MR1954867 (2003j:30071) 30F60 30F35 30F50

Bishop, Christopher J. (1-SUNYS)

Non-rectifiable limit sets of dimension one. (English. English summary)

Rev. Mat. Iberoamericana **18** (2002), *no.* 3, 653–684.

R. Bowen proved that any deformation of a cocompact Fuchsian group gives a quasi-Fuchsian Kleinian group whose limit set is either a circle or has Hausdorff dimension > 1. This was extended to all divergence type groups by the author and was shown to be false for all convergence groups (of the first kind) by K. Astala and M. Zinsmeister. They showed that all such groups have a deformation such that the limit set is a non-circular rectifiable curve. Zinsmeister asked if Bowen's property could fail in a different way, namely, are there quasi-Fuchsian groups whose limit sets are not locally rectifiable, but still have dimension 1? The author shows that there are many such groups by constructing quasiconformal deformations of convergence type Fuchsian groups such that the resulting limit set is a Jordan curve of Hausdorff dimension 1, but having tangents almost nowhere. The main tools in this construction are a characterization of tangent points in terms of Peter Jones' β's, a result of Stephen Semmes that gives a Carleson type condition on a Beltrami coefficient μ which implies rectifiability, and a construction of quasiconformal deformations of a surface which shrink a given geodesic and whose dilatations satisfy an exponential decay estimate away from the geodesic.

Vasily A. Chernecky

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Note: This list, extracted from the PDF form of the original paper, may contain data conversion errors, almost all limited to the mathematical expressions.

MR1954866 (2003j:30070) 30F60 30F35

Bishop, Christopher J. (1-SUNYS)

Quasiconformal mappings of Y-pieces. (English. English summary)

Rev. Mat. Iberoamericana **18** (2002), *no.* 3, 627–652.

The main purpose of the paper is to present an explicit way of deforming a Riemann surface collapsing a given closed geodesic γ. In particular, it is done by a quasiconformal deformation with the complex dilatation μ such that $|\mu|$ decays exponentially fast away from γ. The construction is used in a companion paper in the same volume to construct quasi-Fuchsian groups whose limit sets are non-rectifiable curves of dimension 1. In fact, the author gives precise estimates of $|\mu|$ for generalized Y-pieces that are Riemann surfaces bounded by three closed geodesics (or punctures) which are homeomorphic to a sphere minus three discs (or points). Every finite area Riemann surface can be written as a finite union of such pieces. *Alexander Vasil′ev*

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MR1948883 (2003i:30063) 30F10 30C62 30F35 57M50 57M60

Bishop, Christopher J. (1-SUNYS)

Quasiconformal Lipschitz maps, Sullivan's convex hull theorem and Brennan's conjecture. (English. English summary)

Ark. Mat. **40** (2002), *no.* 1, 1–26.

The purpose of this paper is to point out a connection between three-dimensional hyperbolic geometry and the expanding properties of planar conformal maps. It is shown that a theorem of Dennis Sullivan concerning convex sets in hyperbolic 3-space implies the factorization theorem: Let $\Omega \subsetneq \mathbf{R}^2$ be a simply connected domain and let $C(\partial\Omega) \subset \mathbf{H}^3$ be the hyperbolic convex hull of $\partial\Omega$ (this is the hyperbolic convex hull of the set of all hyperbolic geodesies with endpoints in $\partial\Omega$). Suppose Sullivan's theorem holds with quasiconformal constant K and that $f\colon \mathbf{D} \to \Omega$ is conformal. Then $f = g \circ h$, where $h\colon \mathbf{D} \to \mathbf{D}$ is a K-quasiconformal self-map of the unit disk $\mathbf{D}$ and $g\colon \mathbf{D} \to \Omega$ is expanding in the sense that $|g'(z)| > C|f'(0)|$ for all $z \in \mathbf{D}$. One consequence is that if (even a weak version of) Sullivan's theorem could be proved with its conjectured sharp constant $K = 2$, then the Brennan conjecture would follow. In particular, there is always a Lipschitz homeomorphism from any simply connected Ω (with its internal path metric) to the unit disk. Some other related results and questions arising from Sullivan's theorem are also discussed: sharp constants in Sullivan's theorem, harmonic measure, dimension of the convex hull measure, dimension distortion, integral means, deformations of Fuchsian groups, minimal sets, conformal welding, the action on quasisymmetric maps, quasiconformal maps in higher dimensions, uniformly perfect sets.

Vasily A. Chernecky

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MR1945276 (2003m:20065) 20H10 30F35 37F30

Bishop, Christopher J. (1-SUNYS); **Jones, Peter W.** (1-YALE)

Compact deformations of Fuchsian groups. (English. English summary)

Dedicated to the memory of Thomas H. Wolff.

J. Anal. Math. **87** (2002), 5–36.

Let G be a Kleinian group (a discrete group of hyperbolic isometries of the 3-ball $\mathbb{B}^3$) which is non-elementary, and let $M = B^3/G$ be the quotient orbifold. The limit set $\Lambda \subseteq S^2$ of G has a partition as $\Lambda_c \cup \Lambda_e$, where Λ_c is the set of points x in Λ such that there is a geodesic ray in B^3 ending in x with its projection into M returning to a compact set of M infinitely often (points in Λ_c are called conical limit points, or points of approximation in, for instance, [A. F. Beardon and B. Maskit, Acta Math. **132** (1974), 1–12; MR0333164], or radial limit points in, say, [P. Tukia, Invent. Math. **82** (1985), no. 3, 557–578; MR0811551]), so that its complement Λ_e in Λ can be thought of as the "escaping" limit set. If G is a Fuchsian group, a compact deformation Φ of G is a G-compatible quasiconformal homeomorphism of $\mathbb{C}$ that is analytic on the unit disc $\mathbb{D} \subseteq \mathbb{C}$ and with its dilatation compactly supported modulo G, so that $G' = \Phi G \Phi^{-1}$ is a quasi-Fuchsian group. Here it is shown that any compact deformation Φ of a Fuchsian group preserves the Hausdorff dimension of the escaping limit set Λ_e, and that $\Phi(\Lambda_e)$ is contained in a countable union of rectifiable curves.

The proofs require looking at subsets of $\mathbb{D}$ known as Carlson squares, and then a range of analytic techniques is applied, involving the Schwarzian derivative of Φ and the Green's function of the Riemann surface $\mathbb{D}/G$. *Jack O. Button*

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MR1884352 (2002k:30035) 30C62
Bishop, Christopher J. (1-SUNYS)
BiLipschitz approximations of quasiconformal maps. (English. English summary)
Ann. Acad. Sci. Fenn. Math. **27** (2002), *no.* 1, 97–108.

The author proves the theorem: Let f be a K-quasiconformal mapping of the upper half-plane H onto itself and let $\epsilon > 0$. There exist a constant C and a $(K+\epsilon)$-quasiconformal mapping of H onto itself which is C-bi-Lipschitz with respect to the hyperbolic metric on H and which agrees with f on the boundary of H. *H. Renelt*

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MR1860497 (2002j:30021) 30C62 28A80 37F30
Bishop, Christopher J. (1-SUNYS); **Tyson, Jeremy T.** (1-SUNYS)
Conformal dimension of the antenna set. (English. English summary)
Proc. Amer. Math. Soc. **129** (2001), *no.* 12, 3631–3636.

A map f is said to be quasisymmetric if there is a homeomorphism η of $[0,\infty]$ to itself such that

$$|x-y| \le t|x-z| \implies |f(x)-f(y)| \le \eta(t)|f(x)-f(z)|.$$

For a given compact metric space X define its conformal dimension as $C.\dim(X) = \inf_f \dim(f(X))$ where the infimum is taken over all quasisymmetric maps f of X into some metric space and "dim" denotes Hausdorff dimension. Here the authors are interested in the question of whether the infimum must be attained by some quasisymmetric image of X. It is known that for some sets (e.g. the Cantor middle third set in $\mathbb{R}$) the conformal dimension is zero, whereas any quasisymmetric image must have strictly positive dimension. In this paper the authors show that the self-similar set known as the "antenna set" has the property that $\inf_f \dim(f(X)) = 1$ (the infimum is over all quasiconformal mappings of the plane), but that this infimum is not attained by any quasiconformal map; indeed, is not attained for any quasisymmetric map into any metric space. *G. Lakshma Reddy*

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MR1847593 (2003b:30052) 30F35 30F40 37A99 37D99 37F30

Bishop, Christopher J. (1-SUNYS)

Divergence groups have the Bowen property. (English. English summary)

Ann. of Math. (2) **154** (2001), *no.* 1, 205–217.

