A Brief History of Triangulation
Stony Brook University
Monday, March 2, 2026

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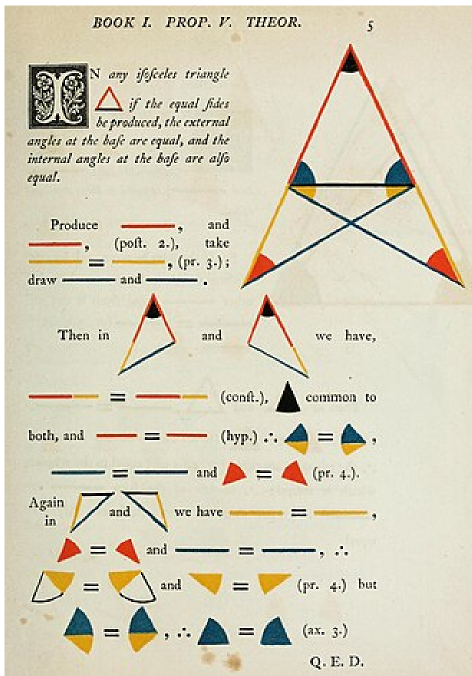

BOOK I. PROP. V. THEOR. 5

In any isosceles triangle if the equal sides be produced, the external angles at the base are equal, and the internal angles at the base are also equal.

Produce , and , (post. 2.), take = , (pr. 3.); draw and .

Then in and we have, = (const.), common to both, and = (hyp.) ∴ = , = and = (pr. 4.).

Again in and we have = , = and = , ∴ = and = (pr. 4.) but = , ∴ = (ax. 3.)

Q. E. D.

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Google definitions of triangulation:

Political science: A strategy where a politician takes positions that seem to be between the left and right sides of the political spectrum, often by adopting some ideas from the opposing side to make their own position seem more centrist.

Psychology: In family therapy, it describes a two-person relationship pulling in a third person (like a child) to reduce tension. This creates a "triangle," which is a more stable emotional system, but the two people in the most stable part of the triangle can shift over time.

Geometry: This is the process of dividing a polygon into triangles by creating a set of triangles with non-overlapping interiors that, when combined, form the original polygon.

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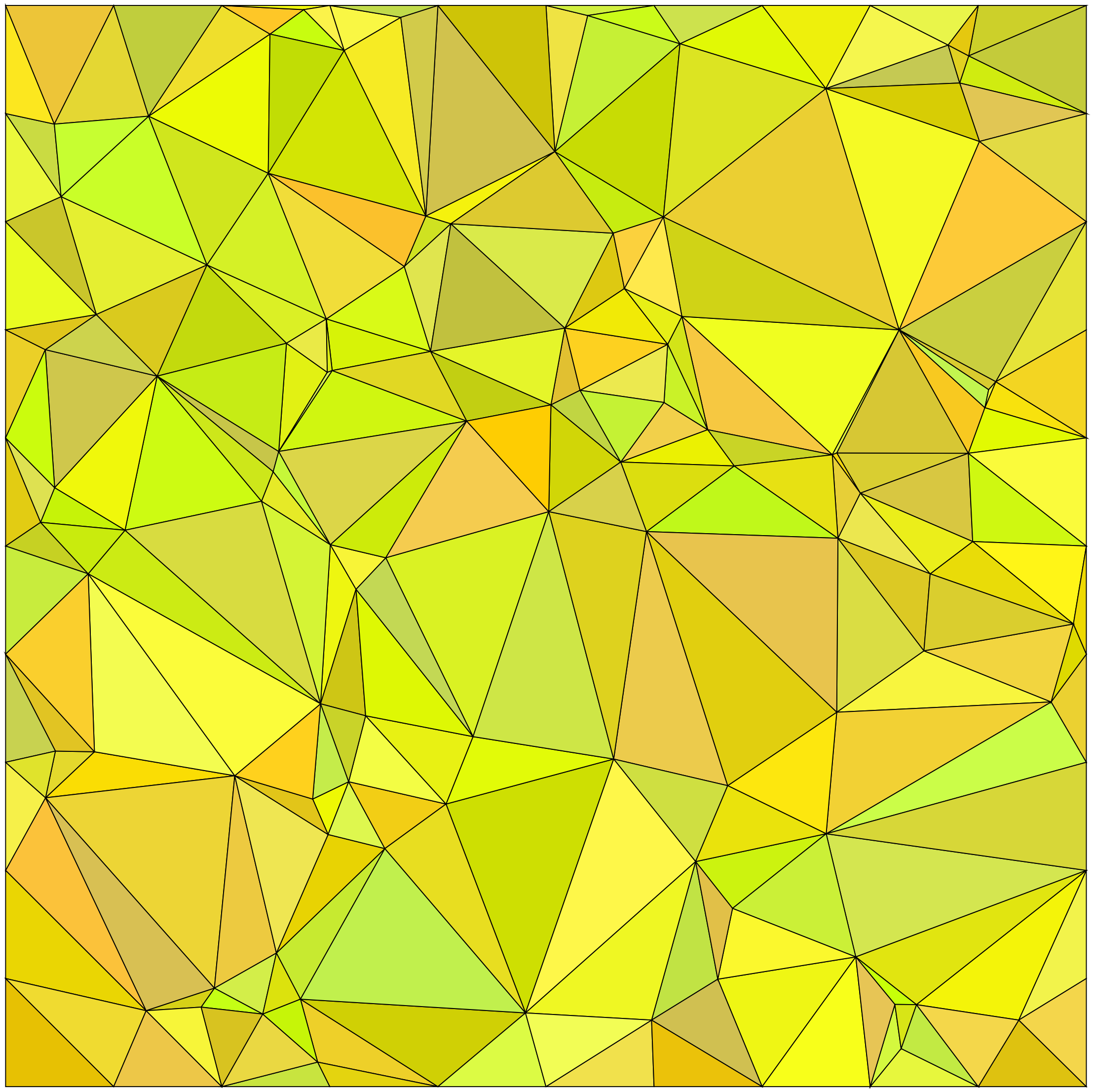
Polygon triangulation: This is the process of dividing a polygon into triangles by creating a set of triangles with non-overlapping interiors that, when combined, form the original polygon.



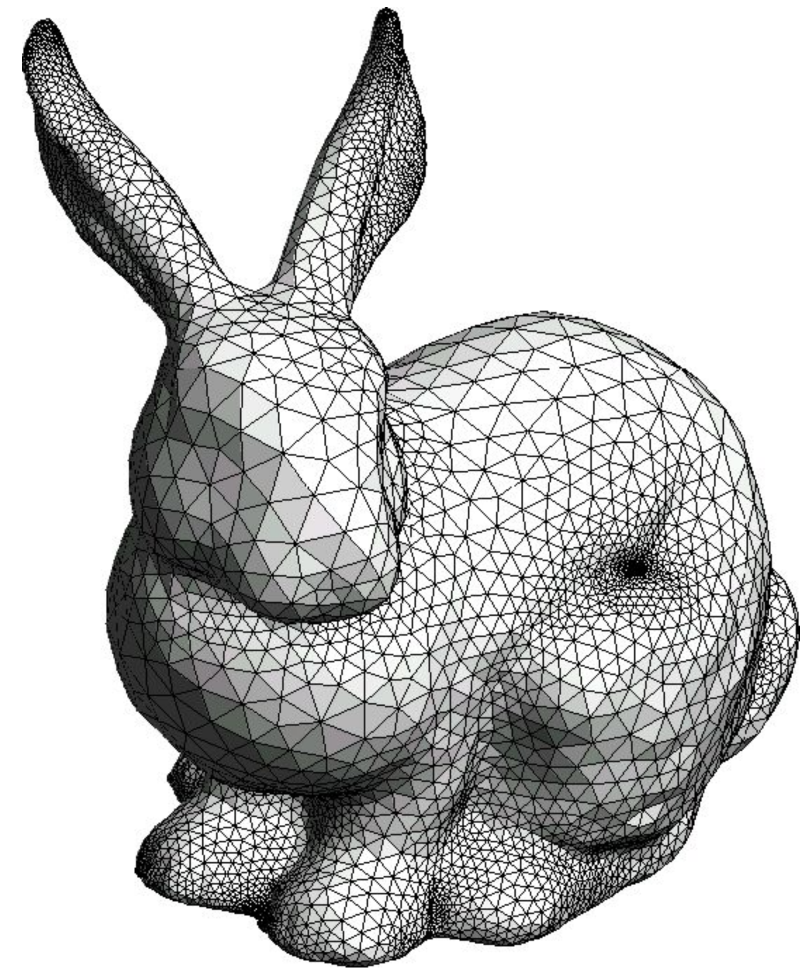
A triangulation

Egyptian triangles

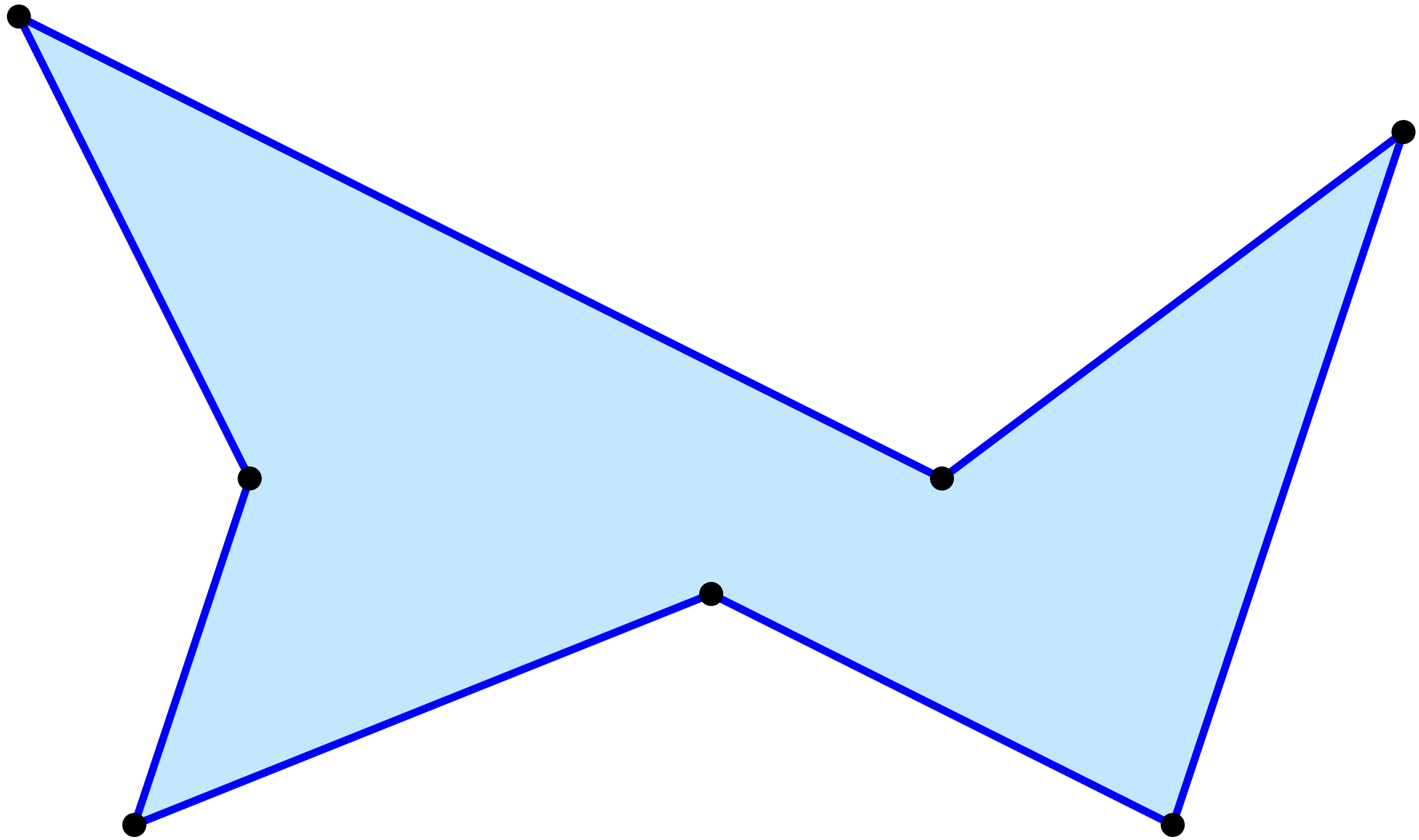
French triangles

American triangles

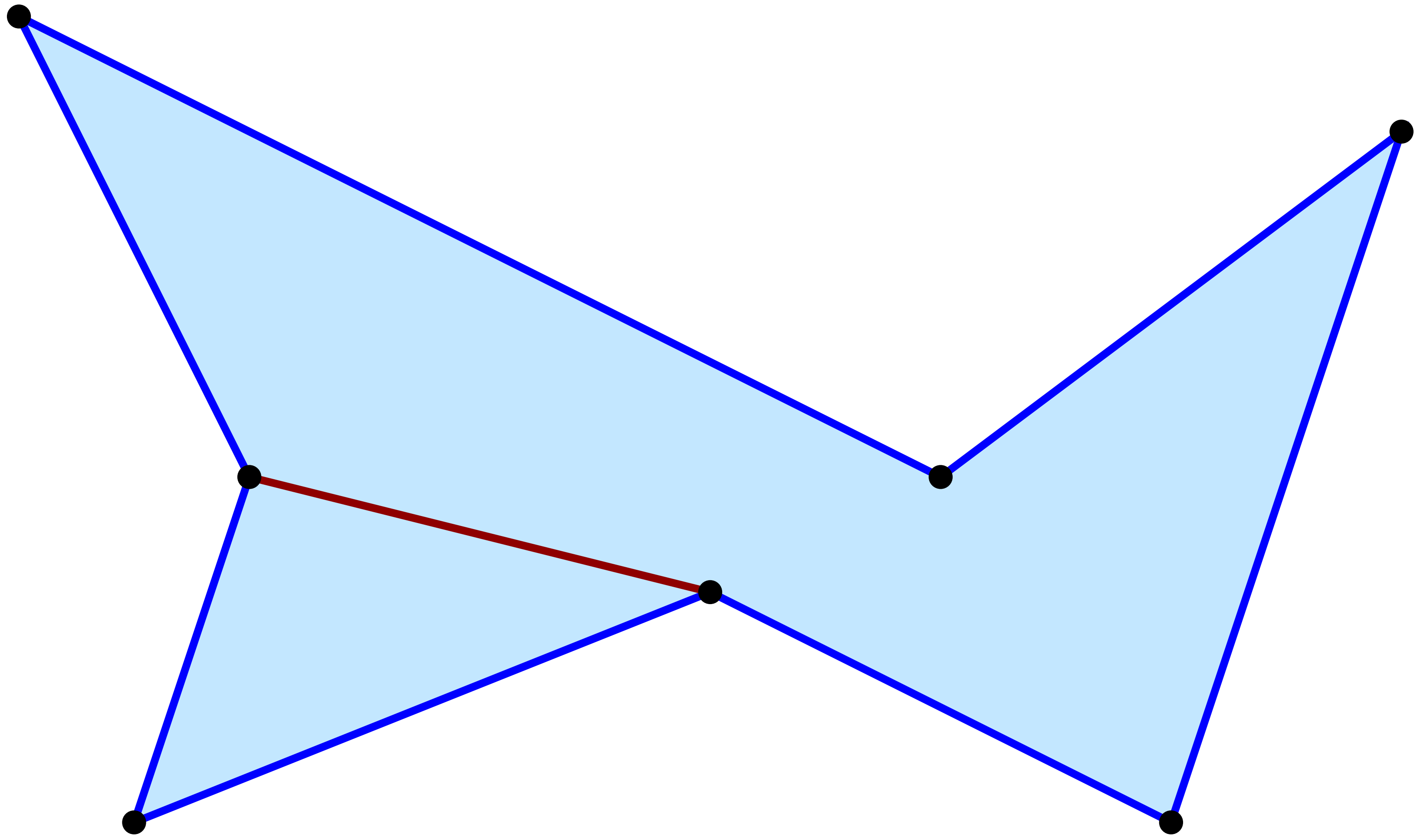
Triangles in engineering



Triangles in computer graphics
(The Stanford Bunny)



Triangulation of a polygon



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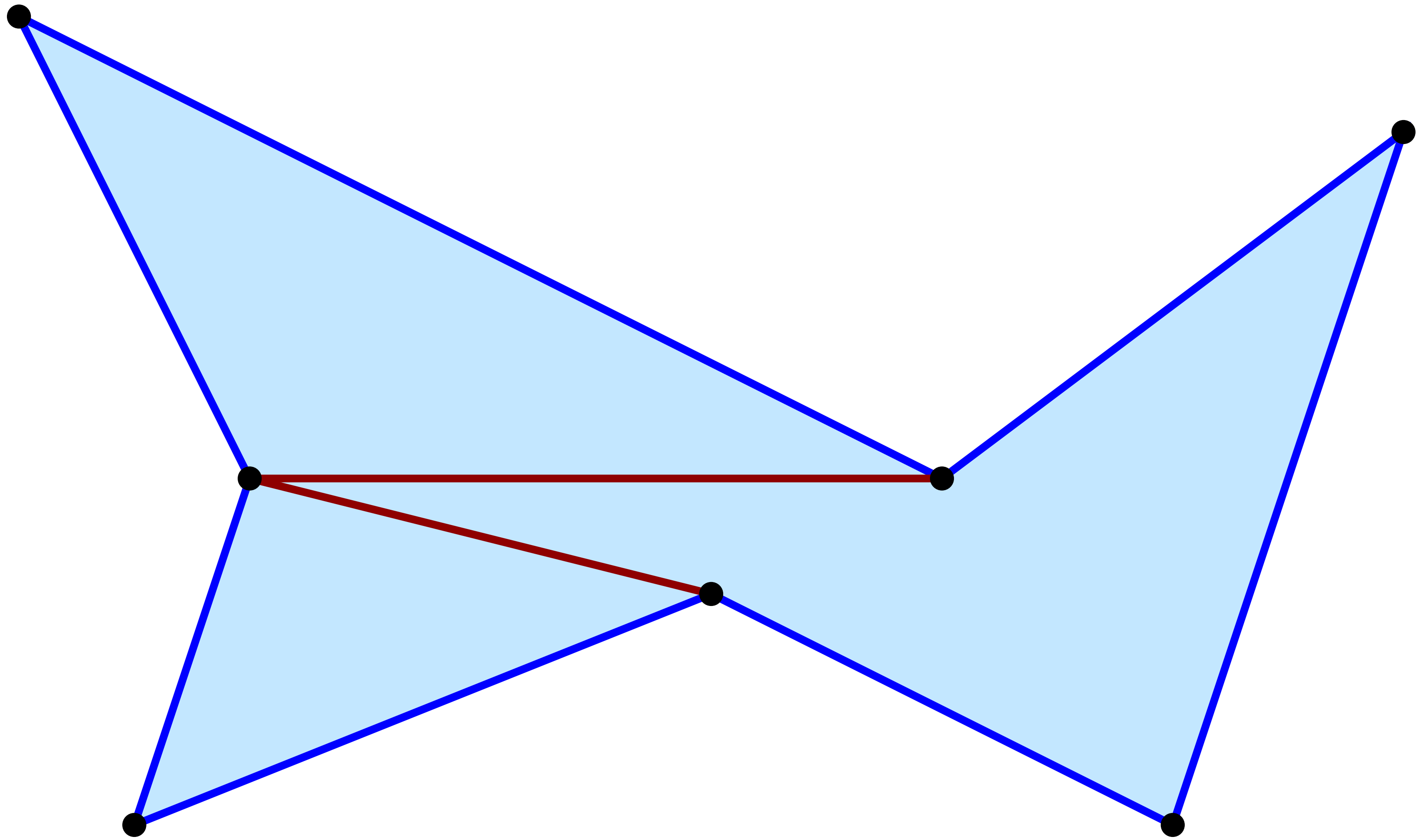
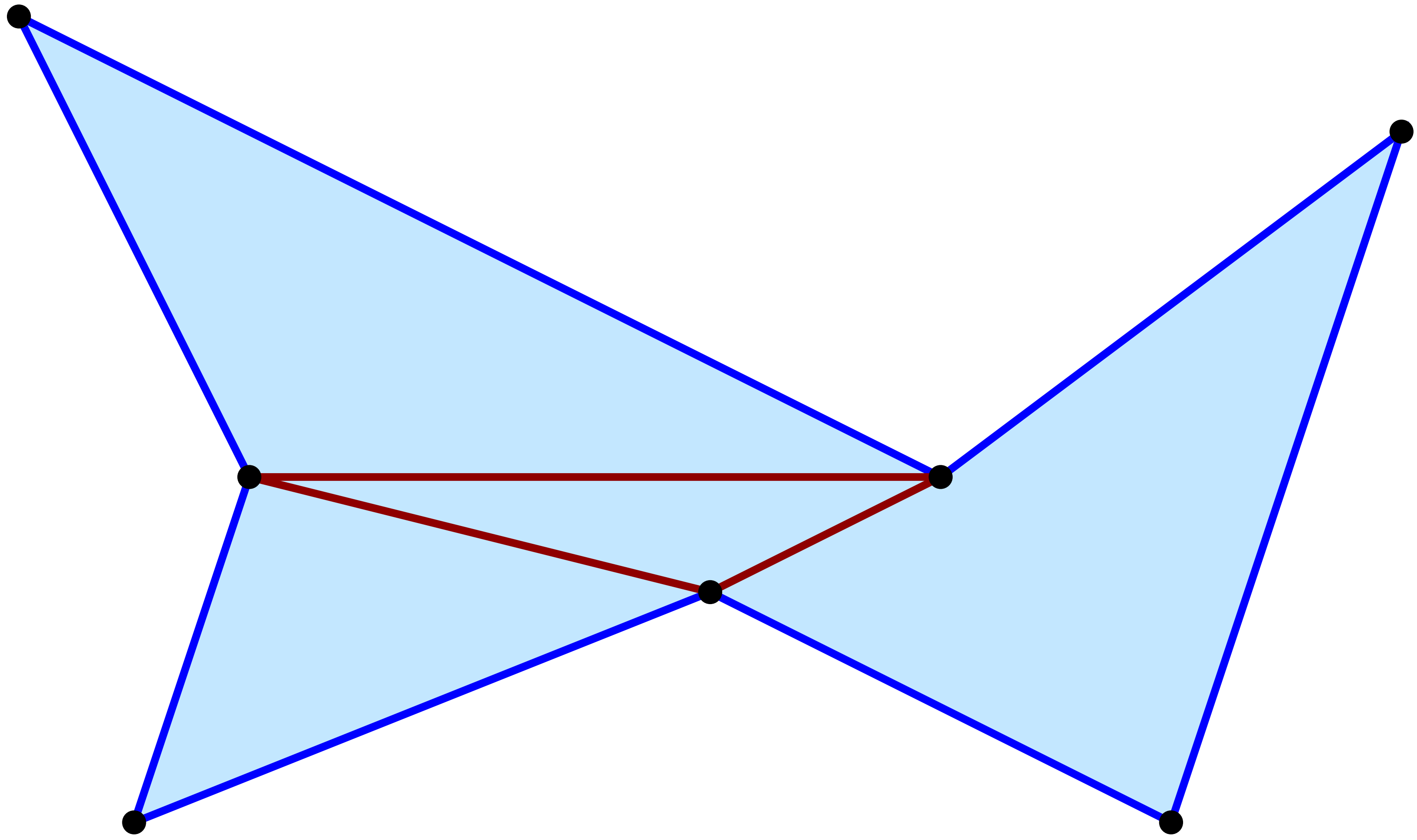
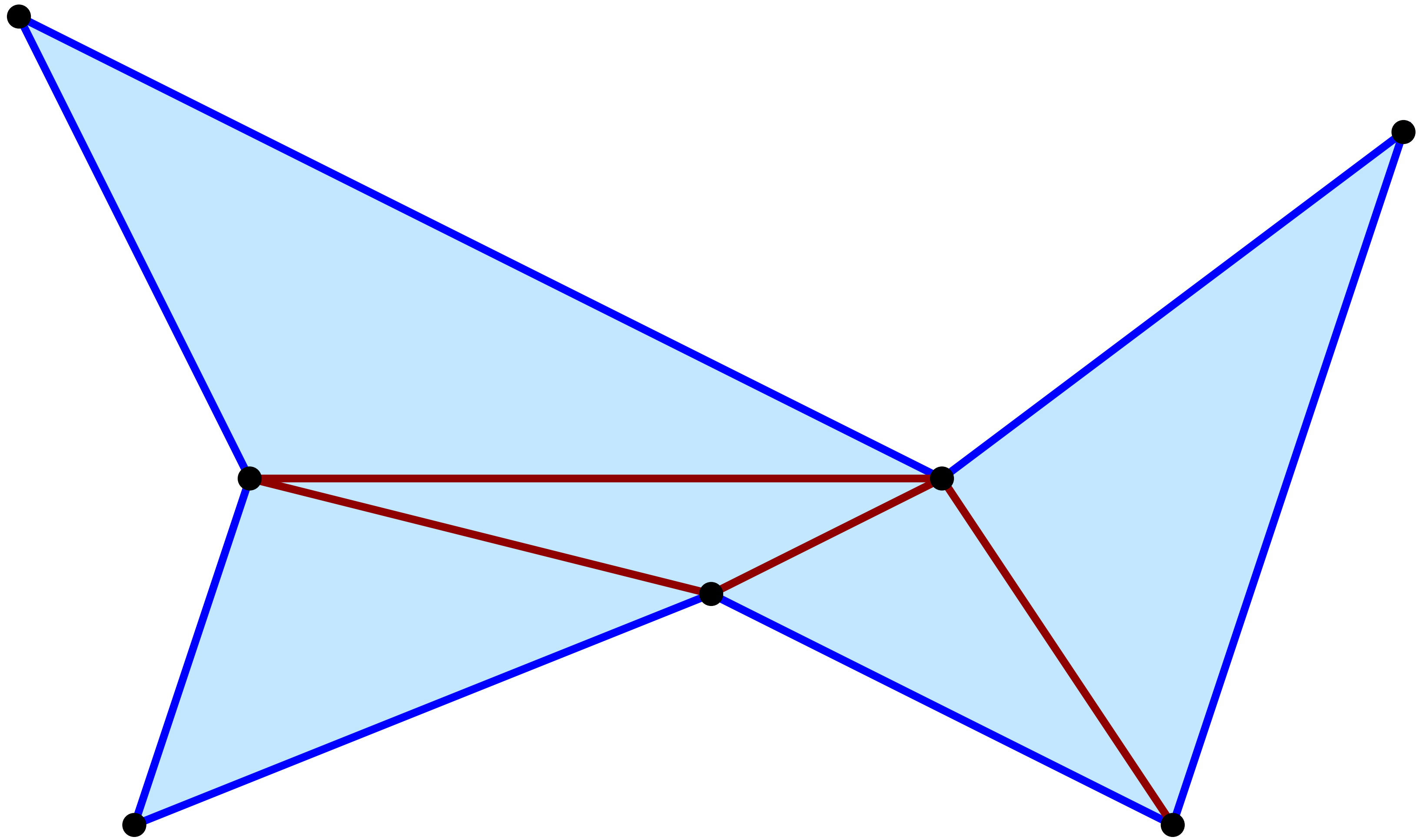


Triangulation of a polygon



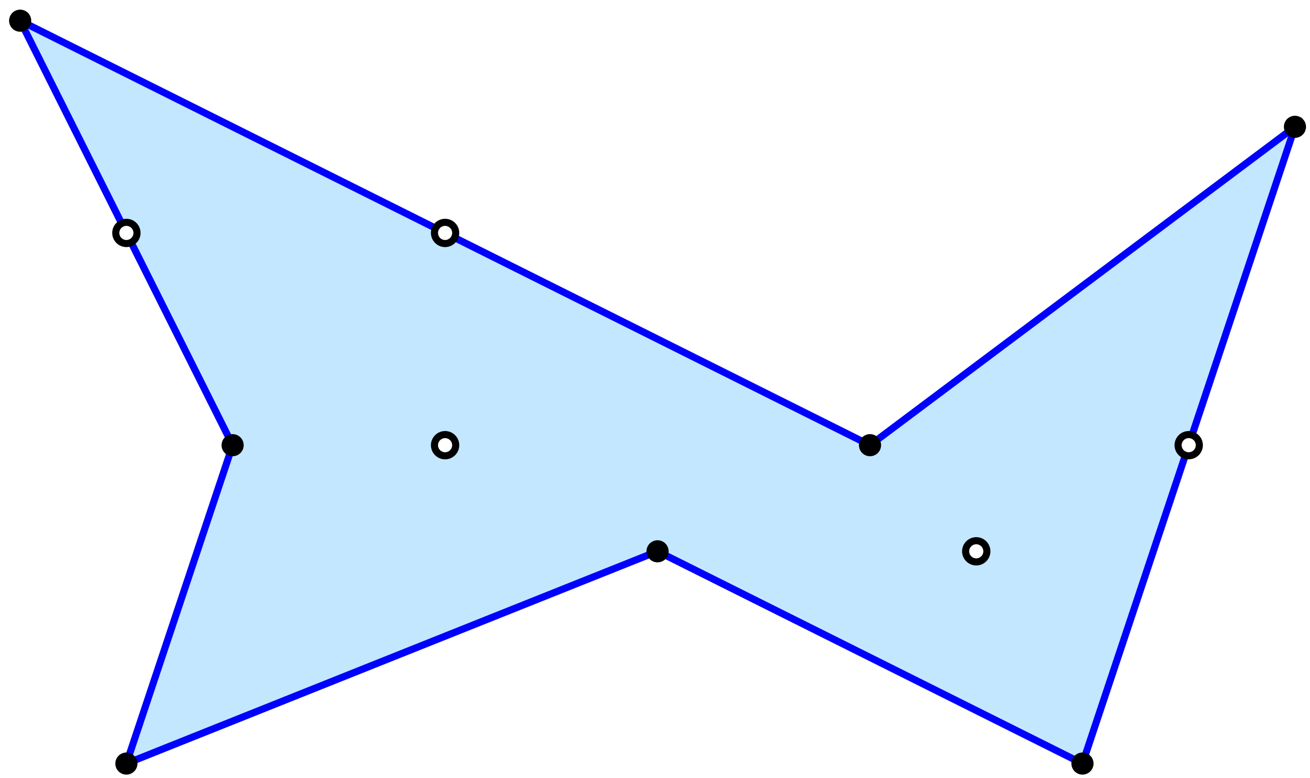
Triangulation of a polygon



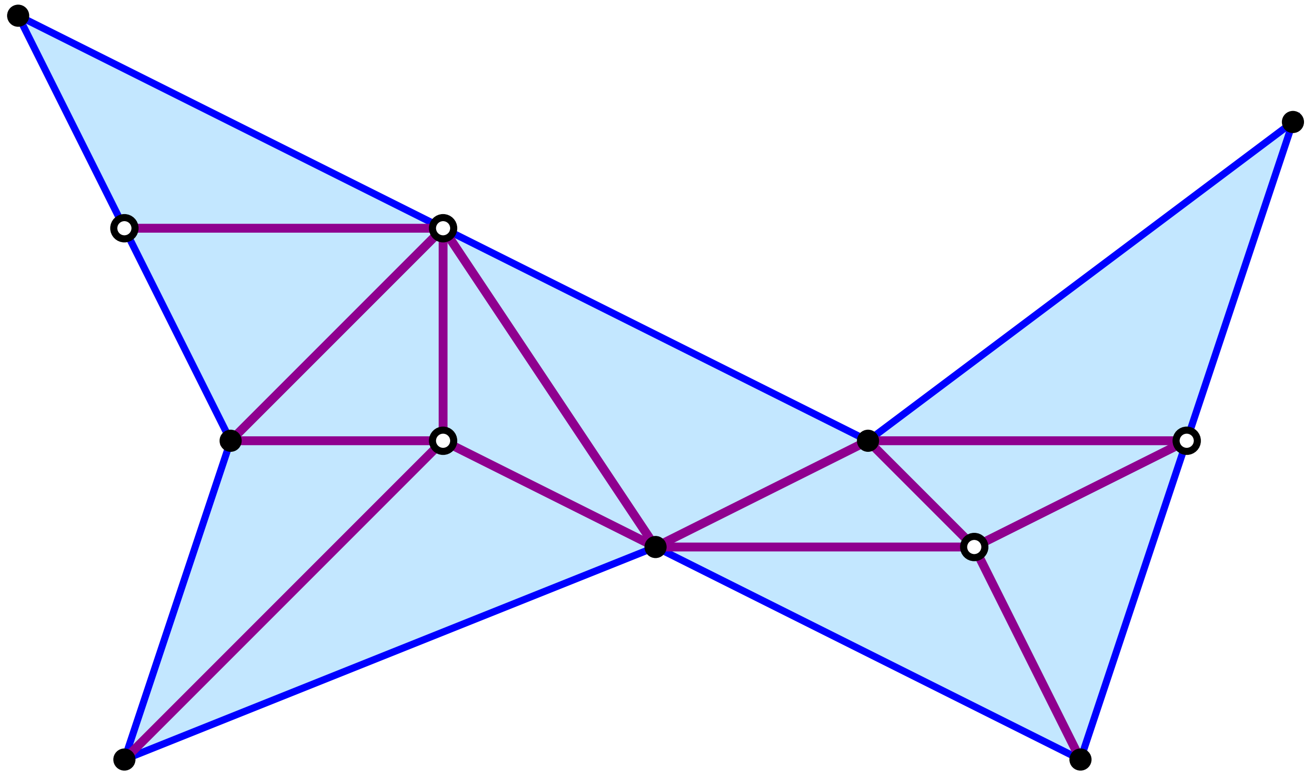
Triangulation of a polygon
using only the vertices of the polygon

There are three types of triangulations:

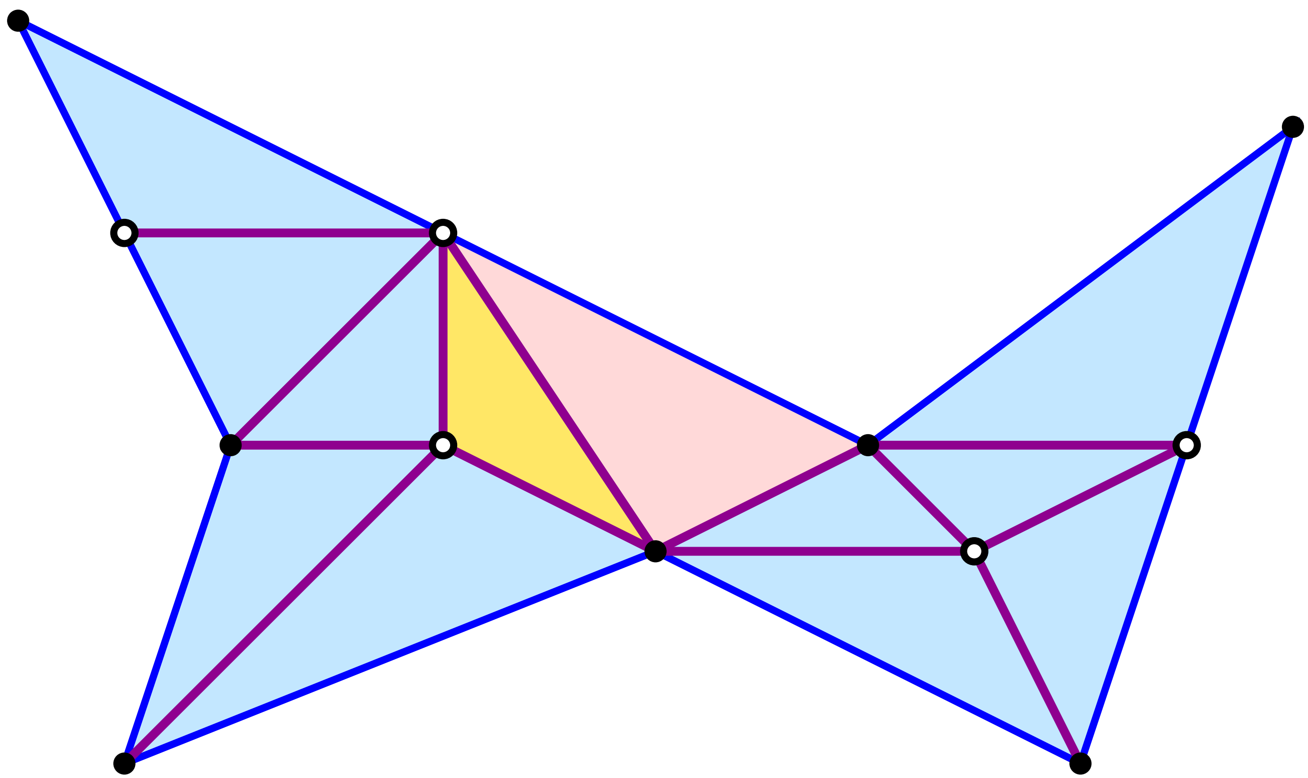
- Triangulation (no extra vertices)
- Steiner triangulation (extra vertices allowed, edges match)
- Triangular dissections (extra vertices allowed, edges need not match)



