A COUNTEREXAMPLE IN CONFORMAL WELDING CONCERNING HAUSDORFF DIMENSION

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1. Introduction. If Γ is a closed Jordan curve on the Riemann sphere $\bar{\mathbf{C}}$ we let Ω_1 and Ω_2 denote the complementary components, and for fixed $z_1 \in \Omega_1$ and $z_2 \in \Omega_2$ we let ω_1 and ω_2 denote the harmonic measures on Γ with respect to these points (see [2]). The object of this note is to prove the following.

THEOREM 1.1. *For any* $1 \le d < 2$ *there is a quasicircle* Γ, *a* $C > 0$, *and points* $z_1 \in \Omega_1$ *and* $z_2 \in \Omega_2$ *such that* $\dim(\Gamma) = d$ *and, for any* (*Borel*) *set* $E \subset \Gamma$,

$$C^{-1} \le \frac{\omega_1(E)}{\omega_2(E)} \le C. \tag{1.1}$$

Here "dim" means Hausdorff dimension (see [4]). Suppose $\{\infty\} \in \Gamma$. Since Ω_1 is simply connected, the Riemann mapping theorem says there is a conformal map Φ_1 from the upper half-plane $H_+ = \{\text{Im}(z) > 0\}$ to Ω_1 with $\Phi_1(i) = z_1$ and $\Phi_1(\infty) = \infty$. Ω_1 is a Jordan domain, so by Carathéodory's theorem Φ_1 extends to a homeomorphism of $\mathbf{R}$ to $\Gamma \setminus \{\infty\}$; similarly for Φ_2 going from the lower half-plane H_- to Ω_2. Hence $\psi \equiv (\Phi_2)^{-1} \circ \Phi_1$ defines an increasing homeomorphism of $\mathbf{R}$ to itself. If (1.1) holds then ψ is bi-Lipschitz. Furthermore, one can show that $\dim(\Gamma) > 1$ implies there is a nonconstant element in the space of bounded, continuous functions on $\bar{\mathbf{C}}$ which are holomorphic off Γ (this space is denoted A_Γ). An element of A_Γ corresponds to a pair of functions $f_1 \in A(H_+)$, $f_2 \in A(H_-)$ (i.e., continuous functions with bounded, holomorphic extensions to H_+ and H_- respectively) which satisfy $f_1 = f_2 \circ \psi$. Thus we obtain the following.

COROLLARY 1.2. *There is a bi-Lipschitz, increasing homeomorphism* ψ *of* $\mathbf{R}$ *to itself and a nonconstant* $f \in A(H_+)$ *such that* $f \circ \psi \in A(H_-)$.

If ψ is a homeomorphism of $\mathbf{R}$ to itself, then Jones showed in [8] that the map $f \to f \circ \psi$ is bounded from BMO to itself if and only if ψ' is in the Muckenhoupt class A_∞. In [9] Semmes asked if such a map must behave well with respect to the decomposition of BMO into holomorphic and antiholomorphic BMO. More precisely, if P denotes the projection from BMO to holomorphic BMO, he asks if there is a constant $C > 0$ such that for any holomorphic $f \in \text{BMO}$

$$\|P(f \circ \psi)\|_* > C\|f\|_*,$$

where $\| \ \|_*$ denotes the BMO norm. Since our homeomorphism satisfies $C^{-1} < \psi' < C$, it is in A_∞, but clearly does not satisfy this inequality.

Before starting the proof we should discuss some related results. For example, recall that Γ is a quasicircle if and only if there is a $C > 0$ such that

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$$C^{-1} \leq \frac{\psi(x+t)-\psi(x)}{\psi(x)-\psi(x-t)} \leq C \tag{1.2}$$

for every $x, t \in \mathbf{R}$ (see [1]). In particular, if ψ is bi-Lipschitz it automatically satisfies this condition. Since a quasicircle must have dimension less than 2, this shows we cannot take $d=2$ in Theorem 1.1. We say Γ is a *chord arc curve* if it is locally rectifiable and there is a $C>0$ such that

$$\sigma(z, w) \leq C|z-w|$$

for all $z, w \in \Gamma$ (σ denotes arclength distance on Γ). By a theorem of David [5], this holds with a constant close to 1 if and only if ψ is absolutely continuous and $\log(\psi')$ is in BMO with small norm. This happens if ψ is bi-Lipschitz with constant close to 1, so Theorem 1.1 fails if C is close to 1. Since A_Γ is trivial if Γ is locally rectifiable, Corollary 1.2 also fails in this case (see [6]). A related example is given in [10], where Semmes constructs a non-locally rectifiable curve satisfying (1.1). Also, in [7] Garnett and O'Farrell give an example of an absolutely continuous ψ on $\mathbf{R}$ and a nonconstant $f \in A(H_+)$ such that $f \circ \psi \in A(H_-)$.