The author gives a new proof of a theorem of D. Sullivan [Bull. Amer. Math. Soc. (N.S.) **6** (1982), no. 1, 57–73; MR0634434] about convex hulls in hyperbolic 3-space. Using this theorem, he establishes that the limit set of every deformation of a divergence type Fuchsian group is either a circle or has dimension > 1. The result finishes the study of a dichotomy (the Bowen) property of Fuchsian groups begun by R. Bowen [Inst. Hautes Études Sci. Publ. Math. No. 50, (1979), 11–25; MR0556580]. *I. Kra*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

MR1833245 (2002c:30027) 30C65 28A78

Bishop, Christopher J. (1-SUNYS); **Tyson, Jeremy T.** (1-SUNYS)

Locally minimal sets for conformal dimension. (English. English summary)

Ann. Acad. Sci. Fenn. Math. **26** (2001), *no.* 2, 361–373.

The conformal dimension of a metric space X [cf. P. Pansu, Ann. of Math. (2) **129** (1989), no. 1, 1–60; MR0979599] is $\mathrm{C}\dim(X)=\inf_j \dim[fd(X)]$, where the infimum is taken over all quasisymmetric maps of X into some metric space and "dim" denotes Hausdorff dimension. A set X is called minimal for conformal dimension if $\mathrm{C}\dim(X)=\dim(X)$. {The authors do not define the "conformal dimension" of a set, e.g., $X\subset\mathbf{R}^d$, but, as I have deduced from their paper, in this case, $\mathrm{C}\dim(X)$ is given by the above formula, where the infimum is taken over all quasisymmetric maps of the set X into a metric space.} Their main result is Theorem 1: If $1\le\alpha<d$ and $K<\infty$, then there is a compact, totally disconnected set $X\subset\mathbf{R}^d$ of Hausdorff dimension α such that (1) $\dim[f(X)]\ge\alpha$ for all K-quasisymmetric maps of X to a metric space, (2) for any $\epsilon>0$ there is a quasiconformal map g of $\mathbf{R}^d$ to itself such that $\dim[g(X)]<\epsilon$. In particular, $\mathrm{C}\dim(X)=0$.

They obtain also, as corollaries, that for every $\alpha\in[1,d)$, there is a totally disconnected set $X\subset\mathbf{R}^d$ minimal for conformal dimension and a set $X\subset\mathbf{R}^d$ with $\mathrm{C}\dim(X)=\alpha$, but this dimension is not attained for any quasisymmetric image of X. Finally, for sets $X\subset\mathbf{R}^d$, they consider also the quasiconformal dimension $\mathrm{QC}\dim(X)=\inf\dim[g(X)]$, where the infimum is taken over all quasiconformal maps g of $\mathbf{R}^d$ to itself. It is possible to have $\mathrm{QC}\dim(X)>\mathrm{C}\dim(X)$ if X is minimal for quasiconformal selfmaps of $\mathbf{R}^d$, but not minimal for quasisymmetric maps which do not extend to the whole space $\mathbf{R}^d$.

P. Caraman

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MR1828465 (2002b:30023) 30C65 30C62 37F30 37F35

Bishop, Christopher J. (1-SUNYS)

Bi-Lipschitz homogeneous curves in $\mathbb{R}^2$ are quasicircles. (English. English summary)

Trans. Amer. Math. Soc. **353** (2001), *no.* 7, 2655–2663.

The aim of the paper under review is to characterize bi-Lipschitz homogeneous curves (BLH curves) in the plane.

The main result is Theorem 1.1: If Γ is a bi-Lipschitz homogeneous closed curve in $\mathbb{R}^2$, then it is a quasicircle. Namely, this theorem says that in $\mathbb{R}^2$, a bi-Lipschitz homogeneous closed curve must satisfy the bounded turning condition.

Ghamsari and Herron showed that if Γ is a rectifiable closed curve in $\mathbb{R}^2$, then it is a quasicircle. And Herron and Mayer proved it when Γ was homogeneous with respect to a group of uniformly bi-Lipschitz mappings. They gave an example of a curve in $\mathbb{R}^3$ which is bi-Lipschitz homogeneous but not bounded turning.

By combining Theorem 1.1 with these results of Herron and Mayer, the author deduces the characterizations of BLH curves in the plane, as Corollary 1.2. *Kiyoko Nishizawa*

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MR1724076 (2000i:30044) 30C65

Bishop, Christopher J. (1-SUNYS)

Quasiconformal mappings which increase dimension. (English. English summary)

Ann. Acad. Sci. Fenn. Math. **24** (1999), *no.* 2, 397–407.

The author gives a proof of the following general theorem which includes some particular previous results: (1.1) For any compact set E in $\mathbf{R}^d$ with $\dim(E) > 0$ and any $0 < \gamma < d$ there is a quasisymmetric map $h\colon \mathbf{R}^d \to \mathbf{R}^d$ such that $\dim(h(E)) > \gamma$. The proof is first reduced to a class of "standard Cantor sets" which are proved to lie on quasiarcs, hence a new reduction to the case $d = 1$ is given which is solved by a modification of the measure; then (1.1) is proved for $d \geq 2$ by a quasiconformal extension. *J. Ferrand*

MR1711562 (2000i:47090) 47G10 30C20 30E20 42A50 45E05 47A53 47L80

Bishop, C. J. (1-SUNYS); **Böttcher, A.** (D-TUCHM);
Karlovich, Yu. I. (D-TUCHM); **Spitkovsky, I.** (1-CWM)

Local spectra and index of singular integral operators with piecewise continuous coefficients on composed curves. (English. English summary)

Math. Nachr. **206** (1999), 5–83.

The authors establish a symbol calculus for deciding whether singular integral operators with piecewise continuous coefficients are Fredholm on the Lebesgue space $L^p(\Gamma, w)$ where $1 < p < \infty$, Γ is a composed Carleson curve, and w is a Muckenhoupt weight in the class $A_0(\Gamma)$. They also provide index formulas for the operators in the closed algebra of singular integral operators with piecewise continuous matrix-valued coefficients. The main theorem is based upon three pillars: identification of the local spectrum of the Cauchy singular integral operator at the endpoints of simple Carleson arcs, an appropriate "N-projections theorem", and results of geometric function theory pertaining to the problem of extending Carleson curves and Muckenhoupt weights. *V. S. Pilidi*

MR1707864 62H25

Tipping, Michael E. (4-MSFT); **Bishop, Christopher M.** (4-MSFT)

Probabilistic principal component analysis. (English. English summary)

J. R. Stat. Soc. Ser. B Stat. Methodol. **61** (1999), *no.* 3, 611–622.

MR1610908 (99j:30023) 30C65

Bishop, Christopher J. (1-SUNYS)

A quasisymmetric surface with no rectifiable curves. (English. English summary)

Proc. Amer. Math. Soc. **127** (1999), *no.* 7, 2035–2040.

The author provides an interesting and clever construction, based on repeated stretchings and foldings, which proves the following theorem and may have further uses later: there is a quasiconformal mapping f of $\mathbf{R}^3$ onto itself such that for a suitable constant $C>1$ and a suitable positive increasing function $\phi(t)$ with $\lim_{t\to 0}+\phi(t)/t=+\infty$, we have

$$1/C\le |f(z)-f(w)|/\phi(|z-w|)\le C$$

for all distinct z, w in the plane $\mathbf{R}^2\times\{0\}$.

This then implies that $f(\mathbf{R}^2\times\{0\})$ contains no rectifiable curves, which gives a negative answer to a question of S. Rohde. More generally, if $E\subset\mathbf{R}^2\times\{0\}$ and $\mathcal{H}^\alpha(f(E))<+\infty$, then $\mathcal{H}^\alpha(E)=0$, where $\mathcal{H}^\alpha(E)$ denotes the α-dimensional Hausdorff measure of E. The author notes that this f also answers some other, more technical questions which we omit in this review. *A. Hinkkanen*

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Note: This list, extracted from the PDF form of the original paper, may contain data conversion errors, almost all limited to the mathematical expressions.

