Optimal angle bounds for Steiner triangulations of polygons

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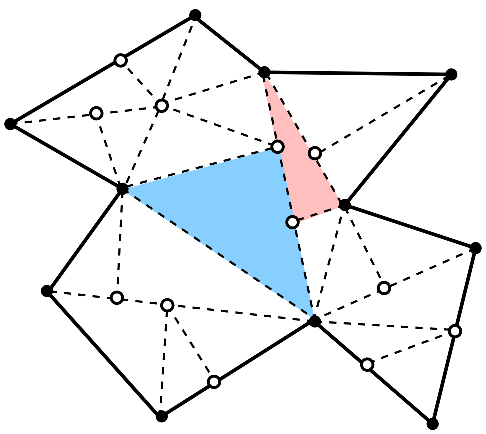
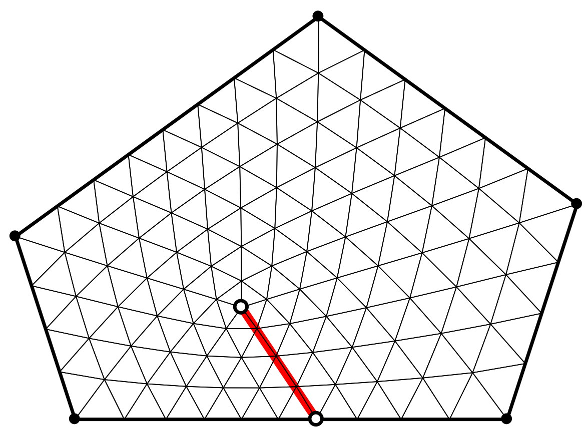
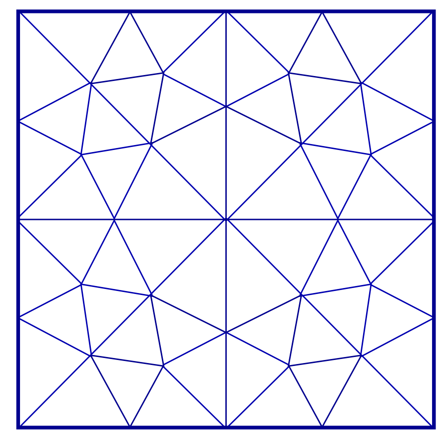
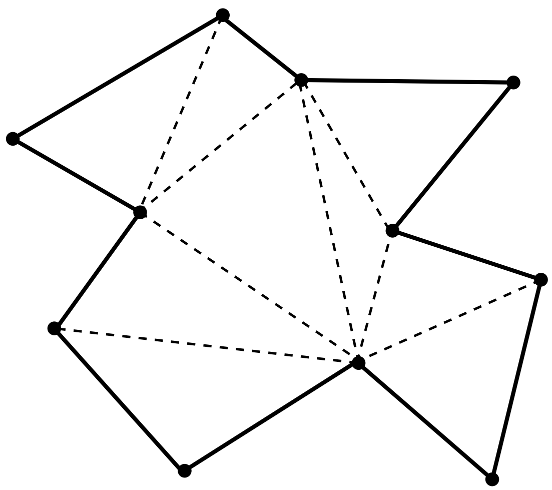
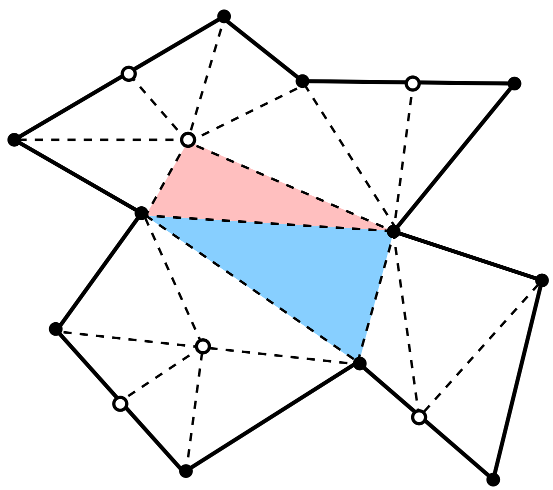




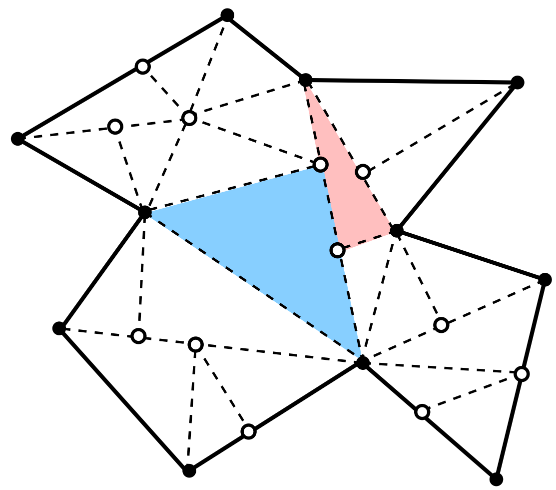




No Steiner Points

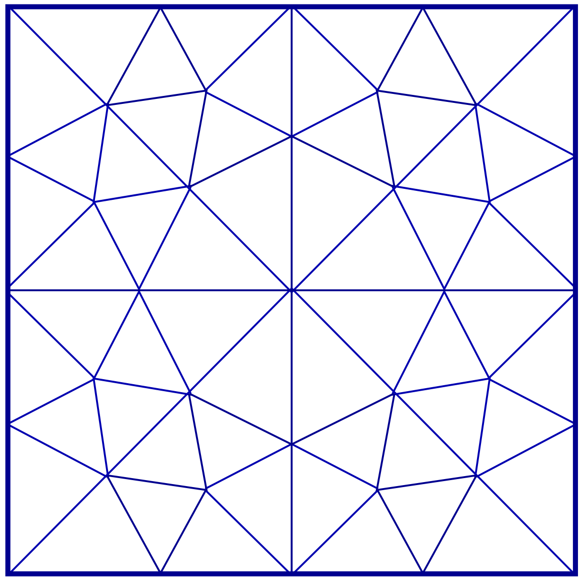


With Steiner Points

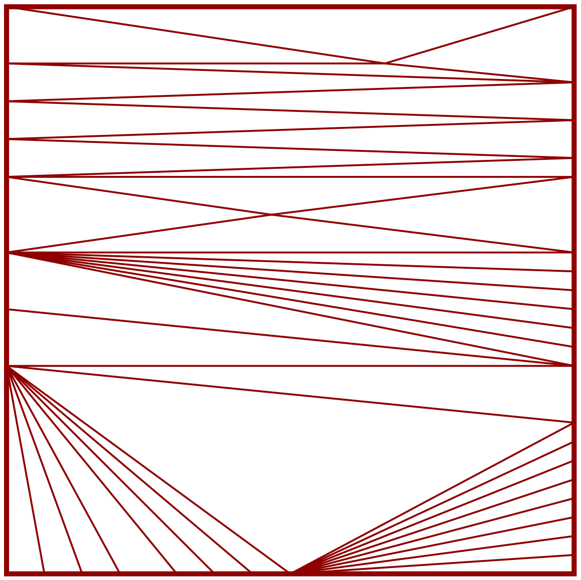


Dissection

Three types of triangulations



Good



Bad

Goal: make pieces as close to equilateral as possible.

Minimize the maximum angle (compute MinMax angle).

"Good" meshes improve performance of numerical methods.

Defn: ϕ-triangulation = all angles $\leq \phi$.