In the next section we motivate the construction, and in Section 3 we recall some simple estimates on harmonic measure. In Section 4 we prove Theorem 1.1.

The results of this paper come from Chapter IV of my doctoral dissertation [3]. I would like to thank my advisor, Peter W. Jones, for all his help and encouragement and Stephen Semmes for many useful conversations concerning these results. I am also grateful to the National Science Foundation for their support in the form of a graduate fellowship and to the referee for his comments.

2. The basic construction. We will build a curve with a given dimension by passing it through a Cantor set with the desired dimension. Start with a unit square. Fix an α, $\frac{1}{4} \leq \alpha < \frac{1}{2}$, and consider the four equal subsquares of side length α obtained by removing a "cross" from the center. Each of these squares is then divided into four squares of side length α^2 in the same way and so on, obtaining at the nth stage 4^n squares of side length α^n. The limiting set is the Cantor set $E = E(\alpha)$. One easily shows

$$\dim(E) = \frac{\log 4}{-\log \alpha} \equiv d_\alpha. \tag{2.1}$$

Now we want a curve which contains $E(\alpha)$. Consider Figure 1. It shows a square of sidelength $(1+\delta)$ with four squares of side length $\alpha(1+\delta)$ removed, one from around each of the first-generation squares of E. In order for these small boxes not to overlap we set $\delta = 1-2\alpha$. We then draw a curve Γ, symmetric with respect to the center of the box, consisting of five arcs connecting the midpoints of the vertical sides of the boxes, as shown in Figure 1.

Now dilate this picture by α and place one copy in each of the smaller squares. Iterating this construction and adding the set $E(\alpha)$ we obtain a curve Γ resembling Figure 2. Since $E(\alpha) \subset \Gamma$ and Γ is smooth away from E,

$$\dim(\Gamma) = \dim(E) = d_\alpha - \frac{\log 4}{\log \alpha}$$

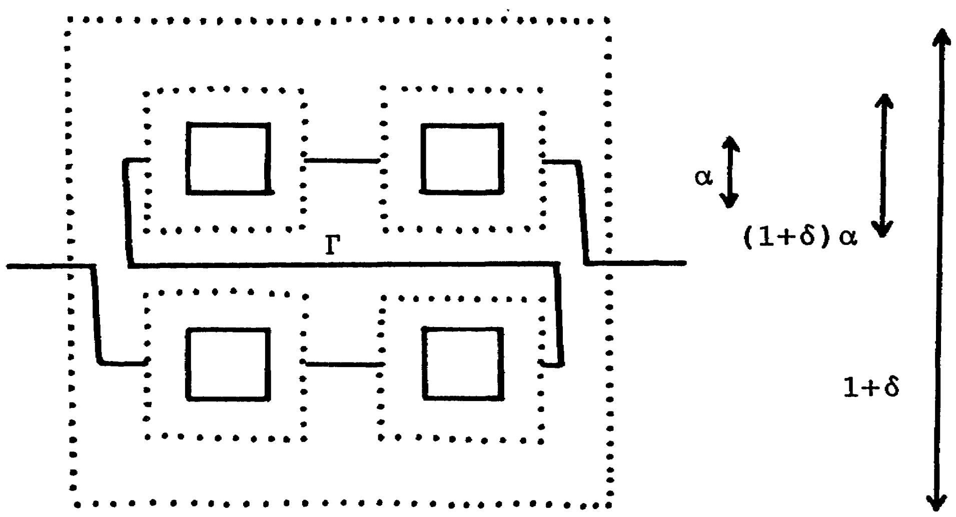

Figure 1 A building block

which varies from 1 to 2 as α goes from $\frac{1}{4}$ to $\frac{1}{2}$. Moreover, it is easy to see that A_Γ is nontrivial, since if μ is the d_α-dimensional Hausdorff measure then

$$F(z) = \frac{1}{z} * \mu = \int \frac{d\mu(w)}{z-w}$$

defines a nonconstant element of A_Γ (see [4] or [6]).