MR1484767 (98k:22043) 22E40 30F40

Bishop, Christopher J. (1-SUNYS); **Jones, Peter W.** (1-YALE)

Hausdorff dimension and Kleinian groups.

Acta Math. **179** (1997), *no.* 1, 1–39.

This is a very remarkable paper which represents an important contribution to the geometry and metrical theory of Kleinian groups, an area with a long tradition as well as a lot of current activity.

The main results of the paper are reflected in two theorems. Let G denote a non-elementary Kleinian group acting on the closure of hyperbolic 3-space $\mathbf{D}^3\cup S^2$. Let $L(G)\subset S^2$ and $\delta(G)\in(0,2]$ denote the corresponding limit set and exponent of convergence, respectively. Recall that G is called analytically finite if $(S^2\backslash L(G))/G$ is a finite union of compact Riemann surfaces with at most finitely many punctures and branch points. Also, let $L_r(G)$ denote the radial limit set, where $\xi\in L_r(G)$ if and only if the geodesic ray from the origin in $\mathbf{D}^3$ to ξ projects onto $M=\mathbf{D}^3/G$ to give a geodesic ray which returns infinitely often to some compact subset of M. Recall that G is called geometrically finite if every limit point is either a radial limit point or else (in the case where G has parabolic elements) one of the countably many parabolic fixed points. Finally, let $\dim_H$ denote Hausdorff dimension. Theorem 1. For any non-elementary Kleinian group G we have that $\dim_H(L_r(G))=\delta(G)$. Theorem 2. If G is analytically finite and not geometrically finite, then $\dim_H(L(G))=2$.

These two theorems of the paper give rise to rather interesting and deep further results. For instance, for analytically finite G, we have that: geometrical finiteness and $\dim_H(L(G))<2$ are equivalent; $\delta(G)<\dim_H(L(G))$ implies $\operatorname{area}(L(G))>0$; the box-counting dimension and Hausdorff dimension of $L(G)$ coincide. For finitely generated G, further consequences are: $L(G)$ is either totally disconnected or a circle, or $\dim_H(L(G))>1$; if G_n converges algebraically to G, then $\dim_H(L(G))\le\liminf_n\dim_H(L(G_n))$.

The proof of Theorem 1 is purely elementary (but nevertheless rather beautiful). The key observation is to interpret in a straightforward geometric manner the divergence of the Dirichlet series for values smaller than the exponent of convergence. The proof of Theorem 2 combines the expertise of the two authors in the theory of harmonic measure with an estimate of E. B. Davies on the heat kernel on M and a theorem of Elstrodt and Patterson concerning the bottom of the spectrum of the Laplacian on M.

The paper significantly generalises previously known results by various authors. Since it appears to be a nearly impossible task to list the complete history of those related works, the reviewer refers for a first account to the representative list of references given in the paper. *Bernd O. Stratmann*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

MR1478778 (99f:30066) 30F40 28A78

Bishop, Christopher J. (1-SUNYS); **Jones, Peter W.** (1-YALE)

Wiggly sets and limit sets. (English. English summary)

Ark. Mat. **35** (1997), *no.* 2, 201–224.

Let E be a subset of $\mathbf{R}^2$. For Q a dyadic square in $\mathbf{R}^2$, let αQ be the square concentric to Q and homothetic of ratio α, let $\mathcal{L}$ be the set of lines intersecting Q and

$$\beta(Q) = (\operatorname{diam} Q)^{-1} \inf_{L\in\mathcal{L}} \sup_{z\in E\cap 3Q} d(z, L),$$

which measures the minimum oscillation of the set E in all directions in Q. Say E is uniformly wiggly if there exists $\beta_0 > 0$ such that $\beta(Q) \geq \beta_0$ for every Q such that $\frac{1}{3}Q$ meets E and $\operatorname{diam} Q \leq \operatorname{diam} E$.

The authors prove that if E is uniformly wiggly, then the Hausdorff dimension of E is at least $1 + c\beta^2$, where c is a universal constant. Improving work of Bowen, of Sullivan, and of Canary and Taylor, they give several applications to limit sets Λ of Kleinian groups G (i.e. discrete subgroups of $\mathrm{PSL}_2(\mathbf{C})$, assumed to contain no abelian subgroup of finite index), where $\Lambda \subset \mathrm{P}_1(\mathbf{C})$ is the accumulation set of any orbit, by proving that many of them are uniformly wiggly. Let $\Omega = \mathrm{P}_1(\mathbf{C}) \smallsetminus \Lambda$ be the domain of discontinuity of G. Say G is analytically finite if Ω/G is a compact Riemann surface with finitely many points removed. Let $\delta(G) = \inf\{s \geq 0\colon \sum_{g\in G} e^{-sd(x_0, gx_0)} < \infty\}$, where d is the hyperbolic distance in the ball with boundary $\mathrm{P}_1(\mathbf{C})$. The authors prove that if G is analytically finite, and Ω_0 is a simply connected invariant component of Ω, then $\dim(\partial\Omega_0) = 1$ if and only if $\delta(G) = 1$ if and only if $\partial\Omega_0$ is a circle. So the limit set of an analytically finite Kleinian group is either totally disconnected, a circle or has Hausdorff dimension > 1, and it satisfies $\delta(G) > 1$ if nongeometrically finite. *Frédéric Paulin*

MR1476980 (98j:30051) 30F40 28A78 30C35 31A15

Bishop, Christopher J. (1-SUNYS); **Jones, Peter W.** (1-YALE)

★**The law of the iterated logarithm for Kleinian groups. (English. English summary)**

Lipa's legacy (*New York*, 1995), 17–50, *Contemp. Math.*, 211, *Amer. Math. Soc.*, *Providence*, *RI*, 1997.

Suppose G is a Kleinian group acting on the hyperbolic three-ball $\mathbf{B}$ and let Λ and Λ_c denote the limit set and conical limit set respectively. The purpose of this paper is to prove the following result: Suppose G is an analytically finite but geometrically infinite Kleinian group such that the injectivity radius of $M = \mathbf{B}/G$ is bounded away from zero. Then $\Lambda \smallsetminus \Lambda_c$ has positive Hausdorff measure with respect to the function $\varphi(t) = t^2\sqrt{\log\frac{1}{t}\log\log\log\frac{1}{t}}$. If, in addition, G is topologically tame and $\Lambda \neq S^2$, then

Λ has finite Hausdorff φ-measure and Λ_c has zero φ-measure. The corollaries on the size of the conical limit set, the existence and ergodicity of conformal densities, and the differentiability of quasiconformal conjugacies are also obtained. *Vasily A. Chernecky*

MR1428819 (97k:60105) 60G17 60J65

Bishop, Christopher J. (1-SUNYS); **Jones, Peter W.** (1-YALE);
Pemantle, Robin (1-WI); **Peres, Yuval** (1-CA-S)

The dimension of the Brownian frontier is greater than 1. (English. English summary)

J. Funct. Anal. **143** (1997), *no.* 2, 309–336.

Let X_t be a two-dimensional Brownian motion and let A be the boundary of the unbounded connected component of the complement of $X[0,1]$. It is proved that the Hausdorff dimension of A is strictly greater than 1. It had been conjectured that the dimension is equal to 4/3. *Krzysztof Burdzy*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

MR1423024 (97k:30055) 30F40 20H10 43A55

Bishop, Christopher J. (1-SUNYS)

Geometric exponents and Kleinian groups. (English. English summary)

Invent. Math. **127** (1997), *no.* 1, 33–50.

Let G be an analytically finite Kleinian group (i.e. the quotient of the domain of discontinuity $\Omega \subset S^2$ by G is a finite union of surfaces of finite type). Let $\{\Omega_j\}$ be an enumeration of the components of Ω, and define $\operatorname{diam}(\Omega_j)$ and $\operatorname{inrad}(\Omega_j)$ as the metric diameter and radius of largest disk inside Ω_j, respectively (using the spherical metric on S^2). The first theorem of this paper states that the series $\sum_j \operatorname{diam}(\Omega_j)^2$ converges. The idea of the proof is based on estimating the diameter of all sets in the G orbit of a particular component Ω_0 (by hypothesis, there are only finitely many G-equivalence classes in $\{\Omega_j\}$). For $\Omega_j = g_j\Omega_0$, the author proves a lemma of Maskit showing that $\operatorname{diam}(\Omega_j)$ is comparable to r_j^2/η_j where r_j is the radius of the isometric circle of g_j and η_j is the distance from $g_j^{-1}(\infty)$ to Ω_0. Elementary estimates finish the proof.

