RANDOM WALKS, TRUE TREES AND EQUILATERAL TRIANGULATIONS

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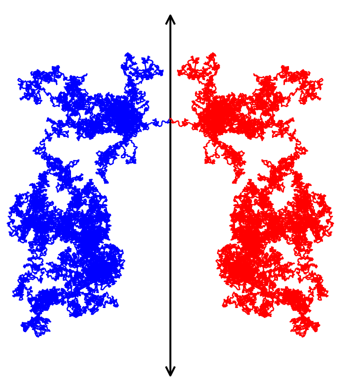
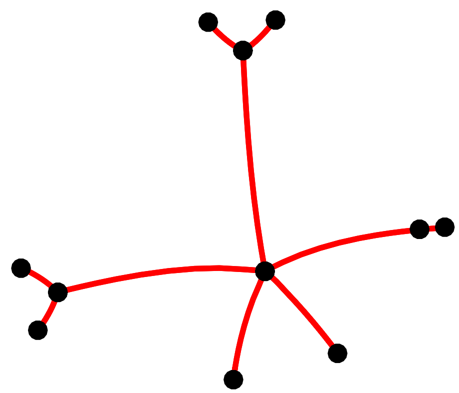
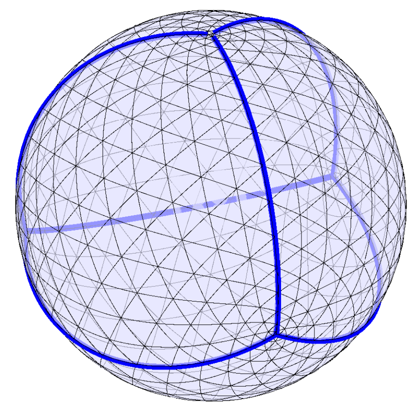
Graphs on surfaces and curves over number fields seminar
November 17, 2021

