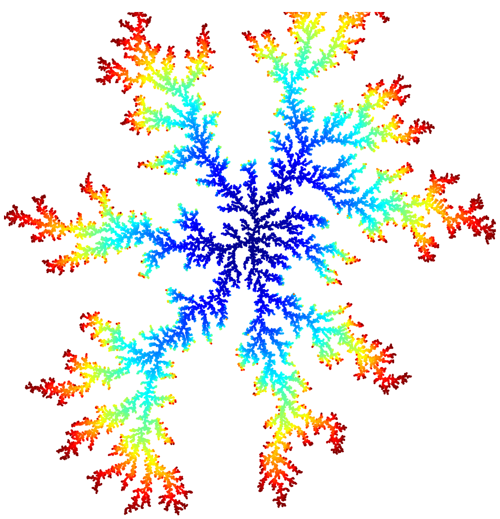
RANDOM THOUGHTS ON RANDOM SETS

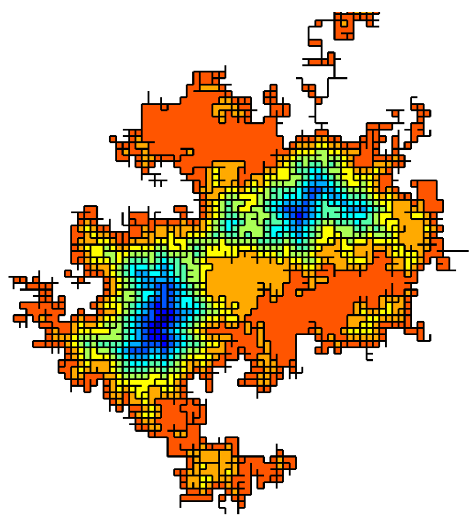
Christopher Bishop, Stony Brook

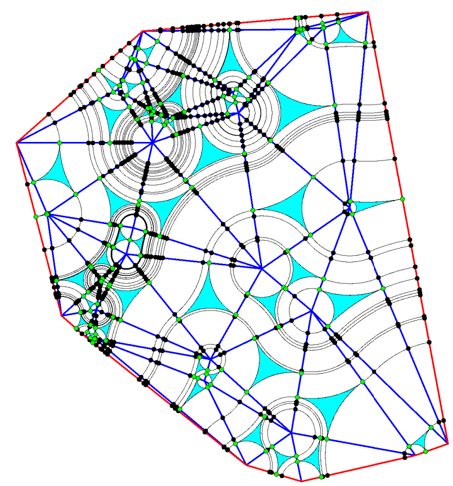
March 2, 2021

St Andrews Analysis Seminar

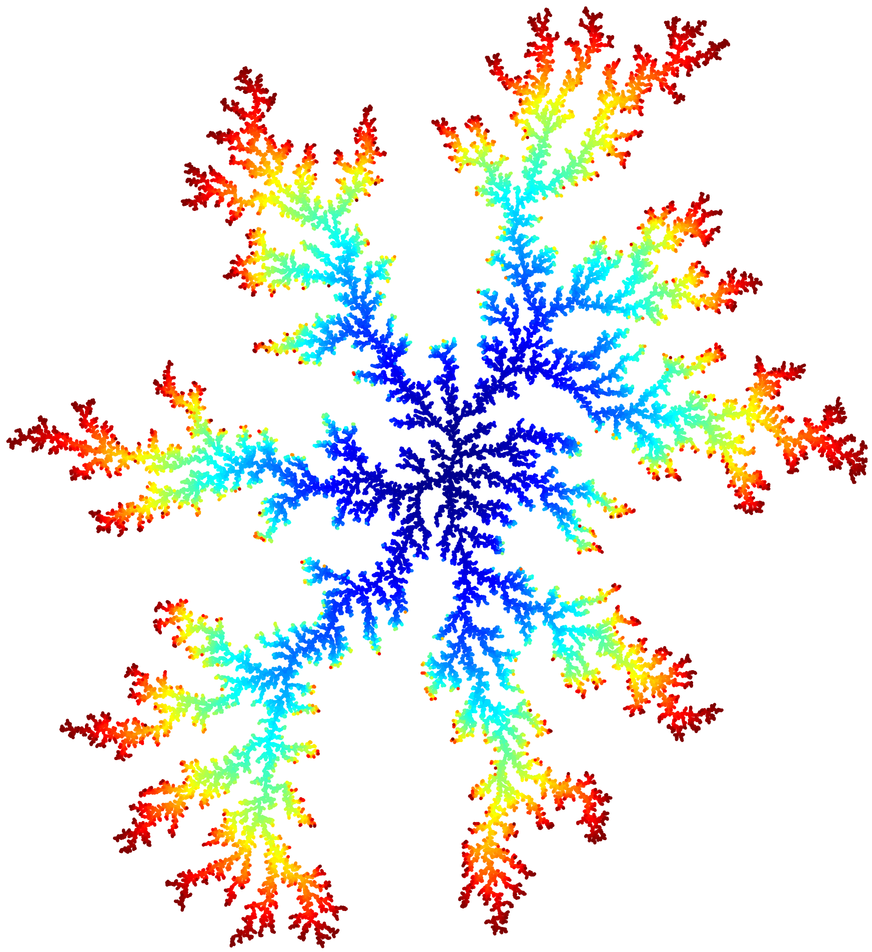
www.math.sunysb.edu/~bishop/lectures

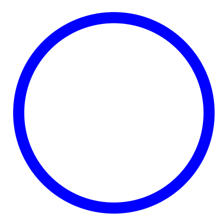


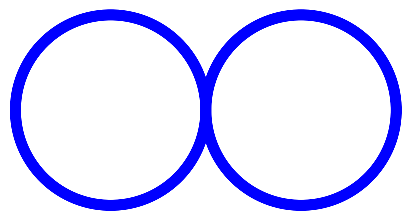


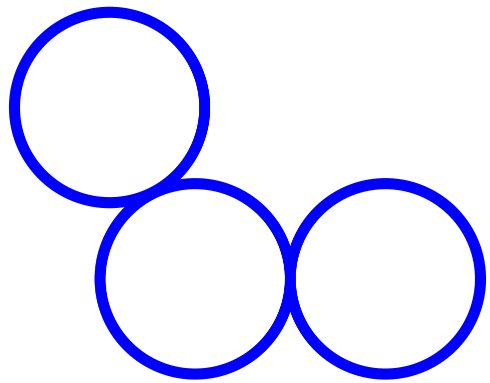


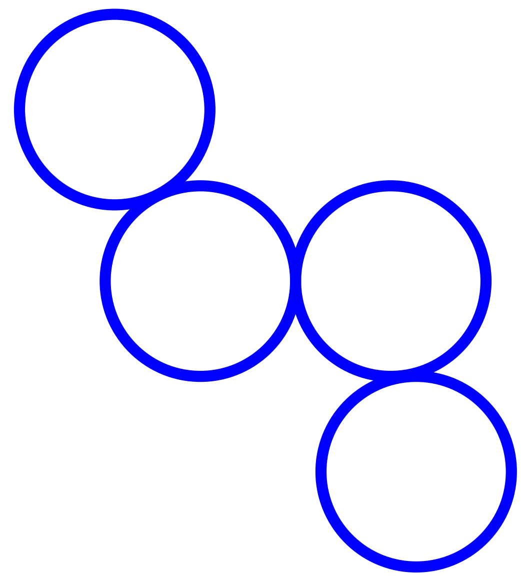
PART I: DIFFUSION LIMITED AGGREGATION (DLA)

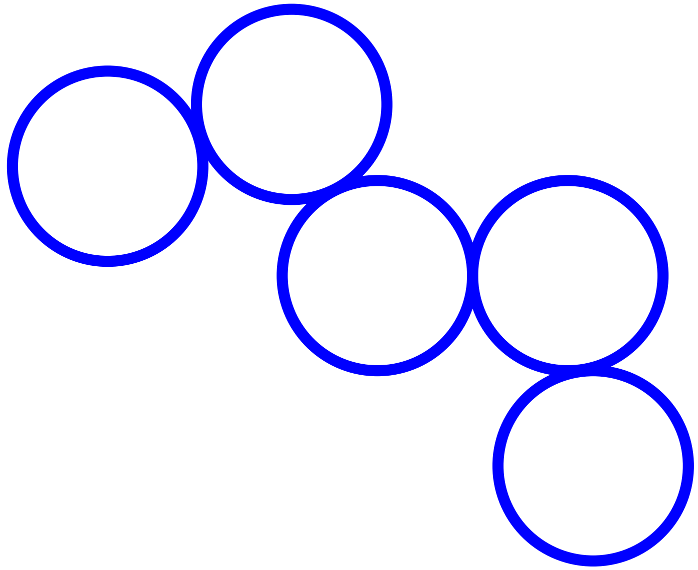


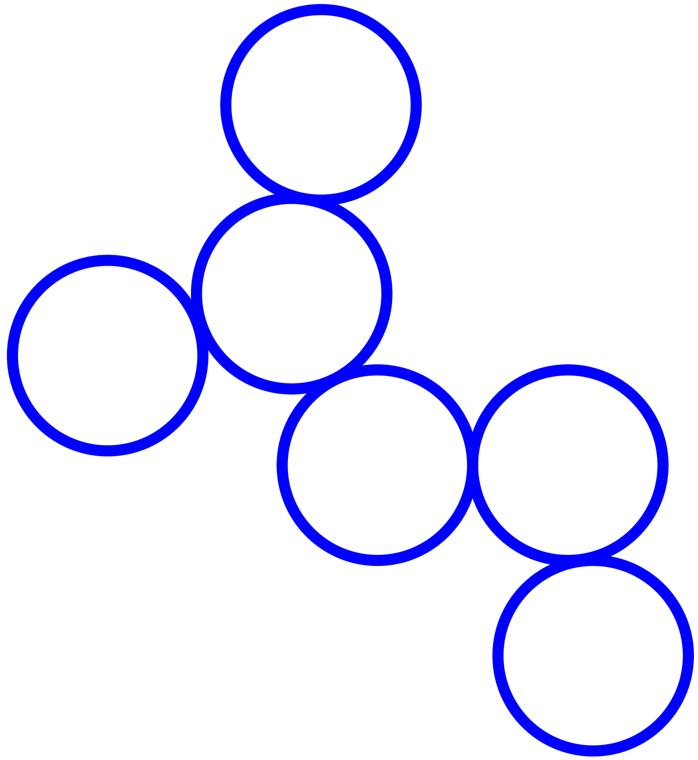


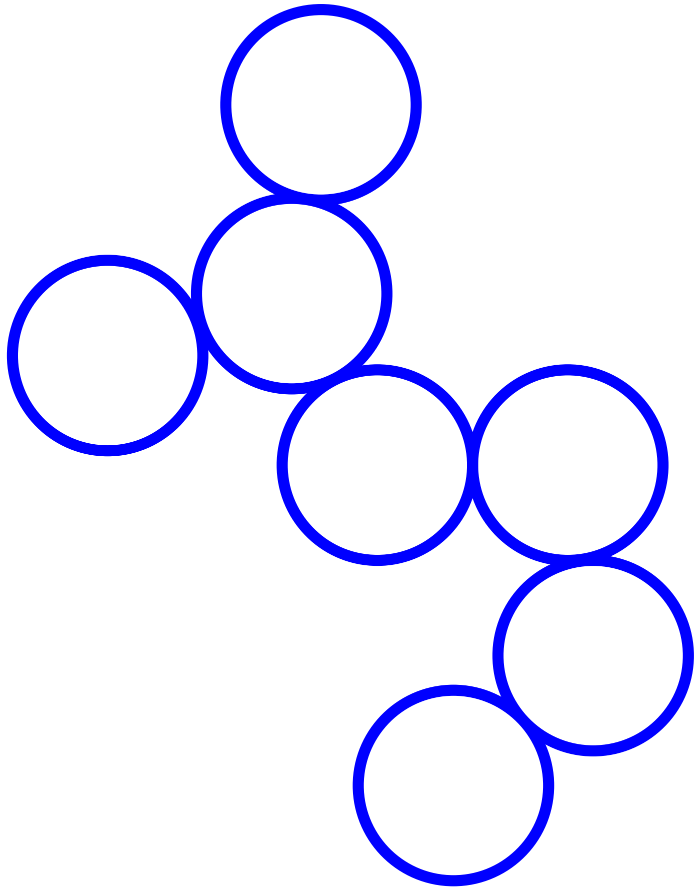


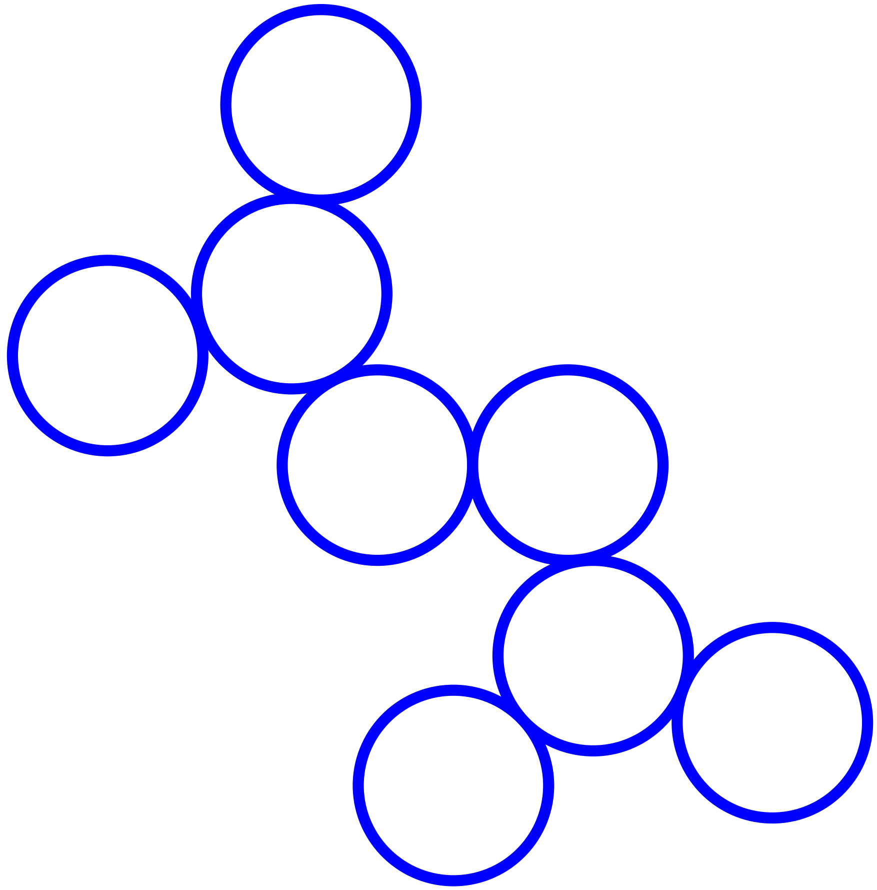


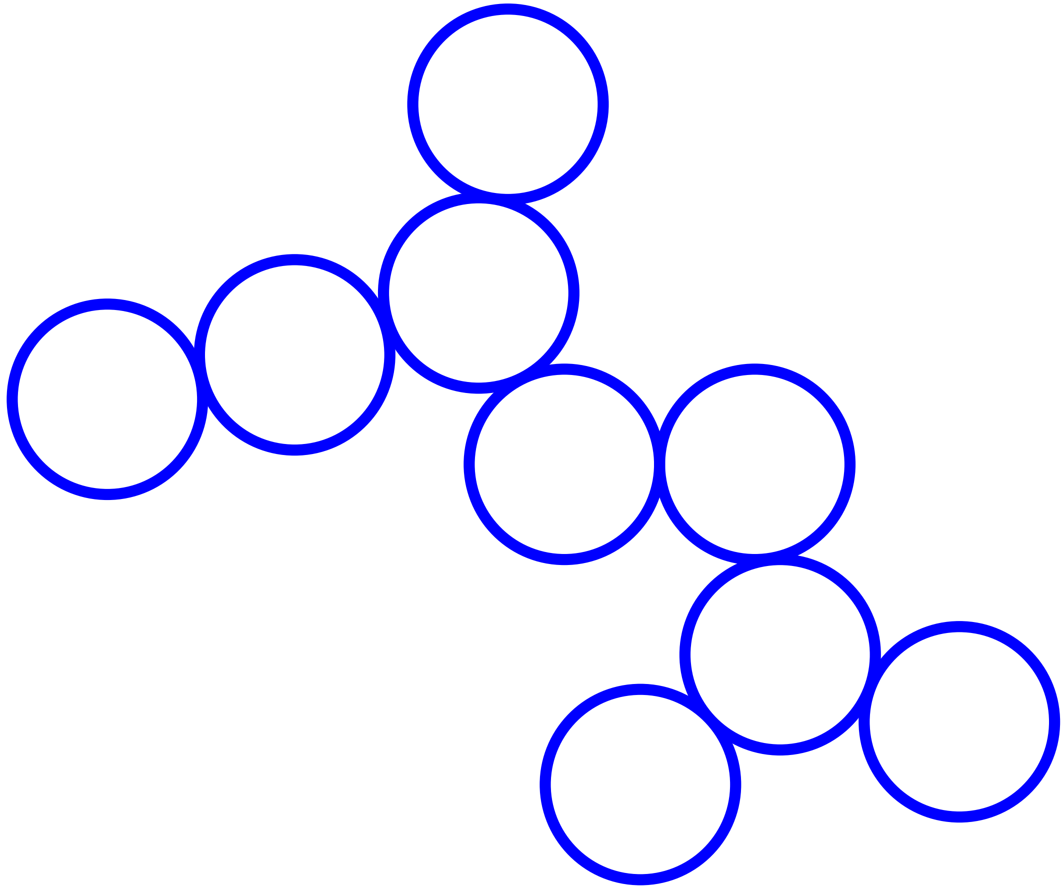


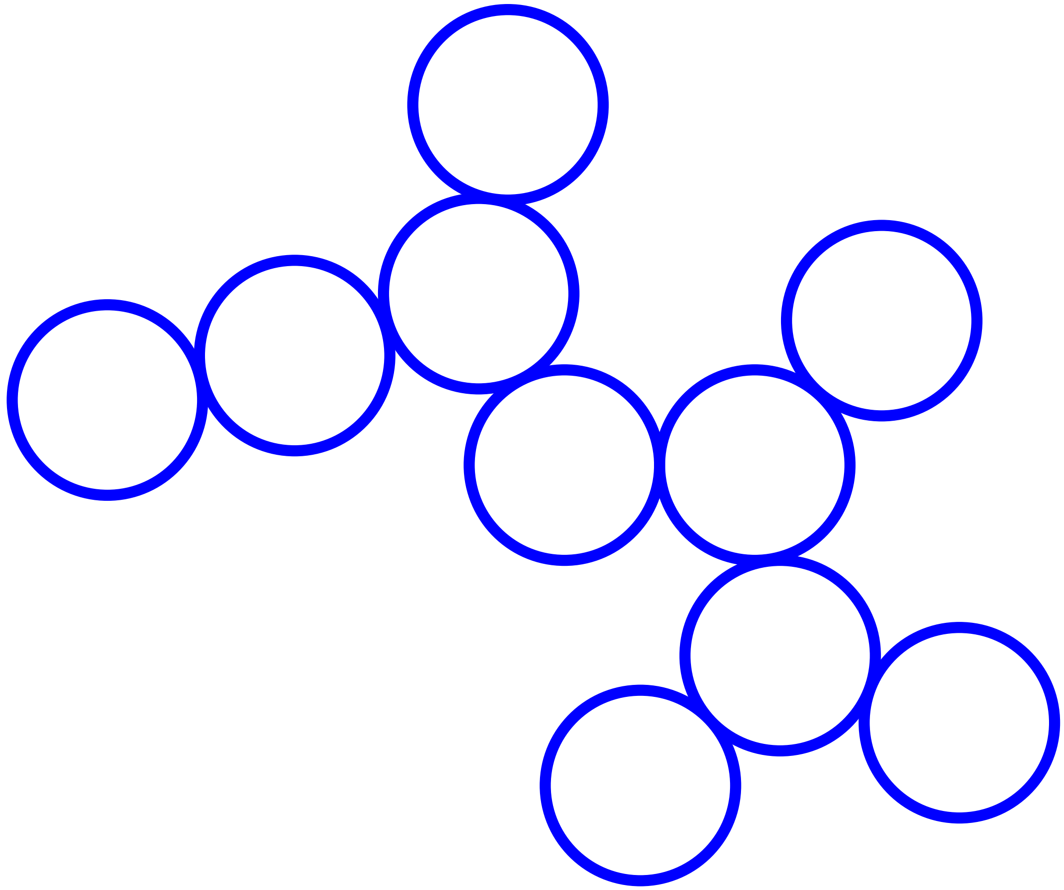


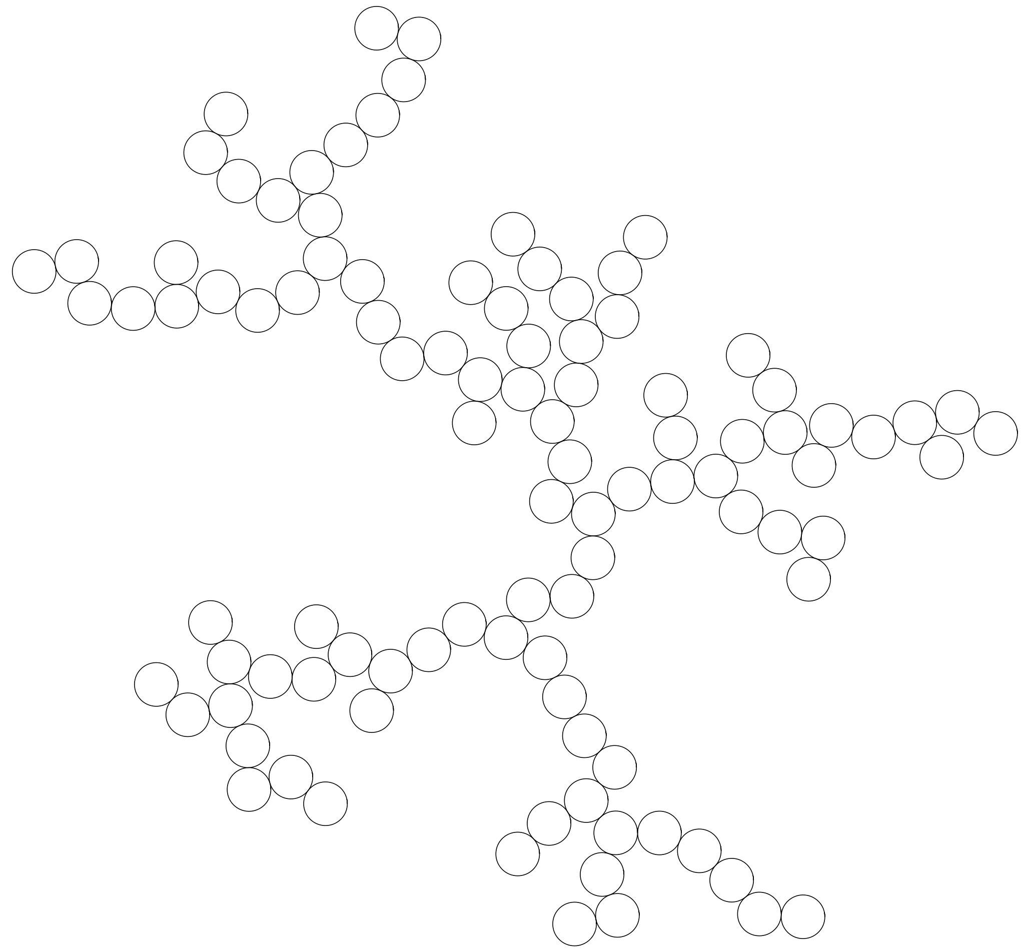


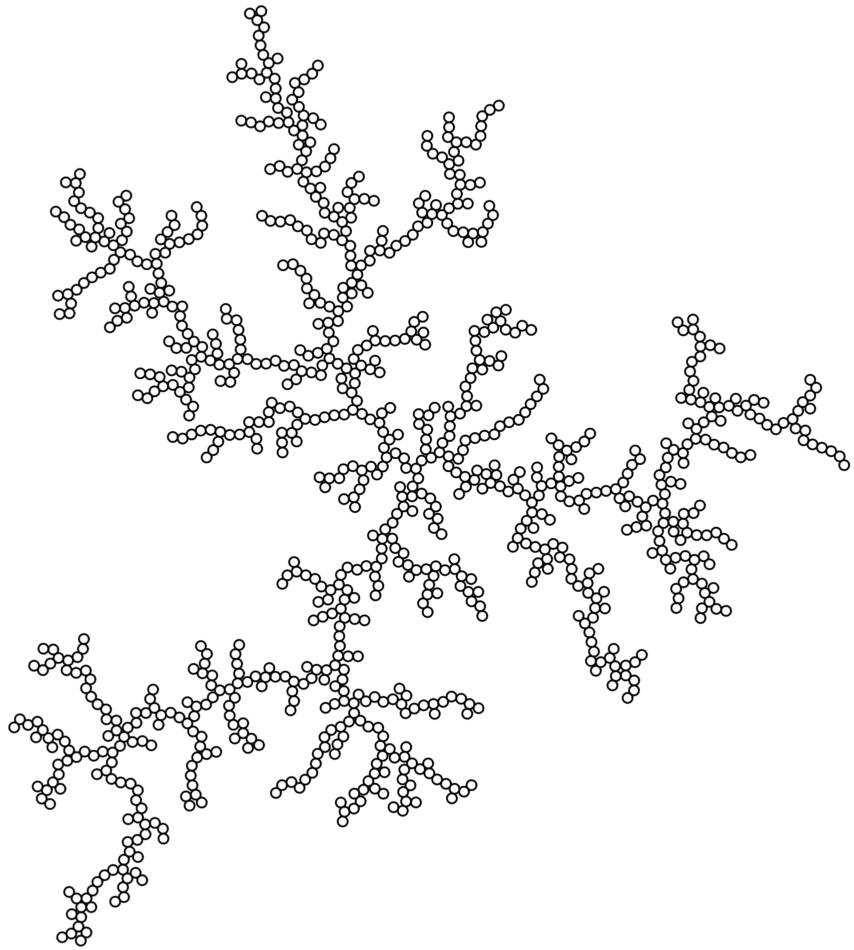


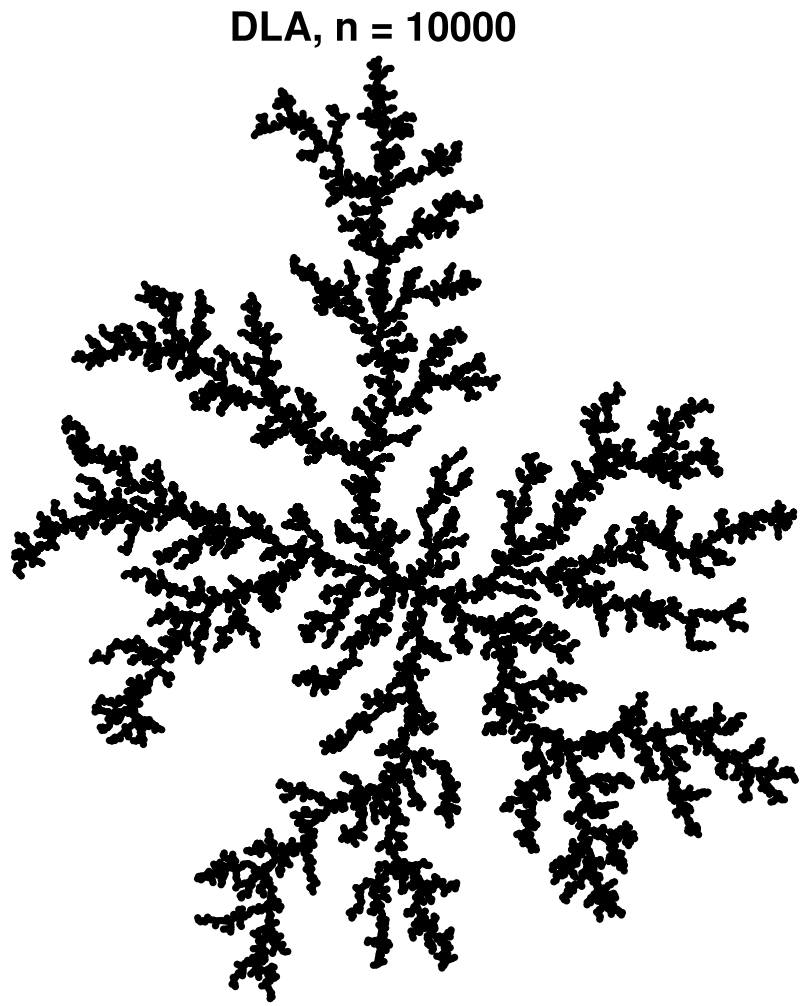




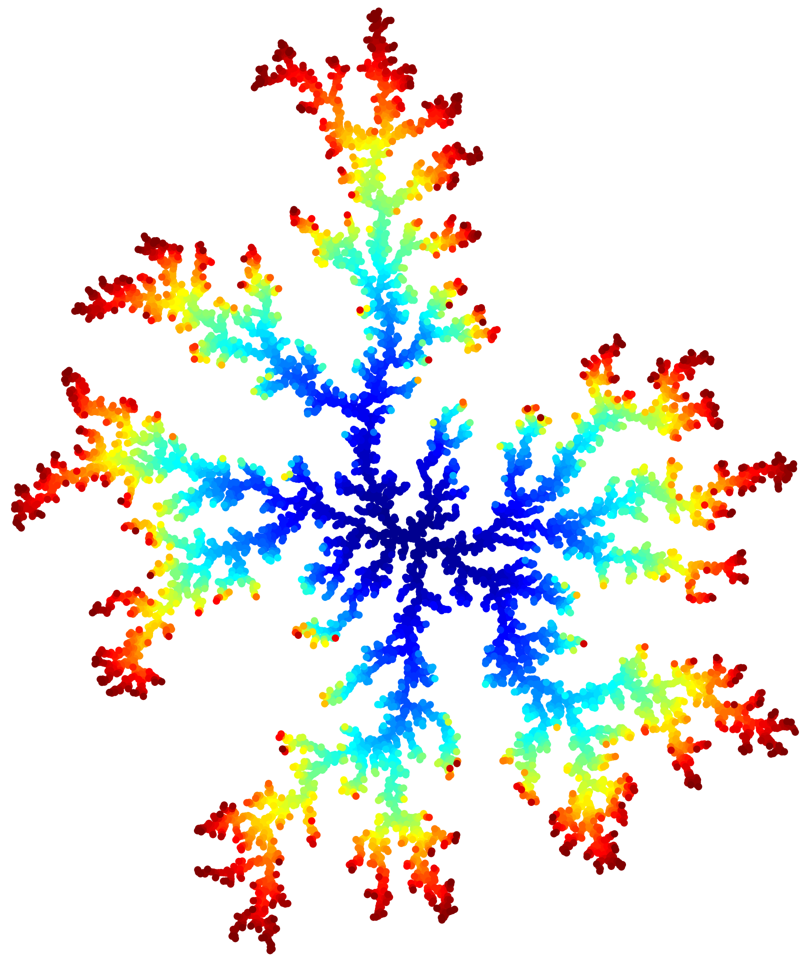




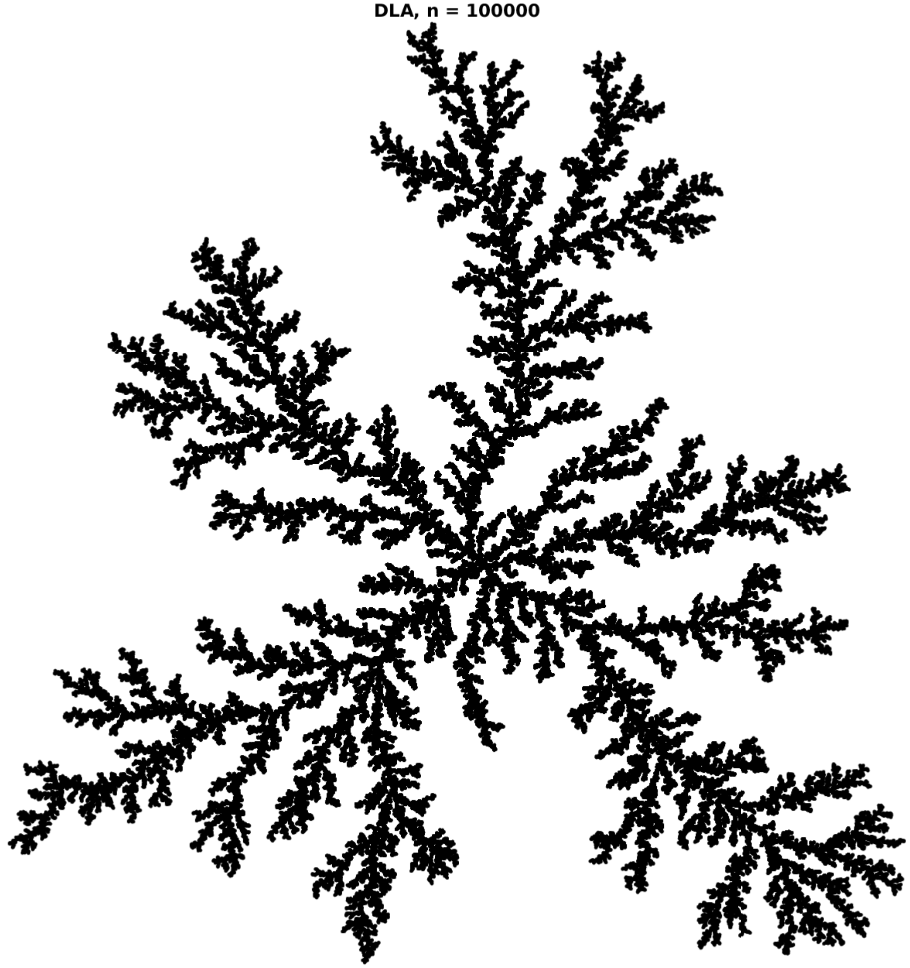


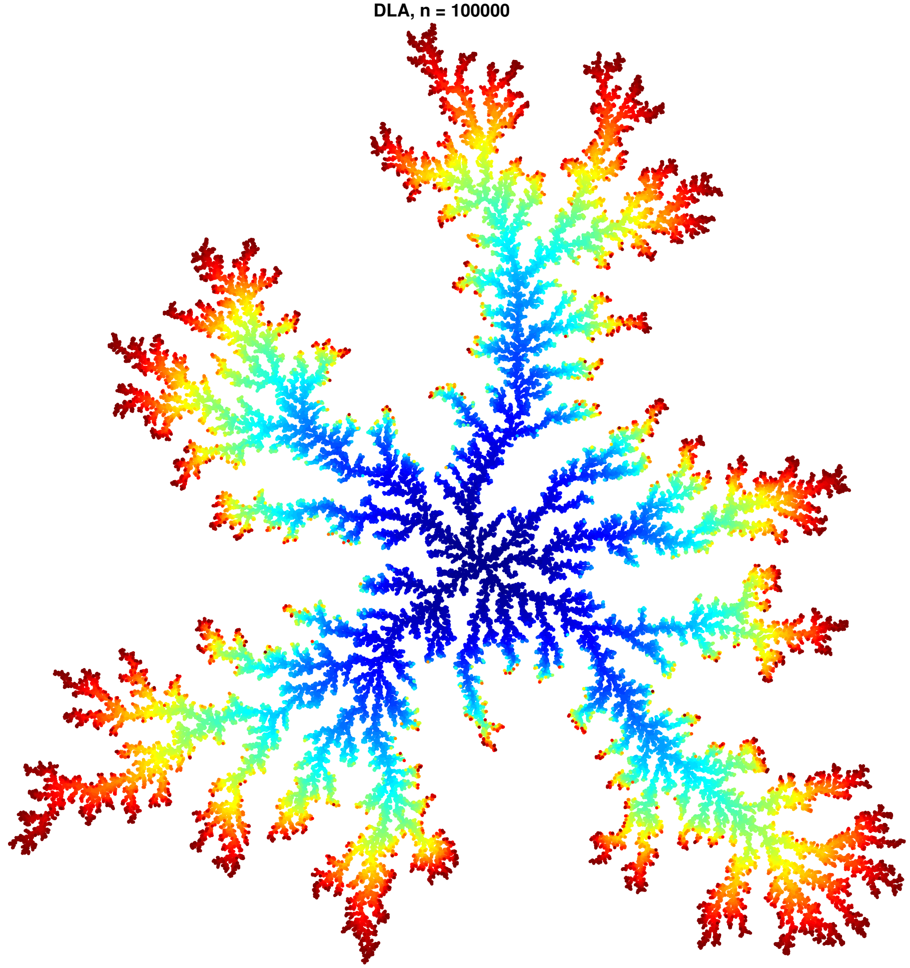

DLA, n = 10000

DLA, n = 10000

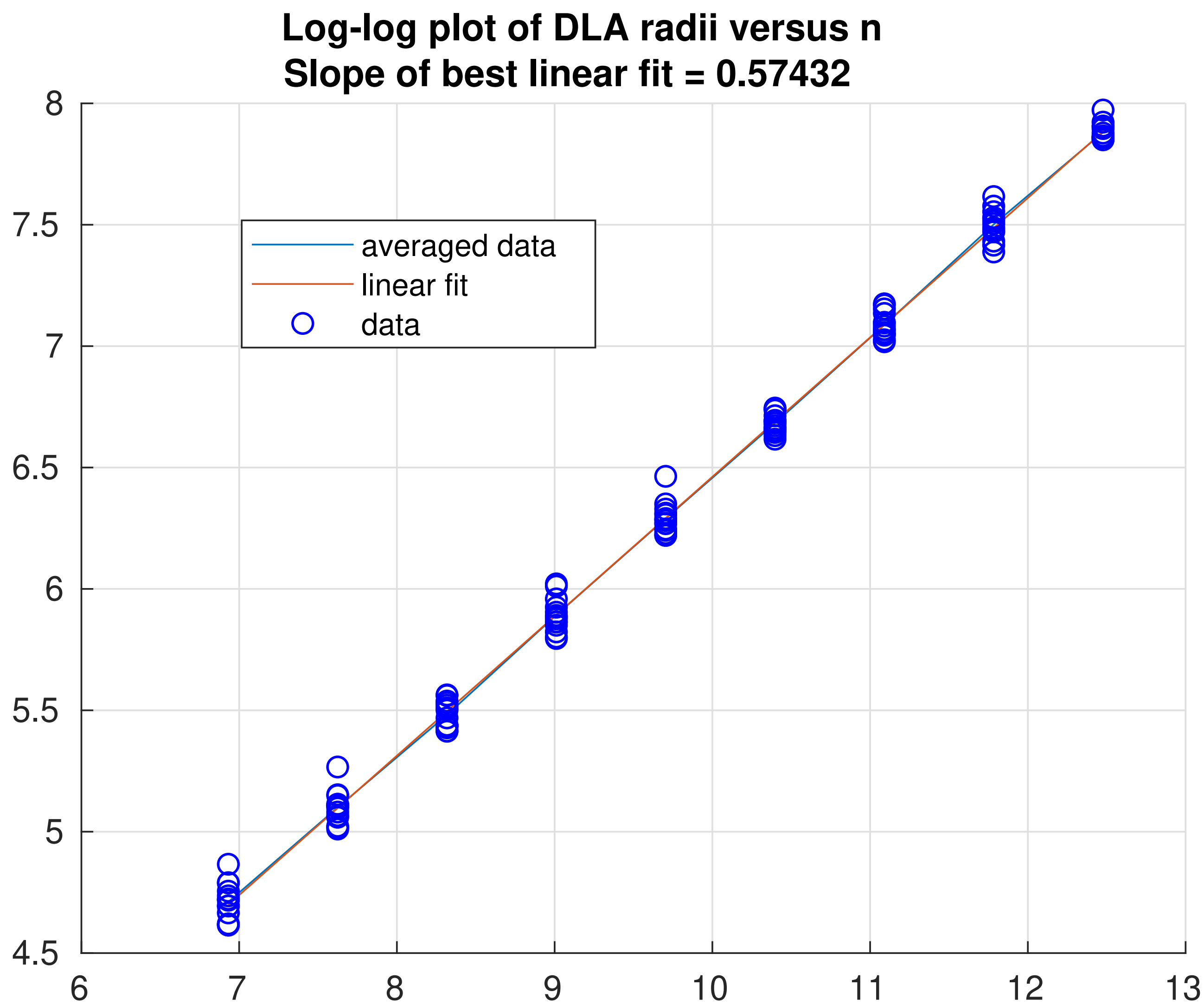


DLA, n = 100000

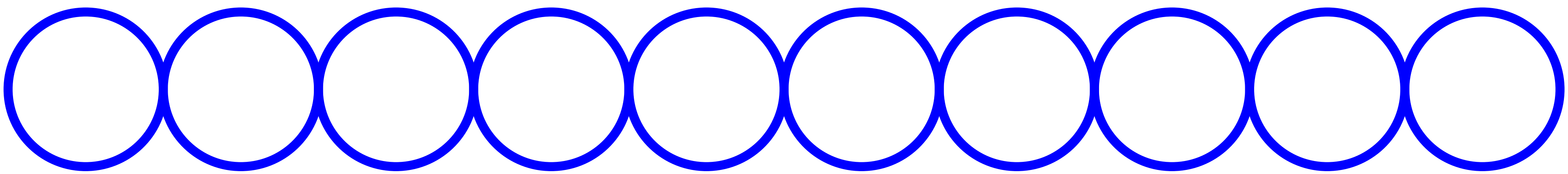




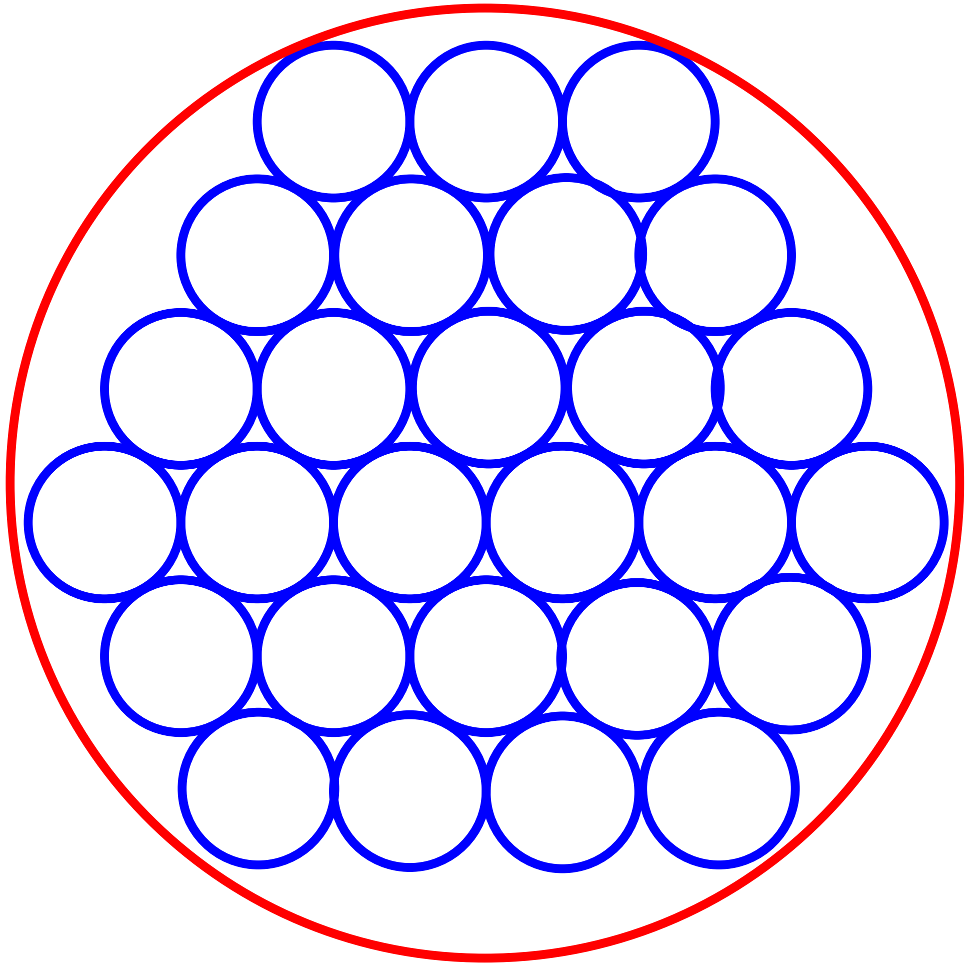
How fast does the diameter grow?