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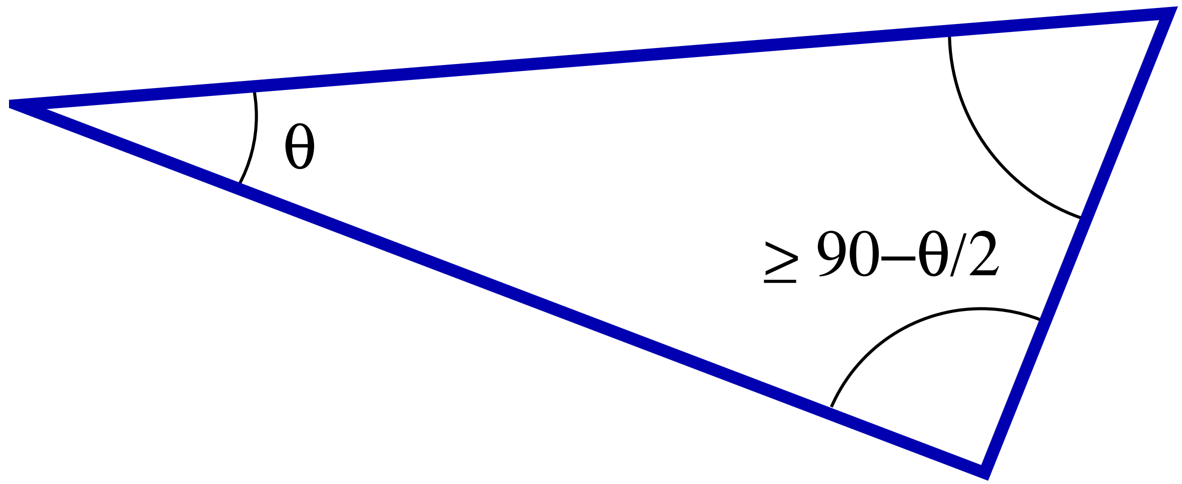

Angles of a triangle sum to $180^\circ \Rightarrow$

$$\Phi(P) \;\geq\; 90^\circ - \frac{\theta_{\min}}{2} \;\geq\; 60^\circ.$$

$\theta_{\min}$ = minimum interior angle of P.

Taking $\theta \to 0 \Rightarrow$ no angle bound $< 90^\circ$ works for all polygons.

Thm (Burago-Zalgaller, 1960): $\Phi(P) < 90^\circ$ all polygons.

"Every polygon has an acute triangulation."

Rediscovered by Baker-Grosse-Rafferty, 1988.

Much work on acute and non-obtuse triangulations by Bern, Edelsbrunner, Eppstein, Erten, Gilbert, Hirani, Itoh, Kopczyński, Maehara, S. Mitchell Pak, Przytycki, Ruppert, Saraf, Shewchuk, Tan, Üngör, VanderZee, Vavasis, Yuan, Zamfirescu, ...

Remaining questions:

- Compute $\Phi(P)$ for a given P?
- Is optimal angle bound attained?
- Can dissections do better than triangulations?
- Give simple estimates of $\Phi(P)$?

Theorem (MinMax angle with Steiner points):

(1) $\Phi(P)$ can be computed in linear time.

(2) Bound is always attained except for some 60°-polygons.

(3) Optimal bound for triangulations is same as for dissections.

(4) $\Phi(P) \leq 72^\circ$ unless $\theta_{\text{min}} \leq 36^\circ$; then $\Phi(P) = 90^\circ - \frac{1}{2}\theta_{\text{min}}$.

(5) $\theta_{\text{min}} \geq 144^\circ \Rightarrow \Phi(P) = 72^\circ$.

60°-polygon = all angles multiples of $60^\circ \Rightarrow \Phi(P) = 60^\circ$.

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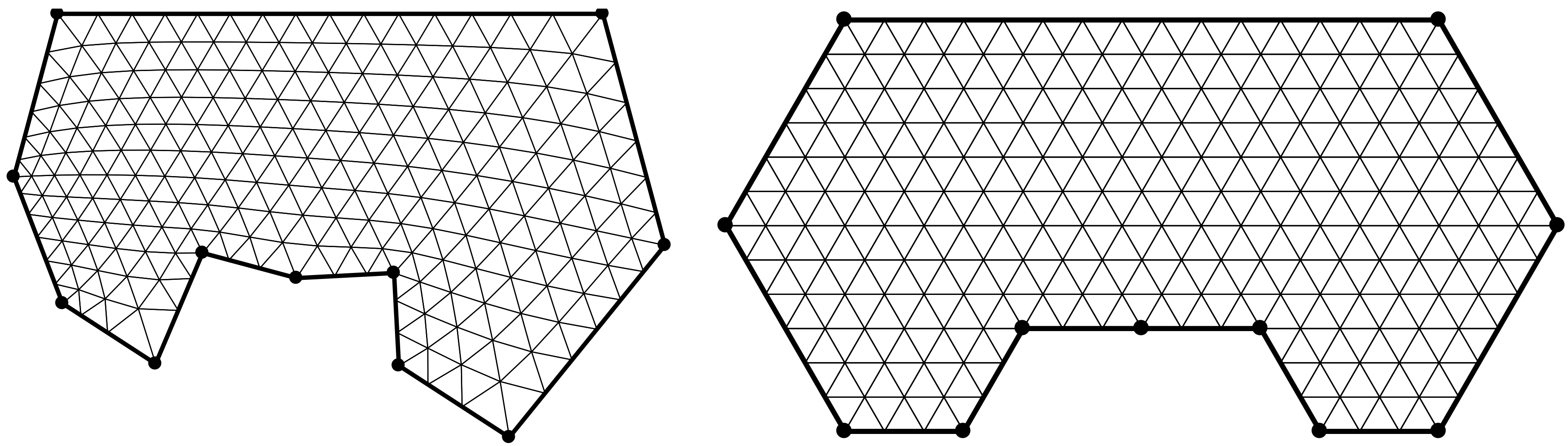
Analogous result holds for computing MaxMin angle; see paper.

If no Steiner points, then Delaunay triangulation gives MaxMin angle. Algorithms for MinMax without Steiner points by Bern, Eppstein, Edelsbrunner, S. Mitchell, Tan, Waupotitsch. $O(n^2 \log n)$.

Idea of Proof:



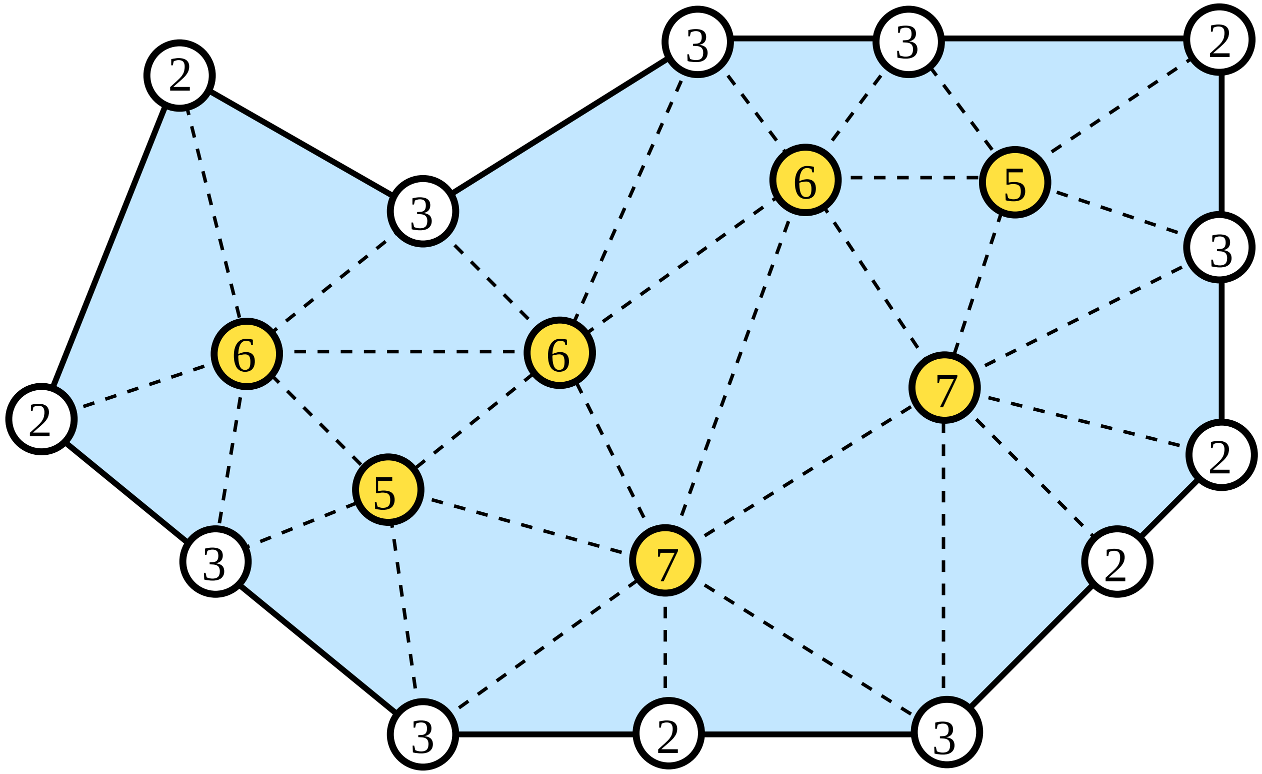
Given P, construct a 60°-polygon P' that "approximates" P.

Conformally map a nearly equilateral triangulation from P' to P.

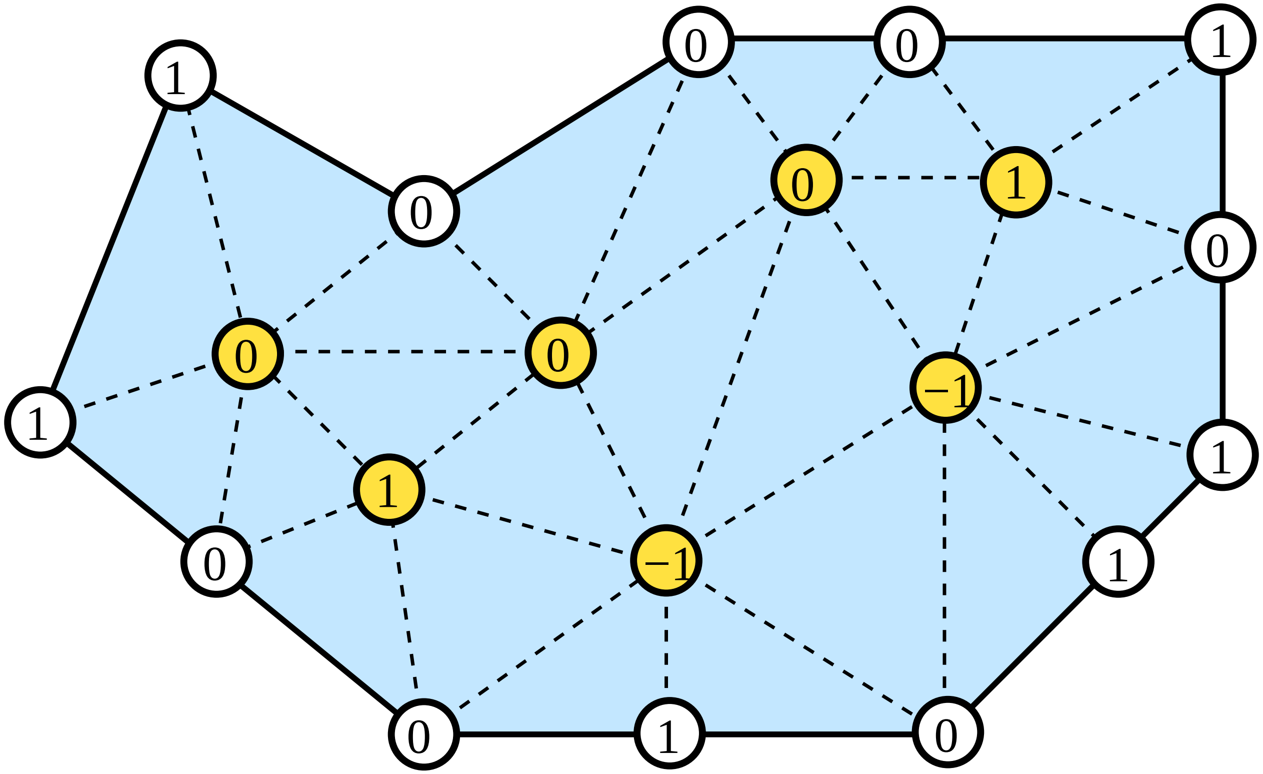
Conformal = 1-1, holomorphic = preserves angles infinitesimally.

Problems: must map vertices to vertices, bound angle distortion, …

Also, Euler's formula sometimes forces vertices of degree 5 or 7.

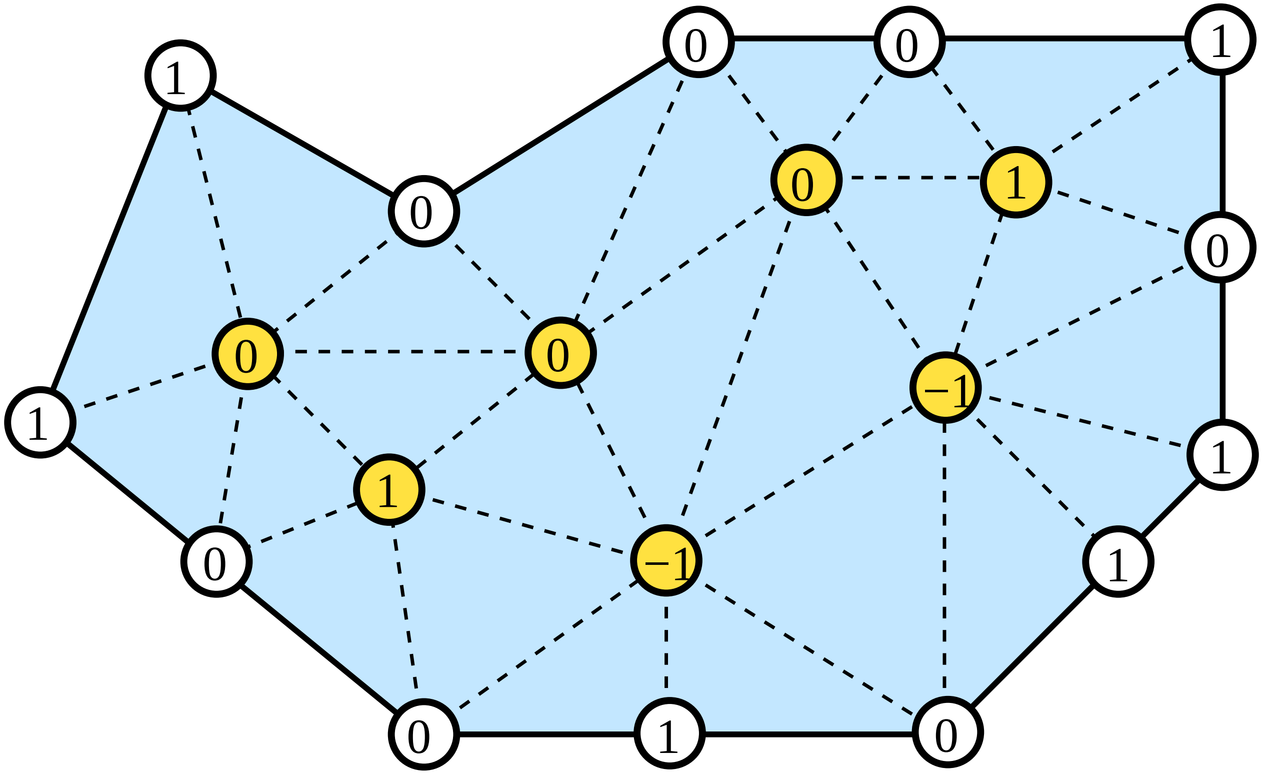


Let $L(v)$ = number of triangles with v as vertex.