www.math.sunysb.edu/~bishop/lectures

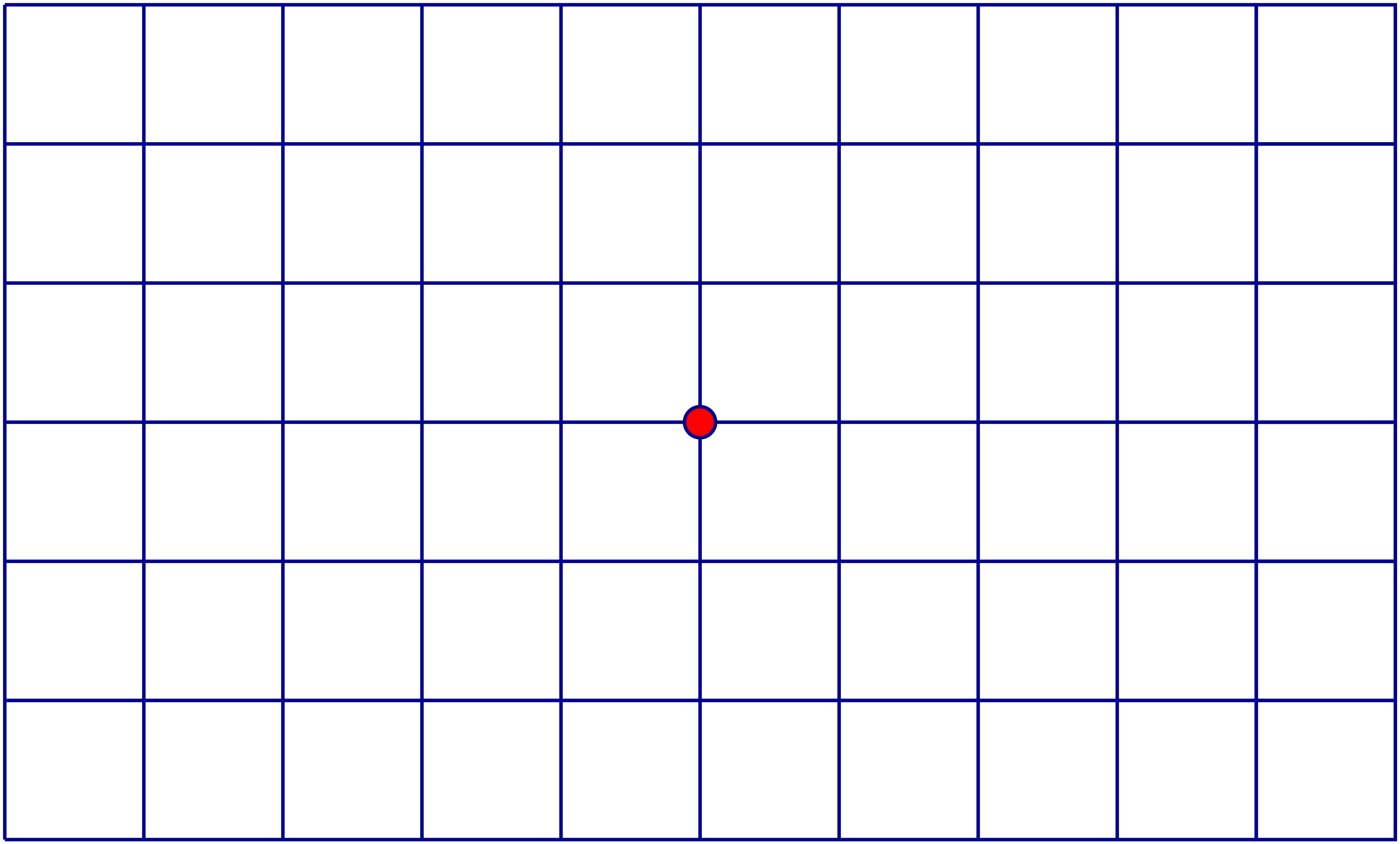


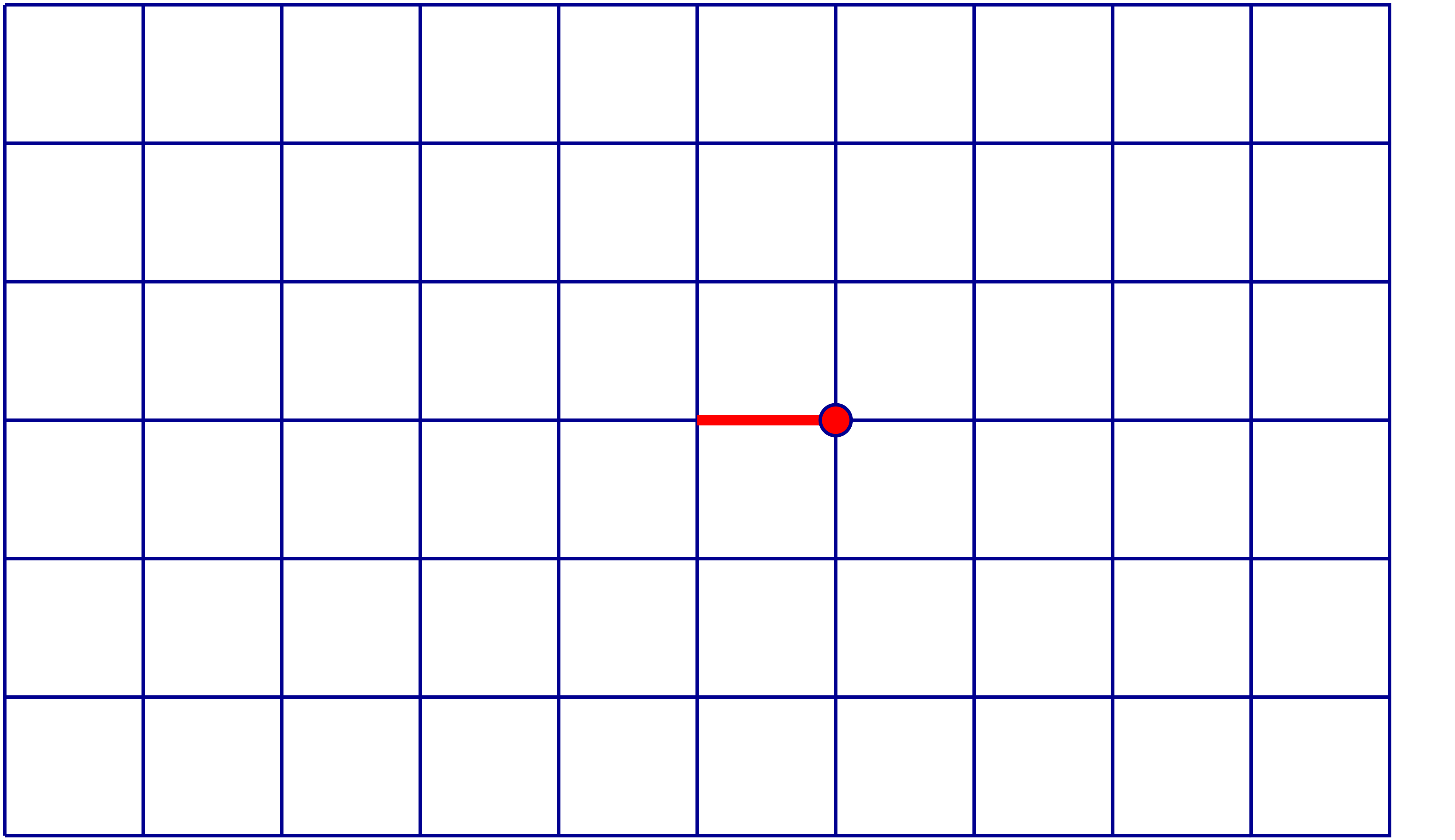


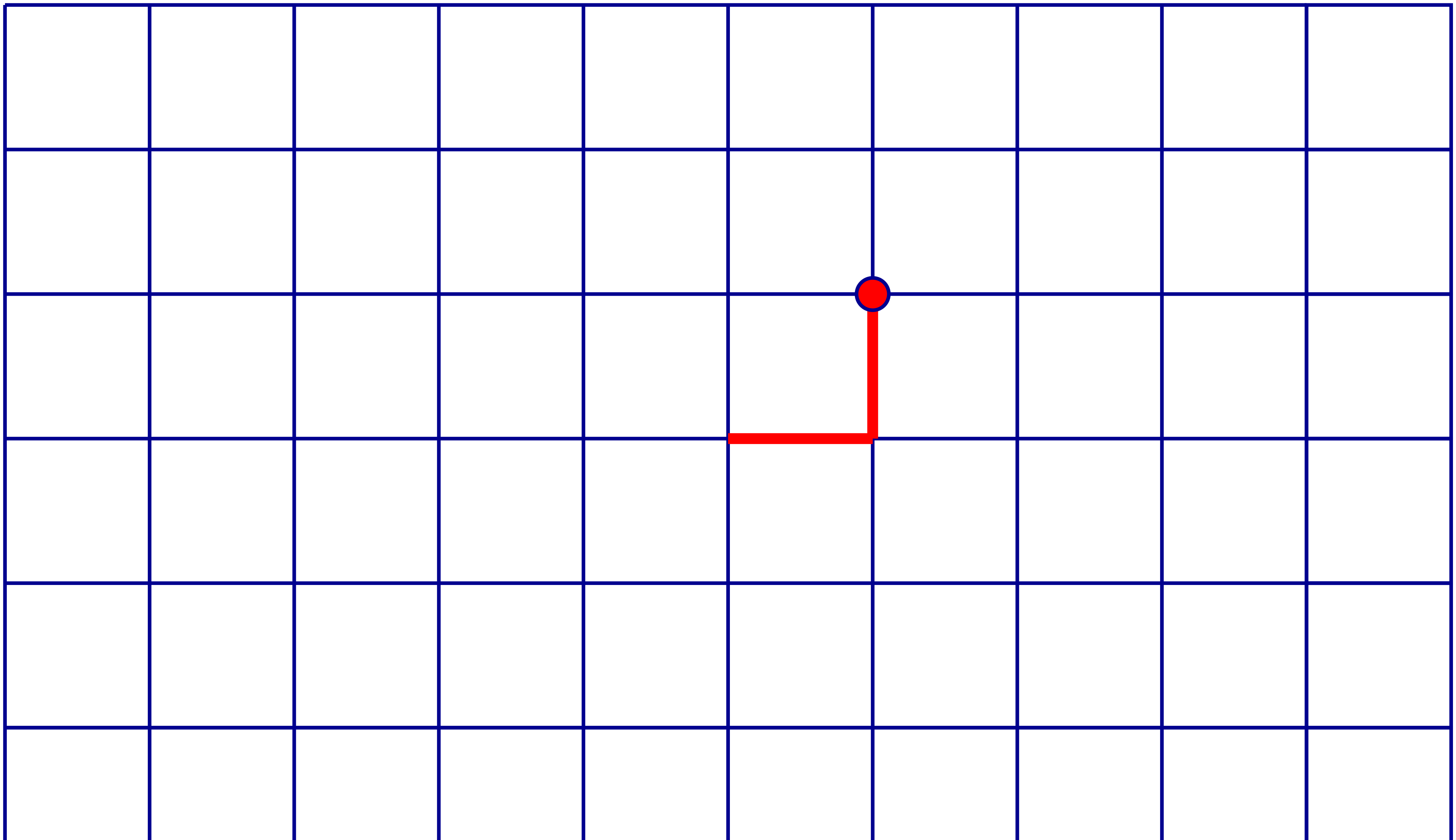