Numerical experiment for growth rate.



Trivial upper bound is $O(n)$.



Trivial lower bound is $\Omega(\sqrt{n})$.

Theorem (Kesten): $\mathrm{diam}(\mathrm{DLA}(n)) = O(n^{2/3})$.

Theorem (Kesten): $\operatorname{diam}(\mathrm{DLA}(n)) = O(n^{2/3})$.

Equivalent: DLA takes $\gtrsim m^{3/2}$ steps to exit ball of radius m.

Suppose current radius is m. How long to reach $2m$?

Theorem (Beurling): If K is connected and has diameter R, the harmonic measure of $D(x, 1) \cap K$ is $\leq C/\sqrt{R}$.

harmonic measure = first hitting distribution of Brownian motion

Proved using conformal maps, extremal length.

Theorem (Beurling): If K is connected and has diameter R, the harmonic measure of $D(x,1) \cap K$ is $\leq C/\sqrt{R}$.

$\Rightarrow$ Given a unit disk in a cluster of diameter m, it takes about $\sqrt{m}$ attempts to attach a new disk it.

$\Rightarrow$ It takes $m \cdot \sqrt{m}$ attempts to grow a path of length m.

Theorem (Beurling): If K is connected and has diameter R, the harmonic measure of $D(x, 1) \cap K$ is $\leq C/\sqrt{R}$.

$\Rightarrow$ Given a unit disk in a cluster of diameter m, it takes about $\sqrt{m}$ attempts to attach a new disk it.

$\Rightarrow$ It takes $m \cdot \sqrt{m}$ attempts to grow a path of length m.

$\Rightarrow$ It takes time $m^{3/2}$ to grow from m to $2m$.

Beurling's theorem is sharp when K is line segment.

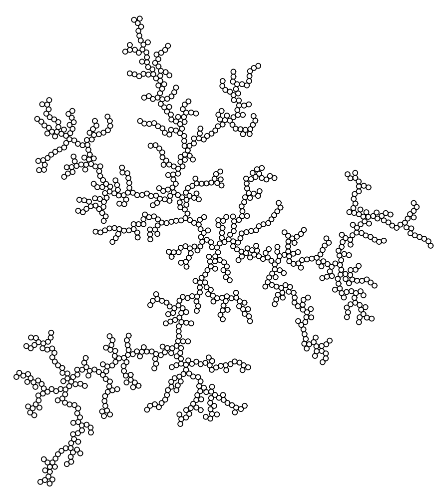
Since DLA never looks like a line segment,

$\Rightarrow$ smaller estimate for harmonic measure,

$\Rightarrow$ longer waiting time to cross disk,

$\Rightarrow$ better upper bound for the diameter of DLA.

How to make this precise? Prove DLA is not a "line segment".



Amazingly, there is no known better lower bound than the trivial $\sqrt{n}$.

Conjecture: Almost surely,

$$\lim_{n\to\infty} \frac{\mathrm{diam}(\mathrm{DLA}(n))}{\sqrt{n}} = \infty.$$

Amazingly, there is no known better lower bound than the trivial $\sqrt{n}$.

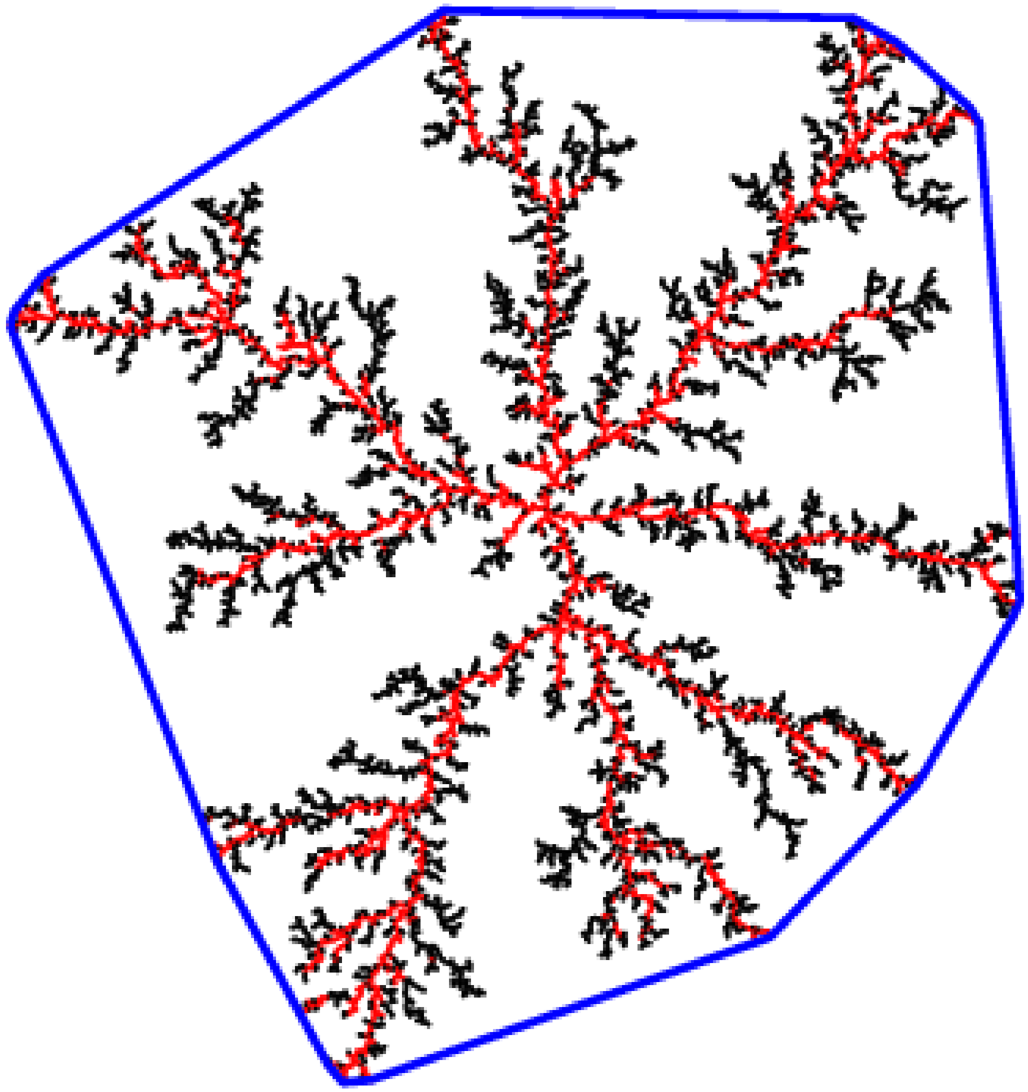
Conjecture: Almost surely,

$$\lim_{n\to\infty} \frac{\operatorname{diam}(\mathrm{DLA}(n))}{\sqrt{n}} = \infty.$$

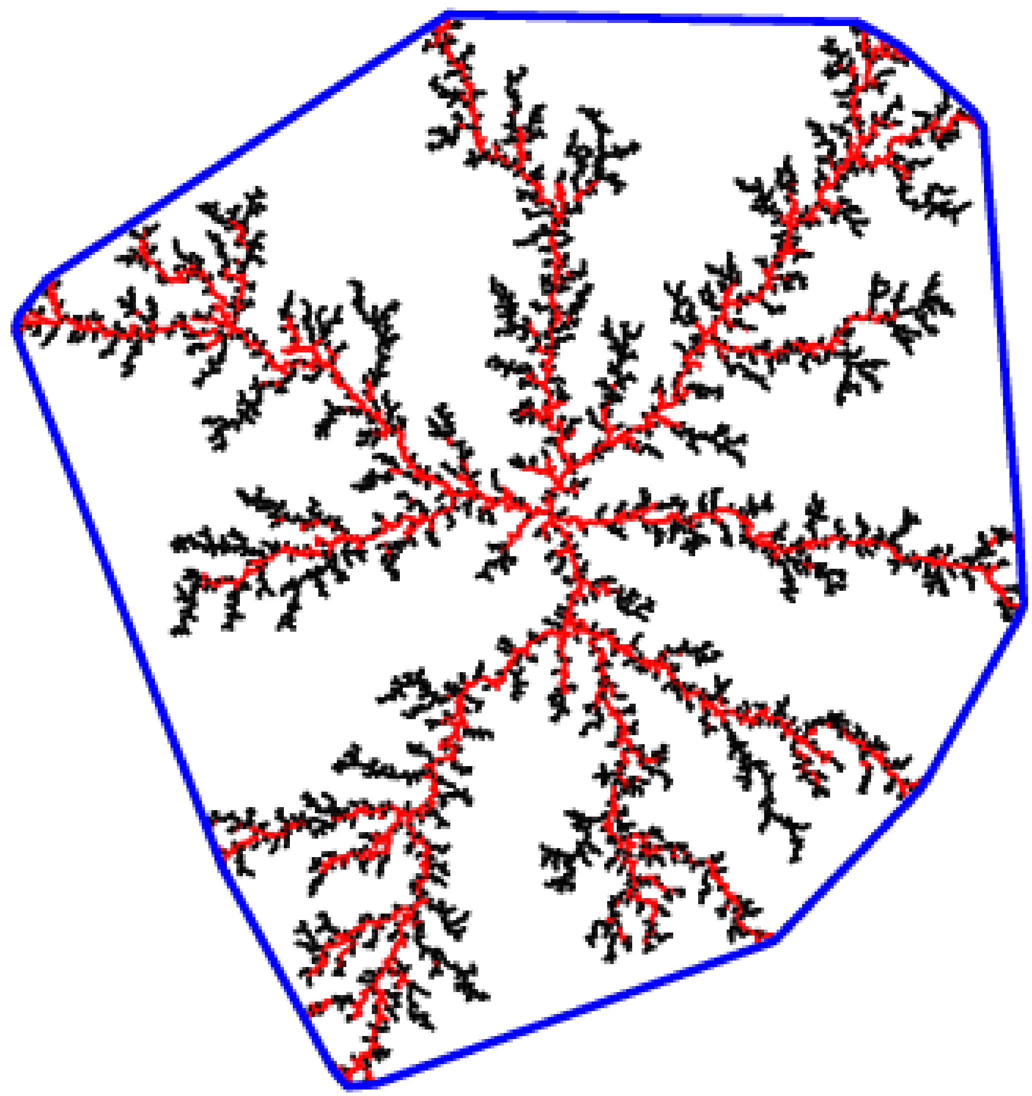
If $\mathrm{DLA}(n)$ is roughly a disk of radius $\sqrt{n}$ then any boundary disk is hit with probability $\simeq 1/\sqrt{n}$, which gives which gives the trivial lower bound.

For non-trivial lower bound, we need to show there are points that get hit with probability $\gg n^{-1/2}$.

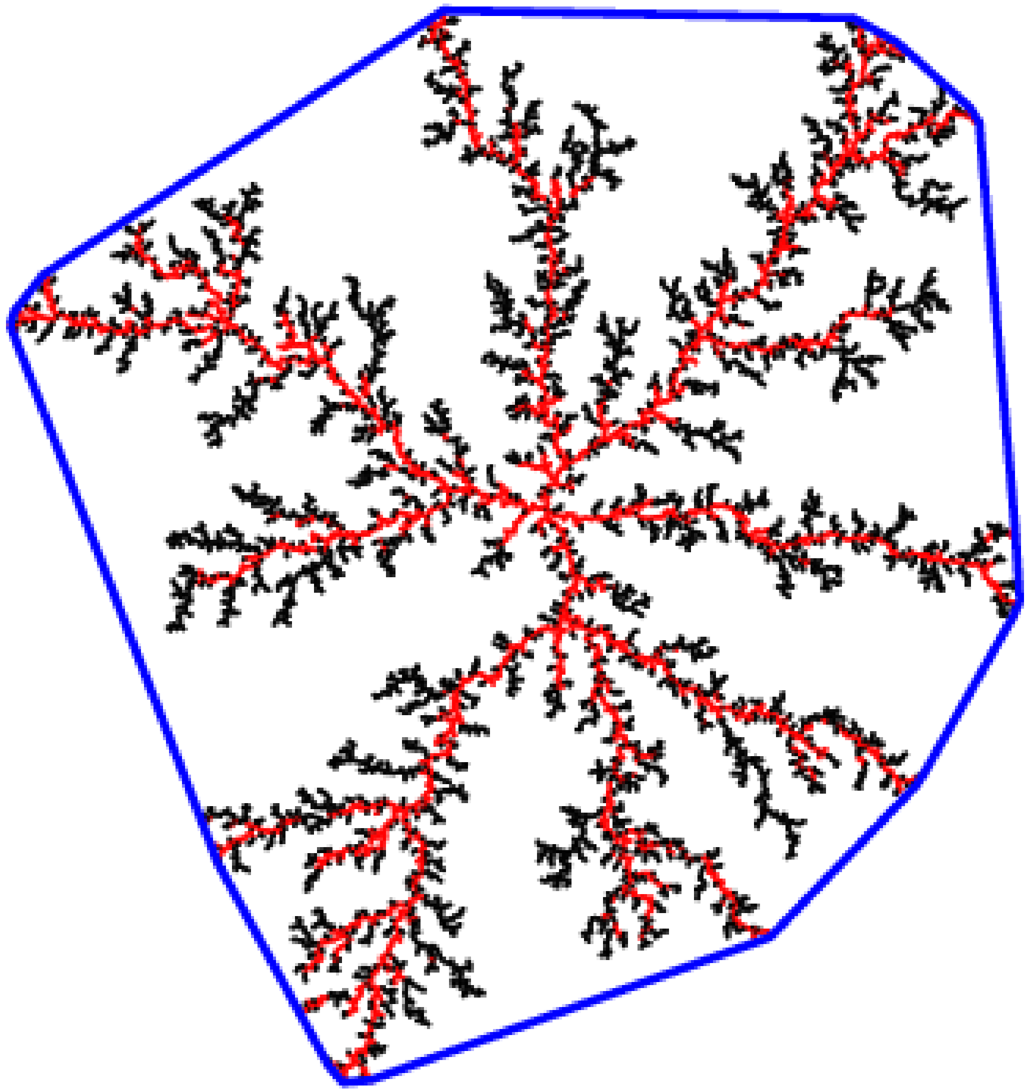
Consider convex hull of the DLA cluster. What is the harmonic measure of the disks that touch the convex hull boundary?

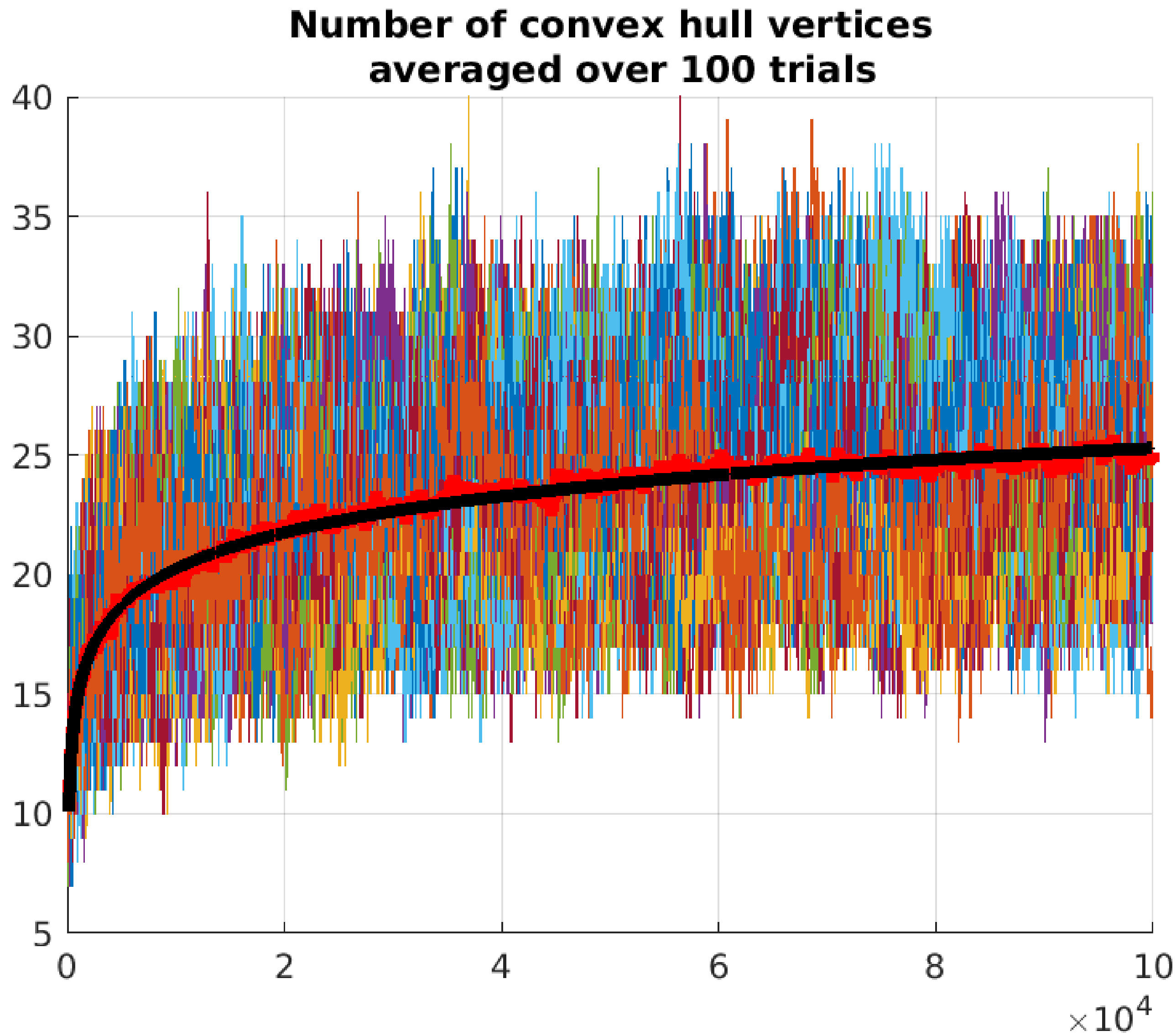


If convex hull has "sharp angles" some vertices have larger than average harmonic measure, implies faster than trivial growth.

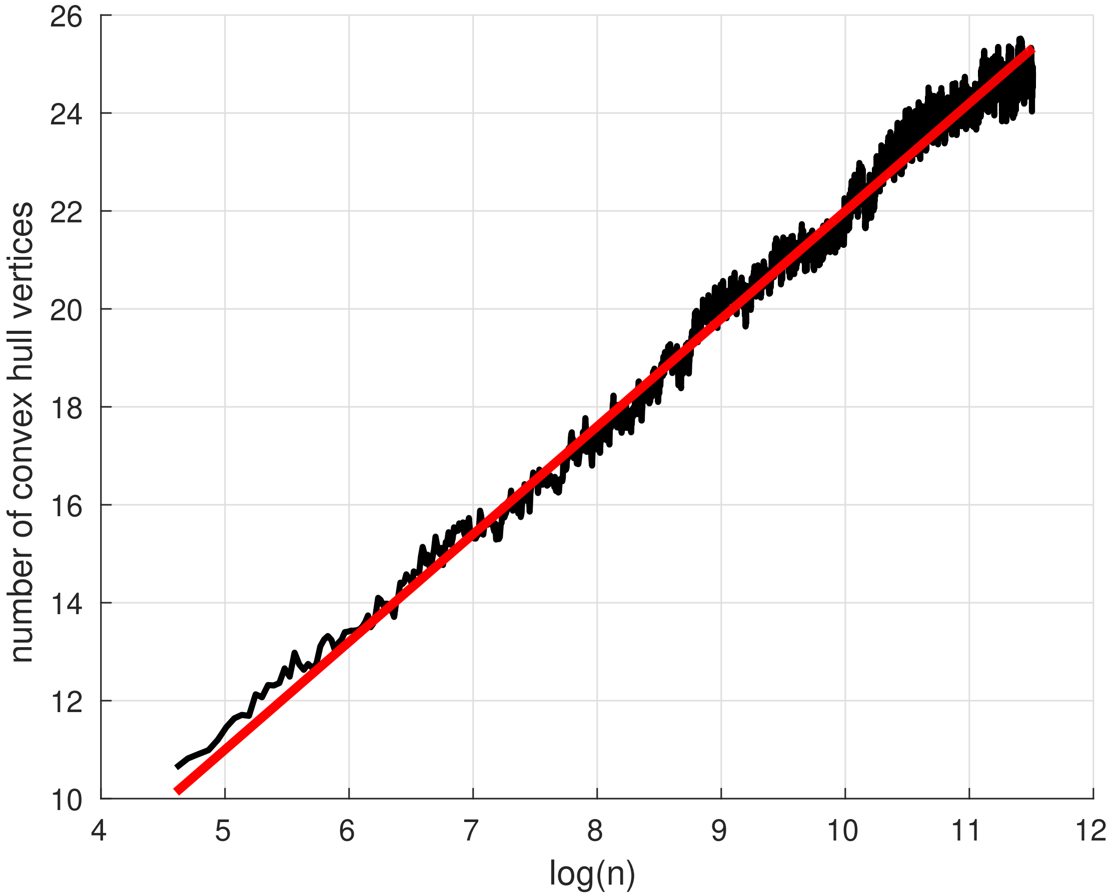


One way to have “sharp angles” is to have few vertices: if the convex hull boundary has few vertices, some of the angles should be large.

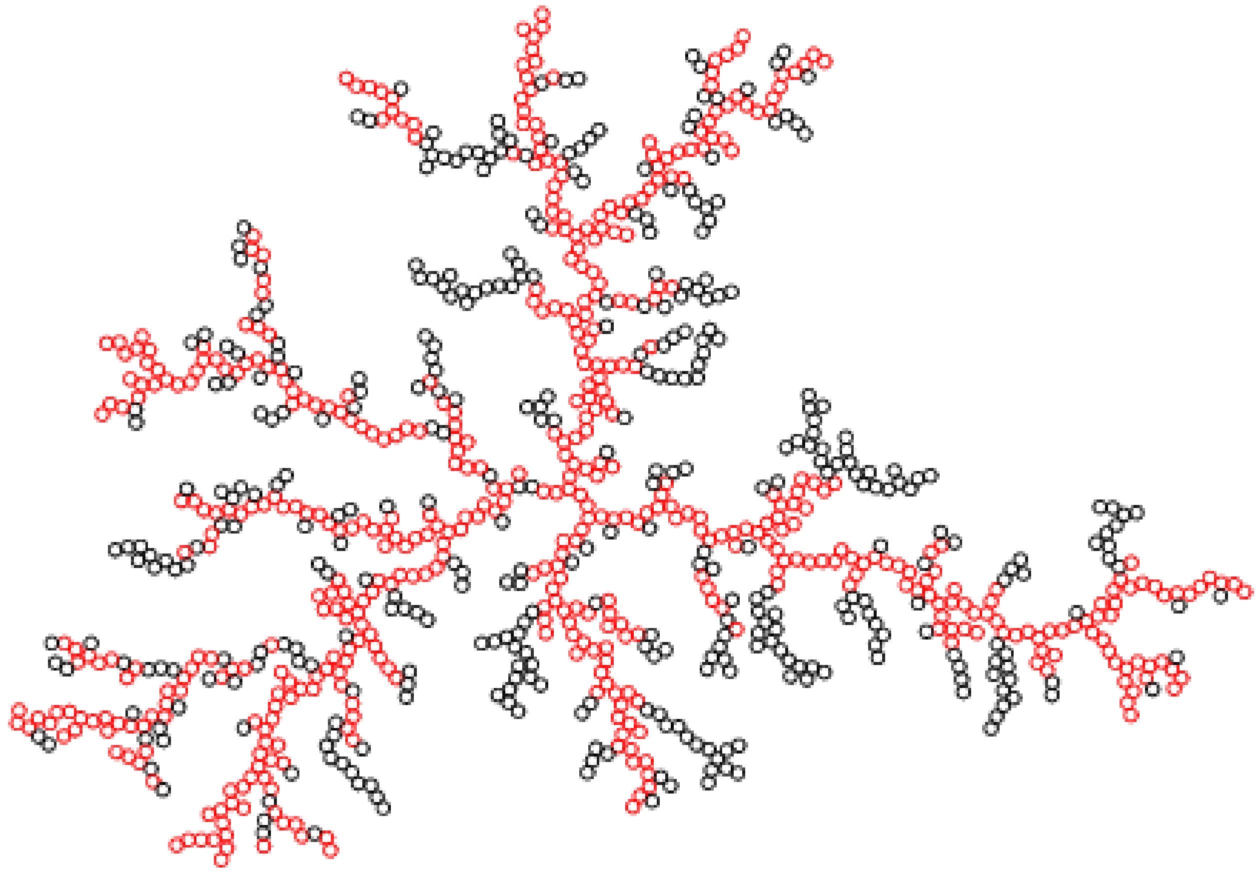




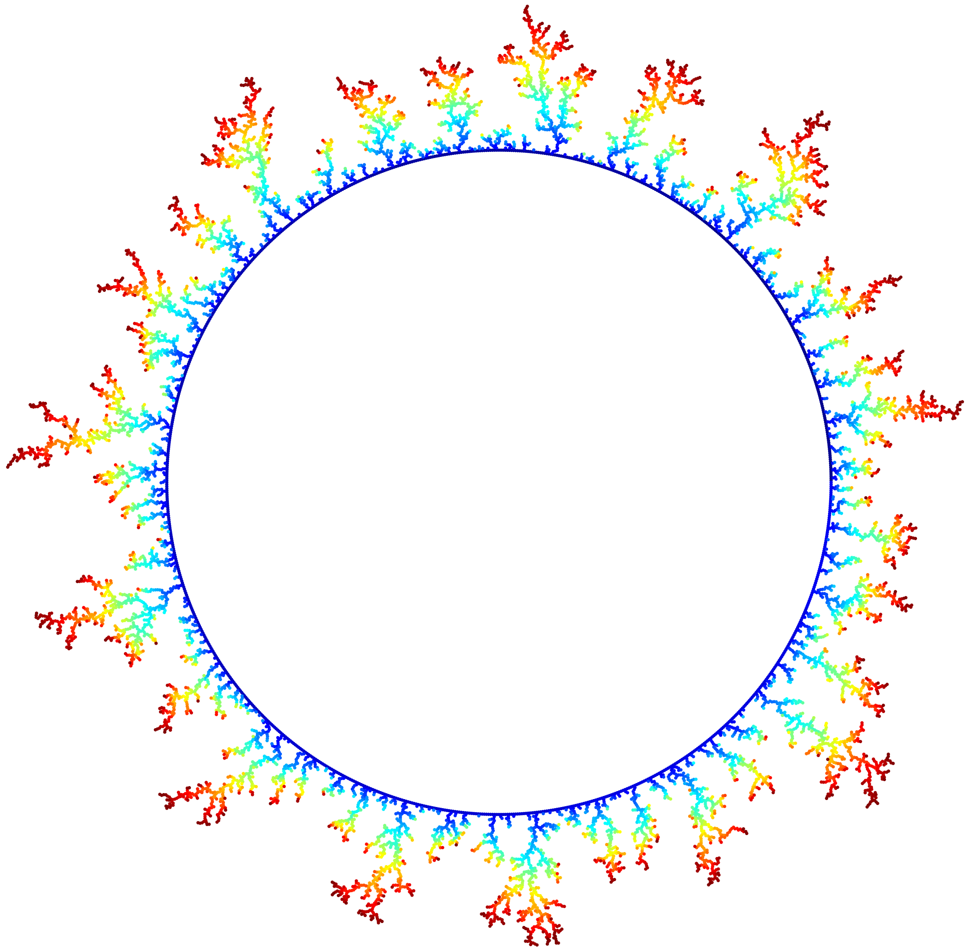
How many convex hull vertices are there at time n?



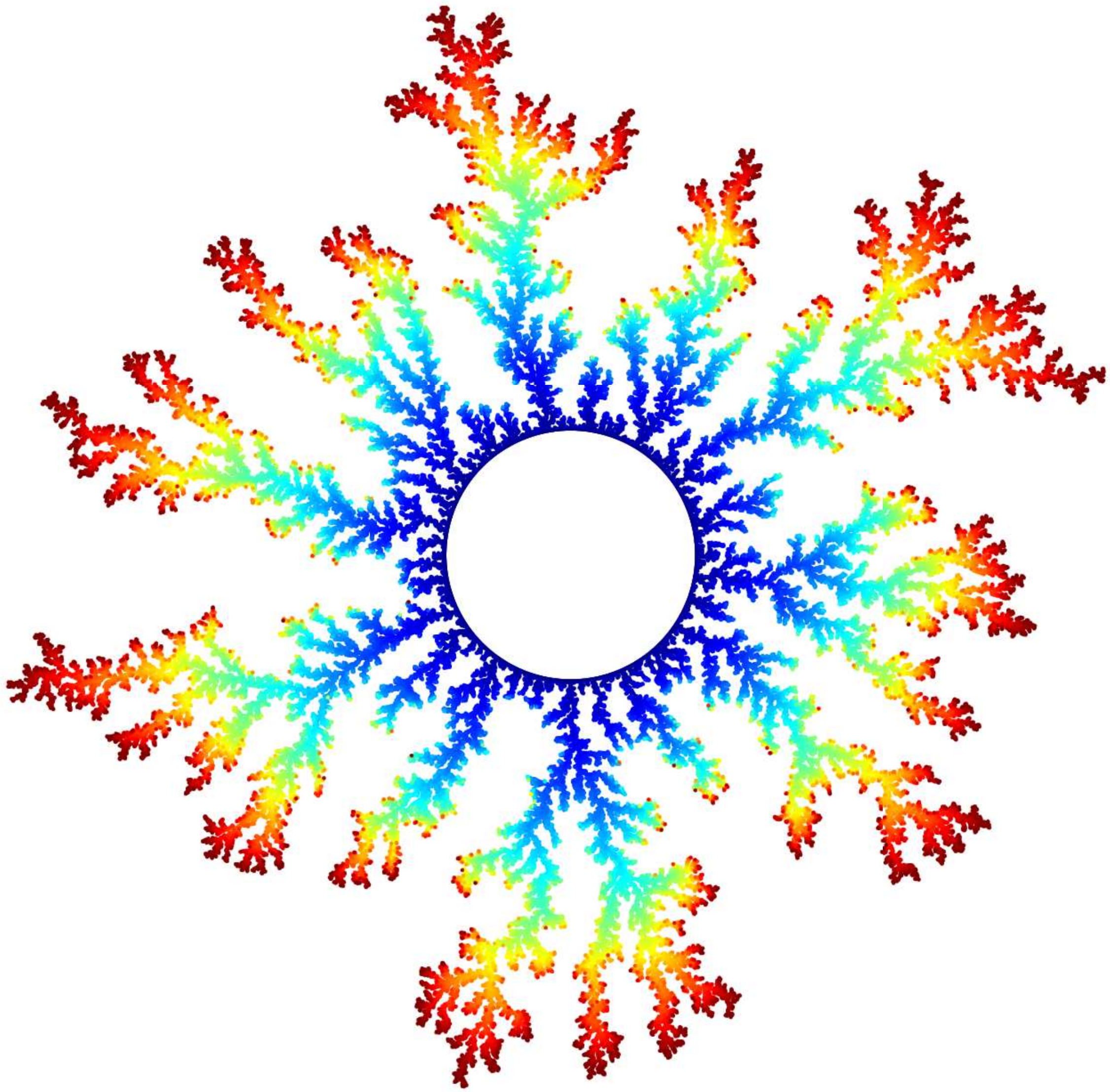
Number of convex hull vertices, averaged over 100 trials.
Plotted versus $\log(n)$, looks linear.
Numerically, $\approx (2.2) \log n$.



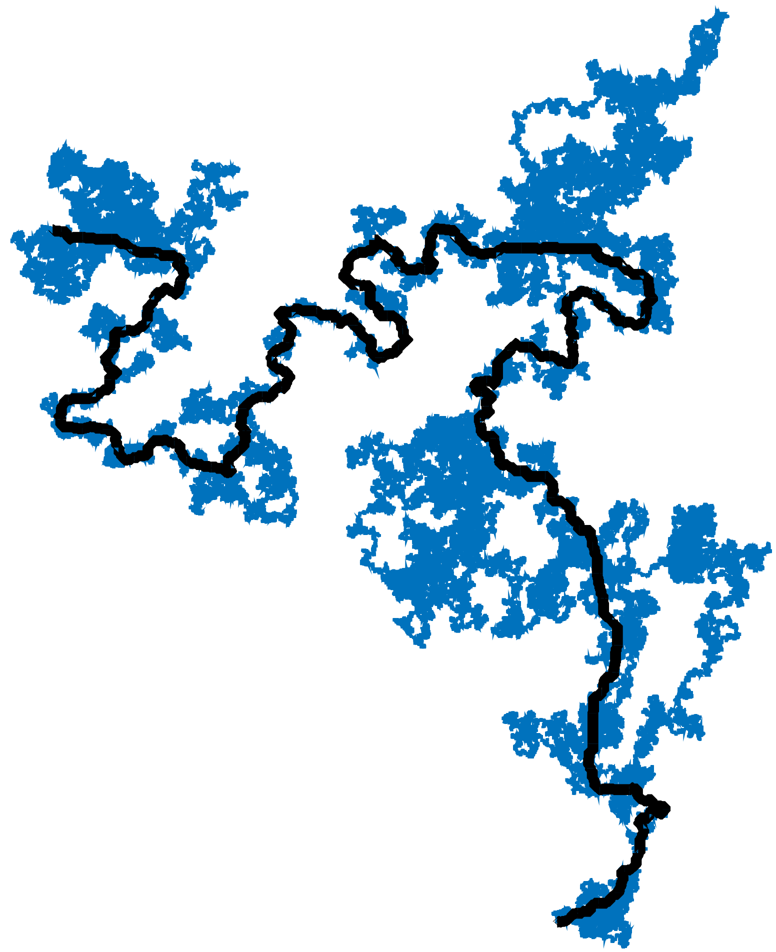
Red disks where on convex hull boundary when added.
Percentage probably tends to zero, but how fast?