The second result is that the Poincaré exponent of convergence for G equals the critical exponents for both $\sum_j \operatorname{diam}(\Omega_j)^s$ and $\sum_j \operatorname{inrad}(\Omega_j)^s$. If, in addition, the limit set Λ of G is a null set (with respect to Lebesgue measure), then these exponents agree with the Hausdorff dimension of Λ. Finally, the author defines a Whitney decomposition for planar regions and shows the corresponding critical exponent for any Whitney decomposition of Λ equals the Poincaré exponent when $\Lambda \neq S^2$. The proofs of these statements require some nontrivial lemmas from harmonic analysis. *Eric M. Freden*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

MR1404092 (97f:30064) 30F40 57M50 57M60

Bishop, Christopher J. (1-SUNYS)

On a theorem of Beardon and Maskit. (English. English summary)

Ann. Acad. Sci. Fenn. Math. **21** (1996), *no.* 2, 383–388.

A Kleinian group G is a discrete (as a group of matrices) subgroup of the group $\mathrm{PSL}(2,\mathbf{C})$. It acts conformally on points in $\mathbf{S}^2$ via homeomorphisms and isometrically on points in the unit sphere $B^3=\{x\in\mathbf{R}^3\colon\ |x|<1\}$ equipped with the Poincaré metric $ds^2 = dx^2/(1-|x|^2)^2$.

We say that G is geometrically finite if its action on points in B^3 has a finite-sided fundamental polyhedron. Since G is discrete, it follows that the orbit of any element under G can only accumulate in $\mathbf{S}^2$; therefore we define the limit set $\Lambda(G)$ of G as the closure of the set $G(0)=\{x\in B^3\colon\ x=g(0),\ g\in G\}$. For $z\in\mathbf{S}^2$ let $\Gamma(x,r)$ be the convex hull (in Euclidean geometry) of $\{z\}$ and $\{x\in B^3\colon\ |x|<r\}$. The point z as above is a conical limit point if there exists an $r<1$ such that z is the accumulation point of the set $\Gamma(x,r)\cap G(0)$.

In this paper the author proves the following theorem: Let G be a discrete subgroup of $\mathrm{PSL}(2,\mathbf{C})$. Then G is geometrically finite if and only if every point in $\Lambda(G)$ is either a conical limit point or a fixed point of a parabolic element of G. (Recall that $g\in G$ is parabolic if it has only one fixed point in $\mathbf{S}^2$.)

The technical step in the proof of the above theorem is based on a result due to F. Bonahon about closed geodesics in hyperbolic 3-manifolds [see Ann. of Math. (2) **124** (1986), no. 1, 71–158; MR0847953].

This result refines a similar theorem of A. F. Beardon and B. Maskit [see Acta Math. **132** (1974), 1–12; MR0333164]. *André C. Rocha*

MR1398371 (97j:46053) 46J15

Bishop, Christopher J. (1-SUNYS)

A distance formula for algebras on the disk. (English. English summary)

Pacific J. Math. **174** (1996), *no.* 1, 1–27.

Let $H^\infty(D)[\mathcal{F}]$ be the closed algebra on the disk generated by $H^\infty(D)$ and a countable collection $\mathcal{F}$ of bounded harmonic functions. For $g\in L^\infty(D)$, the distance from g to $H^\infty(D)[\mathcal{F}]$ is denoted by $\mathrm{dist}(g,H^\infty(D)[\mathcal{F}])$. A recipe is given for calculating $\mathrm{dist}(g,H^\infty(D)[\mathcal{F}])$. Let $\mathcal{F}=\{f_n\}_{n\in I}$, where $I=\mathbf{N}$ or $I=\{1,\cdots,N\}$. For each $f_n=u_n+iv_n$, there corresponds a holomorphic function $h_n=((u_n+v_n^*)-i(v_n-u_n^*))/2$ (possibly unbounded), where u_n^* is the harmonic conjugate of u_n with $u_n^*(0)=0$. Put $\mathcal{H}=\{h_n\}_{n\in I}$. For $\delta>0$ and $\alpha=\{a_n\}_{n\in I}\in\mathbf{C}^I$, define $\Omega_{\mathcal{H}}(\alpha,\delta,m)=\bigcap_{h_n\in\mathcal{H},n\le m}\Omega_{h_n}(a_n,\delta)$, where $\Omega_{h_n}(a_n,\delta)=\{z\in D;\ |h_n(z)-a_n|<\delta\}$. It is proved that

$$\mathrm{dist}(g,H^\infty(D)[\mathcal{F}])=\inf_{\delta>0,\ m\in I}\ \sup_{\alpha\in\mathbf{C}^I}\mathrm{dist}(g,H^\infty(\Omega_{\mathcal{H}},(\alpha,\delta,m))).$$

The author also discusses the Bourgain closure of $H^\infty(D)[\mathcal{F}]$ and gives a new proof of a result of S. Axler and A. L. Shields [Indiana Univ. Math. J. **36** (1987), no. 3, 631–638; MR0905614]. *Keiji Izuchi*

MR1398152 (98h:30061) 30F40 20H10

Bishop, Christopher J. (1-SUNYS)

Minkowski dimension and the Poincaré exponent.

Michigan Math. J. **43** (1996), *no.* 2, 231–246.

In this paper the author obtains the connection of a Poincaré exponent of convergence with the geometrical property of the limit set $\Lambda(G)$. Let $\Omega(G) = S^2\Lambda(G)$ be the ordinary set of a Kleinian group G. If $z_0 \in \Omega(G)$ then the critical exponent (or Poincaré exponent) is defined as

$$\delta(G) = \inf\left\{s\colon \sum_{g\in G} \operatorname{dist}(g(z_0), \Lambda(G))^s < \infty\right\},$$

where distance is in the spherical metric. It is known that this exponent does not depend on the choice of z_0. The upper Minkowski dimension of a compact $K \subset \mathbf{R}^2$ is defined as

$$\overline{\operatorname{Mdim}}(K) = \limsup_{\epsilon\to 0} \log N(K,\epsilon)/\log(1/\epsilon),$$

where $N(K,\epsilon)$ is the minimal number of ϵ-balls needed to cover K. Theorem 1.1: Suppose G is an analytically finite, non-elementary Kleinian group. If $\operatorname{area}(\Lambda(G)) = 0$ then $\delta(G) = \overline{\operatorname{Mdim}}(\Lambda(G))$. {For related results see C. J. Bishop [Invent. Math. **127** (1997), no. 1, 33–50; MR1423024].} *Gregory M. Lyan*

MR1376540 (97c:28015) 28A80 28A75

Bishop, Christopher J. (1-SUNYS); **Peres, Yuval** (1-CA-S)

Packing dimension and Cartesian products. (English. English summary)

Trans. Amer. Math. Soc. **348** (1996), *no.* 11, 4433–4445.

Summary: "We show that for any analytic set A in $\mathbf{R}^d$, its packing dimension $\dim_P(A)$ can be represented as $\sup_B\{\dim_H(A\times B) - \dim_H(B)\}$, where the supremum is over all compact sets B in $\mathbf{R}^d$, and $\dim_H$ denotes Hausdorff dimension. (The lower bound on packing dimension was proved by Tricot in 1982.) Moreover, the supremum above is attained, at least if $\dim_P(A) < d$. In contrast, we show that the dual quantity $\inf_B\{\dim_P(A\times B) - \dim_P(B)\}$ is at least the 'lower packing dimension' of A, but can be strictly greater. (The lower packing dimension is greater than or equal to the Hausdorff dimension.)" *Claude Tricot*

MR1328340 (96m:46052) 46E35 41A63

Bishop, Christopher J. (1-SUNYS)

A counterexample concerning smooth approximation. (English. English summary)

Proc. Amer. Math. Soc. **124** (1996), *no.* 10, 3131–3134.