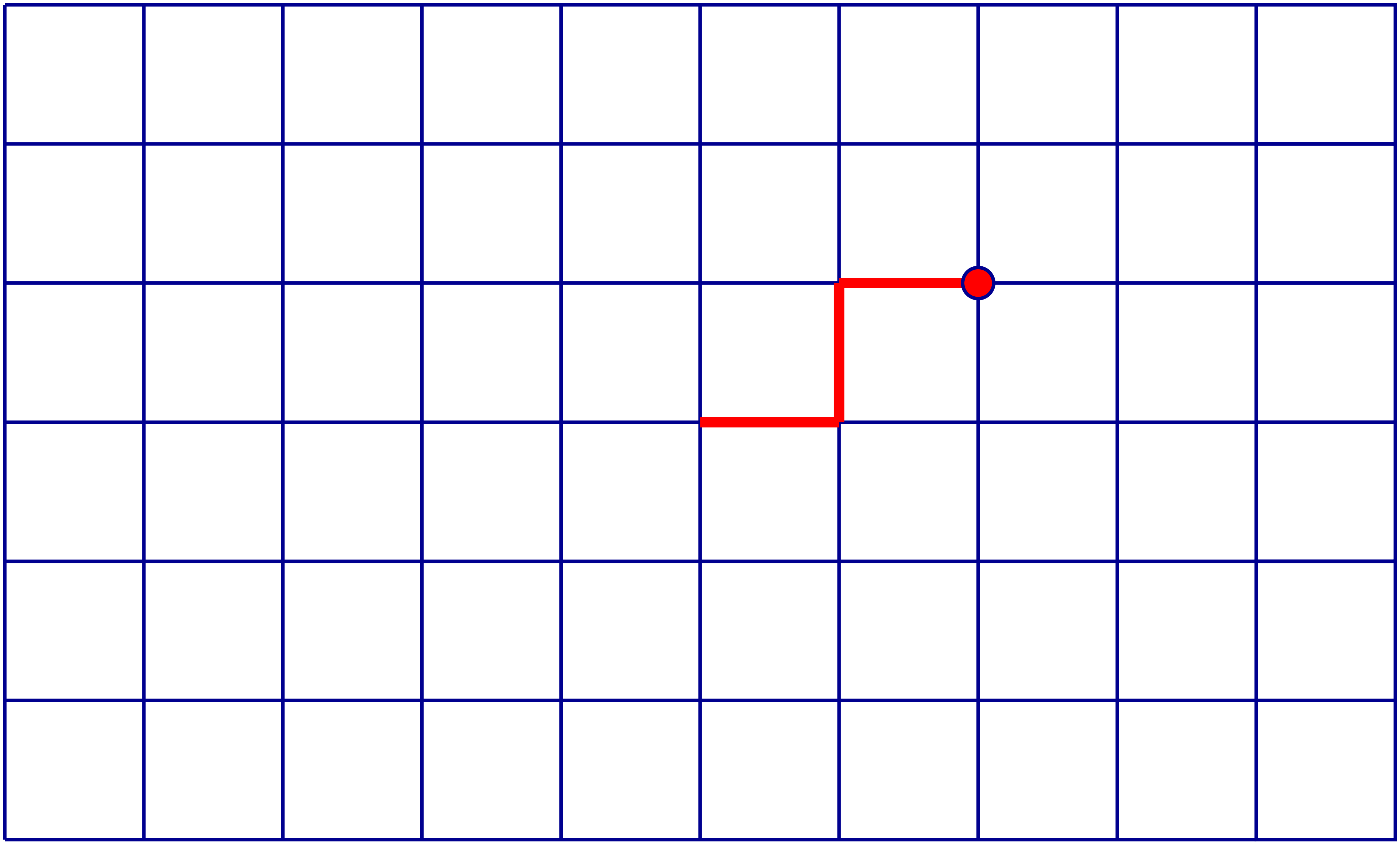
THE PLAN

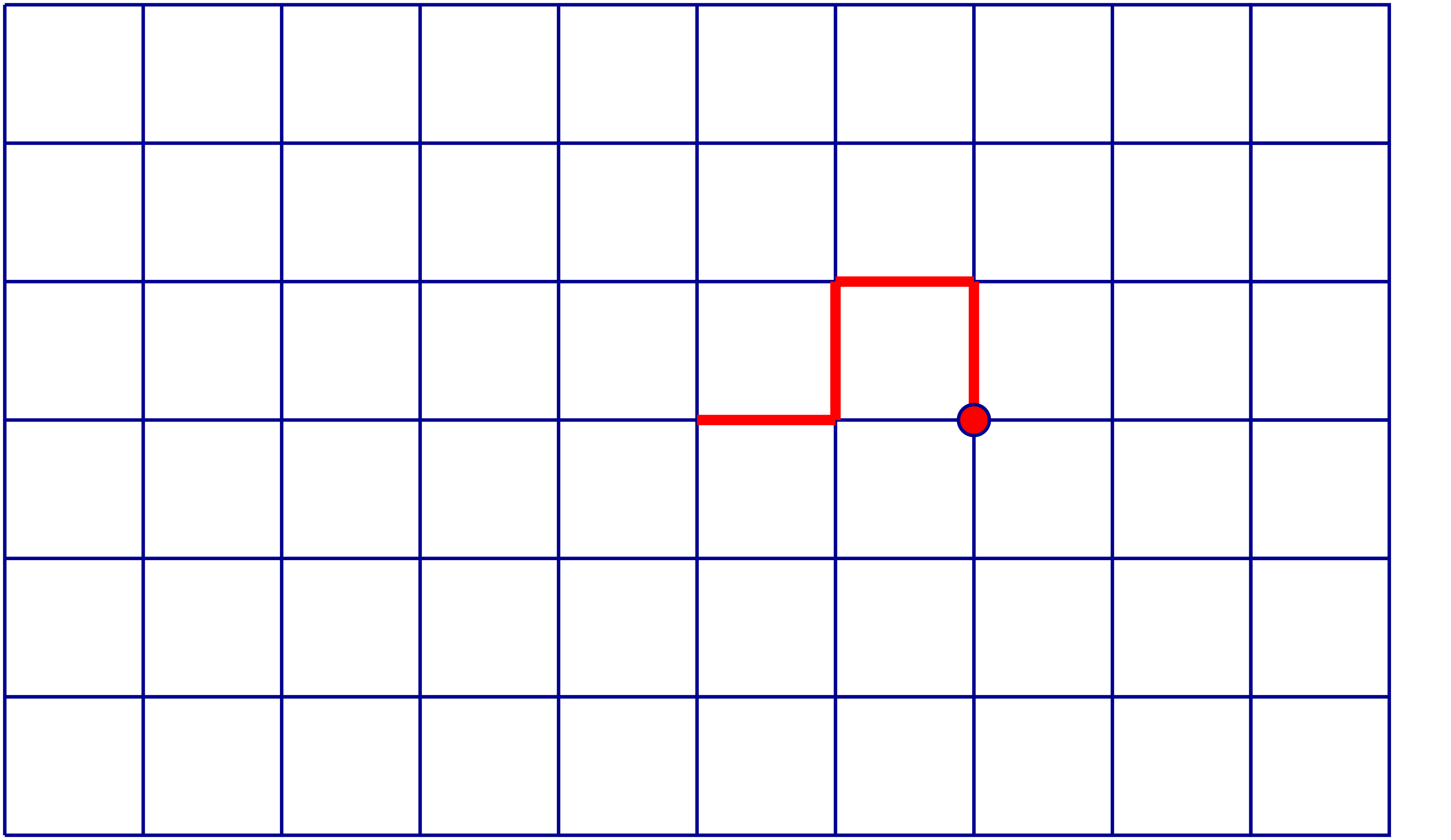
- Harmonic measure
- True trees and Shabat polynomials
- Every planar tree has a true form
- Every planar tree is approximated by a true tree
- Equilateral triangulations of surfaces
- Infinite trees and entire functions