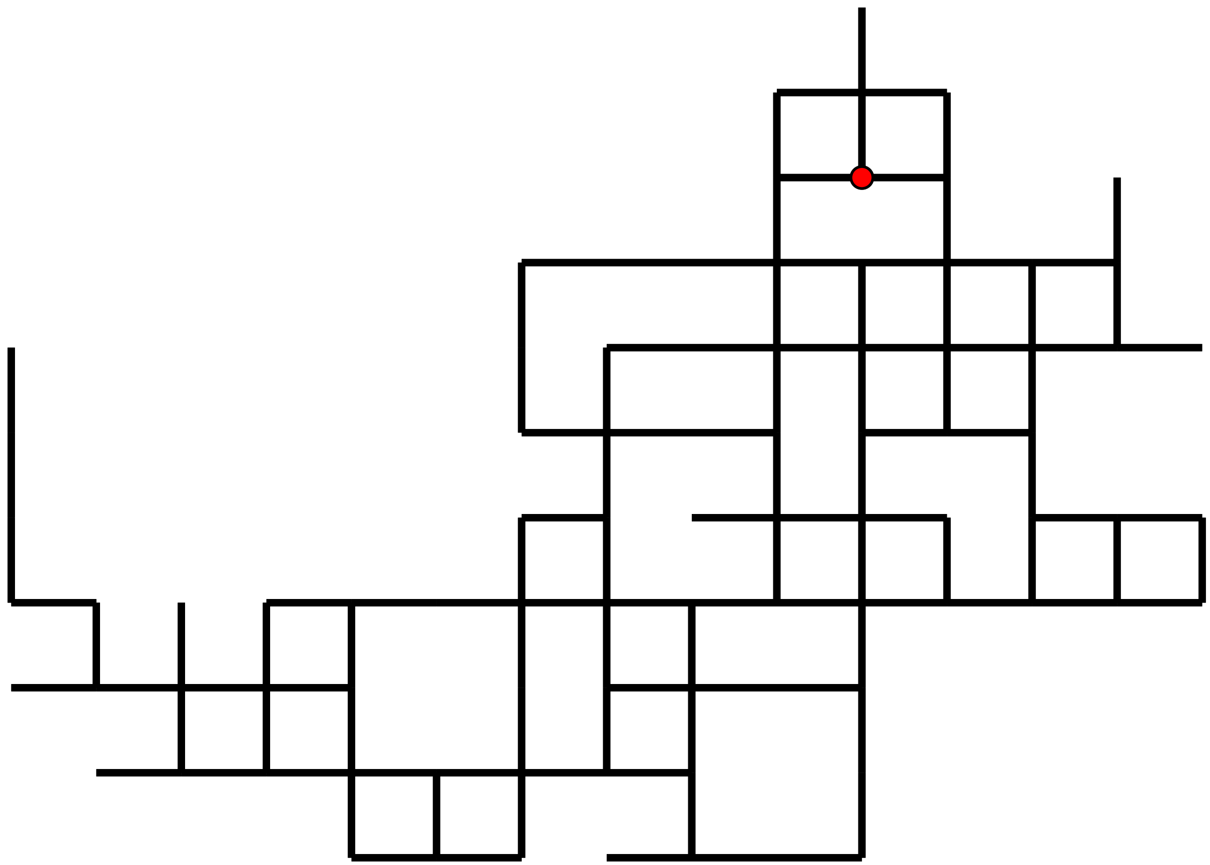


Prove that disk evolves into a “non-disk”

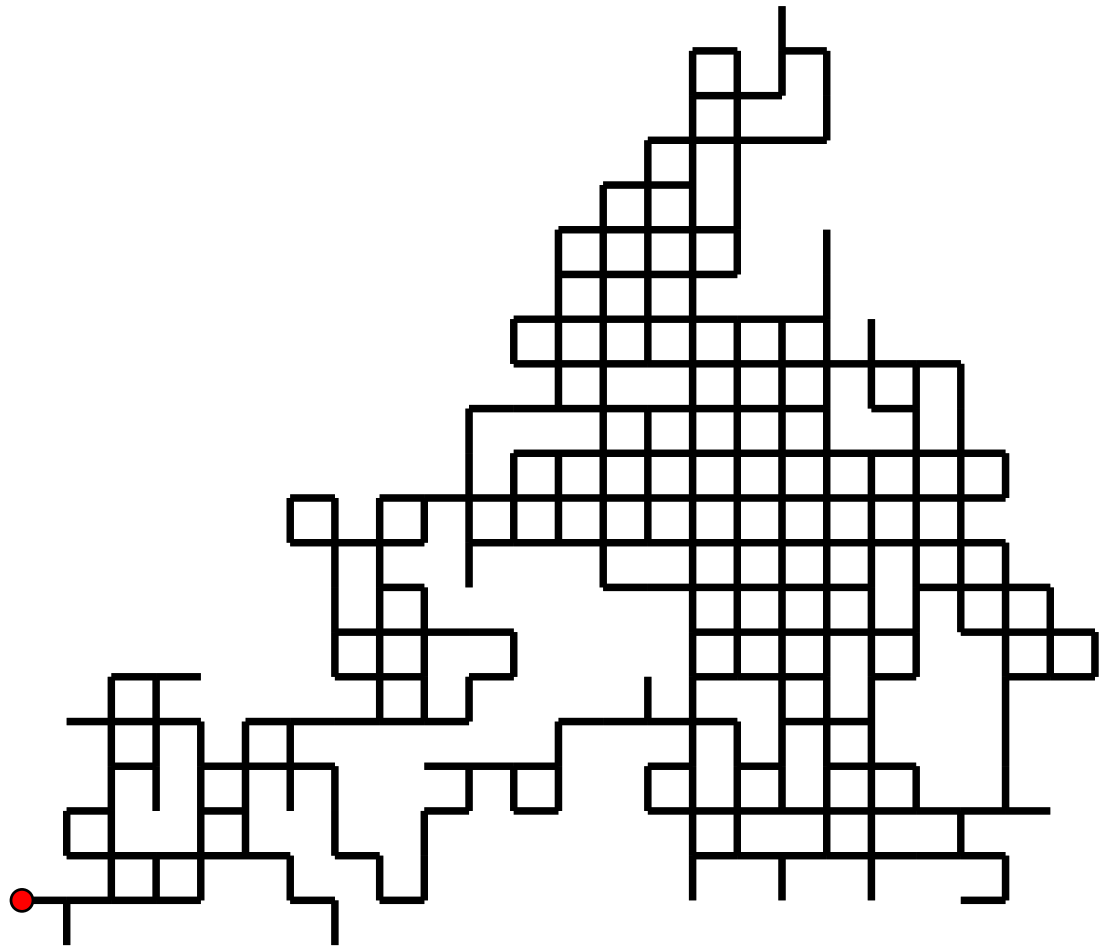


PART II: SHORT PATHS IN THE BROWNIAN TRACE

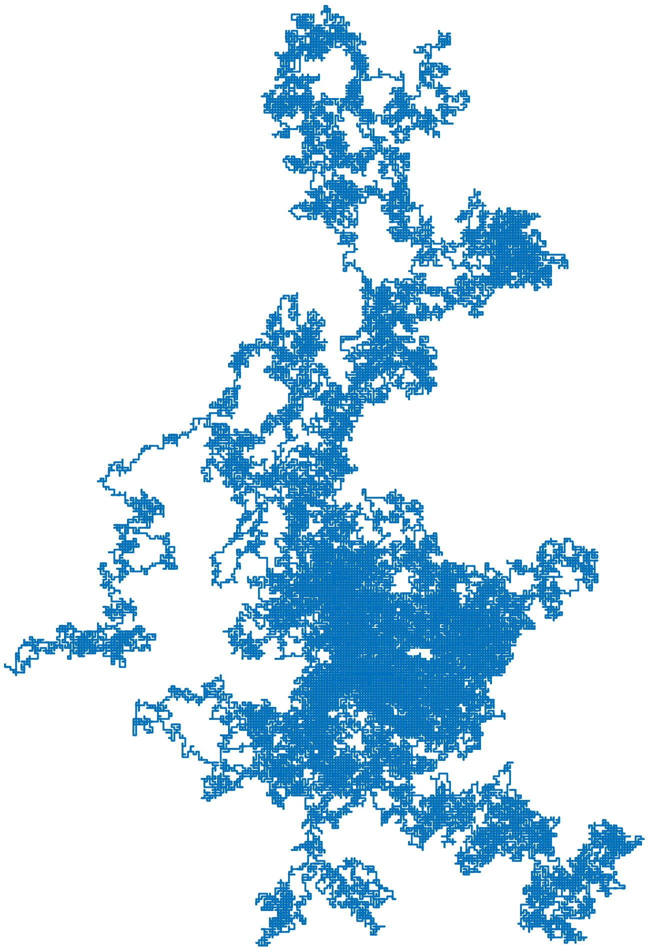




200 step random walk.



1000 step random walk.



100,000 step random walk.

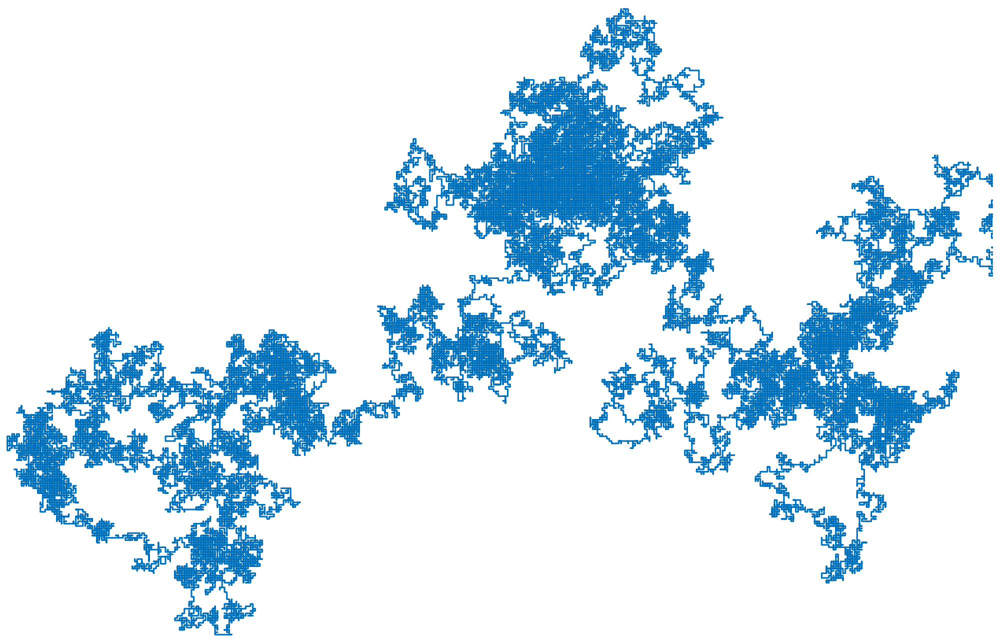
Brownian motion = limit of rescaled random walks

It a “random continuous path” in plane.

Run forever is dense in plane.

Run for unit time gives compact set = Brownian trace.

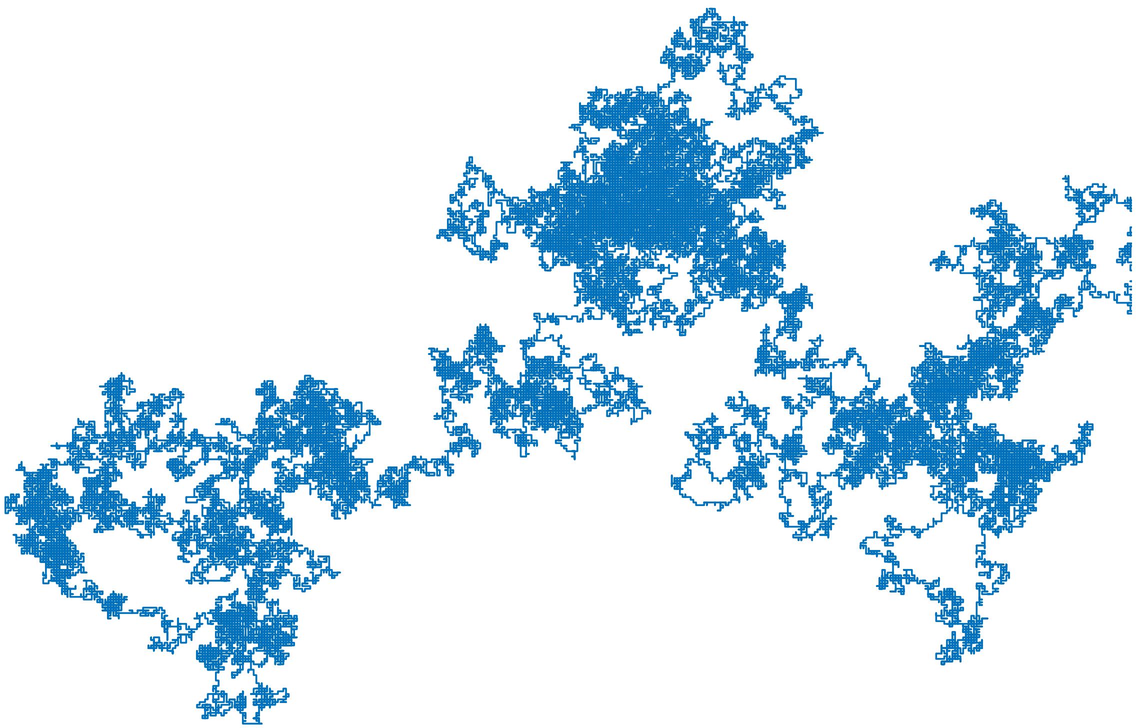
Zero area, dimension 2. Infinitely many complementary components.



Robin Pemantle proved Brownian motion does not cover a line segment.

Question: Does it cover a rectifiable curve?

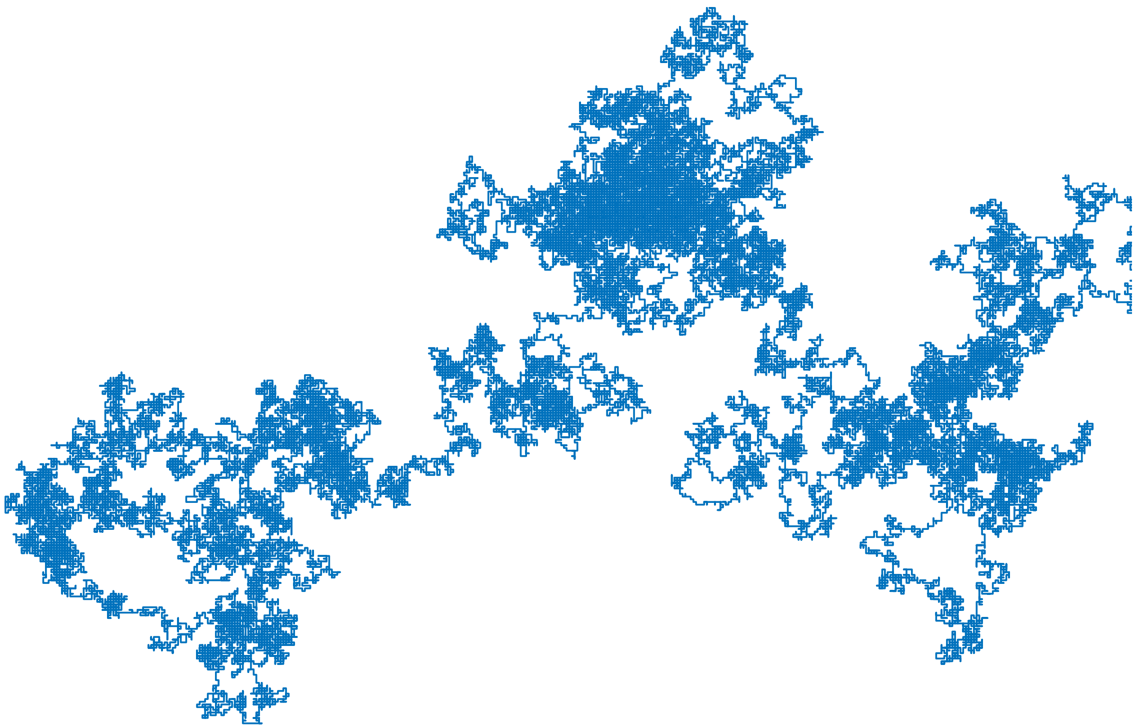
Question: Does it cover a curve of dimension $1 + \epsilon$, any $\epsilon > 0$?

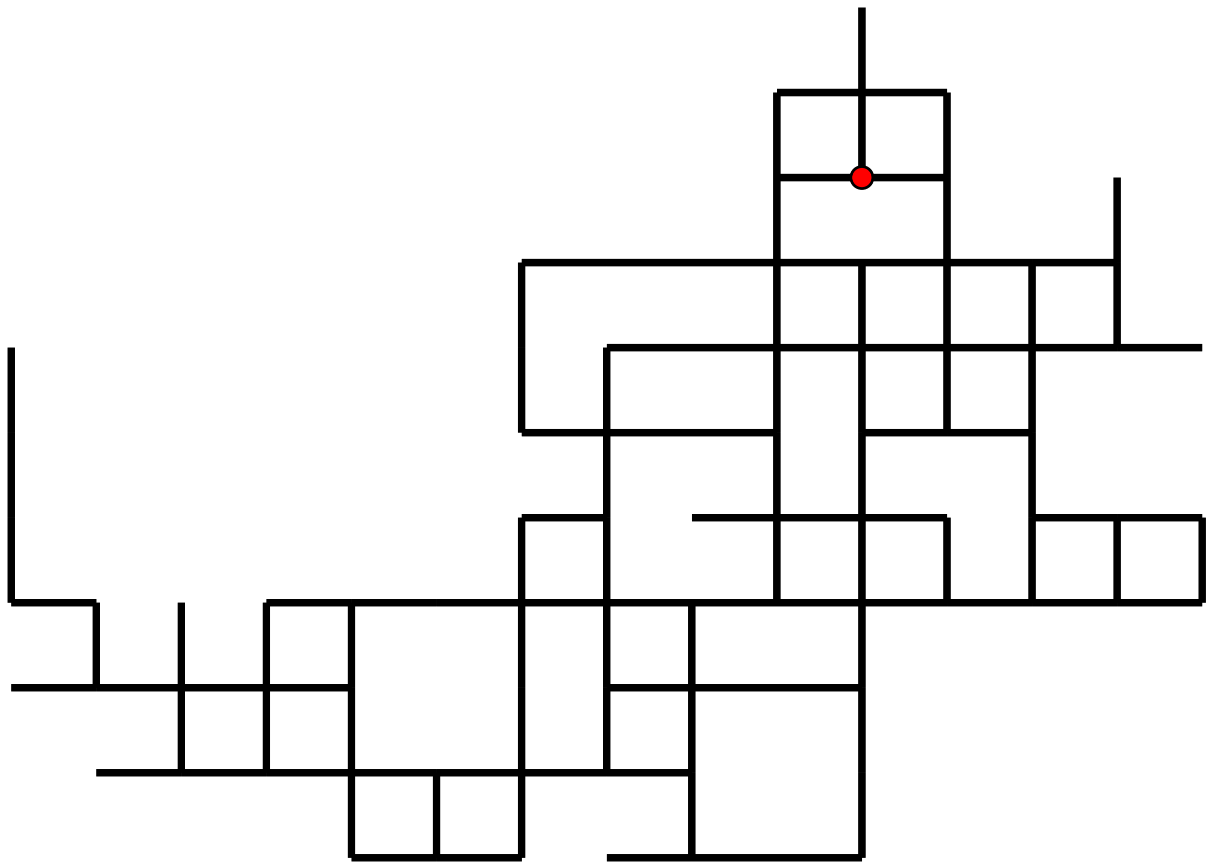


Percolation dimension = minimal dimension of Jordan arc inside the set.

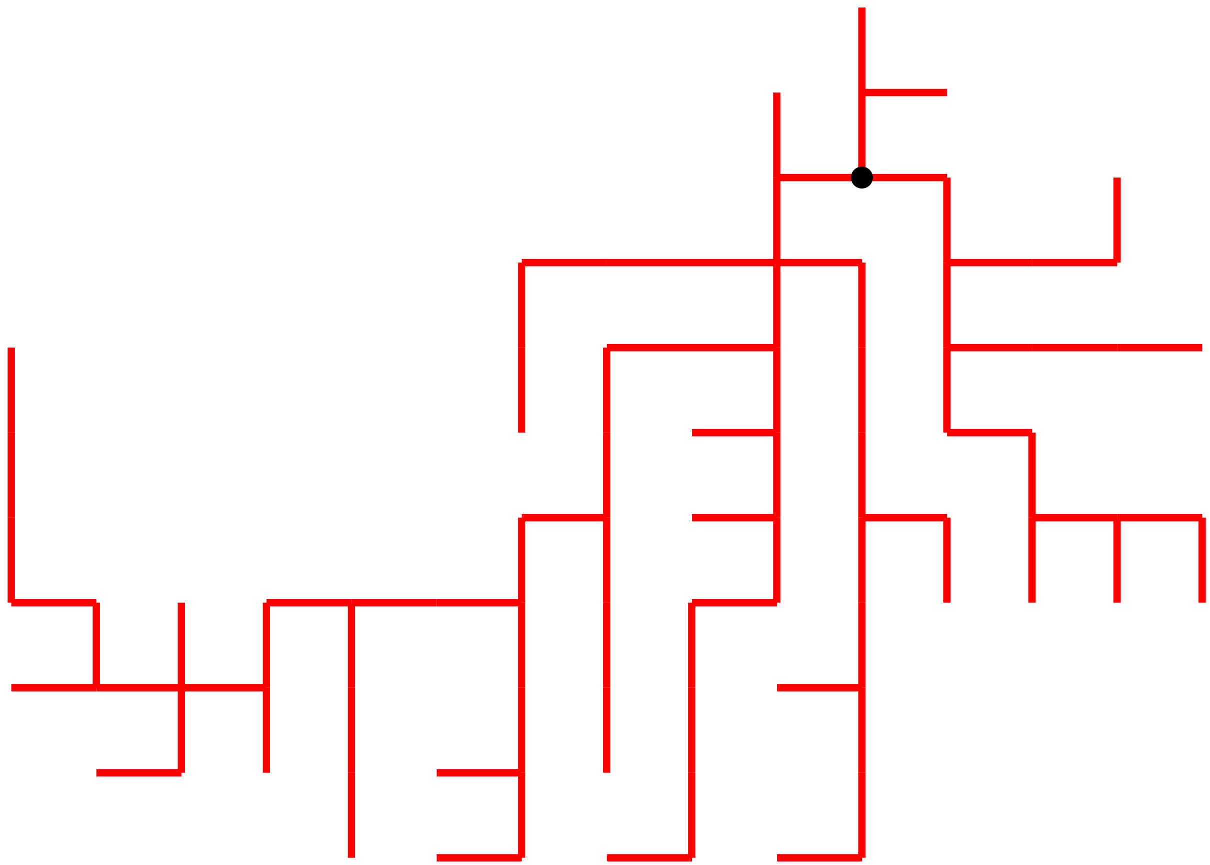
Frontiers have dimension 4/3 (Lawler et. al.).

Brownian trace contains curves of dimension 5/4 (Dapeng Zhan).

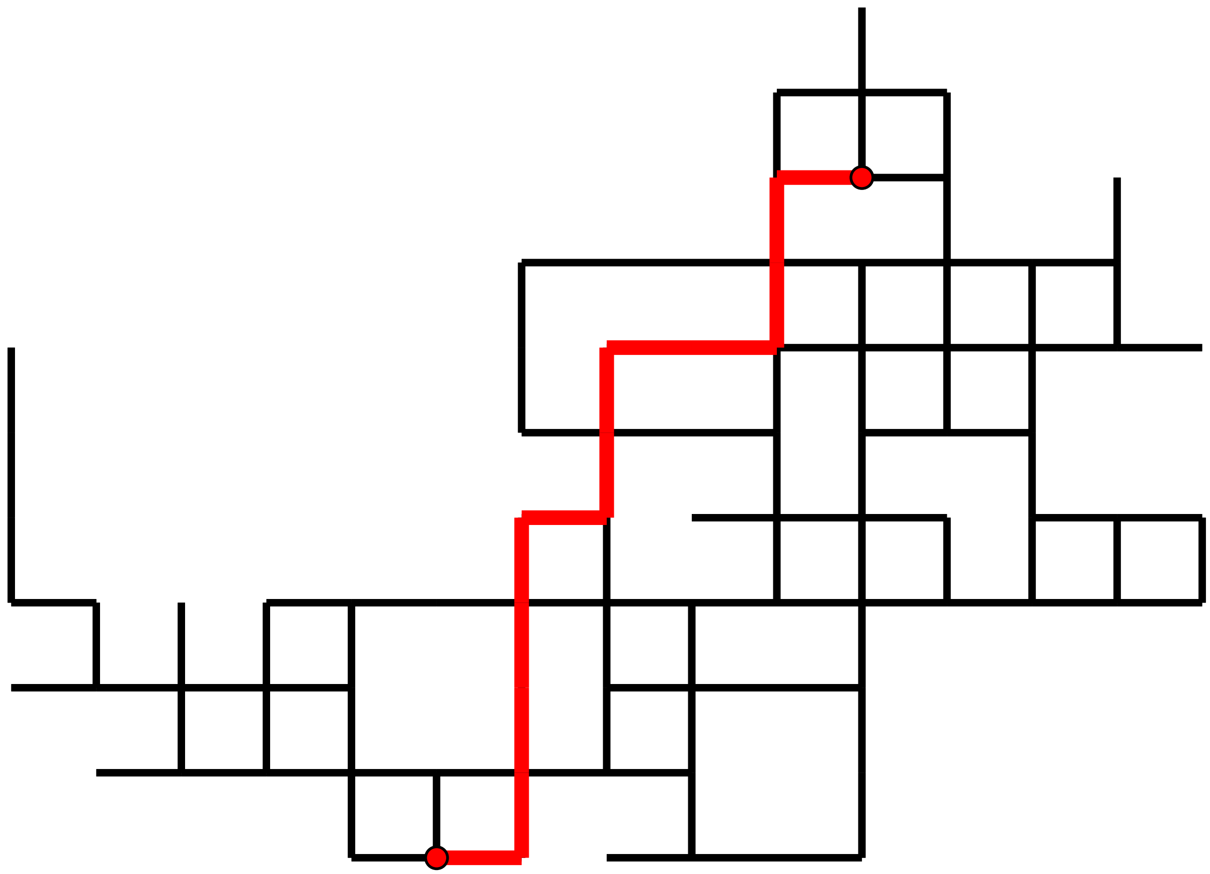




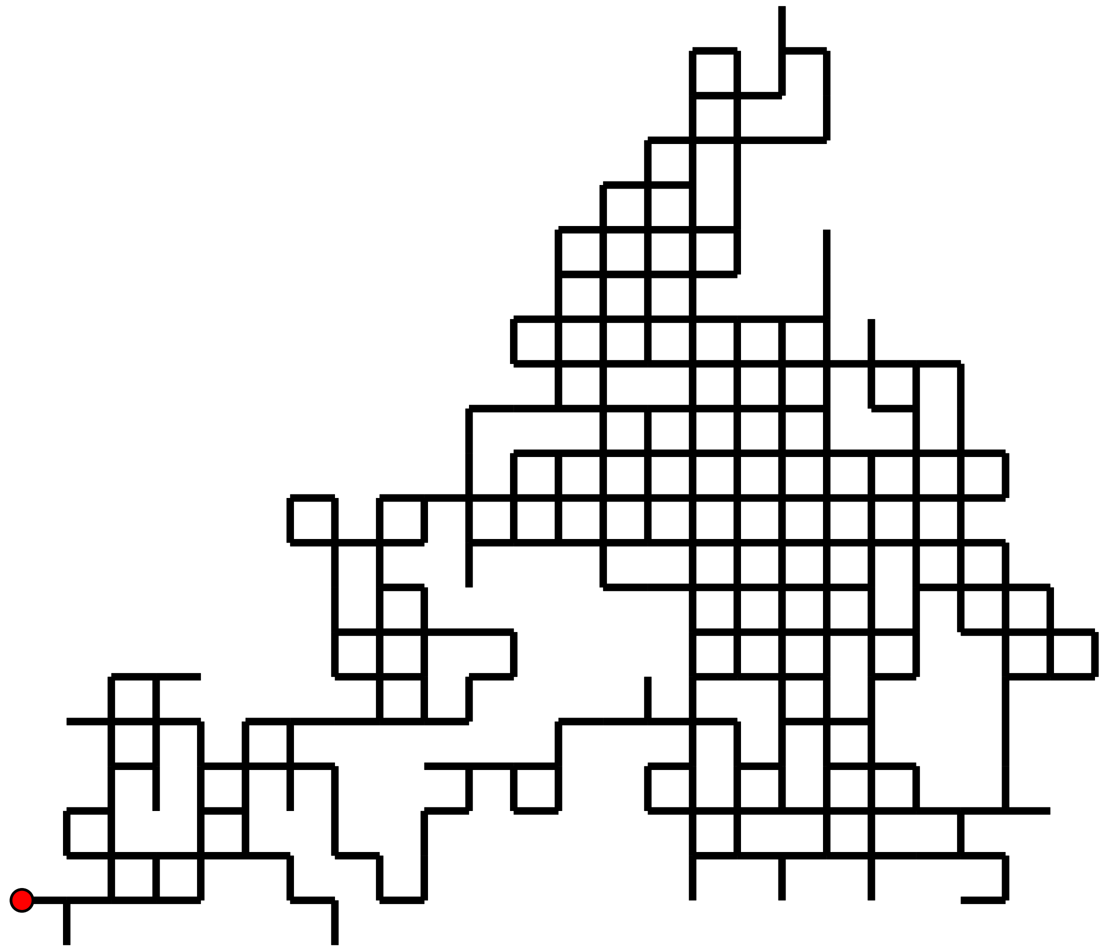
200 step random walk.



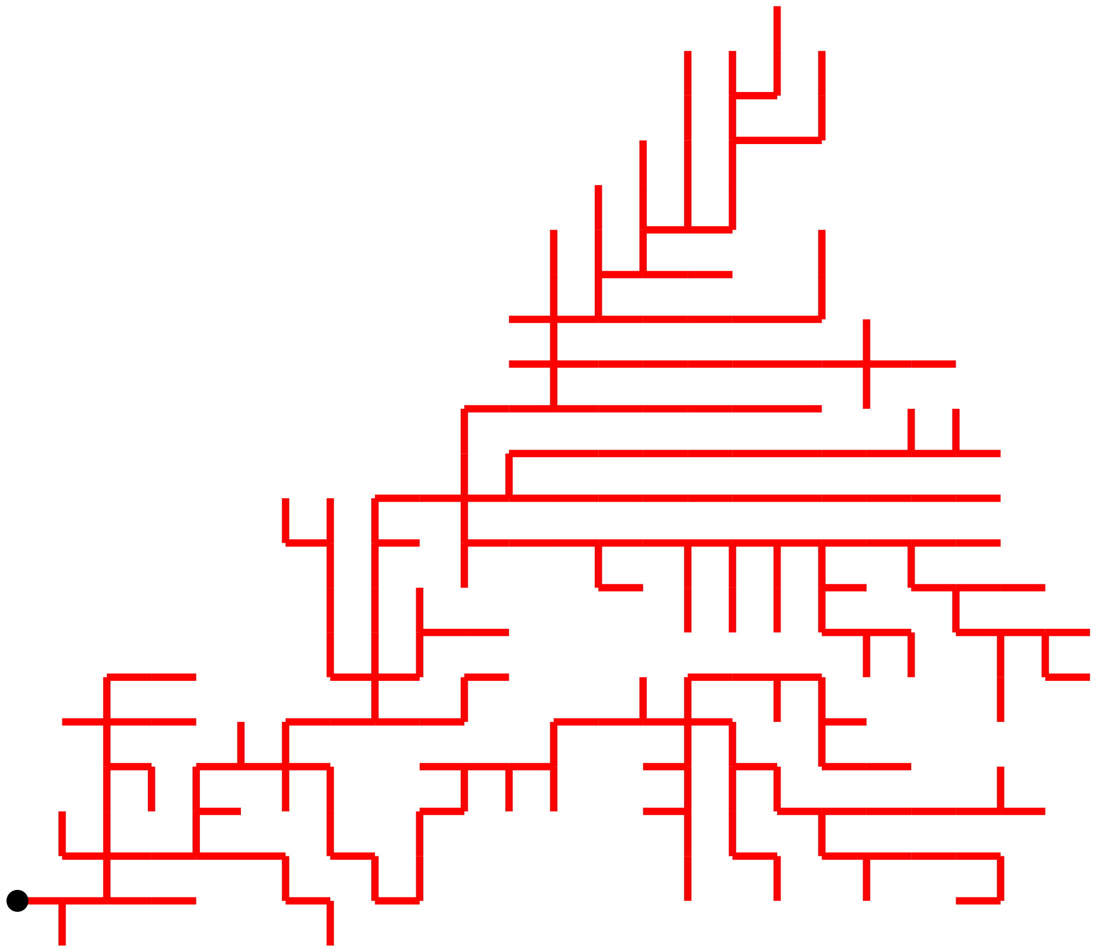
Minimal distance rooted spanning tree.



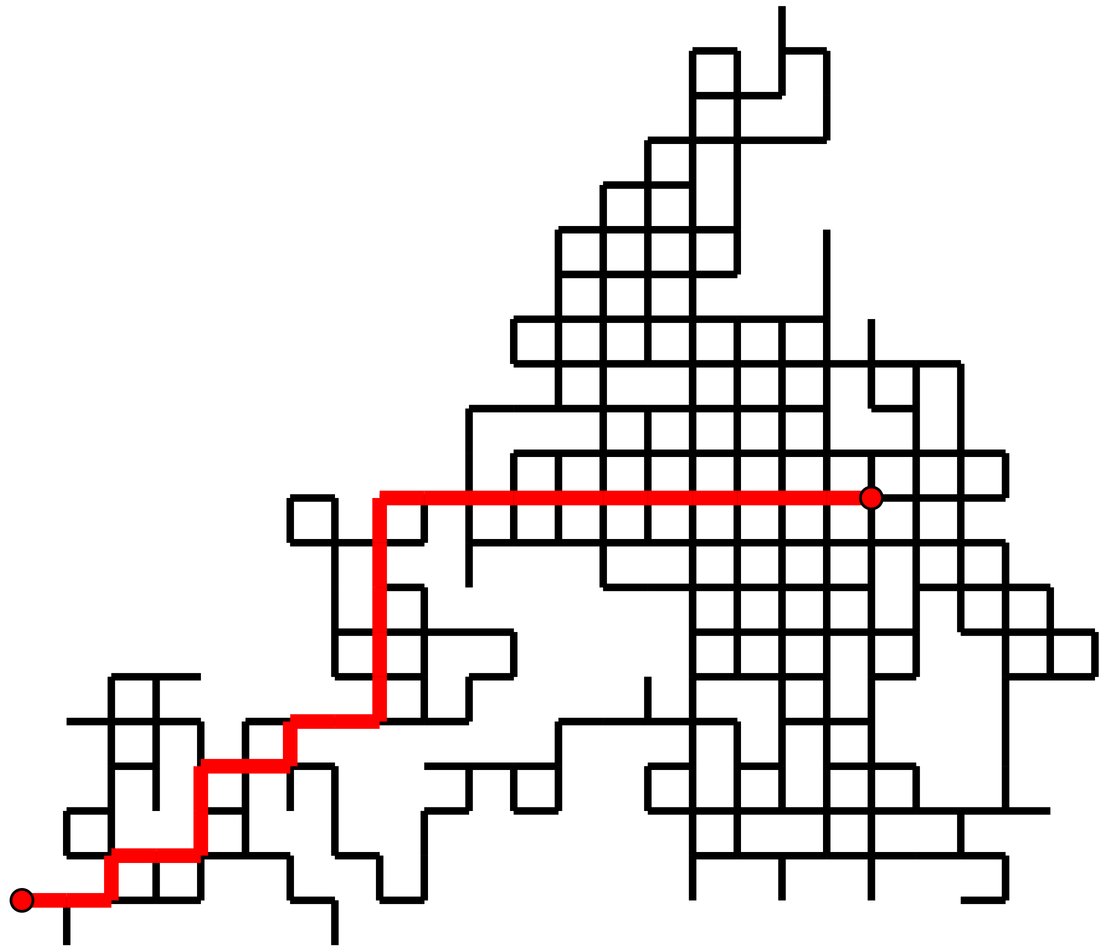
A shortest path from 0 to $\sqrt{n}/2$.



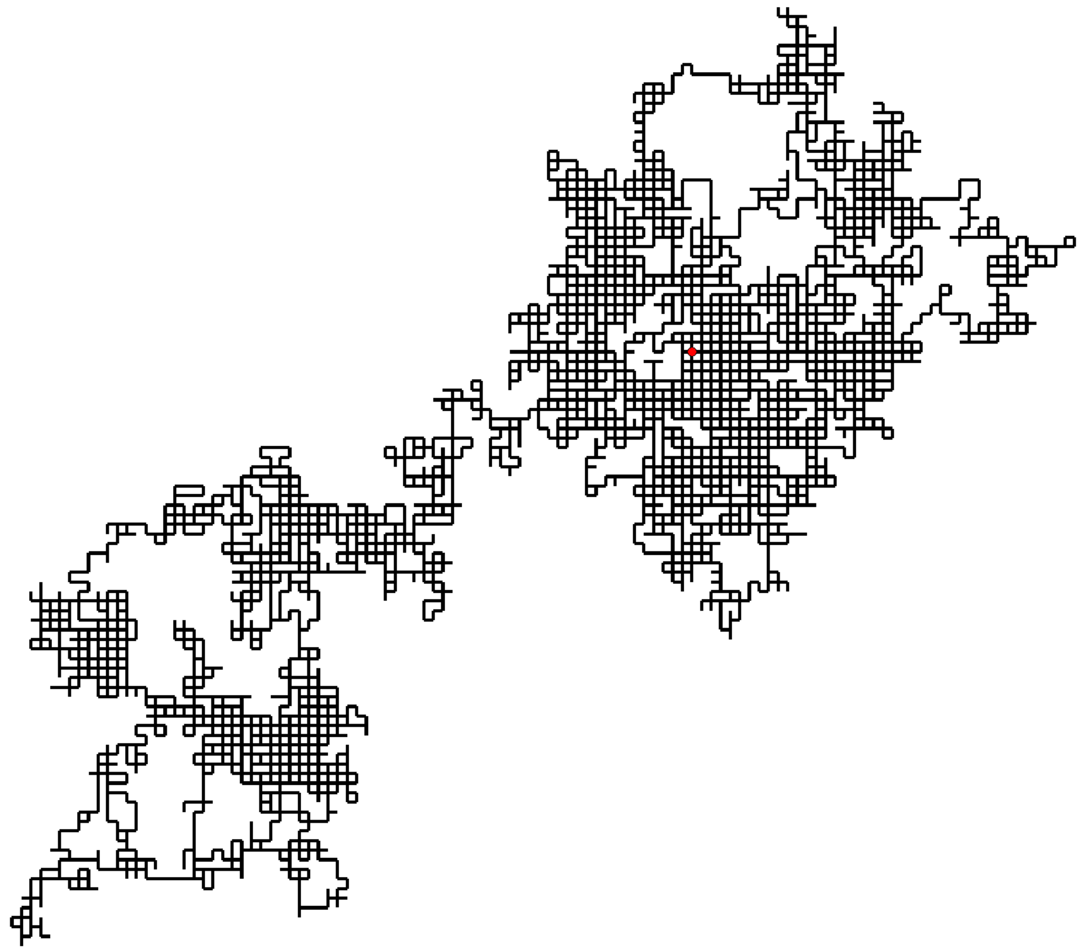
1000 step random walk.



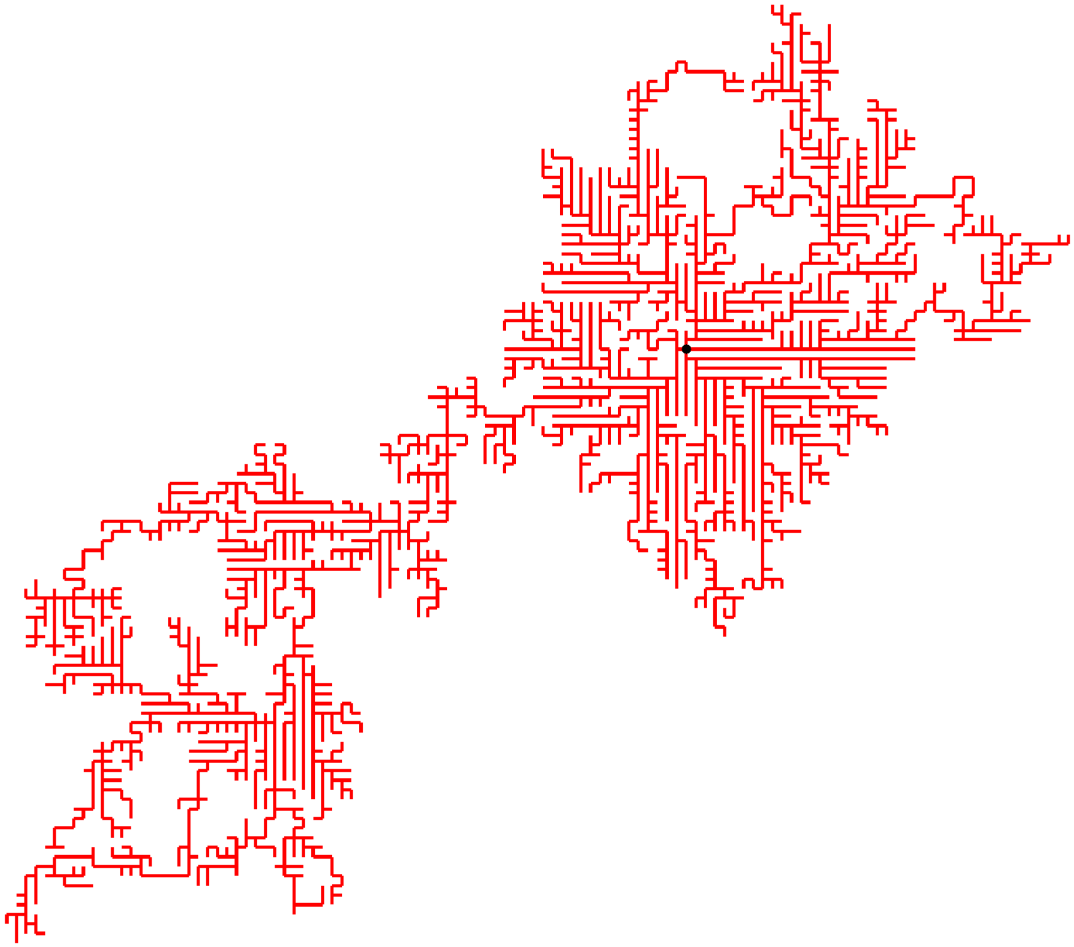
Minimal distance rooted spanning tree.



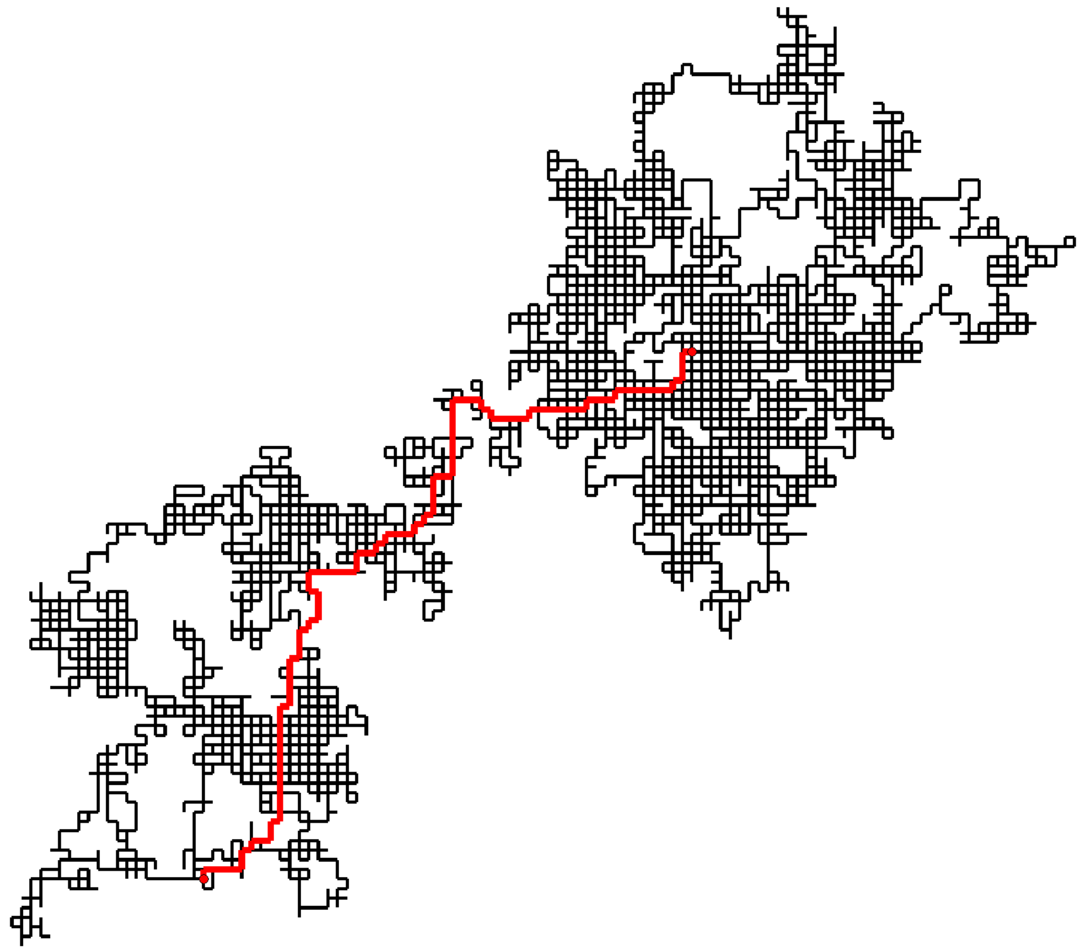
A shortest path from 0 to $\sqrt{n}/2$.



