at most 2π. The argument for negative indices is identical. □

7. Foldings

Now we prove Lemma 3.4. This is the step that makes the gluing procedure a little different from a standard quasiconformal surgery.

Lemma 7.1 (Simple folding)**.** *There is a quasiconformal map* $\psi_3 : U_K \to R_K$ *so that (*ψ_3 *depends on* j *and on* K*, but we drop these parameters from the notation)*

(1) ψ_3 *is the identity on* $\partial R_K \setminus L_1$ *(i.e., it is the identity on the the top, bottom and right side of* R_K*),*
(2) ψ_3^{-1} *extends continuously to the boundary and is linear on each element of* $\mathcal{J}$ *lying in* I_K*,*
(3) ψ_3 *maps* I_K *(linearly) to* J_K*,*

(4) *for each* $z \in I_K$, $\psi_3^{-1}(z) = \psi_3^{-1}(z^*) \in X_k$ *(i.e.,* ψ_3 *maps opposite sides of* X_k *to paired points in* I_k*),*

(5) *the quasiconstant of* ψ_3 *depends only on* $|I_K|/|J_K|$*, i.e., on the number of elements in the block associated to* K*. It is independent of the original model and of the choice of* j *and* K*.*

Proof. The proof is a picture, namely Figure 5. The map is defined by giving compatible finite triangulations of R_k and U_k (compatible means that there is 1-to-1 map between vertices of the triangulations that preserves adjacencies along edges). Such a map defines linear maps between corresponding triangles that are continuous across edges. Since each such map is non-degenerate, it is quasiconformal and hence the piecewise linear map defined between U_k and R_K is quasiconformal (with quasiconstant given by the worst quasiconstant of the finitely many triangles). The other properties are evident. □

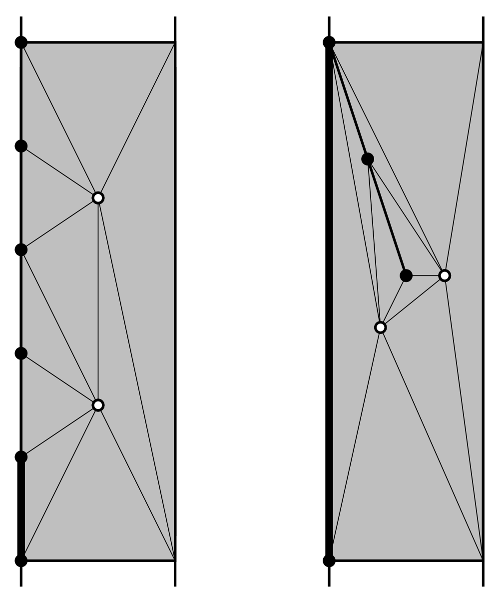

FIGURE 5. The pictorial proof of Lemma 7.1 for $n = 5$.

8. Interpolating between exp and cosh

Lemma 8.1 (exp-cosh interpolation)**.** *There is a quasiregular map* $\sigma_j : S \to D(0, e^2)$ *so that*

$$\sigma_j(z) = \begin{cases} \exp(z), & z \in J \in \mathcal{J}_1^j, \\ e \cdot \cosh(z-1), & z \in J \in \mathcal{J}_2^j, \\ \exp(z), & z \in \mathbb{H}_r + 2. \end{cases}$$

The quasiconstant of σ_j *is uniformly bounded, independent of all our choices.*

Proof. As with the previous lemma, the proof is basically a picture; see Figure 6. Suppose $J \in \mathcal{J}$ and let $R = J \times [1,2]$. The exponential map sends R to the annulus $A = \{e < |z| < e^2\}$, with the left side of R mapping to the inner circle and the top and bottom edges of R mapping to the real segment $[e, e^2]$.