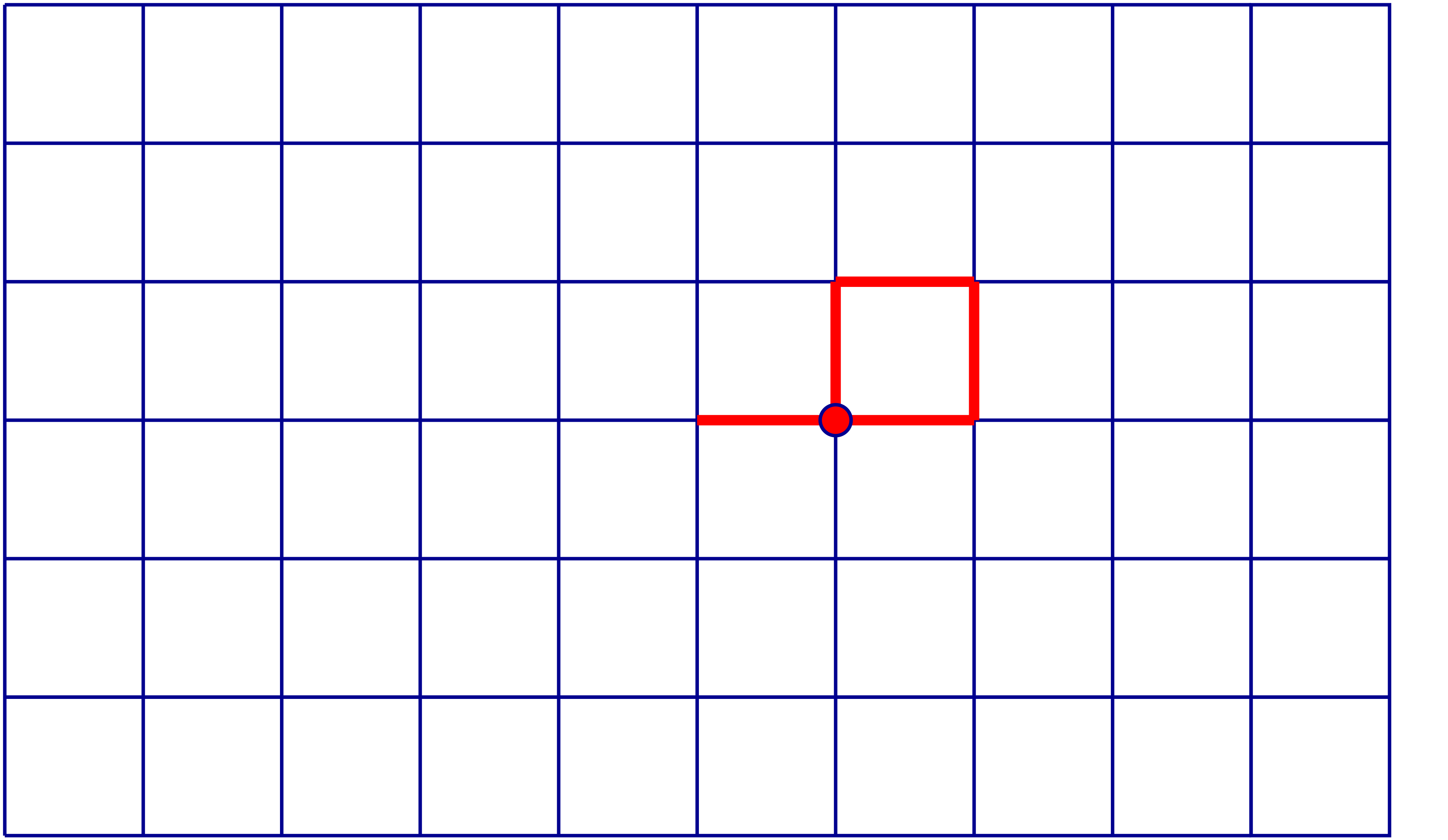


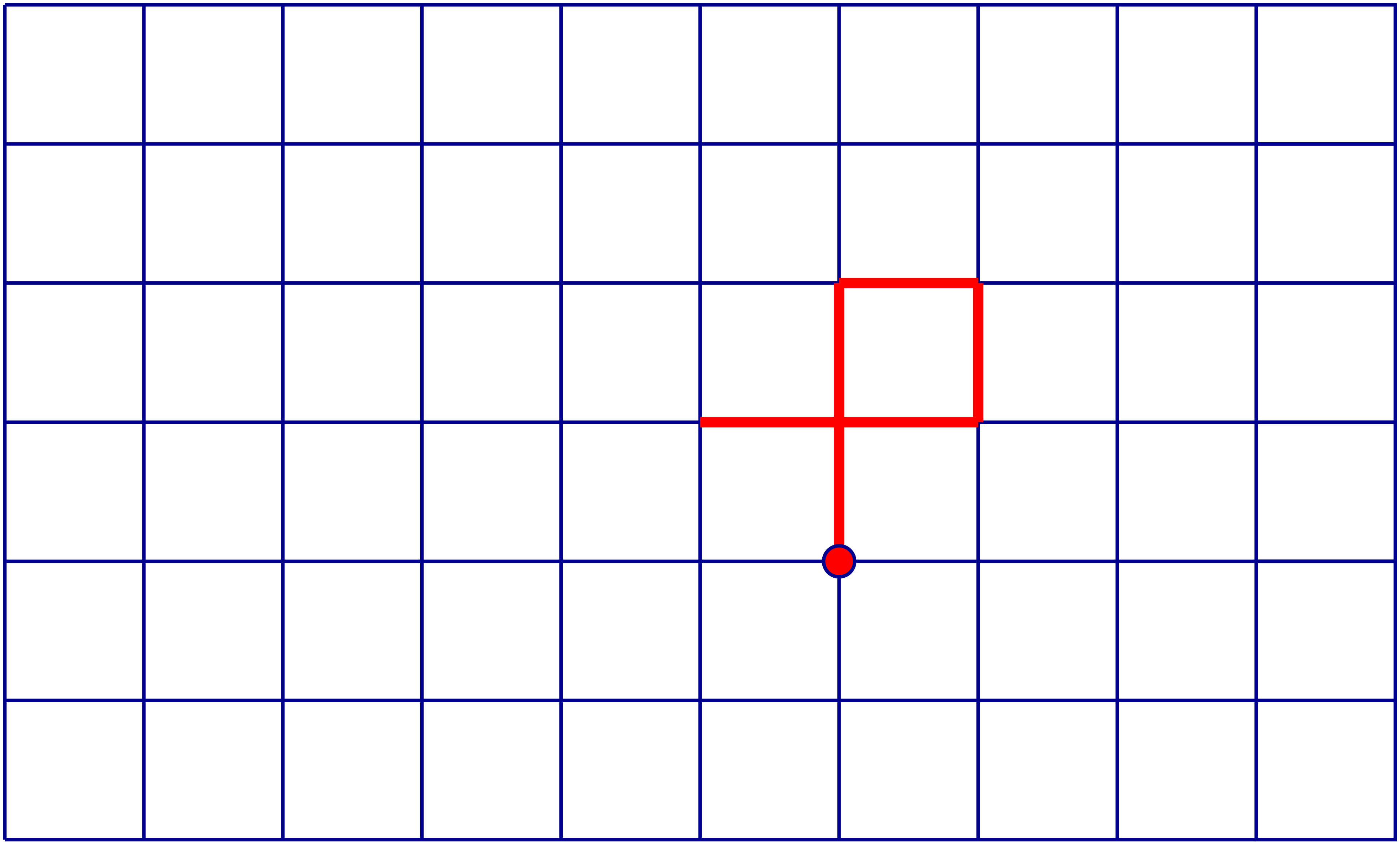


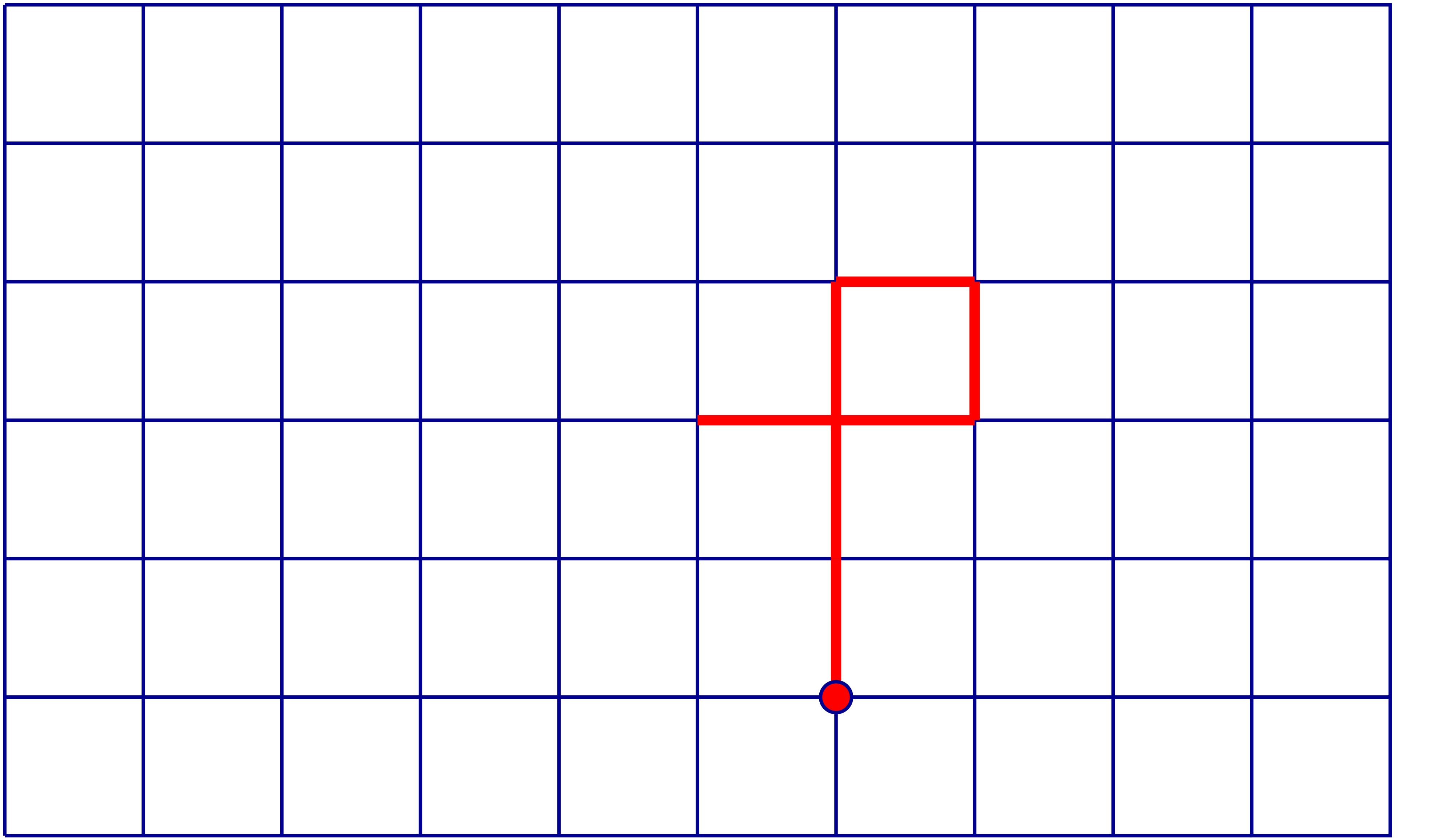