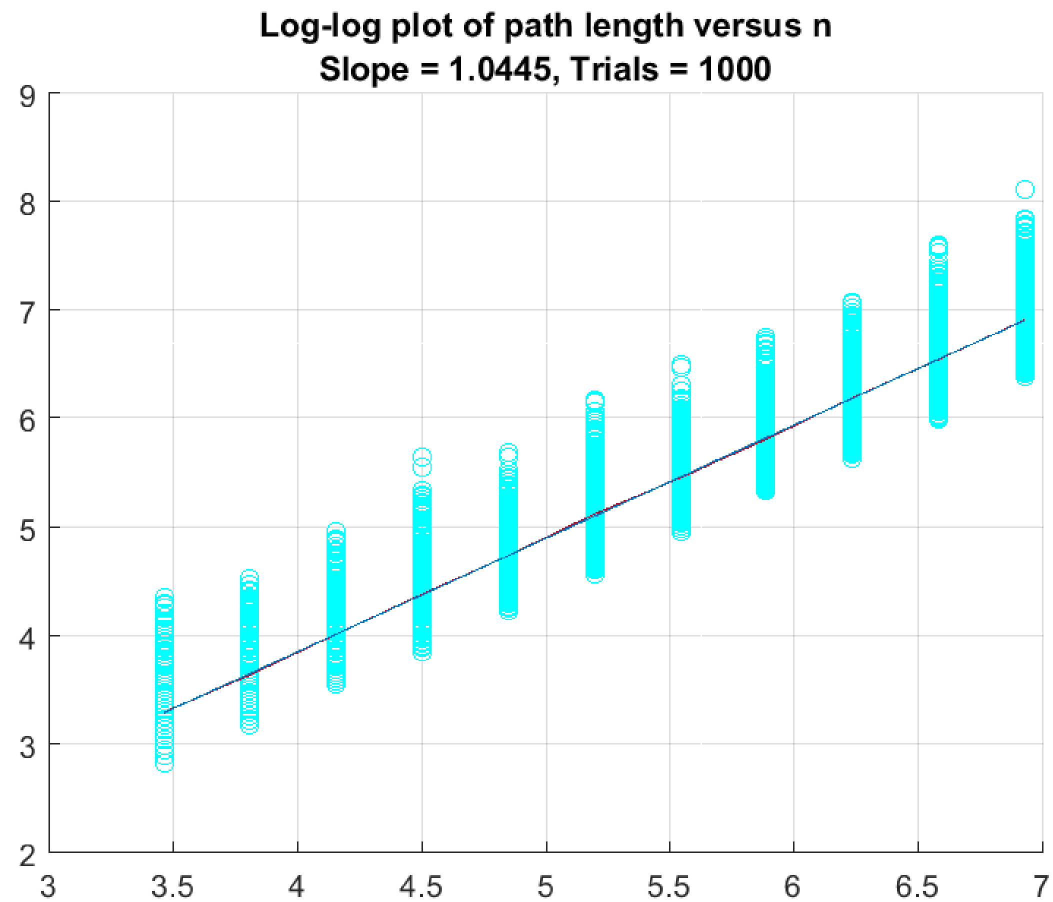
10000 step random walk.



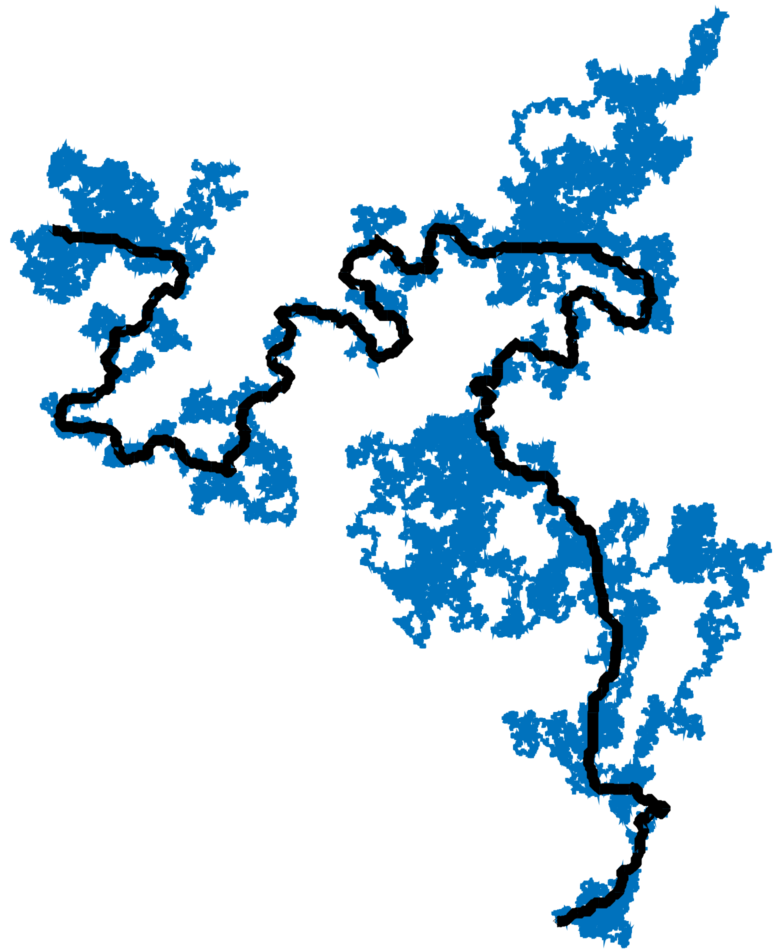
Minimal distance spanning tree, wrt to origin.

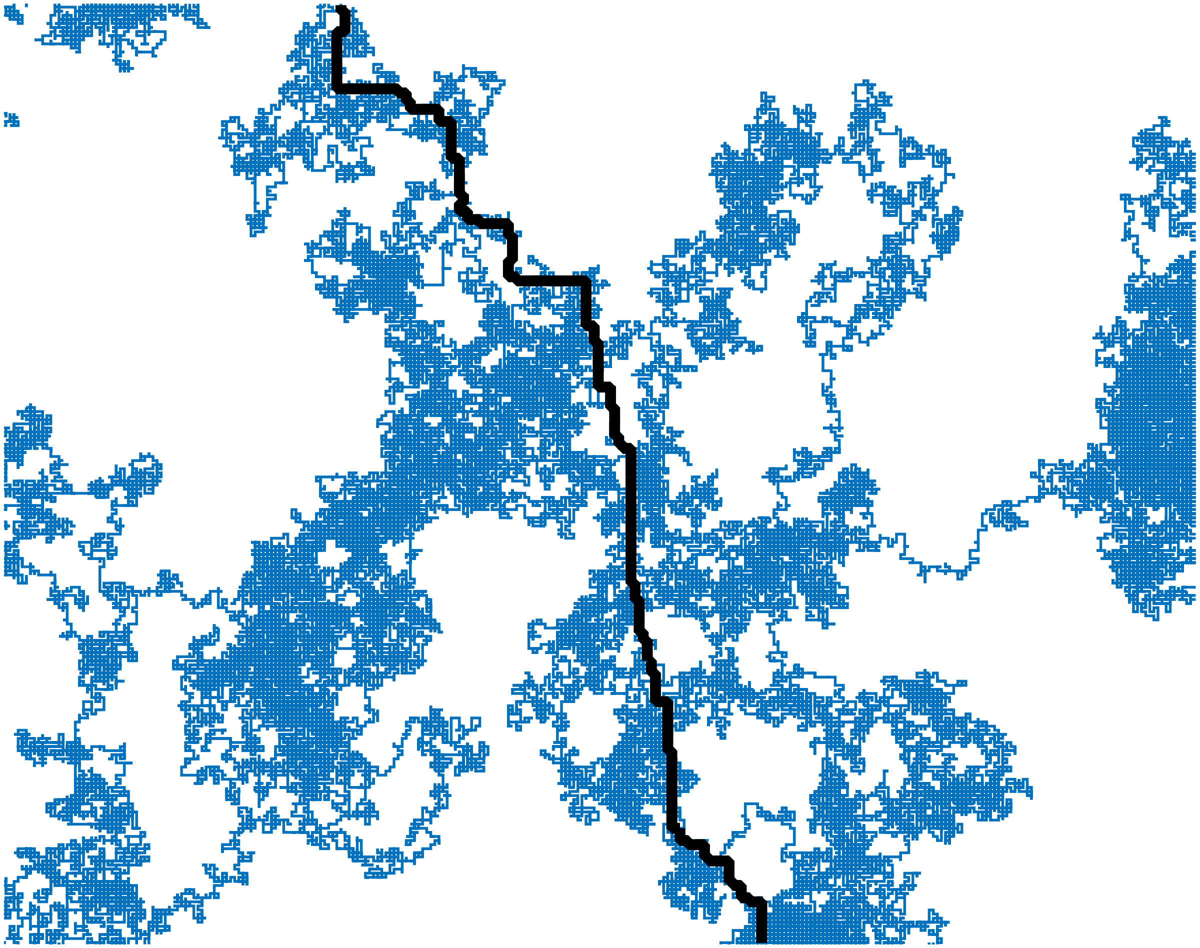


Does this curve converge to a fractal as $n \to \infty$?.

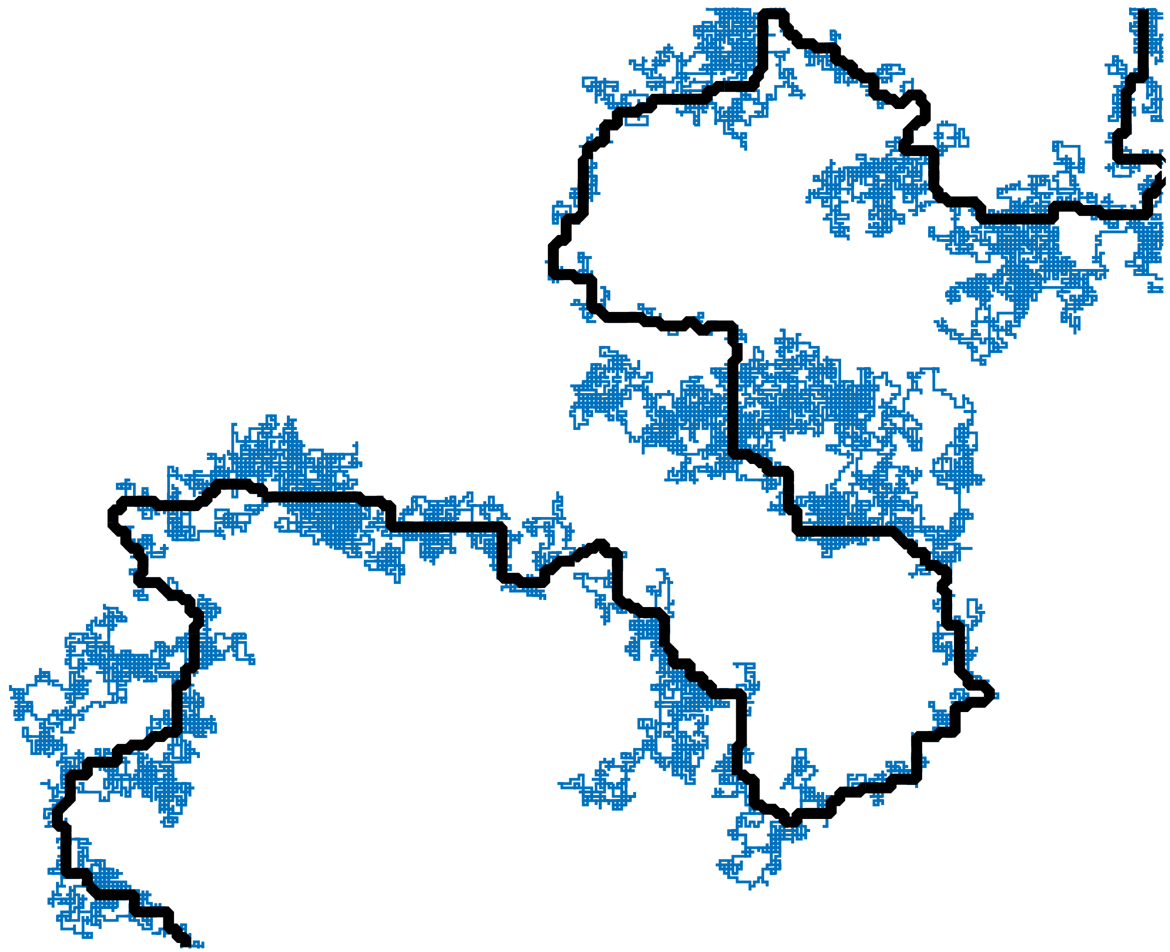


Log-log plot of graph distance 0 to $\{|z| = \frac{1}{2}\sqrt{n}\}$.

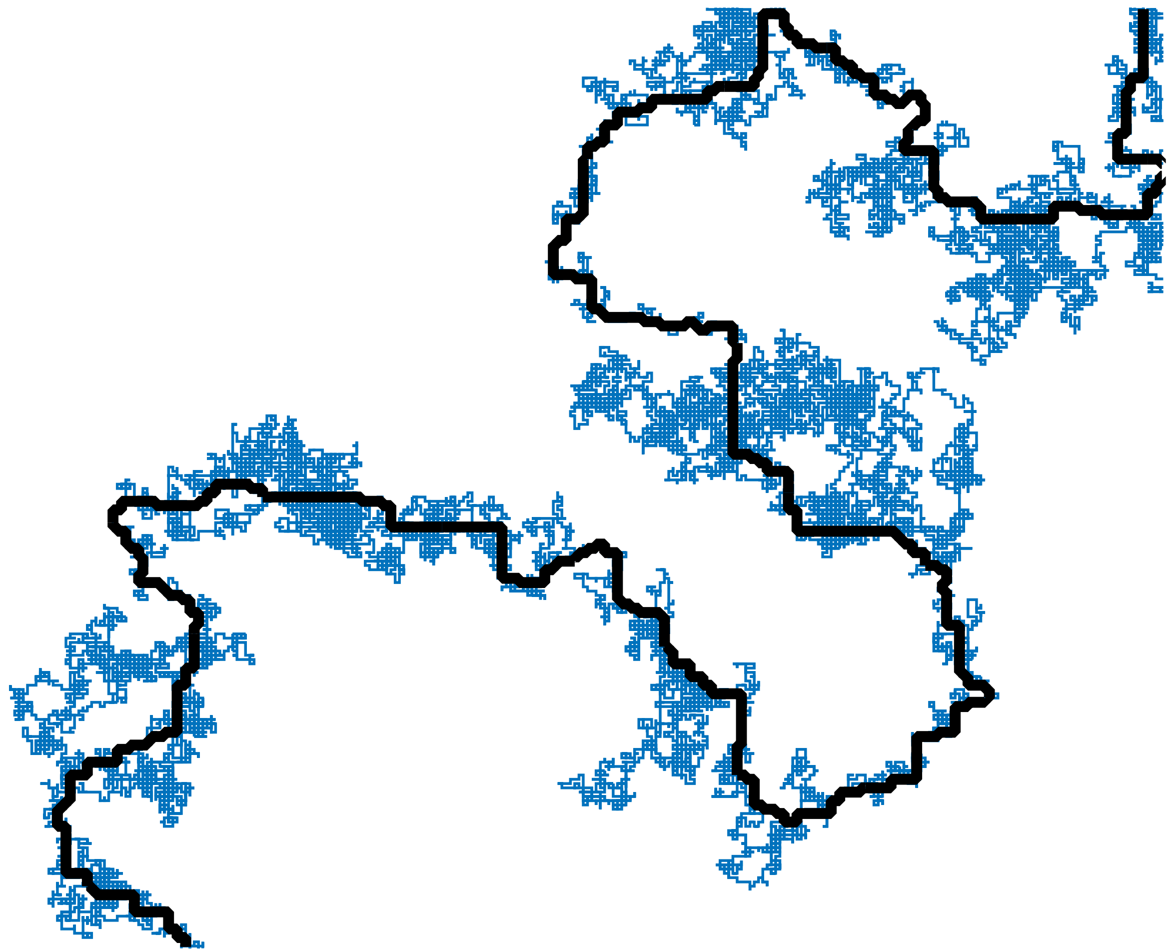




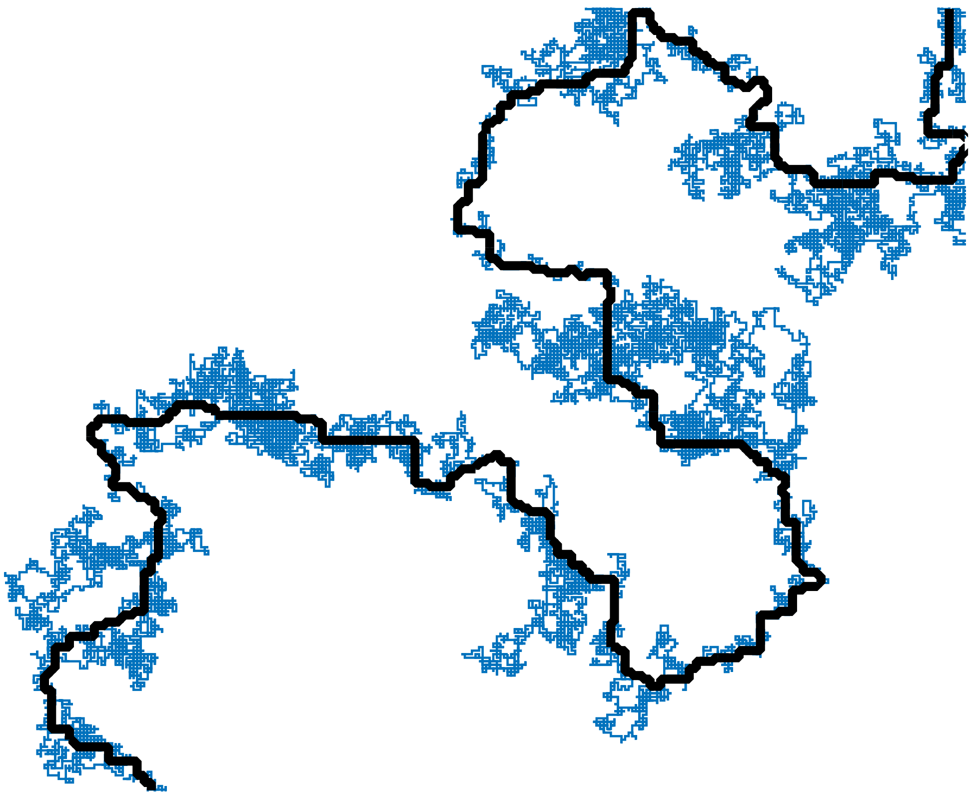
Paths can be straight where trace is dense.



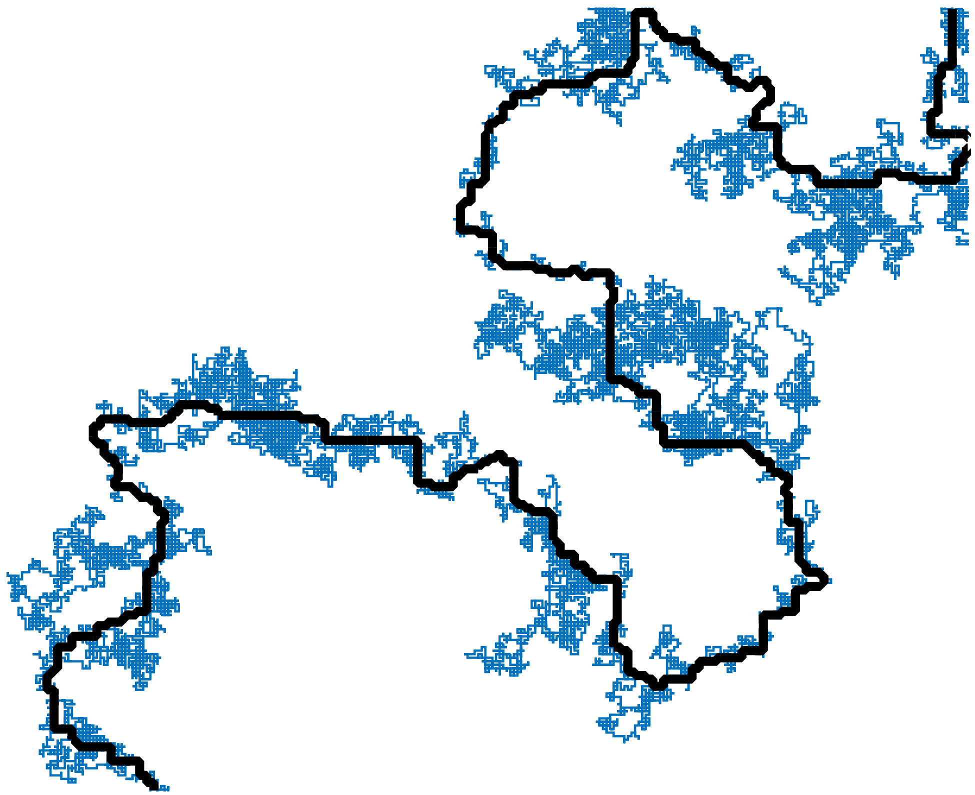
Paths wiggle more when trapped between components.



Can any two components be separated by a rectifiable curve?



Is the intersection of two boundaries rectifiable?



Intersection of component boundaries is empty or dimension 3/4.
Self-similar sets of dimension < 1 are rectifiable.

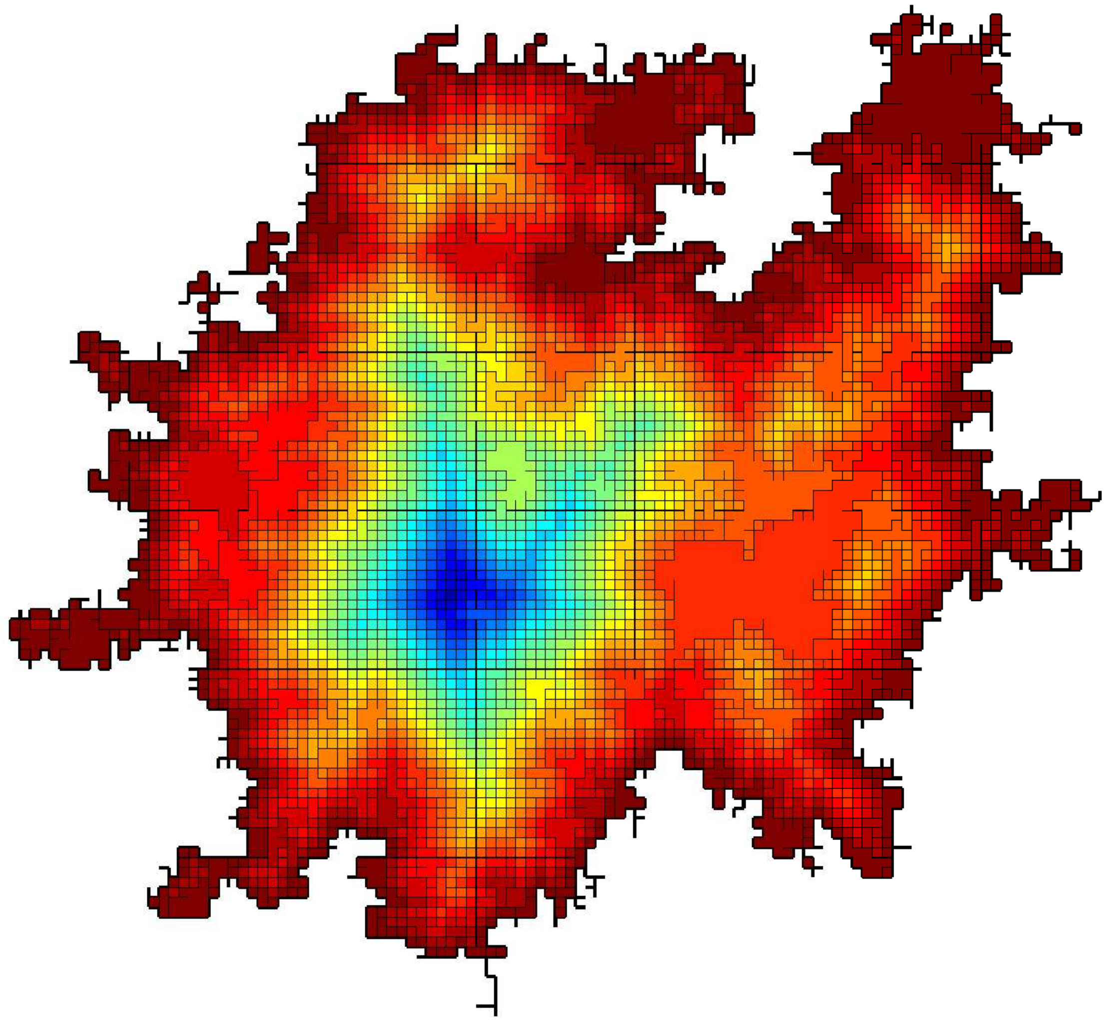
Call a square ϵ-dense for a given trace if the trace comes within ϵ of every point of the square.

Since the trace has Hausdorff dimension 2, there are many such squares at every scale. (Otherwise the trace is porus and has dimension < 2.)

Inside ϵ-dense squares, we can draw paths that are ϵ close to straight. Pemantle's proof shows that such squares cannot "line up".

Can they "percolate", that is, does their union have connected components of macroscopic size (comparable to diameter of the trace)?

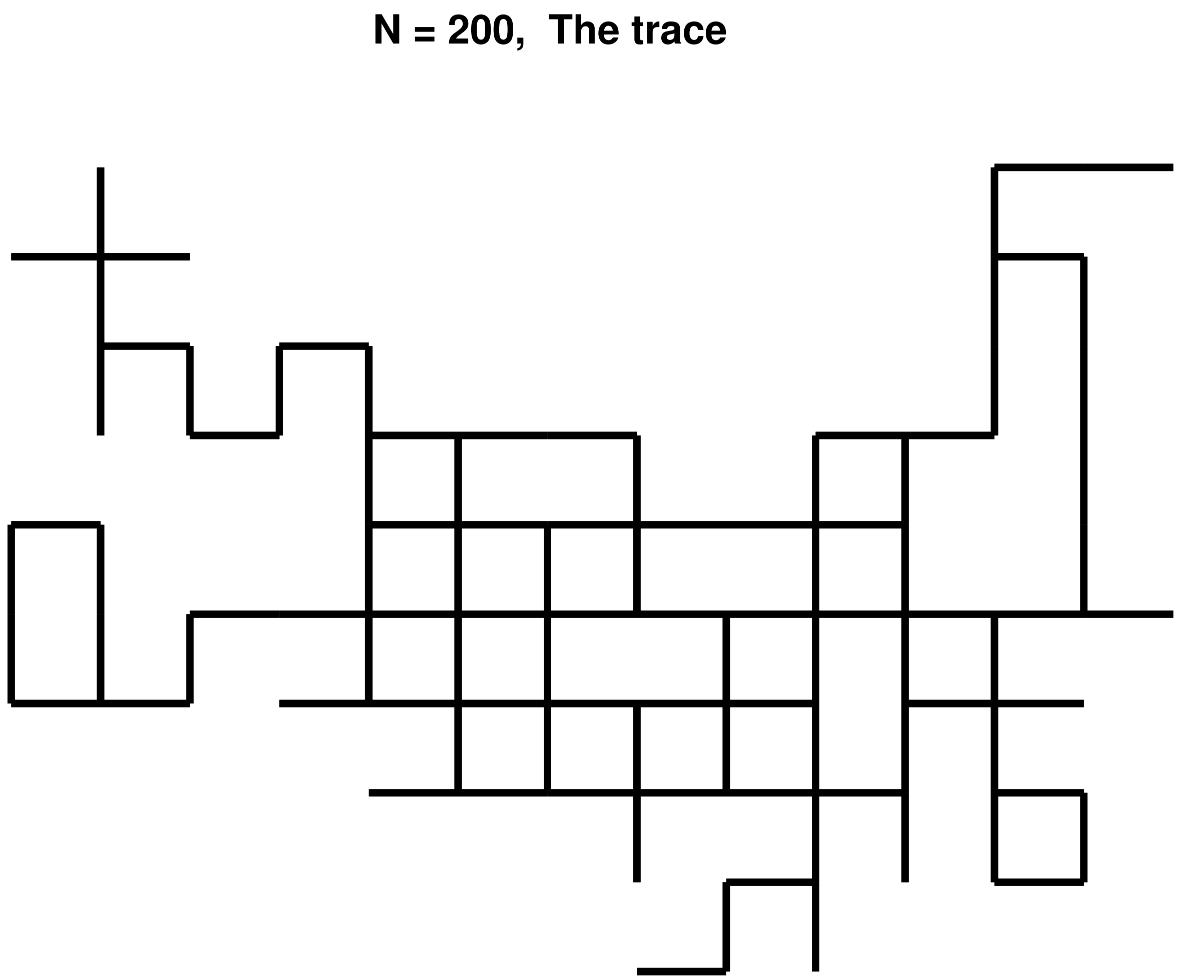
PART III: THE GRAPH OF COMPLEMENTARY COMPONENTS



Consider the complementary components of the Brownian trace as vertices of a graph, with two being adjacent if their boundaries overlap.

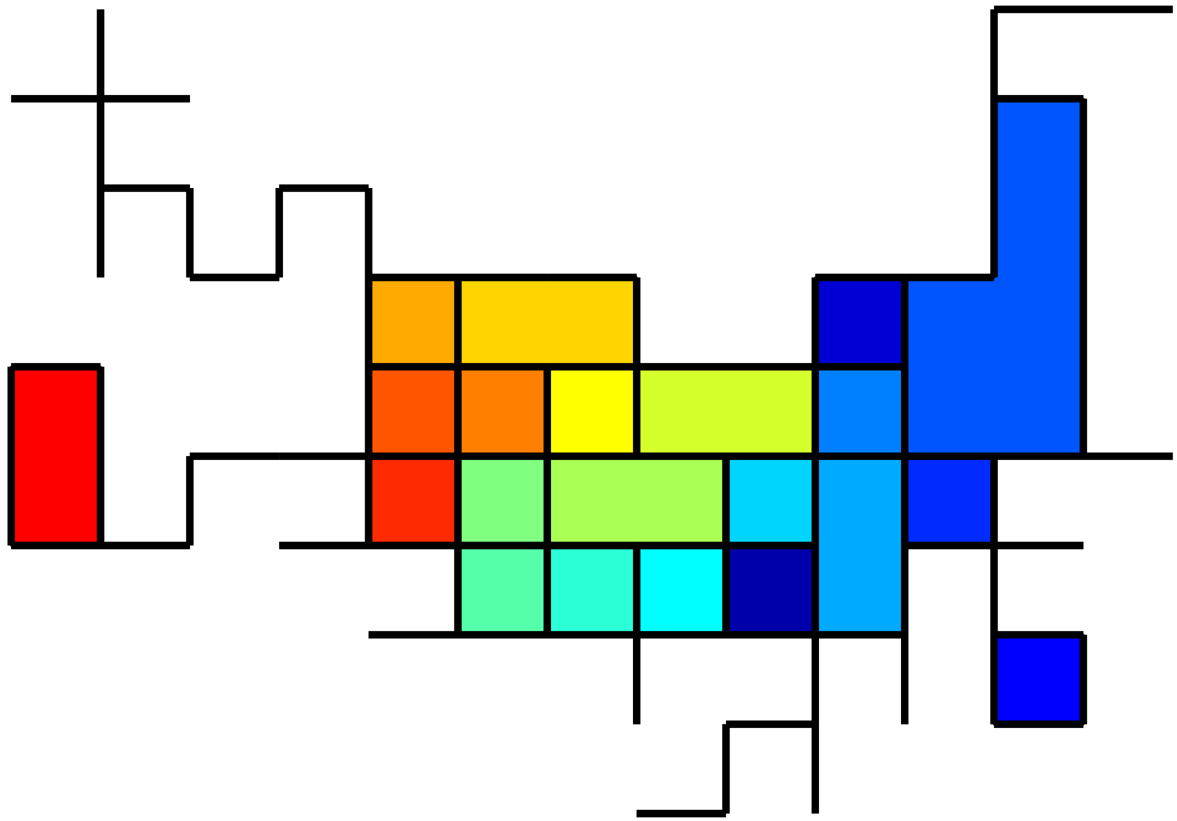
Wendelin Werner conjectured this graph is connected, i.e., any two components are connected by a path hitting the trace only finitely often.

What about a path that hits the trace countably often? Hits in a set of small Hausdorff dimension?



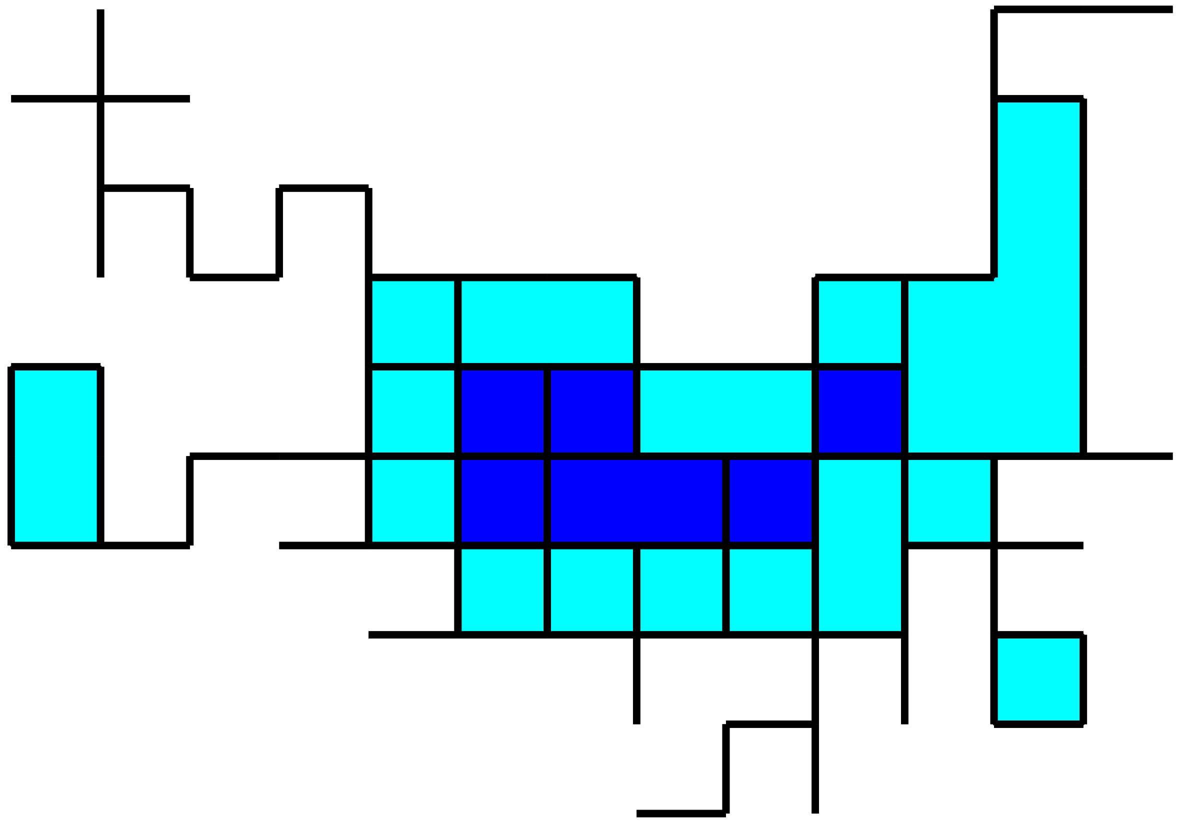
200 random steps on square grid.

N = 200, Number of components = 22

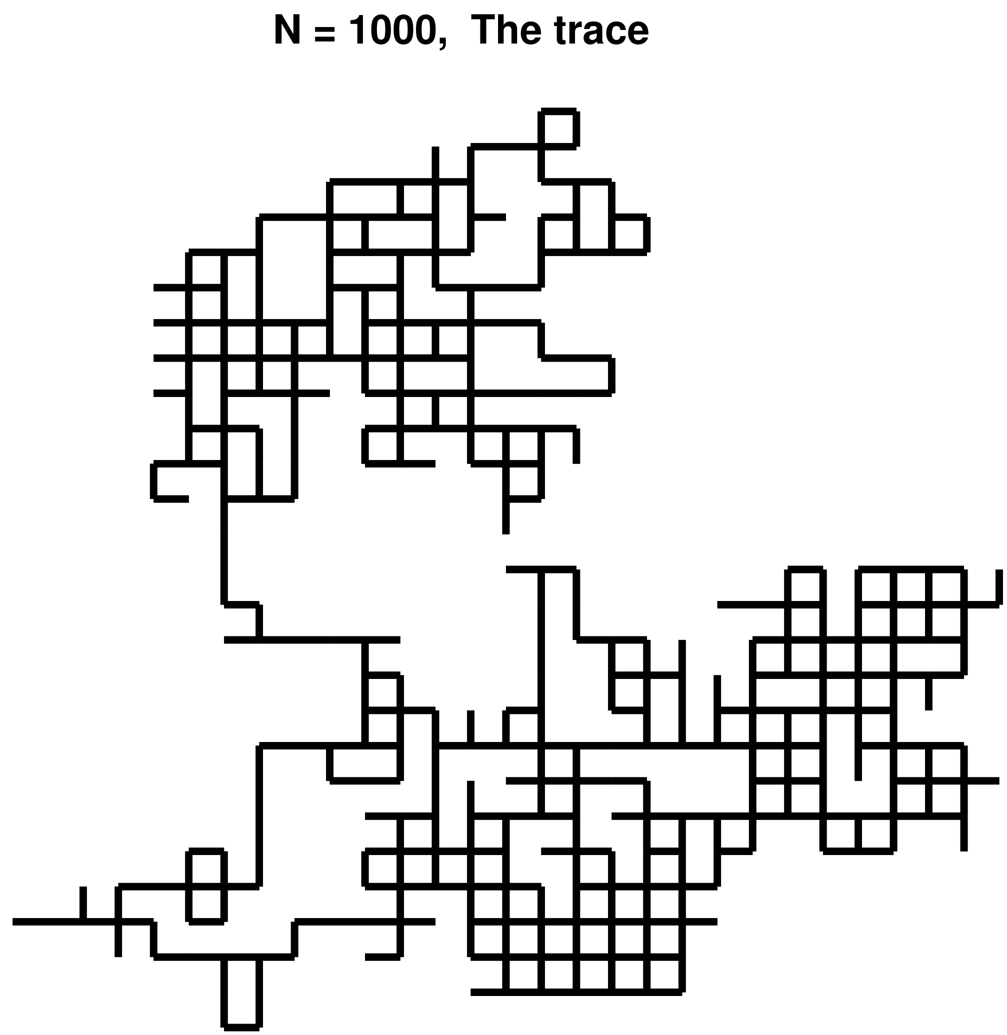


Components form a graph under edge adjacency.

