Examples concerning Abel and Cesàro limits

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Abstract

This note describes examples of all possible equality and strict inequality relations between upper and lower Abel and Cesàro limits of sequences bounded above or below. It also provides applications to Markov Decision Processes.

Keywords: Tauberian theorem, Hardy-Littlewood theorem, Abel limit, Cesàro limit

1 Introduction

For a sequence $\{u_n\}_{n=0,1,\ldots}$ consider lower and upper Cesàro limits

$$\underline{C} = \liminf_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} u_i, \qquad \bar{C} = \limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} u_i$$

and lower and upper Abel limits

$$\underline{A} = \liminf_{\alpha\to 1-}(1-\alpha)\sum_{n=0}^{\infty} u_n\alpha^n, \qquad \bar{A} = \limsup_{\alpha\to 1-}(1-\alpha)\sum_{n=0}^{\infty} u_n\alpha^n.$$

If a sequence $\{u_n\}_{n=0,1,\ldots}$ is bounded above or below then, according to a Tauberian theorem (see, e.g., Sennott [12, pp. 281, 282]),

$$\underline{C} \le \underline{A} \le \bar{A} \le \bar{C}, \tag{1}$$

and, according to the Hardy-Littlewood theorem (see, e.g., Titchmarsh [14, p. 226]), if $\underline{A} = \bar{A}$ then

$$\underline{C} = \underline{A} = \bar{A} = \bar{C}. \tag{2}$$

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In view of the Tauberian and Hardy-Littlewood theorems (1) and (2), either equalities (2) hold or only the following relations can be possible:

$$\underline{C} < \underline{A} < \bar{A} < \bar{C}, \tag{3}$$

$$\underline{C} = \underline{A} < \bar{A} = \bar{C}, \tag{4}$$

$$\underline{C} < \underline{A} < \bar{A} = \bar{C}, \tag{5}$$

$$\underline{C} = \underline{A} < \bar{A} < \bar{C}. \tag{6}$$

Hardy [6], Liggett and Lippman [9], Sznajder and Filar [13, Example 2.2], Sennott [12, p. 286], Keating and Reade [8], and Duren [2, Chapter 7] provided at different levels of details examples of bounded sequences for which inequalities (3) hold. This note demonstrates that inequalities (4)–(6) may also take place for bounded sequences. Example 1 demonstrates the possibility of (4), and Example 2 demonstrates the possibility of (5). Of course, inequalities (6) hold for the sequence $\{-u_n\}_{n=0,1,\ldots}$, if inequalities (5) hold for a sequence $\{u_n\}_{n=0,1,\ldots}$.

The Tauberian and Hardy-Littlewood theorems are important for many applications. For example, they are used to approximate average costs per unit time by total discounted costs for Markov Decision Processes (MDPs) and stochastic games; see e.g., [5, 7, 9, 11, 12]. They are also used to evaluate long-run behavior of stochastic systems by using Laplace-Stieltjes transforms, see e.g., Abramov [1]. This study was motivated by applications to MDPs; see Section 4.

2 Auxiliary facts

Lemma 1. *Let $\{L(n)\}_{n=0,1,\ldots}$ and $\{M(n)\}_{n=0,1,\ldots}$ be two sequences of nonnegative numbers, $f_n(\alpha) = \alpha^{L(n)} - \alpha^{M(n)}$, $n = 0, 1, \ldots,$ and $f^*(\alpha) = \sum_{n=0}^{\infty} f_n(\alpha)$. If $L(n) \to \infty$ and $L(n)/M(n) \to 0$ as $n \to \infty$, then:*

(i) there is a sequence $\alpha_n \to 1-$ such that $f_n(\alpha_n) \to 1$ as $n \to \infty$;

(ii) $\limsup_{\alpha\to 1-} f^*(\alpha) \geq 1$.