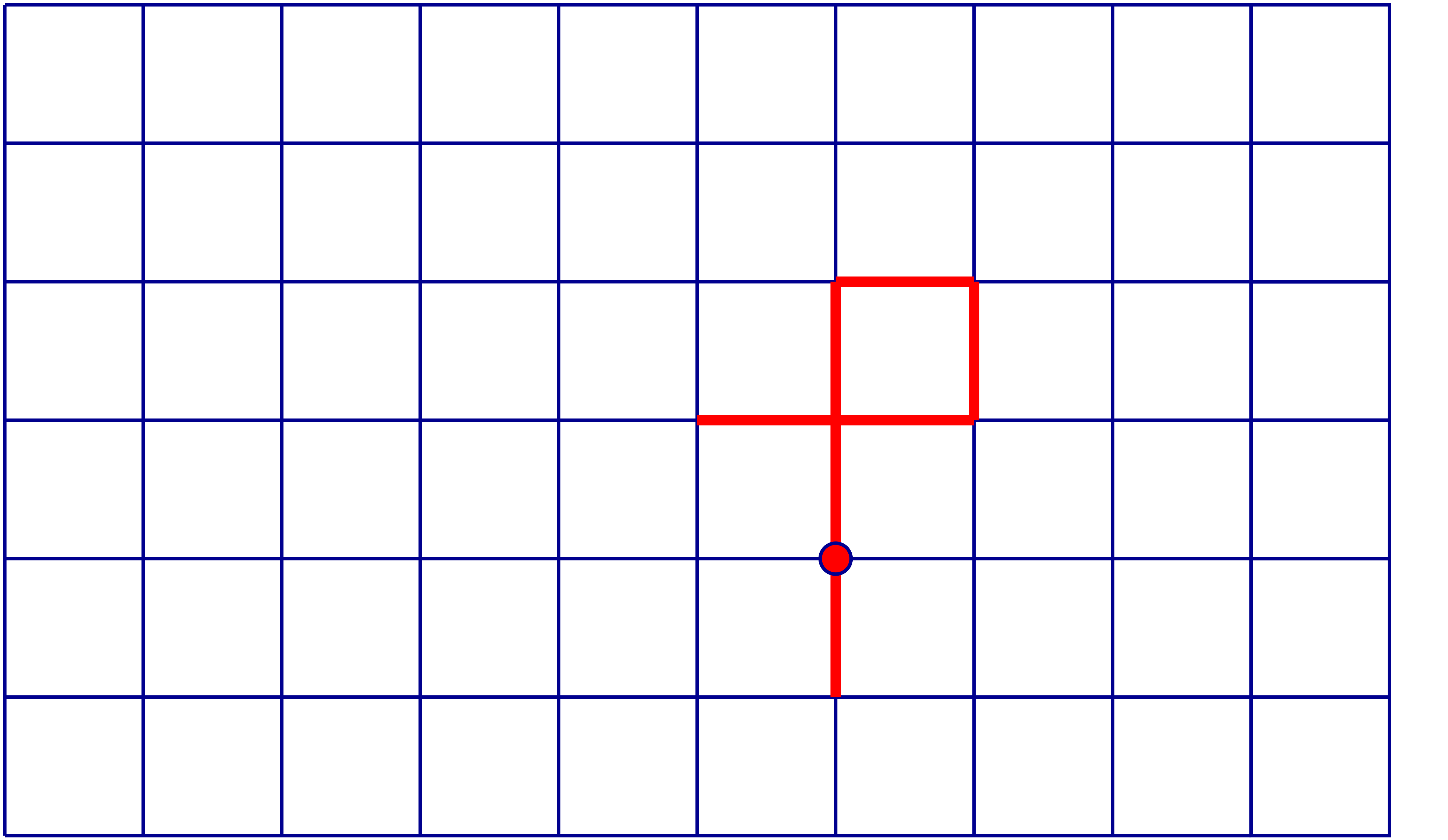


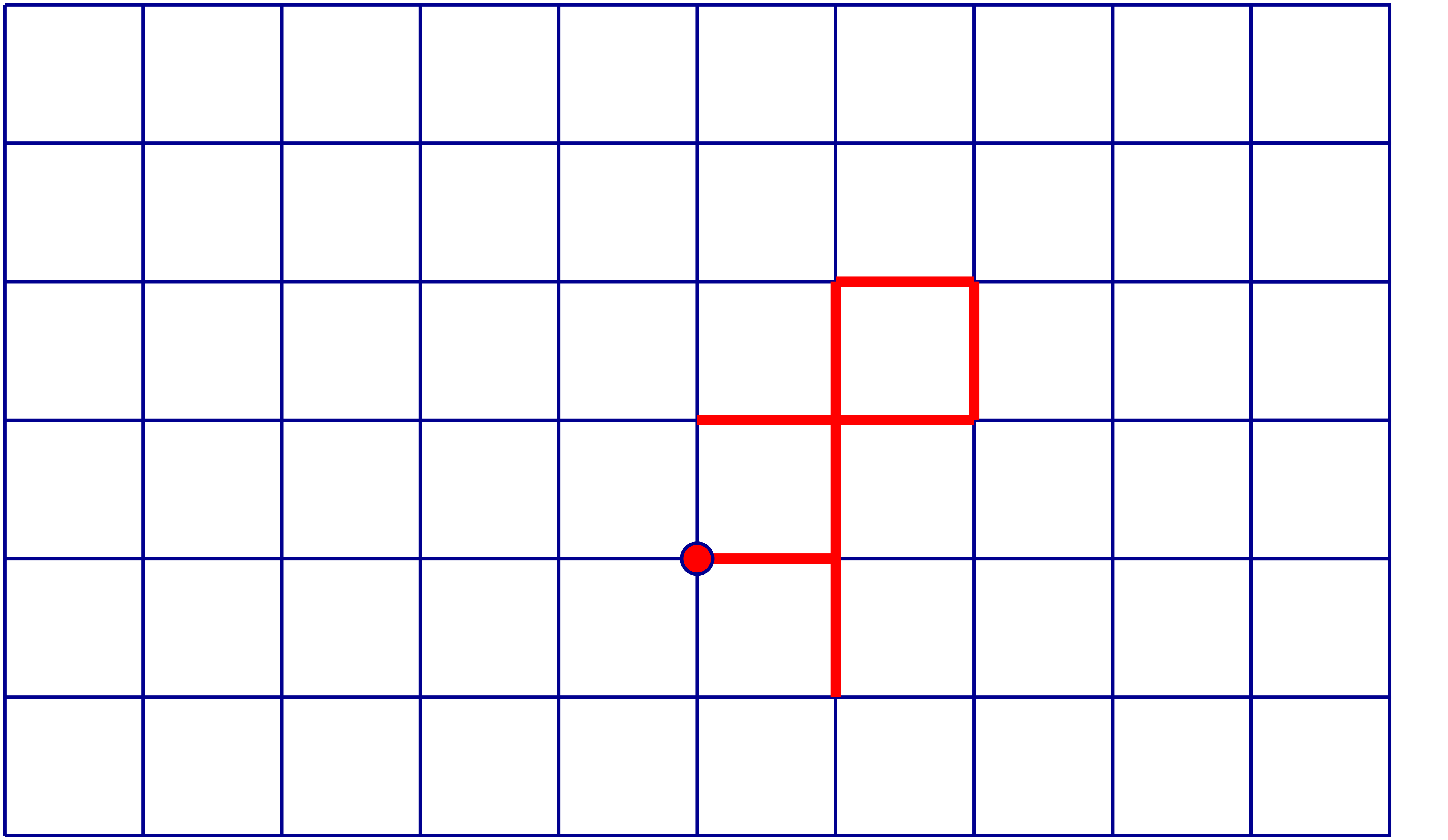