Now define a quasiconformal map $\phi : A \to D(0, e^2)$ that is the identity on $\{|z| = e^2\}$ and on $[e, e^2]$, but that maps $\{|z| = e\}$ onto $[-e, e]$ by $z \to \frac{1}{2}(z + \frac{e^2}{z})$ (this is just a rescaled version of the Joukowsky map $\frac{1}{2}(z + \frac{1}{z})$ that maps the unit circle to $[-1, 1]$, identifying complex conjugate points).

In $\mathbb{H}_r + 2$ and in rectangles of the form $J \times [1,2]$ for $J \in \mathcal{J}_1$ we set $\sigma_j(z) = \exp(z)$. In the rectangles corresponding to elements of $\mathcal{J}_2$ we let $\sigma_j(z) = \phi(\exp(z))$. This clearly has the desired properties. $\square$

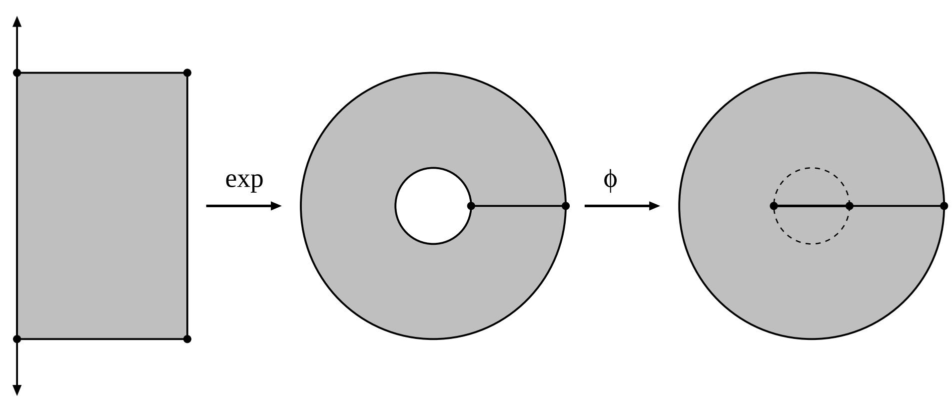

Figure 6. The exponential function maps the rectangle $[1,2] \times J$ conformally to the slit annulus $\{e < |z| < e^2\} \setminus [e, e^2]$. The map ϕ is chosen to map the annulus A=$\{e < |z| < e^2\}$ to the slit disk $\{|z| < e^2\} \setminus [-e, e]$ so that it equals the identity on $\{|z| = e^2\}$ and equals $\frac{1}{2}(z + \frac{e^2}{z})$ on $\{|z| = e\}$.

Actually, the cosh function in the lemma can be replaced by any function $h : J \to [-1, 1]$ that has the property that $h(z)$ only depends on the distance from z to the endpoint of J. This will ensure that after applying a folding map, points that started on opposite sides of some slit X_k will end up being identified by h, which is all we need.

This completes the proof of Theorem 1.1.

9. Proof of Theorem 1.2

Theorem 9.1 (Rigidity for disjoint type)**.** *Suppose (Ω, f) and (Ω', g) are disjoint type models, $\varphi : \mathbb{C} \to \mathbb{C}$ is quasiconformal with $\varphi(\Omega) = \Omega'$ and $f = g \circ \varphi$ on Ω. Then there is a quasiconformal map Φ of the plane so that*

$$\Phi \circ f = g \circ \Phi,$$

on Ω. In particular $\mathcal{J}(g) = \Phi(\mathcal{J}(f))$.

Proof. The statement and proof are due to Lasse Rempe-Gillen [16], but we recreate it here for the convenience of the reader.

Let $W = \mathbb{C} \setminus \overline{\Omega}$ and $W' = \mathbb{C} \setminus \overline{\Omega'}$. We can exhaust W by nested open sets $U_1 \subset U_2 \subset \dots$ with smooth boundaries and Ω' is exhausted by the images $\varphi(U_n)$. Since the union of these open nested sets covers $\overline{\mathbb{D}}$ one of them covers $\overline{\mathbb{D}}$, call it U. Thus we can find a new quasiconformal map $\phi : \mathbb{C} \to \mathbb{C}$ that equals φ outside U and is the identity on $\overline{\mathbb{D}}$.