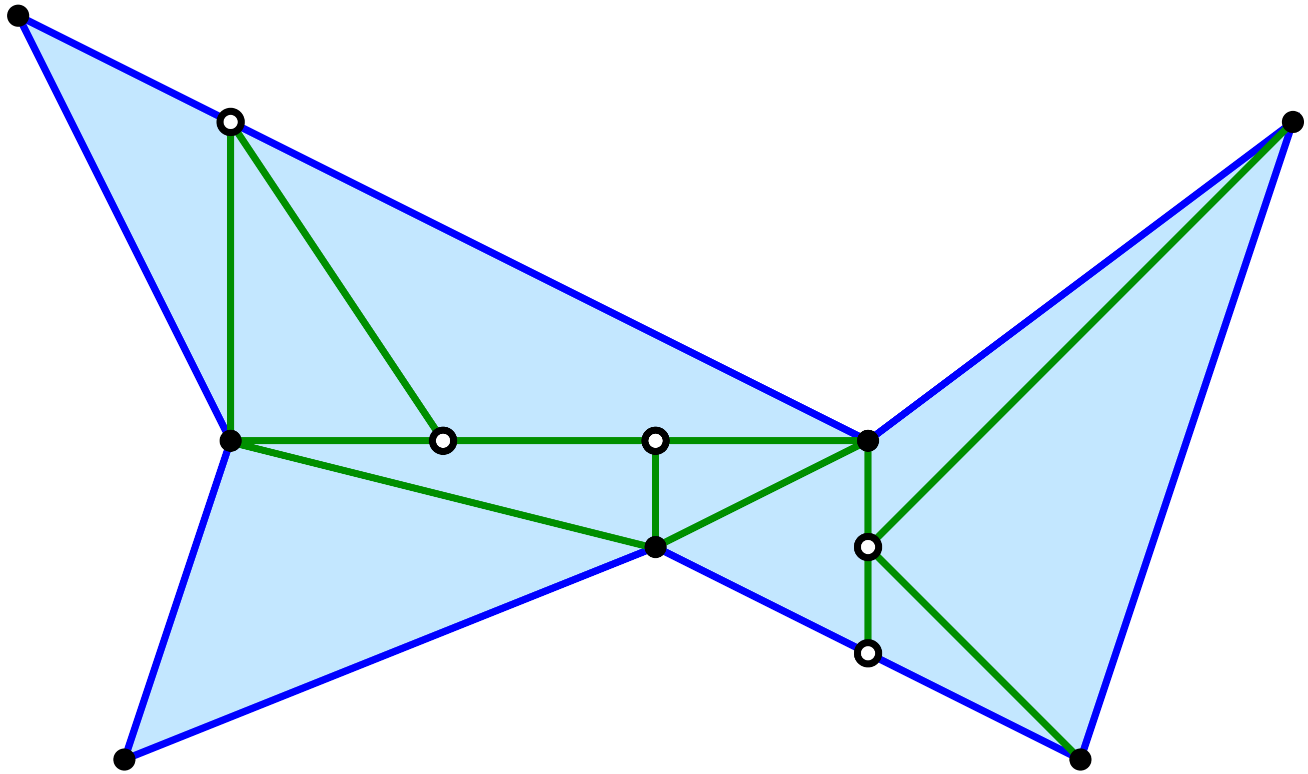
We can add extra points, called Steiner points



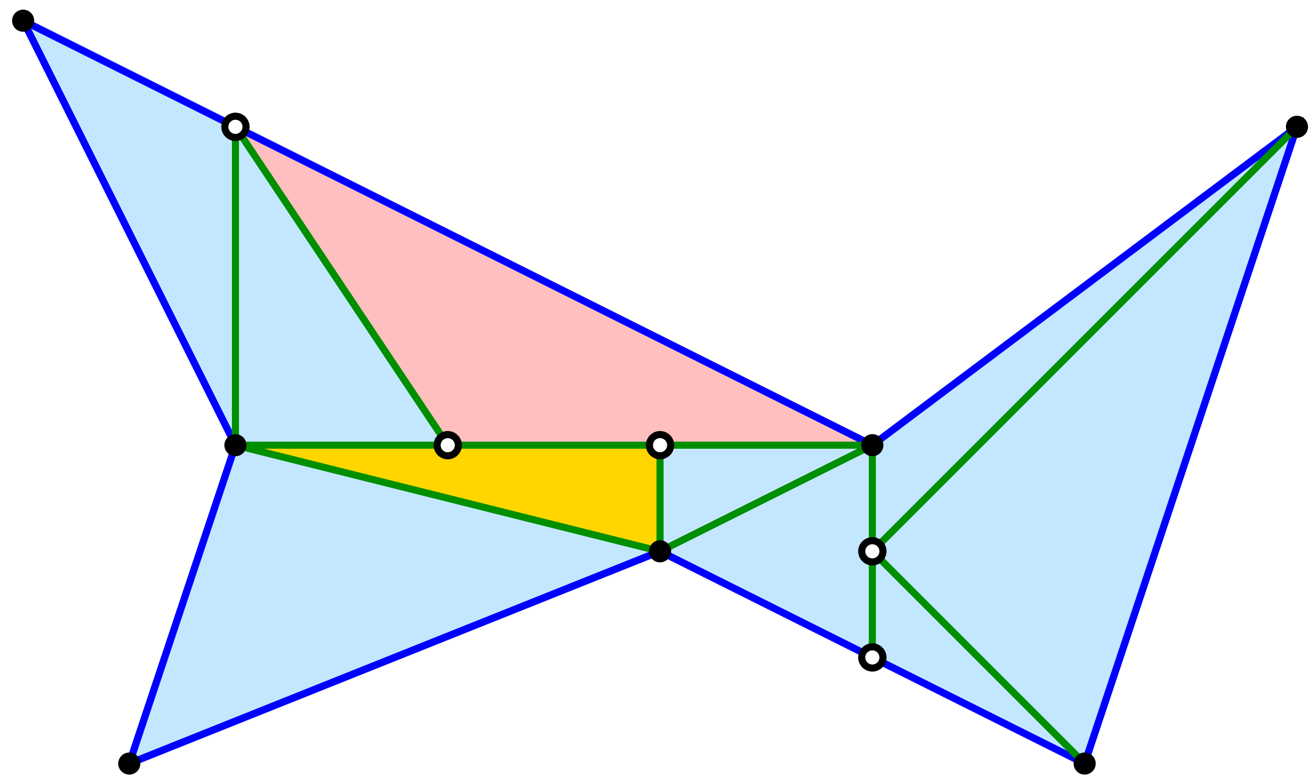
Using these, we get a Steiner triangulation.



Simplex condition: adjacent triangles have matching edges.
Most applications use this condition.

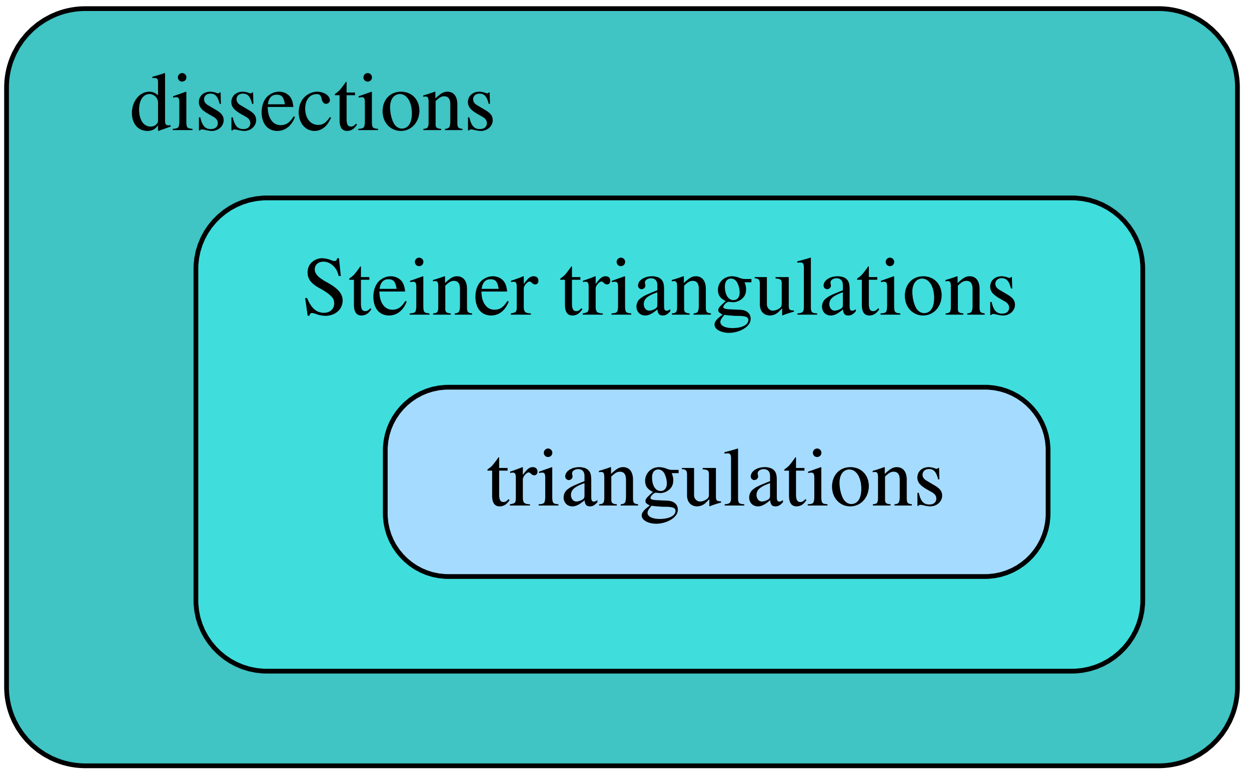


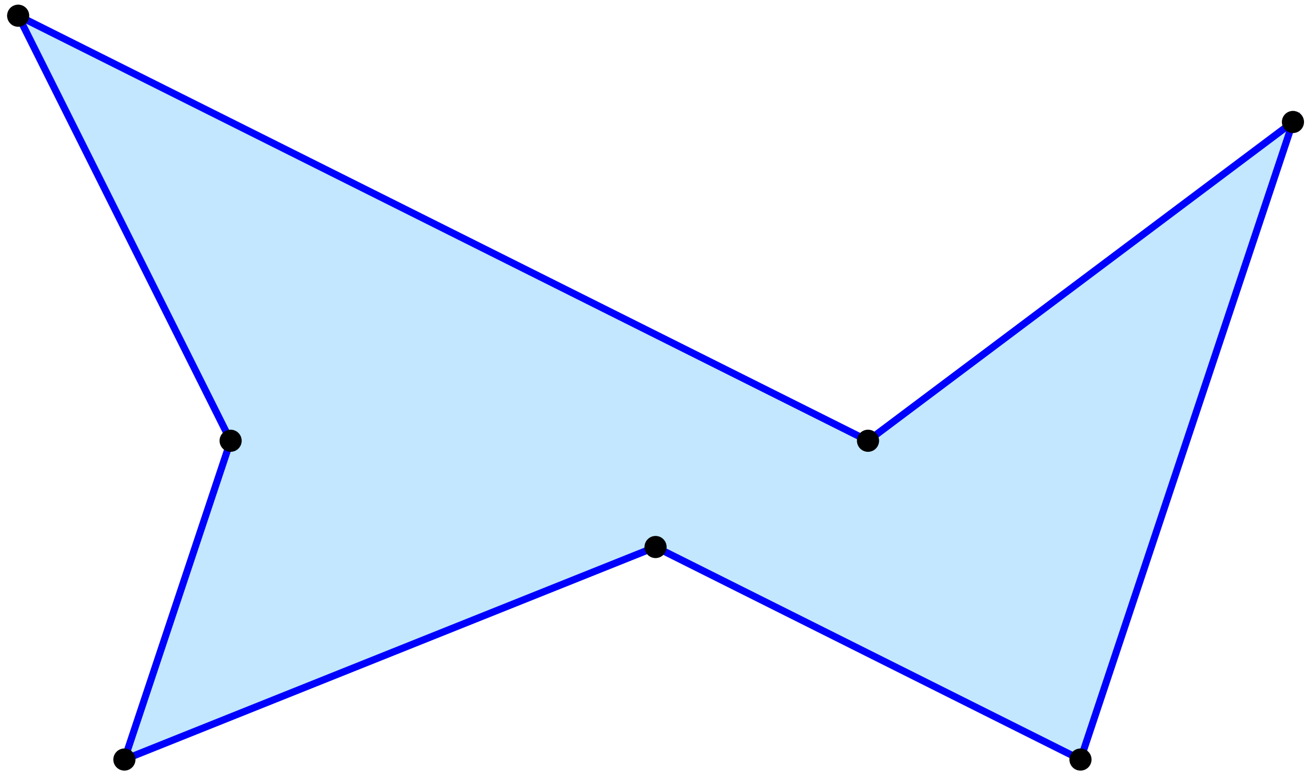
A dissection of a polygon, edges can mis-match



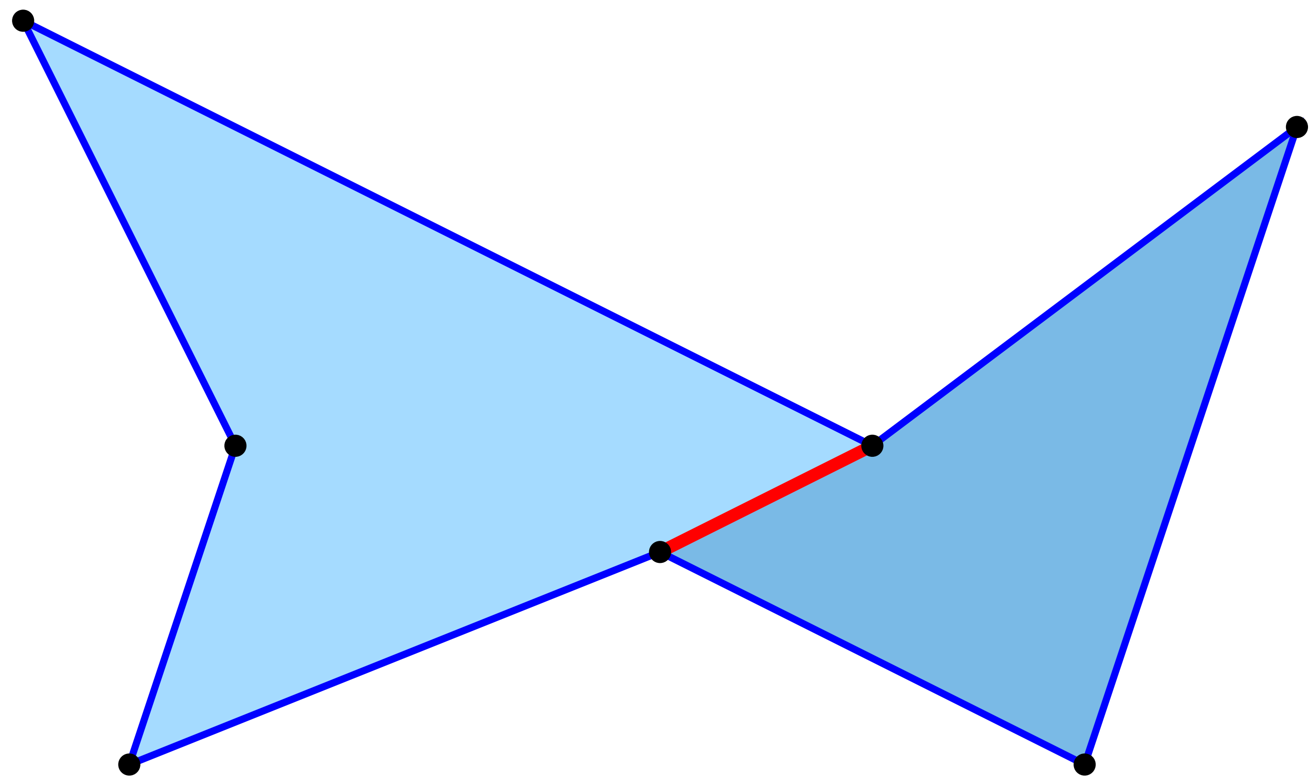
In a **triangulation**, triangles meet only at vertices or full edges.

In a **dissection**, triangles can meet along partial edges.

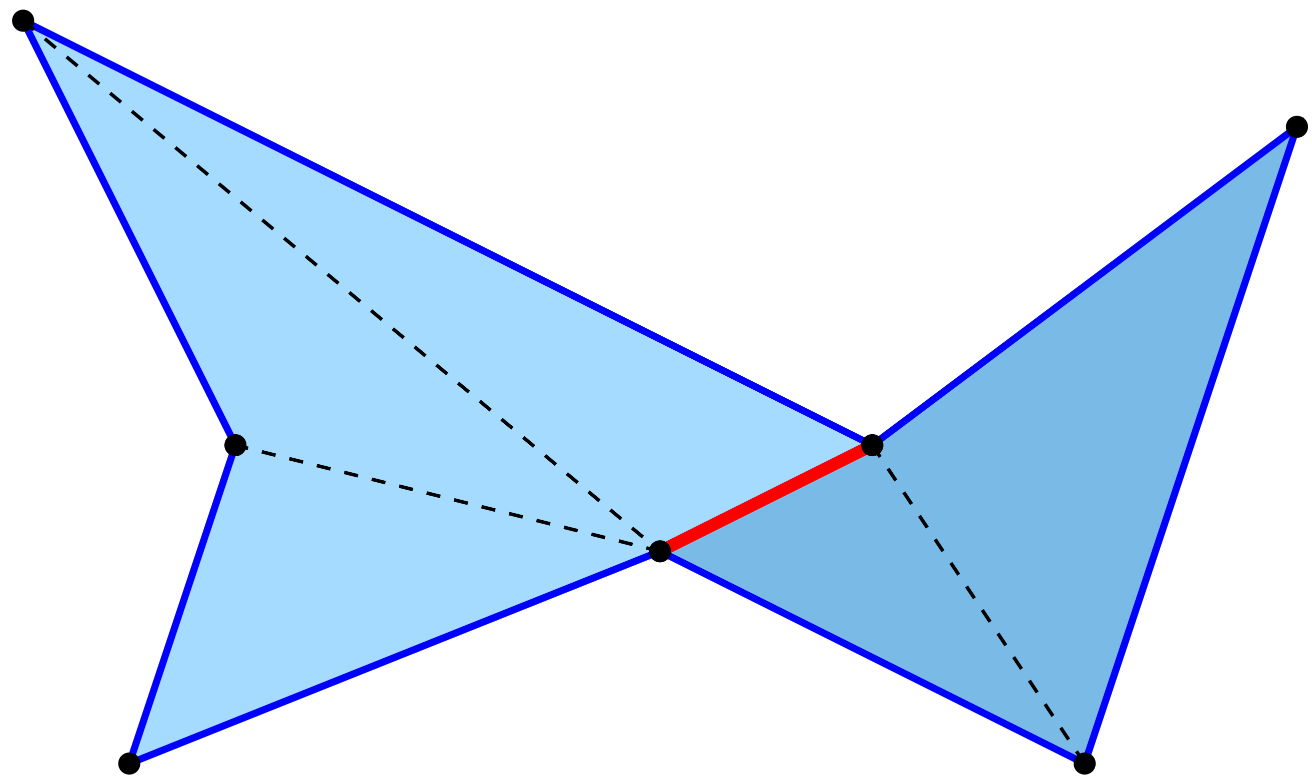

dissections
Steiner triangulations
triangulations



Does every polygon have a triangulation? (no extra vertices)



Yes. Can prove by induction on number of sides N.
$N = 3$ is obvious. For $N > 3$, it suffices to find one diagonal.



This cuts polygon into two pieces each with fewer sides.
Each piece can be triangulated by the induction hypothesis.

Proof that a diagonal always exists:

Claim: every polygon has an interior angle $< 180°$.

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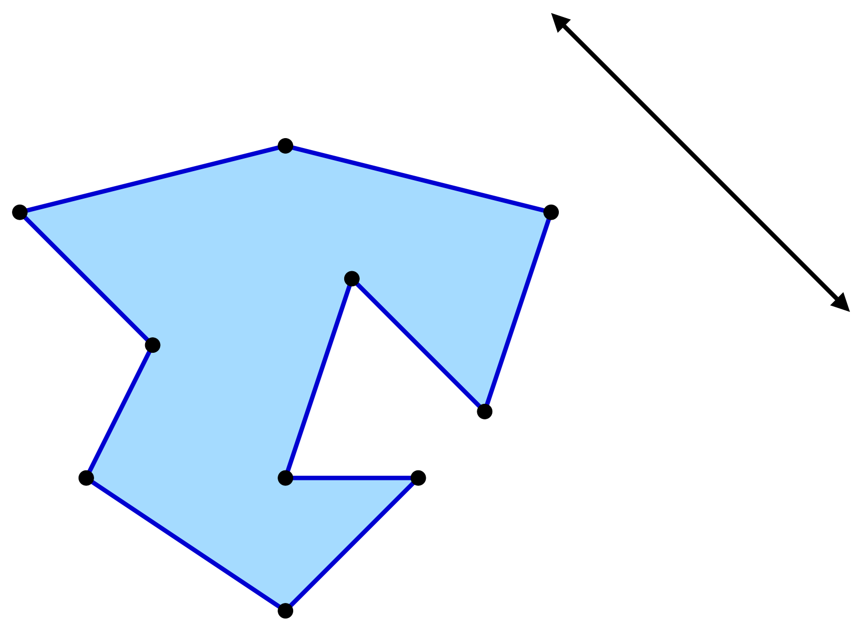
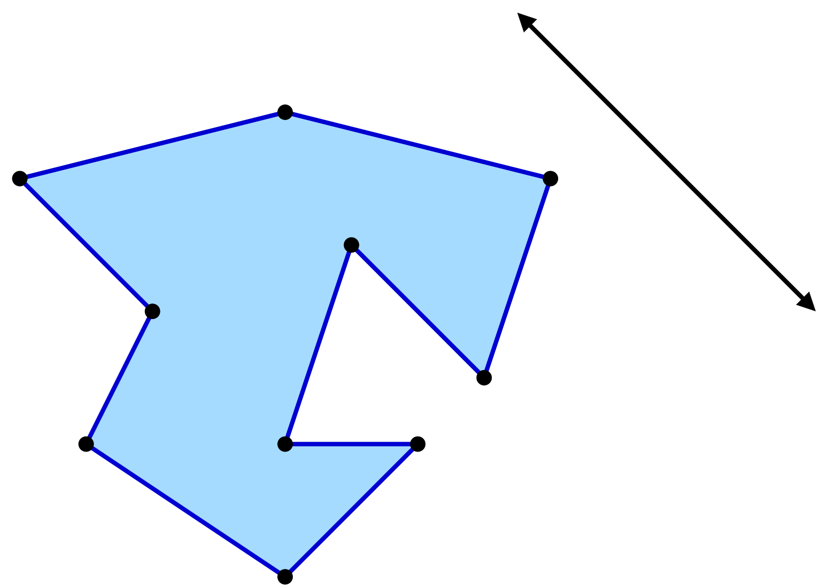


Choose line outside P and not parallel to any side.

Proof that a diagonal always exists:

Claim: every polygon has an interior angle $< 180°$.



Choose line outside P and not parallel to any side.
Slide the line until it first hits P.

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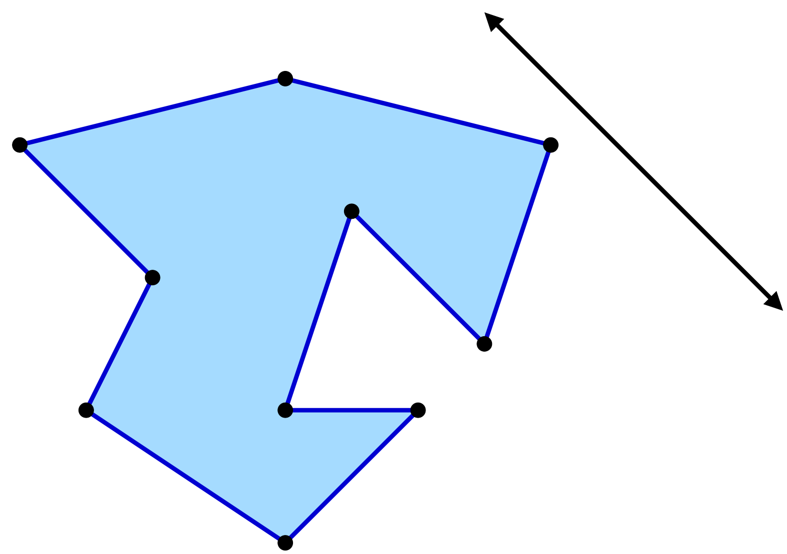

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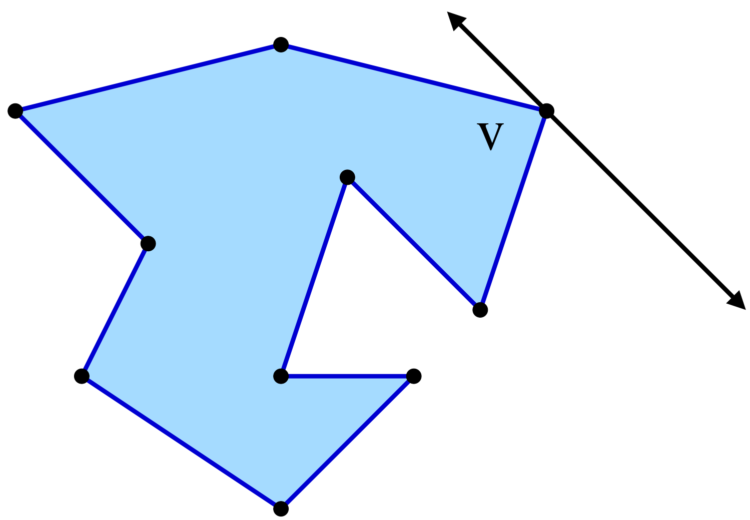
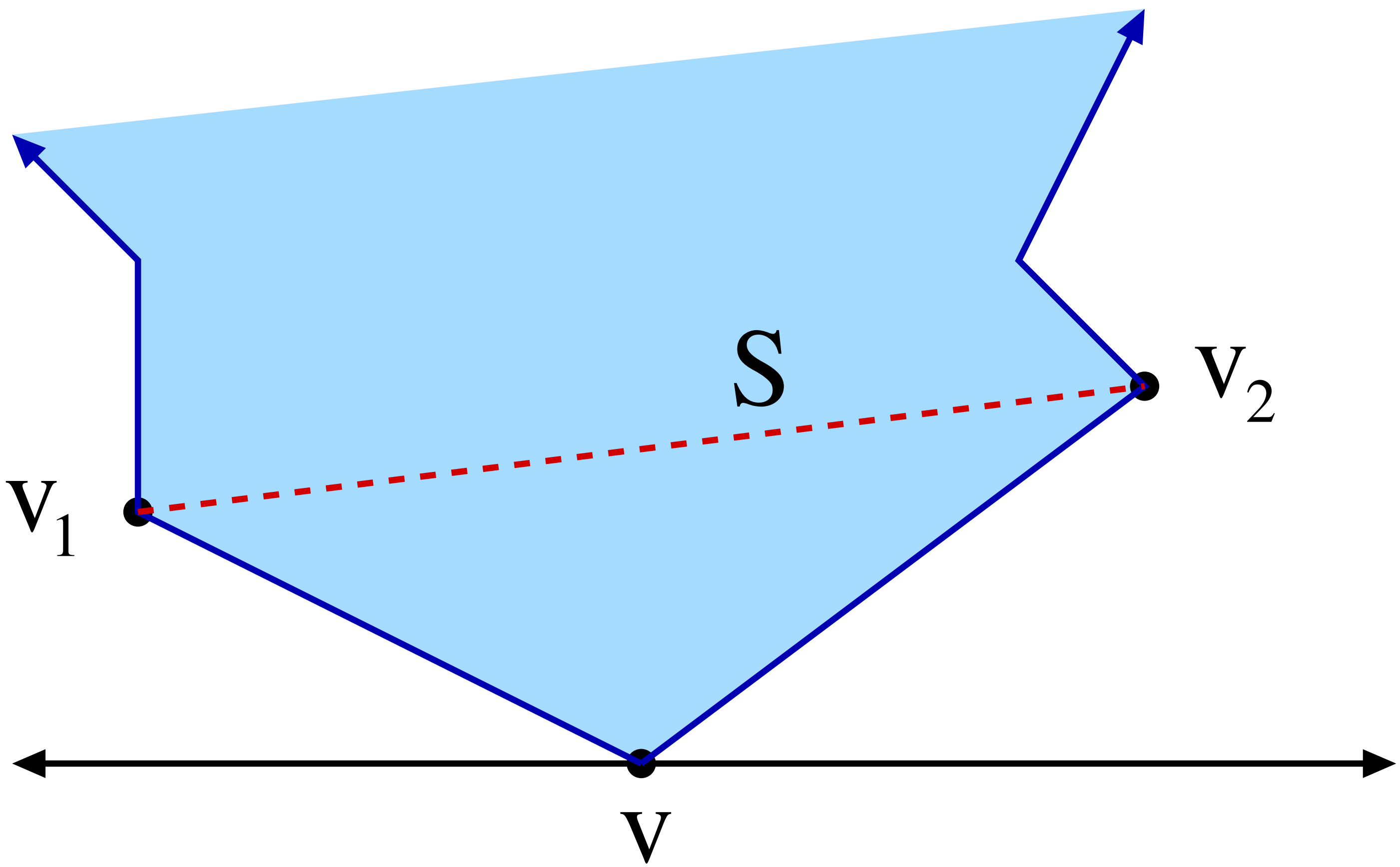


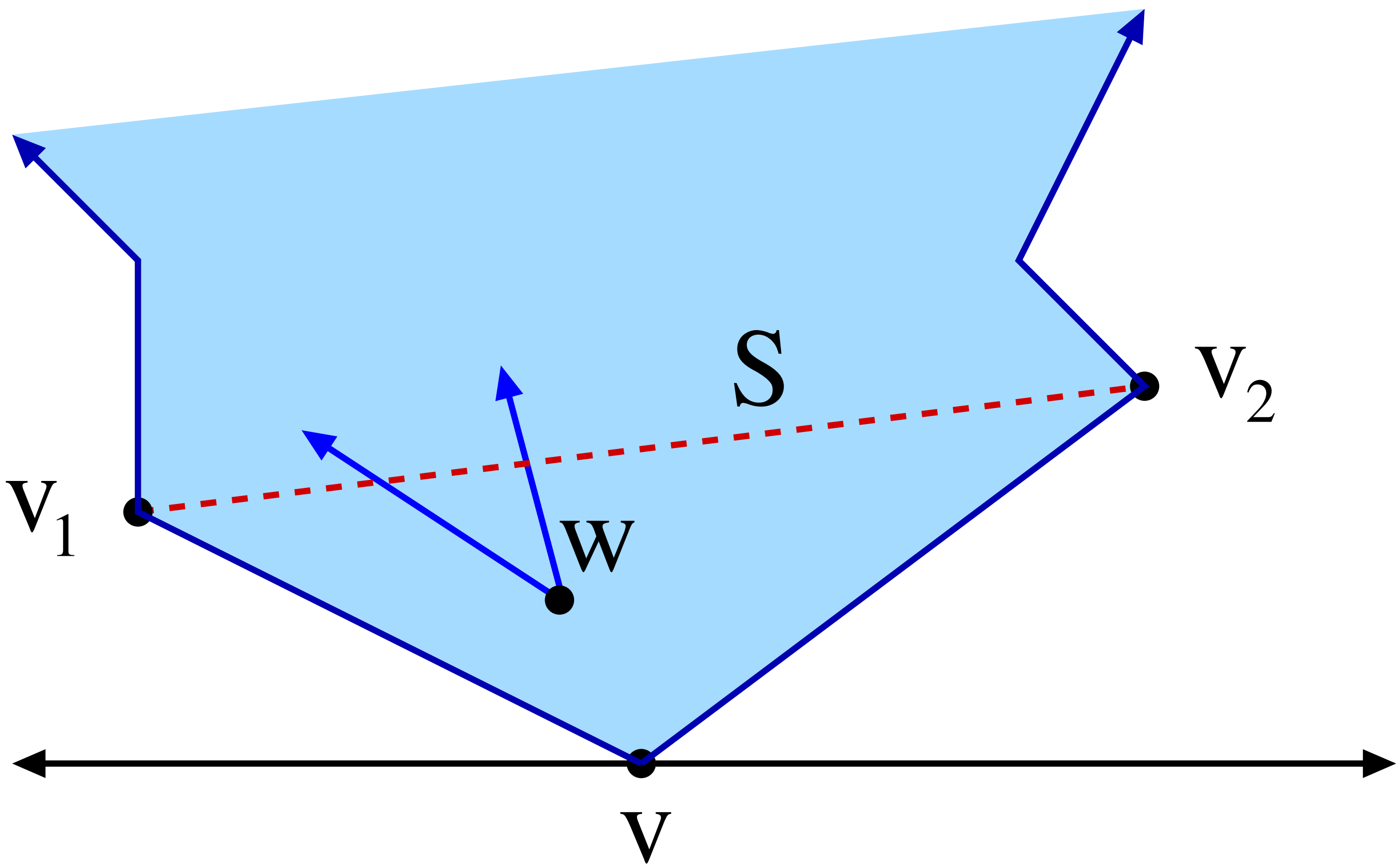
Choose line outside P and not parallel to any side.
Slide the line until it first hits P.
Intersection must contain a vertex v with angle $< 180°$.

Proof that a diagonal always exists, Case 1:



Connect vertices v_1, v_2 adjacent to v by a segment S.
If this segment S inside P, we are done.

Proof that a diagonal always exists, Case 2:



Otherwise, segment $[v_1, v_2]$ hits other edges.

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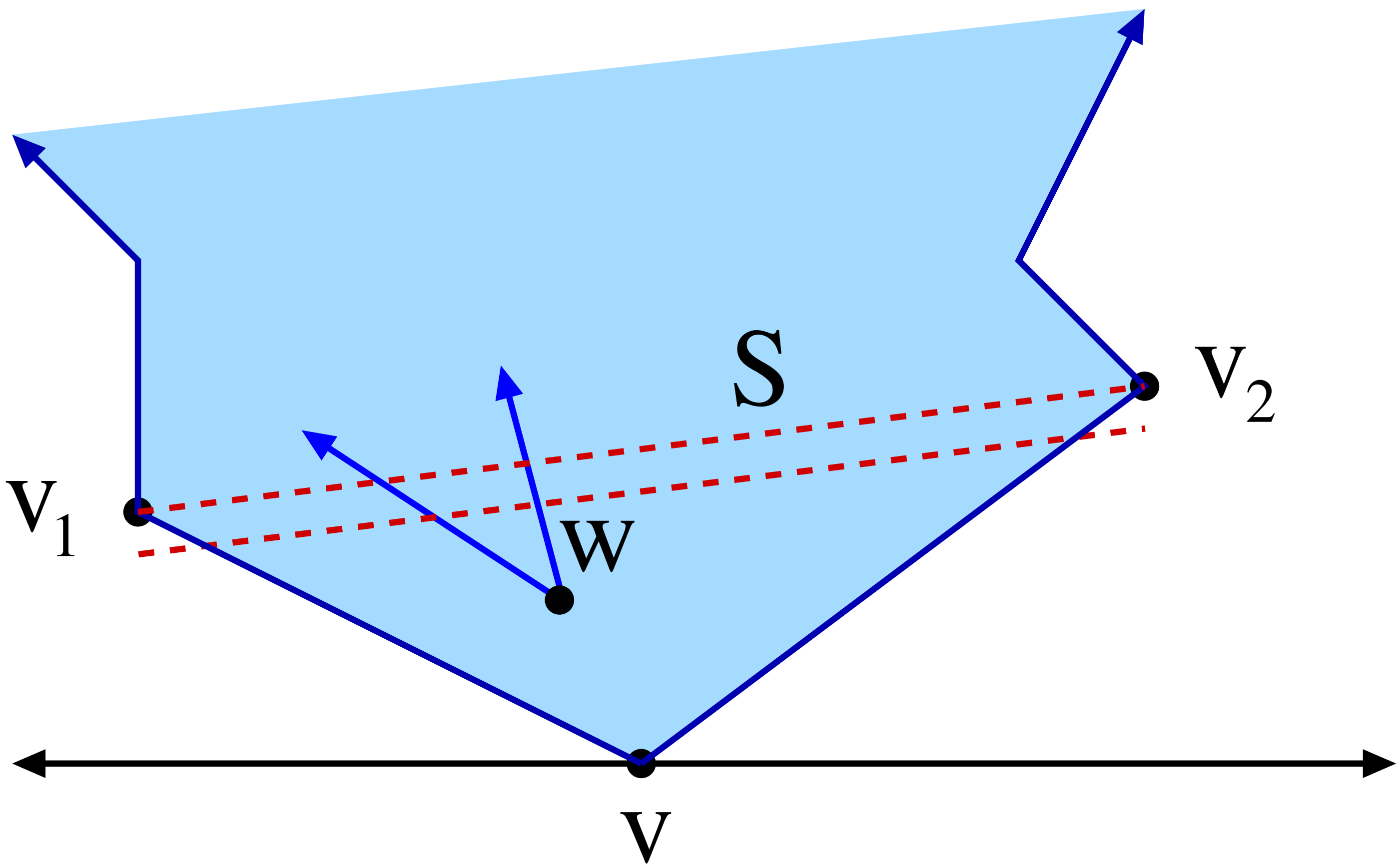

Move S towards v, parallel to itself,
until last time it hits an edge not adjacent to v.