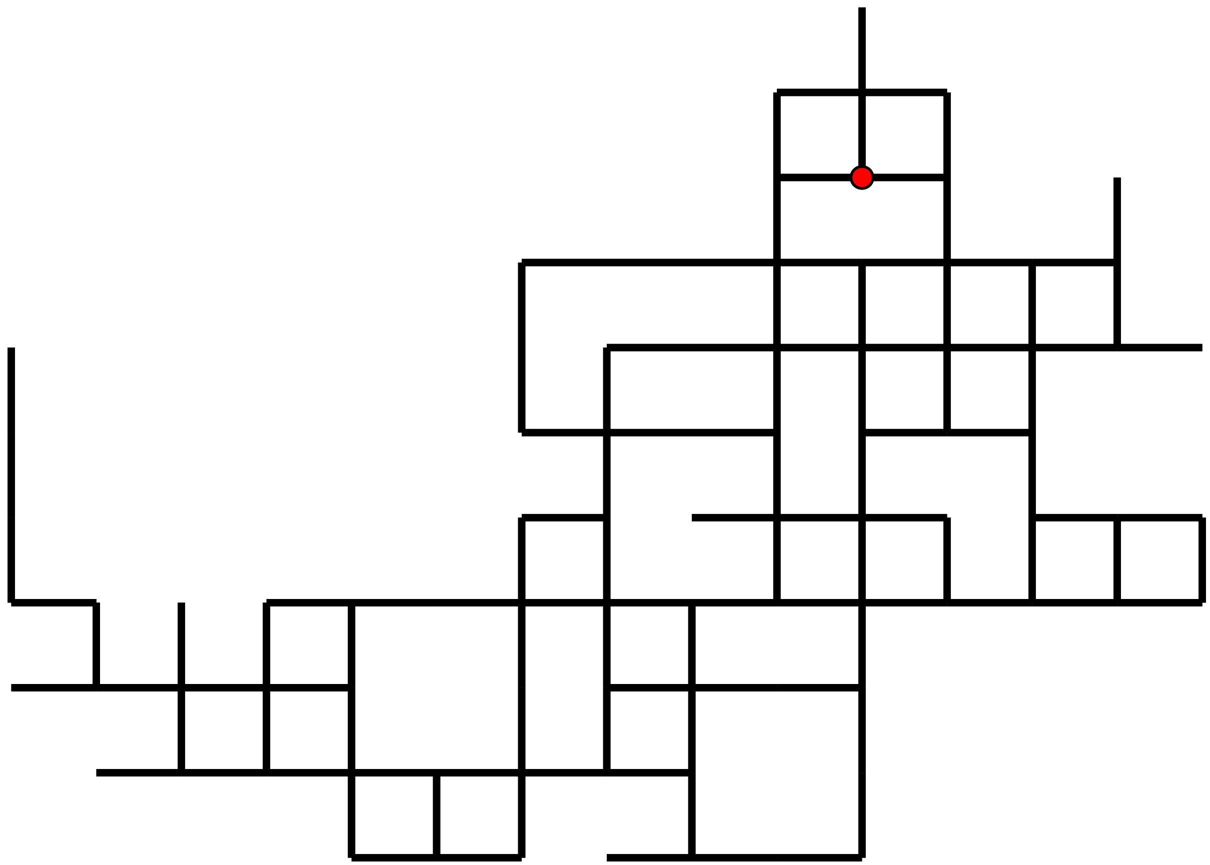




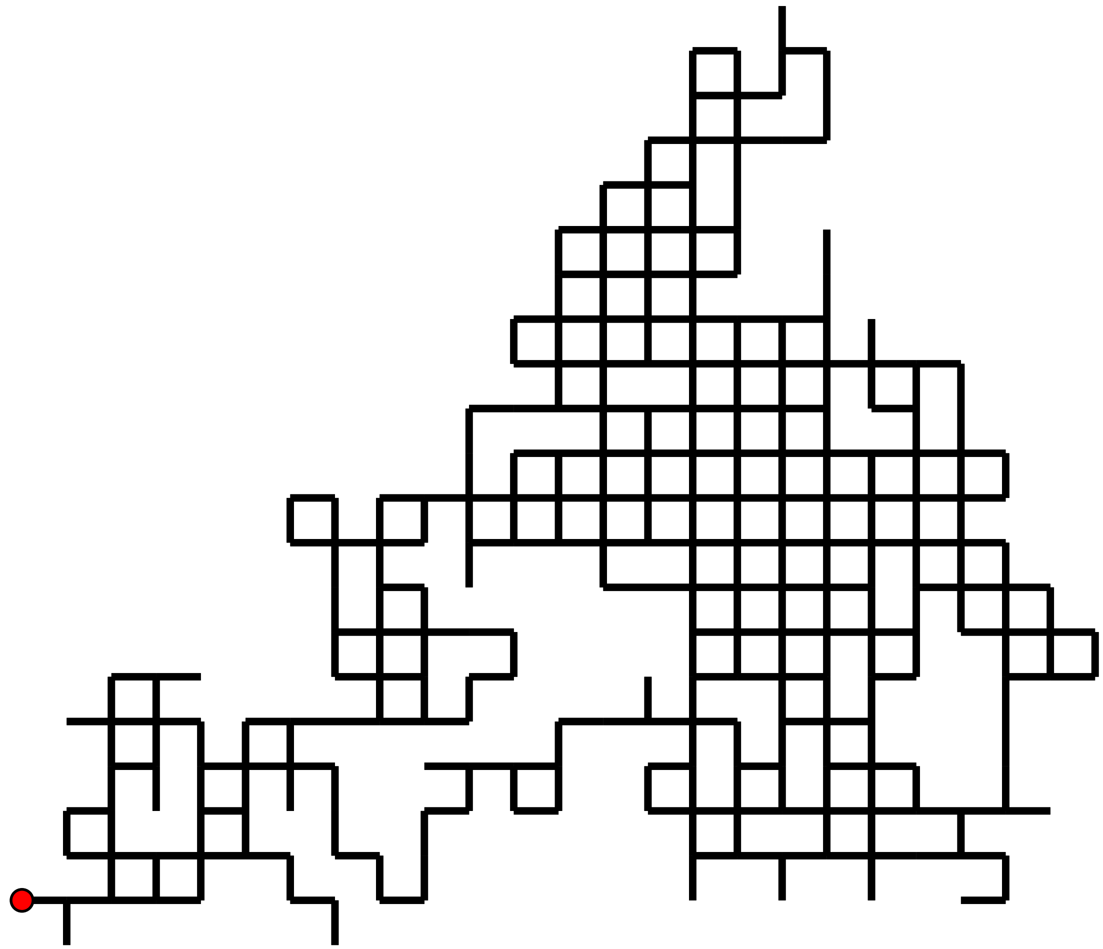




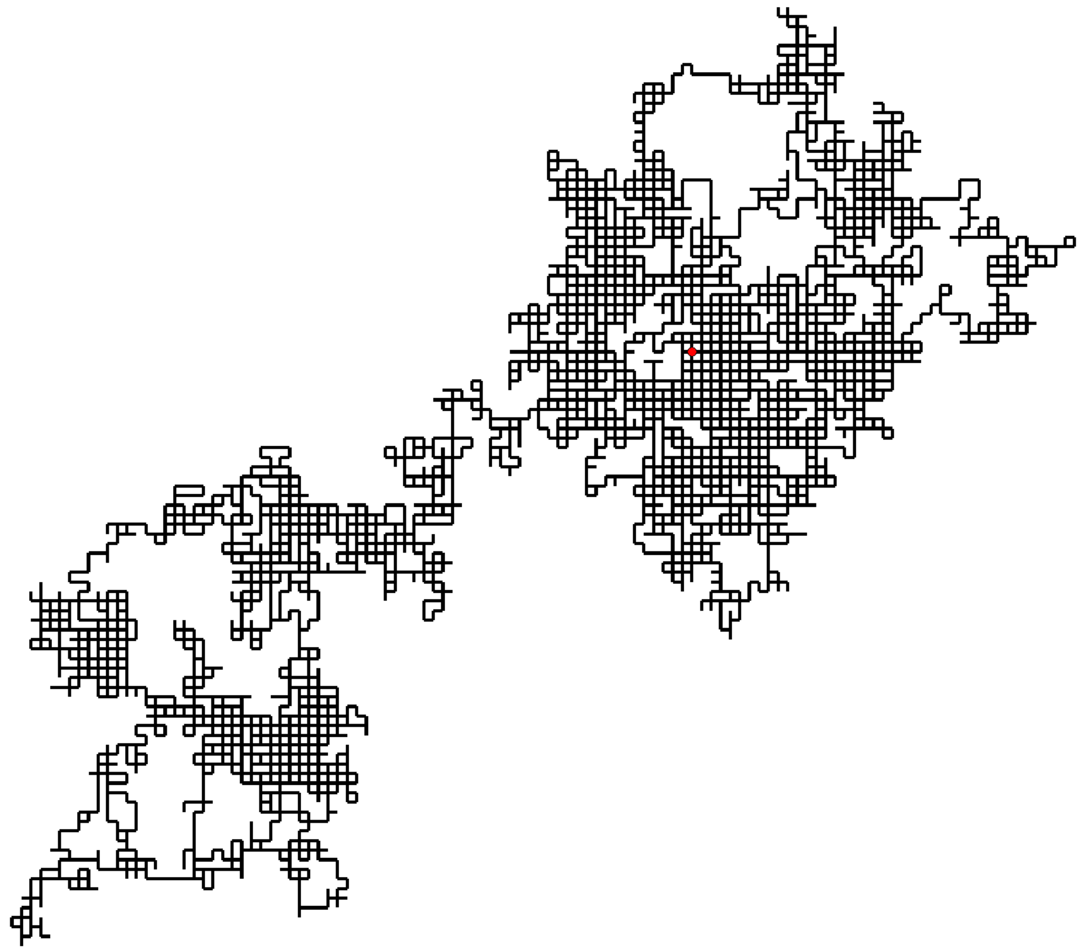