Proof. Since $\lim_{\alpha\to 1-} f_n(\alpha) = 0$ for all n, $\limsup_{\alpha\to 1-} f^*(\alpha) = \limsup_{\alpha\to 1-} \sum_{n=m}^{\infty} f_n(\alpha)$ for any natural m. Choose m such that $L(n) < M(n)$ and $L(n) > 1$ when $n \geq m$.

Observe that (i) implies (ii). So, in the rest of the proof we prove (i).

By differentiating f_n for each $n > m$, observe that this function reaches its maximum on $[0, 1]$ at the point

$$\alpha_n = \left(\frac{L(n)}{M(n)}\right)^{\frac{1}{M(n)-L(n)}}, \tag{7}$$

and the maximum value is

$$f_n(\alpha_n) = \left(\frac{L(n)}{M(n)}\right)^{\frac{L(n)}{M(n)-L(n)}} - \left(\frac{L(n)}{M(n)}\right)^{\frac{M(n)}{M(n)-L(n)}}.$$

Since $\frac{L(n)}{M(n)} \to 0$ as $n \to \infty$, we have $\frac{M(n)}{M(n)-L(n)} \to 1$ as $n \to \infty$. Therefore,

$$\lim_{n\to\infty} f_n(\alpha_n) = \lim_{n\to\infty} \left(\frac{L(n)}{M(n)}\right)^{\frac{L(n)}{M(n)-L(n)}} = \lim_{n\to\infty} \left(\frac{L(n)}{M(n)}\right)^{\frac{L(n)}{M(n)}\frac{M(n)}{M(n)-L(n)}} = 1. \tag{8}$$

In addition, for $n > m$

$$1 \geq \left(\frac{L(n)}{M(n)}\right)^{\frac{1}{M(n)-L(n)}} \geq \left(\frac{L(n)}{M(n)}\right)^{\frac{L(n)}{M(n)-L(n)}} \to 1 \quad \text{as} \quad n \to \infty.$$

Thus, in view of (7) and (8), $\alpha_n \to 1-$ and $f_n(\alpha_n) \to 1$ as $n \to \infty$.

Recall that

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n-1} k!}{n!} = 0 \qquad \text{and} \qquad \lim_{n\to\infty} \frac{\sum_{k=1}^{n} k!}{n!} = 1. \tag{9}$$

Indeed,

$$0 \leq \lim_{n\to\infty} \frac{\sum_{k=1}^{n-1} k!}{n!} = \lim_{n\to\infty} \left[\frac{\sum_{k=1}^{n-2} k!}{n!} + \frac{(n-1)!}{n!}\right] \leq \lim_{n\to\infty} \left\{\frac{(n-2)[(n-2)!]}{n!} + \frac{1}{n}\right\} = 0.$$

3 Examples

For a sequence $\{u_n\}_{n=0,1,\dots}$, define the function

$$f(\alpha) = (1-\alpha)\sum_{n=0}^{\infty} u_n\alpha^n, \qquad \alpha \in [0,1). \tag{10}$$

Example 1. For $D(k) = \sum_{i=1}^{k} i!$, $k = 1, 2, \dots$, let

$$u_n = \begin{cases} 1, & \text{if } D(2k-1) \leq n < D(2k),\ k = 1, 2, \dots, \\ 0, & \text{otherwise.} \end{cases} \tag{11}$$

Proposition 1. *Inequalities (4) hold with $\underline{C} = \underline{A} = 0$ and $\bar{C} = \bar{A} = 1$ for the sequence $\{u_n\}_{n=0,1,\dots}$ defined in (11).*

Proof. By using properties of geometric series, observe that

$$f(\alpha) = \sum_{n=1}^{\infty} f_n(\alpha), \tag{12}$$

where

$$f_n(\alpha) = \alpha^{D(2n-1)} - \alpha^{D(2n)} \geq 0. \tag{13}$$

In view of (9), $D(2n-1) \to \infty$ and $D(2n-1)/D(2n) \to 0$ as $n \to \infty$. Formulas (12), (13) and Lemma 1(ii) imply that $1 \geq \bar{A} = \limsup_{\alpha\to 1-} f(\alpha) \geq 1$. Thus, $\bar{A} = 1$. In view of (1), $\bar{A} \leq \bar{C}$. Since $\bar{C} \leq 1$, then $\bar{C} = 1$.