Now inductively define a sequence of quasiconformal maps $\{\Phi_n\}$ on $\mathbb{C}$ by setting Φ_0 to be the identity and, in general,

$$\Phi_{n+1} = \begin{cases} g^{-1} \circ \Phi_n \circ f, & z \in \Omega \\ \phi, & z \notin \Omega \end{cases}$$

Note that since $f : \Omega \to \{|z| > 1\}$ and $g : \Omega' \to \{|z| > 1\}$ are covering maps, the definition of Φ_{n+1} makes sense as long as Φ_n is a homeomorphism of $\{|z| > 1\}$ to itself. We shall verify this below.

Set $U_0 = U \cap \{|z| > 1\}$ and let $U_n = \cup_{k=1}^{n} \{z \in \Omega : f^k(z) \in \overline{U}\}$. Then $\cup_n U_n$ is the set of all points in Ω that eventually iterate out of Ω. This is the complement of $\mathcal{J}(f)$ in Ω and hence is an open dense set in Ω by Lemma 2.3 of [16]. Let $V_n = \cup_{k=1}^{n} U_k$.

We claim that

(1) for $n \geq 0$, Φ_n maps $\{|z| > 1\}$ to itself,

(2) for $n \geq 0$, Φ_n is quasiconformal with the same quasiconstant as ϕ,

(3) for $n \geq 1$, $\Phi_n = \Phi_{n+1}$ on V_n.

We prove these by induction. The case $n = 0$ for (1) and (2) is trivial since Φ_0 is the identity. For $n = 1$, (3) holds because if $z \in U_1$ then $f(z) \in U_0$

$$\Phi_2(z) \quad = \quad g^{-1}(\Phi_1(f(z)))) = g^{-1}(\Phi_0(f(z)))) = \Phi_1(z).$$

Similarly, for general n (3) holds because if $z \in V_n$, then $z \in U_k$ for some $1 \leq k \leq n$, so $f(z) \in U_k$ for some $0 \leq k \leq n-1$. By the induction hypothesis, $f(z) \in f^{-n-1}(U)$, so

$$\Phi_{n+1}(z) \quad = \quad g^{-1}(\Phi_n(f(z)))) = g^{-1}(\Phi_{n-1}(f(z)))) = \Phi_n(z).$$

Claim (1) follows from (3) for every n since (3) implies Φ_n is the identity on U_0, which contains the unit circle. Since Φ_n is a homeomorphism of the plane that means Φ_n is a homeomorphism of $\{|z| > 1\}$ to itself.

Since $f : \Omega \to \{|z| > 1\}$ and $g : \Omega' \to \{|z| > 1\}$ are holomorphic covering maps, (1) for n implies that the first part of the definition of Φ_{n+1} gives a quasiconformal homeomorphism from Ω to Ω' with the same quasiconstant as Φ_n. By induction, this constant is bounded by the quasiconstant for ϕ. Outside Ω, Φ_{n+1} agrees with ϕ, so again is quasiconformal with constant bounded by that of ϕ. By the Royden gluing lemma (e.g., Lemma 2 of [3], Lemma I.2 of [7] on page 303, [19]), this implies Φ_{n+1} is K-quasiconformal on the whole plane. (In many cases of interest, $\partial\Omega$ will be piecewise smooth, hence removable for quasiconformal mappings, and then the gluing lemma is not needed.) Thus all the claims have been established.

Since the sequence $\{\Phi_n(z)\}$ is eventually constant for every z in the dense set $\cup_n V_n \subset \Omega_0$, and since K-quasiconformal maps form a compact family, we deduce that $\Phi(z) = \lim_n \Phi_n$ defines a K-quasiconformal map of the plane. Moreover,

$$\Phi_{n+1} = g^{-1} \circ \Phi_n \circ f, \quad z \in \Omega$$

becomes

$$\Phi = g^{-1} \circ \Phi \circ f, \quad z \in \Omega$$

in the limit. □

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