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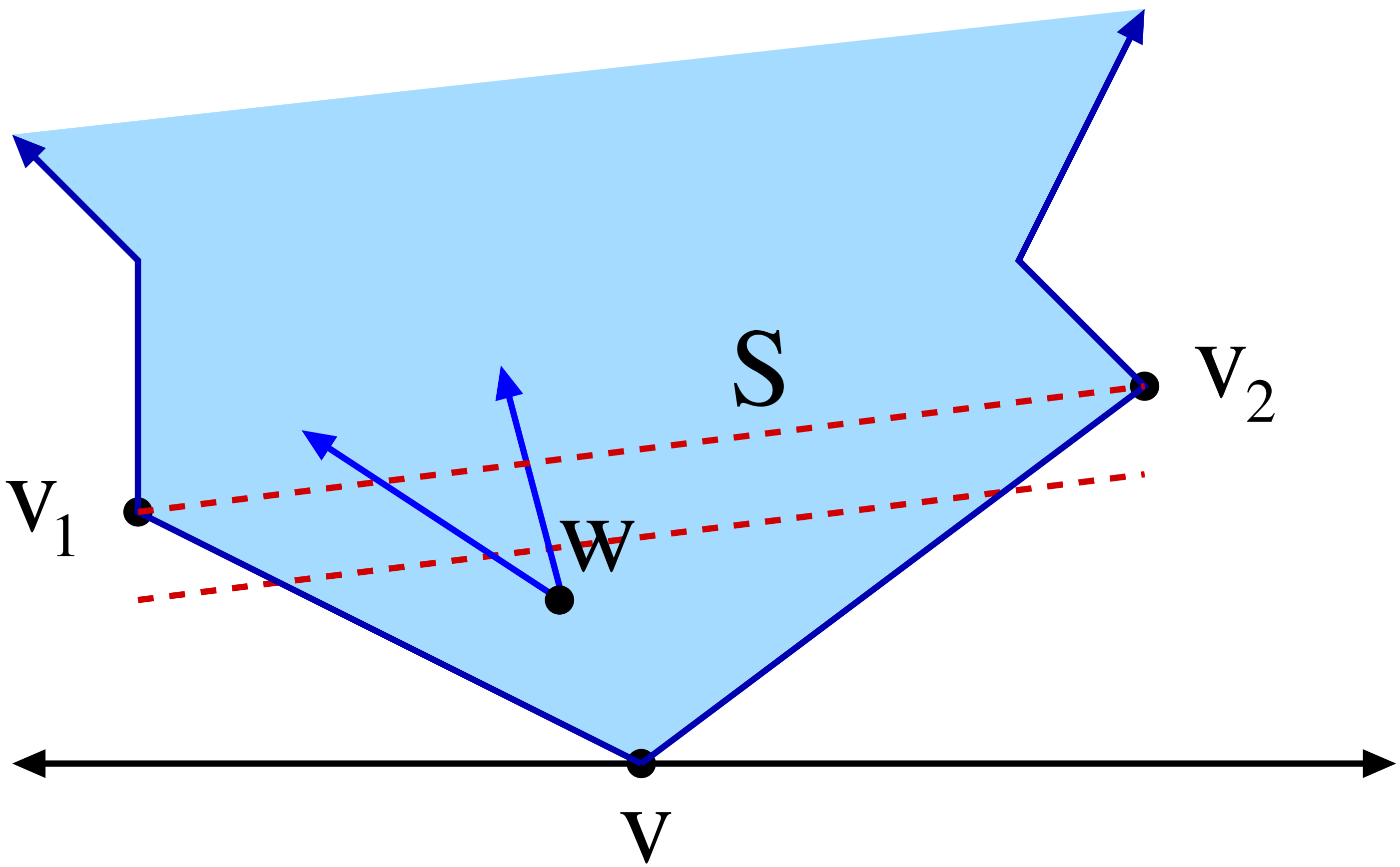

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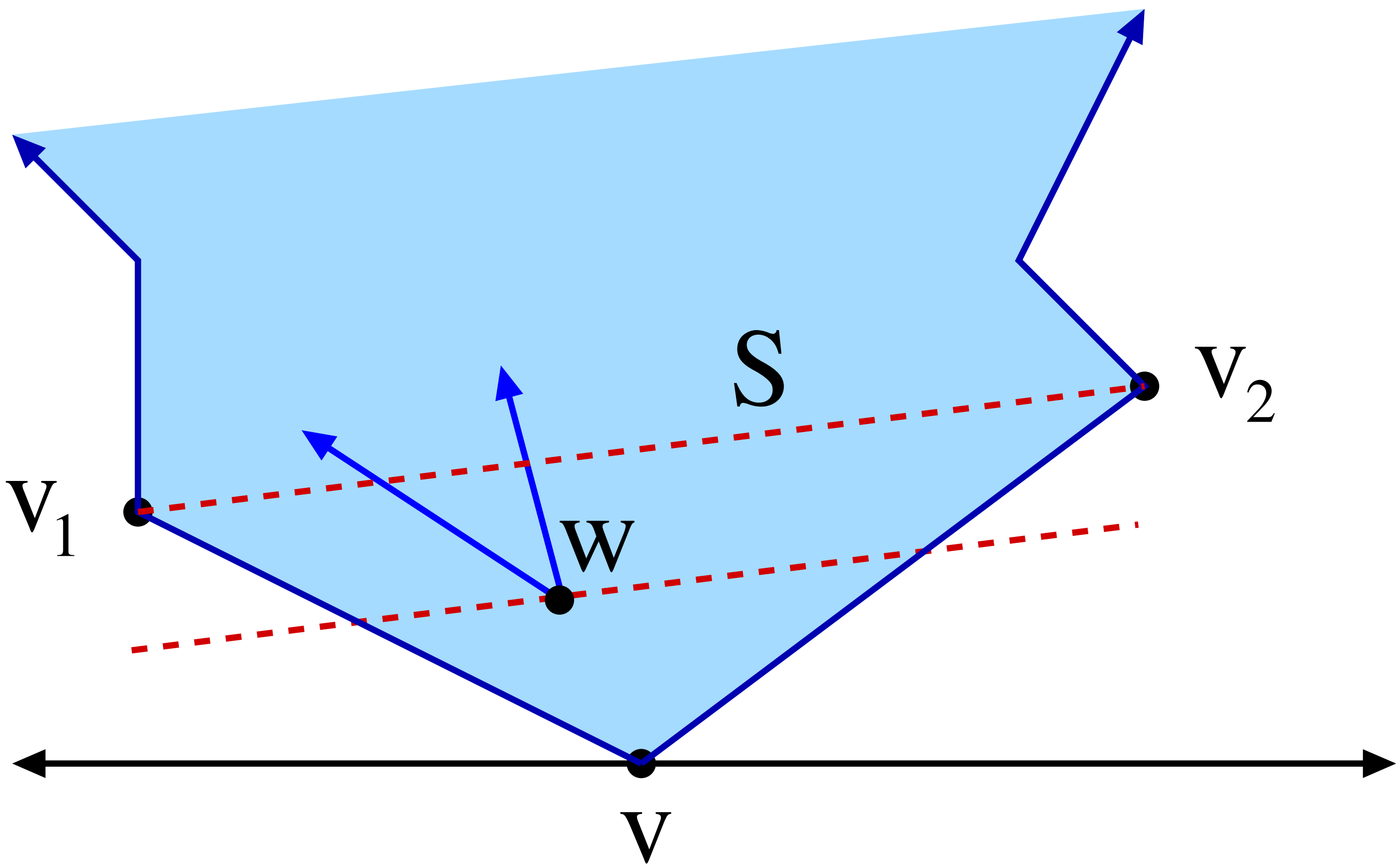

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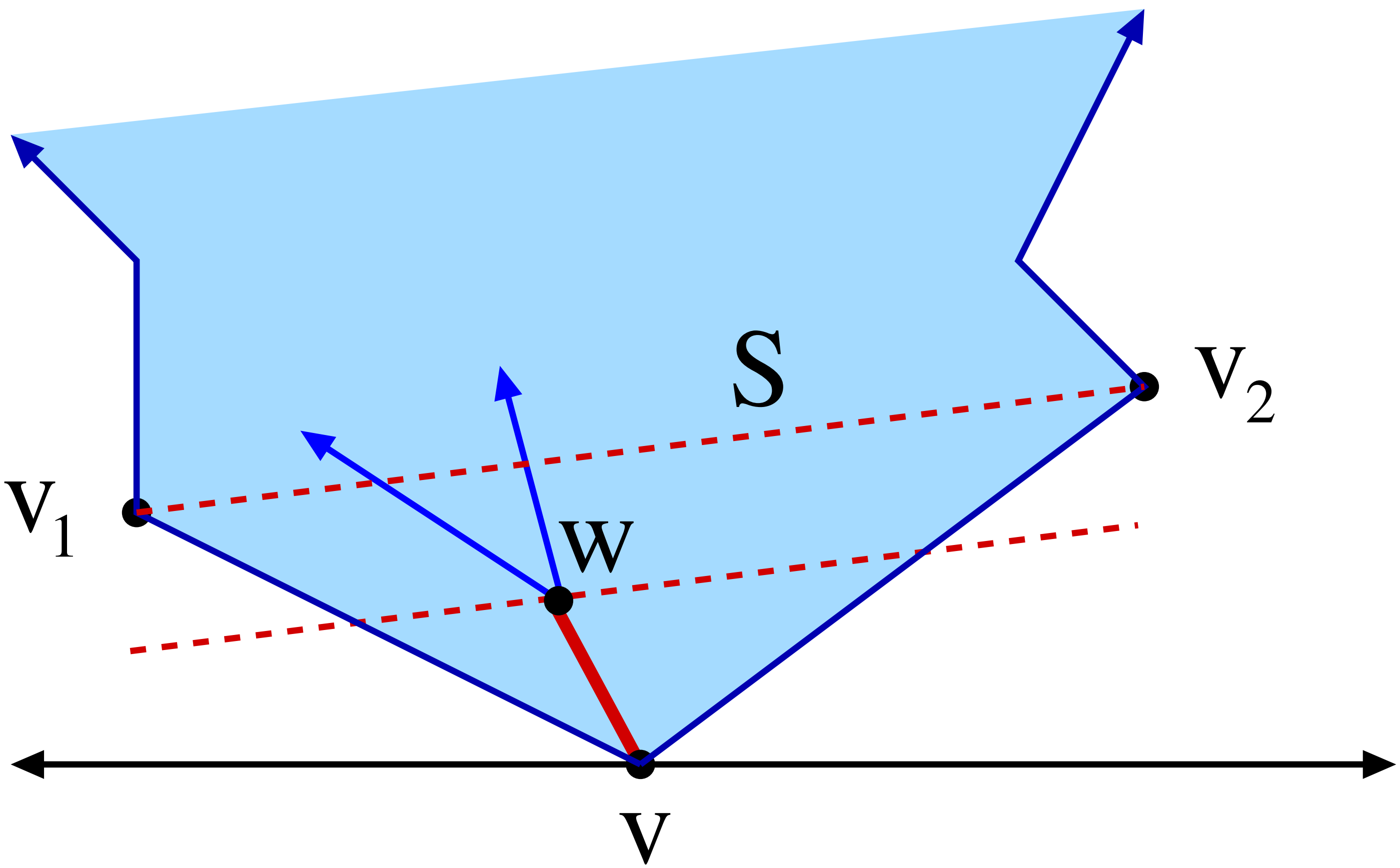
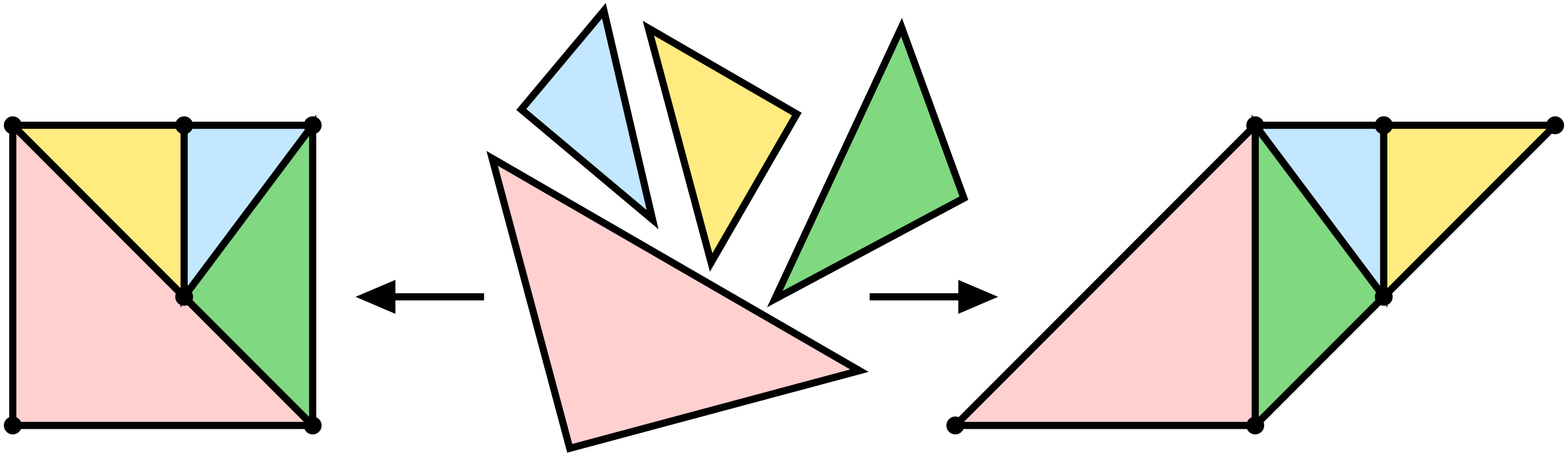
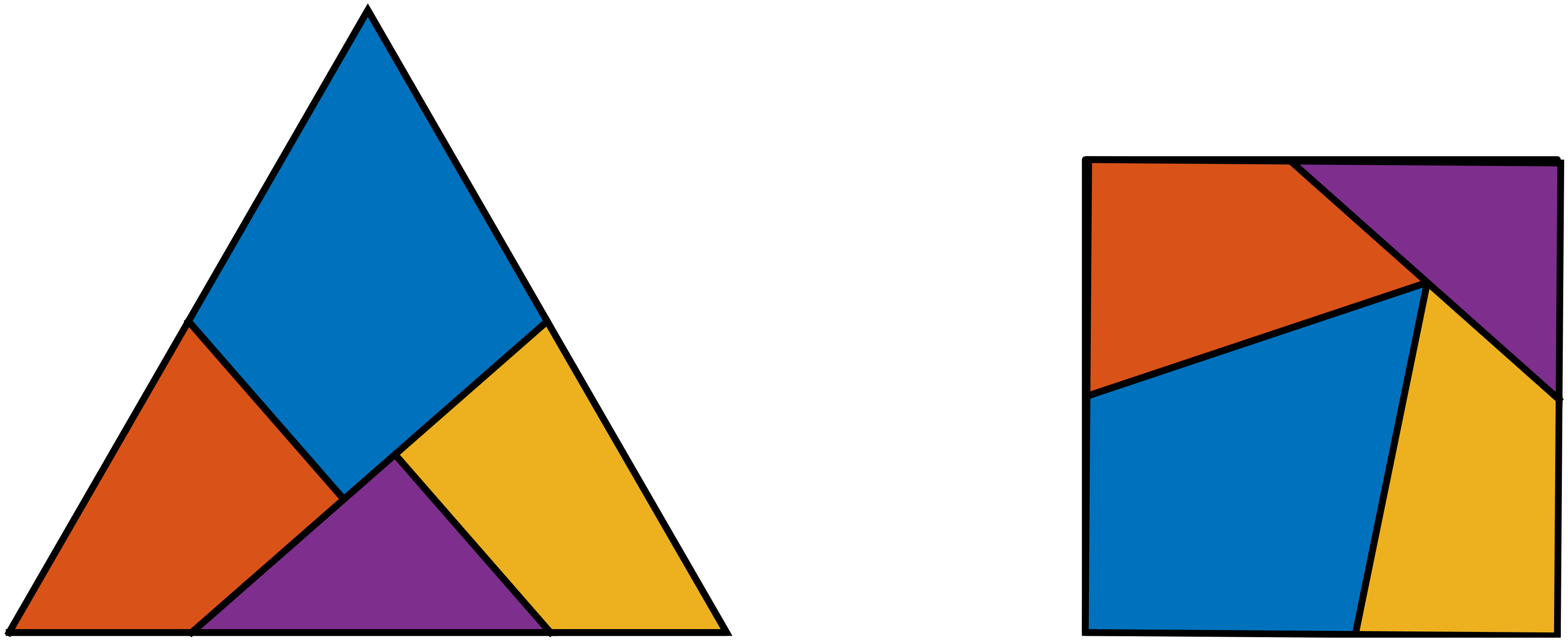


Then S contains a vertex w that can be connected to v. QED



Two polygons cut into same set of triangles must have same area.

Wallace–Bolyai–Gerwien Theorem (early 1800's):
Any two polygons with the same area have dissections into similar sets of sub-polygons (up to rotations and translations).



An equi-dissection of an equilateral triangle and a square.

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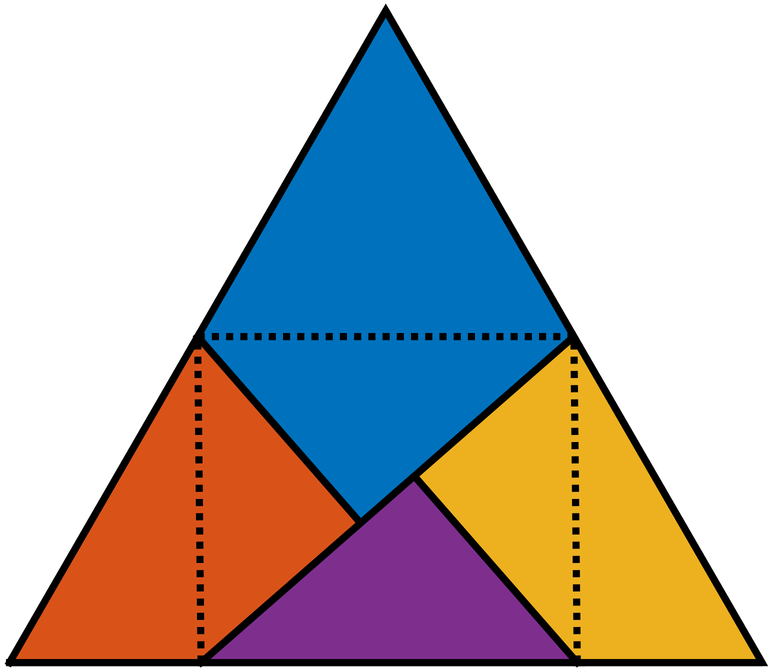
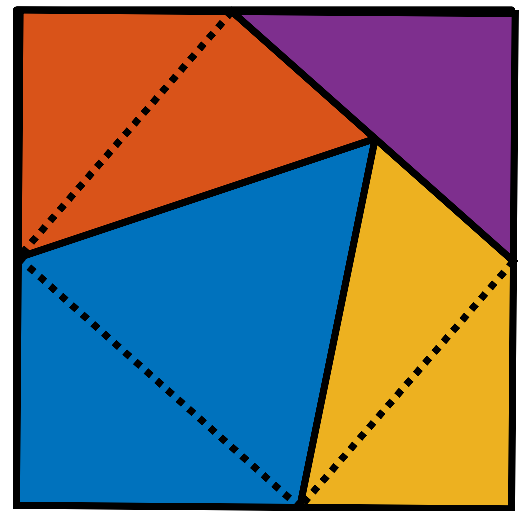




An equi-dissection of an equilateral triangle and a square.

Who proved the WBG theorem?

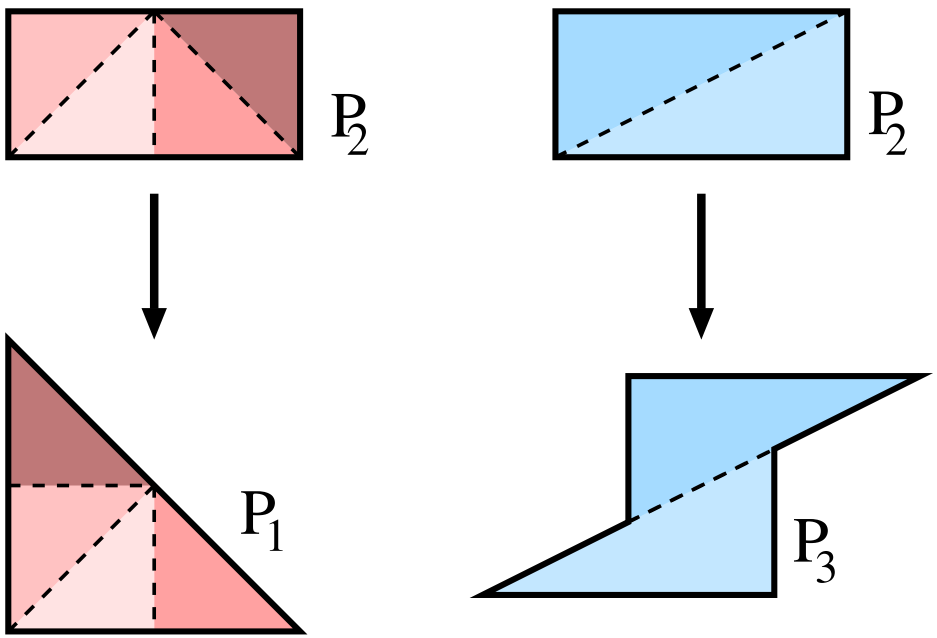
According to Stewart the Wallace-Bolyai-Gerwien theorem was proven by William Wallace around 1808, and independently by Paul Gerwien in 1833 in response to a question of Wolfgang Bolyai.

Giovannini credits John Lowry with a solution in 1814, answering a question of Wallace, with independent proofs by Bolyai (somewhat sketchy) in 1831 and by Gerwien (very detailed) in 1833.

I. Stewart, *From here to infinity*, The Clarendon Press, 1996.

E.N. Giovannini, David Hilbert and the foundations of the theory of plane area, *Arch. Hist. Exact Sci.*, vol 75, 2021, pages 649–698.

Lemma: If P_1 and P_2 have an equi-dissection, and if P_2 and P_3 have an equi-dissection, then so do P_1 and P_3. (Equi-dissection is transitive.)



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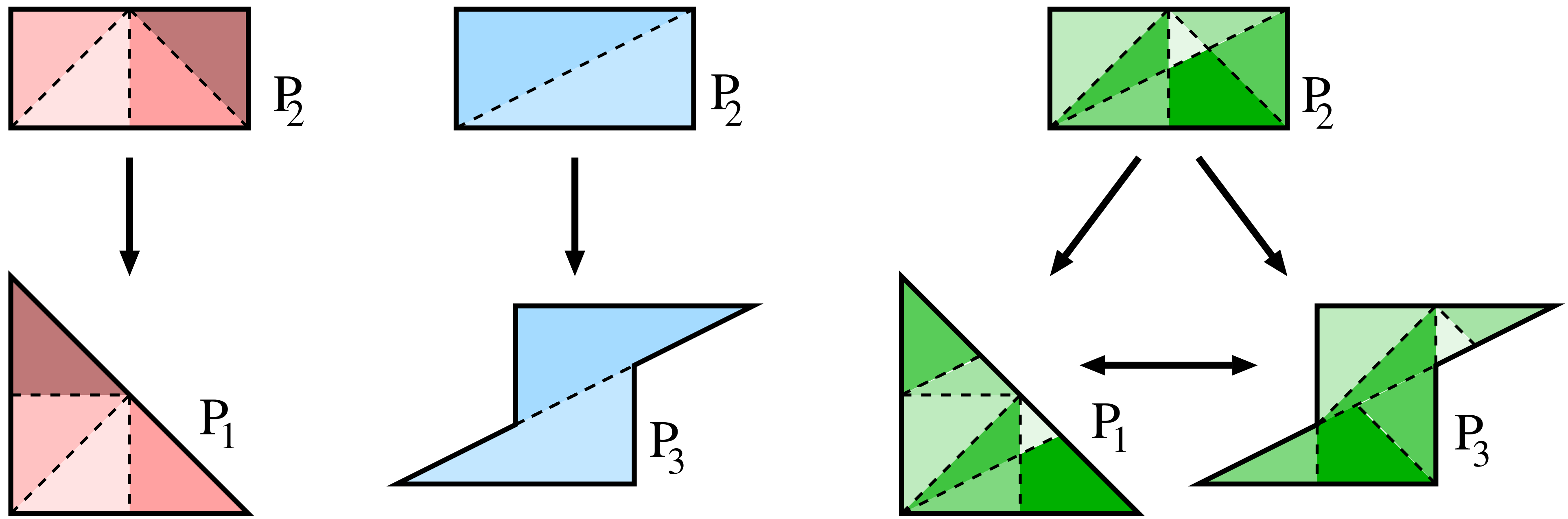
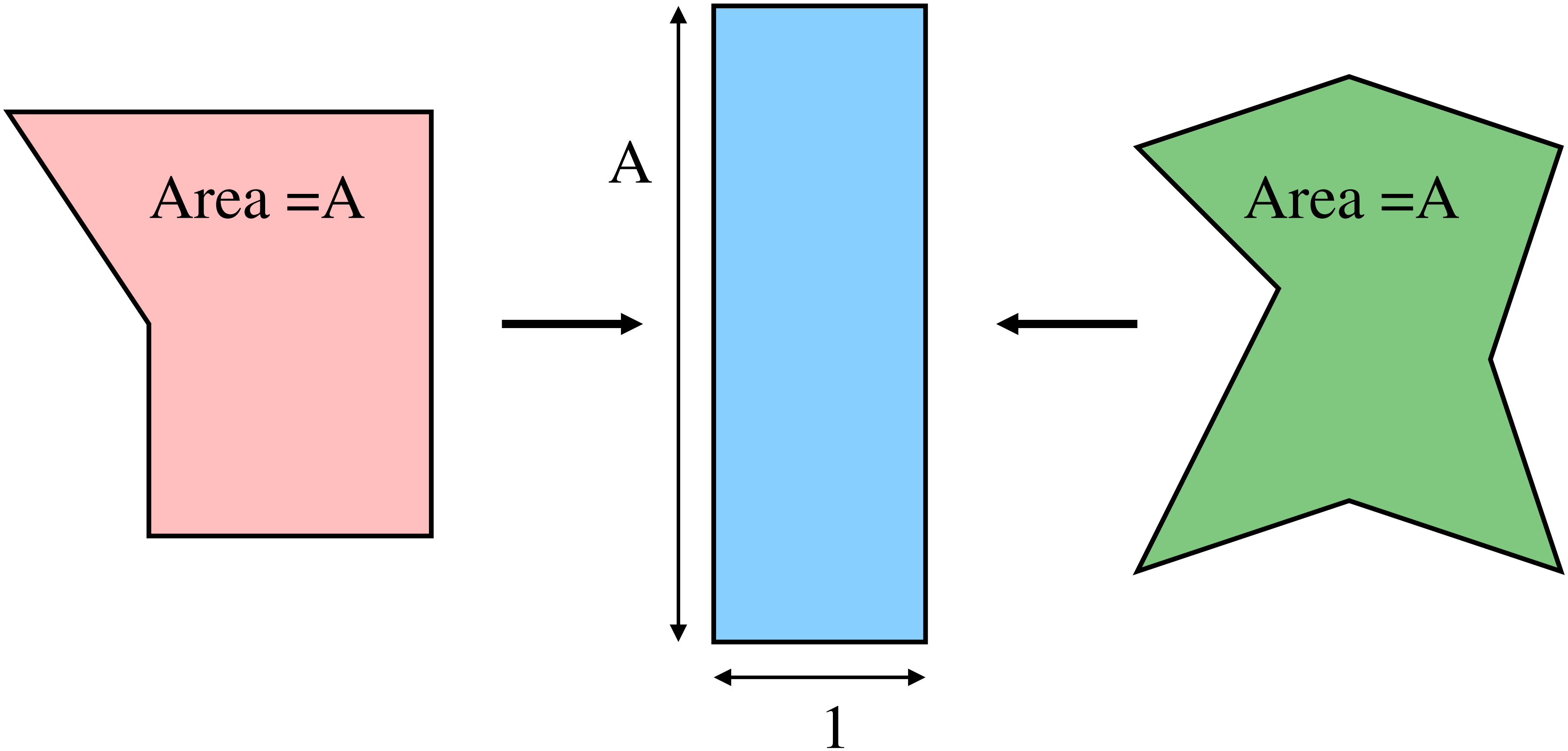


Proof: Intersect the two dissections of P_2. Smaller pieces can be rearranged into either P_1 or P_3.

Proof of WBG theorem.

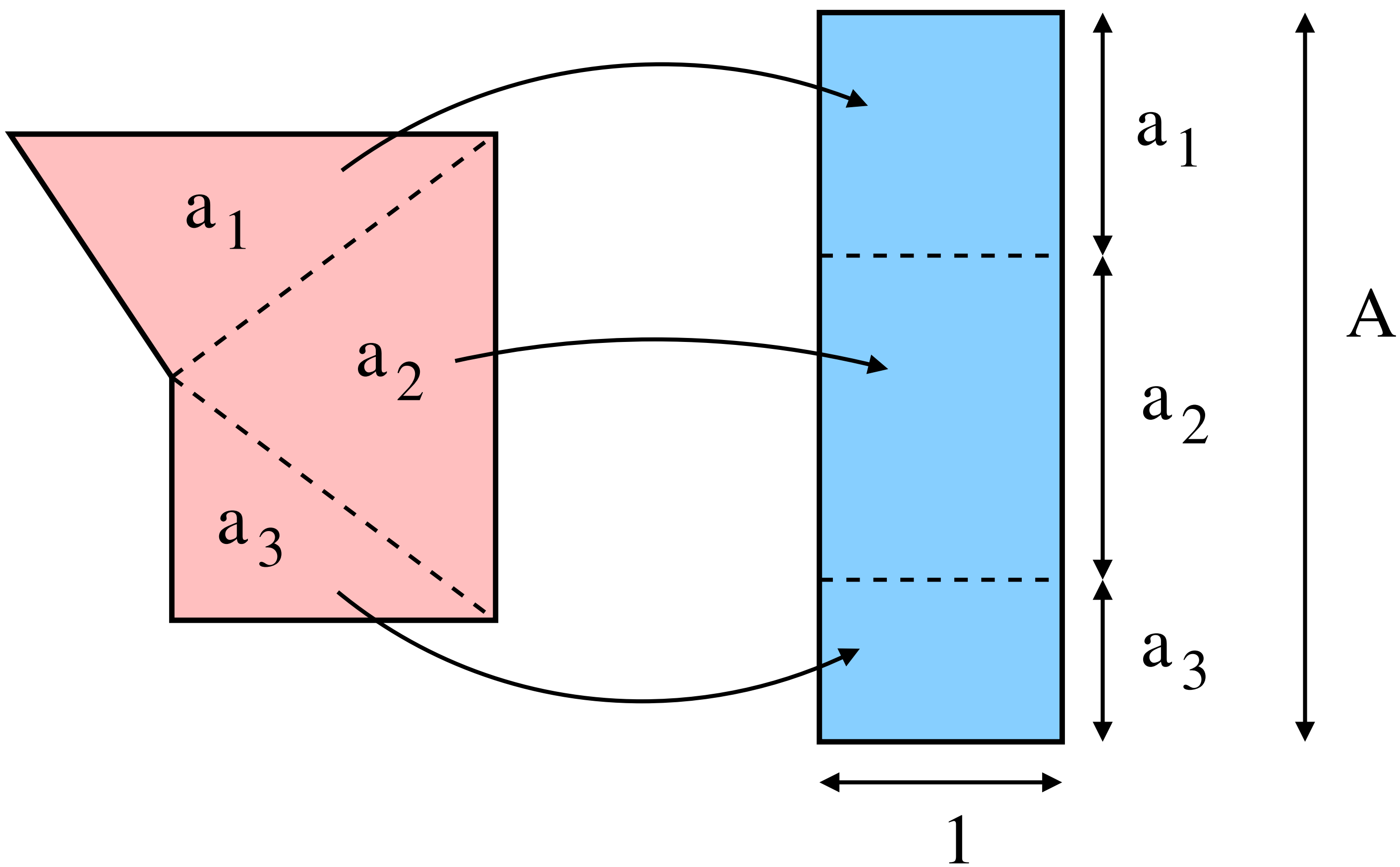
Enough to show a polygon of area A is equivalent to a $1 \times A$ rectangle.



Proof of WBG theorem.

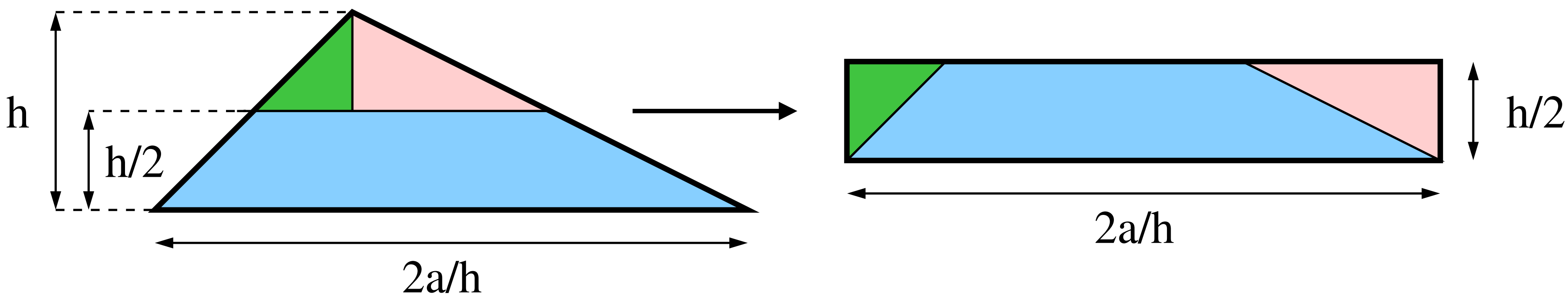
It is enough to show a triangle of area a is equivalent to a $1 \times a$ rectangle.

Triangulate P, convert each triangle to a rectangle and stack them.



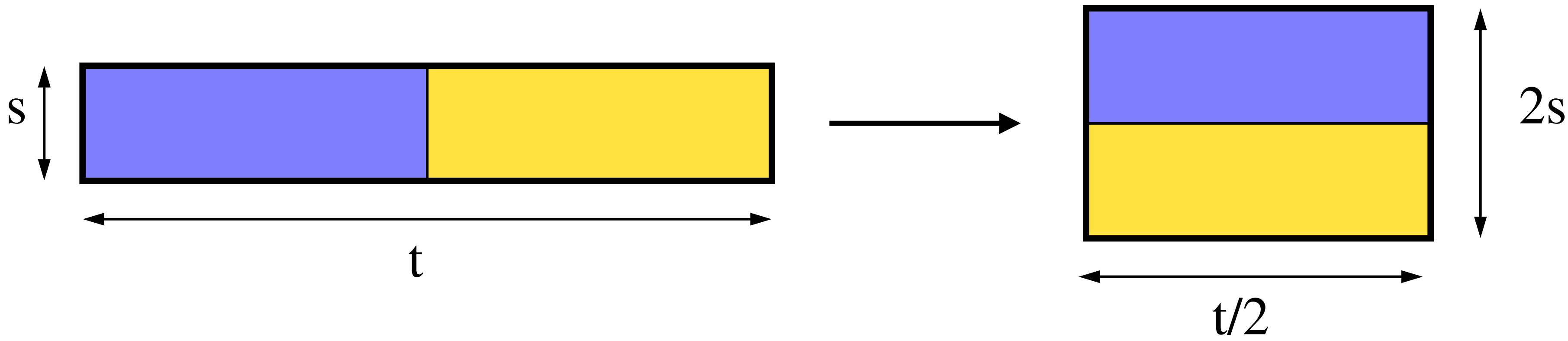
Proof of WBG theorem.

Claim 1: every triangle is dissection equivalent to some rectangle:



Proof of WBG theorem.

Claim 2: a $t \times s$ rectangle is equivalent to a $(t/2) \times (2s)$ rectangle.