Furthermore, one can easily verify that for any $z \notin \Gamma$, $\omega(z, \Gamma \backslash E, \Gamma^c) > \epsilon > 0$ uniformly, so that $\omega(z, E, \Gamma^c) = 0$. Since Γ is smooth elsewhere, the two harmonic measures on Γ are mutually absolutely continuous and so the corresponding

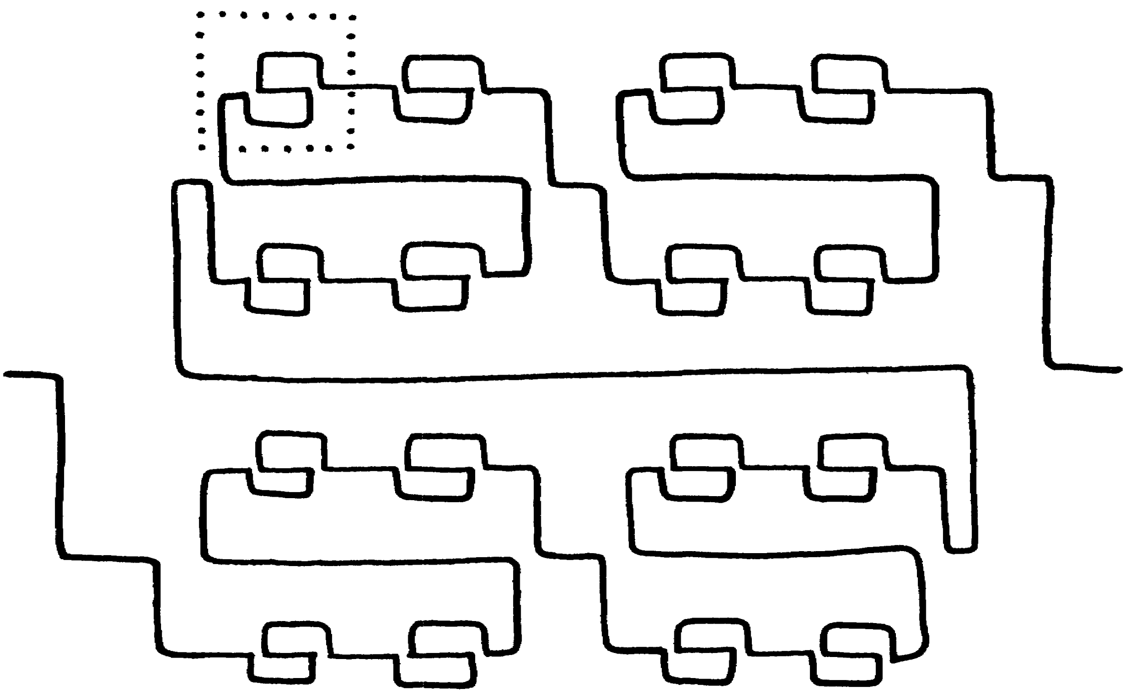

Figure 2 A curve with dim > 1

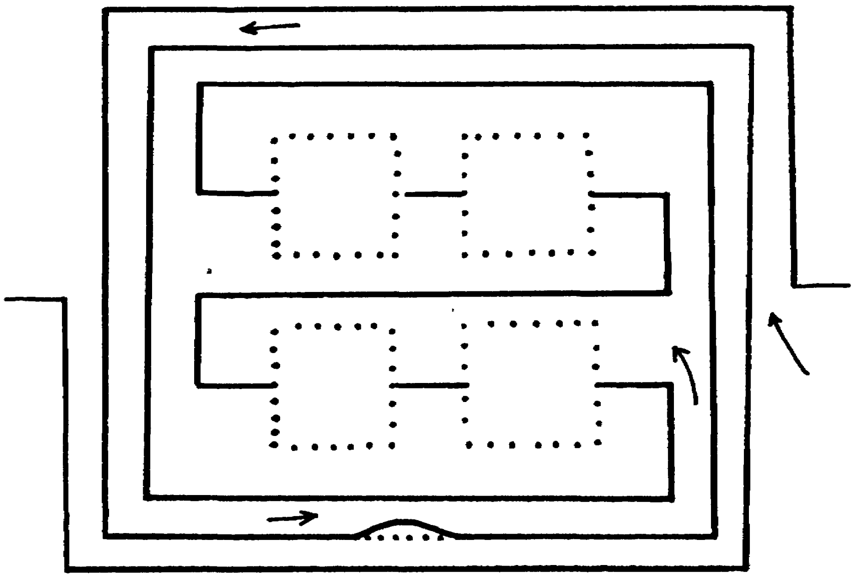

Figure 3 Another building block

homeomorphism ψ is absolutely continuous. This is essentially the construction of Garnett and O'Farrell mentioned in Section 1.