Curvature of boundary vertex v: $\kappa(v) = 3 - L(v)$.

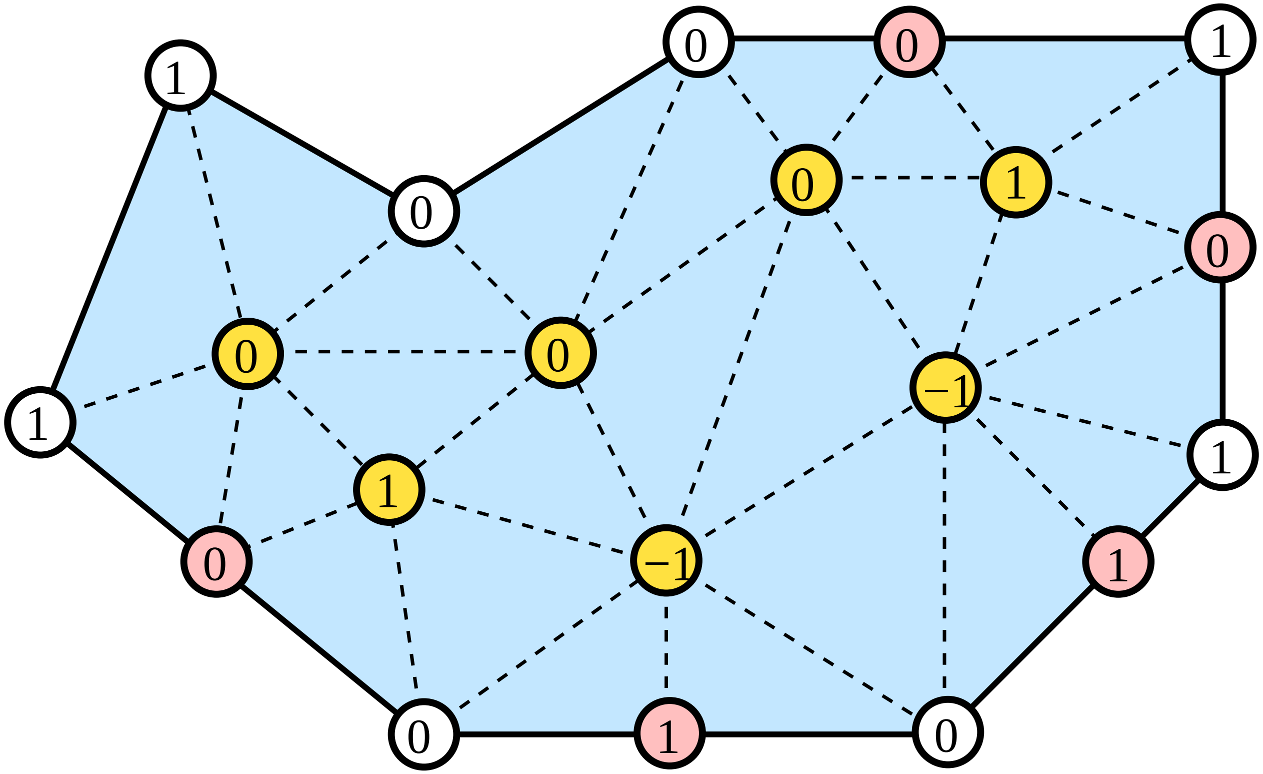
Curvature of interior vertex v: $\kappa(v) = 6 - L(v)$.



Euler's formula can be rewritten to look like Gauss-Bonnet:

$$\sum_{v\in\text{interior}} \kappa(v) = 6 - \sum_{v\in\text{boundary}} \kappa(v)$$

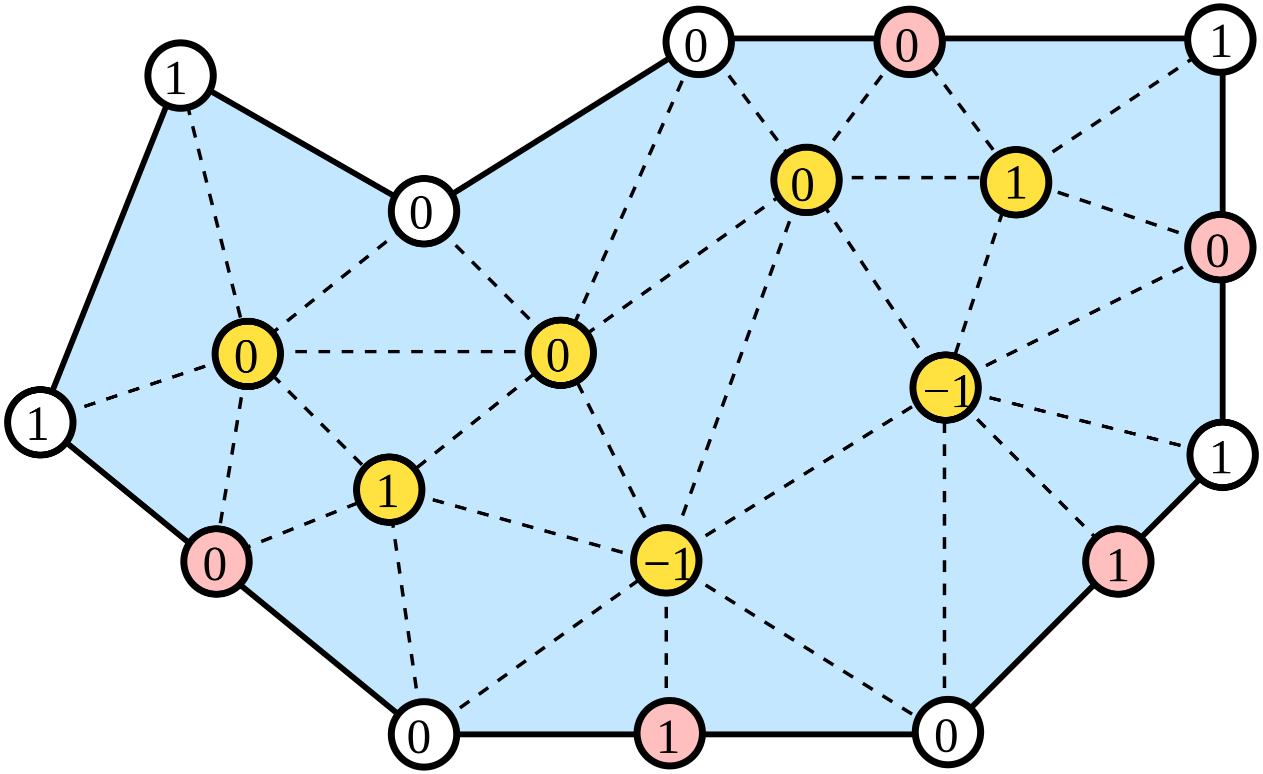
$$\kappa(\mathcal{T}) = 6 - \kappa(\partial\mathcal{T})$$



Define curvature of labeling L of vertices V of P (omit Steiner points):

$$\kappa(L) = 6 - \sum_{v \in P} \kappa(v).$$

Makes sense for any $L : V \to \mathbb{N}$, not just triangulations.