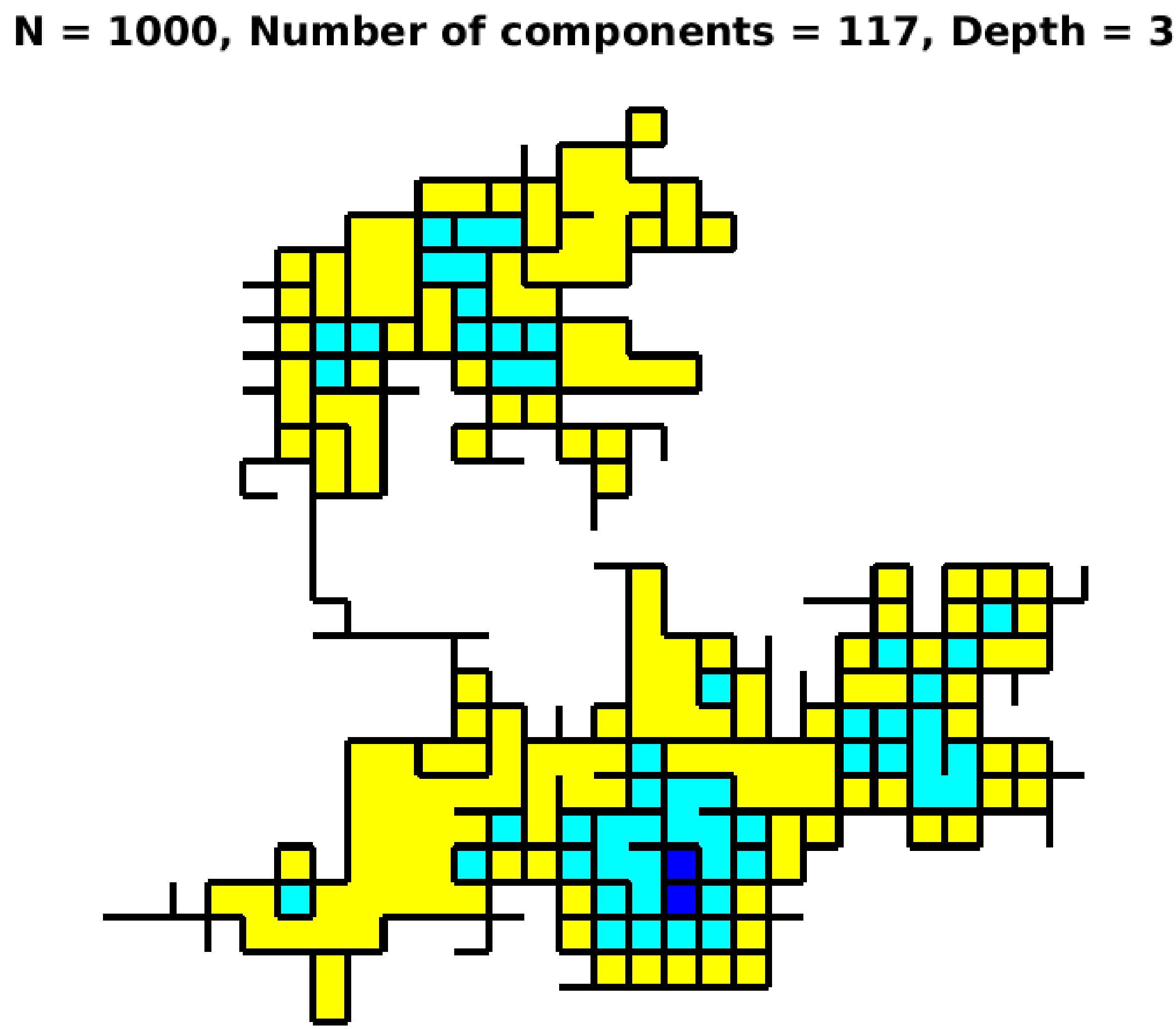
N = 200, Number of components = 22, Depth = 2



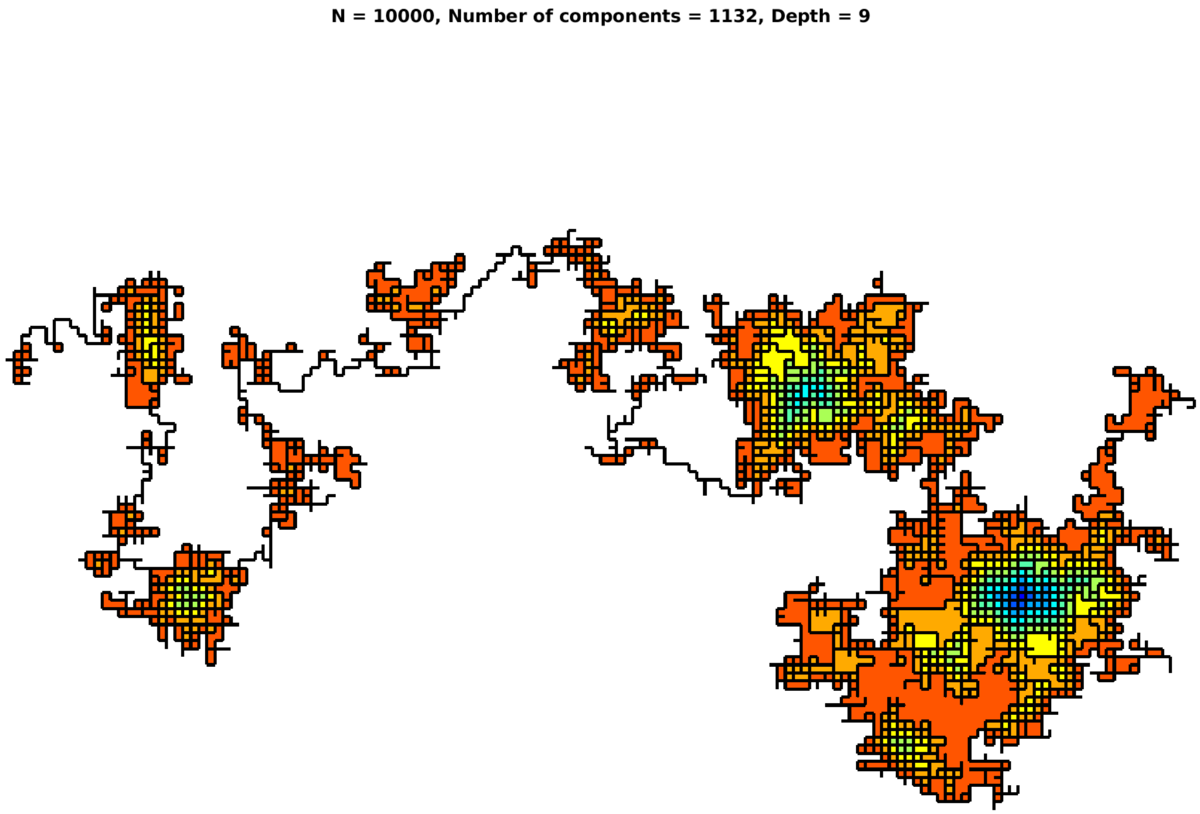
components colored by graph distance to outside.



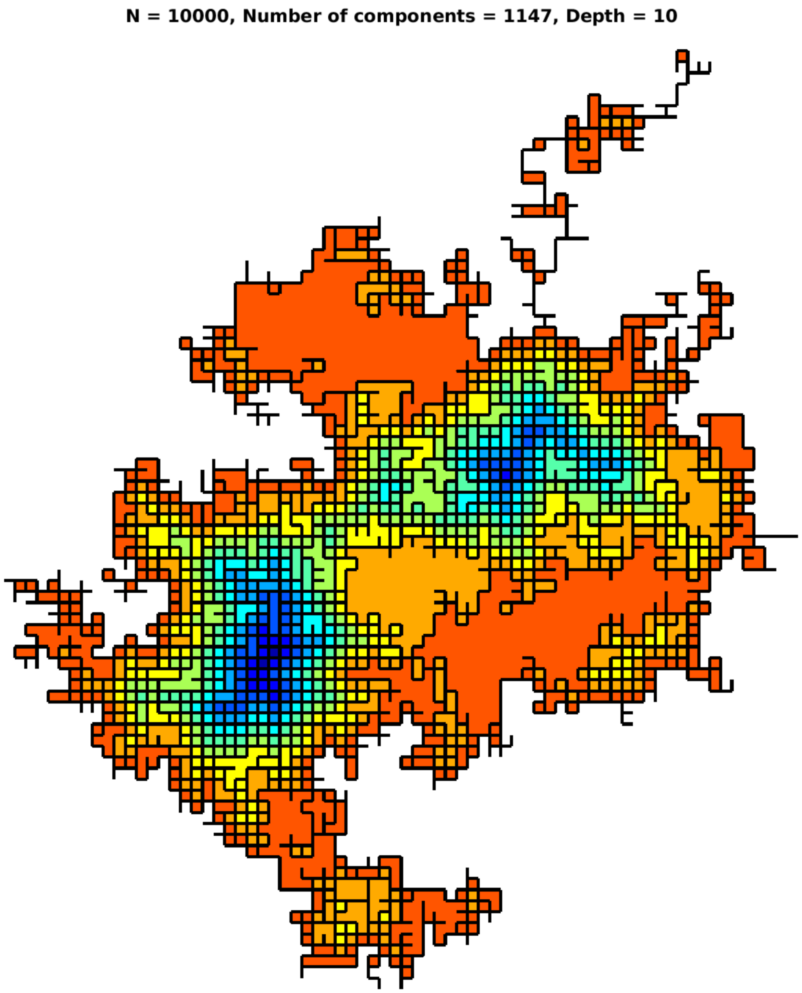
1000 random steps on square grid.



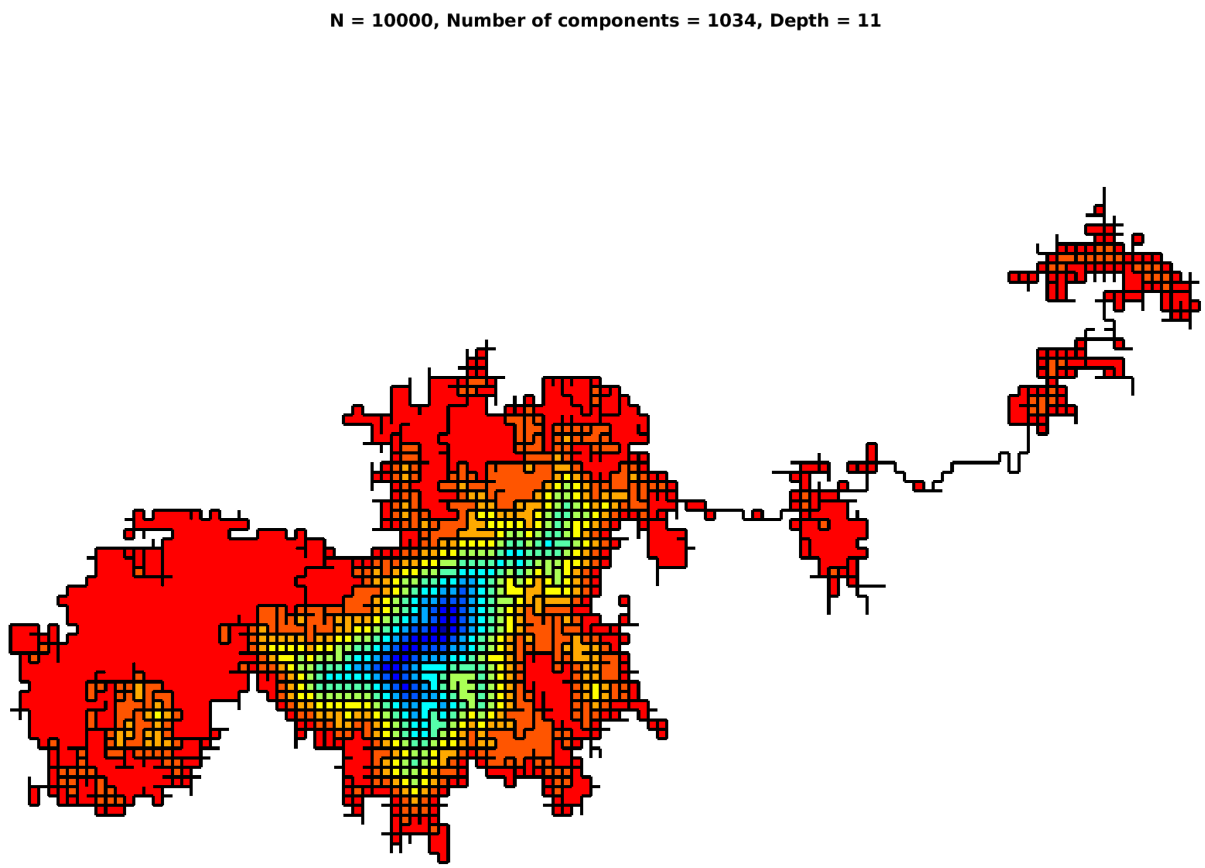
Components colored by graph distance to outer component.



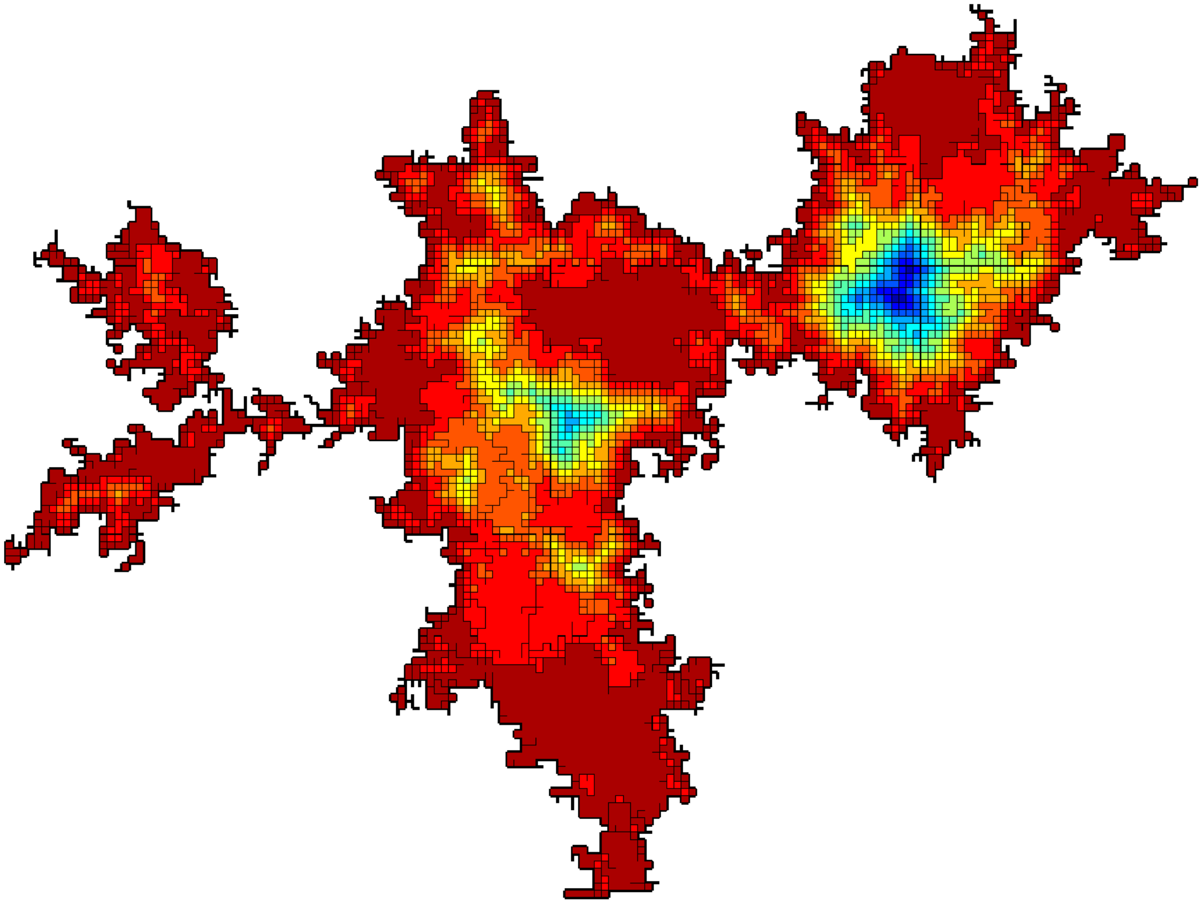
10,000 steps



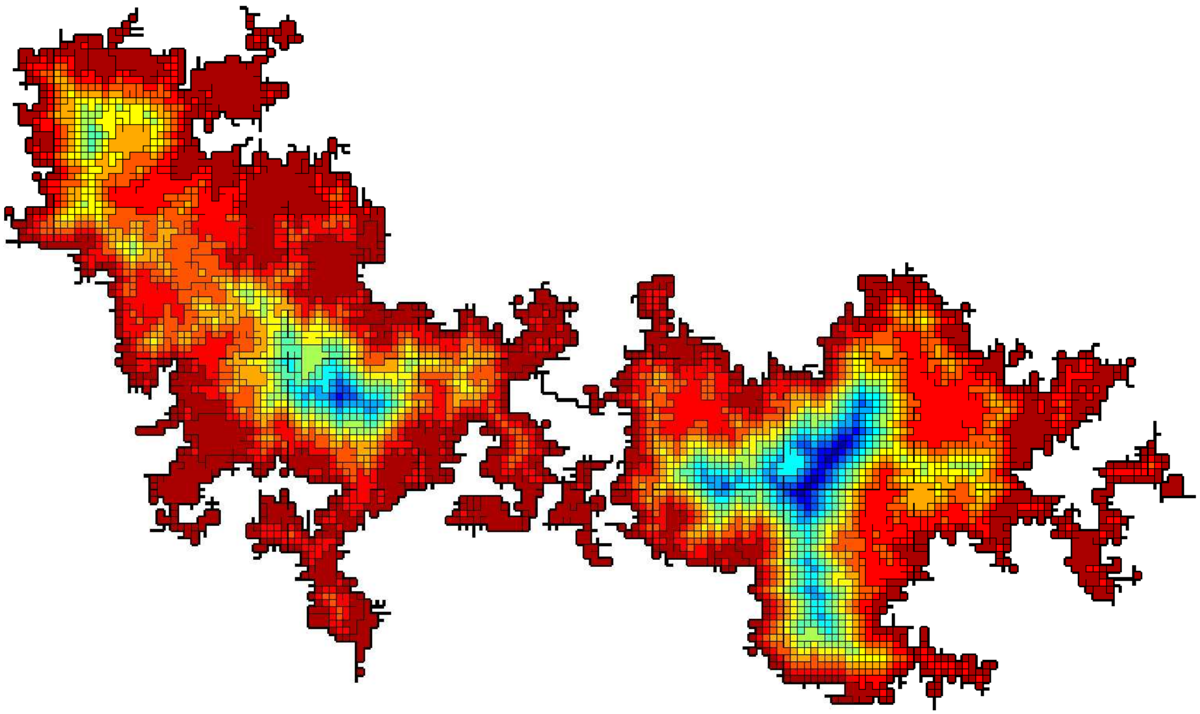
10,000 steps



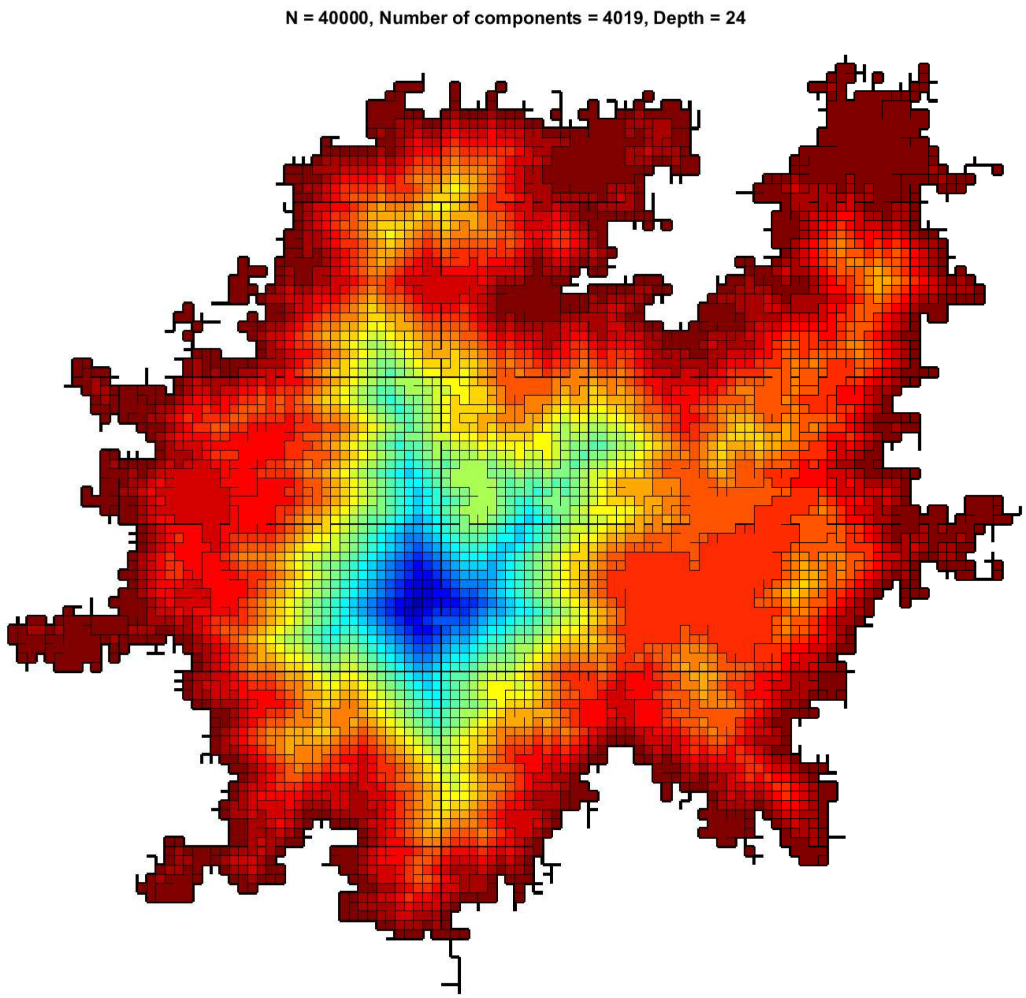
10,000 steps



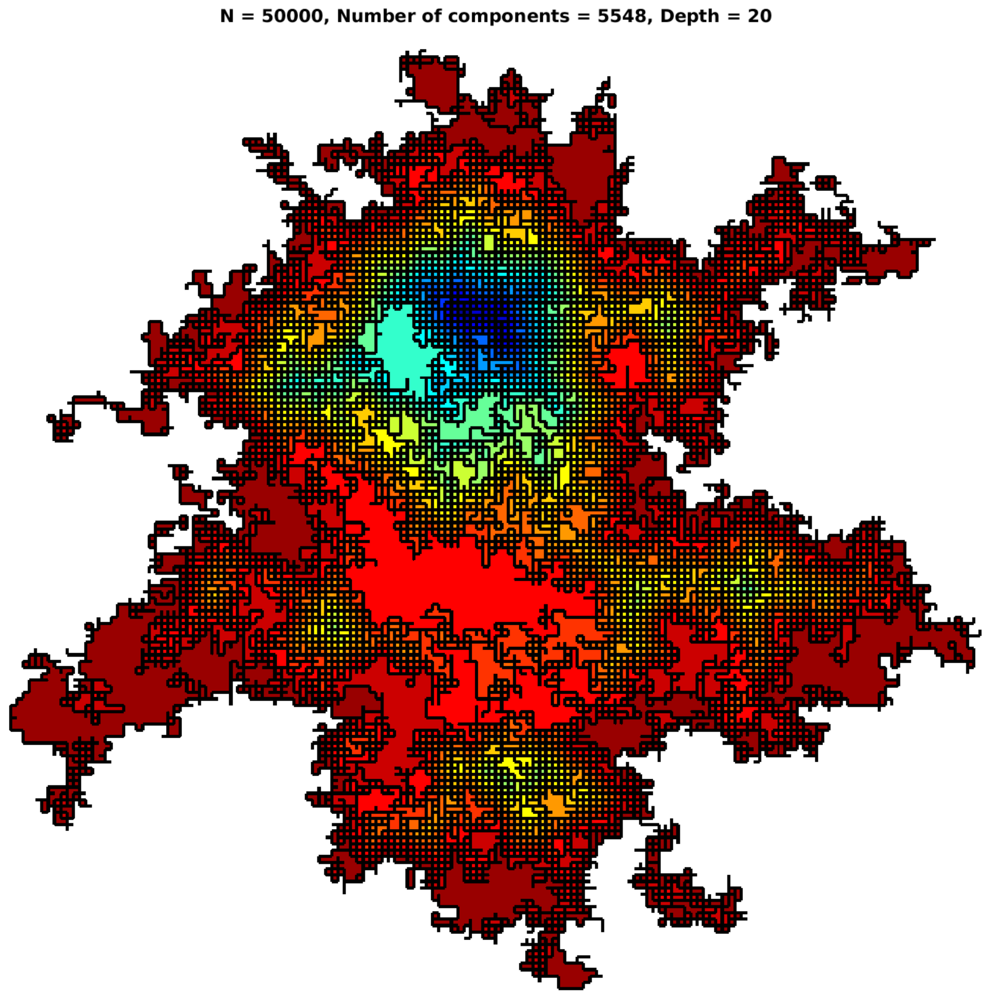
20,000 steps



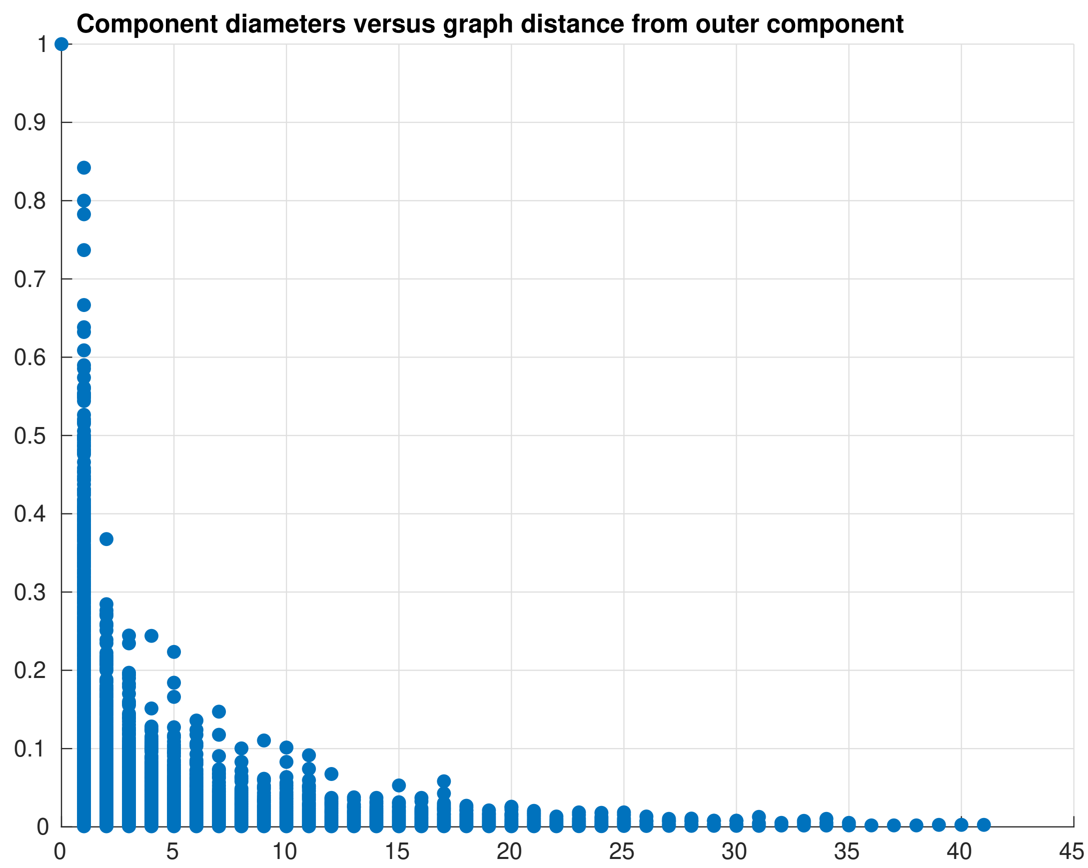
30,000 steps



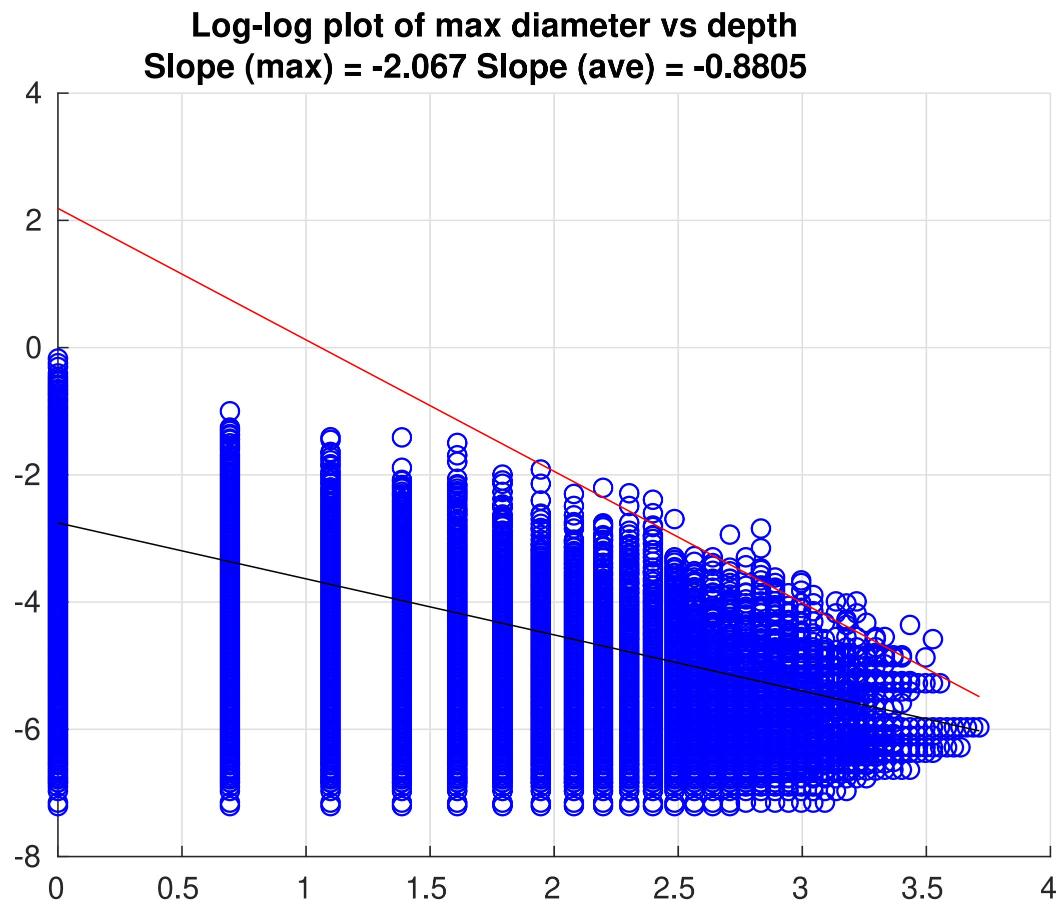
40,000 steps



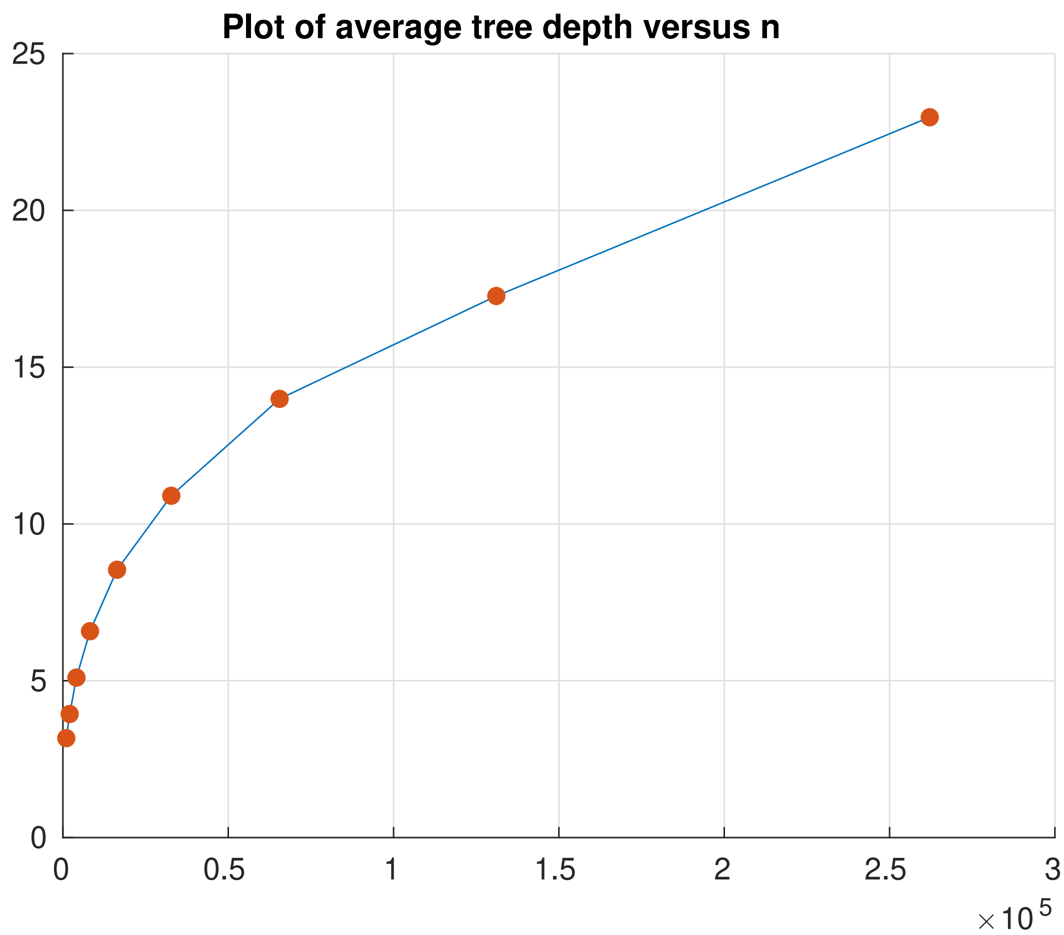
50,000 steps, 10%-component at depth 12



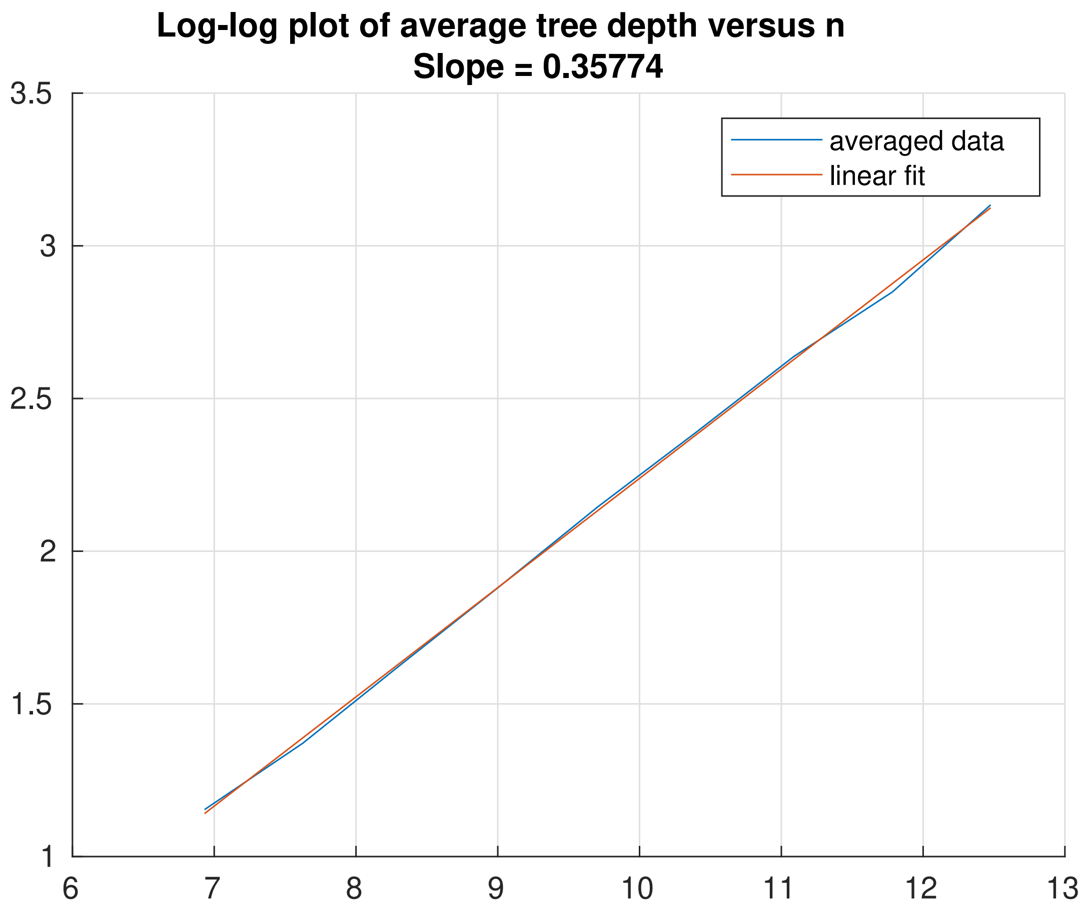
Component diameter versus graph distance from outer component.



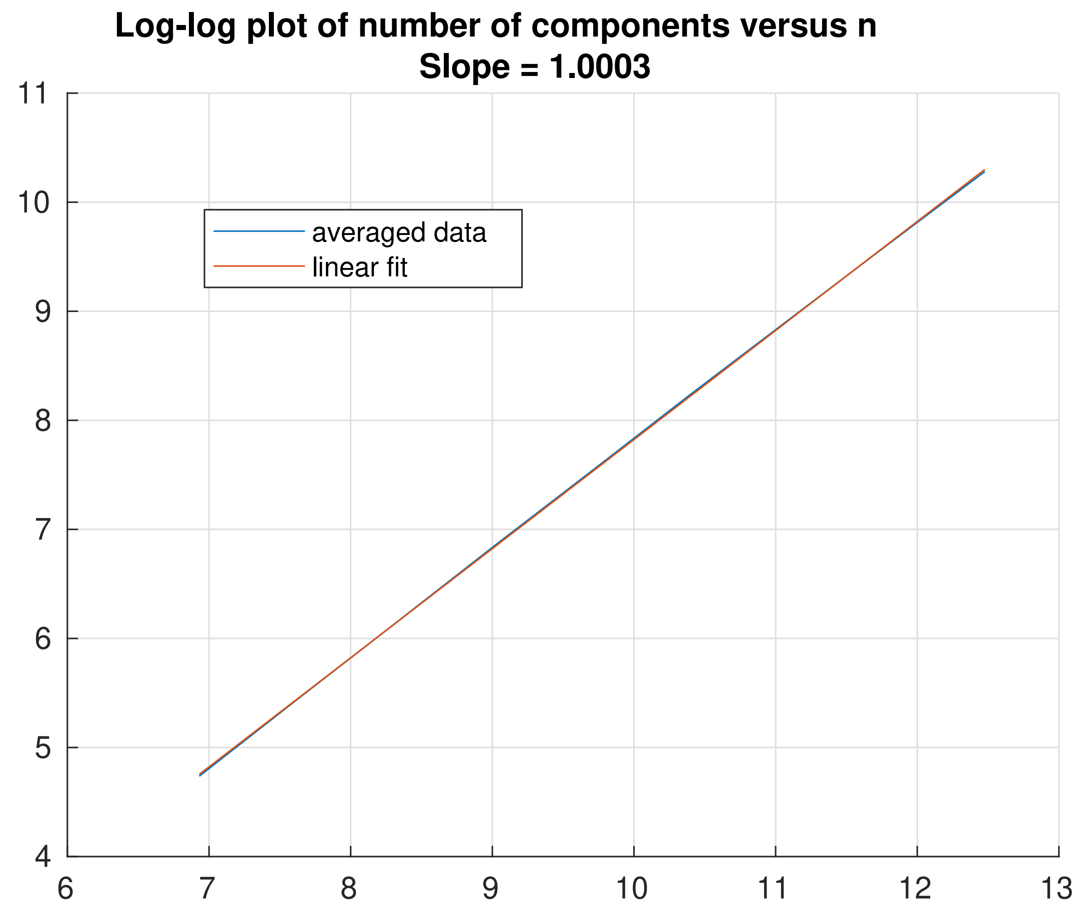
Same plot in log-log coordinates.



Growth of maximal graph distance (depth) to outer component.



Maximum distance from outer component looks like $n^{.36}$.



Number of components appears linear in n
Averaged over 100 trials.

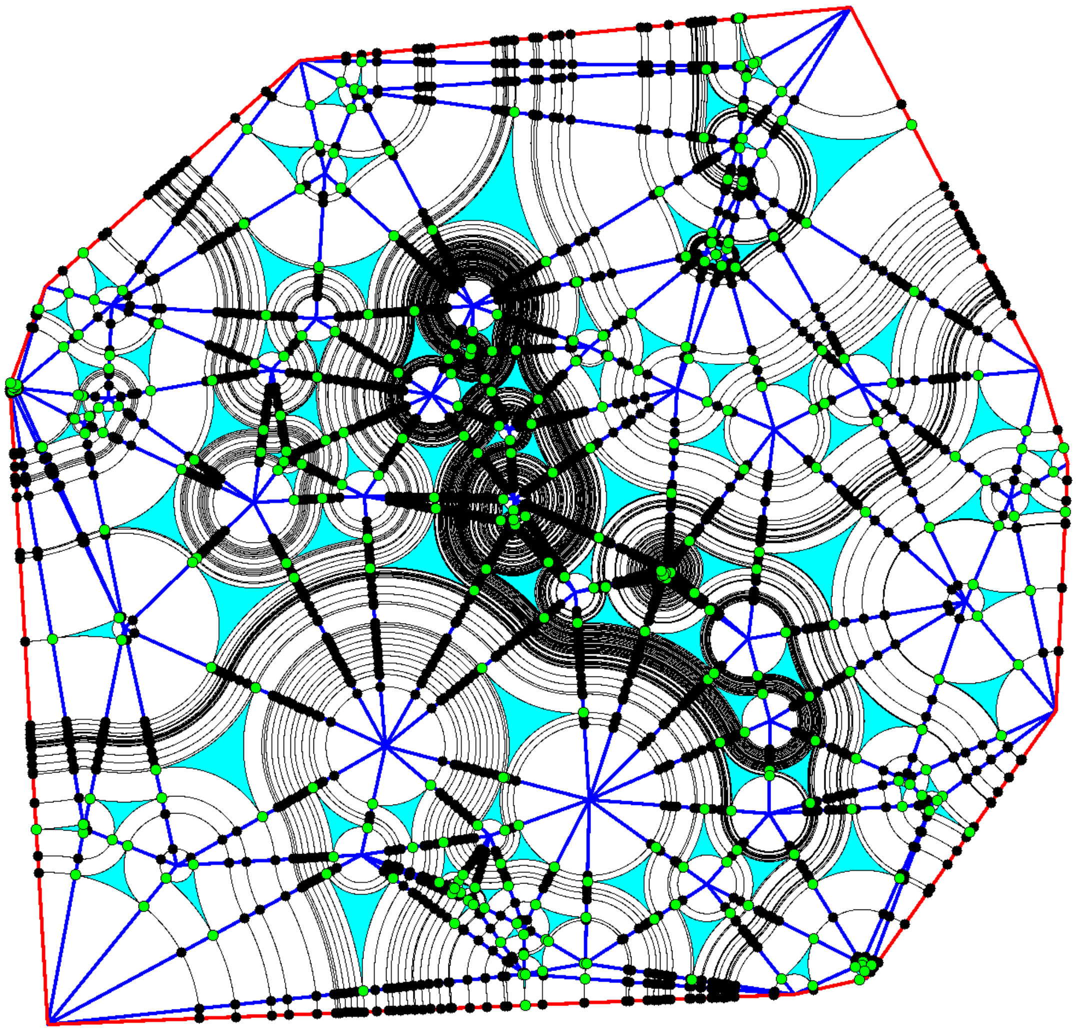
How could Werner's conjecture fail?