Unfortunately, this curve will not satisfy (1.1). For example, if we consider Figure 2 and the portion of the curve in the dotted box, we see it has much larger harmonic measure on one side than on the other, and this gets worse on smaller scales. The problem is that the portions of Γ lying in two different nth-generation boxes can have very different harmonic measures (even from the same side) and this ratio grows with n. We wish to modify the construction of Γ so that the harmonic measure of any two nth-generation boxes is roughly the same, uniformly in n. We begin by replacing Figure 1 with the more complicated "building block" pictured in Figure 3 and iterating to get Γ as before. Assume the corners in Figure 3 are slightly rounded so that Γ is smooth away from E.

We think of the nth-stage square as being connected to a $(n-1)$th-stage square by a long narrow tube (see arrows in Figure 3). If a particular nth-generation box is getting more than its "fair share" of the harmonic measure, we "pinch off" the tube leading to that box (see Figure 3), thus reducing the amount of harmonic measure it gets. The main problem is to show that perturbing the curve to fix the harmonic measure locally does not ruin the estimates elsewhere.

3. Some estimates on harmonic measure. In this section we will state some very simple lemmas which quantify the idea that a local perturbation of Γ does not greatly effect harmonic measure far away.

We will only need to consider certain special domains. Take two disjoint squares Q_1 and Q_2 (possibly of different sizes) with centers z_1 and z_2. Denote one side of Q_i by σ_i, and let C_i (for "collar") denote the middle thirds of the two adjacent sides. Now let Ω_0 be as in Figure 4, a Jordan domain bounded by arcs σ_1, σ_2, Γ_1, and Γ_2, where each Γ_i hits each square exactly along an edge. We obtain another domain Ω

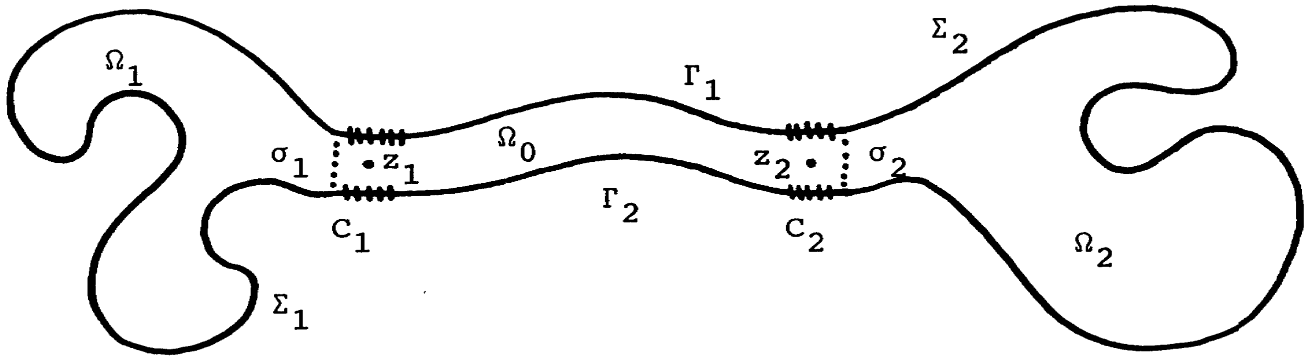

Figure 4 The domain Ω

from Ω_0 by replacing σ_1 and σ_2 by disjoint arcs Σ_1 and Σ_2 which do not hit Ω_0. We let Ω_1 be the subdomain of Ω bounded by σ_1 and Σ_1 and similarly for Ω_2 (see Figure 4).

LEMMA 3.1. *There is a $C>0$ such that*

$$C^{-1} \leq \frac{\omega(z,E,\Omega)}{\omega(z,C_2,\Omega)\omega(z_2,E,\Omega)} \leq C$$

for any $z \in \Omega_1$ and (Borel) $E \subset \Sigma_2$.