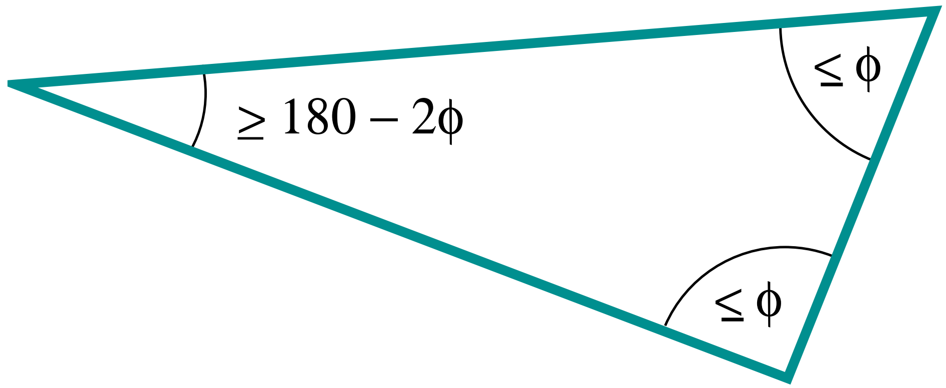


For acute triangulations (angles $< 90^\circ$) it is easy to see

$$\kappa(L) \leq \kappa(\mathcal{T})$$

since omitted boundary Steiner points have $L(v) \geq 3 \Rightarrow \kappa(v) \leq 0$.

If a triangle has all angles $\leq \phi$, then all angles are $\geq 180^\circ - 2\phi$.



If a ϕ-triangulation has $L(v)$ triangles at vertex $v \in P$ of angle θ_v, then

$$L(v) \cdot (180^\circ - 2\phi) \;\leq\; \theta_v \;\leq\; L(v) \cdot \phi.$$

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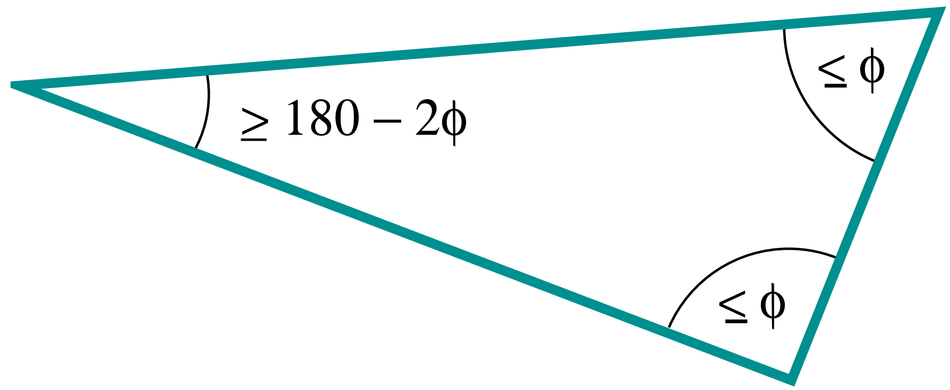

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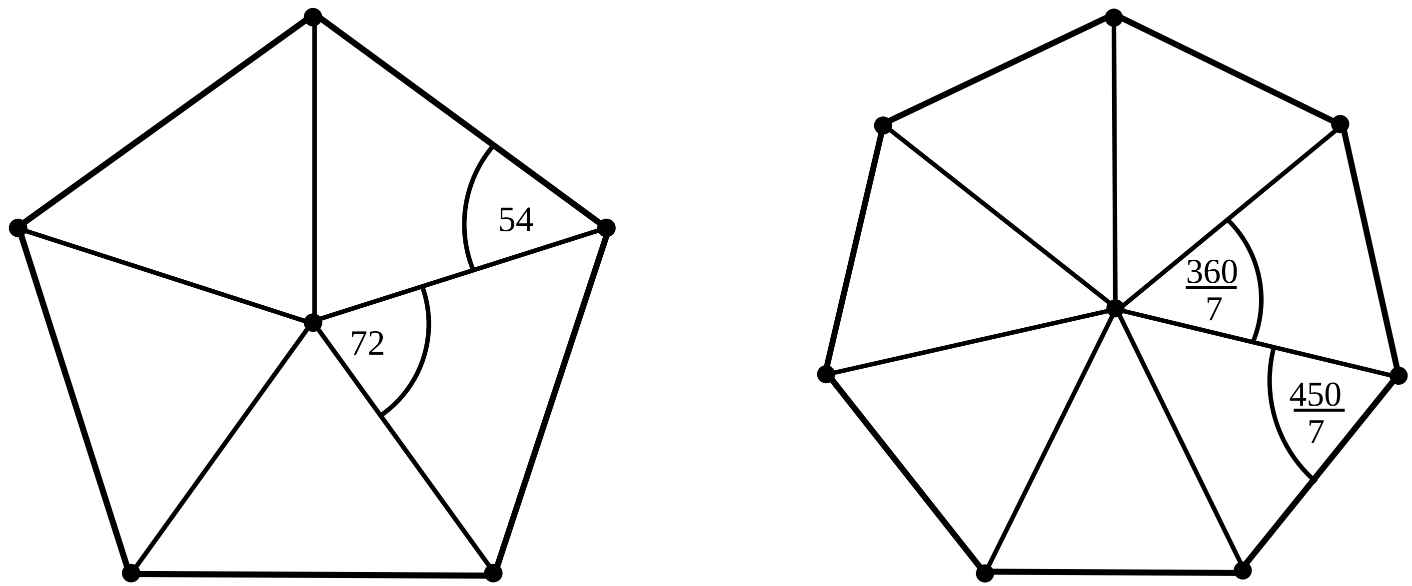
$$L(v) \cdot (180° - 2\phi) \;\leq\; \theta_v \;\leq\; L(v) \cdot \phi.$$

Defn: A labeling L of P is a **ϕ-labeling** if these inequalities hold, i.e.,

$$\frac{\theta_v}{\phi} \;\leq\; L(v) \;\leq\; \frac{\theta_v}{180° - 2\phi}.$$

Every ϕ-triangulation gives a ϕ-labeling. Converse true? Not quite.



Suppose labeling L corresponds to a ϕ-triangulation. Easy to check that:

- If $\phi < 72^\circ$, then $\kappa(L) \leq \kappa(\mathcal{T}) \leq 0$.
- If $\phi < (450/7)^\circ \approx 64.28^\circ$, then $\kappa(L) = \kappa(\mathcal{T}) = 0$.

Remarkably, these necessary conditions are also sufficient.

Theorem: For $60^\circ < \phi < 90^\circ$, a polygon P has a ϕ-triangulation **iff**

1. $72^\circ \leq \phi < 90^\circ$ and P has a ϕ-labeling L of V_P,
2. $\frac{5}{7} \cdot 90^\circ \leq \phi < 72^\circ$, and P has a ϕ-labeling with $\kappa(L) \leq 0$,
3. $60^\circ < \phi < \frac{5}{7} \cdot 90^\circ$, and P has a ϕ-labeling with $\kappa(L) = 0$.