Now we show that $\underline{A} = 0$. For each $k = 1, 2, \ldots$, consider the sequence $\{u_n^k\}_{n=0,1,\ldots}$

$$u_n^k = \begin{cases} 1, & \text{if } n < D(2k) \text{ or } n \geq D(2k+1), \\ 0, & \text{otherwise.} \end{cases}$$

Let f^k be the function f from (10) for the sequence $\{u_n^k\}_{n=0,1,\ldots}$. Since $u_n \leq u_n^k$ for each $n = 0, 1, \ldots$, $f(\alpha) \leq f^k(\alpha)$ for all $\alpha \in [0, 1)$, $k = 1, 2, \ldots$. Therefore, to prove that $\underline{A} = 0$, it is sufficient to show the existence of a sequence $\alpha_k \to 1-$ as $k \to \infty$ such that $\lim_{k\to\infty} f^k(\alpha_k) = 0$.

Observe that

$$f^k(\alpha) = (1-\alpha)\left[\sum_{n=0}^{D(2k)-1} \alpha^n + \sum_{n=D(2k+1)}^{\infty} \alpha^n\right] = 1 - \alpha^{D(2k)} + \alpha^{D(2k+1)}.$$

In view of Lemma 1(i), there exist $\alpha_k \to 1-$ such that $\alpha_k^{D(2k)} - \alpha_k^{D(2k+1)} \to 1$ as $k \to \infty$. Thus, $f^k(\alpha_k) \to 0$ as $k \to \infty$, and $\underline{A} = 0$. This implies $\underline{C} = 0$ since $0 \leq \underline{C} \leq \underline{A}$. □

Example 2. Let

$$u_n = \begin{cases} 0, & \text{if } k! \leq n < 2k!,\ k = 1, 2, \ldots, \\ 1, & \text{otherwise.} \end{cases} \tag{14}$$

Proposition 2. *Inequalities (5) hold with $\underline{C} = \frac{1}{2}$, $\underline{A} = \frac{3}{4}$, and $\bar{C} = \bar{A} = 1$ for the sequence $\{u_n\}_{n=0,1,\ldots}$ defined in (14).*

Proof. By (9)

$$\underline{C} = \lim_{n\to\infty} \frac{1 + \sum_{k=2}^{n-1}[(k+1)! - 2k!]}{2n!} = \lim_{n\to\infty} \frac{\sum_{k=3}^{n}(k)! - \sum_{k=2}^{n-1} 2k!}{2n!} = \frac{1}{2},$$

and

$$\bar{C} = \lim_{n\to\infty} \frac{n! - \sum_{k=1}^{n-1} k!}{n!} = 1.$$

By using the formula for the sum of geometric series,

$$f(\alpha) = 1 - \alpha + \sum_{n=1}^{\infty} \left(\alpha^{2n!} - \alpha^{(n+1)!} \right).$$

By Lemma 1(ii), $\bar{A} \geq 1$. However, $\bar{A} \leq \bar{C} = 1$. Thus, $\bar{A} = 1$.

To compute $\underline{A}$, define

$$g(\alpha) = 1 - f(\alpha) = \sum_{n=1}^{\infty} \left(\alpha^{n!} - \alpha^{2n!} \right)$$

and $\bar{B} = \limsup_{\alpha \to 1-} g(\alpha)$. Then $\underline{A} = 1 - \bar{B}$.

We compute $\bar{B}$ first. Let $g_n(\alpha) = \alpha^{n!} - \alpha^{2n!}$, $n = 1, 2, \ldots$. When $\alpha \in [0,1]$, the function $g_n(\alpha)$ reaches its maximum at $\alpha_n = 2^{-\frac{1}{n!}}$ and $g_n(\alpha_n) = \frac{1}{4}$. In addition, this function increases on the interval $[0, \alpha_n]$ and decreases on the interval $[\alpha_n, 1]$.