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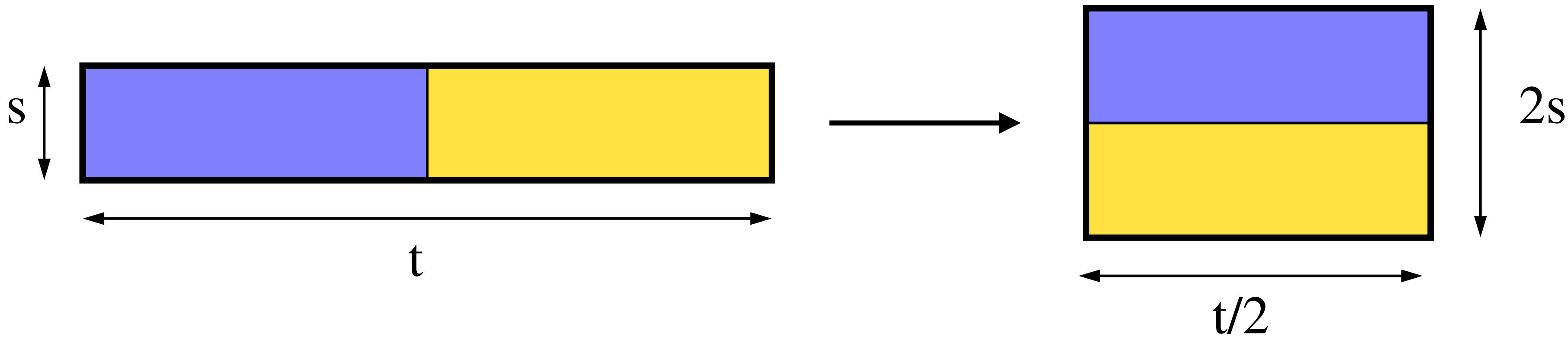


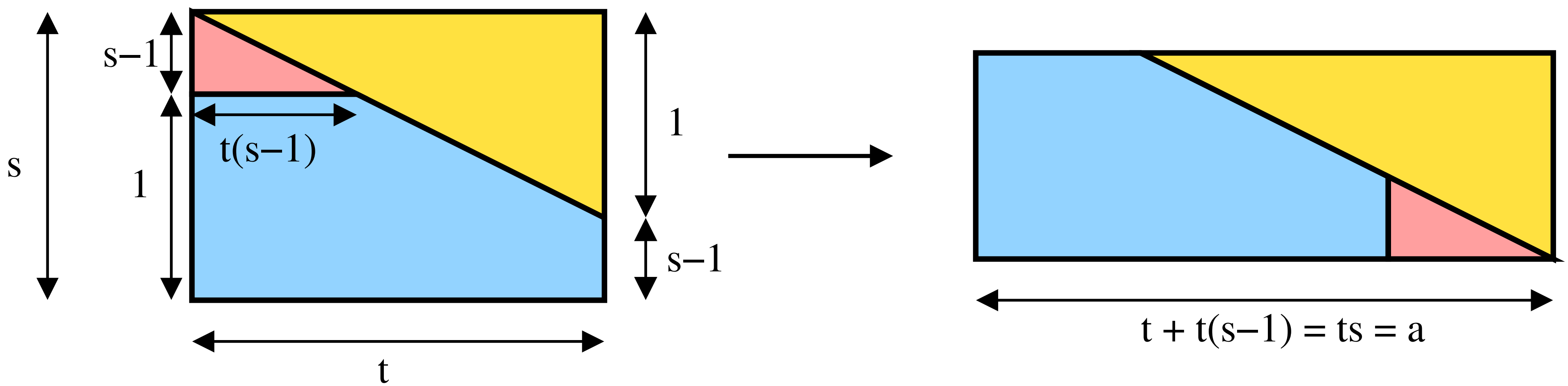
Claim 3: a rectangle of area a is equivalent to a rectangle with side lengths $s \in [1, 2)$ and $t \in (a/2, a]$.

Keep doubling (or halving) sides until this is true.

Proof of WBG theorem.

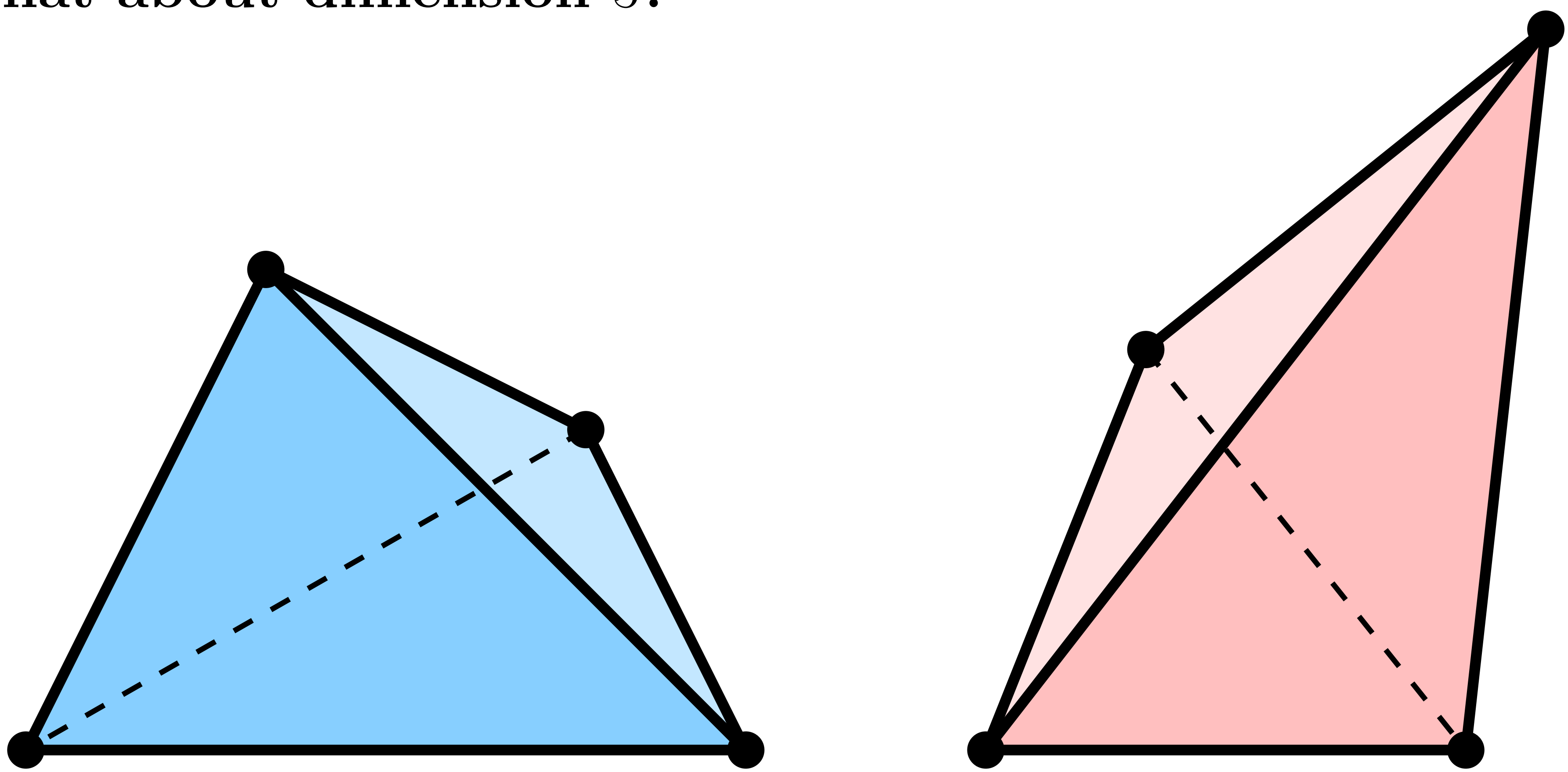
Claim 4: a rectangle of area a is equivalent to a $1 \times a$ rectangle.

By previous steps we may assume $t \times s$ with $1 \le s < 2$.



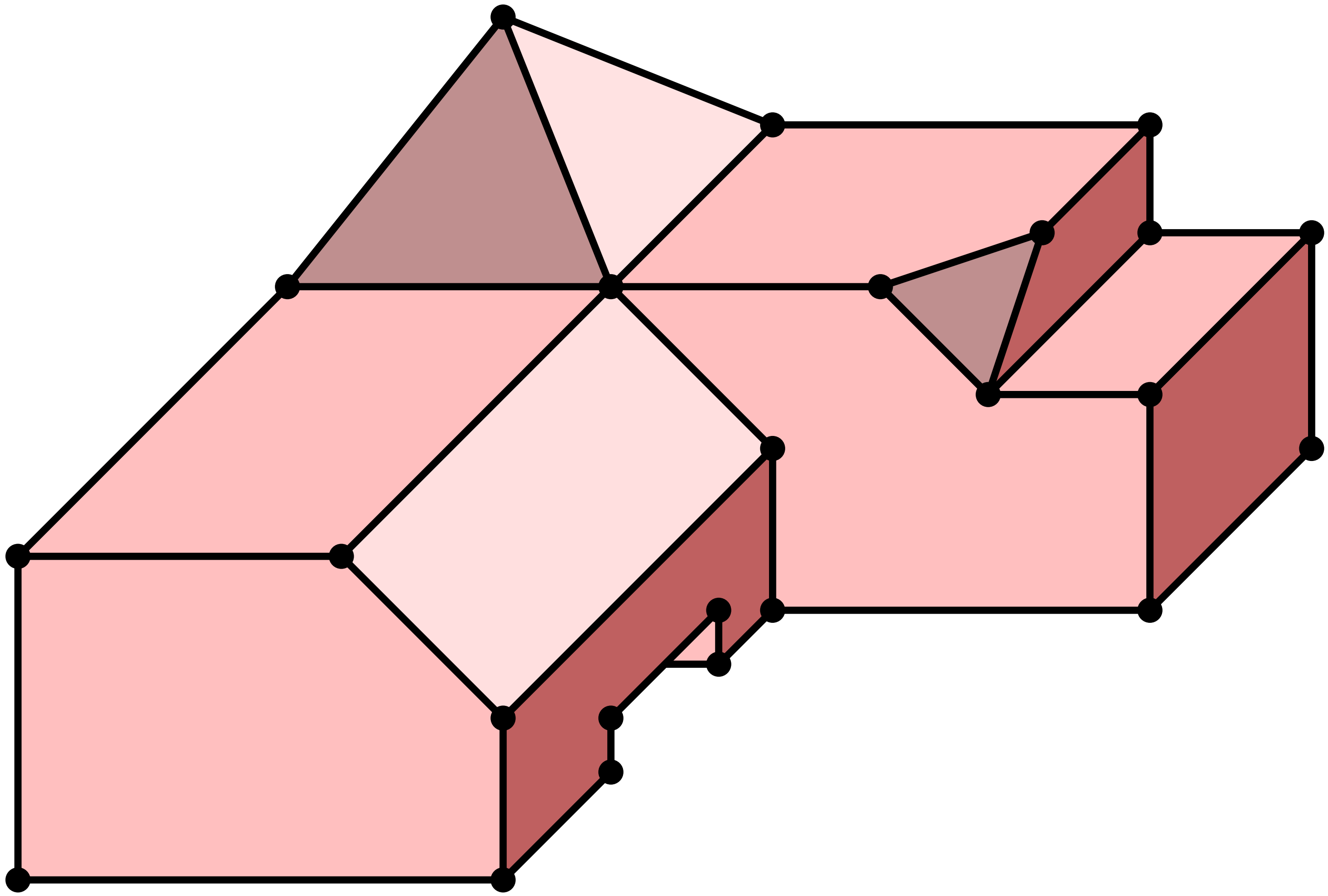
This completes the proof of the WBG theorem.

What about dimension 3?



3D version of a triangle is a tetrahedron = four faces, all triangles

What about dimension 3?



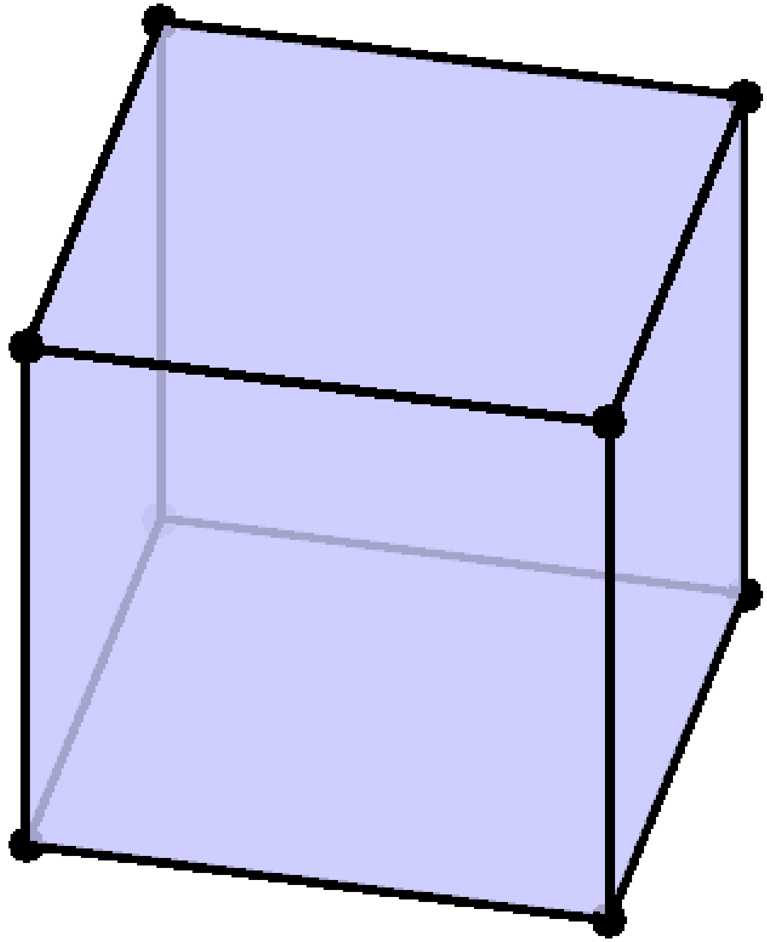
3D version of a polygon is a polyhedron.

So far we have proven

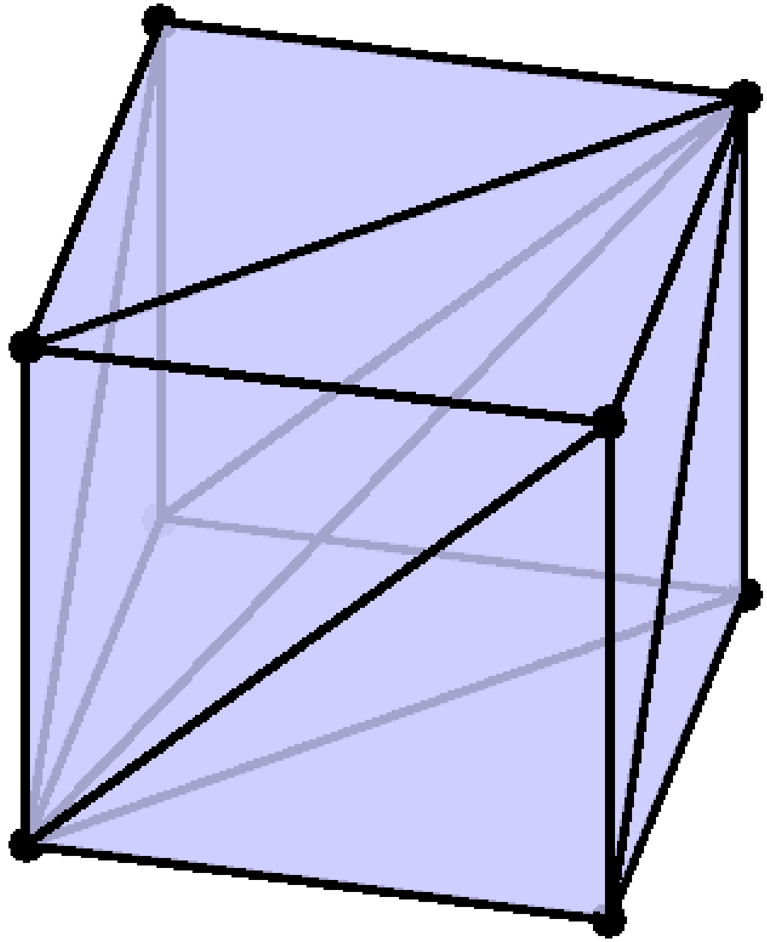
- Every 2D polygon has a triangulation with no extra vertices.
- The Wallace-Bolyai-Gerwien theorem.

Are these true in 3 dimensions?

- Is every polyhedron a union of tetrahedron using no extra vertices?
- Can two equal volume polyhedra be cut into the same set of tetrahedra?



Some polyhedra can be cut into tetrahedra.
A cube can be cut into six tetrahedra.



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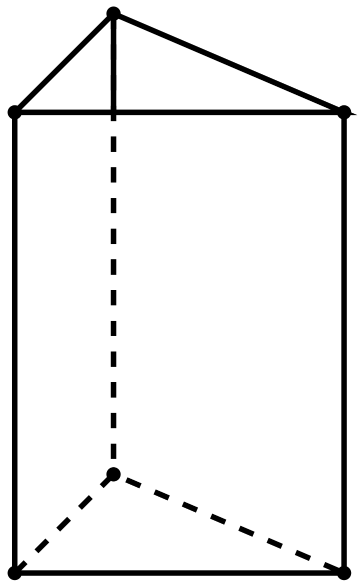
3D counterexample due to Erich Schönhardt in 1928.



The Schönhardt polyhedron does not contains any tetrahedra defined by four of it vertices.

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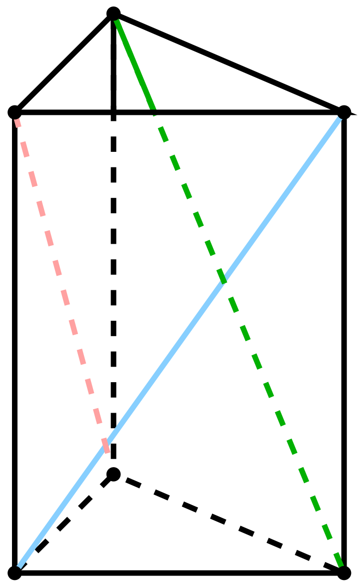

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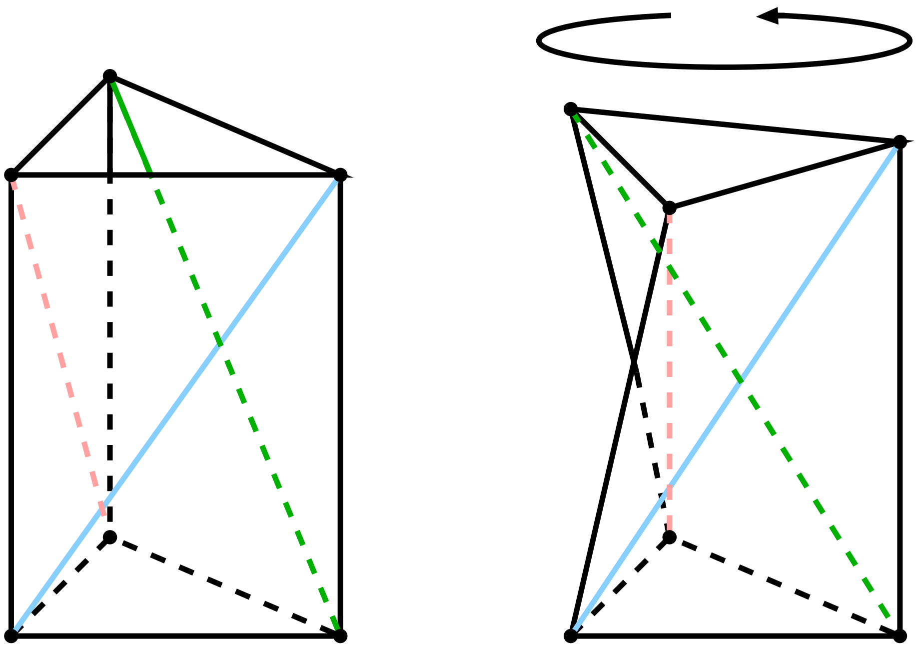
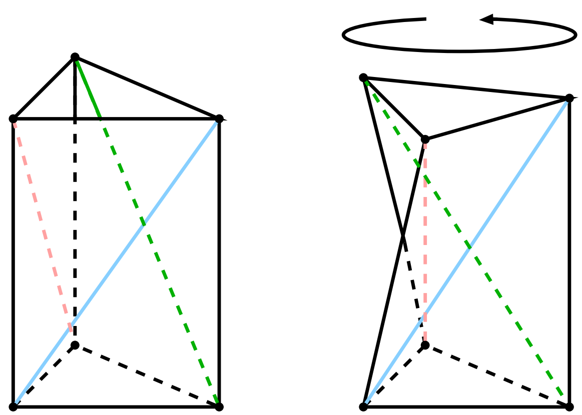
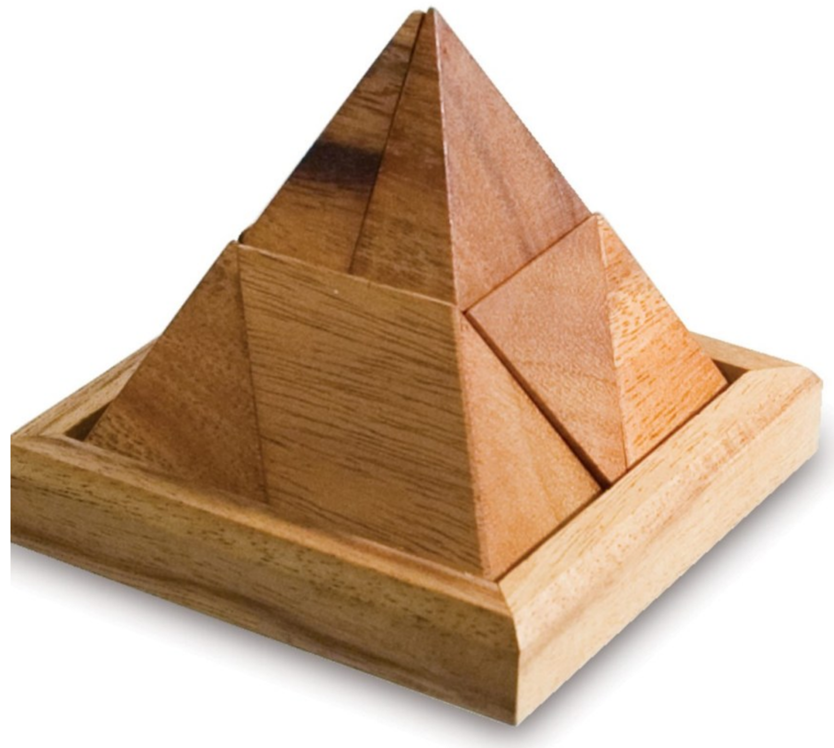


The Schönhardt polyhedron does not contains any tetrahedra defined by four of it vertices.

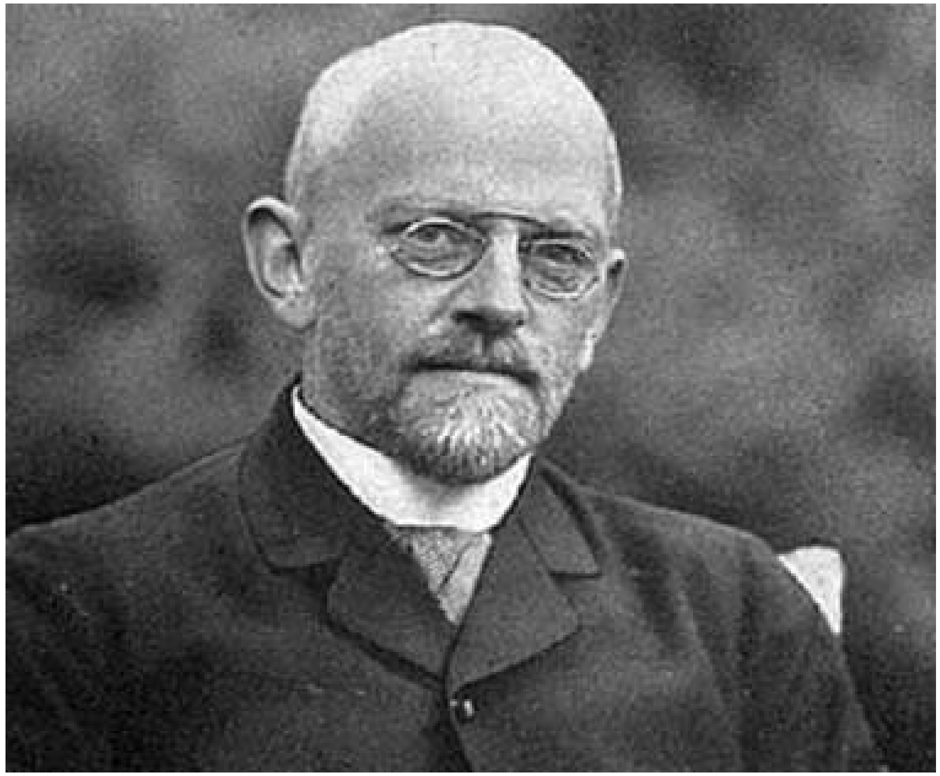
Proof:

- Let P be the Schönhardt polyhedron.
- Suppose T is a tetrahedron using 4 of the 6 vertices of P.
- T is inside P iff none of its vertices can "see" another one.
- Every vertex v on top side of P, can see one vertex on bottom side.
- If T has 3 vertices on top, the vertex on bottom sees one of these. $\Rightarrow\Leftarrow$
- If T has 2 vertices on top, only one on bottom is "invisible" $\Rightarrow\Leftarrow$



Is the WBG Theorem true in 3 dimensions?
Can we cut a cube into tetrahedra and re-assemble as a pyramid?

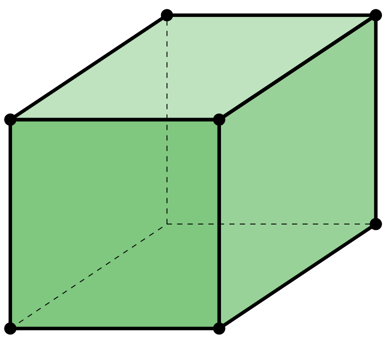


This is “**Hilbert’s third problem**”. Ten problems given at the 1900 International Congress of Mathematicians; 23 later appeared in print.

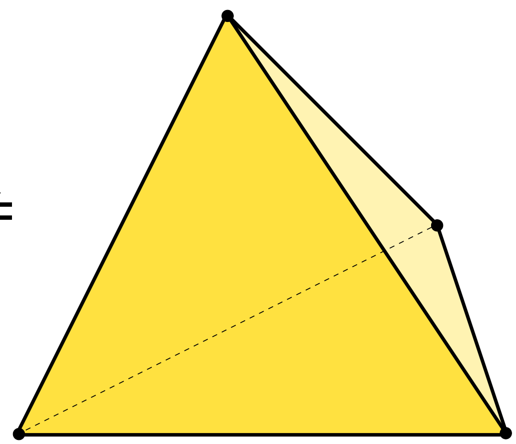
The Dehn Invariant is a "number" associated to each polyhedron.

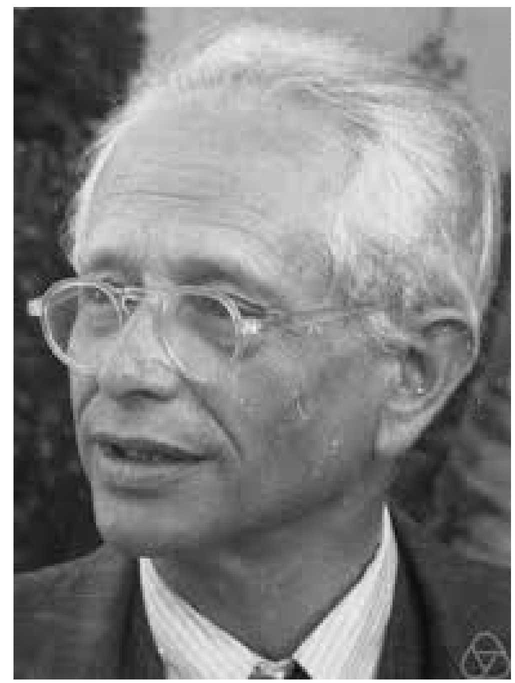
Dehn proved that dissection equivalent polyhedra have same value.

But unit cube and regular tetrahedron have different Dehn invariants.



$\neq$





Max Dehn

In 1965 Jean-Pierre Sydler proved converse:

Theorem: Two polyhedra are dissection equivalent iff equal same Dehn invariants.

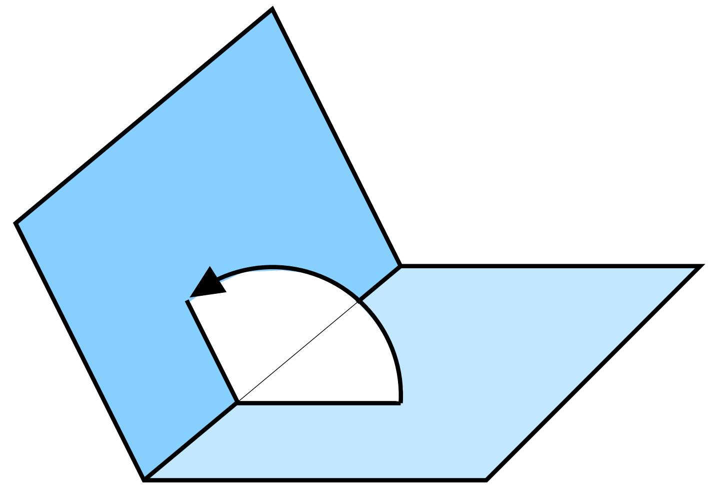
Easier algebraic proof given in 1990 by Johan Dupont and Han Sah.

Proof is related to topology and the homology the 3-D rotation group.

Defining the Dehn invariant:

• "Angles" means dihedral angles between faces. One per edge.

• The real numbers form a vector space over the rationals.



(1) Find a basis for the dihedral angles over the rationals. Add π.

(2) Write each dihedral angle as a sum of basis elements, i.e., a vector.

(3) Multiply each vector by corresponding edge length.

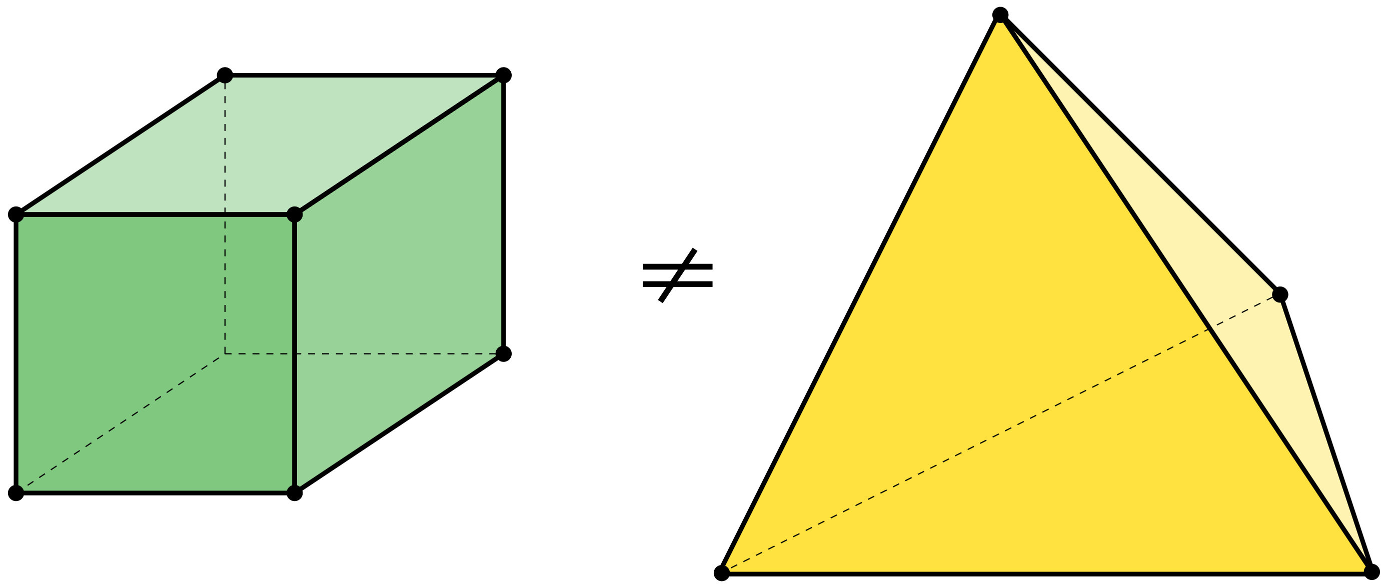
(4) Dehn Invariant = sum vectors over all edges and drop the π coordinate.

Platonic solids have only one angle, so Dehn invariant is 1-dimensional.

Dihedral angles: Cube $= \pi/2$, Regular Tetrahedron $= \arccos(1/3)$.

Fact: $\arccos(1/3)$ is not a rational multiple of π.

$\Rightarrow$ Cube and Tetrahedron and not dissection equivalent.

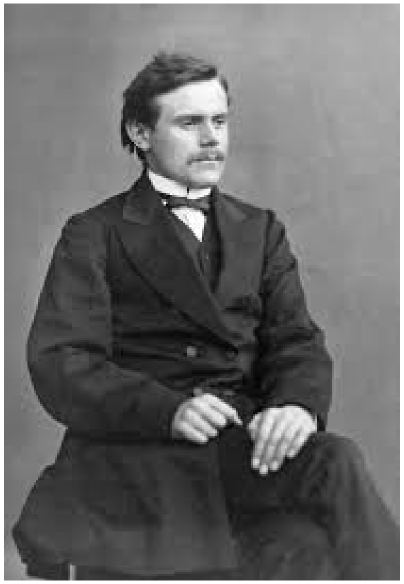


$$\arctan(1/3) =$$

$$
\begin{aligned}
0.\ & 32175055439664219340140461435866131902075529555765 6\\
& 19143280305935675623740581054435640842235064137443 9\\
& 00716937712973914826764297076263440245980928208801 4\\
& 65864323475951060452874388715516781143321574160193 9\\
& 72450662994456761410639485896268183547961536219139 0\\
& 50842437121499800384958349779119321359905775318847 3\\
& 69947112143303502292964634334341703392619240418776 9\\
& 57717919951092074730194665593740393248965373380807 8\\
& 77953088209329624680964433347534811177901171693438 2\\
& 42975386142447747187992961455706172367117050921217 8\\
& 52574822353537803204890063748153835659393183859309 9\\
& 73095300481869959584984714363855355032925020993692 5\\
& 58194102051392680820665435725607752212571661061663 0\ldots
\end{aligned}
$$

Surprising historical note:

Hilbert's problem had been solved even before he stated it.



Władyslaw Kretkowski

Ludwig Birkenmajer

In 1882, when a Polish nobleman Władysław Kretkowski offered a prize of 500 French francs (about \$10,000 today) for its solution.

Awarded by the Polish Academy of Arts and Sciences to a 28-year-old math teacher, Ludwig Birkenmajer, later a professor at Jagiellonian University.

See: "Equidecomposability of polyhedra: a solution of Hilbert's third problem in Kraków before ICM 1900"

by Danuta and Krzysztof Ciesielski, in *Mathematical Intelligencer* vol 40, 2018.

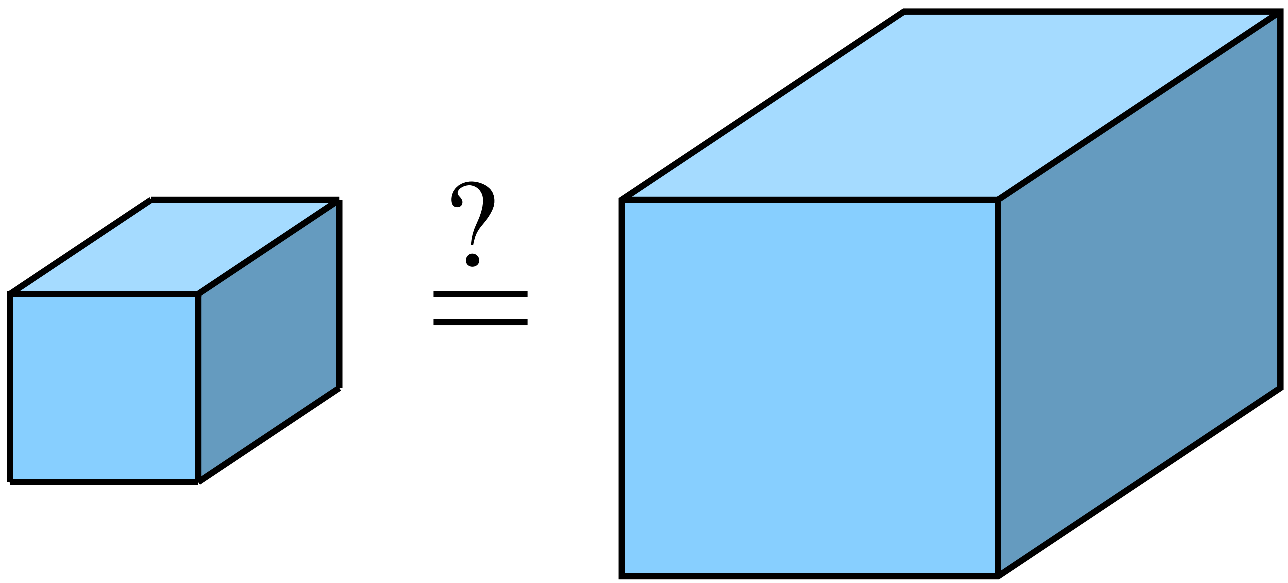
Side remark:

Equal volume polyhedra can't always be equi-partitioned into tetrahedra.

What if we replace tetrahedra by more general sets?

Can two polyhedra of equal volume be decomposed into finite collections of sets that are the same, up to rotation and translation?

Banach-Tarski paradox (1926): Given any two polyhedra in 3-space, the first can be decomposed into finitely many disjoint subsets, that can be rotated and translated to give the second as their disjoint union.