This is quite easy to prove using the maximum principle and Harnack's inequality (e.g., see Lemma IV.3.1 of [3]). We can think of Ω_0 as a "tube" connecting Ω_1 and Ω_2. We quantify this by assuming

$$\omega(z_1, C_2, \Omega) \leq \delta, \qquad \omega(z_2, C_1, \Omega) \leq \delta$$

for some very small δ. The next lemma makes precise the notion that perturbing Ω_2 does not greatly change harmonic measure in Ω_1.

LEMMA 3.2. *Suppose $\tilde{\Omega}$ is obtained from Ω by replacing Σ_2 by some other curve $\tilde{\Sigma}_2$. Then there is a $C>0$ such that if $\delta>0$ is sufficiently small, $z \in \Omega_1$, and $E \subset \Sigma_1$, then*

$$(1+C\delta)^{-1} \leq \frac{\omega(z,E,\Omega)}{\omega(z,E,\tilde{\Omega})} \leq 1+C\delta.$$

For a proof see Lemma IV.3.3 of [3]. Next recall that for a C^∞ Jordan domain, harmonic measure is some smooth function times arclength on the boundary. The following fact is based on this idea and is also in [3].

LEMMA 3.3. *Suppose $\Gamma_1 \cup \Sigma_2 \cup \Gamma_2$ is a C^∞ arc. Then there is a C (depending on Γ_1, Σ_2, and Γ_2) such that*

$$C^{-1} \leq \frac{\omega(z,E,\Omega)}{|E|\omega(z,C_2,\Omega)} \leq C$$

for any $z \in \Omega_1$ and $E \subset \Sigma_2$ ($|E|$ denotes the arclength of E).

This still holds if $\Gamma_1 \cup \Sigma_2 \cup \Gamma_2$ is replaced by a compact family of smooth curves, with C depending on the family. Finally, suppose we replace Ω_0 by a family of domains Ω_0^t satisfying the same conditions as Ω_0 and also

$$\omega(z_1, C_2, \Omega_t) \leq t\delta, \qquad \omega(z_2, C_1, \Omega_t) \leq t\delta,$$

where $\Omega_t = \Omega_1 \cup \Omega_0^t \cup \Omega_2$.

LEMMA 3.4. *For each $\epsilon > 0$ there is a $\tau = \tau(\epsilon)$ such that*

$$\omega(z, E, \Omega_\tau) \leq \epsilon \omega(z, E, \Omega)$$

for all $z \in \Omega_1$ and $E \subset \Sigma_2$.

The proof is easy. The point is that τ depends on ϵ but not on Ω.

4. Proof of the theorem. We will now use the elementary observations of the previous section to prove Theorem 1.1. The argument is quite simple, but the notation is a bit awkward and I apologize for any discomfort it may cause. We will index parts of the nth generation of the construction by the sets $S_n = \{1, 2, 3, 4\}^n$; that is, $j \in S_n$ is a sequence of length n with values in $\{1, 2, 3, 4\}$. For $j \in S_n$ and $1 \leq m \leq n$ we let $j^m \in S_m$ be the sequence whose terms are the first m terms of j. We define a metric on S_n by

$$|k - j| = n - \max\{m: k^m = j^m\}.$$

Suppose we have constructed Γ as in Section 2 by iterating Figure 3 inside a unit square centered at the origin, adding E and letting $\Gamma = \mathbf{R}$ away from the square. We fix $z_1 = i$ and $z_2 = -i$, and let Ω^1 and Ω^2 be the corresponding domains. We divide these domains into generations, writing

$$\Omega^i = \Omega_0^i \cup \bigcup_{n=1}^{\infty} \bigcup_{j \in S_n} (T_j^i \cup \Omega_j^i)$$

for $i = 1, 2$, where Ω_0^i, T_j^i, and Ω_j^i are as pictured in Figure 5. We think of $\{\Omega_j^i\}$ with $j \in S_n$ as the 4^n pieces of the nth generation of Ω^i. We index them so that T_j^i is the "tube" connecting Ω_j^i to Ω_k^i, $k = j^{n-1} \in S_{n-1}$.