Let $\beta_k = 2^{-\frac{1}{(k-1)!\sqrt{k}}}$, $k = 1, 2, \ldots$. When $\alpha \in [\beta_k, \beta_{k+1}]$, $k = 1, 2, \ldots$, then, if $n < k$, the function $g_n(\alpha)$ decreases and reaches its maximum on this interval at the point β_k; if $n > k$ then it increases and reaches the maximum at the point β_{k+1}; and, if $n = k$, it achieves the maximum at α_k. Thus,

$$g(\alpha) = \sum_{n=1}^{k-1} g_n(\alpha) + g_k(\alpha) + \sum_{n=k+1}^{\infty} g_n(\alpha) < \sum_{n=1}^{k-1} g_n(\beta_k) + g_k(\alpha) + \sum_{n=k+1}^{\infty} g_n(\beta_{k+1}). \tag{15}$$

Observe that

$$\sum_{n=1}^{k-1} g_n(\beta_k) = \sum_{n=1}^{k-1} \beta_k^{n!} \left(1 - \beta_k^{n!}\right) < \sum_{n=1}^{k-1} \left(1 - \beta_k^{n!}\right) < \frac{\sum_{n=1}^{k-1} n!}{(k-1)!\sqrt{k}} \ln 2 \to 0 \quad \text{as} \quad k \to \infty, \tag{16}$$

where the last inequality follows from $2^{-x} > 1 - x \ln 2$ for $x > 0$, and

$$\sum_{n=k+1}^{\infty} g_n(\beta_{k+1}) < \sum_{n=k+1}^{\infty} \beta_{k+1}^{n!} = \sum_{n=k+1}^{\infty} 2^{-\frac{n!}{k!\sqrt{k+1}}} = 2^{-\sqrt{k+1}} \sum_{n=k+1}^{\infty} 2^{-\frac{n!-(k+1)!}{k!\sqrt{k+1}}} \leq 2^{1-\sqrt{k+1}}, \tag{17}$$

where the last inequality holds because $\frac{n!-(k+1)!}{k!\sqrt{k+1}} \geq n - (k+1)$ when $n \geq (k+1)$. Thus (16) and (17) imply that

$$\lim_{k \to \infty} \left(\sum_{n=1}^{k-1} g_n(\beta_k) + \sum_{n=k+1}^{\infty} g_n(\beta_{k+1}) \right) = 0. \tag{18}$$

In conclusion,

$$\bar{B} = \limsup_{k \to \infty} \sup_{\alpha \in [\beta_k, \beta_{k+1}]} g(\alpha) \leq \lim_{k \to \infty} \left(g_k(\alpha_k) + \sum_{n=1}^{k-1} g_n(\beta_k) + \sum_{n=k+1}^{\infty} g_n(\beta_{k+1}) \right) = \frac{1}{4},$$

where the first equality holds since $\beta_k \to 1$, the inequality holds because of (15) and because the function g_k reaches its maximum at α_k on the interval $[0,1]$, and the last equality holds because of $g_k(\alpha_k) = \frac{1}{4}$ and (18). In addition, $\bar{B} \geq \lim_{k\to\infty} g(\alpha_k) \geq \lim_{k\to\infty} g_k(\alpha_k) = \frac{1}{4}$. Thus $\bar{B} = \frac{1}{4}$ and $\underline{A} = 1 - \bar{B} = \frac{3}{4}$. $\square$

4 On approximations of average costs per unit time by normalized discounted costs for MDPs

Average costs for an MDP can be defined either as upper or as lower limits of expected costs per unit time over finite time horizons as the time horizon lengths tend to infinity. For each of these two definitions of average costs, the minimal value is the infimum of average costs taken over the set of all policies. As shown below, if the state space is infinite, Example 2 implies that it is possible that one of these two minimal values can be approximated by normalized total expected discounted costs, while such approximations for another one are impossible.