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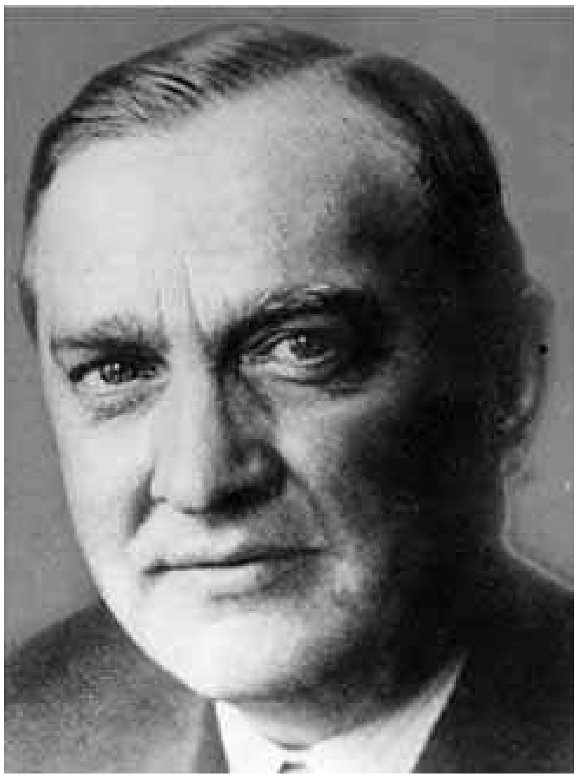
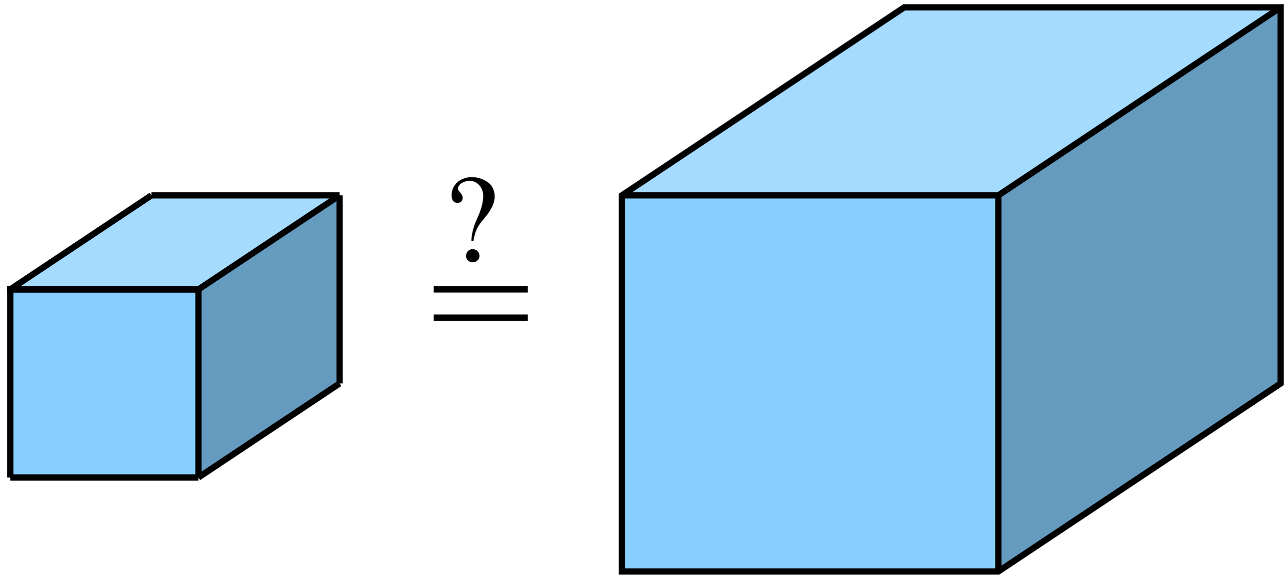


Stephen Banach

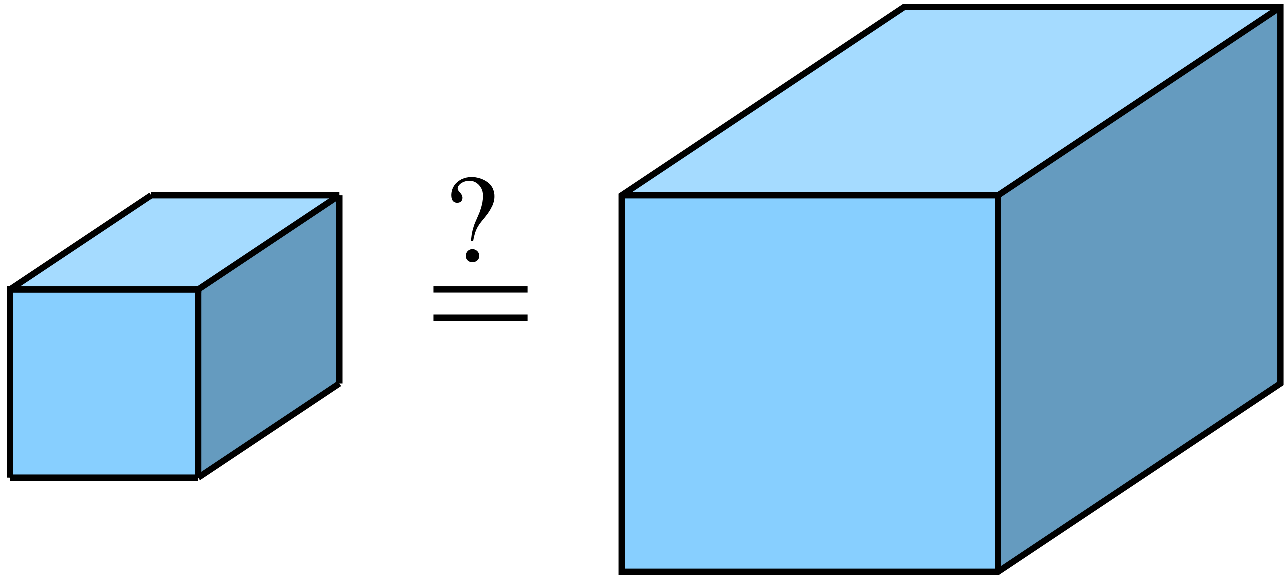
Alfred Tarski



This violates our intuition that

(1) volume does not change under translations or rotations,

(2) the volume of a disjoint union should be the sum of the volumes.

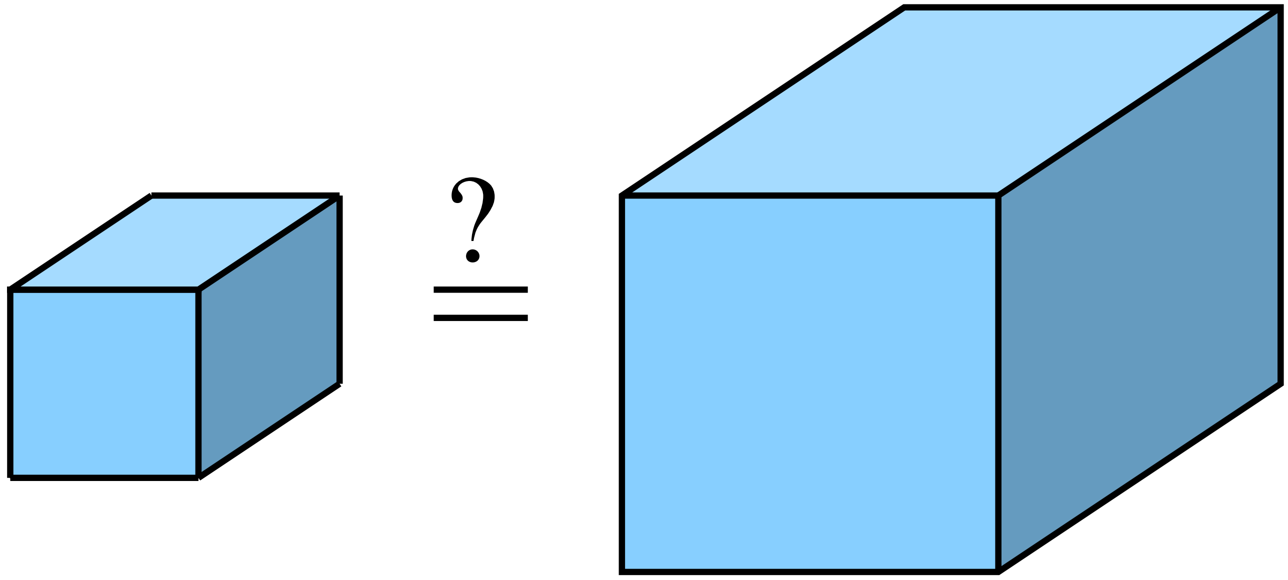


This violates our intuition that

(1) volume does not change under translations or rotations,

(2) the volume of a disjoint union should be the sum of the volumes.

We can't define 3D-volume so it has both properties for all sets.

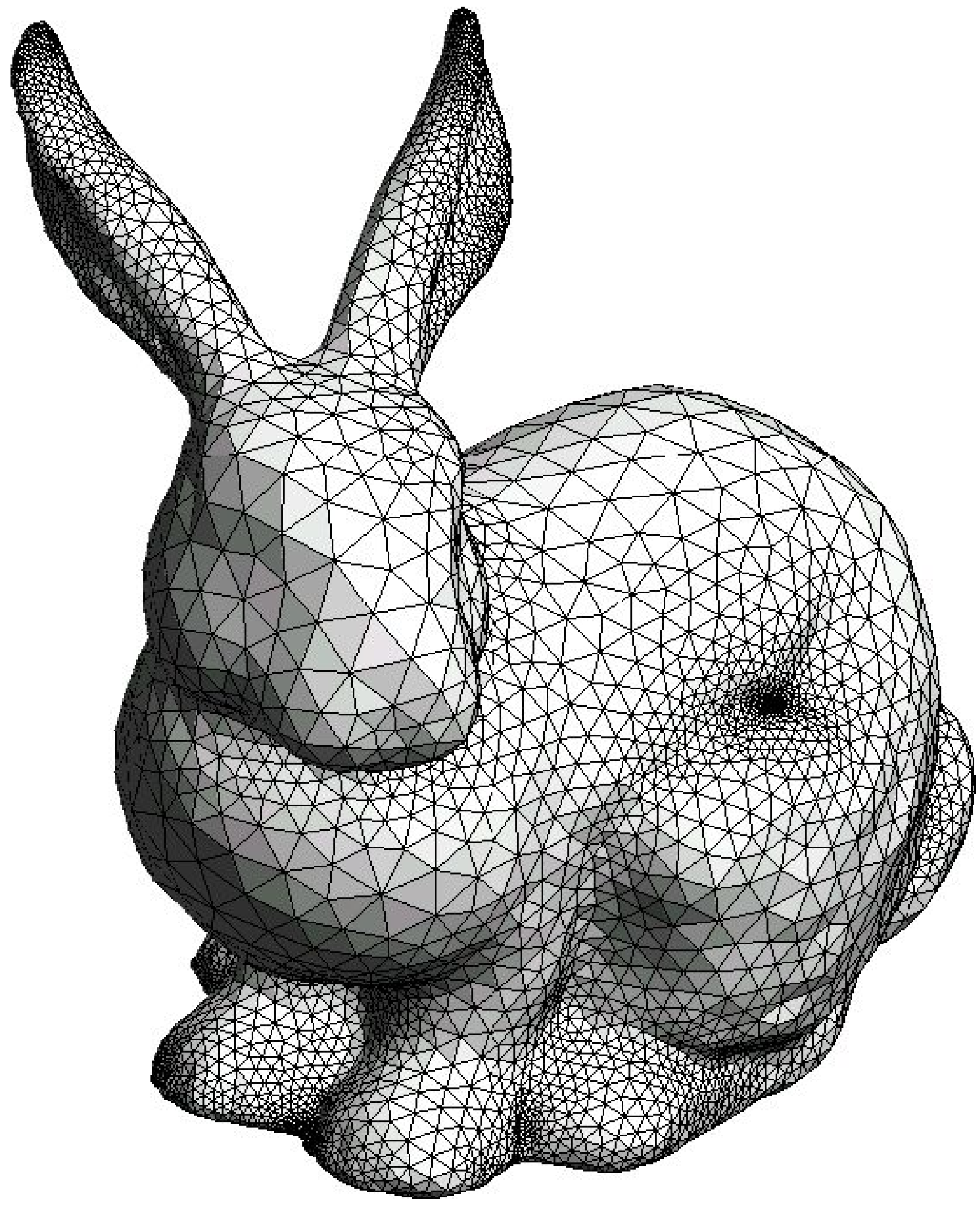


This is why a field called "Measure Theory" exists.

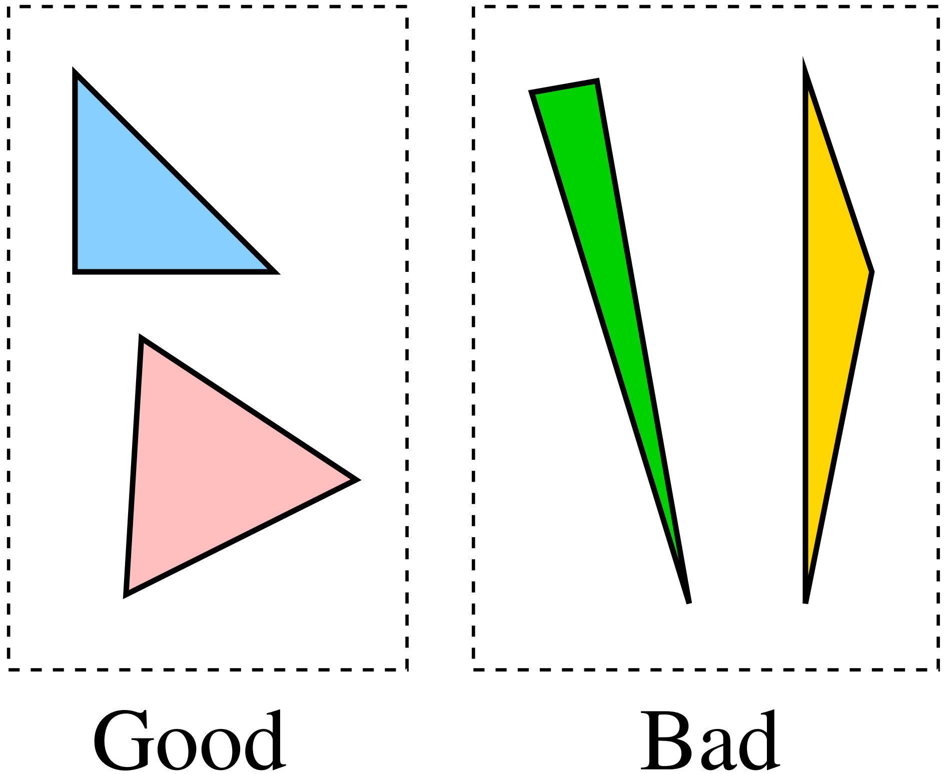
Volume can only be defined for special "measurable sets".

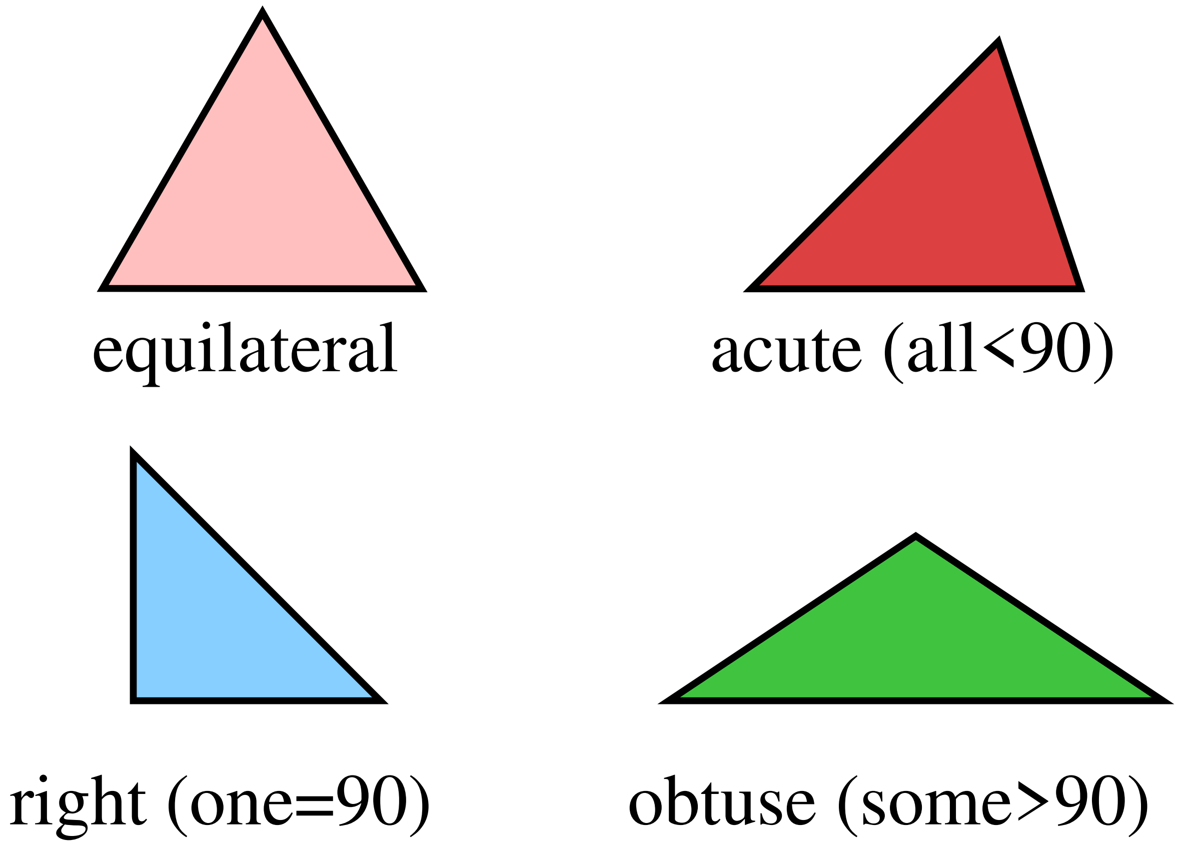
"Measurable" includes every set you can explicitly describe.

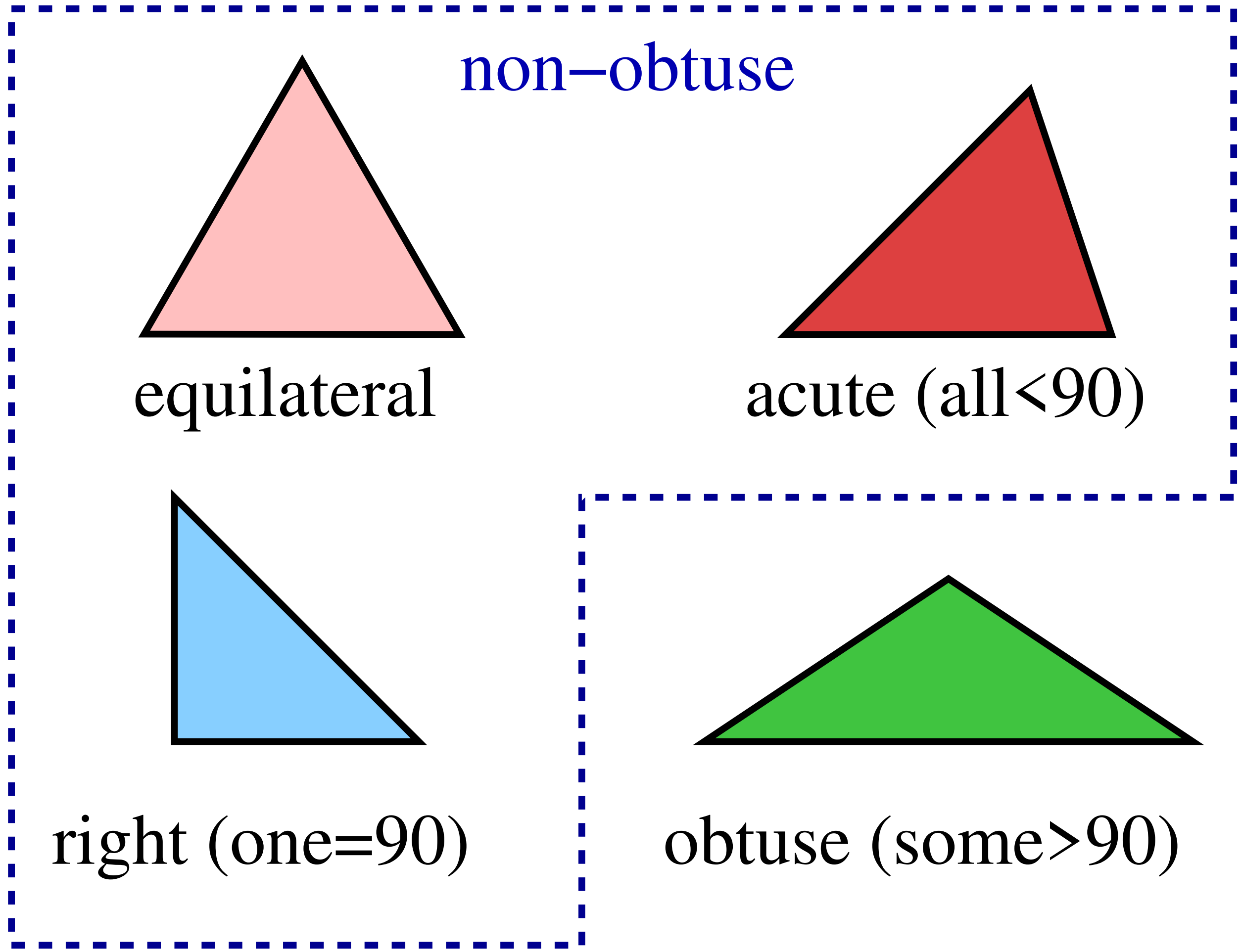
Non-measurable sets require the Axiom of Choice to construct.



For many applications, it is important to have triangulations with "nice" triangles. ("nice" = angles not too close to 0 or 180.)

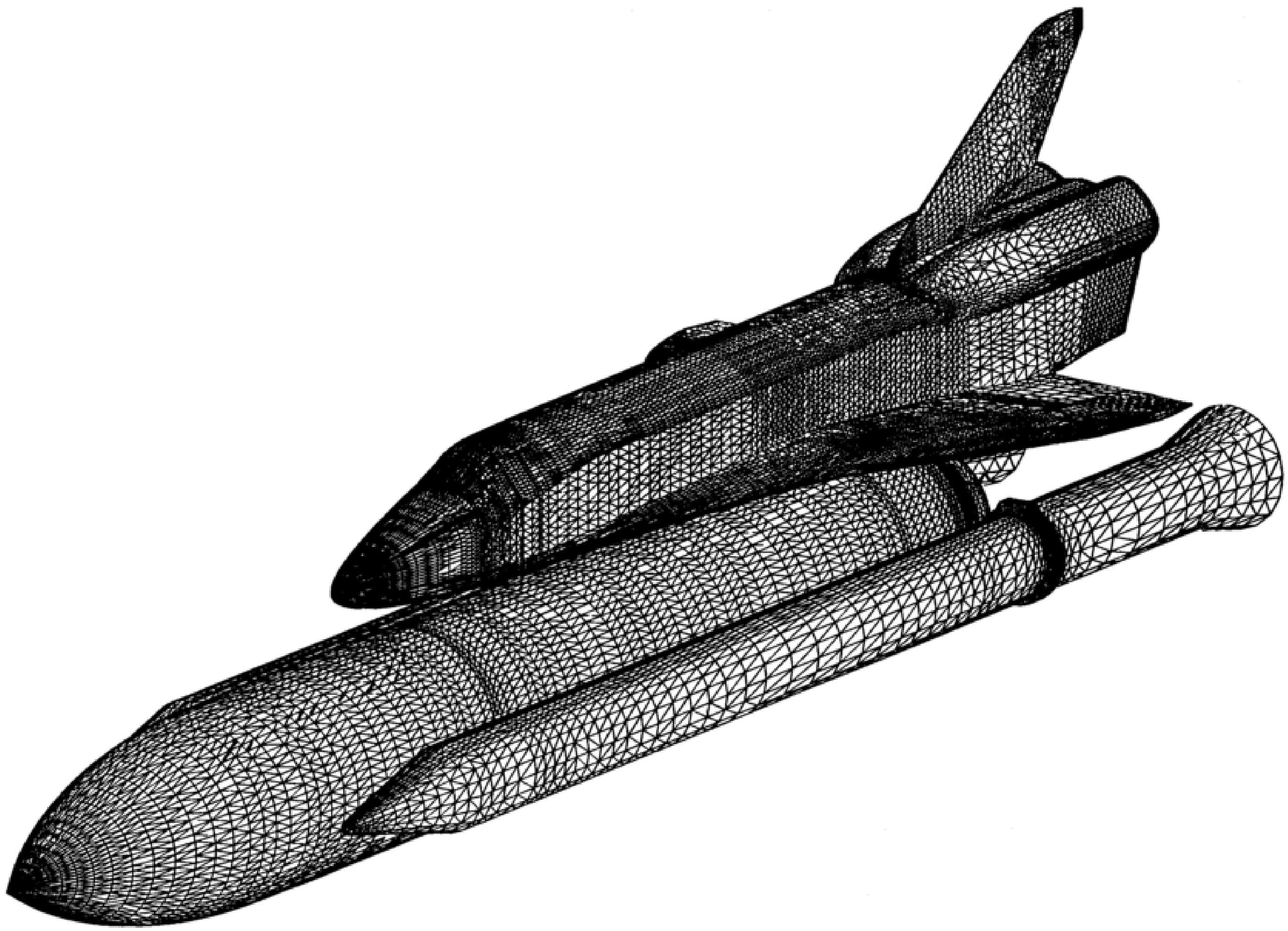



equilateral
acute (all<90)
right (one=90)
obtuse (some>90)



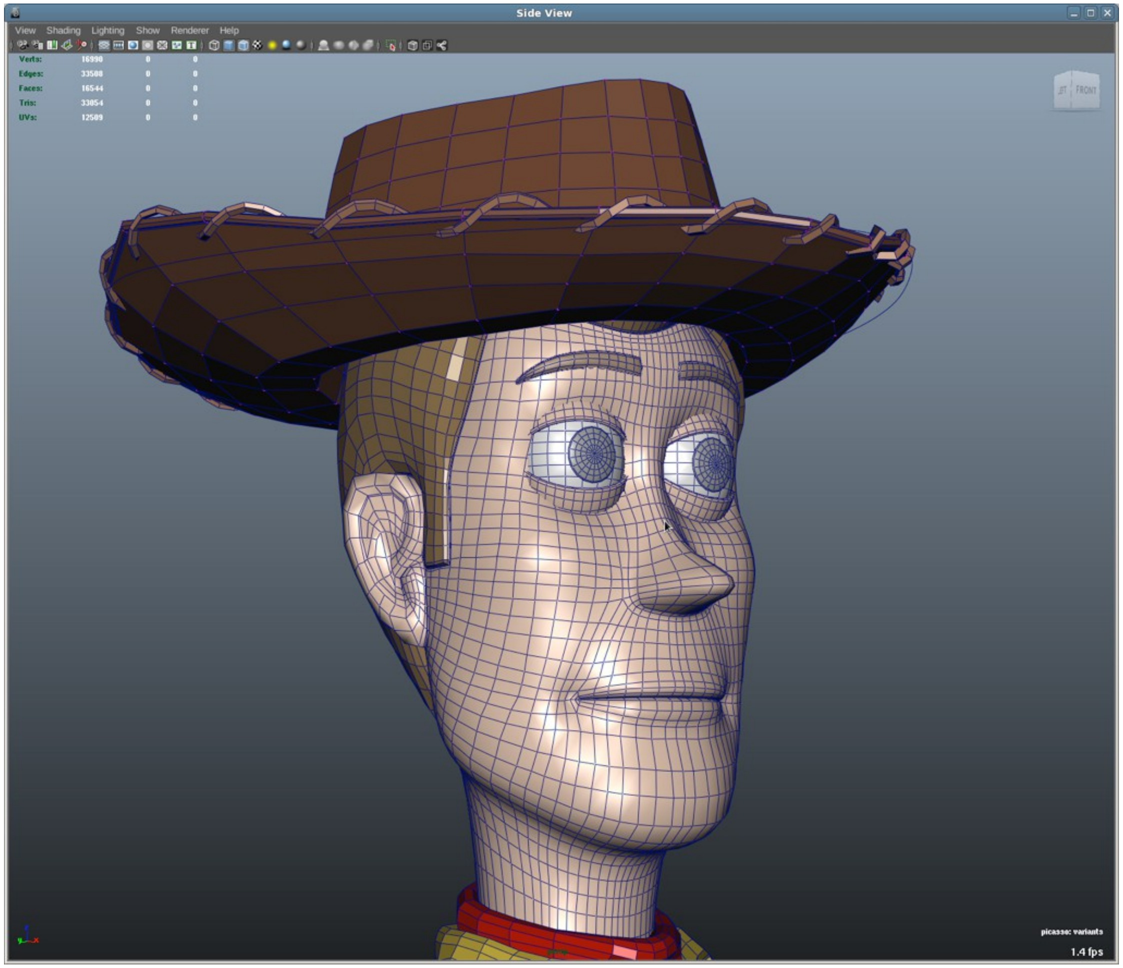
non–obtuse
equilateral
acute (all<90)
right (one=90)
obtuse (some>90)

Acute triangles are often better for computer graphics and engineering.



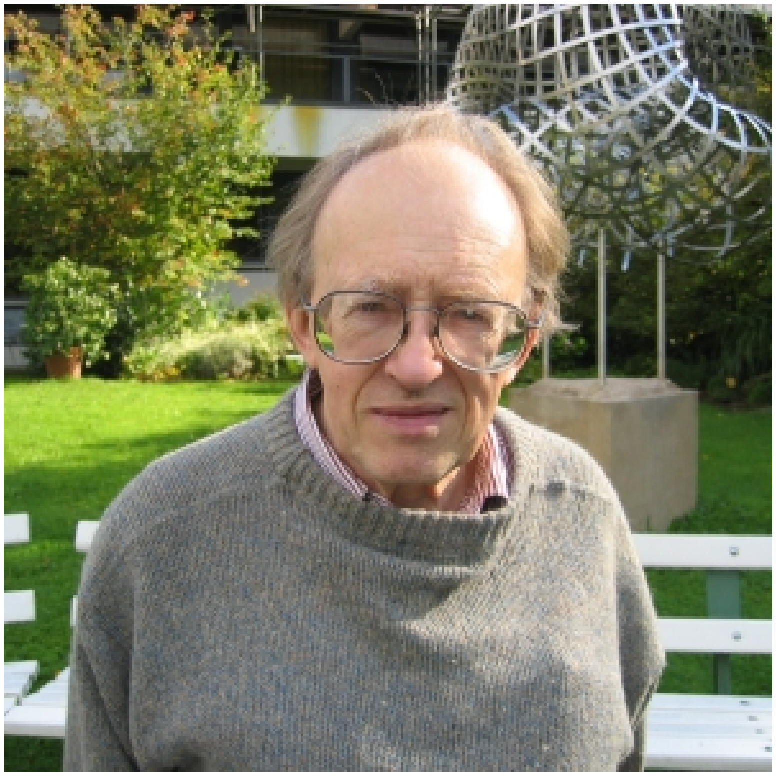
"An essential element of the numerical solution of partial differential equations (PDEs) on general regions is the construction of a grid (mesh) on which to represent the equations in finite form ... it can take orders of magnitude more man-hours to construct the grid than it does to perform and analyze the PDE solution on the grid."

Quoted from the introduction *Delaunay Mesh Generation*, by Cheng, Dey and Shewchuk, Chapman & Hall, 2013.



The same text points out that motion pictures
are now the most economically significant consumers
of high quality meshing algorithms.

Burago-Zalgaller (1960): all polyhedral surfaces have acute triangulations.

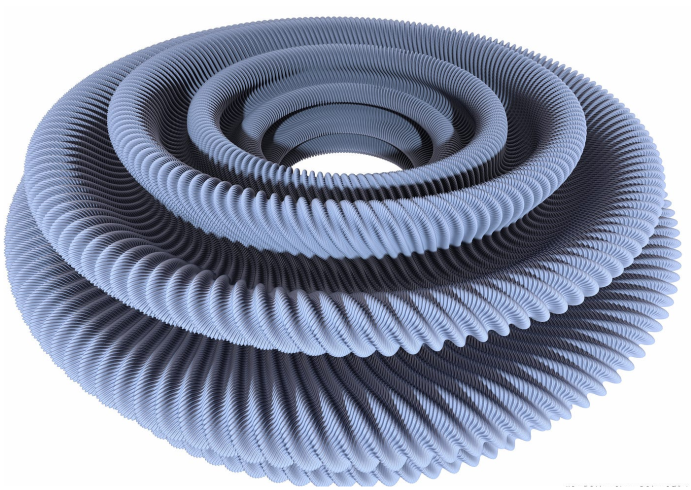


Yuri Burago (1936 -)

Victor Zalgaller (1920 - 2020)

However, their result was a lemma buried inside a Russian language paper on topology and went unnoticed in the West. First reference I know to Burago-Zalgaller in CS papers is in 2004.

They were proving a polyhedral version of the Nash Embedding Theorem.



Nash embedding

John Nash

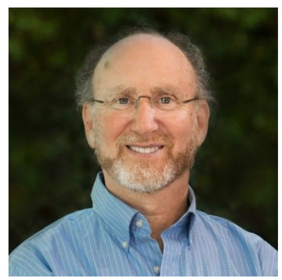
not John Nash

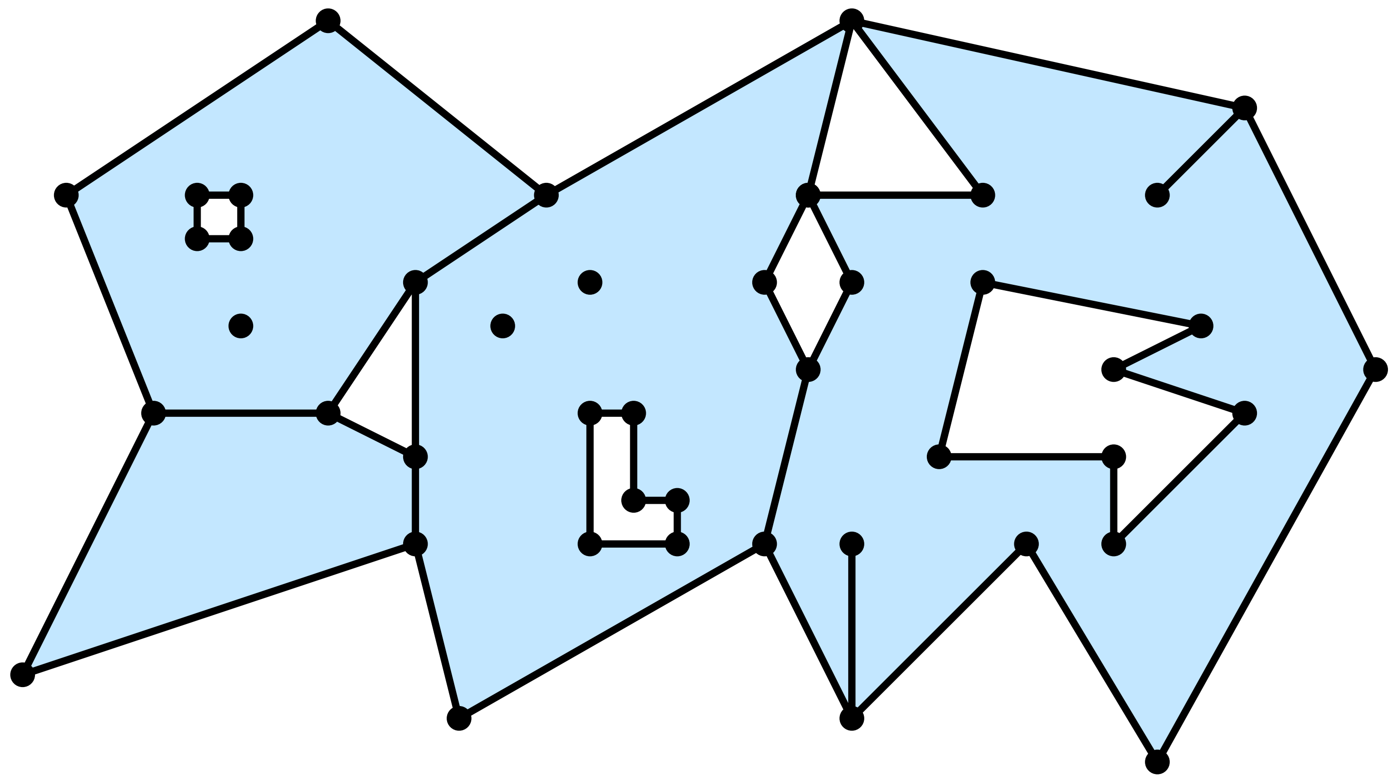
Computational complexity = number of triangles needed.

- Baker-Grosse-Rafferty (1987) proved every n-gon has a non-obtuse triangulation (weaker than the Burago-Zalgaller theorem).

- Bern-Mitchell-Ruppert (1995): Cn triangles suffice for simple polygons.

- Maehara (2002): any nonobtuse triangulation, can be subdivided to give an acute triangulation $\Rightarrow$ Burago-Zalgaller + linear complexity.



Non-simple polyon = PSLG = Planar Straight Line Graph

- Bishop (2016): $Cn^{2.5}$ triangles suffice for PSLGs.

Some examples need $\simeq n^2$. Sharp power remains unknown.