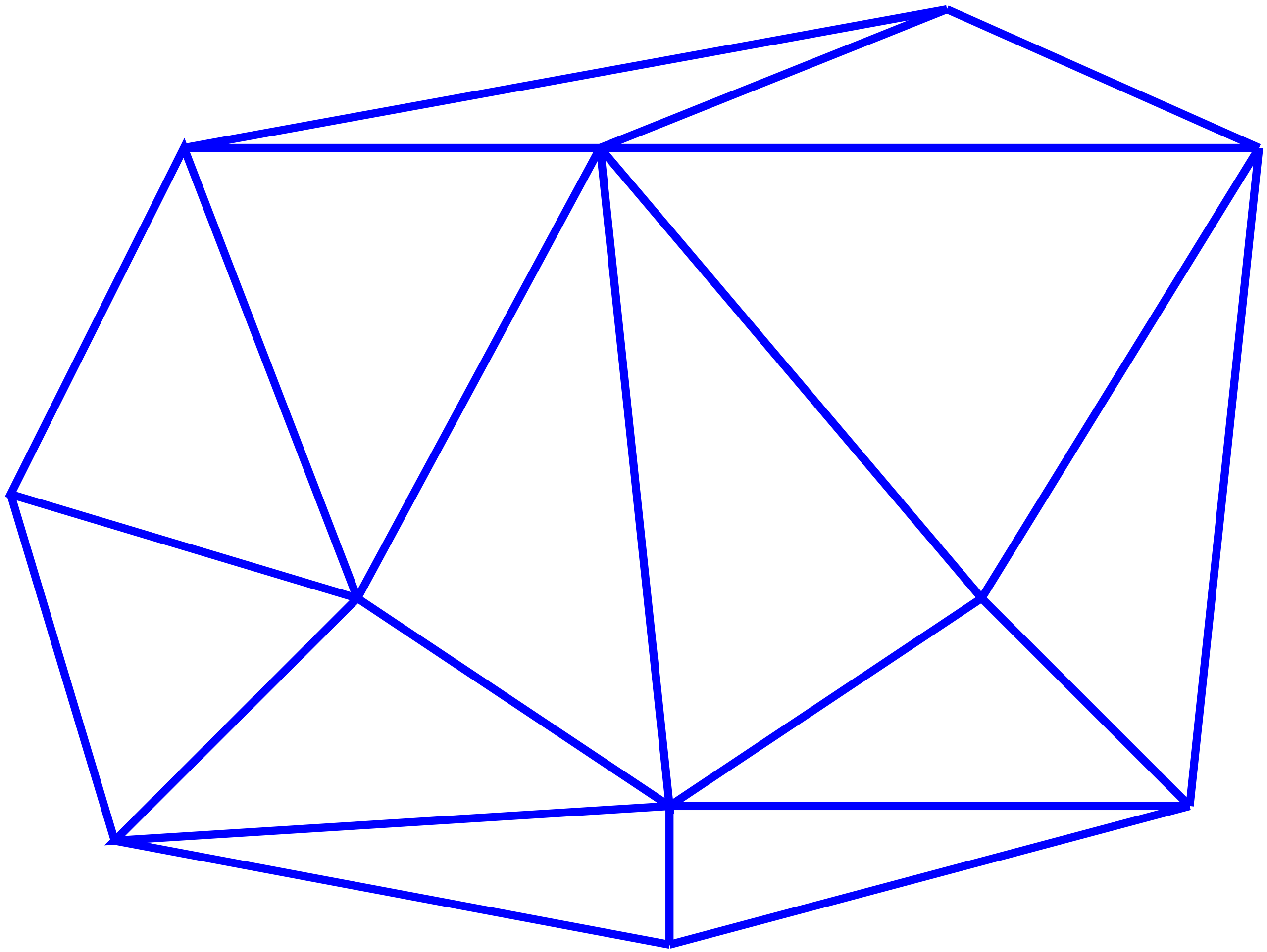
Need large component surrounded by much smaller components. Would happen if it was surrounded by ϵ-dense squares for every ϵ.

If ϵ-dense squares "percolate", they might form macroscopic loops that cause Werner's conjecture to fail.

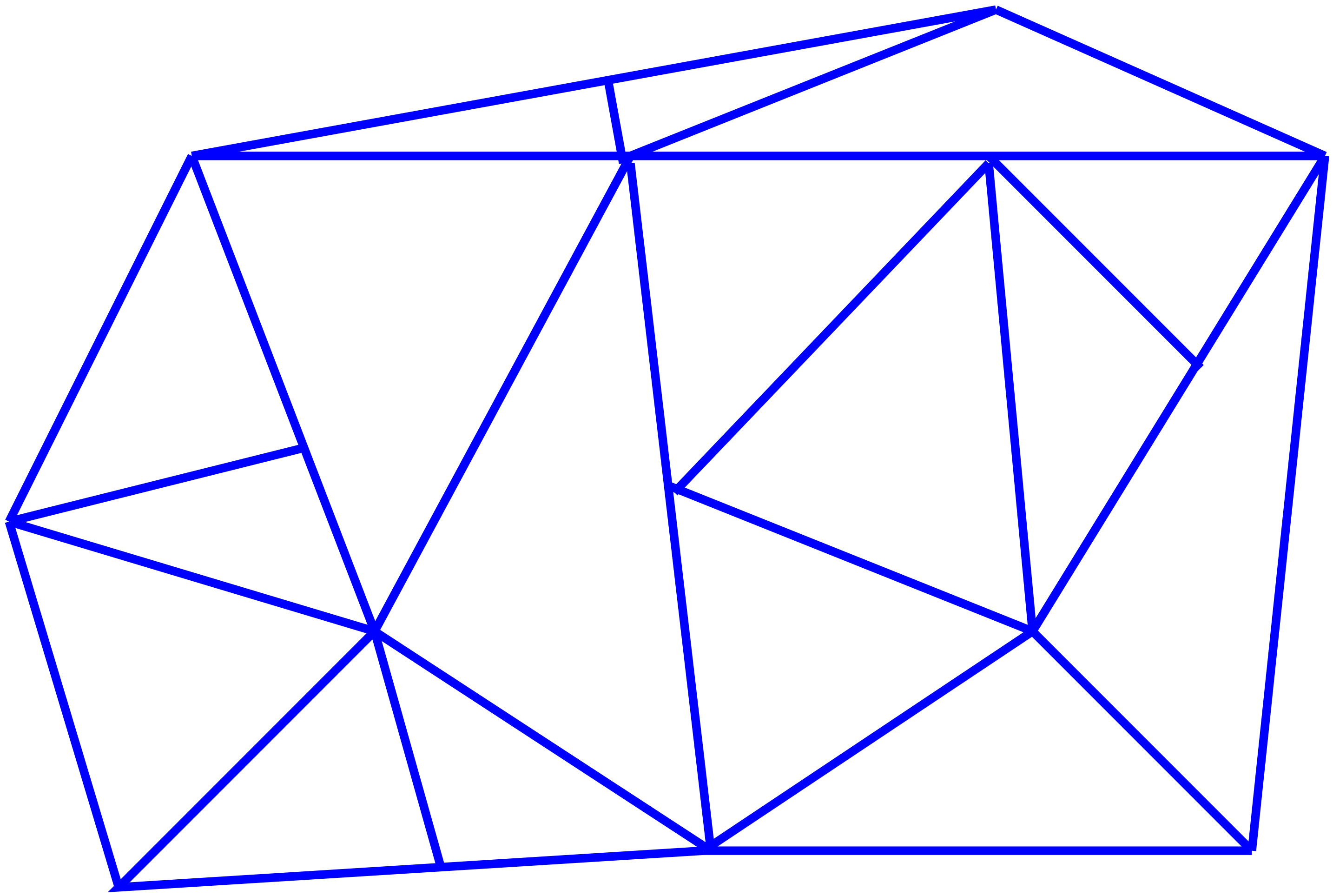
Can we prove that either Werner's conjecture fails or there are curves of small dimension in the trace, without knowing which occurs?

PART IV: TRIANGULATION FLOWS

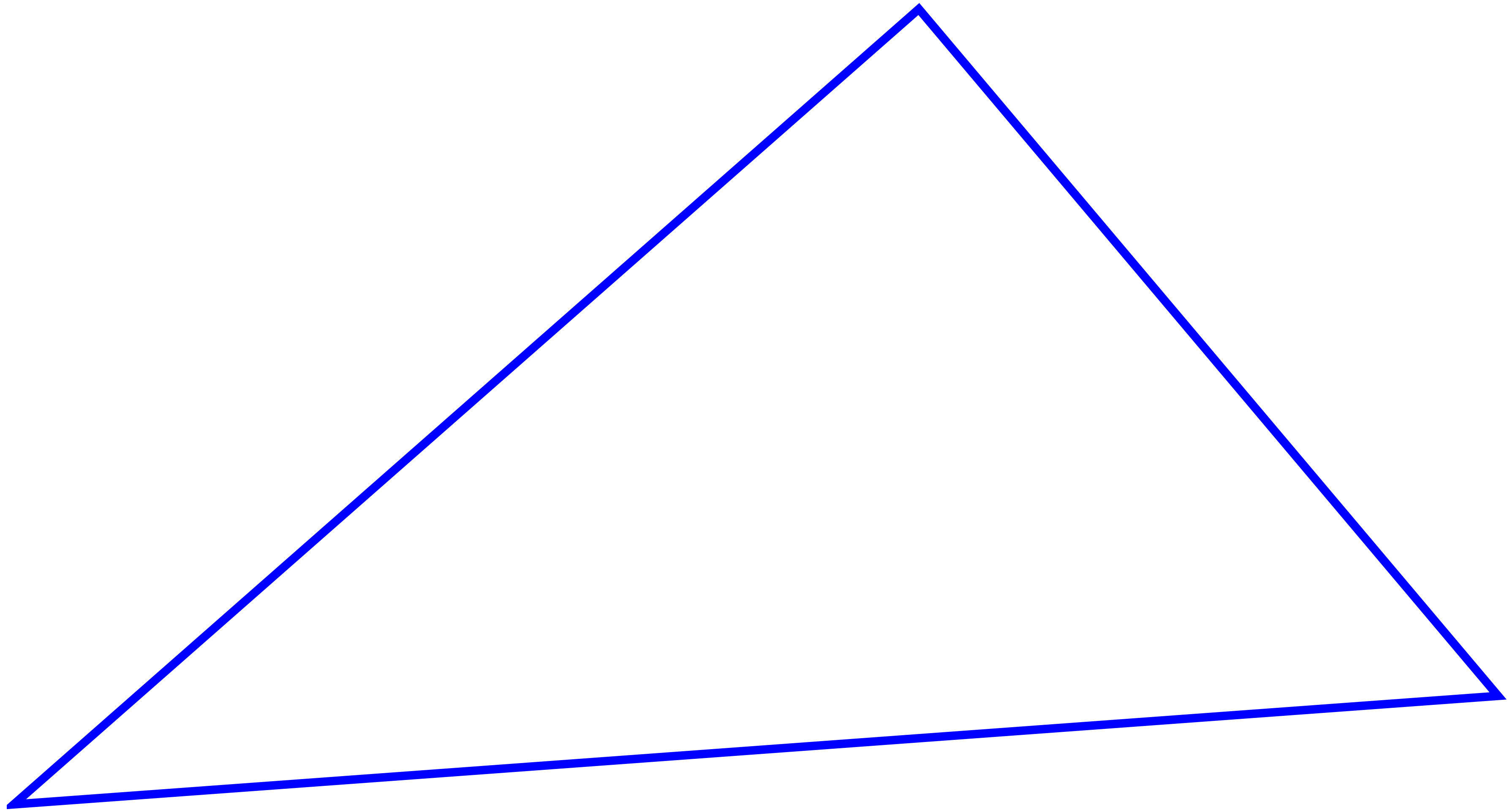




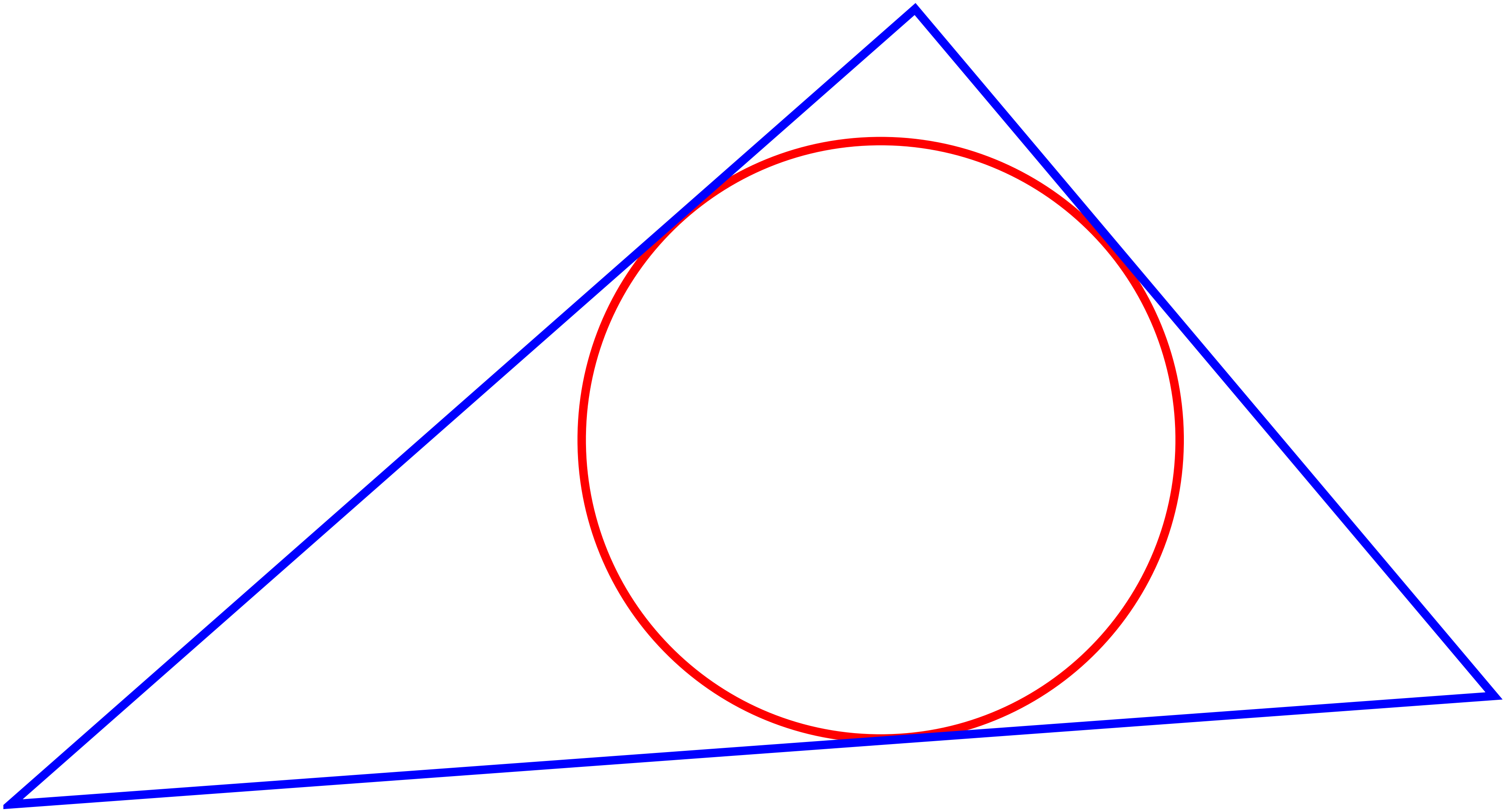
A triangulation: overlapping edges agree.



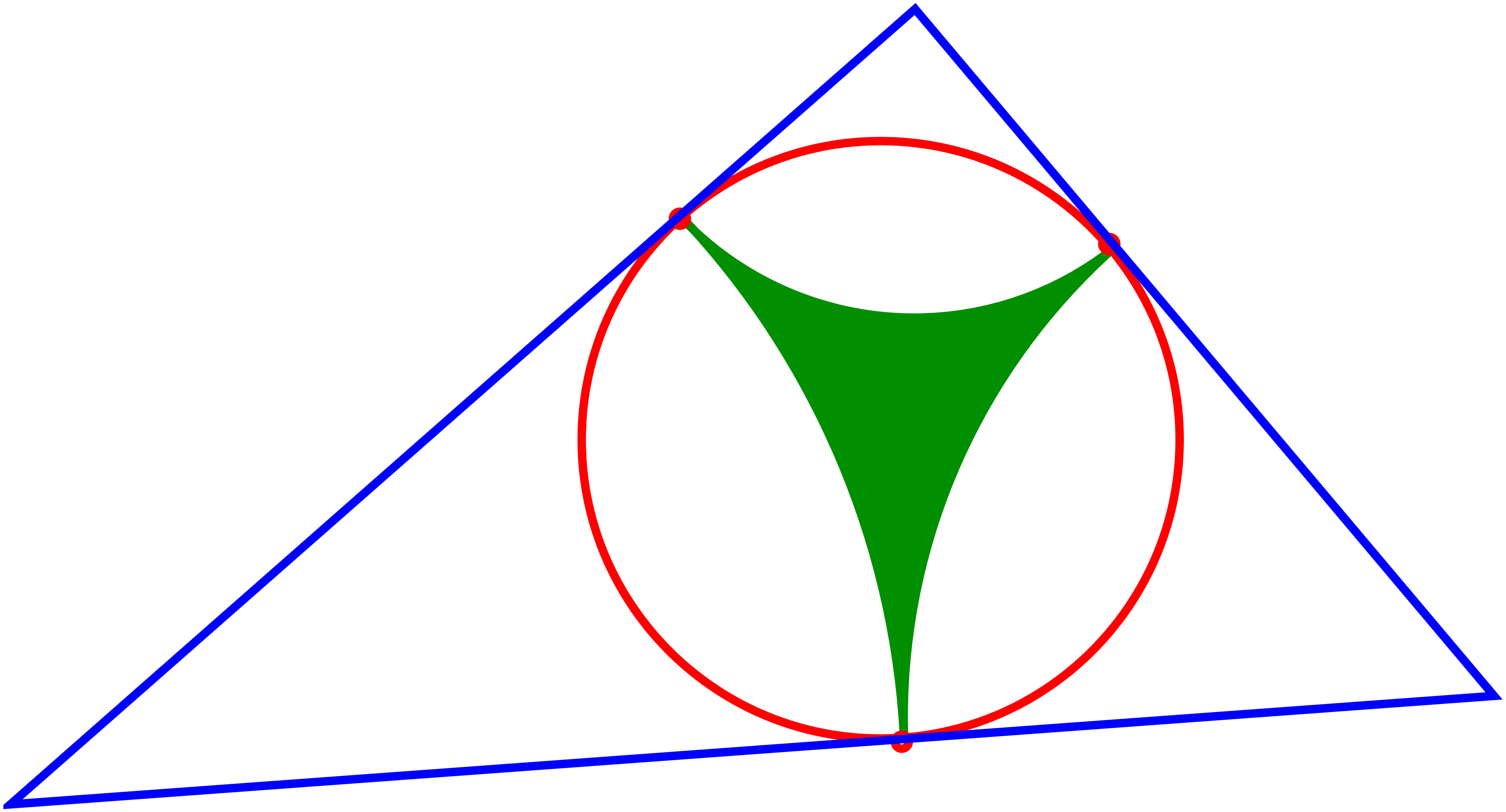
A dissection.



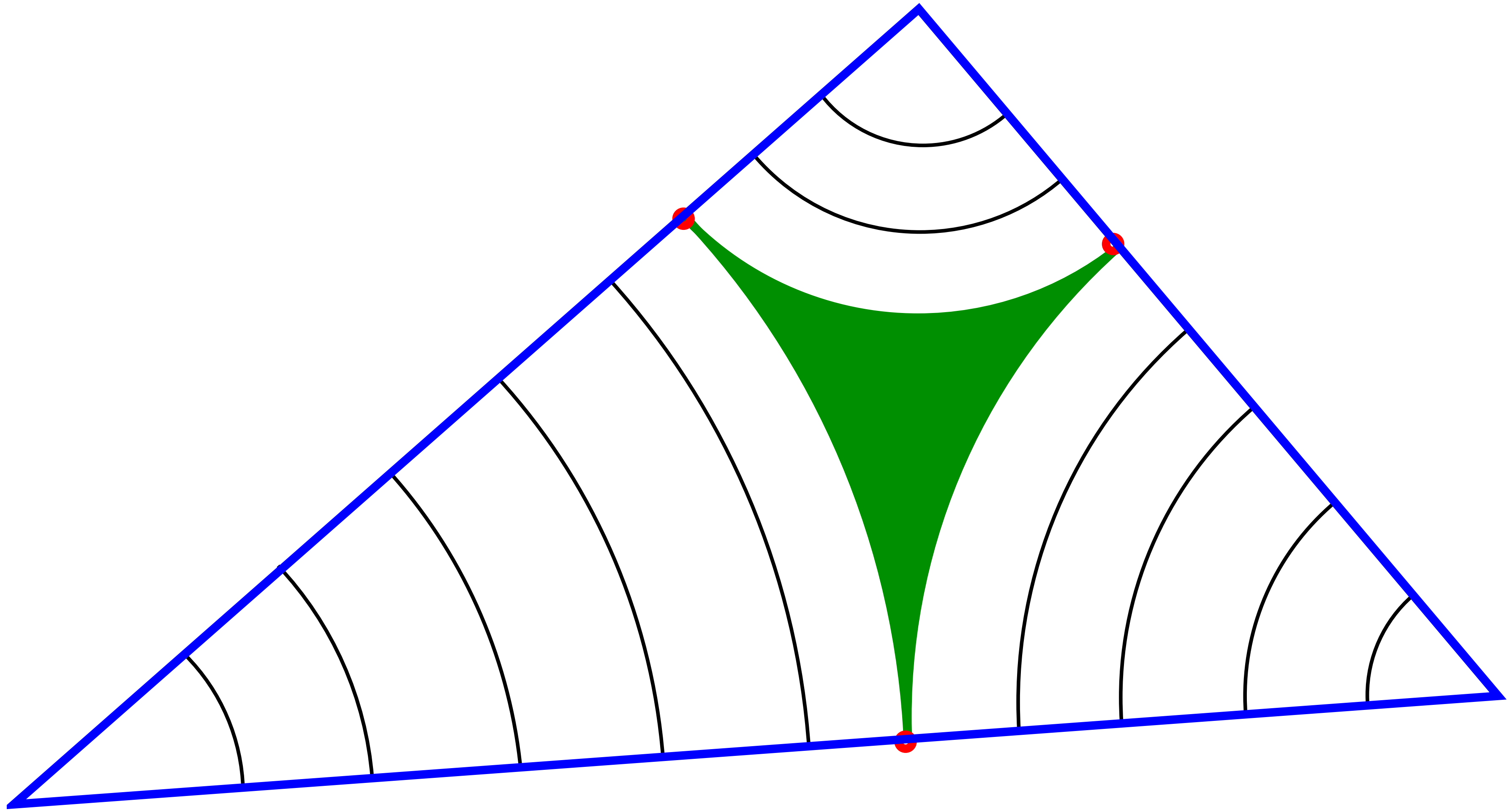
A triangle.



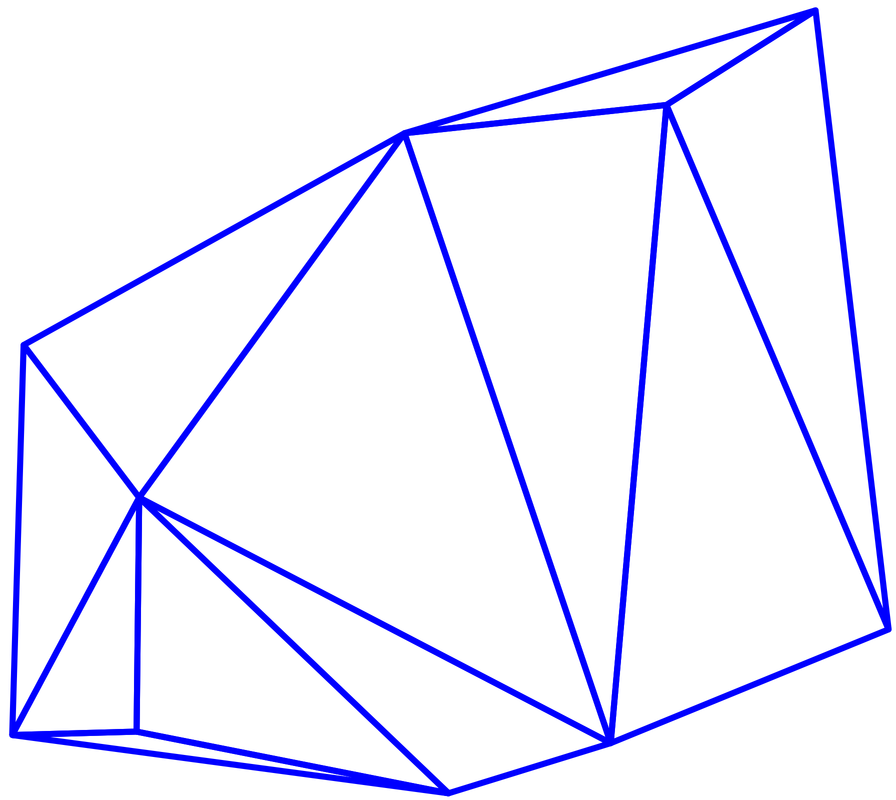
Its in-circle.



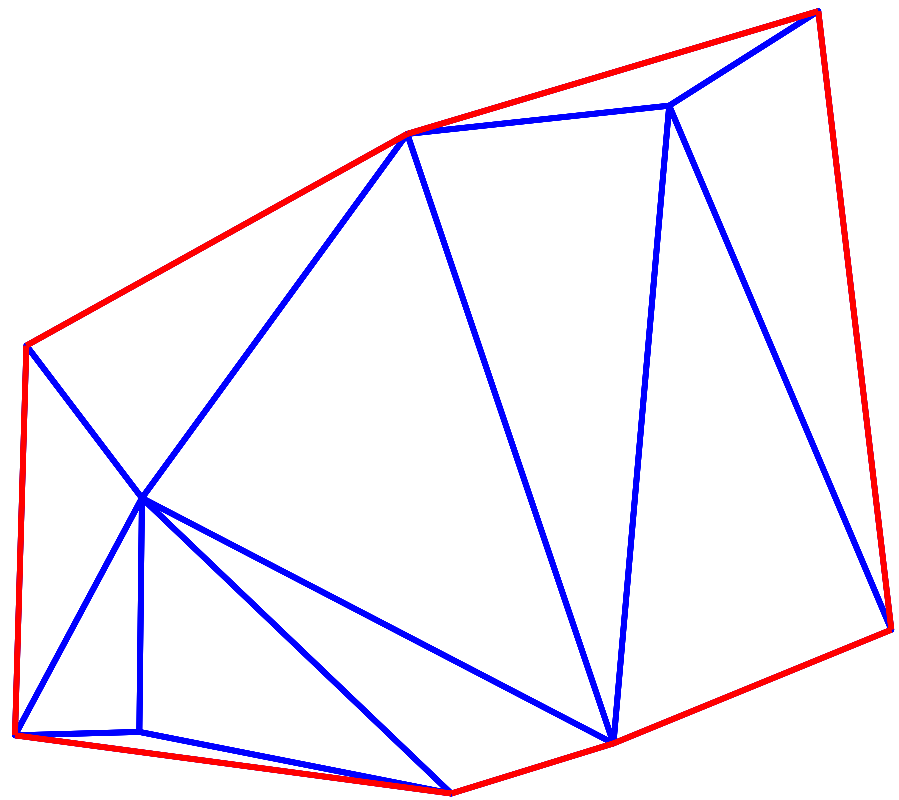
The central region and three sectors (thin version).



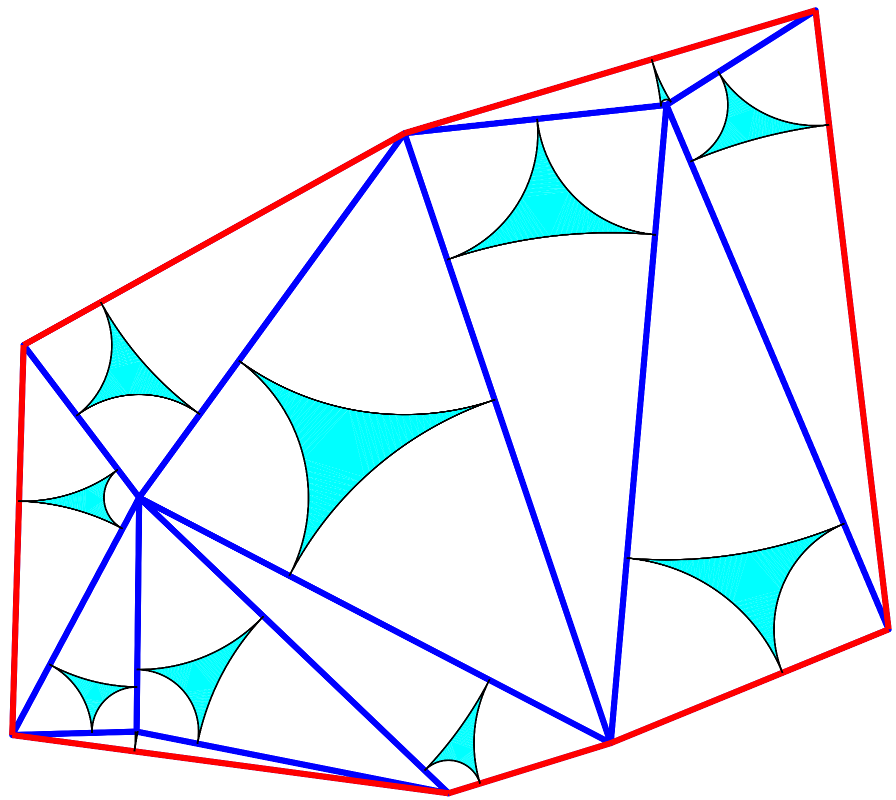
The three sectors are foliated by circular arcs.
Defines flow on a triangulation that stops at boundary or cusp point.



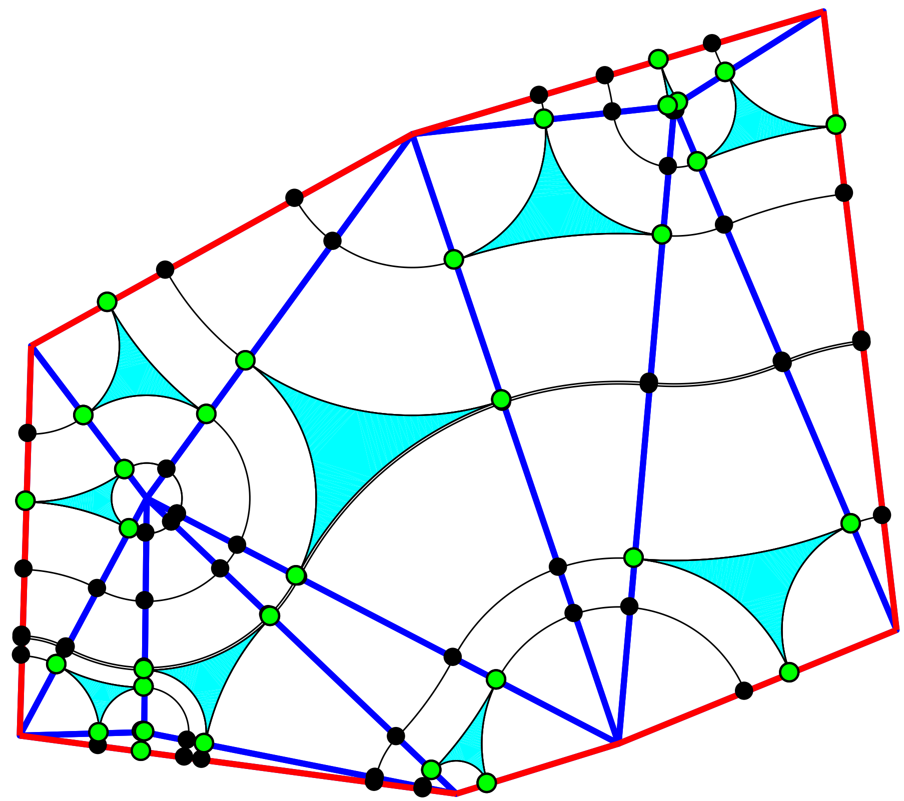
Delaunay triangulation of 10 random points,



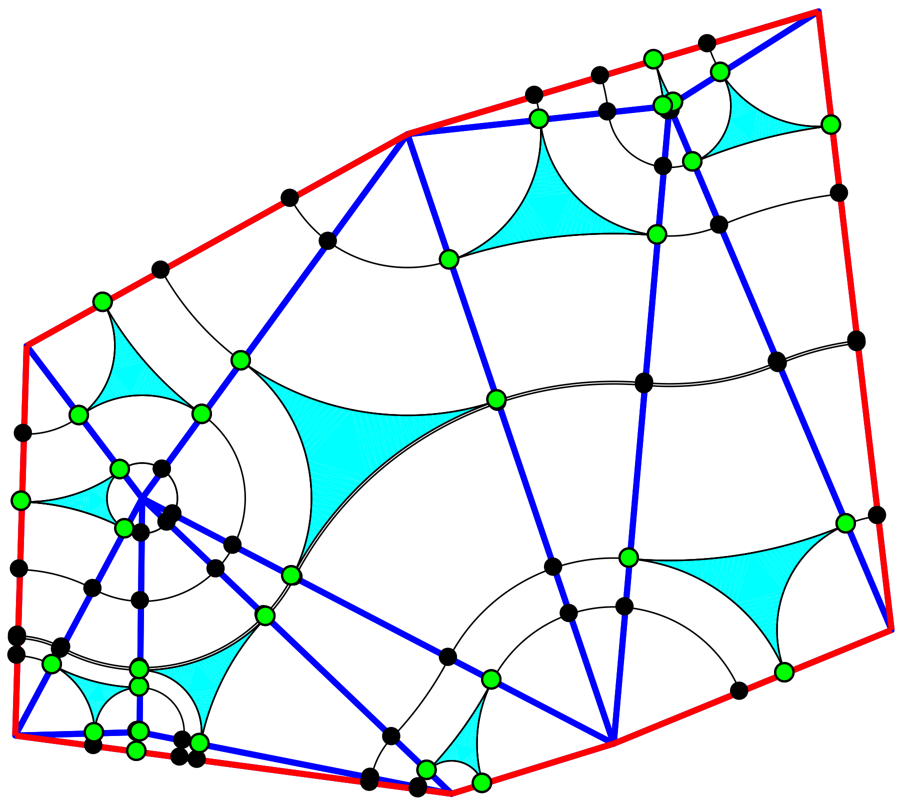
The boundary of the triangulation



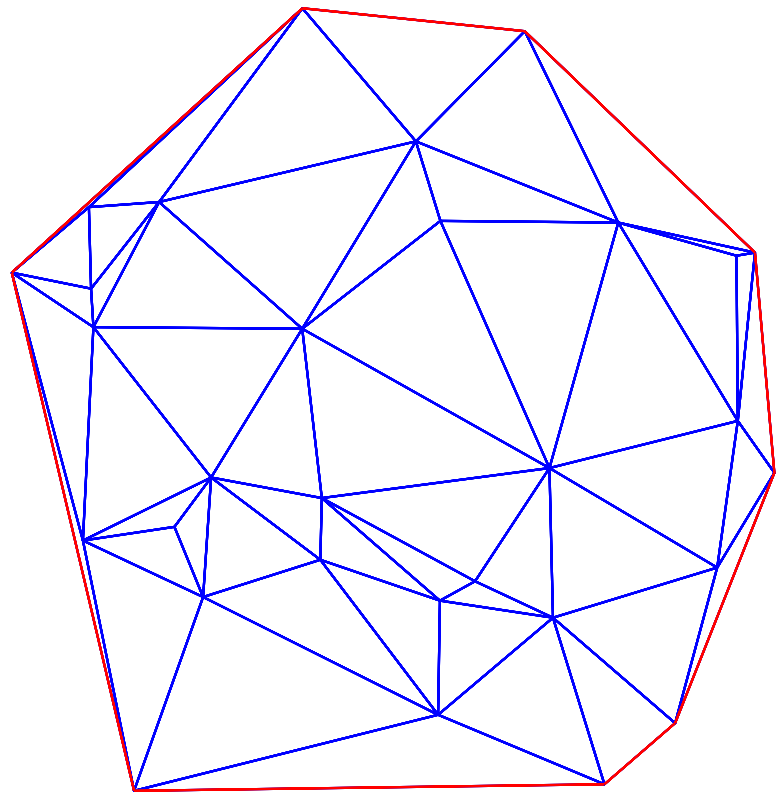
The central regions.



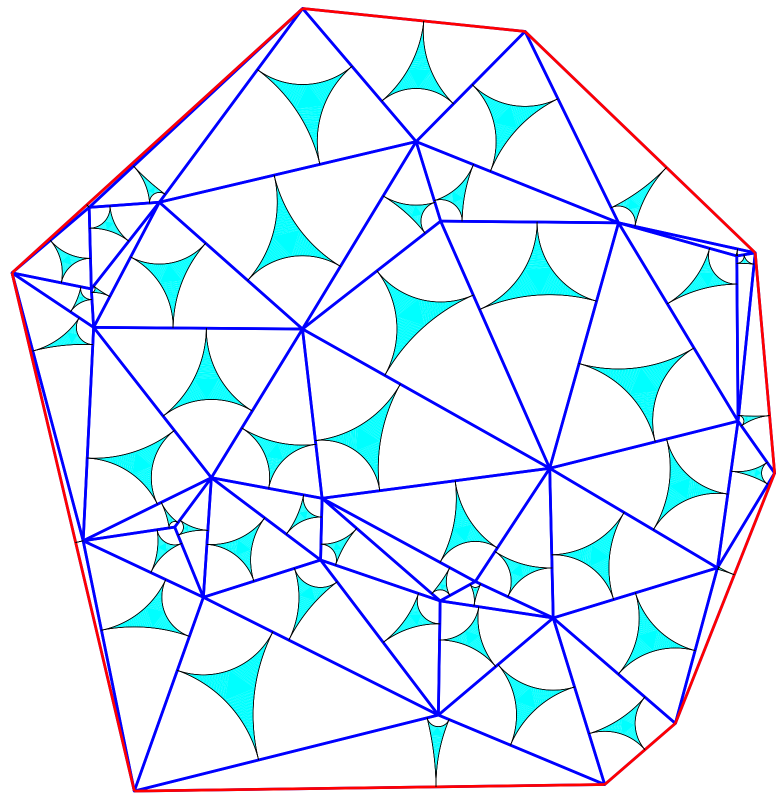
Propagation lines starting at all cusp points.



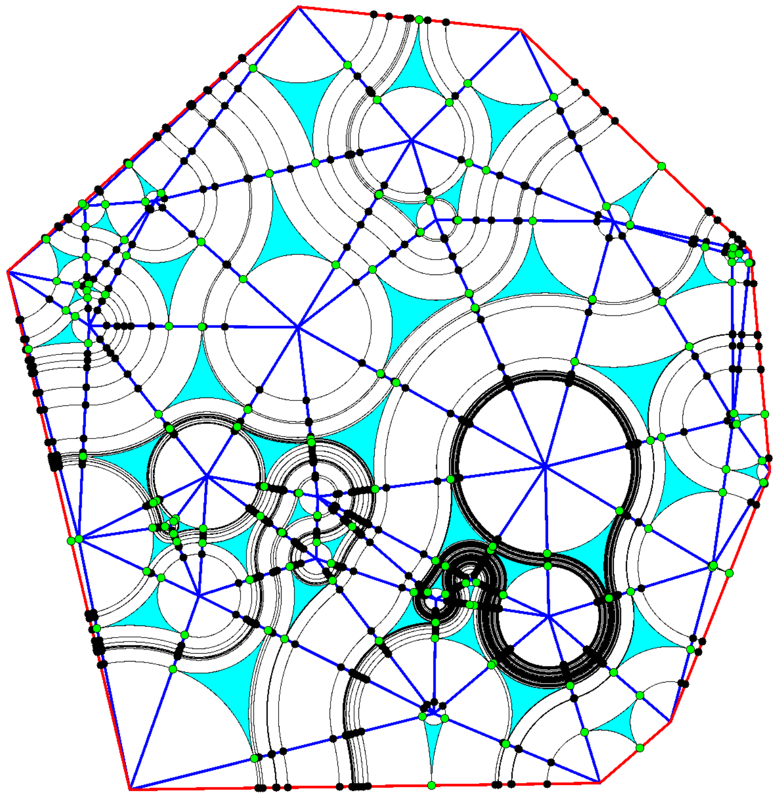
Propagation lines identify boundary points; induces tree.
Discontinuous, but piecewise length preserving.