Summary: "We answer a question of W. S. Smith, A. Stanoyevitch and D. A. Stegenga [J. London Math. Soc. (2) **49** (1994), no. 2, 309–330; MR1260115] in the negative by constructing a simply connected planar domain Ω with no two-sided boundary points and for which every point on Ω^c is an m_2-limit point of Ω^c and such that $C^\infty(\overline{\Omega})$ is not dense in the Sobolev space $W^{k,p}(\Omega)$."

MR1326997 (96k:46091) 46J15 30D55 46J20

Bishop, Christopher J. (1-SUNYS)

Some characterizations of $C(\mathcal{M})$. (English. English summary)

Proc. Amer. Math. Soc. **124** (1996), *no.* 9, 2695–2701.

Let H^∞ denote the algebra of bounded analytic functions on the open unit disc $\mathbf{D}$, let $M(H^\infty)$ denote its maximal ideal space, and let X denote the Shilov boundary of H^∞. In this article, the author gives necessary and sufficient conditions for a bounded continuous function g defined on $\mathbf{D}$ to extend continuously to $M(H^\infty)$. The author obtains several corollaries, one of which is related to results of Axler and Shields, and of Ivanov.

S. Axler and A. L. Shields [Pacific J. Math. **145** (1990), no. 1, 1–15; MR1066396] showed that a continuous function on $\mathbf{D}\cup X$ with values in $\mathbf{C}\cup\{\infty\}$ has a nontangential limit at almost every point of the unit circle $\mathbf{T}$. In this paper, the author shows that a bounded continuous function f on $\mathbf{D}$ has a continuous extension to X if and only if f has nontangential limits almost everywhere on $\mathbf{T}$. O. V. Ivanov has several related results in his paper [Dokl. Akad. Nauk Ukrain. SSR **1991**, no. 7, 5–8, 177; MR1157940], as well as several other related papers. *Pamela Gorkin*

MR1385195 (97m:68172) 68T10 92B20

Bishop, Christopher M. (4-ASTN-AM)

★**Neural networks for pattern recognition. (English. English summary)**

With a foreword by Geoffrey Hinton.

The Clarendon Press, *Oxford University Press*, *New York*, 1995. xviii+482 *pp.* $98.00; $45.00 paperbound. *ISBN* 0-19-853849-9; 0-19-853864-2

There has been an acute need for authoritative textbooks in neural networks that explain the main ideas clearly and consistently using the basic tools of linear algebra, calculus, and simple probability theory. There have been many attempts to provide such a text, but until now, none has succeeded. This is a serious attempt at providing such an ideal textbook.

By concentrating on pattern recognition aspects of neural networks, the author is able to treat many important topics in much greater depth. The most important contribution of the book is the solid statistical pattern recognition approach, a sign of increasing maturity of the field.

The first chapter provides an introduction to the principal concepts of pattern recognition. Chapter 2 deals with the problem of modeling the probability distribution of a set of data, and reviews conventional parametric and non-parametric methods, as well as discussing more recent techniques based on mixture distributions. Single-layer neural networks are introduced in Chapter 3. Chapter 4 provides a comprehensive treatment of the multi-layer perceptron. Radial basis function networks are discussed in Chapter 5. Several error functions can be used for training neural networks, and these are examined in Chapter 6. Chapter 7 reviews many of the most important algorithms for optimizing the values of the parameters in a network. Chapter 8 covers a range of issues associated with data-preprocessing, and describes practical techniques related to dimensionality reduction and the use of prior knowledge. Chapter 9 provides a number of insights into the problem of generalization, and describes methods for addressing the central issue of model order selection. The final chapter discusses the treatment of neural networks from a Bayesian perspective.

This book is aimed at researchers in neural computing as well as those wishing to apply neural networks to practical applications. It is also intended to be used as the primary text for a graduate-level course on neural networks. Exercises are provided at the end of each chapter.

The book is largely self-contained as far as the subject of neural networks is concerned, although some prior exposure to the subject may be helpful to the reader. It is assumed that the reader has a good working knowledge of vector and matrix algebra, as well as integral and differential calculus for several variables. *Răzvan Andonie*

MR1303516 (96d:30028) 30C85 30D40
Bishop, Christopher J. (1-SUNYS)
How geodesics approach the boundary in a simply connected domain.
J. Anal. Math. **64** (1994), 291–325.
This paper is an excellent contribution to the theory of "fine structure" of harmonic measure and other conformal invariants which has been developed in recent work by N. G. Makarov, P. Jones, the author, and others. Let ω be a bounded Jordan domain in the plane. Consider the set of hyperbolic geodesics of ω emanating from some fixed interior point z_0. Each geodesic ends at some point of $\partial\omega$, and the correspondence between the geodesics and their terminal boundary points is bijective. The author addresses questions of the following type: If a geodesic is close to its terminal point at some time t, how likely is the geodesic to stray far away from it later?

It turns out that this and related questions can be given satisfyingly precise answers. Let f be a Riemann map of the unit disk onto ω with $f(0)=z_0$. Then the geodesics from z_0 are indexed by points $e^{i\theta}$ on the unit circle, and are parameterized by the formula $\gamma(t,\theta)=f(e^{i\theta})$, $0\le t\le 1$. Set

$$e(t,\theta)=|\gamma(t,\theta)-\gamma(1,\theta)|,\quad E(t,\theta)=\sup_{t\le s\le 1} e(s,\theta).$$

Let ϕ be a positive decreasing function on $(0,1),$ subject to some mild restrictions. Define

$$I(\phi)=\int_0^1 \phi^{-9/2}(t)\frac{dt}{t},\quad L(\theta)=\limsup_{t\to 1}\frac{E(t,\theta)}{e(t,\theta)\phi(e(t,\theta))}.$$

Theorem. If $I(\phi)<\infty,$ then $L(\theta)=0$ for a.e. θ. But if $I(\phi)=\infty$ there exists an ω for which $L(\theta)=\infty$ for a.e. θ.

As one might imagine, the proofs are complicated. They rely on ingenious decompositions dictated by the geometry of geodesics and numerous estimates of harmonic measure. The domains illustrating sharpness have fractal boundaries of just the right type. The magic exponent $2/9$ arises as follows: if E_i, $i=1,2,3,$ are three disjoint subsets of a circle, with respective longest arcs of angular measure ${\theta_i}^*,$ then

$$\sum_{i=1}^{3}\frac{1}{{\theta_i}^*}\le\frac{9}{2\pi}.$$

There are also results in which the distance $e(t,\theta)$ from a point on the geodesic to its endpoint is replaced by the shortest distance $b(t,\theta)$ from that point to $\partial\omega$.

Albert Baernstein, II

MR1274085 (95j:30008) 30C35 30C62 30C85

Bishop, Christopher J. (1-SUNYS)

Some homeomorphisms of the sphere conformal off a curve. (English. English summary)

Ann. Acad. Sci. Fenn. Ser. A I Math. **19** (1994), *no.* 2, 323–338.

A closed set E in the plane is said to be removable for conformal homeomorphisms if any homeomorphism of the Riemann sphere which is conformal off E must be conformal on the whole sphere (i.e., a Möbius transformation). On the other hand, a closed curve Γ is said to be flexible (for conformal homeomorphisms) if for any other closed curve Γ' and any $\epsilon > 0$ there is a homeomorphism Φ of the sphere which is conformal off Γ and such that $\Phi(\Gamma)$ is in the ϵ-neighborhood of Γ' with respect to the Hausdorff metric. The main result of this paper is to show that there exists a closed Jordan curve which is flexible. It is proved by an iterative construction based on the following idea. If the curve Γ was actually a thin strip of some positive width, there would be no problem constructing Φ by letting it be any conformal map on either side of the strip and interpolating these maps continuously across the strip. If the strip is "filled up" by a highly oscillating curve, then conformal maps on either side of the curve can still be continuously interpolated in some sense. The construction shows that, for any Hausdorff measure function h with $h(t) = o(t)$ as $t \to 0$, one can make the flexible curve to have zero Hausdorff measure with respect to h. This extends a result of R. P. Kaufman on non-removable sets which are close to σ-finite length [Ann. Acad. Sci. Fenn. Ser. A I Math. **9** (1984), 27–31; MR0752389].