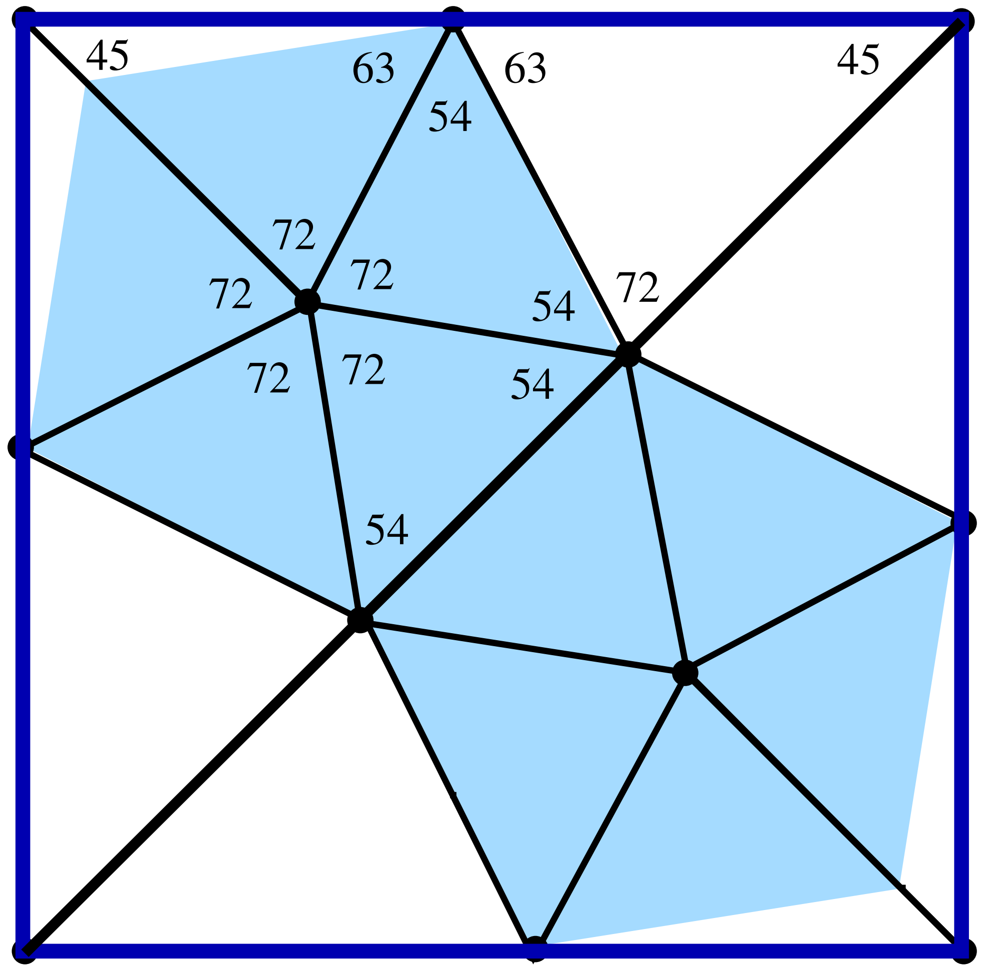
Defn: A θ-triangulation is one where every angle is $\leq \theta$.

Burago-Zalgaller $\Rightarrow$ every simple polygon has a θ-triangulation for some $\theta < 90°$.

How does θ depends on the polygon?

From here on, "triangulation" means "Steiner triangulation".

For example, a square has a 72°-triangulation, and this is optimal.



For general polygons, we have (Bishop, 2026):

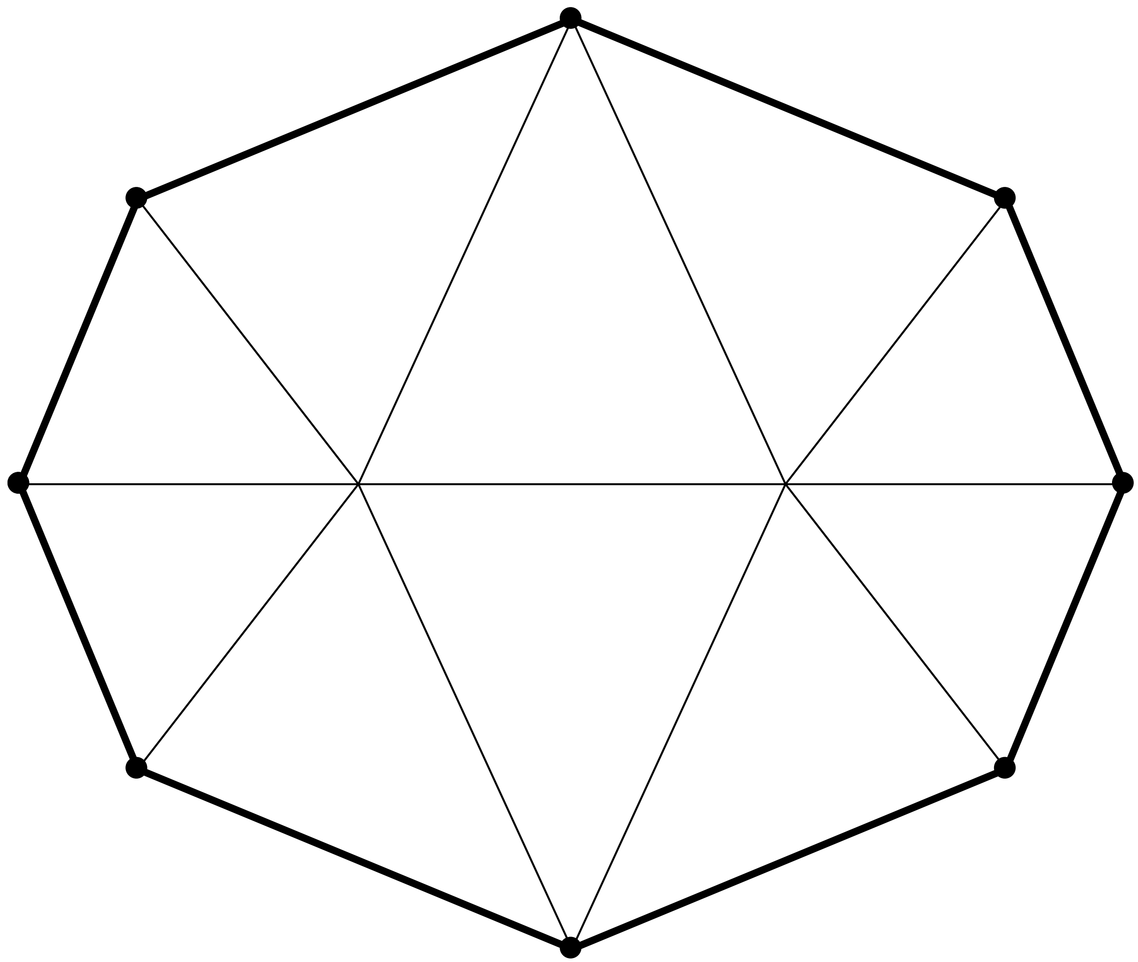
Theorem: The optimal angle bound for triangulating an n-gon can be computed in time $O(n)$.

Theorem: The optimal angle bound only depends on the set of angles, not their order or the edge lengths.

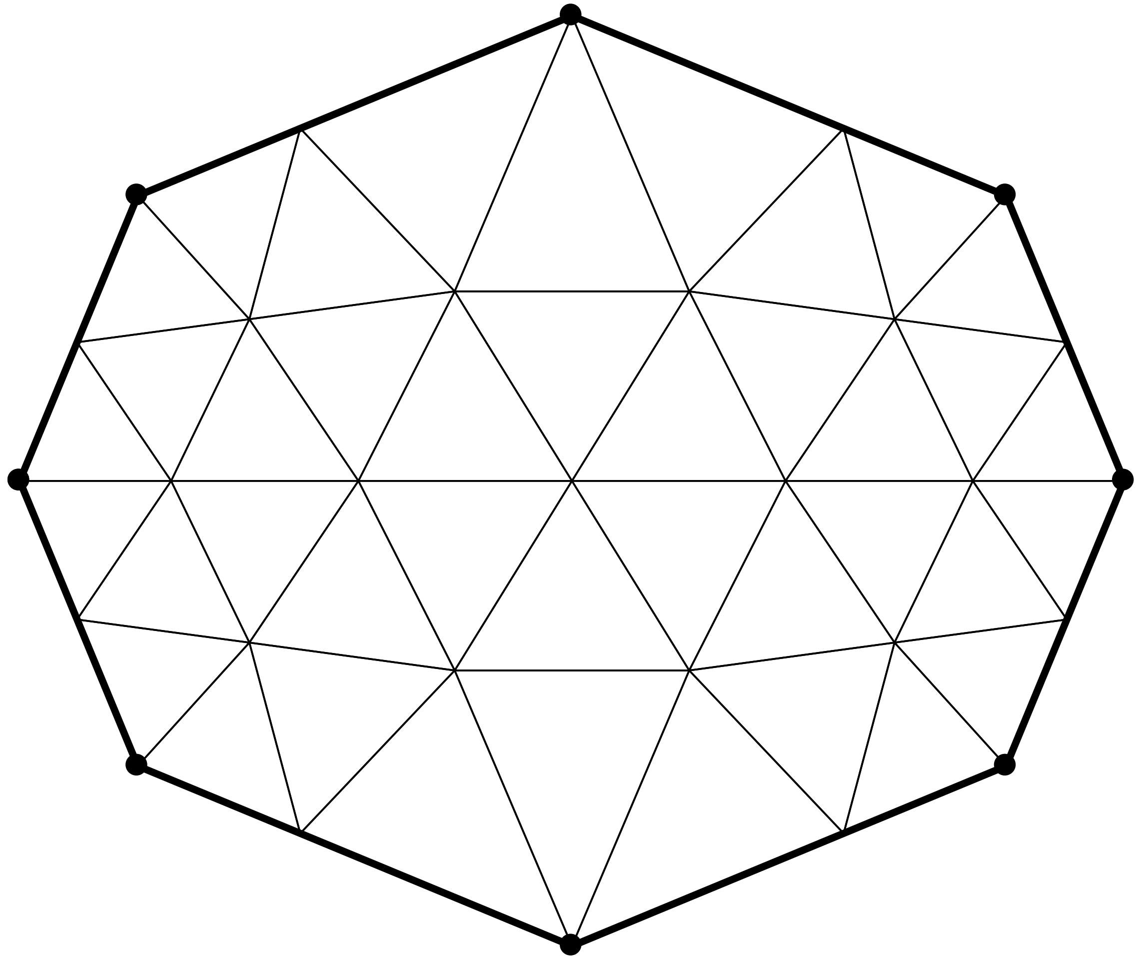
Theorem: If P has minimal angle $\geq 36^\circ$ then it has a 72°-triangulation.

Theorem: The optimal angle bound for triangulations is the same as for triangular dissections.

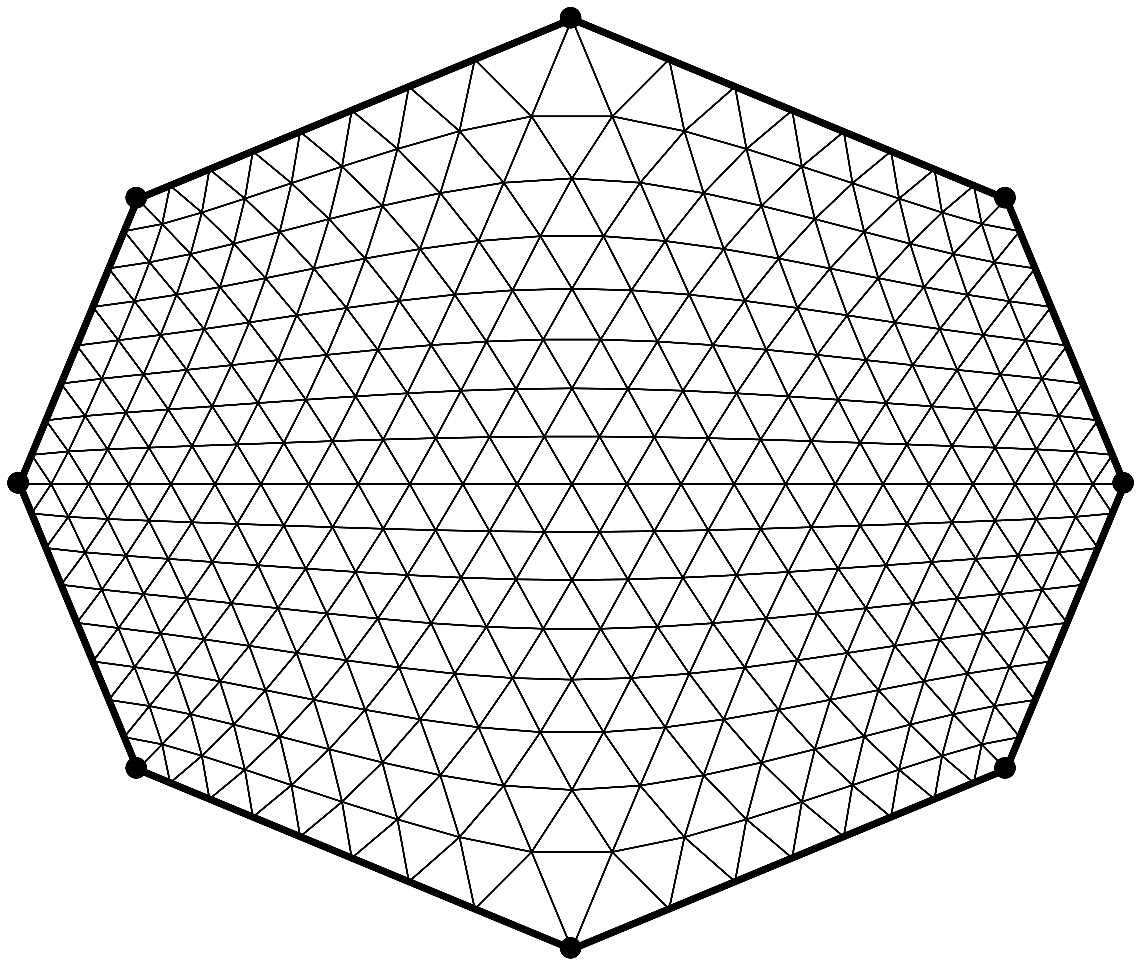
Theorem: An optimal triangulation "usually" exists.



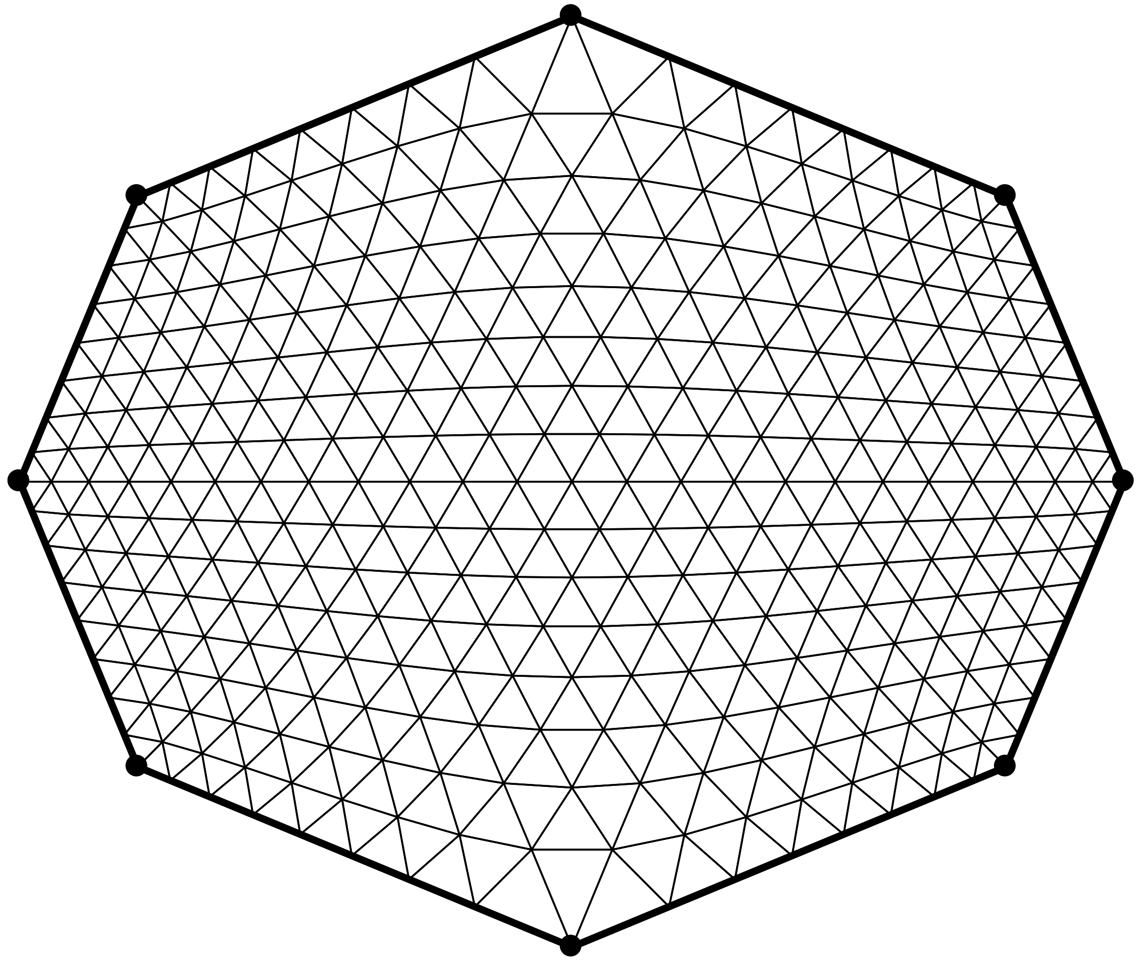
Maximum angle ≈ 74.7482.



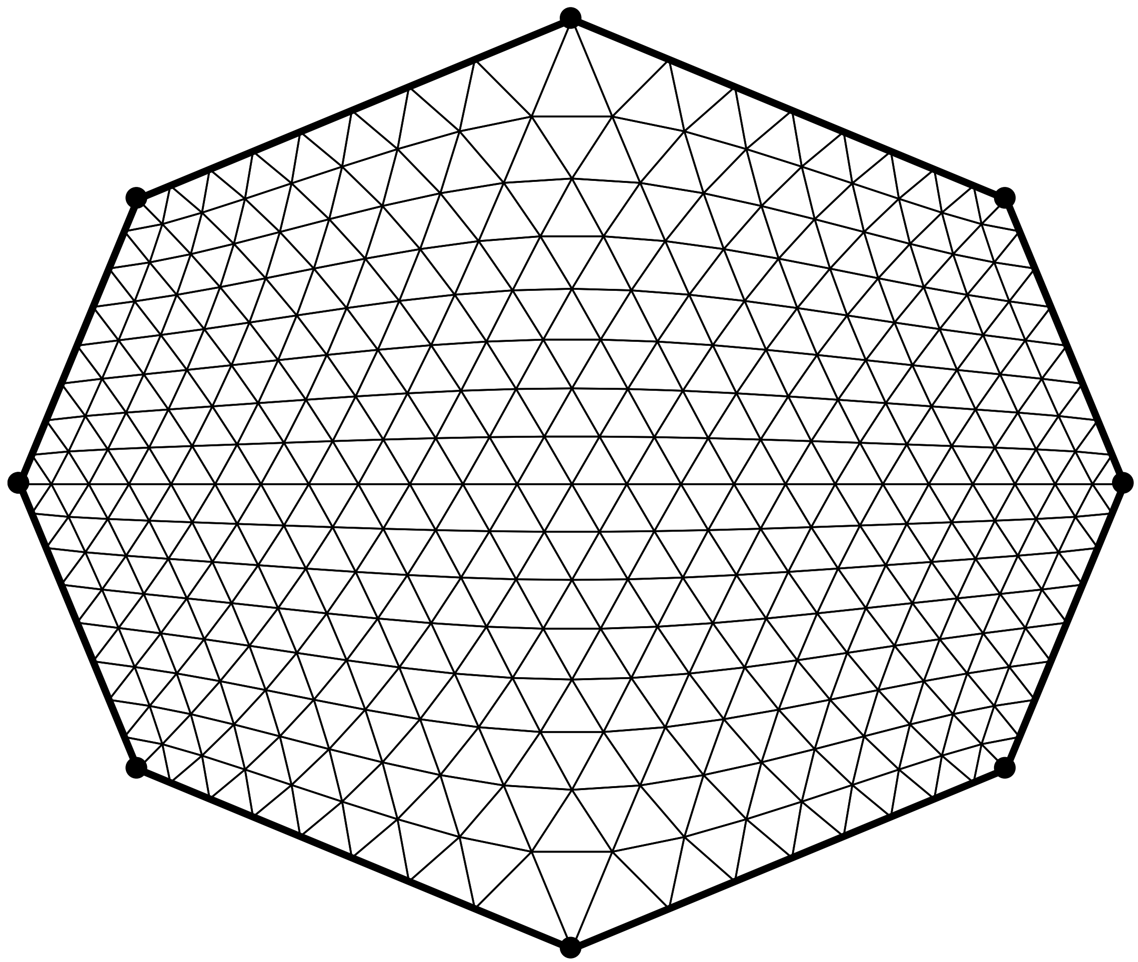
Maximum angle ≈ 70.3590.



Maximum angle ≈ 67.8690.

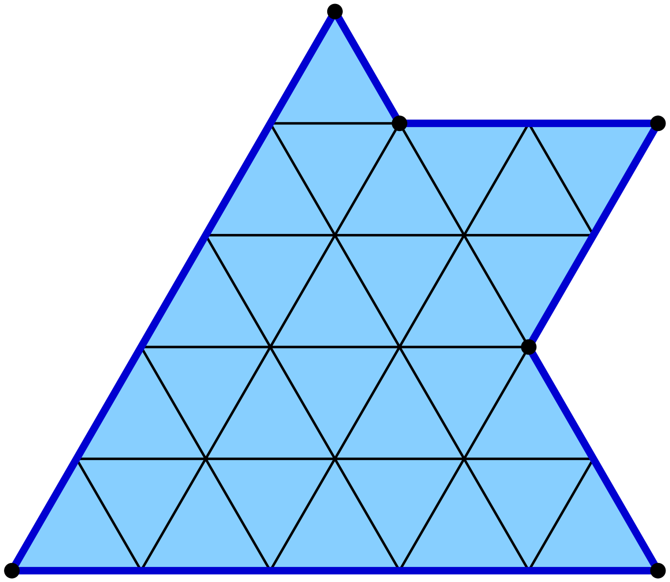


Optimum bound is $= 67.5^\circ$. Can it be attained?



Optimum bound is $= 67.5°$. Can it be attained?

Yes in this case, but not for all polygons.



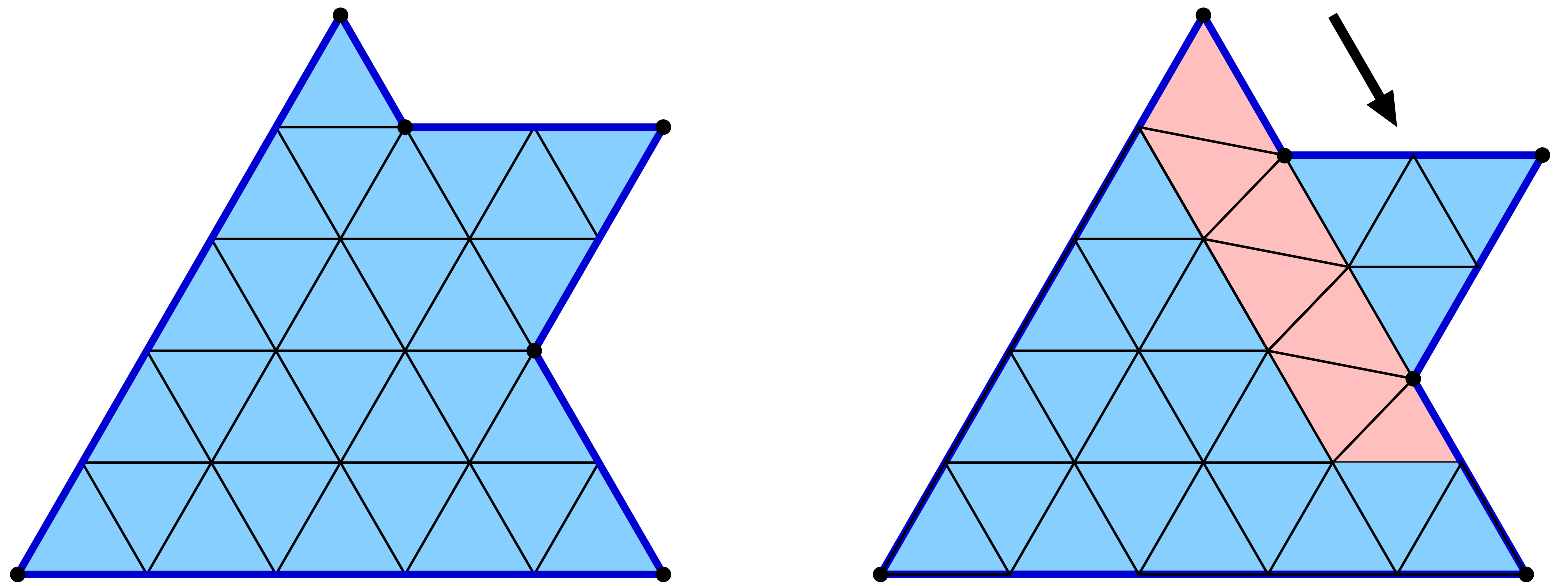
If a polygon P has an equilateral triangulation, then:

(1) all angles of P are multiples of $60°$ (P is a $60°$-polygon) .

(2) all triangles are same size.

$\Rightarrow$ all edges of P are multiples of this length

$\Rightarrow$ length ratios of any two edges is rational



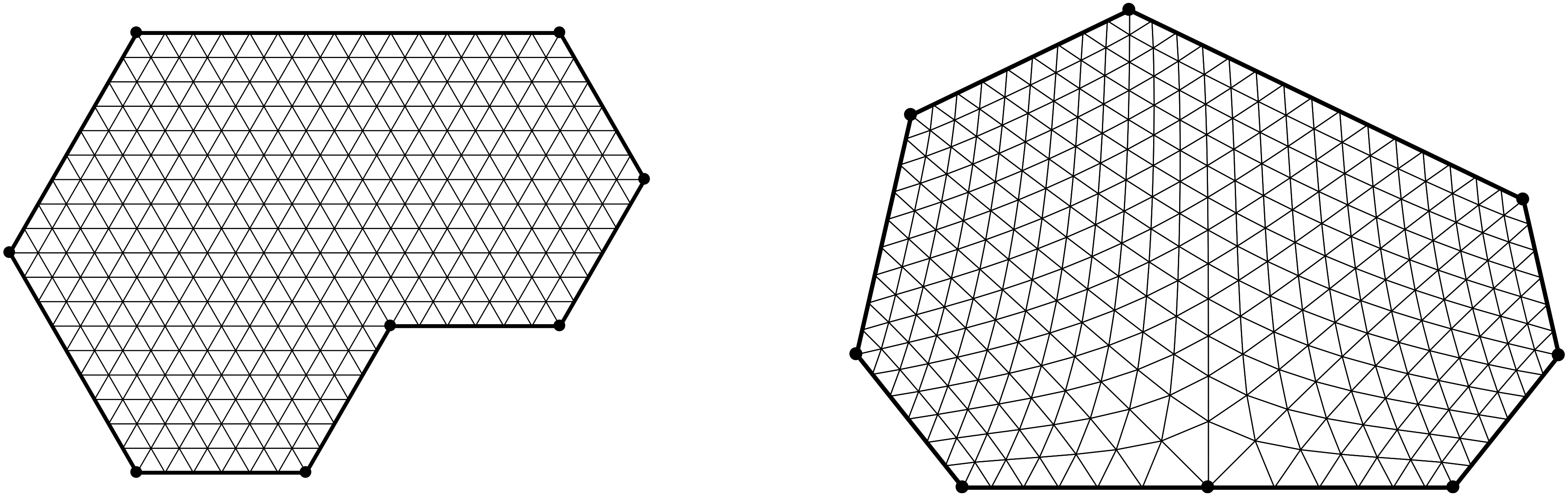
Lemma: every 60°-polygon has a θ-triangulation for every $\theta > 60^\circ$.

But if some edge ratio is irrational no equilateral triangulation exists.

Conclusion: for such polygons no optimal triangulation exists.

Thm: An optimum triangulation exists for every other polygon.

Very, very rough idea behind proofs:



- Given P "approximate" it by a $60°$-polygon P'.
- Use conformal map $P' \to P$ to transfer equilateral triangulation.
- Prove worst angle distortion is near vertices.

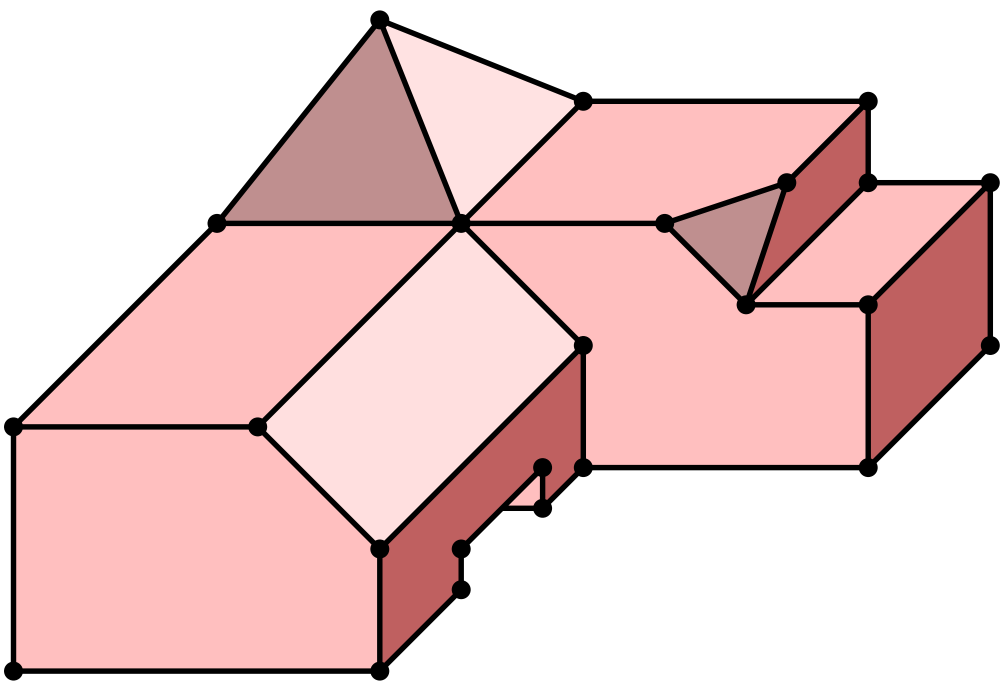
Choosing the correct P' is the hard part.

The Future (open problems)

Acute Triangulation in Dimension 3:

Question: Does a polyhedron with n vertices have a polynomial sized acute tetrahedralization?

(dihedral angles $< 90^\circ$, number of elements less than a power of n.)



This is pretty hard even for the unit cube.

The “smallest” known acute triangulation of a cube uses **1370** tetrahedra.

Found by VanderZee, Hirani, Zharnitsky and Damrong in 2010, using a computer search. A more conceptual construction with **2715** tetrahedra was given in 2012 by Kopczyński, Pak and Przytycki.

446 E. VanderZee et al. / Computational Geometry 43 (2010) 445–452

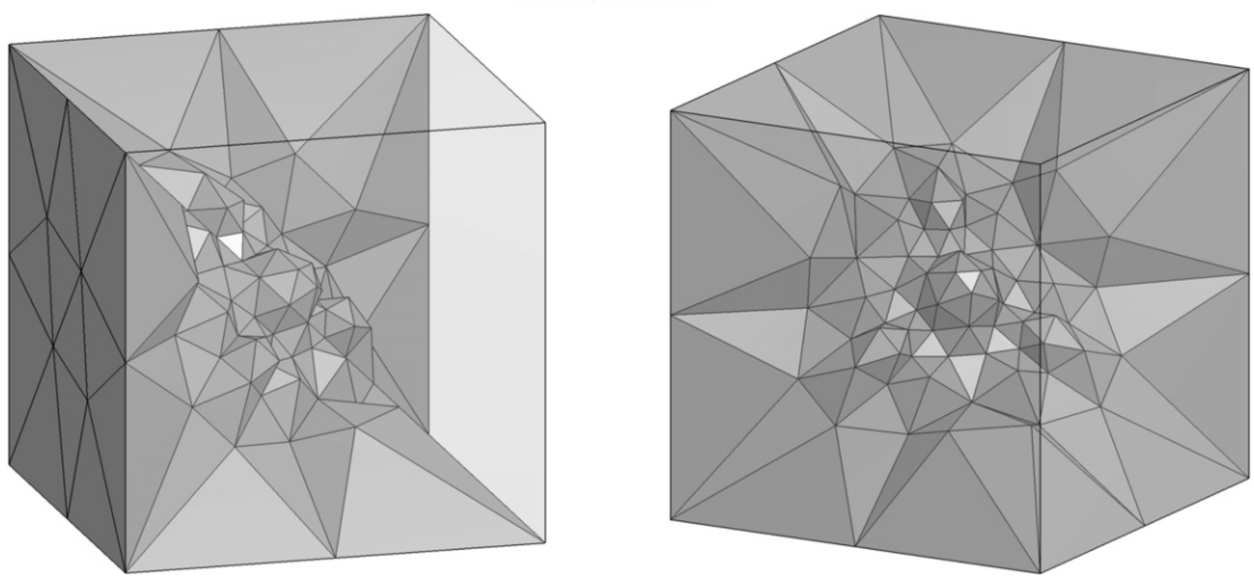


Fig. 1. Two views of a cutaway section of the first-known acute triangulation of the cube. The view at right is a 45° rotation about the z-axis from the view at left. On the left a 14-triangle triangulation of one of the square faces of the cube is visible. This 14-triangle triangulation of the square is used on each face of the cube.

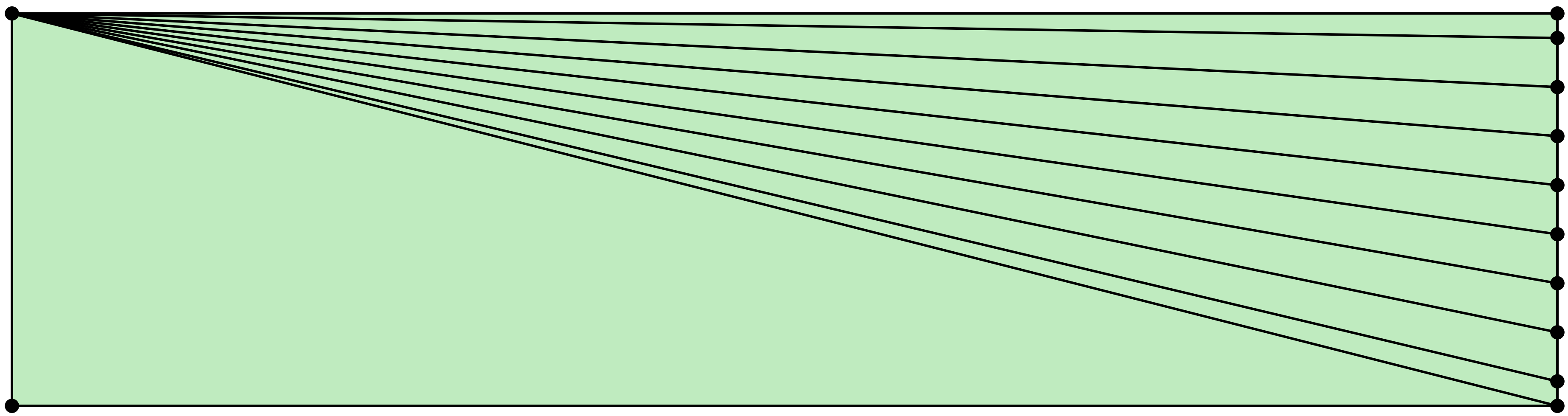
Minimal Weight Triangulations

Find triangulation that minimizes total edge length .

$1 \times R$ rectangle with n extra points on right side.

Minimal Weight Triangulations

Find triangulation that minimizes total edge length.



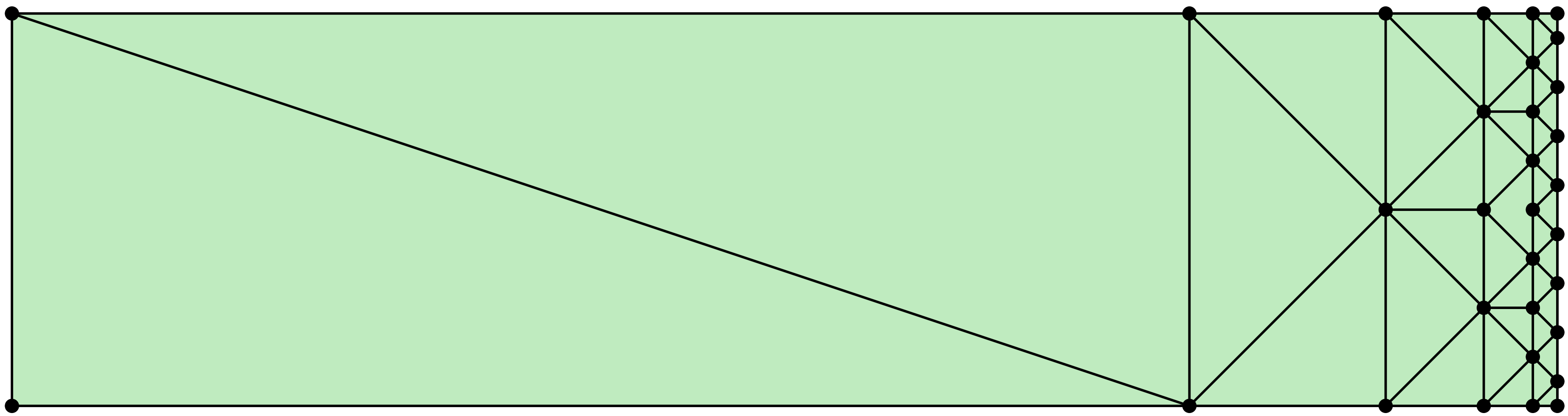
$1 \times R$ rectangle with n extra points on right side.