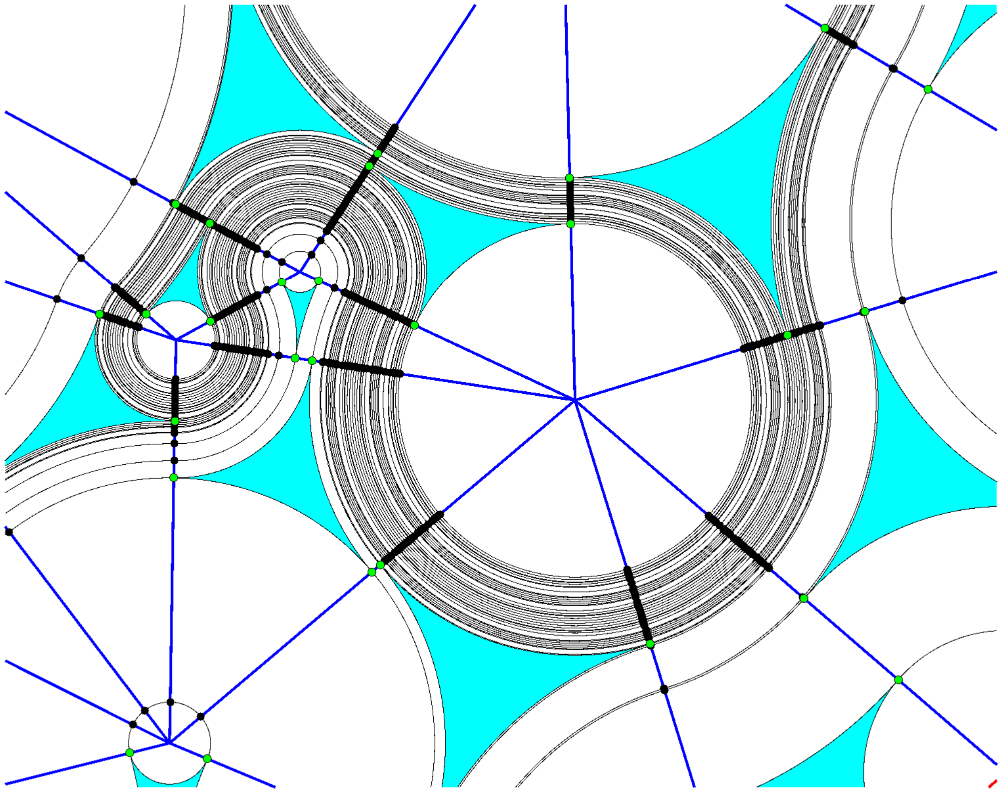
Delaunay triangulation of 30 random points in disk.



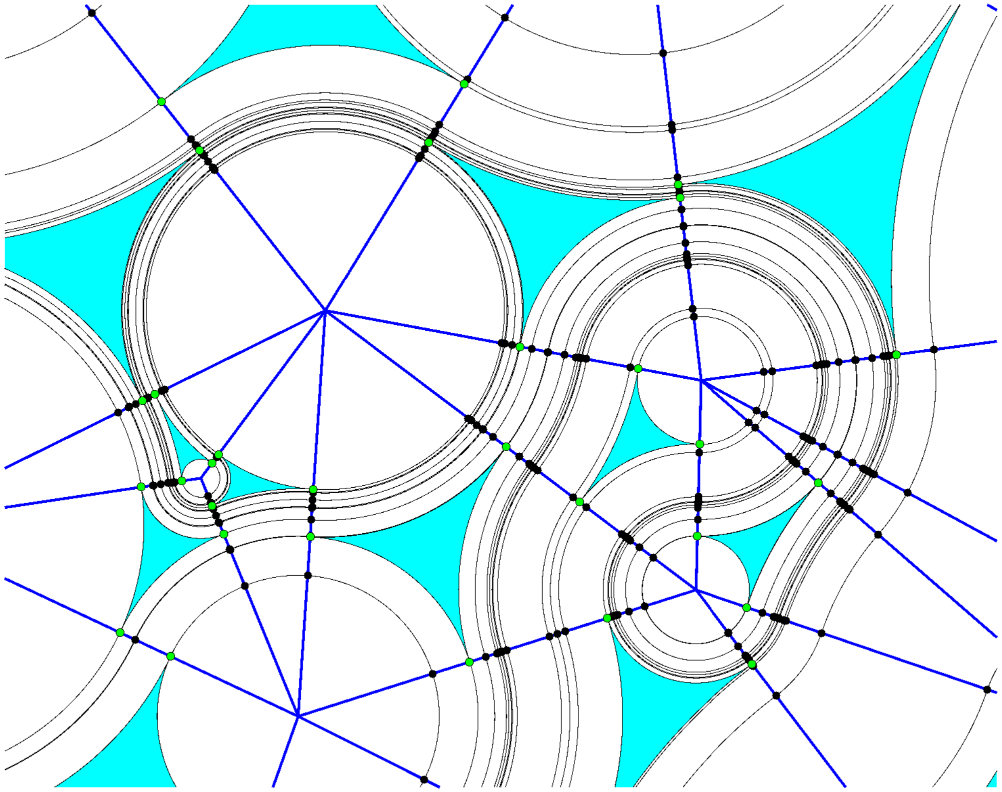
The central regions.



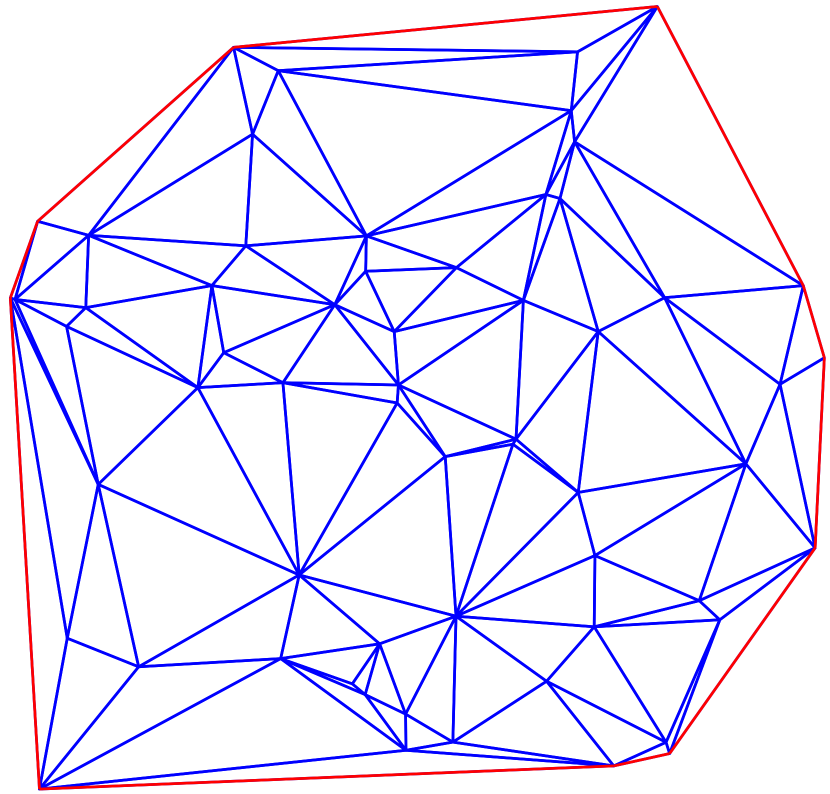
Propagation lines starting at all cusp points.



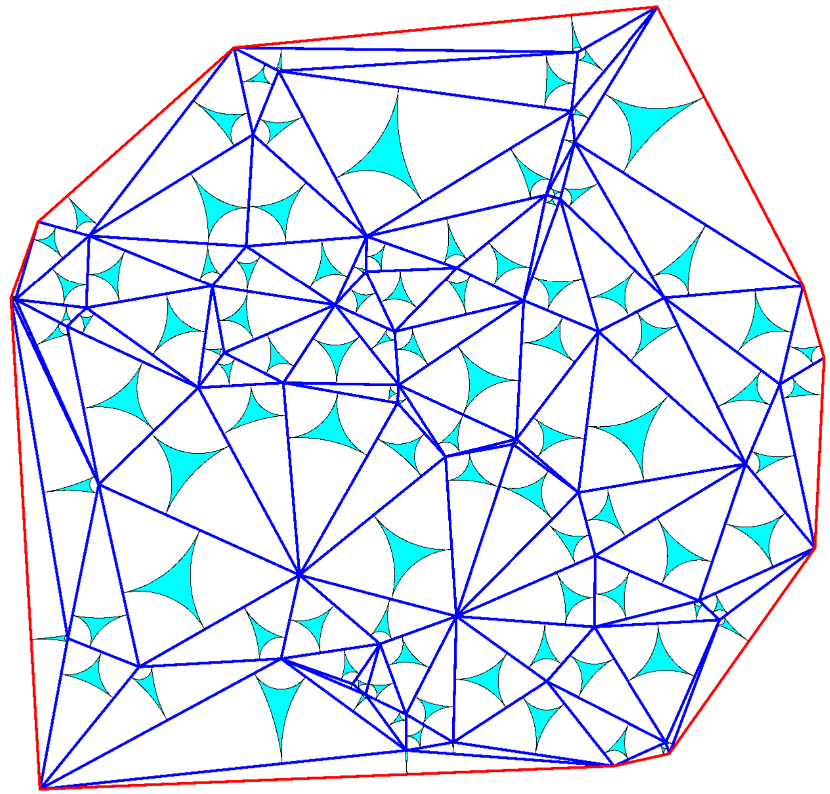
Enlargement 1.



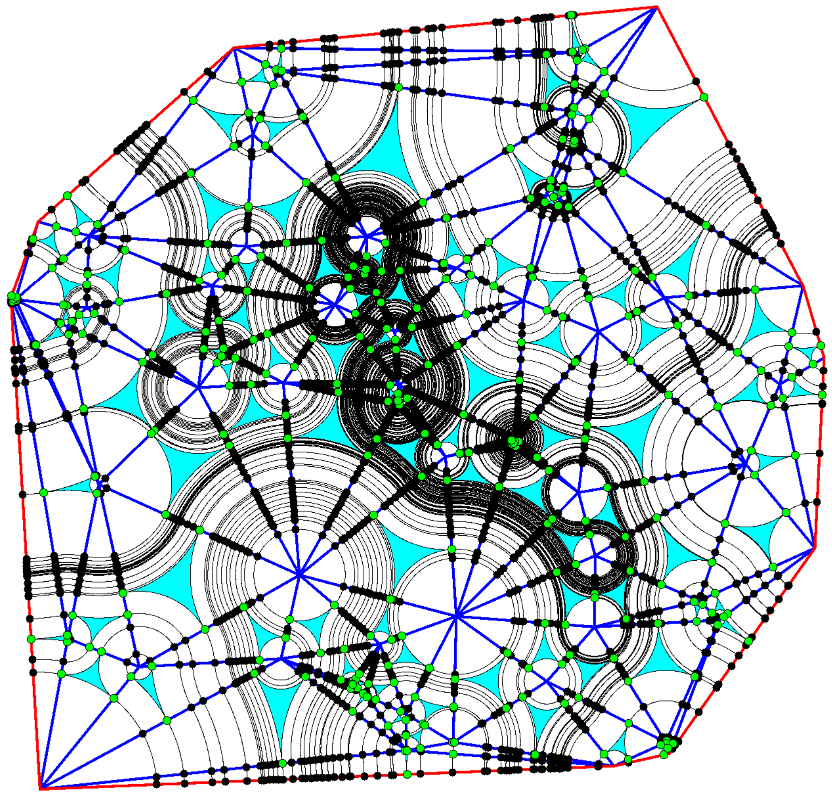
Enlargement 2.



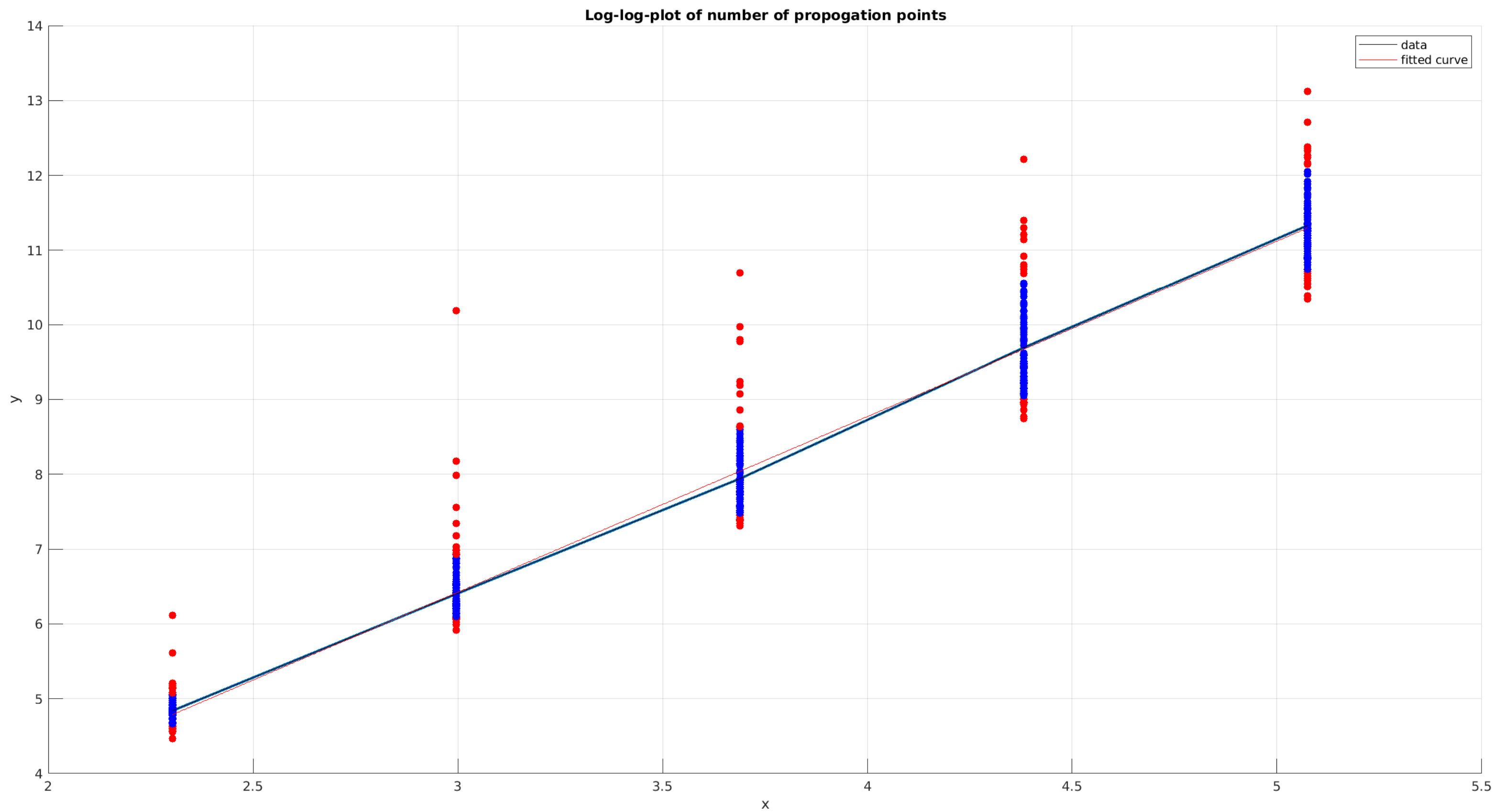
60 points



The central regions.



Propagation lines starting at all cusp points.

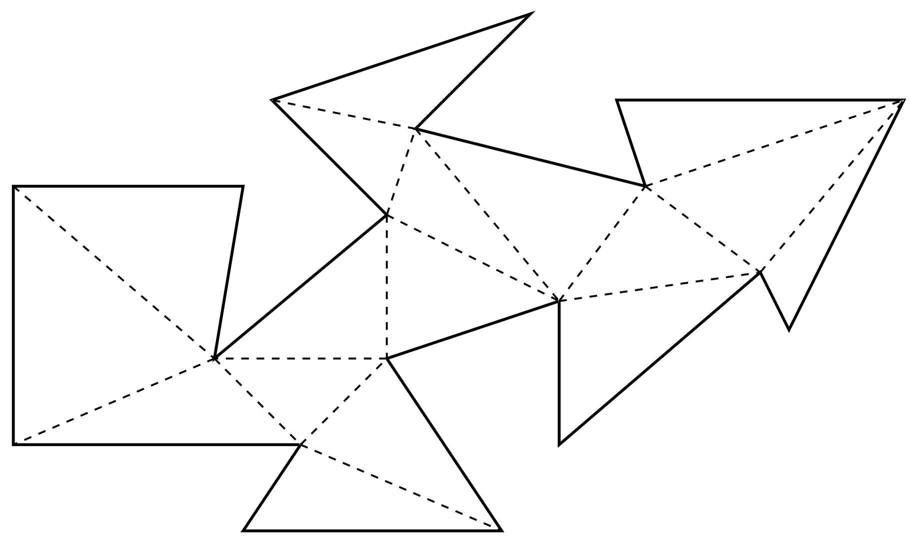


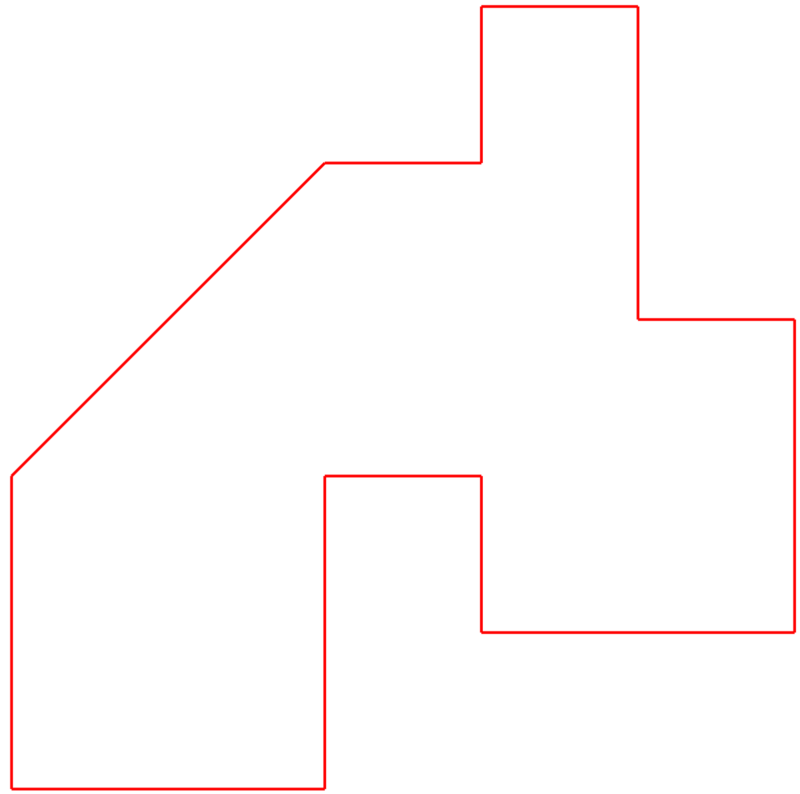
Log-log plot of number points created versus n.
Slope ≈ 2.5

Theorem: For a triangulation of a simple polygon by diagonal, at most $O(n^2)$ points are created.

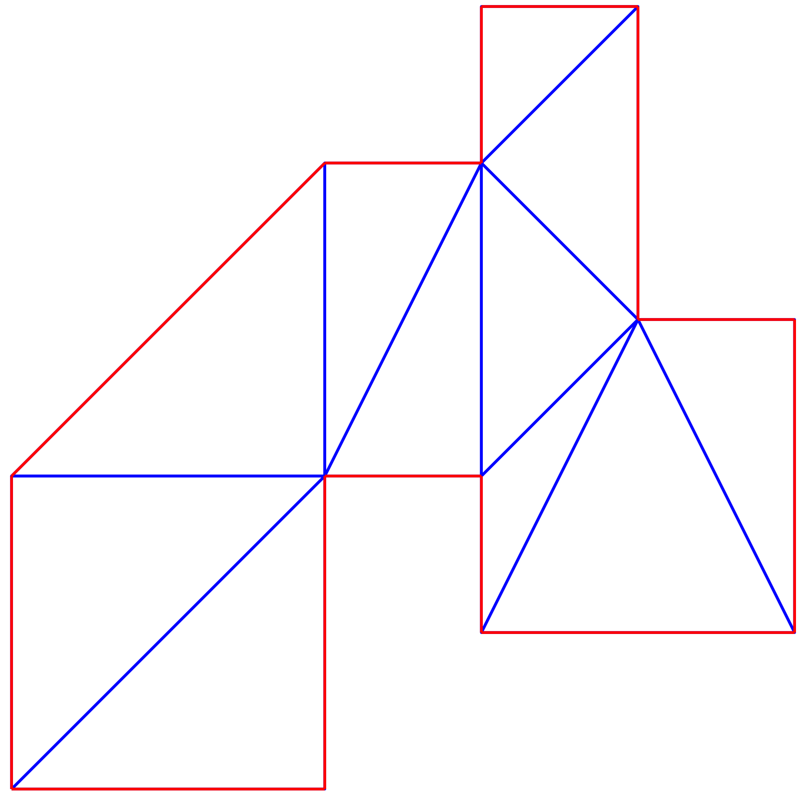
Proof: In this case, the triangles form the vertices of a tree where adjacency means sharing an edge.

Since a flow line never re-enters a triangle, it visits at most n triangles, so at most $O(n^2)$ points are generated.

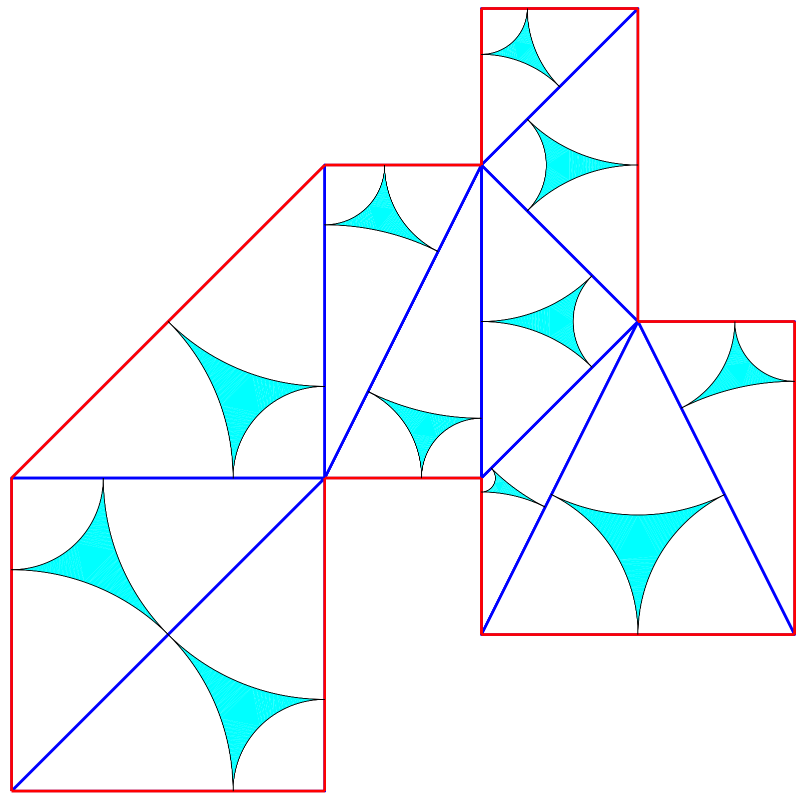




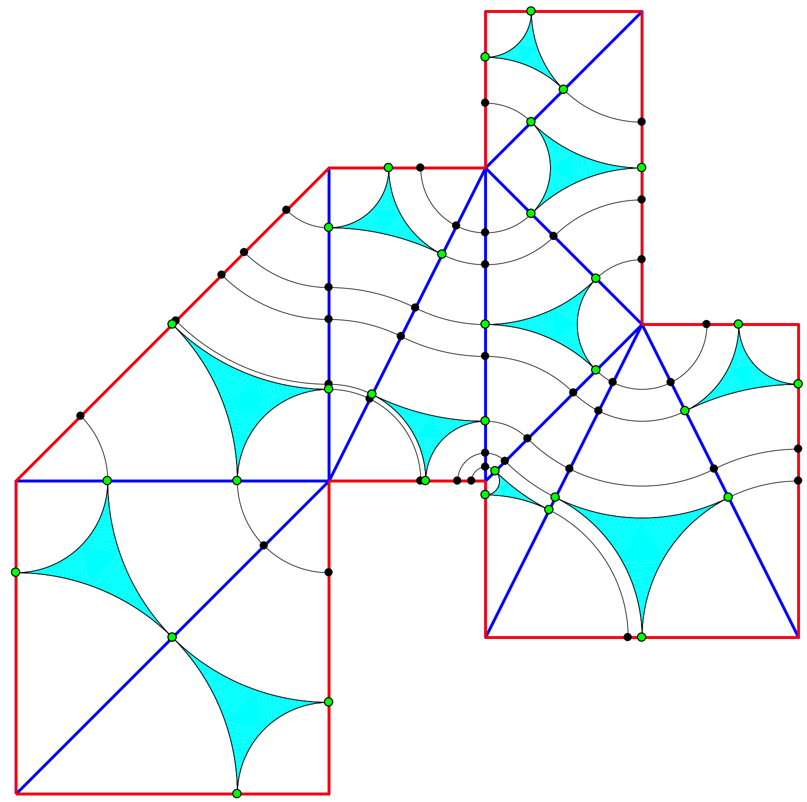
A simple polygon



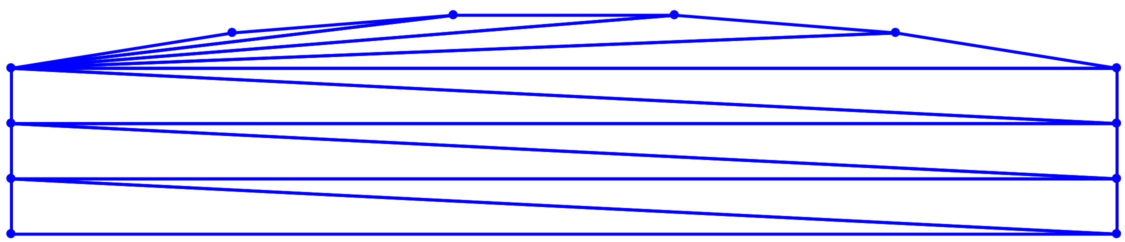
A triangulation of the polygon using diagonals.



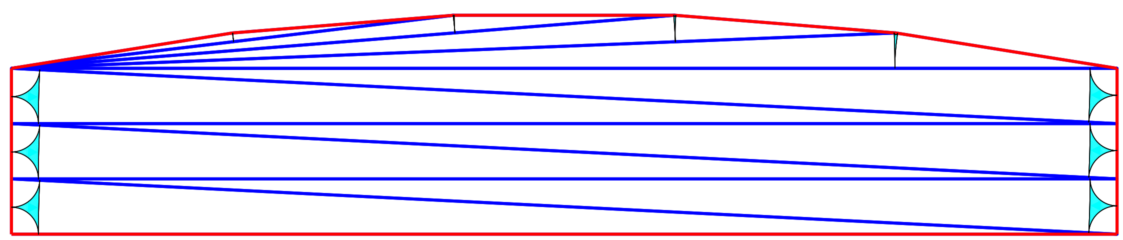
The central cusp regions.



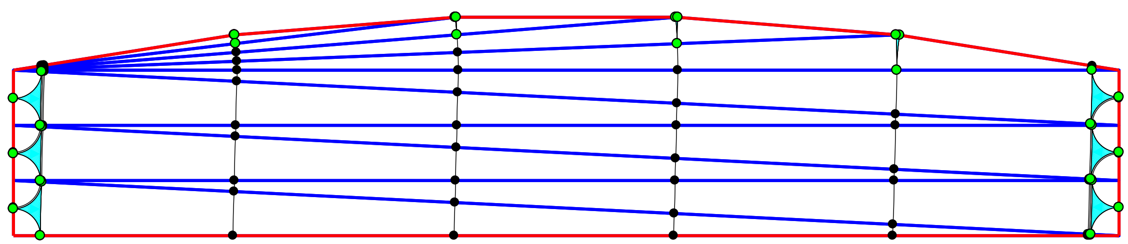
No flow line returns to a triangle, so each path creates at most n new points, for a total of $O(n^2)$.



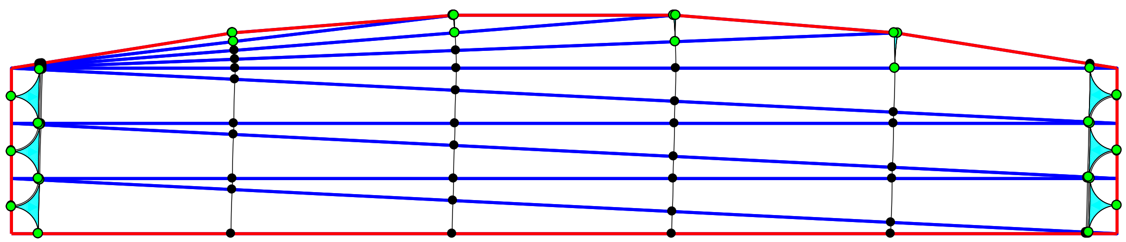
The n^2 is sharp.



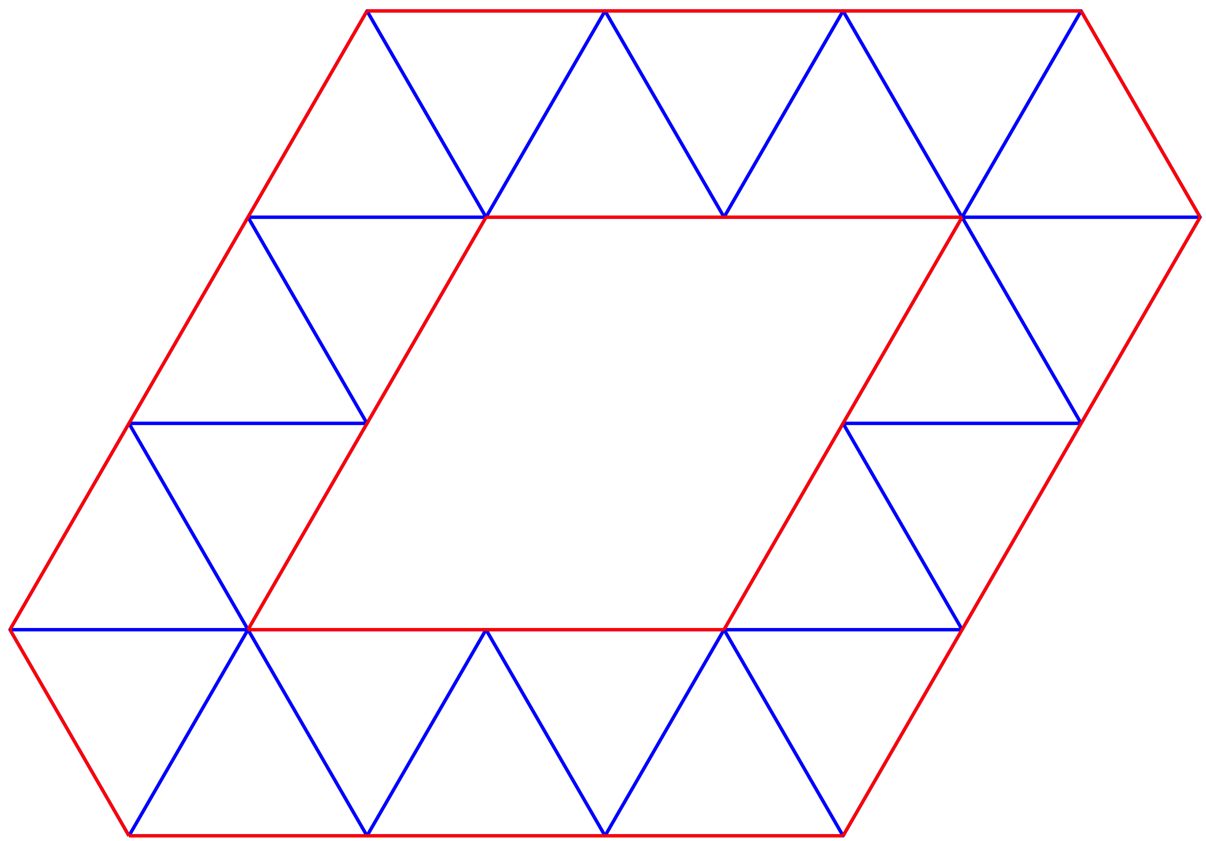
The n^2 is sharp.



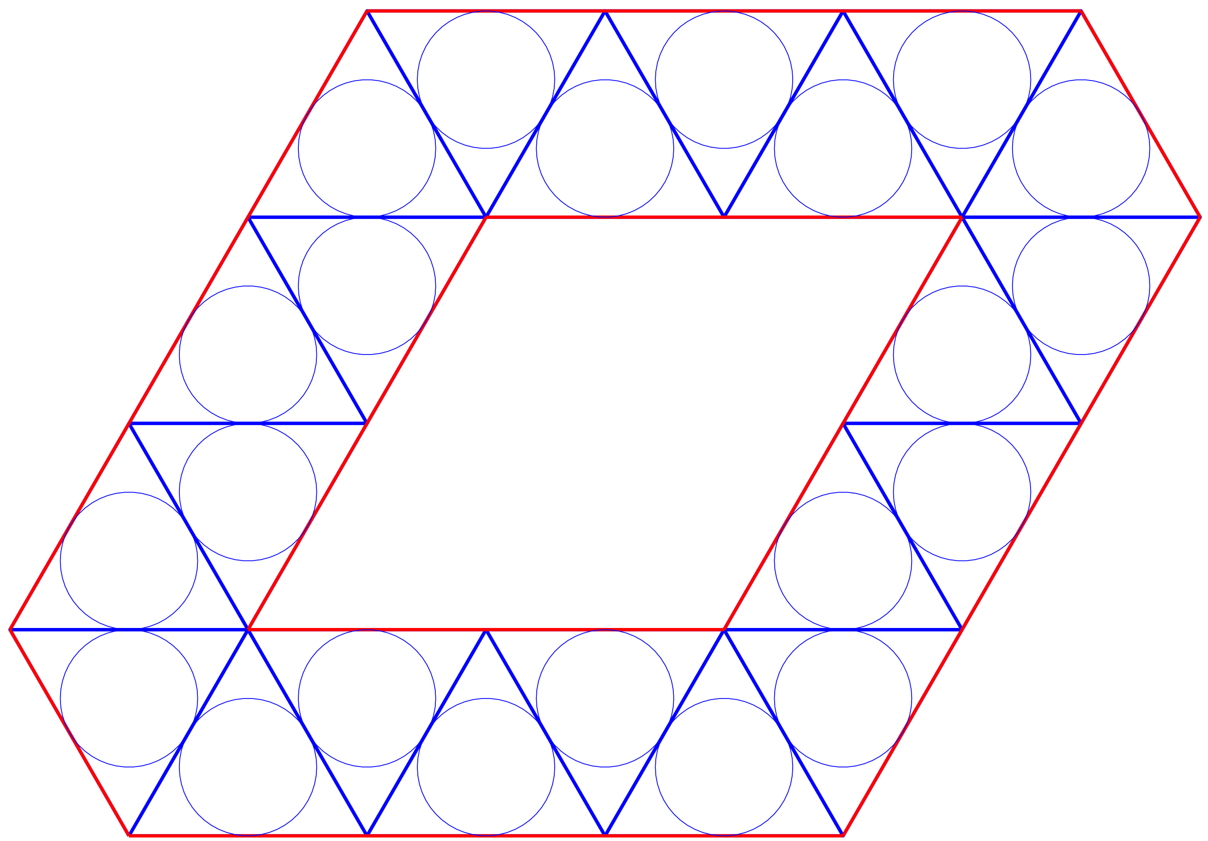
Each point "along top" creates a path that hits $\simeq n$ triangles.



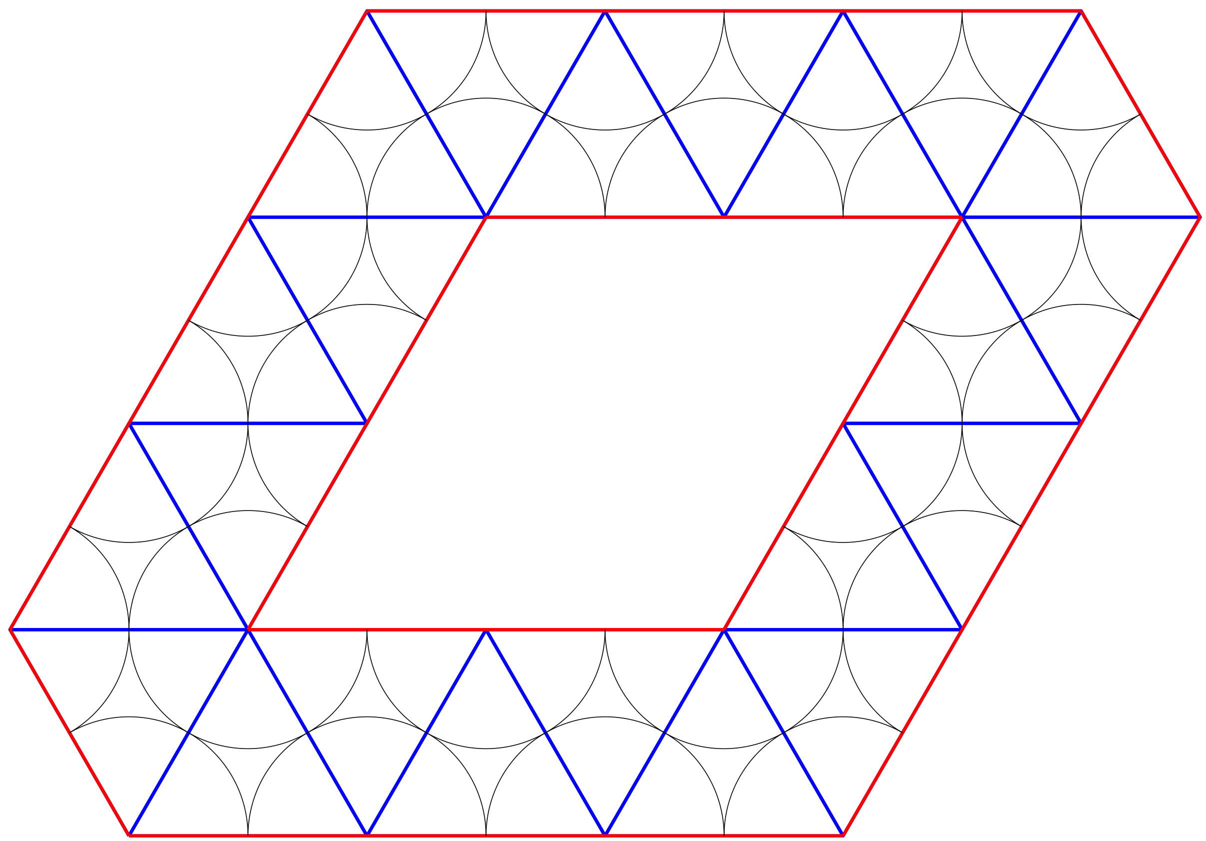
The n^2 is stable under small perturbations of vertices.