This paper also gives a nice survey of some known results and open problems about removable sets for several different classes of functions such as analytic functions with finite energy and quasiconformal maps. *Shan Shuang Yang*

MR1269200 (95f:30034) 30C85 30C62 31A15

Bishop, Christopher J. (1-SUNYS); **Jones, Peter W.** (1-YALE)

Harmonic measure, L^2 estimates and the Schwarzian derivative. (English. English summary)

J. Anal. Math. **62** (1994), 77–113.

This paper is part of the recent program, pursued by a number of people, where classical analytic "L^2-techniques" are put to work to register the geometry of sets, rather than the behavior of functions. This point of view is lucidly explained in the introduction. For instance, if Γ is a Jordan curve in the plane, then, for $x \in \Gamma$ and $t > 0$, write $\epsilon(x,t) = \max\{|\pi - \theta_1(t)|, |\pi - \theta_2(t)|\}$, where $\theta_i(t)$ is the angle measure of the longest arc in the circle, centered at x with radius t, also contained in Ω_i, one of the two complementary domains of Γ. Carleson has conjectured that the Dini type condition $\int_0 \epsilon(x,t)^2\,dt/t < \infty$ characterizes, up to a set of linear measure zero, those points x on Γ that admit a tangent.

The first result of this paper is a proof of a statement that is slightly weaker than Carleson's conjecture, which to my knowledge is still open. In this version $\epsilon(x,t)$ is replaced with $\beta(x,t)$, which measures the "smoothness" of Γ at x at the scale t as in the second author's work on the traveling salesman problem and rectifiable sets [Invent. Math. **102** (1990), no. 1, 1–15; MR1069238].

In the second part of the paper the authors sharpen and quantify their earlier work [Ann. of Math. (2) **132** (1990), no. 3, 511–547; MR1078268] on harmonic measure and rectifiable curves. They present an integrability condition on the derivatives of the Riemann mapping (this involves the Schwarzian of the mapping, among other things) which assures that the boundary of the image domain is reasonably well approximable by rectifiable curves. From this result one can derive Lavrent′ev-type estimates for the harmonic measure in domains whose boundary is not necessarily (locally) rectifiable. Many of the techniques used here are elaborations on the ideas of the difficult aforementioned 1990 paper by the authors.

The paper contains a wealth of related results connecting the geometry of a simply connected domain to the behavior of the Schwarzian of its Riemann mapping function. Particularly appealing is Theorem 4, which gives characterizations, both geometric and analytic, of the domains whose Riemann mapping function has the property that the logarithm of its derivative is in BMO on the circle. This characterization complements an earlier work of K. Astala and M. Zinsmeister [Math. Ann. **289** (1991), no. 4, 613–625; MR1103039]. One of the geometric characterizations of these BMO-domains immediately gives their bi-Lipschitz invariance, a property which a priori is far from obvious.

Juha Heinonen

MR1240926 (94j:30032) 30D50 30D45
Bishop, Christopher J. (1-SUNYS)
An indestructible Blaschke product in the little Bloch space. (English. English summary)
Publ. Mat. **37** (1993), *no.* 1, 95–109.

In this paper, the author constructs an infinite Blaschke product B in $\mathcal{B}_0$, the little Bloch space, that is indestructible, i.e., such that $(B-a)/(1-\overline{a}B)$ is also a Blaschke product for every point a in the unit disk. In a previous paper [Pacific J. Math. **142** (1990), no. 2, 209–225; MR1042042], the author showed how to produce an infinite Blaschke product in $\mathcal{B}_0$ by constructing its zero set, but the method there seems unsuited for producing an indestructible Blaschke product. Here the author employs a "cut and paste" scheme based on a method of K. Stephenson [Trans. Amer. Math. Soc. **308** (1988), no. 2, 713–720; MR0951624] for constructing inner functions in B_0. One builds a simply connected Riemann surface lying over the unit disk by pasting together copies of the disk along suitably chosen radial slits. One obtains the desired Blaschke product by composing a Riemann map from the disk to the surface with the projection from the surface to the disk. The details are formidable. By a similar construction, the author produces a function in VMOA of sup-norm 1 which, in every interior neighborhood of each point on the unit circle, assumes each value in the disk infinitely often.

D. Sarason

MR1208564 (94g:22023) 22E40 30F35
Bishop, Christopher (1-UCLA); **Steger, Tim** (1-CHI)
Representation-theoretic rigidity in $\mathrm{PSL}(2,\mathbf{R})$**.**
Acta Math. **170** (1993), *no.* 1, 121–149.

Let Γ be an abstract group and G a noncompact connected simple Lie group with trivial center. Let ι_1, and ι_2 be two inclusions of Γ in G such that $\iota_1(\Gamma)$ and $\iota_2(\Gamma)$ are lattices in G, i.e., are discrete subgroups of G of finite covolume. ι_1 and ι_2 are said to be equivalent if there is an automorphism ρ of G such that $\iota_2 = \rho\cdot\iota_1$. According to the strong rigidity theorem of G. D. Mostow, if G is not isomorphic to $\mathrm{PSL}(2,\mathbf{R})$, then ι_1 and ι_2 are equivalent. This is false if $G=\mathrm{PSL}(2,\mathbf{R})$. In the paper under review, the authors prove the following nice theorem: If π_1 and π_2 are irreducible unitary representations of $\mathrm{PSL}(2,\mathbf{R})$ not in discrete series, then $\pi_1\cdot\iota_1$ and $\pi_2\cdot\iota_2$ are equivalent representations of Γ if and only if ι_1 and ι_2 are equivalent and, moreover, π_1 and π_2 are equivalent representations of $\mathrm{PSL}(2,\mathbf{R})$.

The key step in the proof of this theorem is an interesting criterion for the equivalence of ι_1 and ι_2 which the authors prove using some geometric considerations. Recall that $\mathrm{PSL}(2,\mathbf{R})$ acts on the upper half-plane $\mathbf{H}$ by Mobius transformations. Let $d(\cdot,\cdot)$ denote the hyperbolic metric on $\mathbf{H}$ and for $g\in\mathrm{PSL}(2,\mathbf{R})$, let $h(g)=\exp(-d(g(i),i))$. Then the criterion is that ι_1 and ι_2 are equivalent if and only if $\sum_{\gamma\in\Gamma} h^s(\iota_1(\gamma))h^{1-s}(\iota_2(\gamma)) = \infty$, for some s, $0 < s < 1$. *Gopal Prasad*

MR1159563 (93a:60125) 60J65 31A20 60G17

Bishop, Christopher J. (1-SUNYS)

Brownian motion in Denjoy domains.

Ann. Probab. **20** (1992), *no.* 2, 631–651.

A susbset Ω of $\mathbf{R}^2$ is called a Denjoy domain if it is open and its complement E is a subset of $\mathbf{R}$. If Ω is a Denjoy domain then, for almost every $x \in E$ (with respect to harmonic measure) and $\varepsilon > 0$, Brownian motion in Ω, conditioned to exit at x, hits $[x-\varepsilon, x)$ with probability 1 if and only if it hits $(x, x+\varepsilon]$ with probability 1. If the Hausdorff dimension of E is less than 1 then a.s. Brownian motion killed upon exiting Ω makes a closed loop around its exit point. Several other related results are proved. *Krzysztof Burdzy*

MR1155854 (93f:30023) 30C85 30C35 30C62 31A15

Bishop, Christopher J. (1-UCLA)

★**Some questions concerning harmonic measure.**

Partial differential equations with minimal smoothness and applications (*Chicago*, *IL*, 1990), 89–97, *IMA Vol. Math. Appl.*, 42, *Springer*, *New York*, 1992.

This article is a survey covering 12 interesting open questions associated to the behaviour of harmonic measure on the boundary of a planar domain. Most of the problems arise from attempts to extend in natural ways the fundamental theorems of N. G. Makarov [Proc. London Math. Soc. (3) **51** (1985), no. 2, 369–384; MR0794117]. It should prove useful to young workers looking for serious problems to get started with. *T. J. Lyons*

MR1115072 (93a:31011) 31B20 30C85 31A15

Bishop, Christopher J. (1-UCLA)

A characterization of Poissonian domains.

Ark. Mat. **29** (1991), *no.* 1, 1–24.

Given a domain Ω in $\mathbf{R}^n$, the harmonic measures $d\omega_x$, $x \in \Omega$, define a map of L^∞ of the boundary $\partial\Omega$ (in $\mathbf{R}^n \cup \{\infty\}$) into the bounded harmonic functions on Ω. If this map is onto, the domain is said to be Poissonian.