Without Steiner points, all triangulations length $> R \cdot n$.

For general polygon finding (non-Steiner) optimum is NP hard.

Minimal Weight Triangulations

Find triangulation that minimizes total edge length.

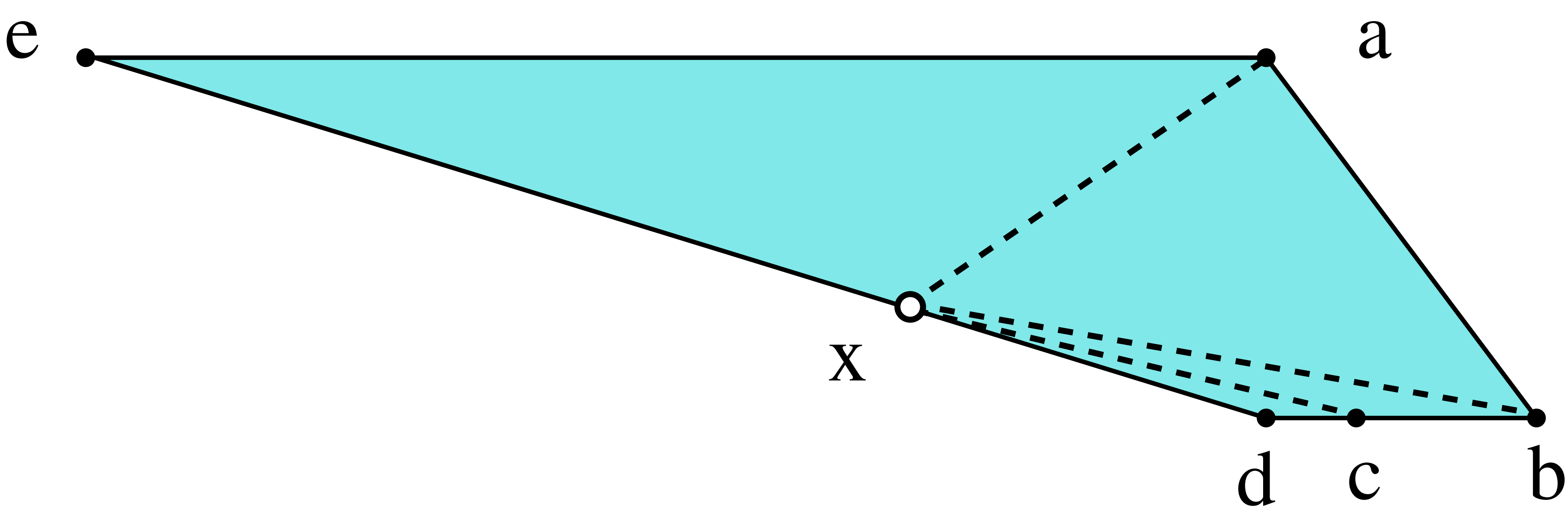


$1 \times R$ rectangle with n extra points on right side.

With Steiner points, length $\leq R + O(\log n)$ is possible.

What is optimal? Does an optimal triangulation exist?

Minimal Weight Triangulations



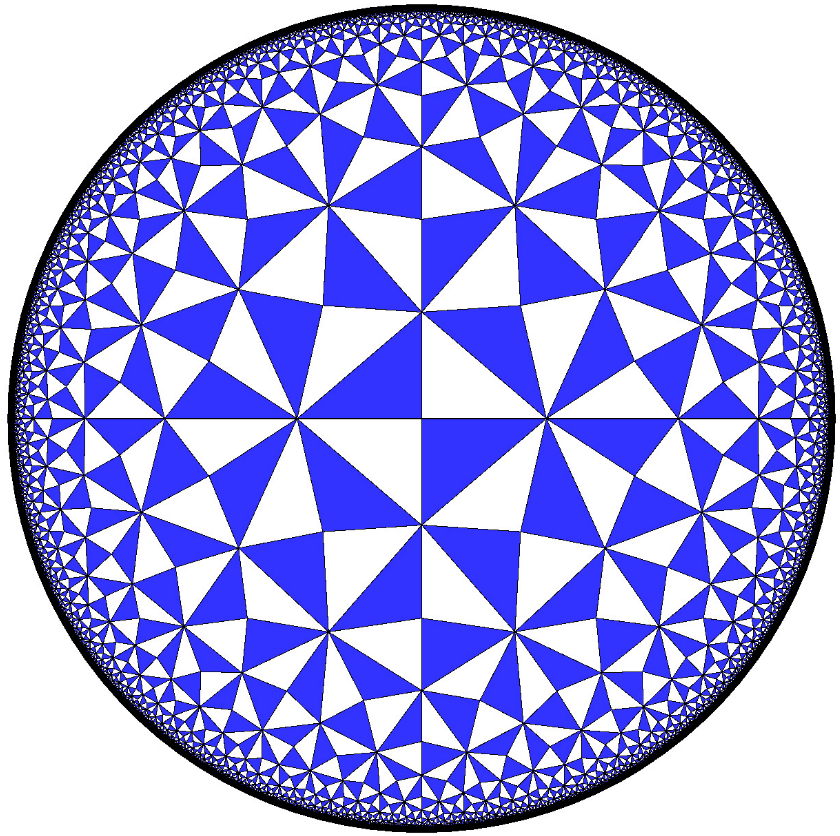
As x approached d, total length tends to infimum.

A minimal weight Steiner triangulation (MWST) does not exist.

But there are three co-linear points (b, c, d) .

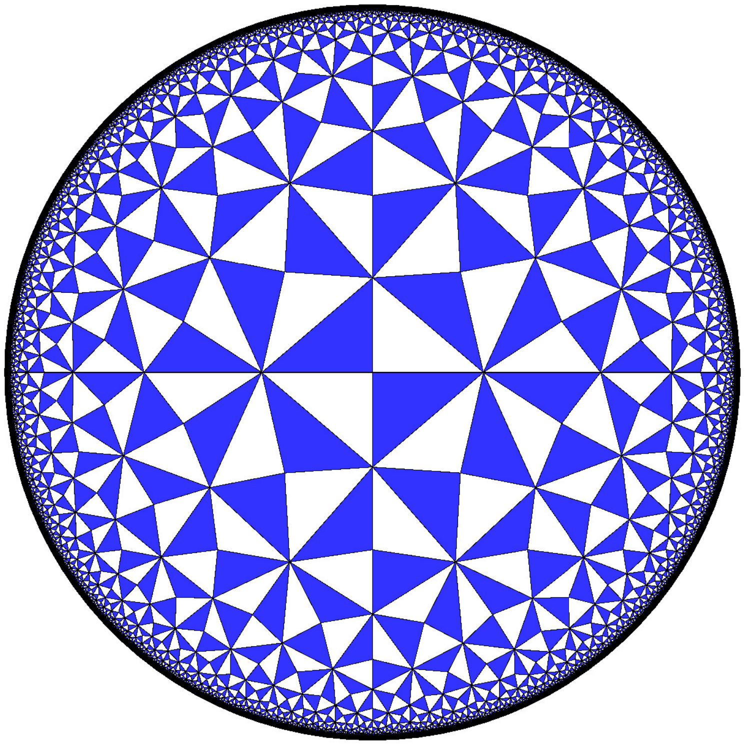
Question: Does a MWST always exist for general polygons?

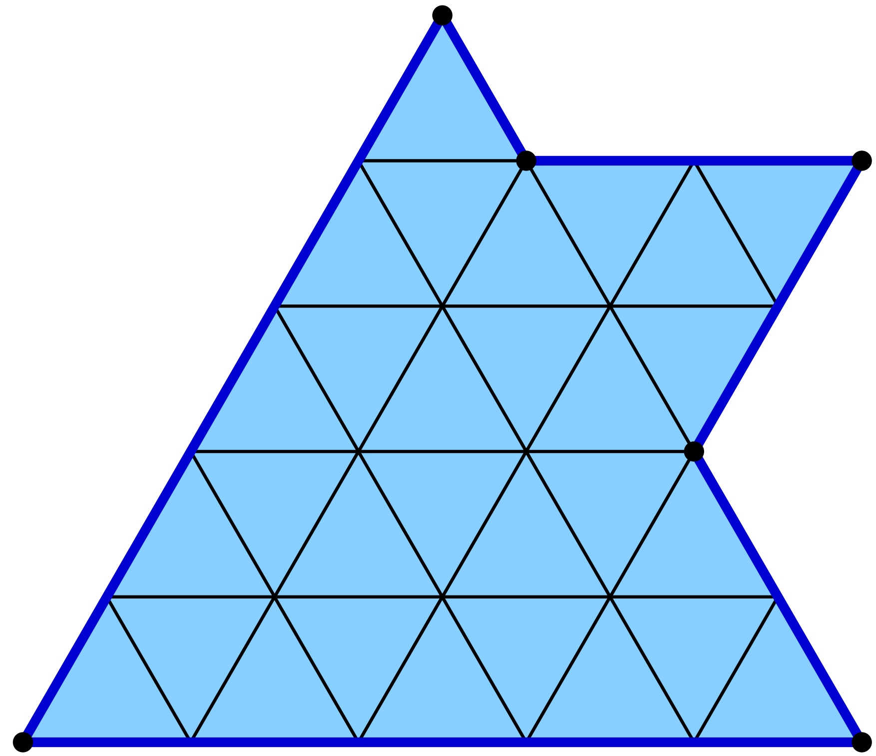
Hyperbolic triangulations:



Equilateral triangulation of disk in hyperbolic metric.

Hyperbolic triangulations:

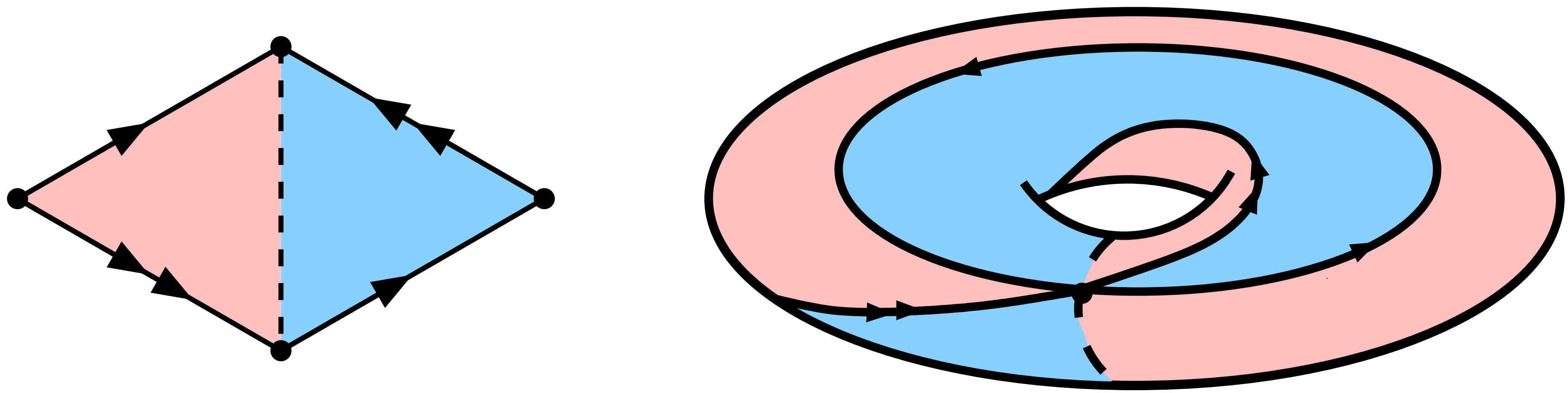




We say a triangulation is equilateral if any two triangles sharing an edge are reflections of each other across that edge.

This works in either geometry (if we know how to define reflections).

Hyperbolic triangulations:

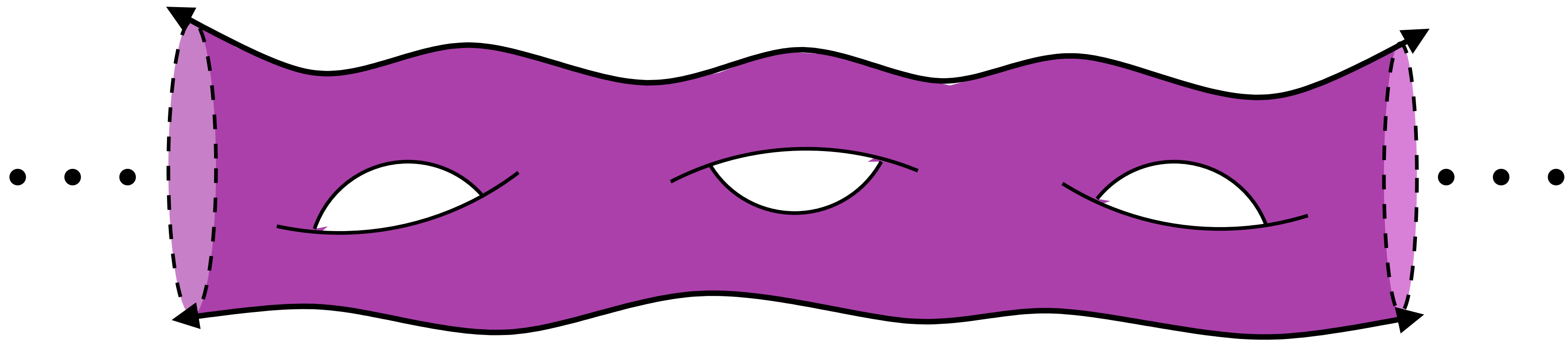


Some compact surfaces have an equilateral triangulation.

Most do not; only finitely many ways to glue together n triangles.

Compact surfaces with equilateral triangles have been intensely studied: related to Grothendieck's theory of *dessins d'enfants*, graph theory, probability, algebraic number theory, Galois theory, ...

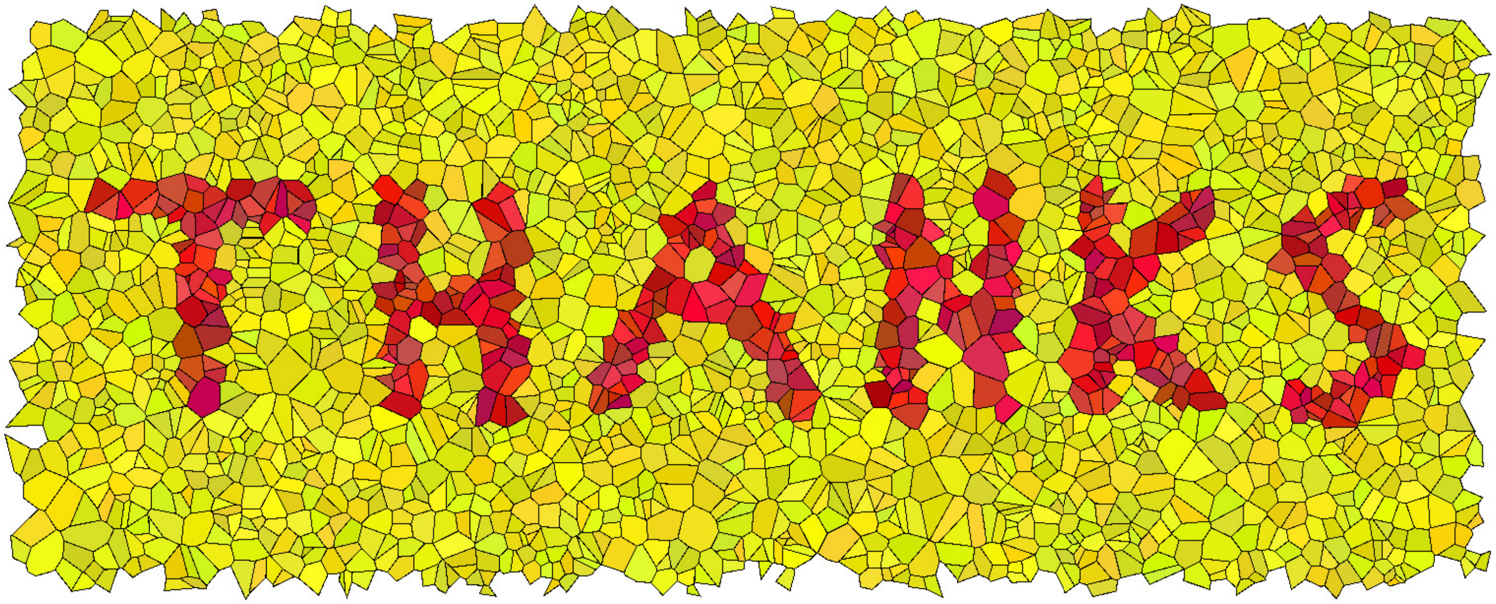
Hyperbolic triangulations:

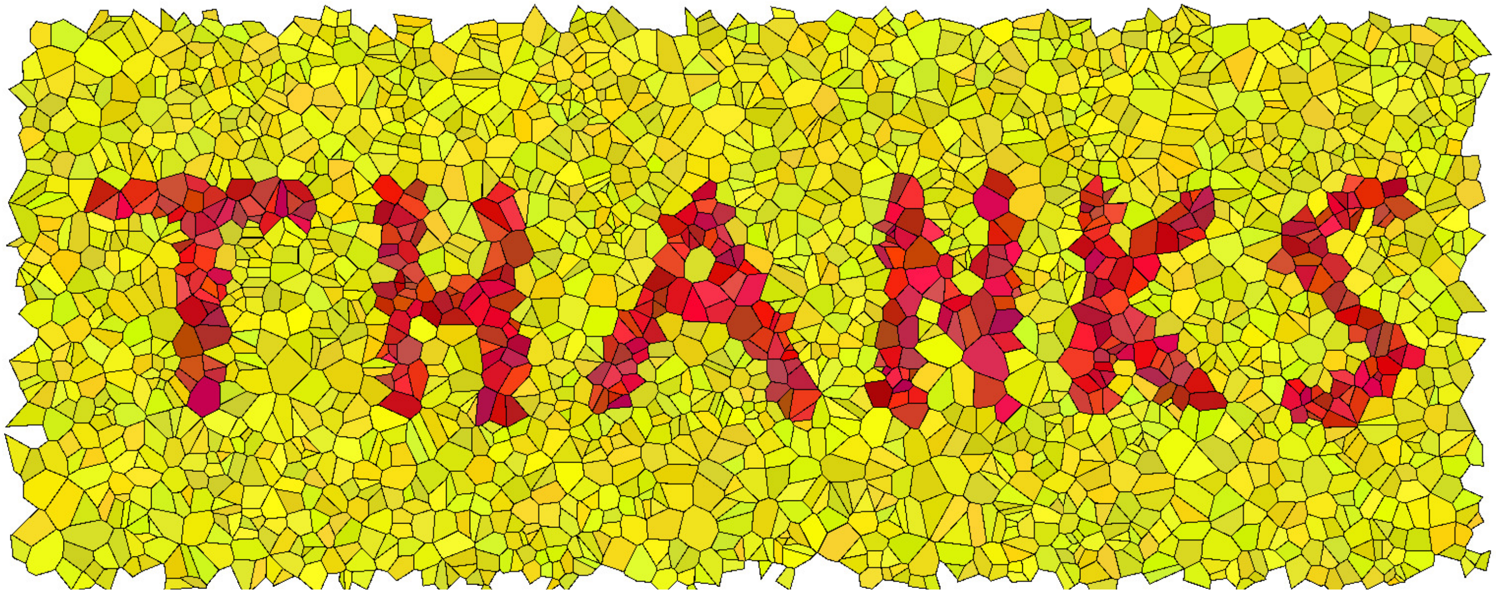


Thm (Bishop-Rempe, 2026): Every non-compact hyperbolic surface has an equilateral triangulation.

Proof combines differential equations and planar acute triangulation.

Many questions about hyperbolic triangulations remain open.


THANKS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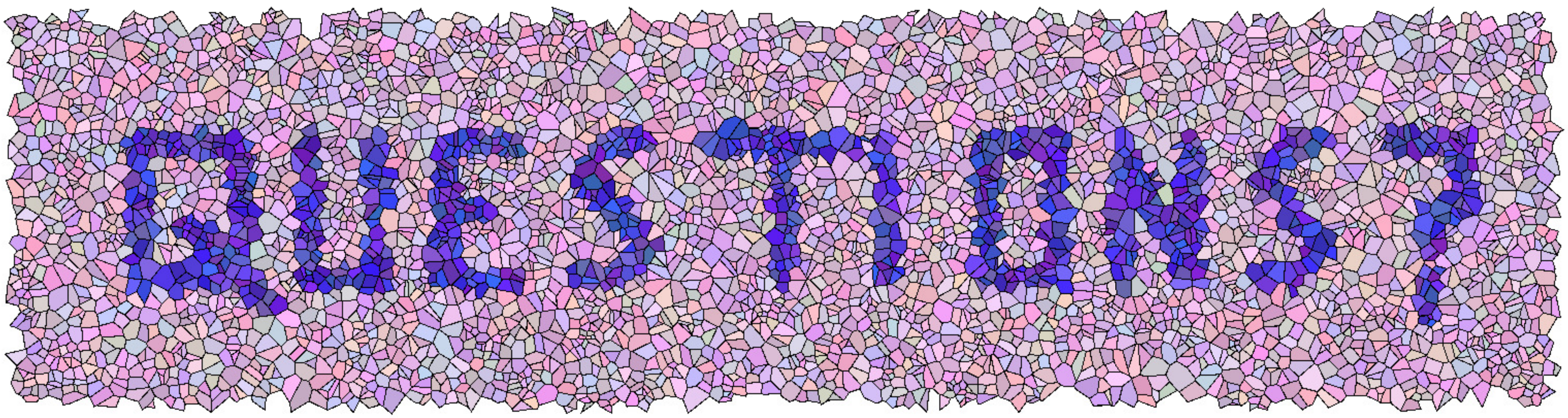
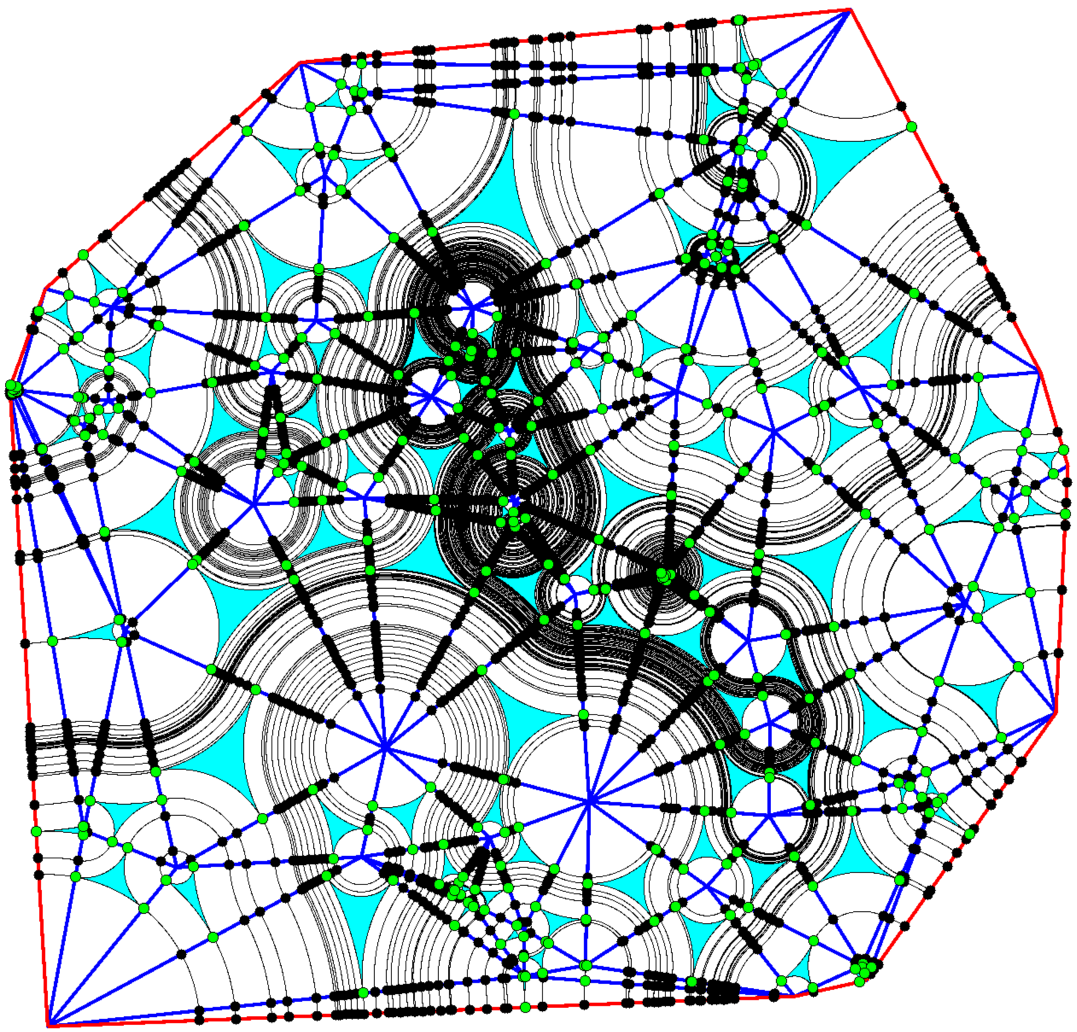

QUESTIONS?

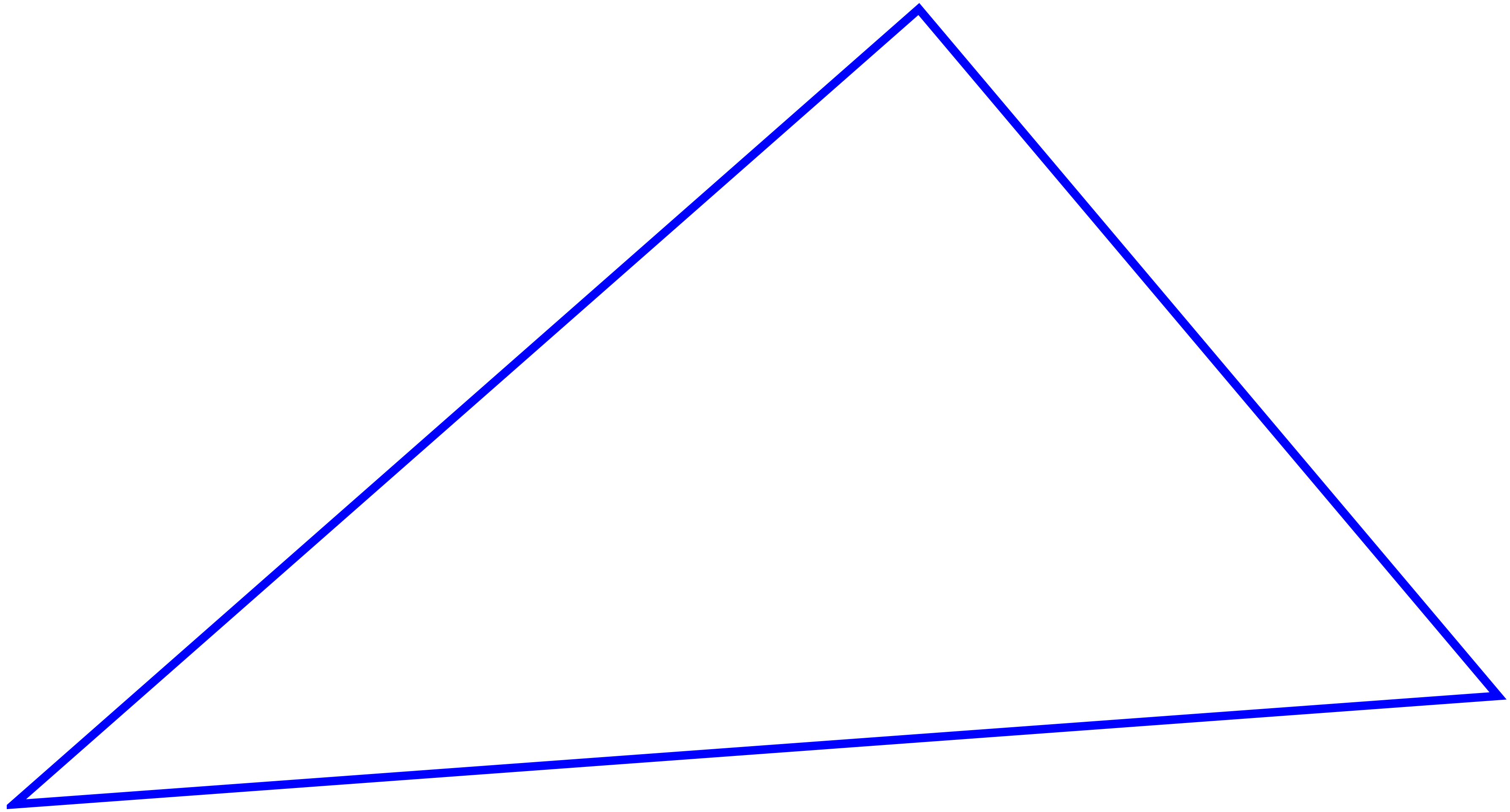
Exercise: Prove that $\theta = \arccos(1/3)$ is not a rational multiple of π.

Solution: We give a proof by contradiction. Suppose $\theta = m\pi/n$. Then $\sin(n\theta) = \sin(m\pi) = 0$, and hence $\cos(n\theta) = \pm 1$. Let $z = \cos\theta + i\sin\theta$. Then $z^n = \cos(n\theta) + i\sin(n\theta) = \pm 1$, so z is an algebraic integer (the root of an equation with integer coefficients, $z^n \pm 1$).

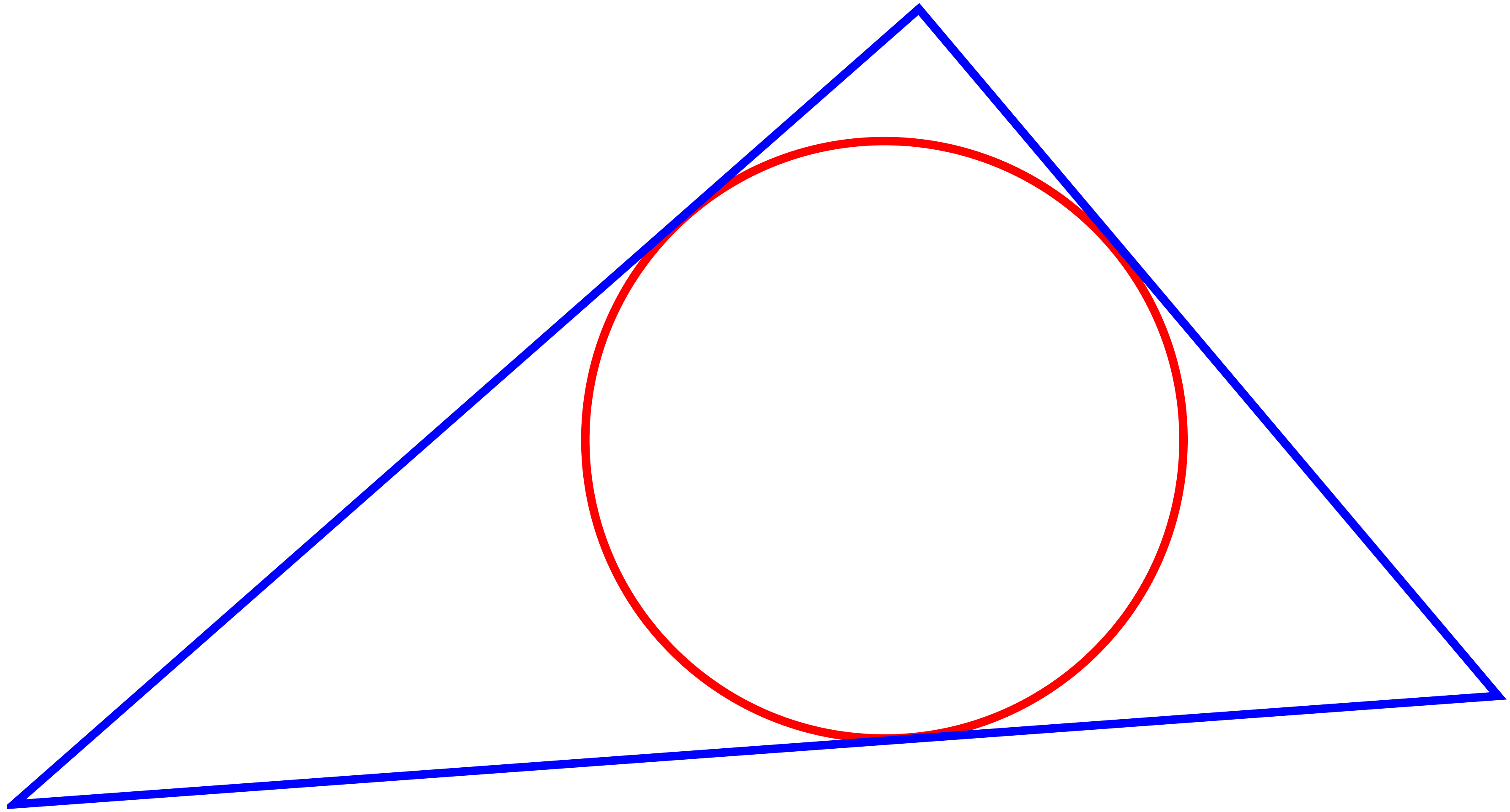
Similarly, $(1/z)^n = \pm 1$, so $1/z$ is also an algebraic integer.