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Gerver (1984) proved necessity when P has ϕ-dissection.

Corollary: For $\phi > 60^\circ$, the following are equivalent:

(1) P has a ϕ-dissection.
(2) P has a ϕ-triangulation.

$\Rightarrow$ **Dissections and triangulations give same angle bound.**

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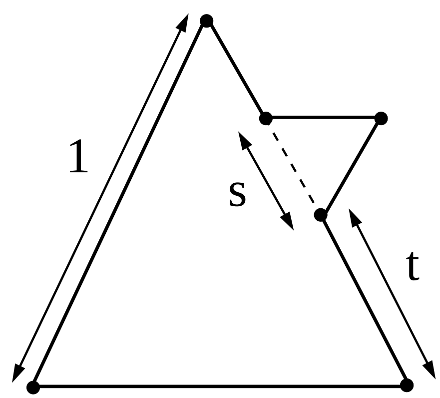
Corollary: If $\Phi(P) > 60^\circ$ this bound is attained.

Corollary: $\Phi(P) = 60^\circ$ iff $P = 60^\circ$-polygon. Attained iff all side length ratios are rational.



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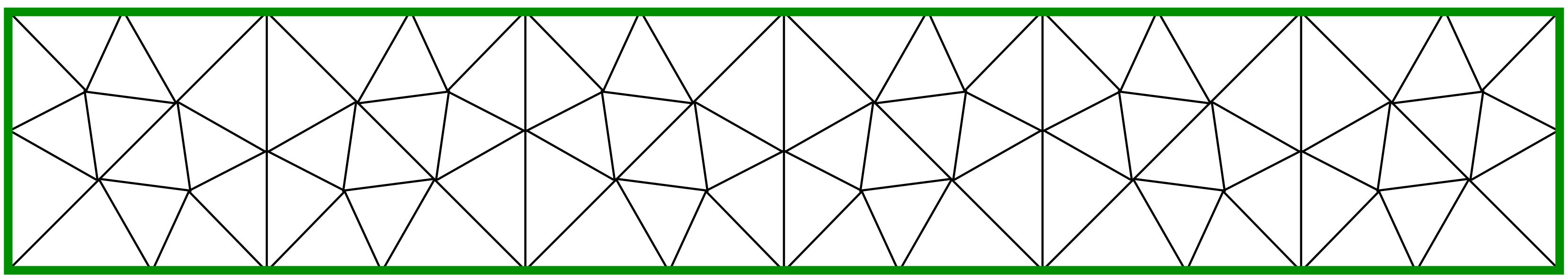
Cor: For an N-gon $\Phi(P)$ can be computed in time $O(N)$.

However, $1 \times R$ rectangle needs $\gtrsim R$ triangles.



$\Rightarrow$ no bound for number of triangles in terms of N.

Idea for $O(N)$ computation of $\Phi(P)$:

If $\theta_{\text{min}} \leq 36^\circ$ then $\Phi(P) = 90^\circ - \theta_{\text{min}}/2$. Find θ_{min} in $O(N)$.

Otherwise, finding $\Phi(P)$ (eventually) reduces to computing

$$\begin{aligned}\phi_0 &= \inf\{\phi : \exists\ \phi\text{-labeling with } \kappa(L) = 0\} \leq 72^\circ \\ &= \inf\{\phi : \min(f(\phi), 0) + \max(g(\phi), 0) = 0\}\end{aligned}$$

where f, g are the monotone step functions:

$$f(\phi) = \sum_{v \in P} \inf\{k : 180 - 2\phi \leq \frac{\theta_v}{k} \leq \phi\}$$

$$g(\phi) = \sum_{v \in P} \sup\{k : 180 - 2\phi \leq \frac{\theta_v}{k} \leq \phi\}$$

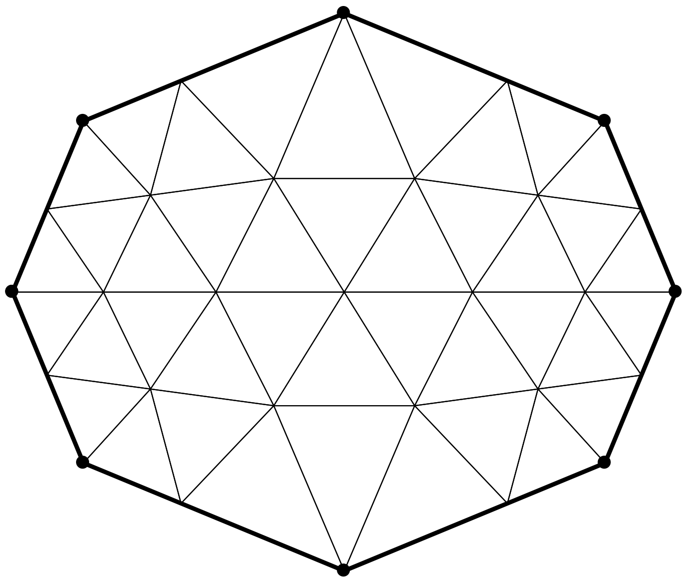
Note that $\phi_0 \in J =$ the $O(N)$ (known) jump points of f, g.
$O(N^2)$ work to find ϕ_0? Evaluate N-sums for $O(N)$ values?

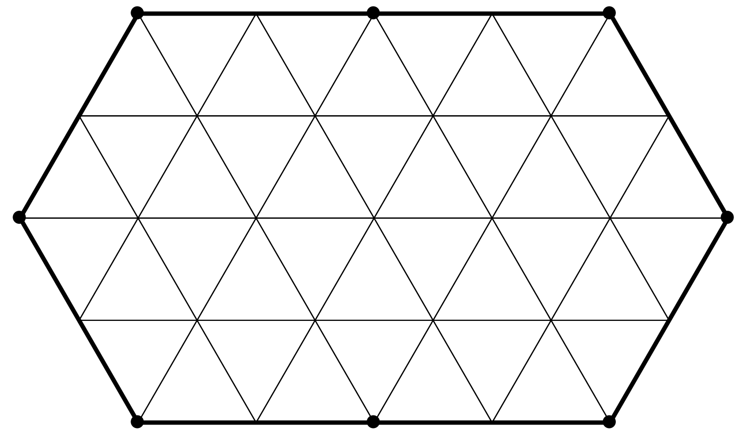
However, we can find $\phi_0 \in J$ in time $O(N)$ as follows:

- Find smallest, largest elements of J. Evaluate f, g.
- Find median of J by median-of-medians algorithm. Evaluate f, g.
- Decide if ϕ_0 is $\geq$ or $\leq$ median. Delete half of J.
- Repeat last two steps until ϕ_0 is found.
- Monotonicity implies new evaluations only use remaining points.

$\Rightarrow$ Work diminishes geometrically. Total is $O(N)$.