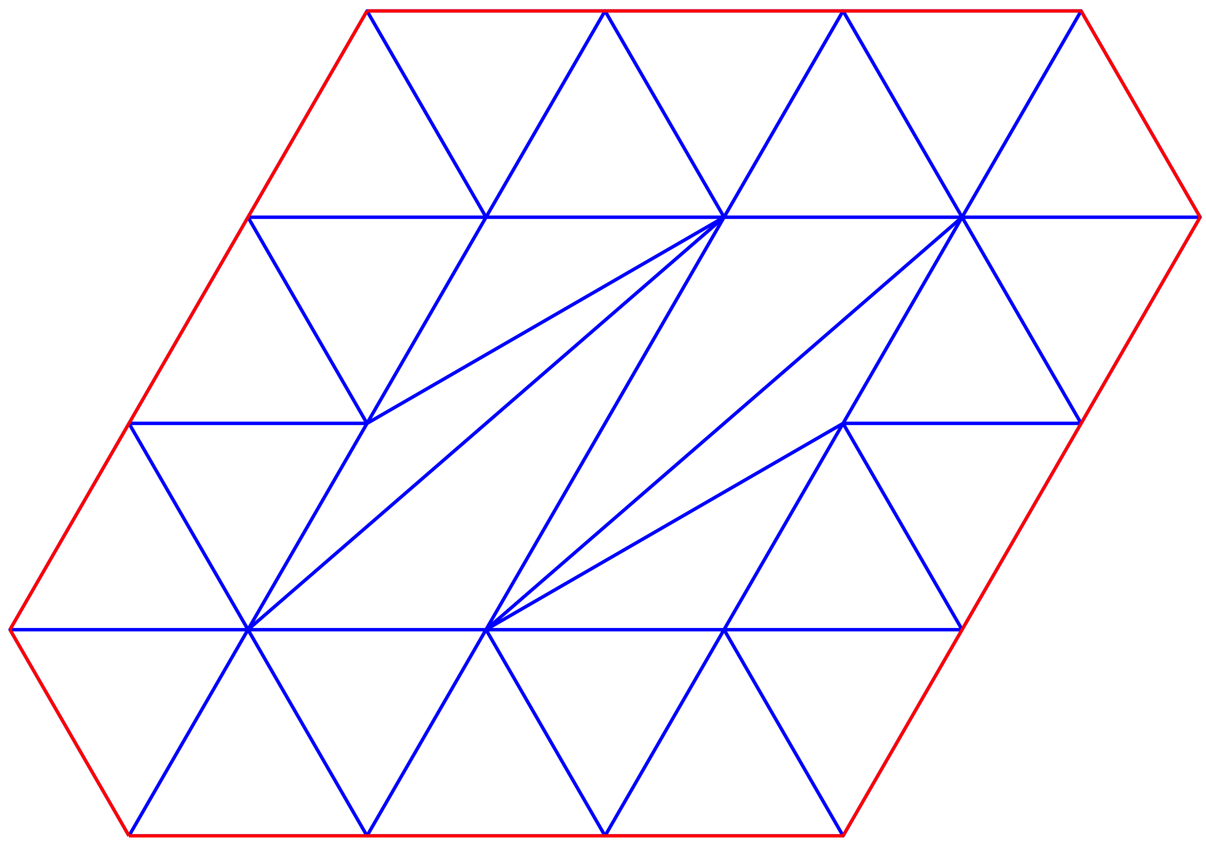
A ring of equilateral triangles.



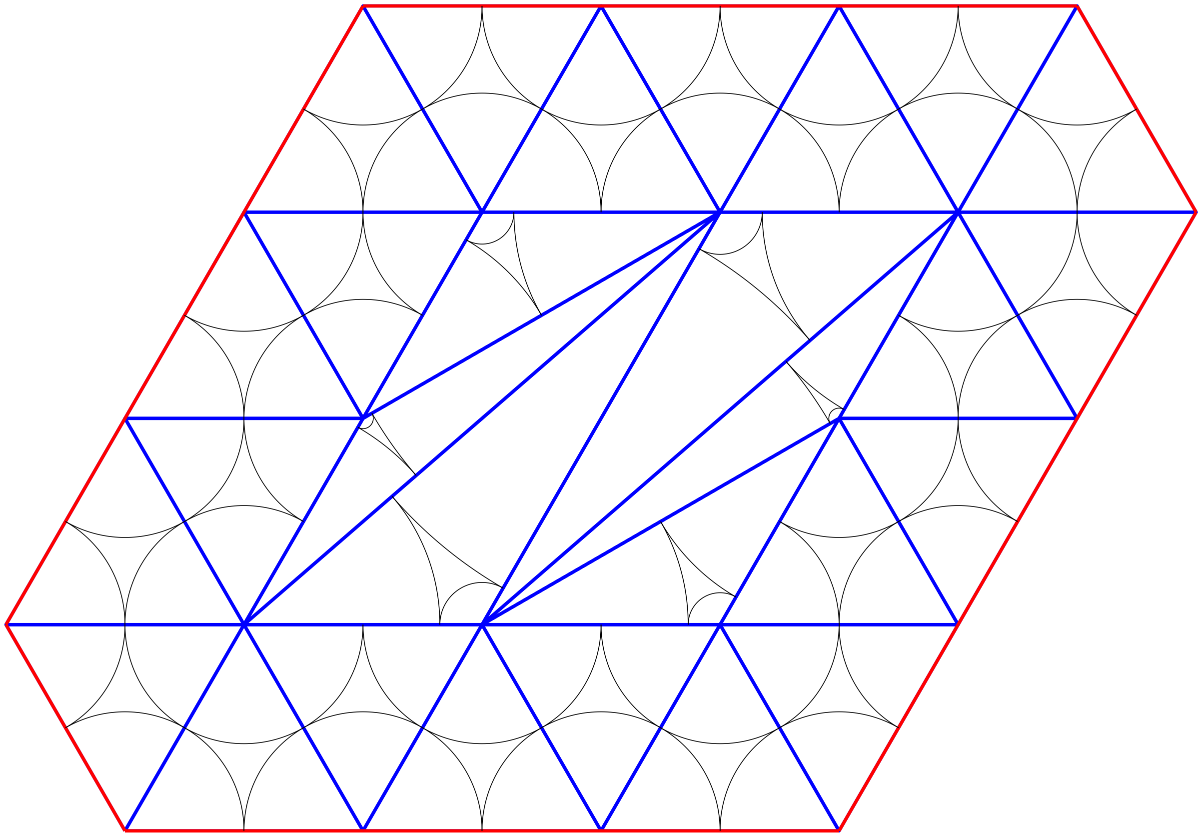
The in-circles are tangent.



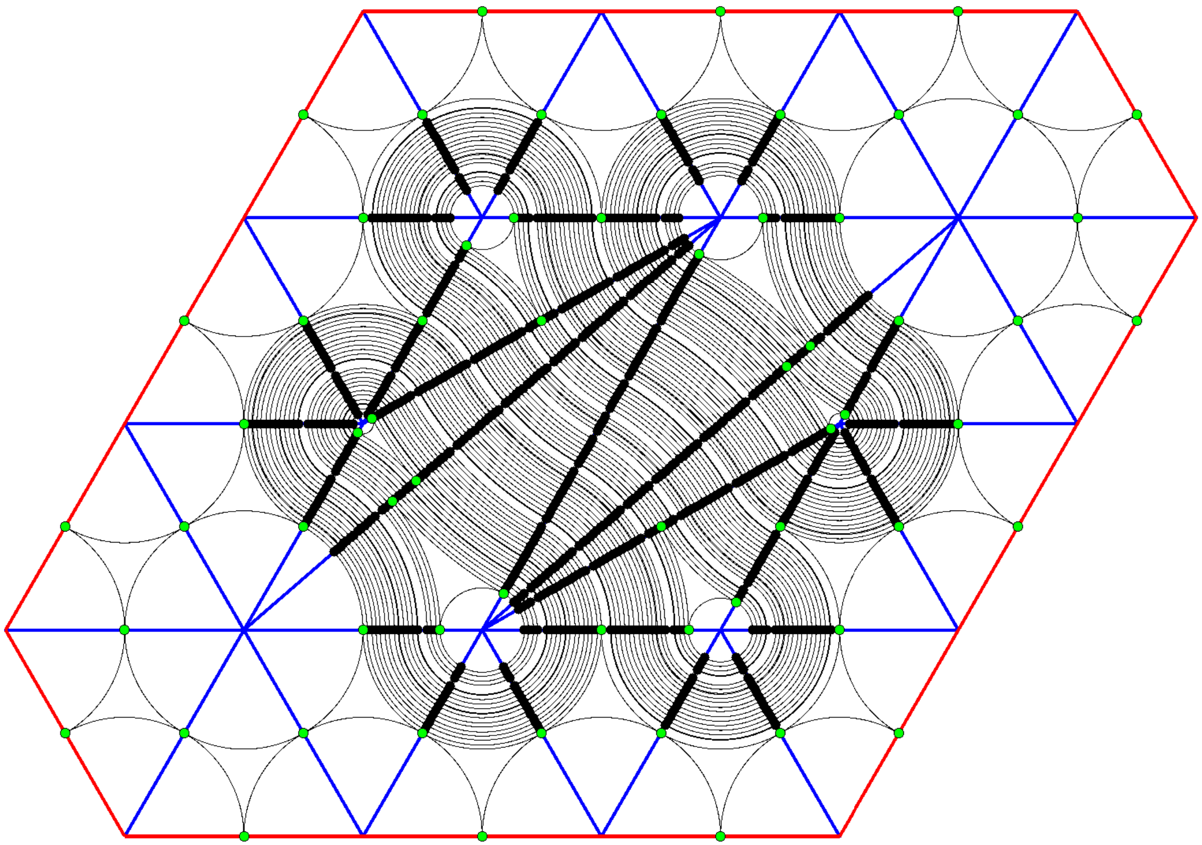
The the cusps touch; for a closed flow line.



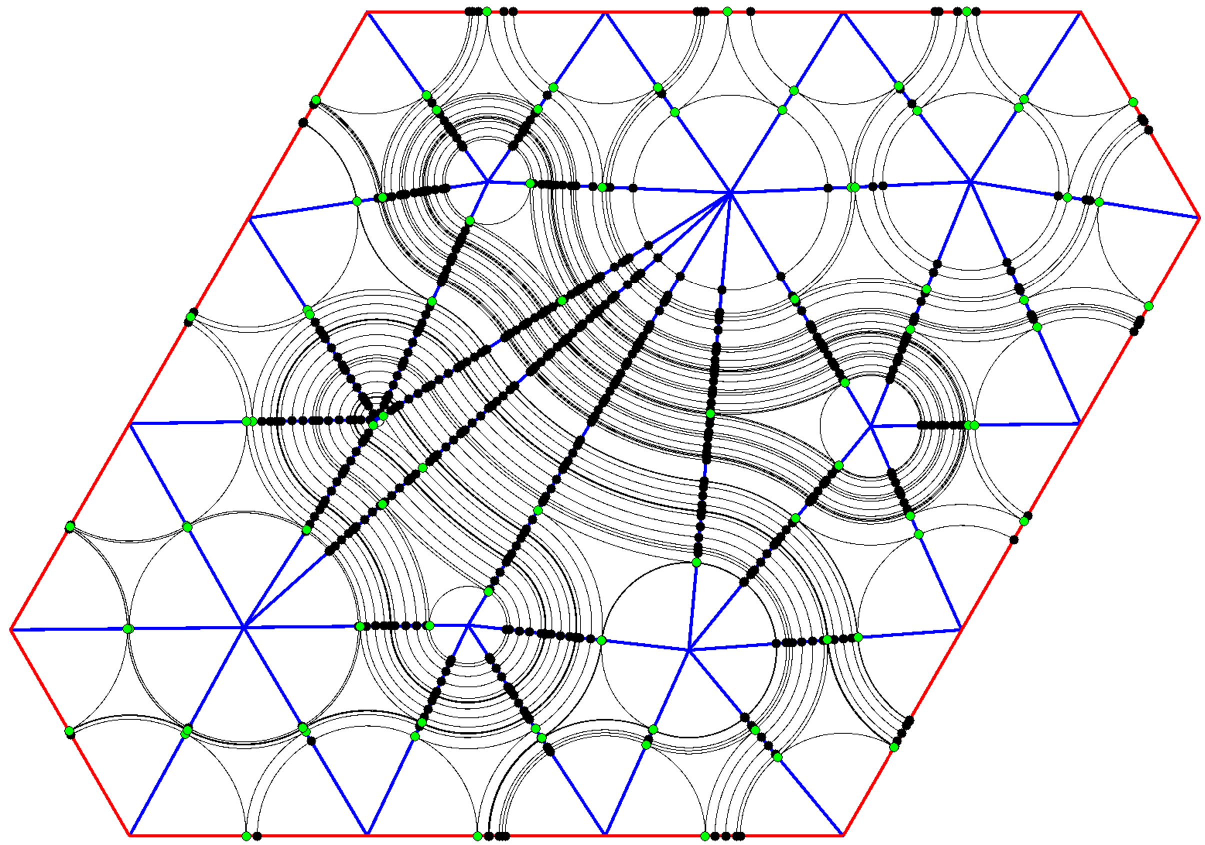
No matter how we triangulate interior, flow lines never exit.



No matter how we triangulate interior, flow lines never exit.



No matter how we triangulate interior, flow lines never exit.



Randomly perturb some vertices; flow lines “leak” to the boundary.

The triangulation flow arises from applications.

A **non-obtuse triangulation (NOT)** has angles $\leq \pi/2$.

NOTs important in applications, give better numerical results.

E.g., Vavasis showed that matrices arising from finite element method for a certain PDE have conditions numbers that grow exponentially (in number of triangles) for general triangulations, but only linearly for non-obtuse triangulations.

NOT Theorem, (B. 2016) Any n-triangulation has a non-obtuse refinement with $O(n^{2.5})$ elements.

NOT Theorem, (B. 2016) Any n-triangulation has a non-obtuse refinement with $O(n^{2.5})$ elements.

First polynomial time bound. 2.5 sharp?

Crossing points of triangle flow give triangulation vertices.

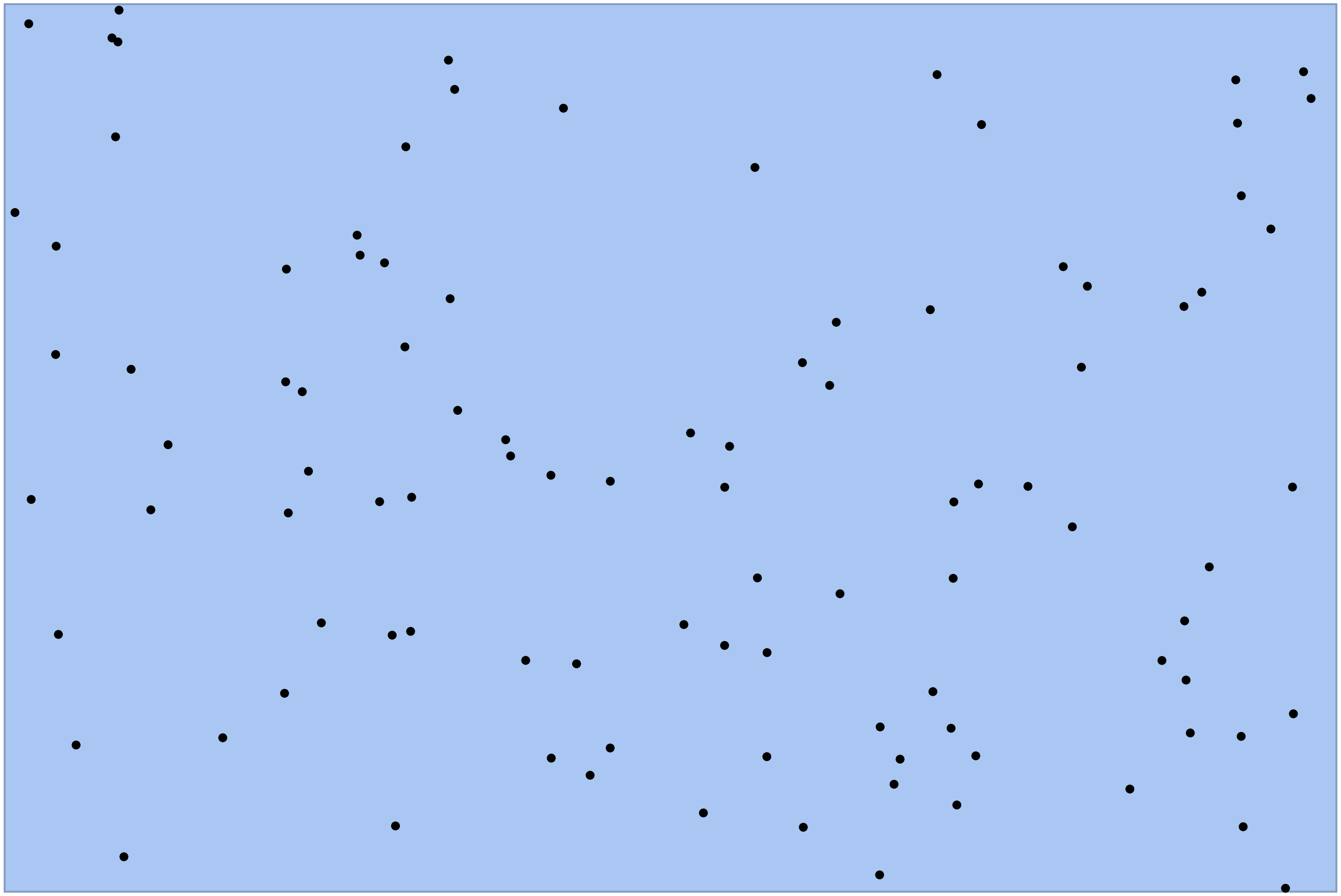
If too many crossings, perturb flow to create closed loops.

Reminiscent of Pugh's closing lemma in dynamics.

Gives $O(n^{2.5})$ in worst case. What is average behavior?

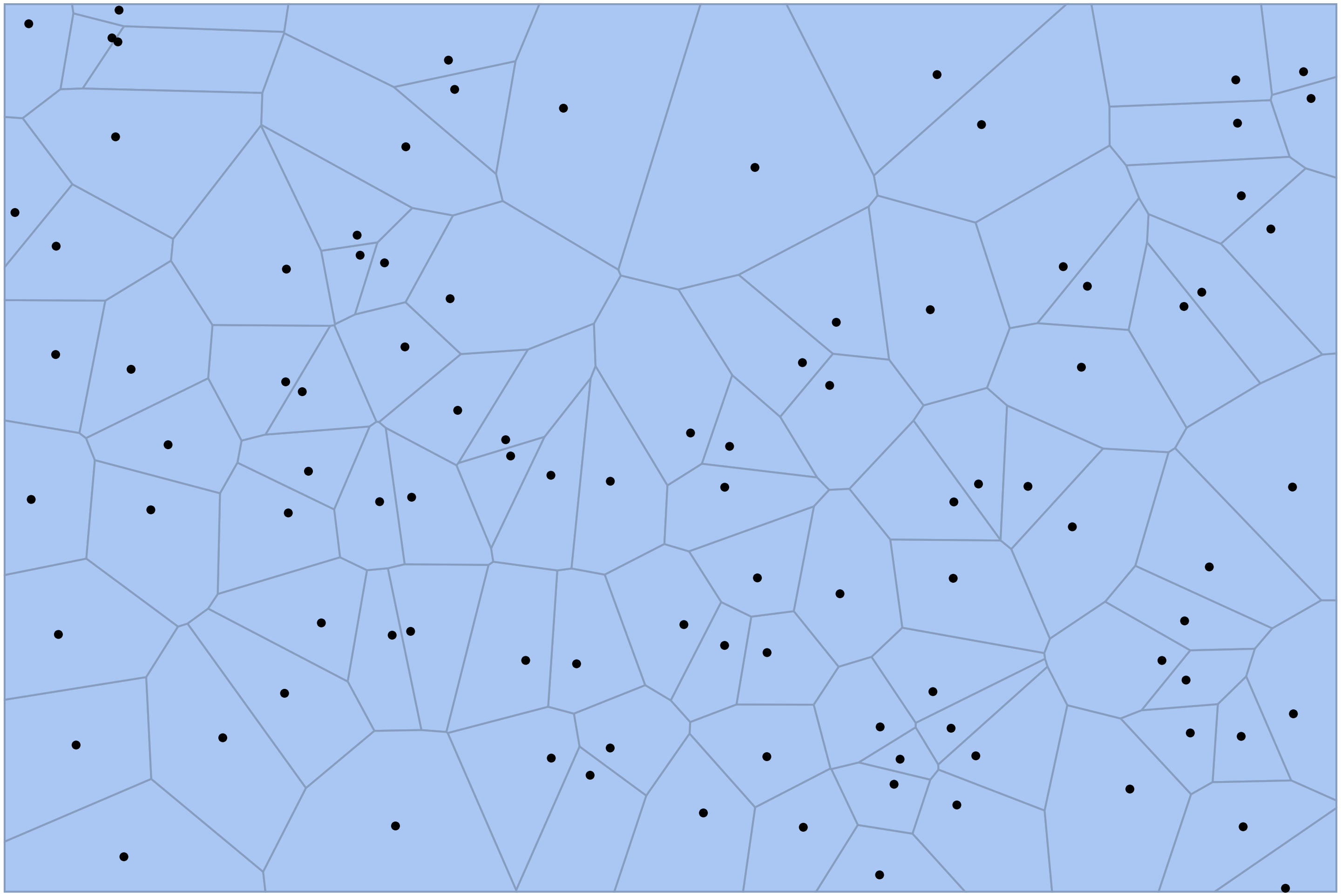
THANKS FOR LISTENING

QUESTIONS?

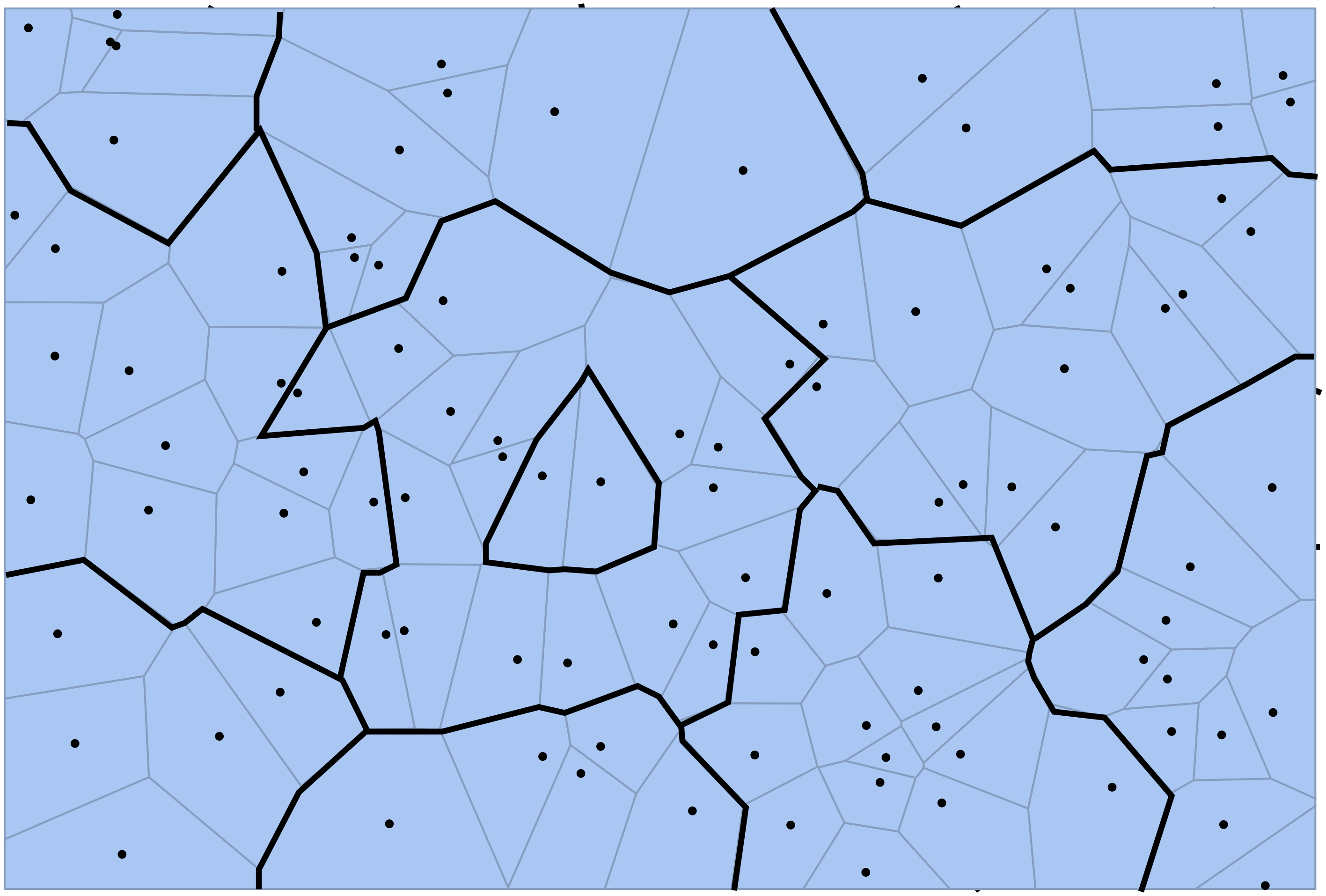


An application of the NOT theorem

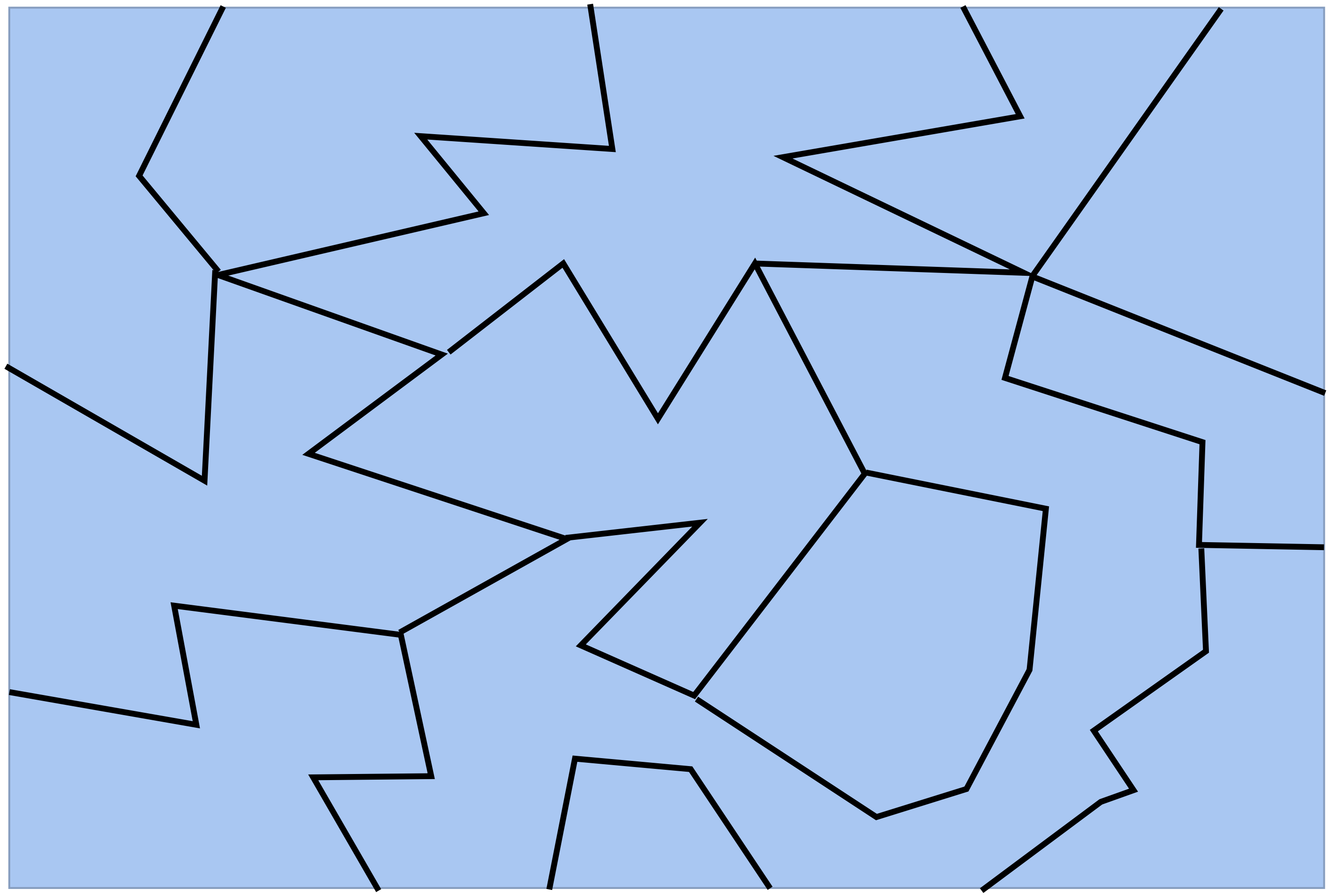
Consider a finite set of points in the plane.



Voronoi cells (think of cell phone connecting to closest tower).

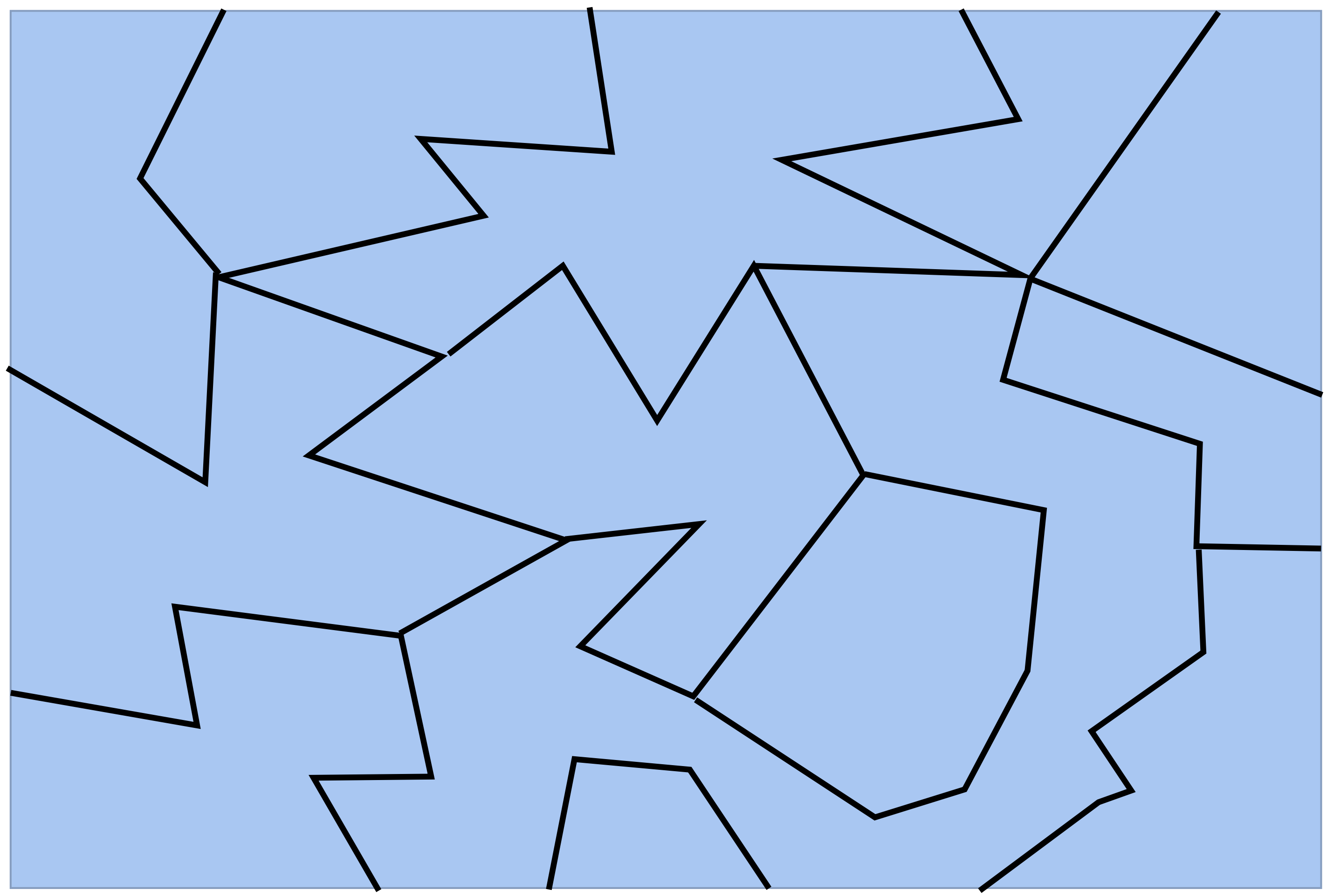


If region boundaries conform to cell boundaries, then a phone always connects to a tower in the same region.



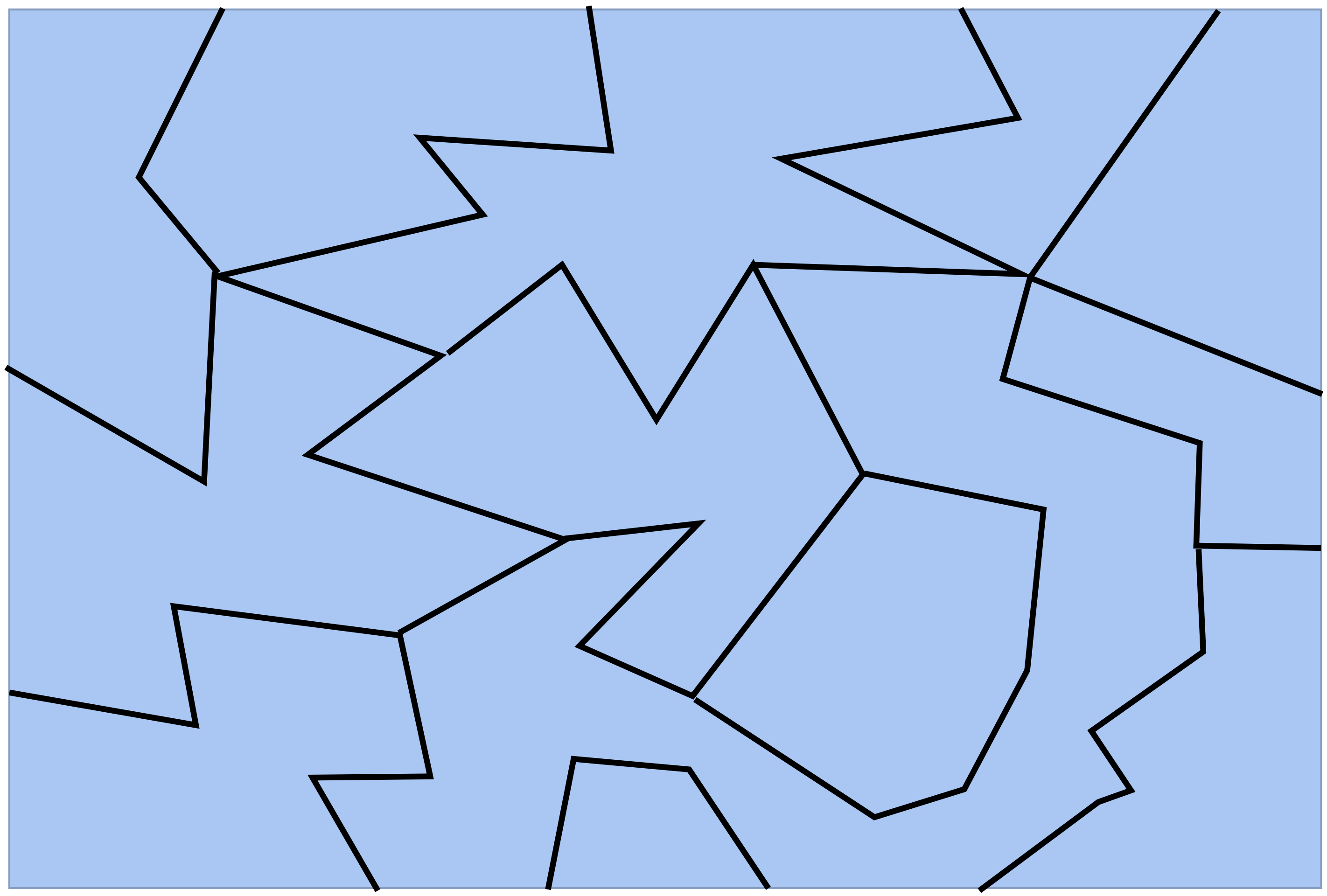
Given countries, can we place towers so this happens?

Do a polynomial number of towers suffice?



Given countries, can we place towers so this happens?

Do a polynomial number of towers suffice? Yes (B 2016)



Proof: It's easy to place points explicitly if regions are all non-obtuse triangles. In general, triangulate the regions, then non-obtusely refine the triangulation.

Theorem (Kesten): $\mathrm{diam}(\mathrm{DLA}(n)) = O(n^{2/3})$.

Equivalent: DLA takes $\gtrsim m^{3/2}$ steps to exit ball of radius m.

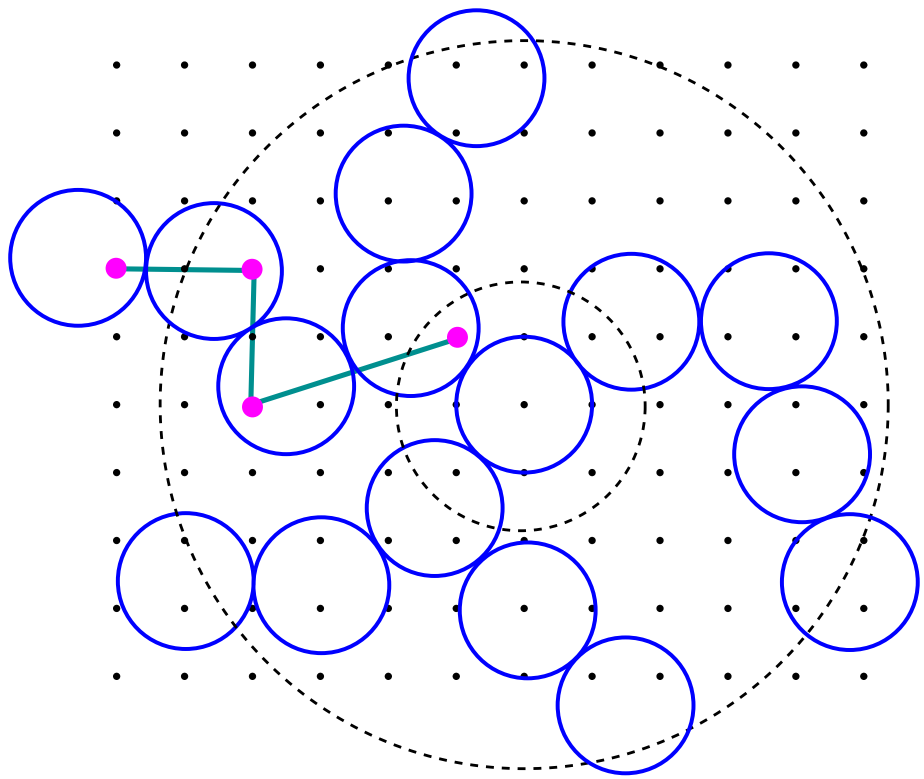
Suppose current radius is m. How long to reach $2m$?

Sketch (following Lawler). Suppose $\beta m^{3/2}$ disks suffice to exit, $\beta > 0$.

Then cluster contains a chain of lattice points $\mathbf{z} = \{z_1, \ldots, z_k\}$ so that

$$|z_1| < m/2, \qquad |z_k| > m,$$

$$|z_j - z_{j+1}| \leq 4, j = 1, \ldots, k, \qquad m/4 \leq k \leq \beta m^{3/2}.$$



Let $W_m(\mathbf{z})$ be all clusters associated to a chain $\mathbf{z}$. Let $W_m = \cup_{\mathbf{z}} W_m(\mathbf{z})$.

At most $O(m^2 80^k)$ chains: $O(m^2)$ starting points and 80 choices per step.

Claim: $\text{Prob}(W_m(\mathbf{z})) \leq (C\beta)^k$.

Let $W_m(\mathbf{z})$ be all clusters associated to a chain $\mathbf{z}$. Let $W_m = \cup_{\mathbf{z}} W_m(\mathbf{z})$.

At most $O(m^2 80^k)$ chains: $O(m^2)$ starting points and 80 choices per step.

Claim: $\mathrm{Prob}(W_m(\mathbf{z})) \le (C\beta)^k$.

Assuming claim, Kesten's theorem follows: if β small,

$$\sum_m \mathrm{Prob}(W_m) \le \sum_m \sum_{\mathbf{z}} \mathrm{Prob}(W_m(\mathbf{z})) \le C \sum_m m^2 80^m C^m \beta^m < \infty.$$

By Borel-Cantelli a.s. W_m occurs only finitely often.

Eventually, exit time from radius m is $> \beta m^{3/2}$. QED

Proof of claim:

Given a chain $\mathbf{z} = z_1, \ldots, z_m$, let D_j is disk covering z_j.

How long do we wait between adding D_j and D_{j+1}?

Probability that next disk lands less than distance 4 of D_j is $\lesssim m^{-1/2}$.

Why?

Well known result in conformal mapping.

Beurling's thm: If $\Omega = \mathbb{C} \setminus K$, K compact and connected, $x \in K$,

$$\omega(\infty, D(x,1) \cap K, \Omega) \leq \frac{C}{\sqrt{\operatorname{diam}(K)}}.$$

Suppose X_k is event "D_{j+1} is added to cluster $\geq k$ steps after D_j".

Then $\text{Prob}(X_k) \leq (1-p)^t$ where $p \lesssim m^{1/2}$.

Fact from Probability: If $X_1, \ldots, X_n$ are independent geometric random variables with parameter p, then

$$\text{Prob}(\sum_{k=1}^{n} X_k < \frac{\beta n}{p}) \le (2e^2\beta)^n.$$

$$n = \text{ number of points in chain } \in [m, m^{3/2}]$$

$$\sum_{k=1}^{n} X_k = \text{ time to cover all points in chain}$$

$$\frac{\beta n}{p} \ge \frac{\beta m}{m^{-1/2}} = \beta m^{3/2}$$

$$\text{Prob(chain is covered in} \le \beta m^{3/2} \text{ steps)} \le (2e^2\beta)^n.$$

$\Rightarrow$ Kesten's theorem.