Consider an MDP with a state space $\mathbb{X}$, action space $\mathbb{A}$, sets of available actions $A(x)$, transition probabilities p, and one-step cost c. Here we assume that:

(i) the state space $\mathbb{X}$ is a nonempty countable set,

(ii) the action space $\mathbb{A}$ is a measurable space $(\mathbb{A}, \mathcal{A})$ such that all its singletons are measurable subsets, that is, $\{a\} \in \mathcal{A}$ for each $a \in \mathbb{A}$;

(iii) for each state $x \in \mathbb{X}$ the set of available actions $A(x)$ is nonempty and belongs to $\mathcal{A}$;

(iv) if an action $a \in A(x)$ is chosen at a state $x \in \mathbb{X}$, then $p(y|x, a)$, where $y \in \mathbb{X}$, is the probability that y is the state at the next step; it is assumed that $p(\cdot|x, a)$ is a probability mass function on $\mathbb{X}$ and $p(y|x, \cdot)$ is a measurable function on $A(x)$;

(v) if an action $a \in A(x)$ is selected at a state $x \in \mathbb{X}$, then the one-step cost $c(x, a)$ is incurred; it is assumed that the values $c(x, a)$ are uniformly bounded below, and the function $c(x, \cdot)$ is measurable on $A(x)$ for each $x \in \mathbb{X}$.

Let $\mathbb{H}_n = \mathbb{X} \times (\mathbb{A} \times \mathbb{X})^n$ be the set of trajectories up to the step $n = 0, 1, \ldots$. For $n = 1, 2, \ldots,$ consider the sigma-field $\mathcal{F}_n$ on $\mathbb{H}_n$ defined as the products of the sigma-fields of all subsets of $\mathbb{X}$ and $\mathcal{A}$. A policy π is a sequence $\{\pi_n\}_{n=0,1,\ldots}$ of transition probabilities from $\mathbb{H}_n$ to $\mathbb{A}$ such that: (i) for each $h_n = x_0a_0x_1...a_nx_n \in \mathbb{H}_n$, $n = 0, 1, \ldots,$ the probability $\pi_n(\cdot|h_n)$ is defined on $(\mathbb{A}, \mathcal{A})$, and it satisfies the condition $\pi_n(A(x_n)|h_n) = 1$, and (ii) $\pi_n(B|\cdot)$ is a measurable function on $(\mathbb{H}_n, \mathcal{F}_n)$ for each $B \in \mathcal{A}$. A policy π is called stationary if there is a mapping $\phi : \mathbb{X} \to \mathbb{A}$ such that $\phi(x) \in A(x)$ for all $x \in \mathbb{X}$ and $\pi_n(\{\phi(x_n)\}|x_0a_0x_1 \ldots x_n) = 1$ for all $n = 0, 1, \ldots,$ $x_0a_0x_1 \ldots x_n \in \mathbb{H}_n$. Since a stationary policy is defined by a mapping ϕ, it is also denoted by ϕ with a slight abuse of notations. Sometimes in the literature, a stationary policy is called nonrandomized stationary, deterministic stationary, or deterministic. Let Π be the set of all policies.

The standard arguments based on the Ionescu Tulcea theorem [10, Chapter 5, Section 1] imply that each initial state x and policy π define a stochastic sequence on the sets of trajectories $x_0a_0x_1a_1,\dots$. We denote by $\mathbb{E}_x^\pi$ expectations for this stochastic sequence.

For an initial state $x \in \mathbb{X}$ and for a policy π, the average cost per unit time is

$$w^*(x,\pi) = \limsup_{N\to\infty} \frac{1}{N}\mathbb{E}_x^\pi \sum_{n=0}^{N-1} c(x_n,a_n) = \limsup_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \mathbb{E}_x^\pi c(x_n,a_n).$$

In general, if the performance of a policy π is evaluated by a function $g(x,\pi)$ with values in $[-\infty,\infty]$, where $x \in \mathbb{X}$ is the initial state, we define the value function $g(x) = \inf_{\pi\in\Pi} g(x,\pi)$. For $\epsilon \geq 0$, a policy π is called ϵ-optimal, if $g(x,\pi) \leq g(x)+\epsilon$ for all $x \in \mathbb{X}$. A 0-optimal policy is called optimal.