A number of interesting results are proved, of which Theorem 1.1, which characterizes such domains, is the most striking; Ω is Poissonian if and only if, for every pair of disjoint subdomains Ω_1 and Ω_2 with $\partial\Omega_1 \cap \partial\Omega_2 \subset \partial\Omega$, the harmonic measures ω_i for Ω_i are mutually singular. As a consequence, every component of the intersection of two Poissonian domains is Poissonian. Also, if $E \subset \mathbf{R}^n$ is a closed subset of a Lipschitz graph, then $\Omega = \mathbf{R}^n \setminus E$ is Poissonian if and only if E has zero $(n-1)$-dimensional measure.

The author also characterises Poissonian plane domains in terms of a Wiener type condition which is a weakening of the double cone condition used by Bishop et al. [Pacific J. Math. **138** (1989), no. 2, 233–236; MR0996199].

{See also the following review [MR1115078].} *J. C. Taylor*

MR1065010 (92k:22018) 22E40
Bishop, Christopher (1-UCLA)**; Steger, Tim** (1-CHI)
Three rigidity criteria for $\mathrm{PSL}(2,\mathbf{R})$**.**
Bull. Amer. Math. Soc. (*N.S.*) **24** (1991), *no.* 1, 117–123.
From the text: "Let G be $\mathrm{PSL}(2,\mathbf{R})$, the quotient of the group of 2×2 real matrices with determinant one by its two-element center, $\{\pm I\}$. By a lattice subgroup of G we mean a discrete subgroup such that the space of cosets G/Γ has finite volume. A familiar example of a lattice subgroup is $\mathrm{PSL}(2,\mathbf{Z})$, the subgroup of matrices in $\mathrm{PSL}(2,\mathbf{R})$ with integer entries. Let Γ be an abstract group and let ι_1 and ι_2 be two inclusions of Γ in G, each having a lattice subgroup as its image. We say ι_1 and ι_2 are equivalent if there is some (continuous) automorphism α of G such that $\iota_2=\alpha\circ\iota_1$. This paper describes three closely related criteria for the equivalence of ι_1 and ι_2: one analytic, one representation-theoretic, and one geometric."

MR1081289 (92c:30008) 30C35 30C62
Bishop, Christopher J. (1-UCLA)
Conformal welding of rectifiable curves.
Math. Scand. **67** (1990), *no.* 1, 61–72.
Suppose that D_1 and D_2 are Jordan domains on the Riemann sphere $\overline{\mathbf{C}}$ and that $\psi\colon\Gamma_1\to\Gamma_2$ is a homeomorphism, where $\Gamma_i=\partial D_i$. Then ψ admits a conformal welding if there exist a Jordan curve Γ with complementary domains Ω_1 and Ω_2 and conformal mappings $\Phi_i\colon D_i\to\Omega_i$ whose homeomorphic extensions to $\overline{D}_i$ satisfy $\psi=\Phi_2^{-1}\circ\Phi_1$.

A. Hubershowed that a conformal welding may not exist even when Γ_1 and Γ_2 are rectifiable and ψ is an isometry [Comment. Math. Helv. **51** (1976), no. 3, 319–331; MR0425110]. On the other hand, it was hoped that under these circumstances the curve Γ would be well behaved; for example, rectifiable if a welding did exist [J. M. Anderson, K. F. Barthand D. A. Brannan, Bull. London Math. Soc. **9** (1977), no. 2, 129–162; MR0440018]. In the article under review the author shows that this is not the case by constructing a pair of rectifiable Jordan curves Γ_1 and Γ_2 and an isometry $\psi\colon\Gamma_1\to\Gamma_2$ for which a welding exists and the curve Γ has positive area.

Two consequences of the author's construction are as follows. First, the curve Γ corresponding to a welding for an isometry ψ between rectifiable curves need not be uniquely determined up to a Möbius transformation. Second, for each $1\le d<2$ there exists an isometry ψ between chord arc curves which admits a conformal welding where Γ has Hausdorff dimension greater than d. *F. W. Gehring*

MR1078268 (92c:30026) 30C85
Bishop, Christopher J. (1-UCLA)**; Jones, Peter W.** (1-YALE)
Harmonic measure and arclength.
Ann. of Math. (2) **132** (1990), *no.* 3, 511–547.
This paper provides affirmative answers to two well-known conjectures on harmonic measure and conformal mappings. Theorem 1 states that if Ω is a simply connected domain in the plane and Γ is a rectifiable curve, then a set $E\subset\partial\Omega\cap\Gamma$ has positive harmonic measure only if it has positive length. This result was conjectured by Øksendal and it is impressive when one remembers that harmonic measure quite often can be supported on a set of zero length; it follows that then it is impossible to draw a rectifiable curve through every point in the support. In Theorem 2 the authors establish a quantitative version of Theorem 1. The so-called Hayman-Wu theorem [W. K. Hayman and J.-M. G. Wu, Comment. Math. Helv. **56** (1981), no. 3, 366–403; MR0639358] asserts that for any line L and any conformal mapping f from the unit disk into $\mathbf{R}^2$ the length of $f^{-1}(L)$ is bounded by an absolute constant, now known to be less than 4π [K. Øyma,

"Harmonic measure and conformal length", Proc. Amer. Math. Soc., to appear]. Later J. L. Fernándezand D. H. Hamilton[Comment. Math. Helv. **62** (1987), no. 1, 122–134; MR0882968] proved that if the line is replaced by a chord arc curve L then the length of $f^{-1}(L)$ is bounded by a constant $C(L)$. They also conjectured that the sufficient condition for such a result to hold is the (obviously necessary) Ahlfors-David regularity: the part of L inside any disk has length no more than a fixed constant times the radius of the disk. The authors here use the harmonic measure device introduced by J. B. Garnett, F. W. Gehringand Jones [Indiana Univ. Math. J. **32** (1983), no. 6, 809–829; MR0721565] and point out that the above level set conjecture follows from Theorem 2. The introduction in the paper contains a clear account of the history and background of these appealing problems.

In proving Theorems 1 and 2 the authors' method is to reduce them to the case where Ω itself has rectifiable boundary, for then classical results apply. This is done in several astute steps. First, results of Pommerenke are used to show that Ω can be replaced by the image of the normal fundamental domain $\mathcal{F}$ under a universal covering $\Phi\colon D \to \mathbf{R}^2 \setminus E$ (here D is the unit disk and E is assumed to be compact). The main bulk of the paper consists of the construction of a Lipschitz subdomain $\mathcal{L}$ of $\mathcal{F}$ such that $\Phi(\mathcal{L})$ has rectifiable boundary and E positive harmonic measure in $\Phi(\mathcal{L})$; it is in this construction that the essential assumption "E lies on a rectifiable curve" is used. Jones, in his traveling salesman paper [Invent. Math. **102** (1990), no. 1, 1–15; MR1069238], found a quantitative way to measure the rectifiability of a set in terms of geometric square functions, and this leads to an estimate for the Schwarzian derivative of the covering map Φ. The desired Lipschitz domain is then built by using this estimate with potently applied L^2-techniques in good and bad cubes. The estimate for the Schwarzian, Lemma 3.1, is interesting in its own right and the authors have successfully employed it in new problems in their forthcoming work.

Regrettably, this ingenious paper does not lend itself to easy reading. The reviewer soon surrendered in an effort to verify all the details. There is an unfortunate misstatement in the proof of Lemma 3.1. As easy examples show, it is not true that the domains $\{V_n\}$ lie in $\mathbf{R}^2 \setminus E$ (p. 520, line 26). It is thus impossible to define the Green's function G on V_N in terms of Φ^{-1}, and the proof rests on these assumptions. Bishop informed the reviewer that Lemma 3.1 can be salvaged by replacing V_N by the domain $\mathcal{D}$ (defined on p. 524) and then integrating $G - G_N$ over the boundary of $\mathcal{D}$. It seems that in this case the desired estimate (3.2) can be obtained by using the reasoning on pp. 523–524, and, e.g., Lemma 3.5 becomes redundant. *Juha Heinonen*

MR1042042 (91b:30101) 30D45 30D55 46E15 46J15

Bishop, Christopher J. (1-UCLA)

Bounded functions in the little Bloch space.