We let C_j^i denote the "collar" joining T_j^i to Ω_j^i; that is, $C_j^i = \partial T_j^i \cap \partial \Omega_j^i$ (see Figure 5). We also define the "pinched" versions of T_j^i by taking a C^∞ family of do-

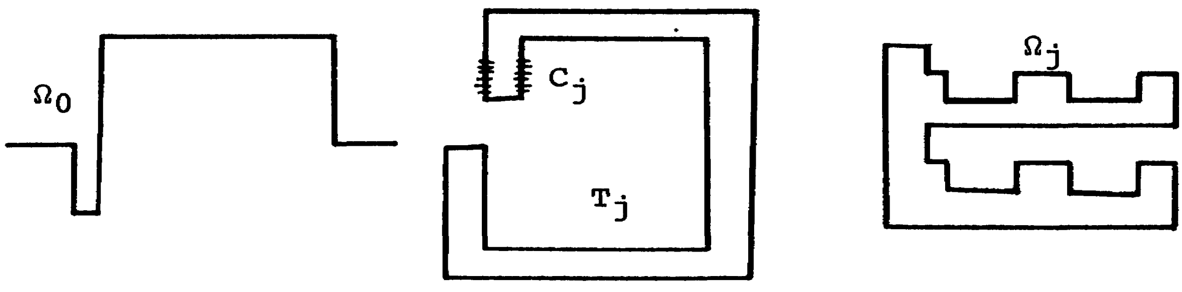

Figure 5 Ω_0, T_j, and Ω_j

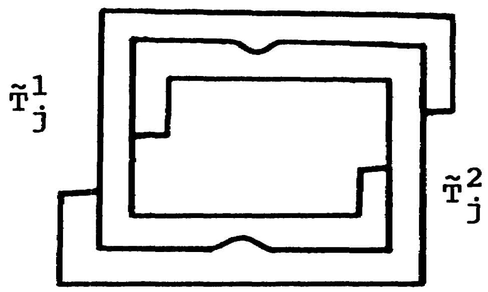

Figure 6 The "bottlenecks" $\tilde{T}_j^i$

mains $\{T_j^i(t)\}$, $0 \le t \le 1$, with $T_j^i(1) = T_j^i$ and so that the bottleneck closes completely as $t \to 0$ (see Figure 6). Note that perturbing T_j^1 only changes the shape of itself and T_j^2, but no other part of the domains. Once we have chosen values of t for both T_j^1 and T_j^2 we denote the two resulting tubes $\tilde{T}_j^1$ and $\tilde{T}_j^2$.

Now suppose we have already constructed

$$\Omega_{N-1}^i = \Omega_0^i \cup \bigcup_{n=1}^{N-1} \bigcup_{j \in S_n} (\tilde{T}_j^i \cup \Omega_j^i).$$

For $j \in S_N$ consider all $2 \cdot 4^N$ values of

$$\omega(z_i, C_j^i, \Omega_{N-1} \cup T_j^i \cup \Omega_j^i)$$

and let $a > 0$ be the minimum value. For each i and j choose t_j^i so that

$$\omega(z_i, C_j^i, \Omega_{N-1} \cup T_j^i(t_j^i) \cup \Omega_j^i) = a.$$

Now set

$$\Omega_N^i = \Omega_{N-1}^i \cup \bigcup_{j \in S_N} (\tilde{T}_j^i \cup \Omega_j^i).$$

Then $\tilde{\Omega}^i = \bigcup_N \Omega_N^i$, $i = 1, 2$, defines two Jordan domains with common boundary Γ which we claim satisfies Theorem 1.1.

First we verify that

$$\begin{aligned} a &= \omega(z_i, C_j^i, \Omega_{N-1} \cup T_j^i(t) \cup \Omega_j^i) \\ &\sim \omega(z_i, C_j^i, \Omega_N). \end{aligned} \tag{4.1}$$

Note that (dropping the i's)

$$\omega(z, C_j, \Omega_{N-1} \cup T_j(t) \cup \Omega_j) \sim \omega(z, C_j, \Omega_{N-1} \cup \tilde{T}_j \cup \Omega_j),$$

since replacing $T_j(t)$ by $\tilde{T}_j$ (the perturbation due to the "other" tube T_j) increases the harmonic measure of C_j by at most a uniform factor. Now let

$$D_j = \bigcup_{n=1}^{N} (\tilde{T}_{j^n} \cup \Omega_{j^n}) \cup \Omega_0,$$

so D_j is the subdomain of Ω_N consisting of all the $\tilde{T}_k$'s and Ω_k's "above" T_j and Ω_j. Also, for $j, k \in S_n$ let

$$\Omega_{kj} = \Omega_{N-1} \cup \tilde{T}_k \cup \Omega_k \cup \tilde{T}_j \cup \Omega_j.$$