For a constant $\alpha \in [0,1)$, called the discount factor, the expected total discounted costs are

$$v_\alpha(x,\pi) = \mathbb{E}_x^\pi \sum_{n=0}^{\infty} \alpha^n c(x_n,a_n) = \sum_{n=0}^{\infty} \alpha^n \mathbb{E}_x^\pi c(x_n,a_n).$$

In general, proofs of the existence of stationary optimal policies for expected average costs per unit time are more difficult than for expected total discounted costs. Average costs per unit time are often analyzed by approximating $w^*(x,\pi)$ with $(1-\alpha)v_\alpha(x,\pi)$ for the values of α close to 1. Let

$$\bar{w}(x,\pi) = \limsup_{\alpha\to 1-} v_\alpha(x,\pi).$$

In addition to the upper limit of the average expected costs (4), consider the lower Cesàro limit

$$w_*(x,\pi) = \liminf_{N\to\infty} \frac{1}{N}\mathbb{E}_x^\pi \sum_{n=0}^{N-1} c(x_n,a_n) = \liminf_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \mathbb{E}_x^\pi c(x_n,a_n)$$

and the lower Abel limit

$$\underline{w}(x,\pi) = \liminf_{\alpha\to 1-} v_\alpha(x,\pi).$$

In view of the Tauberian theorem

$$w_*(x,\pi) \leq \underline{w}(x,\pi) \leq \bar{w}(x,\pi) \leq w^*(x,\pi), \qquad x \in \mathbb{X},\ \pi \in \Pi.$$

Therefore, the same inequalities hold for the values,

$$w_*(x) \leq \underline{w}(x) \leq \bar{w}(x) \leq w^*(x), \qquad x \in \mathbb{X}.$$

The natural questions are whether $w^*(x) = \bar{w}(x)$ and whether $w_*(x) = \underline{w}(x)$?

Let the state space $\mathbb{X}$ be finite. Then, according to Dynkin and Yushkevich [3, Chapter 7, Section 3], for each stationary policy ϕ

$$w_*(x,\phi) = \underline{w}(x,\phi) = \bar{w}(x,\phi) = w^*(x,\phi), \qquad x \in \mathbb{X}. \tag{19}$$

Though for some $\epsilon > 0$ stationary ϵ-optimal policies may not exist for MDPs with finite state and arbitrary action sets (see Dynkin and Yushkevich [3, Chapter 7, Section 8, Example 2]), as proved in Feinberg [4, Corollary 1],

$$w_*(x) = \underline{w}(x) = \bar{w}(x) = w^*(x), \qquad x \in \mathbb{X}. \tag{20}$$

Equalities (19) may not hold, when a stationary policy π is substituted with an arbitrary policy π. In fact, all four situations presented in (3)–(6) are possible with $\bar{C} = w^*(x,\pi)$, $\bar{A} = \bar{w}(x,\pi)$, $\underline{A} = \underline{w}(x,\pi)$, and $\underline{C} = w_*(x,\pi)$. Indeed, consider an MDP with a single state and two actions, that is, $\mathbb{X} = \{x\}$ and $\mathbb{A} = A(x) = \{a,b\}$. Let also $c(x,a) = 1$ and $c(x,b) = 0$. In addition, $p(x|x,a) = p(x|x,b) = 1$ since the process is always at state x. Let at each step $n = 0,1,\dots$ a policy π select actions a and b with probabilities $\pi_n(a)$ and $\pi_n(b)$ respectively. For a sequence $\{u_n\}_{n=0,1,\dots}$, let $\pi_n(a) = u_n$. Then $\mathbb{E}_x^\pi c(x_n,a_n) = u_n$, $n = 0,1,\dots$, and the values of $w^*(x,\pi)$, $\bar{w}(x,\pi)$, $\underline{w}(x,\pi)$, and $w_*(x,\pi)$ are equal to the corresponding Cesàro and Abel limits for the sequence $\{u_n\}_{n=0,1,\dots}$. Since all the inequalities (3)–(6) are possible for Cesàro and Abel limits of bounded sequences $\{u_n\}_{n=0,1,\dots}$, these inequalities are also possible for $\bar{C} = w^*(x,\pi)$, $\bar{A} = \bar{w}(x,\pi)$, $\underline{A} = \underline{w}(x,\pi)$, and $\underline{C} = w_*(x,\pi)$. .