Pacific J. Math. **142** (1990), *no.* 2, 209–225.

A necessary and sufficient condition is given for a bounded holomorphic function in the unit disk to belong to the little Bloch space $\mathcal{B}_0$. This condition is in terms of the measure which arises in the canonical factorization theorem. In particular, this provides an answer to D. Sarason's question about the characterization of the Blaschke products in $\mathcal{B}_0$ in terms of the distribution of their zeros and the author gives an explicit example of such a Blaschke product. *David Bekolle*

MR1015126 (90k:30064) 30D55 30H05

Biship, Kristofer (1-MSRI)

An element of the disk-algebra that is stationary on a set of positive length. (Russian)

Algebra i Analiz **1** (1989), *no.* 3, 83–88; *translation in Leningrad Math. J.* **1** (1990), *no.* 3, 647–652.

We say that the function φ is stationary on the set $E\subset\partial D=\{z\colon|z|=1\}$ if there exists an absolutely continuous function ψ on ∂D such that $\psi(e^{i\theta})=\varphi(e^{i\theta})$ and $\psi'=0$ a.e. on E. The author constructs a virtuoso example of a nonconstant function f from a disk algebra, which is stationary on a set of positive length. Thus he answers a question of V. P. Khavin, B. Jörickeand N. G. Makarov. The result obtained has interesting applications in operator theory and function theory. As the author notes, the general problem of describing the measurable sets $E\subset\partial D$, for which there does not exist a nonconstant function from $H^1(D)$ that is stationary on E, is far from being solved. *Oleg V. Ivanov*

MR0996199 (90d:30069) 30C85 31A15

Bishop, C. J. (1-UCLA); **Carleson, L.** (1-UCLA); **Garnett, J. B.** (1-UCLA); **Jones, P. W.** (1-YALE)

Harmonic measures supported on curves.

Pacific J. Math. **138** (1989), *no.* 2, 233–236.

This article exhibits necessary and sufficient conditions for the mutual singularity of two harmonic measures associated with two disjoint simply connected plane domains. More precisely, the following geometric characterization is established. If ω_1 and ω_2 are harmonic measures associated with the complements of a Jordan curve Γ, then $\omega_1\perp\omega_2$ if and only if $\Lambda_1(T)=0$, where T is the set of all points on Γ which have a tangent. In addition to a lemma credited to Beurling, the proof employs a rcent refinement of N. G. Makarov'swork [Proc. London Math. Soc. (3) **51** (1985), no. 2, 369–384; MR0794117] due to Ch. Pommerenke[J. Analyse Math. **46** (1986), 231–238; MR0861701]. Interpretations and modifications to the proof are provided for the case where Γ is a Jordan arc as well as the general case of two disjoint simply connected plane domains. *David A. Herron*

MR0976315 (90a:46134) 46J15 30H05 46J10

Bishop, Christopher J. (1-MSRI)

Constructing continuous functions holomorphic off a curve.

J. Funct. Anal. **82** (1989), *no.* 1, 113–137.

In this paper the author provides new and interesting proofs of some known results about the algebra of a plane domain complementary to a compact set.

More precisely, let K be a compact set in the Riemann sphere $\overline{\mathbf{C}}$ and let $A_K=A(\overline{\mathbf{C}}\setminus K)$ be the algebra of functions analytic and bounded on $\overline{\mathbf{C}}\setminus K$ and continuous on the closure of $\overline{\mathbf{C}}\setminus K$. It is not known when A_K contains nontrivial functions, but one can characterize the sets K for which A_K is big enough in some sense, for instance when it is a Dirichlet algebra over K.

The following results are proven. Theorem (Browder and Wermer): Let Γ be a curve; then A_Γ is Dirichlet on Γ if and only if $\omega_1\perp\omega_2$, where ω_1 and ω_2 are the harmonic measures on each component of $\overline{\mathbf{C}}\setminus\Gamma$. Theorem (Davie): Let K be a compact and connected set; then A_K is Dirichlet on K if and only if the harmonic measures for different components of $\overline{\mathbf{C}}\setminus K$ are mutually singular and each component of $\overline{\mathbf{C}}\setminus K$ is nicely connected (nicely connected means that, on the boundary of the component, there is a set of full harmonic measure where the conformal map on the unit disc is injective). Theorem: If K is compact and connected then the following are equivalent:

(i) A_K is Dirichlet on K; (ii) A_K is pointwise boundedly dense in $H^\infty(\overline{\mathbf{C}} \setminus K)$; (iii) A_K is strongly pointwise boundedly dense in $H^\infty(\overline{\mathbf{C}} \setminus K)$.

The proofs given in the paper replace the use of functional analysis, mainly of the Hahn-Banach theorem, by explicit constructions involving the $\overline{\partial}$ problem. In particular, this gives explicit constructions of nontrivial functions in the algebra A_K. Some related results about the action of singular homeomorphisms of the unit circle on the disc algebra are also obtained as a consequence of the previous proofs. *Julià Cufí*

MR0961619 (89j:30051) 30E10 31A05 46J15

Bishop, Christopher J. (1-UCLA)

Approximating continuous functions by holomorphic and harmonic functions.

Trans. Amer. Math. Soc. **311** (1989), *no.* 2, 781–811.

Let $H^\infty(\Omega)$ denote the algebra of bounded holomorphic functions on an open connected subset Ω of the extended complex plane. The main result of the present work can be described as follows: Let f be a bounded harmonic function on Ω which is not holomorphic. Then the algebra generated by $H^\infty(\Omega)$ and f contains any uniformly continuous function on Ω.

This result is proved for a rather general class of sets including any finitely connected domain. The special case where Ω is the unit disc was previously proved by S. Axlerand A. Shields[Indiana Univ. Math. J. **36** (1987), no. 3, 631–638; MR0905614].

The paper is very well written and contains many interesting results and methods. First the author proves the above result when f is the complex conjugate of a nontrivial function in $H^\infty(\Omega)$. This proof is rather short, it motivates the proof of the general case, and there are no restrictions on Ω in this special case. The general case is much harder. The problem is "pulled back" from Ω to the unit disc D by a universal covering map. In D the main methods involve solution of a $\overline{\partial}$ problem with L^∞ estimates and a delicate analysis of level sets of certain analytic functions in D. These analytic functions are assumed to have (Fatou) boundary values belonging to BMO, the class of functions having bounded mean oscillation.

Finally the problem is transferred back to Ω by an averaging procedure, and it is here that some restrictions on Ω are needed. There is no doubt that a detailed study of the present paper will pay off well for the interested reader. *Arne Stray*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

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A counterexample in conformal welding concerning Hausdorff dimension.

Michigan Math. J. **35** (1988), *no.* 1, 151–159.

In this very beautiful paper, the author settles two important problems. Theorem: For any $1 \le d < 2$ there exists a quasicircle Γ of dimension d such that the corresponding welding homeomorphism is bi-Lipschitz. Corollary: There exists a bi-Lipschitz homeomorphism $\psi\colon \mathbf{R} \to \mathbf{R}$ and a nonconstant $f \in A(H_+)$ such that $f \circ \psi \in A(H_-)$. (Here $A(H_\pm)$ denote the spaces of continuous functions with bounded holomorphic extensions to $H_\pm$ respectively; $H_\pm = \{\mathrm{Im} z \gtrless 0\}$.)

The theorem improves a result of S. Semmes[Ark. Mat. **24** (1986), no. 1, 141–158; MR0852832]. It is worth mentioning that the bi-Lipschitz constant cannot be taken close to 1, for by a result of G. David, Γ must be a chord-arc curve in this case. The corollary answers a question of Semmes to the effect that the operator $f \mapsto P(f \circ \psi)$ with $\psi' \in A_\infty$ (the Muckenhoupt class) need not be invertible in BMOA. (P denotes the projection from BMO to BMOA.) An example of an absolutely continuous ψ on $\mathbf{R}$ such that $f \circ \psi \in A(H_-)$ for a nonconstant f in $A(H_+)$ was originally given by J. B. Garnettand A. G. O'Farrell[Pacific J. Math. **65** (1976), no. 1, 55–63; MR0419775], whose construction is lurking in the background of the paper under review. *N. G. Makarov*

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★**HARMONIC MEASURES SUPPORTED ON CURVES.**

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