Now apply Lemma 3.2 with $\Omega = \Omega_{kj}$, $\Omega_1 = D_j$, $\Omega_2 = D_k$, and $\delta = \eta^{|j-k|}$, where $\eta > 0$ corresponds to the harmonic measure of one end of a "tube" from the other end. It depends only on the dimension of the Cantor set (i.e., our choice of α in Section 2) and is as small as we wish depending on our choice of curves in Figure 3. We obtain:

$$1 \le \frac{\omega(z, C_j, \Omega_{jk})}{\omega(z, C_j, D_j)} \le 1 + C\eta^{|j-k|}.$$

Applying the lemma to all pairs $j, k \in S_n$ grouped together according to the size of $|j-k|$, we have

$$1 \le \frac{\omega(z, C_j, \Omega_N)}{\omega(z, C_j, D_j)} \le \prod_{l=1}^{N} (1 + C\eta^l)^{4^l}.$$

If η is small enough, the right-hand side is bounded independently of N. This proves (4.1).

Thus, for any N, all $2 \cdot 4^N$ values of $\omega(z_i, C_j^i, \Omega_N^i)$ are comparable (with constants independent of N). Since this was also true for Ω_{N-1} in the preceding construction, all $2 \cdot 4^{N-1}$ values of

$$\omega(z, C_j, \Omega_{N-1} \cup T_j \cup \Omega_j)$$

were comparable with some constant independent of N, by Lemma 3.1. Hence, by Lemma 3.4 our choices of $\tilde{T}_j$ did not involve arbitrarily small choices of t but only $t \ge t_0 > 0$ (independent of N). So by Lemma 3.3, applied to the compact family of possible choices of $\tilde{T}_j$, we see that

$$C^{-1} \le \frac{\omega(z, E, \Omega_N)}{|E|\omega(z, C_j, \Omega)} \le C$$

for all $E \subset \partial(\tilde{T}_j \cup \Omega_j)$ and some $C > 0$ independent of N. In particular,

$$\omega(z_1, E, \Omega_N^1) \sim \omega(z_2, E, \Omega_N^2).$$

By another application of Lemma 3.2 we see that, for $E \subset \partial\Omega \cap \partial(\tilde{T}_j \cup \Omega_j)$,

$$\omega(z, E, \Omega_N) \sim \omega(z, E, \Omega).$$

By dividing any subset $\Gamma = \partial\Omega$ into "generations" we see that (1.1) holds in the unit square. Since it is easy to verify that it holds away from the unit square, we have proven Theorem 1.1. □

Added in proof: A result related to this note is given by Charles Moore in his 1986 UCLA thesis. He independently answers the question in [9] (discussed in the introduction) using a construction similar to that in [10].

REFERENCES

1. L. V. Ahlfors, *Lectures on quasiconformal mappings,* Van-Nostrand, Toronto, 1966.
2. ———, *Conformal invariants,* McGraw-Hill, 1983.

3. C. J. Bishop, *Harmonic measures supported on curves,* Thesis, University of Chicago, 1987.
4. L. Carleson, *Selected problems on exceptional sets,* Van-Nostrand, Princeton, N.J., 1967.
5. G. David, *Courbes corde-arc et espaces de Hardy généralisés,* Ann. Inst. Fourier (Grenoble) 32 (1982), 227–240.
6. J. B. Garnett, *Analytic capacity and measure,* Lecture Notes in Math., 297, Springer, Berlin, 1972.
7. J. B. Garnett and A. G. O'Farrell, *Sobolev approximation by a sum of subalgebras on the circle,* Pacific J. Math. 65 (1976), 55–63.
8. P. W. Jones, *Homeomorphisms of the line which preserve BMO,* Ark. Mat. 21 (1983), 229–231.
9. S. W. Semmes, *Is this operator invertible?* Linear and complex analysis problem book, Lecture Notes in Math., 1043, Springer, 1984.
10. ———, *A counterexample in conformal welding concerning chord-arc curves,* Ark. Mat. 24 (1986), 141–158.

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