Now let $\mathbb{X}$ be countably infinite. For each sequence $\{u_n\}_{n=0,1,\dots}$ consider the MDP with the state space $\mathbb{X} = \{0,1,\dots\}$, a single action a, that is $\mathbb{A} = \{a\}$, transition probabilities $p(x+1|x,a) = 1$, and one-step costs $c(x,a) = u_x$, $x \in X$. For this MDP, there is only one policy, and this policy is stationary. We denote this policy by ϕ and observe that $\mathbb{E}_0^\phi c(x_n,a_n) = u_n$, $n = 0,1,\dots$. Thus, equalities (19) and (20) may not hold. In addition, all the inequalities (3)–(6) are possible with $\bar{C} = w^*(x,\phi)$, $\bar{A} = \bar{w}(x,\phi)$, $\underline{A} = \underline{w}(x,\phi)$, $\underline{C} = w_*(x,\phi)$ and with $\bar{C} = w^*(x)$, $\bar{A} = \bar{w}(x)$, $\underline{A} = \underline{w}(x)$, $\underline{C} = w_*(x)$. In particular, $w^*(x) = \bar{w}(x)$ does not imply $w_*(x) = \underline{w}(x)$, and $w_*(x) = \underline{w}(x)$ does not imply $w^*(x) = \bar{w}(x)$. $\square$

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References

[1] V.M. Abramov, Optimal control of a large dam, J. Appl. Probab. 44(1) (2007) 249-258.

[2] P.L. Duren, Introduction to Classical Analysis, American Mathematical Society, Providence, R.I., 2012.

[3] E.B. Dynkin, A.A. Yushkevich, Controlled Markov Processes, Springer-Verlag, New York, 1979.

[4] E.A. Feinberg, An ϵ-optimal control of a finite Markov chain, Theor. Probab. Appl., 25(1)(1980) 70-81.

[5] E.A. Feinberg, P.O. Kasyanov, N.V. Zadoianchuk, Average cost Markov decision processes with weakly continuous transition probabilities, Math. Oper. Res. 37(4)(2012) 591-607.

[6] G.H. Hardy, On certain oscillating series, Quarterly J. Math. 38(1907) 269-288.

[7] O. Hernández-Lerma, Average optimality in dynamic programming on Borel spaces - Unbounded costs and controls, Syst. Control Lett. 17(3)(1991) 237-242.

[8] J.P. Keating, J.B. Reade, Summability of alternating gap series, P. Edinburgh Math. Soc. 43(1) (2000) 95-101.

[9] T.M. Liggett, S.A. Lippman, Stochastic games with perfect information and time average payoff, SIAM Rev. 11(4) (1969) 604-607.

[10] J. Neveu, Mathematical Foundations of the Calculus of Probability, Holden-Day, San Francisco, 1965.

[11] M. Schäl. Average optimality in dynamic programming with general state space, Math. Oper. Res. 18(1) (1993) 163-172.

[12] L.I. Sennott, Stochastic Dynamic Programming and the Control of Queueing Systems, John Wiley & Sons, New York, 1999.

[13] R. Sznajder, J.A. Filar, Some comments on a theorem of Hardy and Littlewood, J. Optimiz. Theory App. 75(1) (1992) 201-208.

[14] E.C. Titchmarsh, The Theory of Functions (2nd ed.), Oxford University Press, Oxford, 1939.