Idea behind main theorem: conformal maps





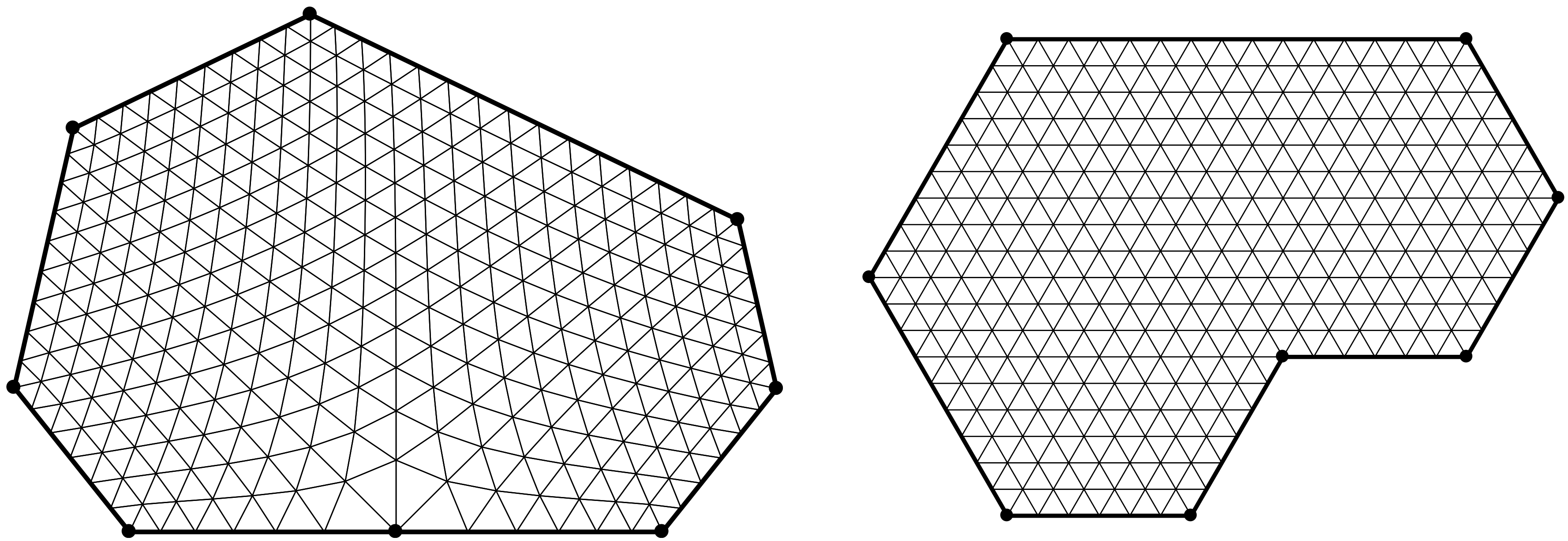
Given P, build P' with angles $\psi_k = L(k) \cdot 60^\circ \approx \theta_k$ and

$$\sum \psi_k = (N-2) \cdot 180^\circ.$$

This requires the labeling of P' to have curvature zero.

If this is a ϕ-labeling of P, the conformal transfer idea works.

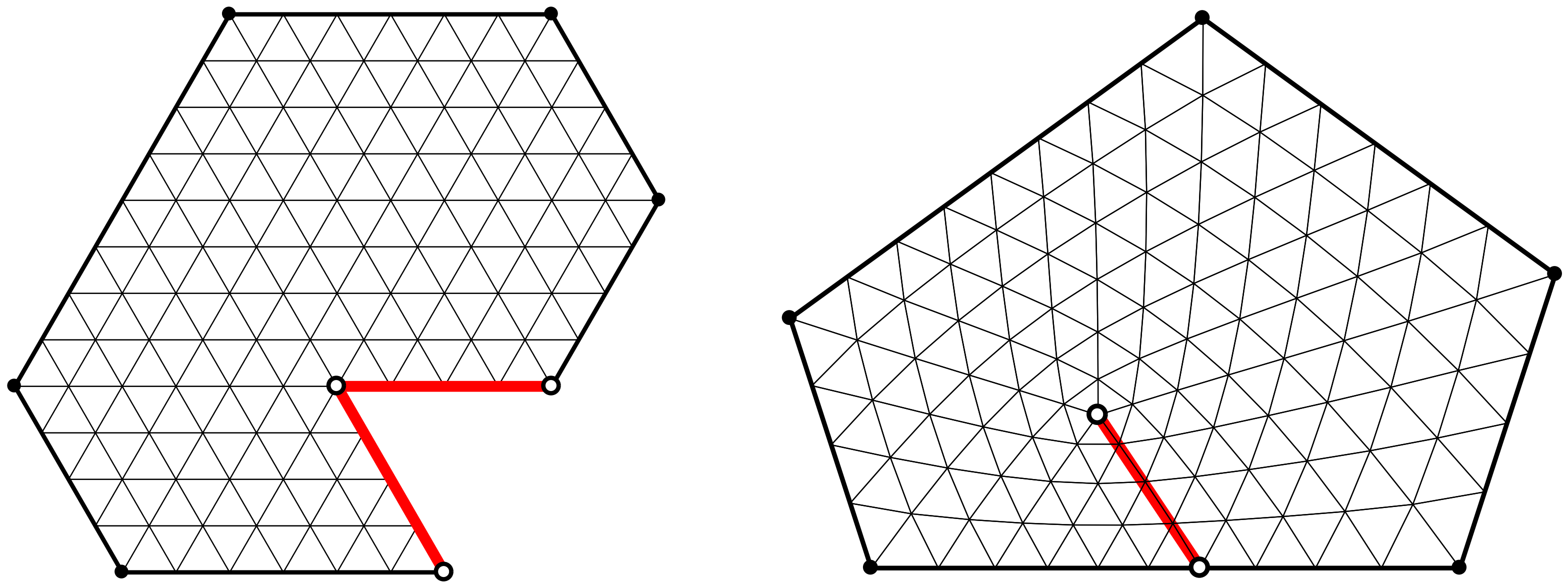
Idea behind main theorem: conformal maps



In this situation, interior vertices of all have degree 6.

$\Rightarrow$ This doesn't work if all ϕ-labelings have non-zero curvature κ.

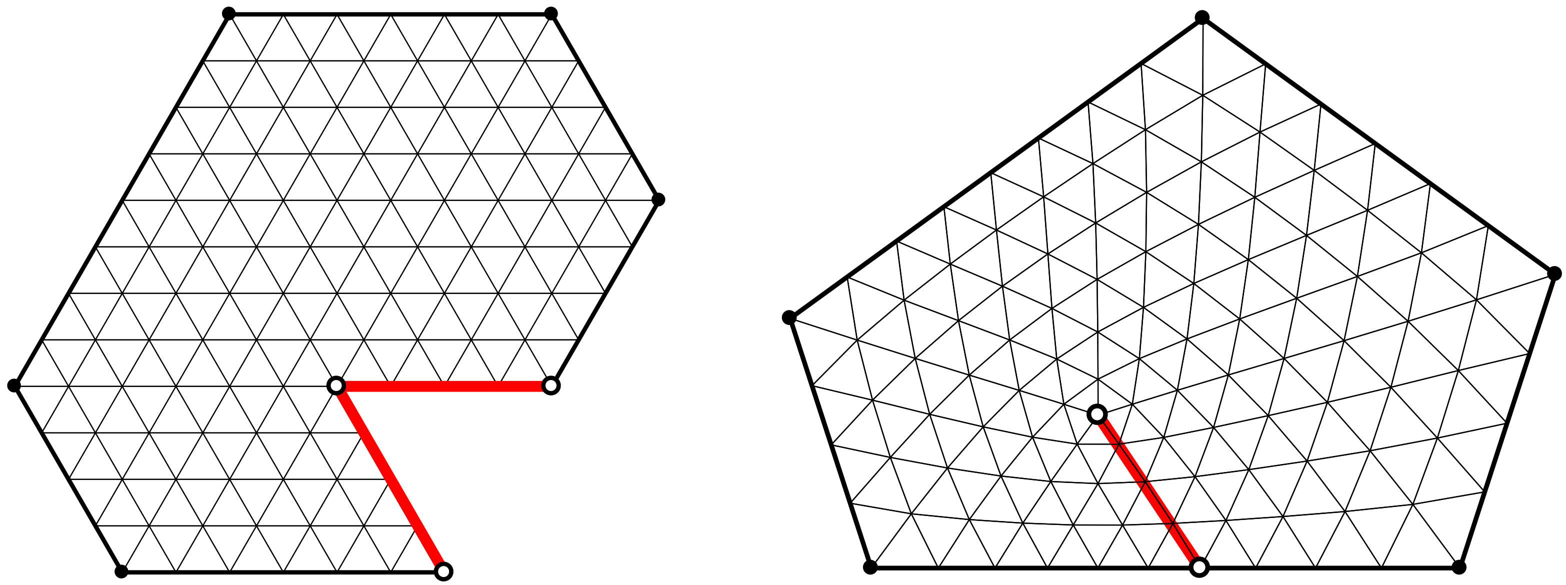
E.g., acute triangulation of regular pentagon must contain degree 5 vertex.