Thus $z + 1/z = 2\cos(\theta) = 2/3$ is an algebraic integer, which is impossible since the only real rational numbers that are algebraic integers are the regular integers. $\square$

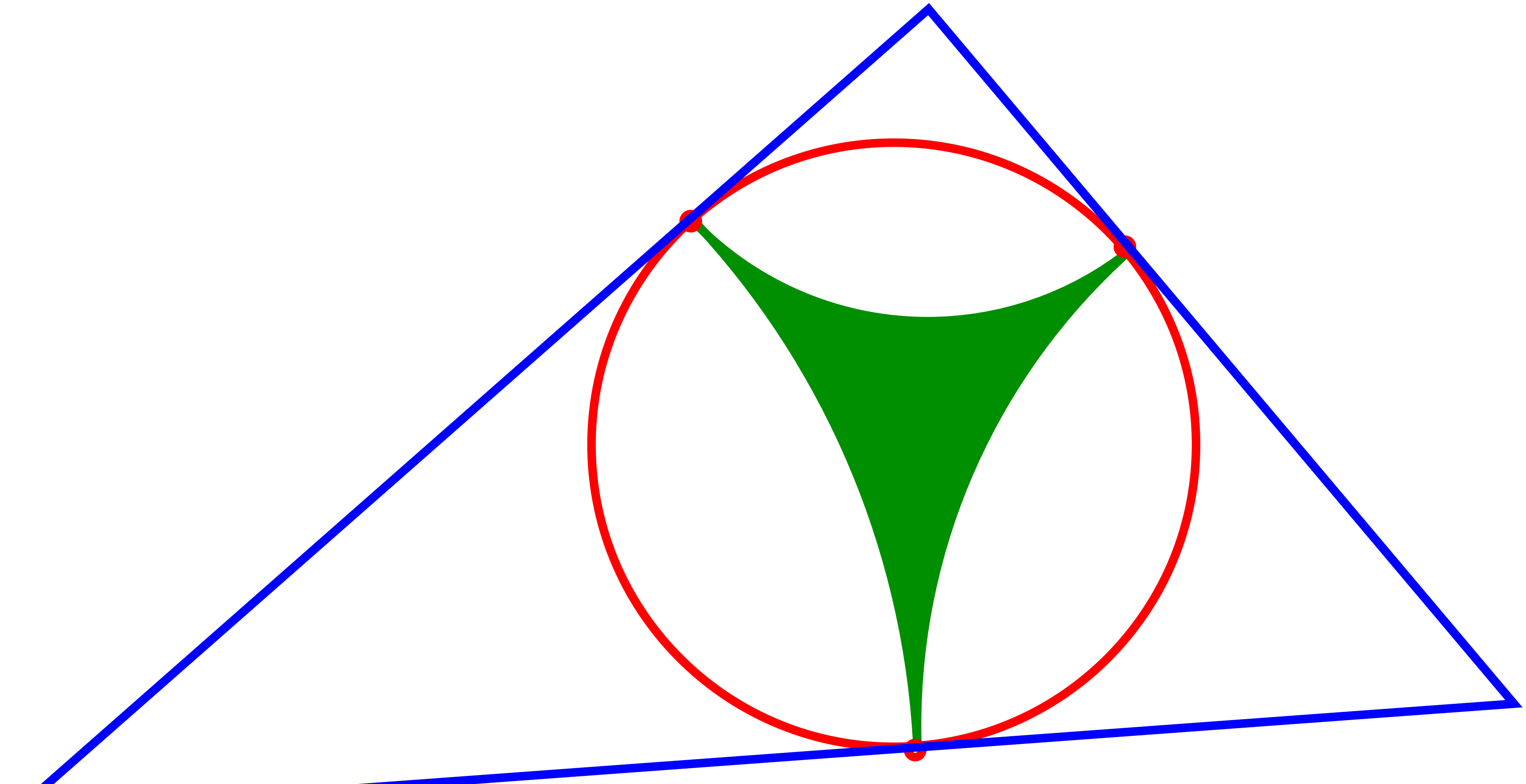




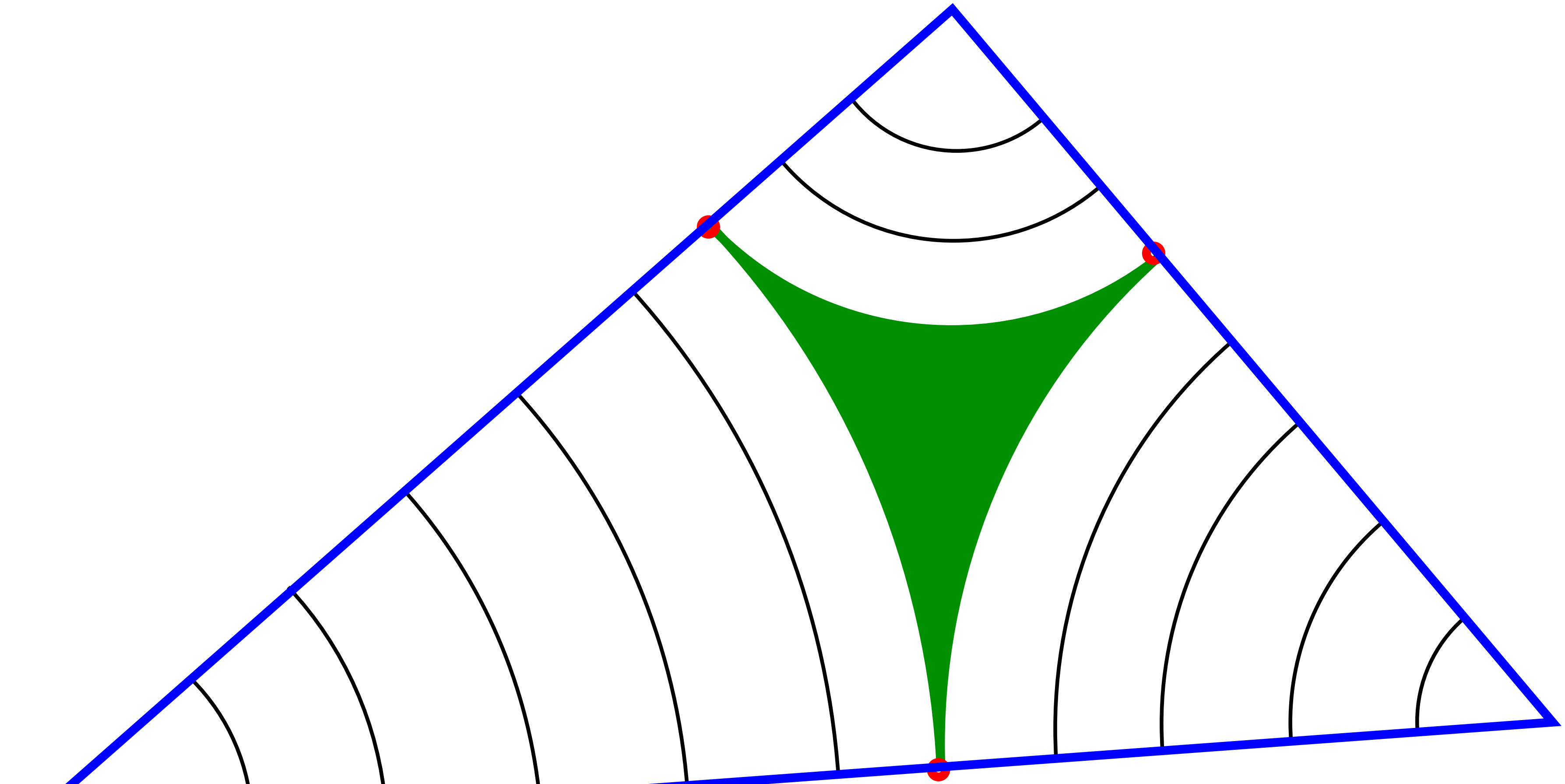
A triangle.



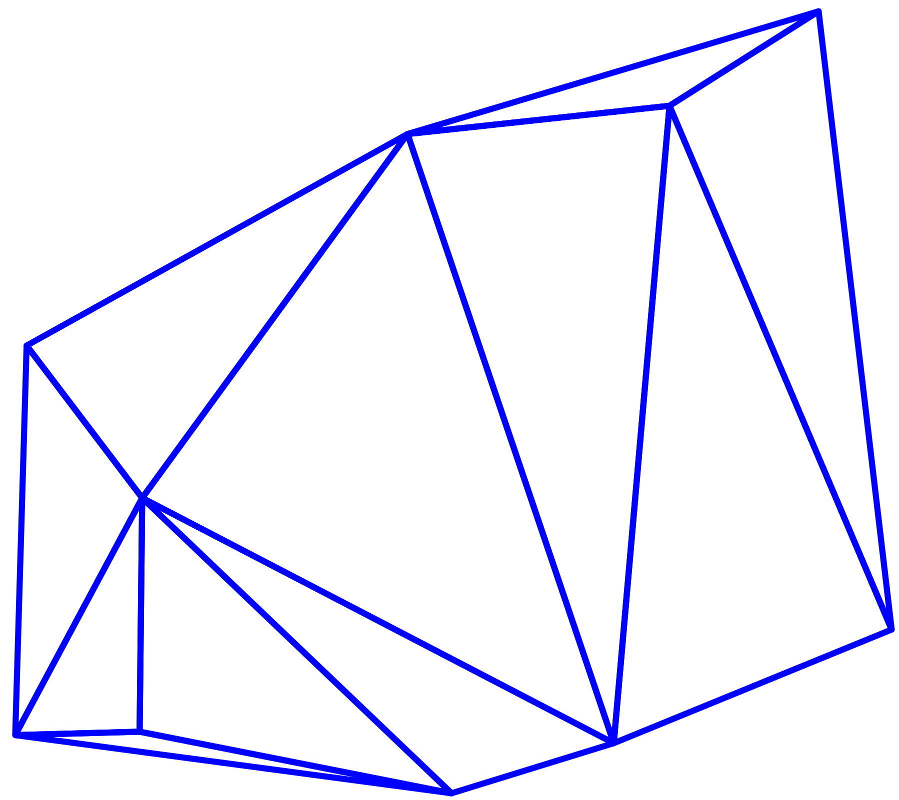
Its in-circle.



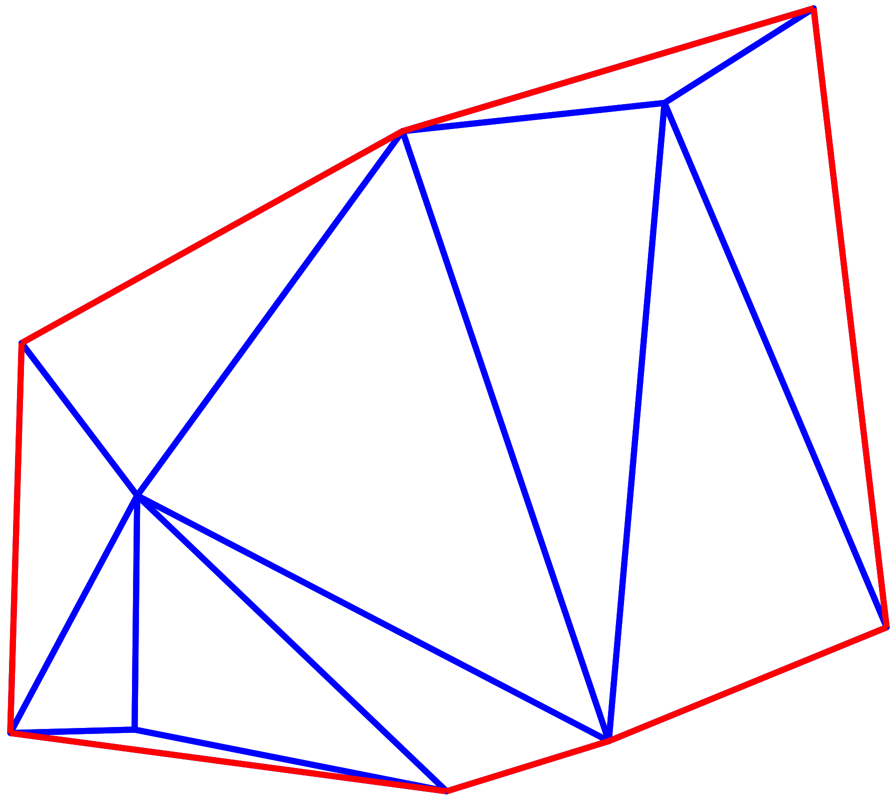
The central region and three sectors (thin version).



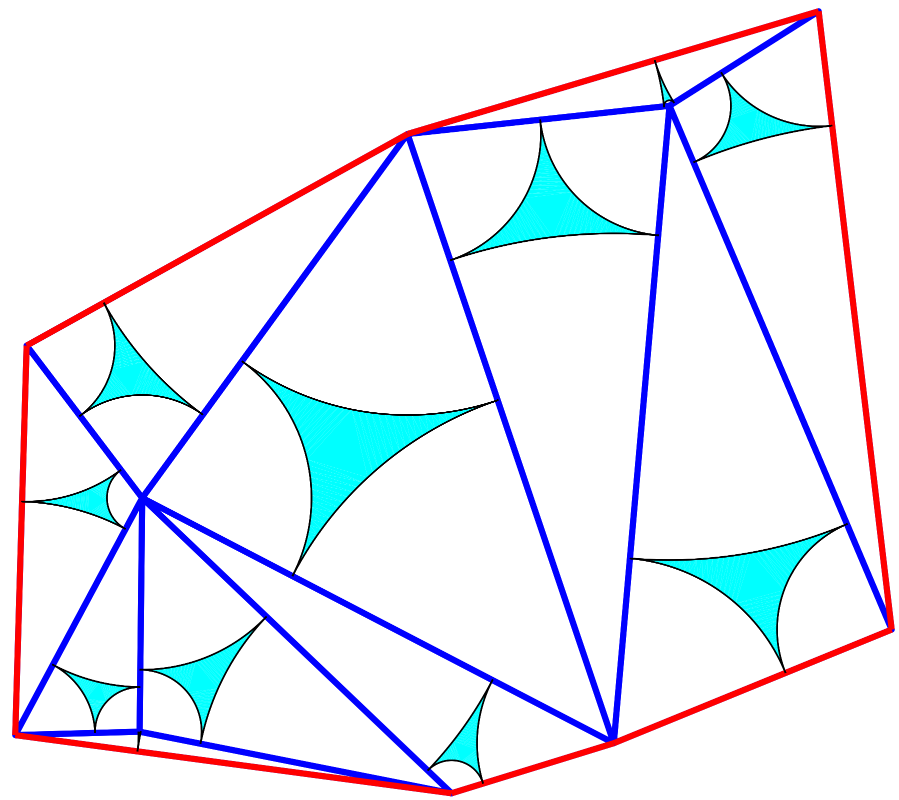
The three sectors are foliated by circular arcs.
Defines flow on a triangulation that stops at boundary or cusp point.



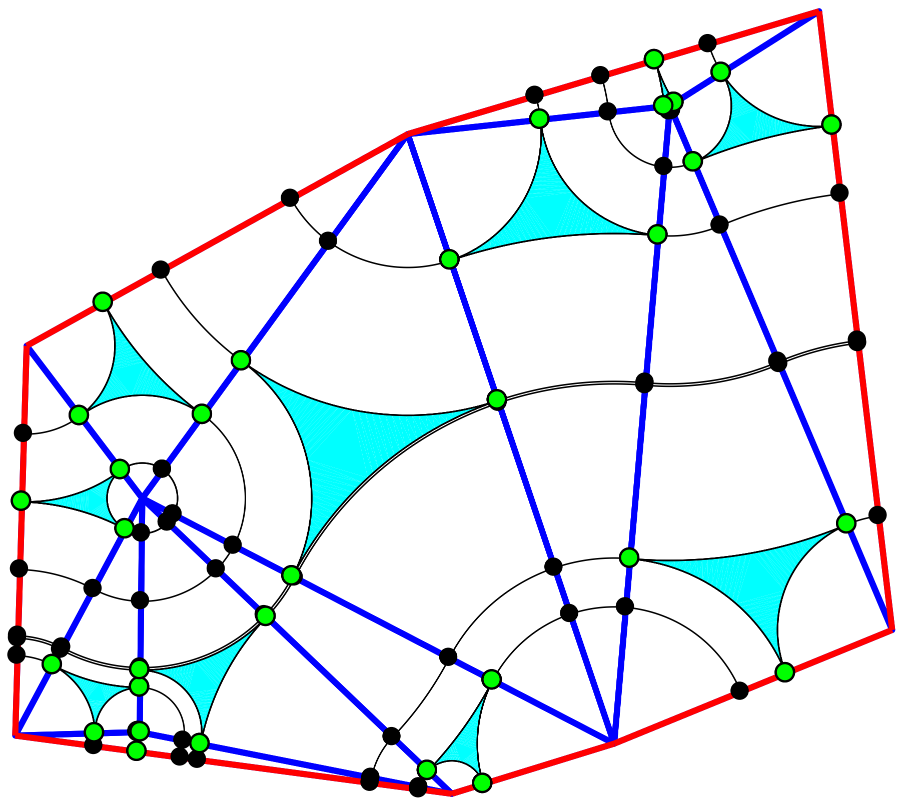
Delaunay triangulation of 10 random points,



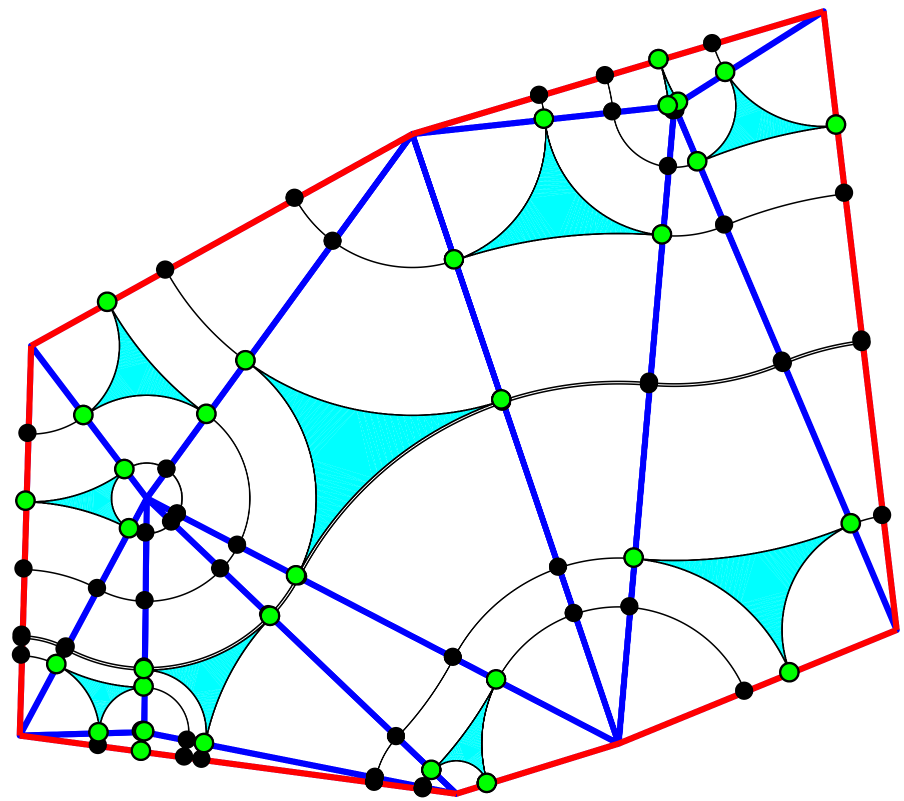
The boundary of the triangulation



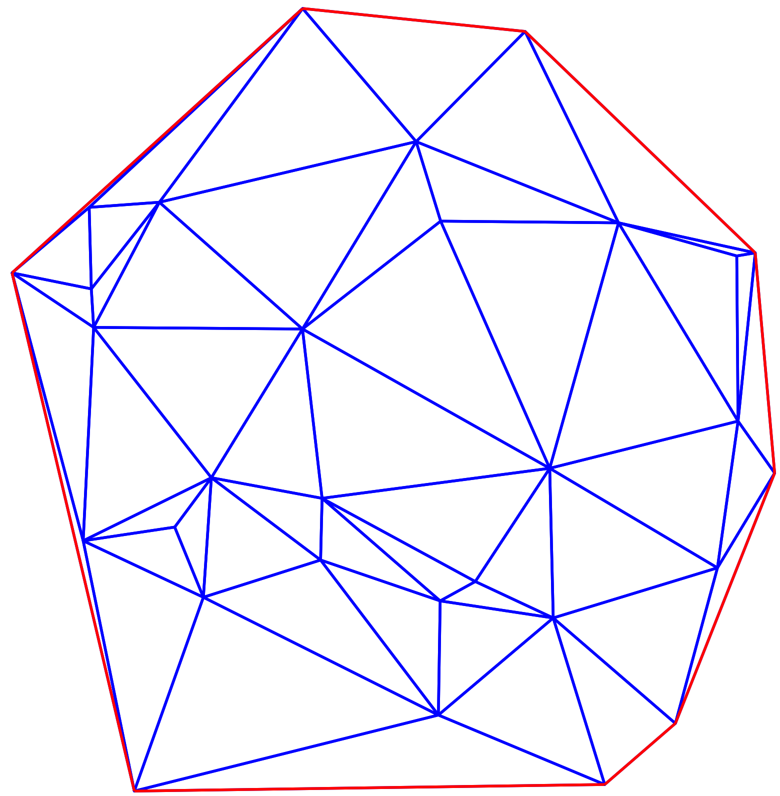
The central regions.



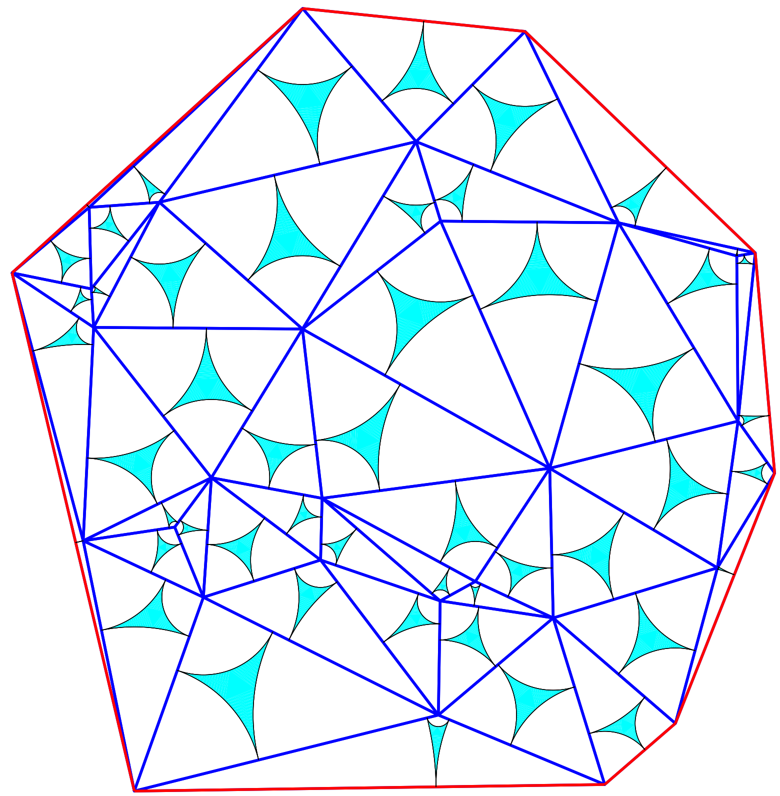
Propagation lines starting at all cusp points.



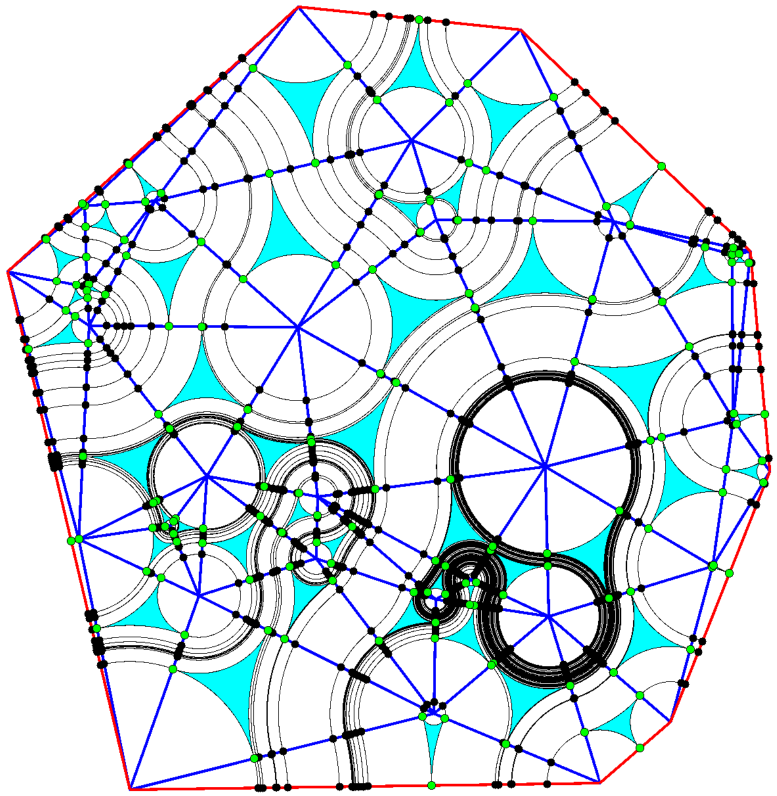
Propagation lines identify boundary points; induces tree.
Discontinuous, but piecewise length preserving.



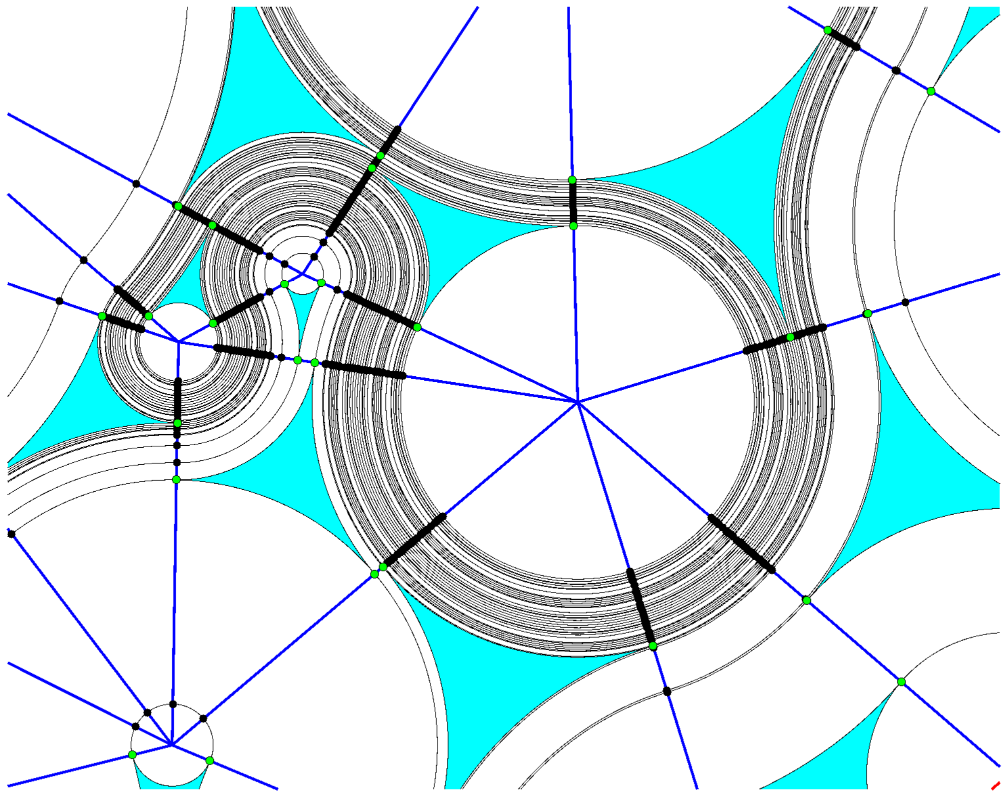
Delaunay triangulation of 30 random points in disk.



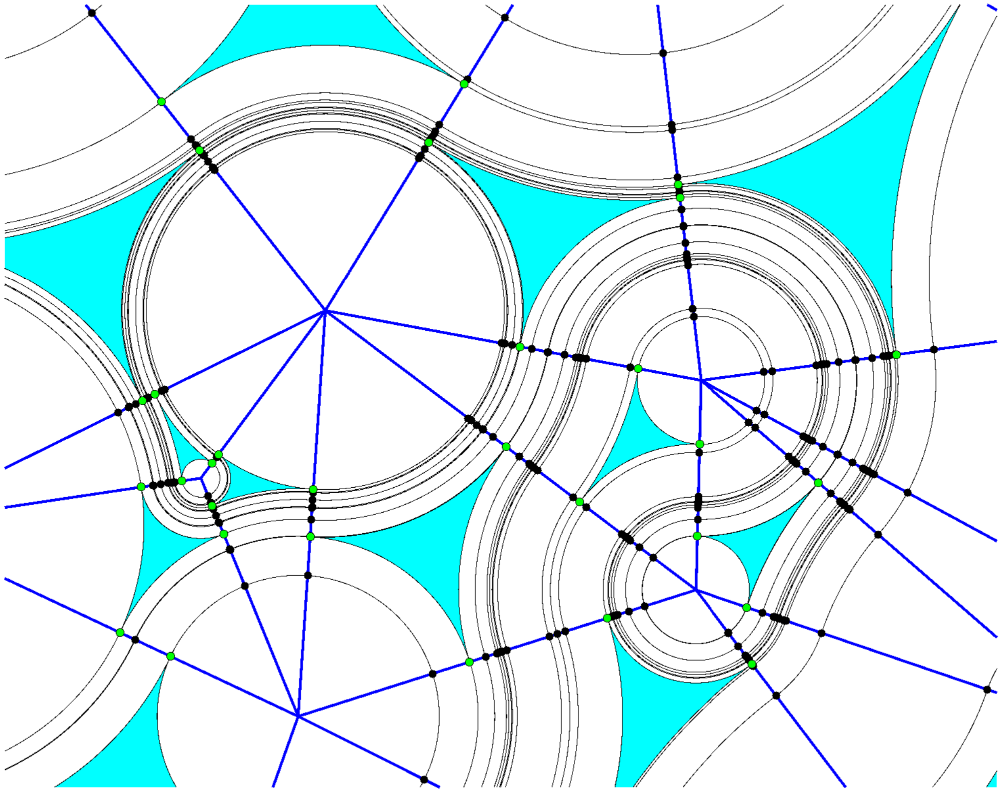
The central regions.



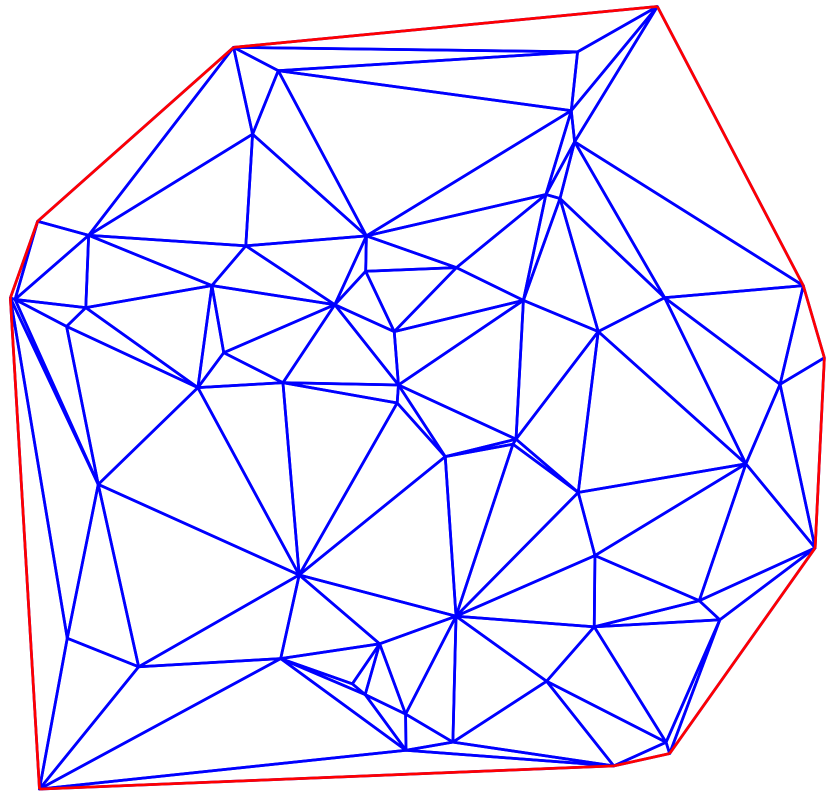
Propagation lines starting at all cusp points.



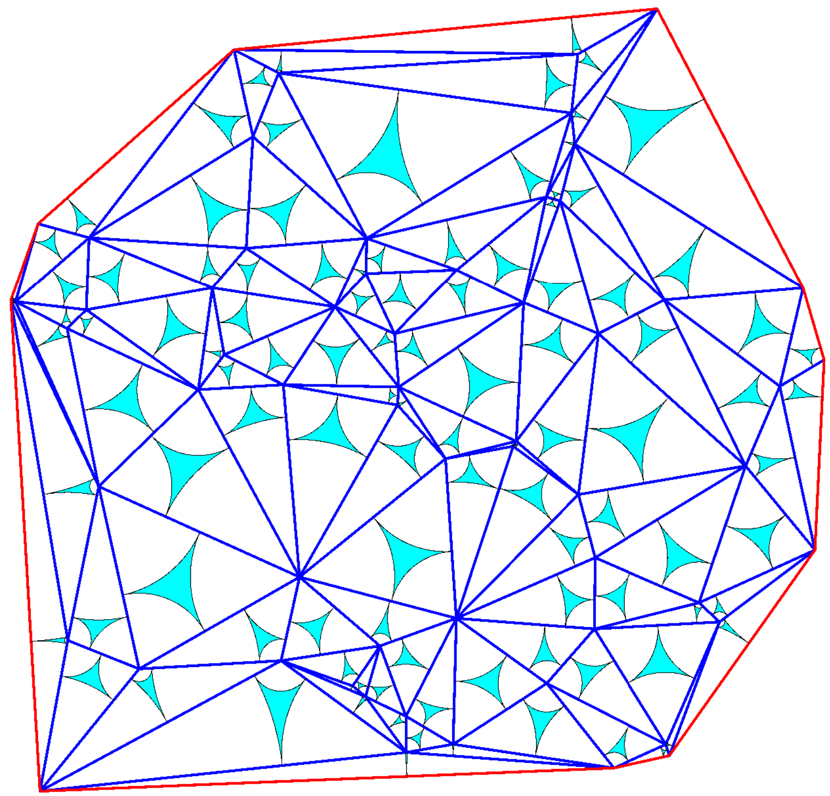
Enlargement 1.



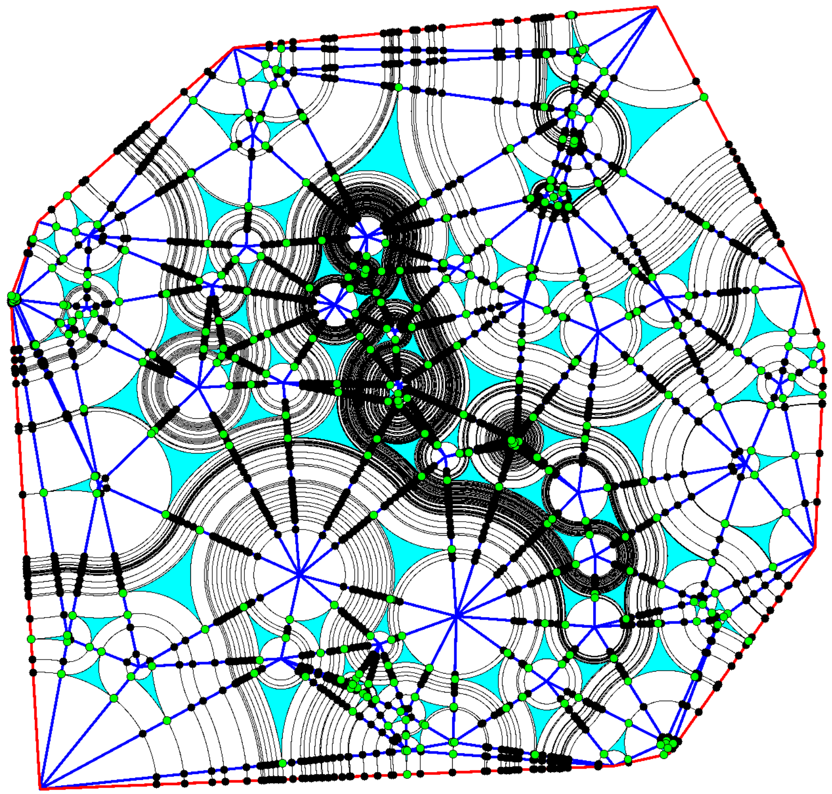
Enlargement 2.



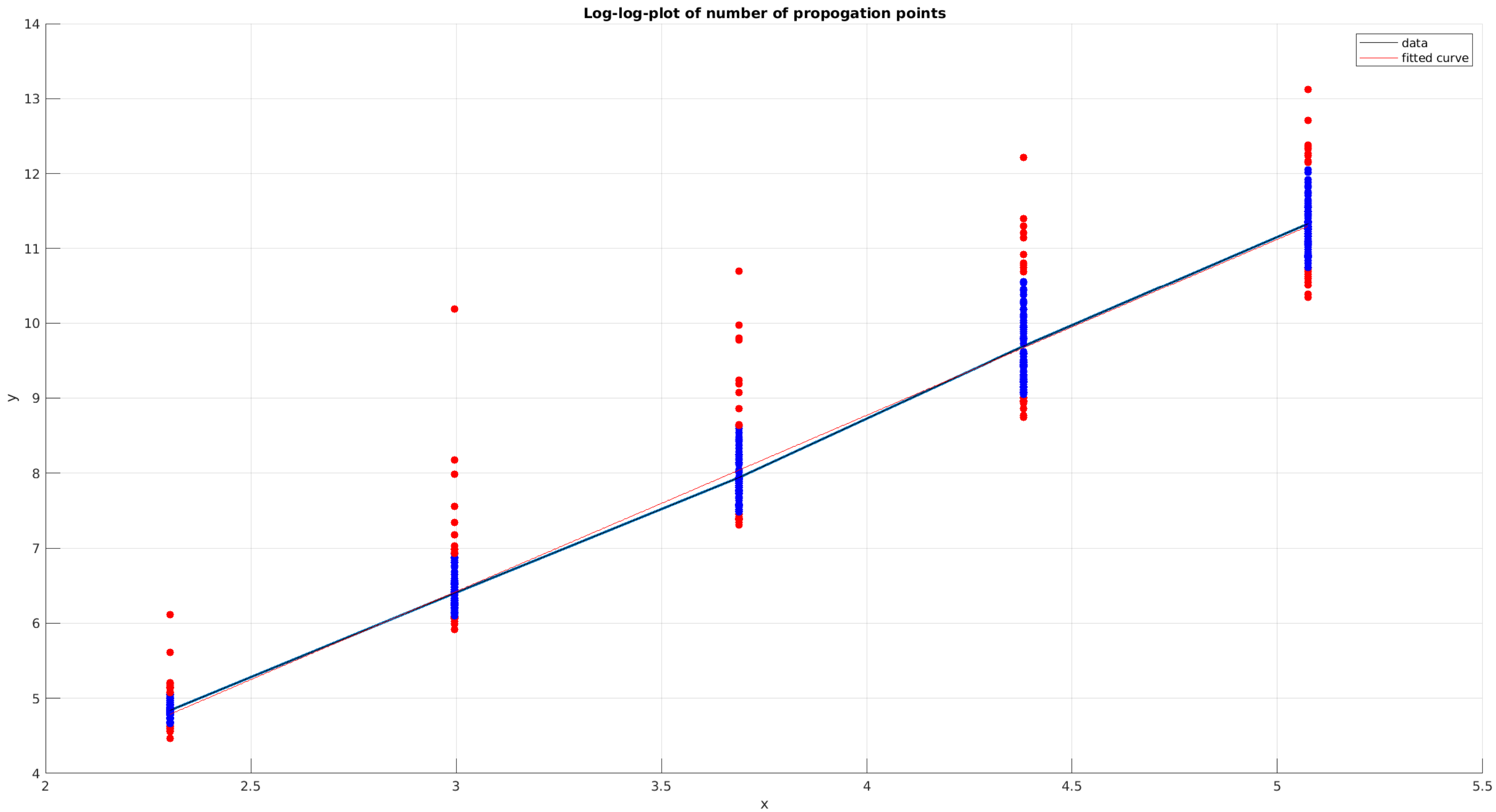
60 points



The central regions.



Propagation lines starting at all cusp points.



Log-log plot of number points created versus n.

Slope ≈ 2.5