Map $f : P' \to P$ can identify boundary segments.

Boundary vertices of P' become interior vertices of P.

In this figure, a degree 5 interior vertex is created.

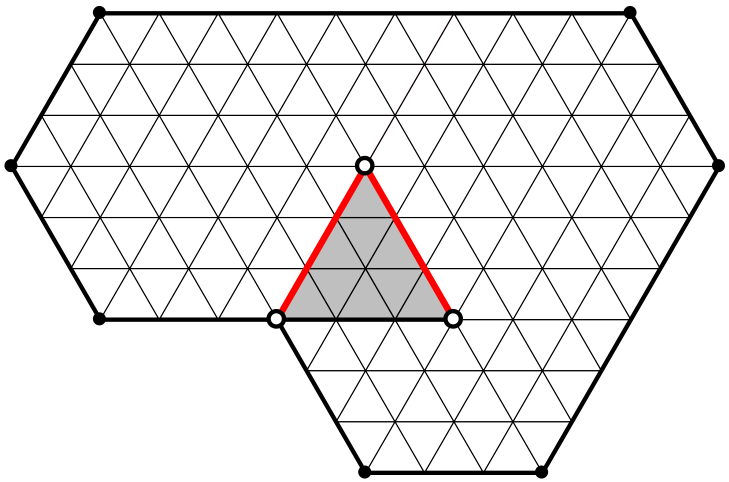


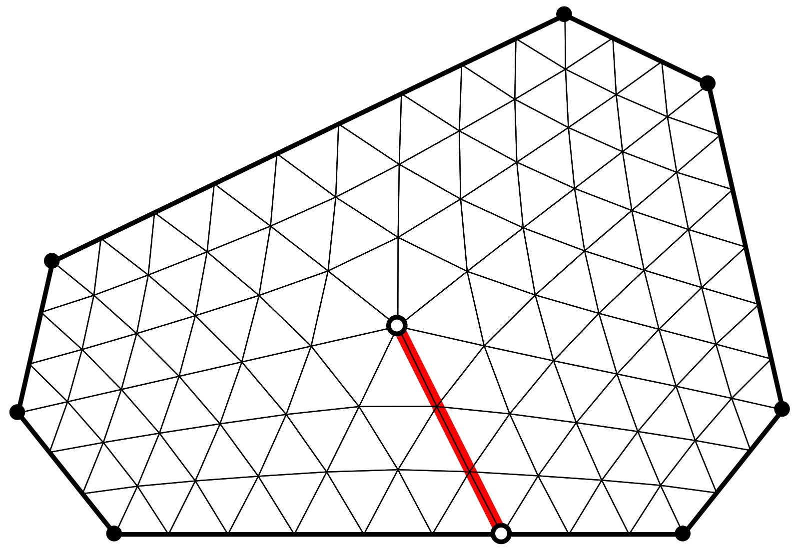
Technical difficulty: slit is not straight (within 3°).

Triangulations must match up across slit.

This occurs if $|f'(w)| = |f'(z)|$ whenever $f(w) = f(z)$.

Differential equation can be solved explicitly (= conformal welding).

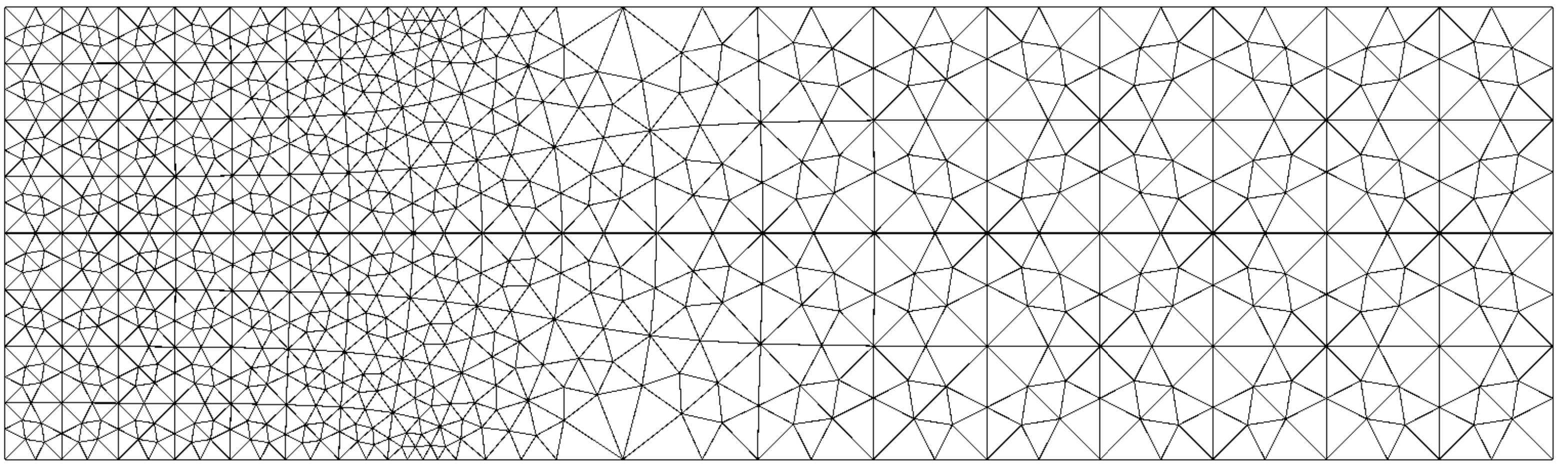




Creating a degree 7 vertex requires P' to be Riemann surface.

All cases can be handled with these “tricks”.

Thanks for listening



Lecture, slides and related papers are posted at
`https://www.math.stonybrook.edu/~bishop`

Email questions to `lastname@math.stonybrook.edu`

Open Problems

- Proof doesn't give "practical" meshes. Benchmark existing methods?
- Construct triangulations within a bounded factor of optimal size?
- Minimal number of triangles needed to get optimal angles? NP hard?
- Compute optimal angle bound for conforming triangulation of a PSLG.
- If a PSLG has minimal angle θ does it have a ϕ-triangulation with $\phi = \max(72°, 90° - \theta/2)$? (Yes, if $72°$ is replaced by some $\theta_0 < 90°$.)
- Two equal area polygons can be dissected into isometric sets of triangles (Wallace-Bolyai-Gerwien Thm). Compute the optimal angle bounds.
- An open set of polygons has optimal bound $72°$. What is the probability a random polygon has this bound? What is a random polygon?