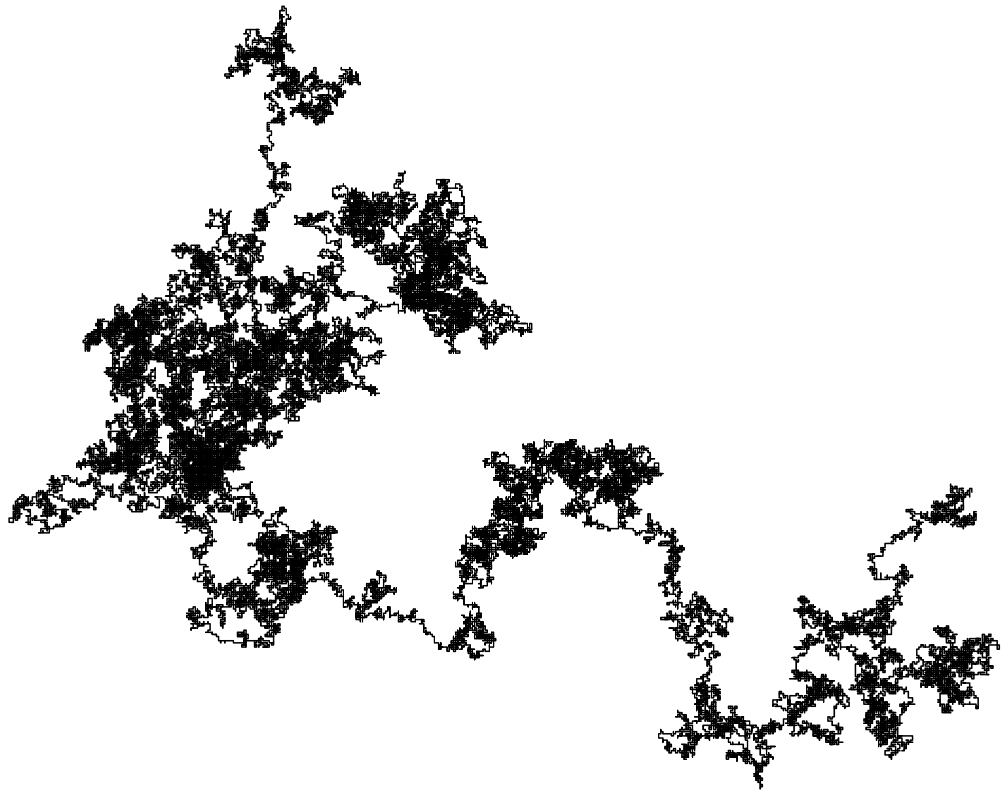
200 step random walk.



1000 step random walk.

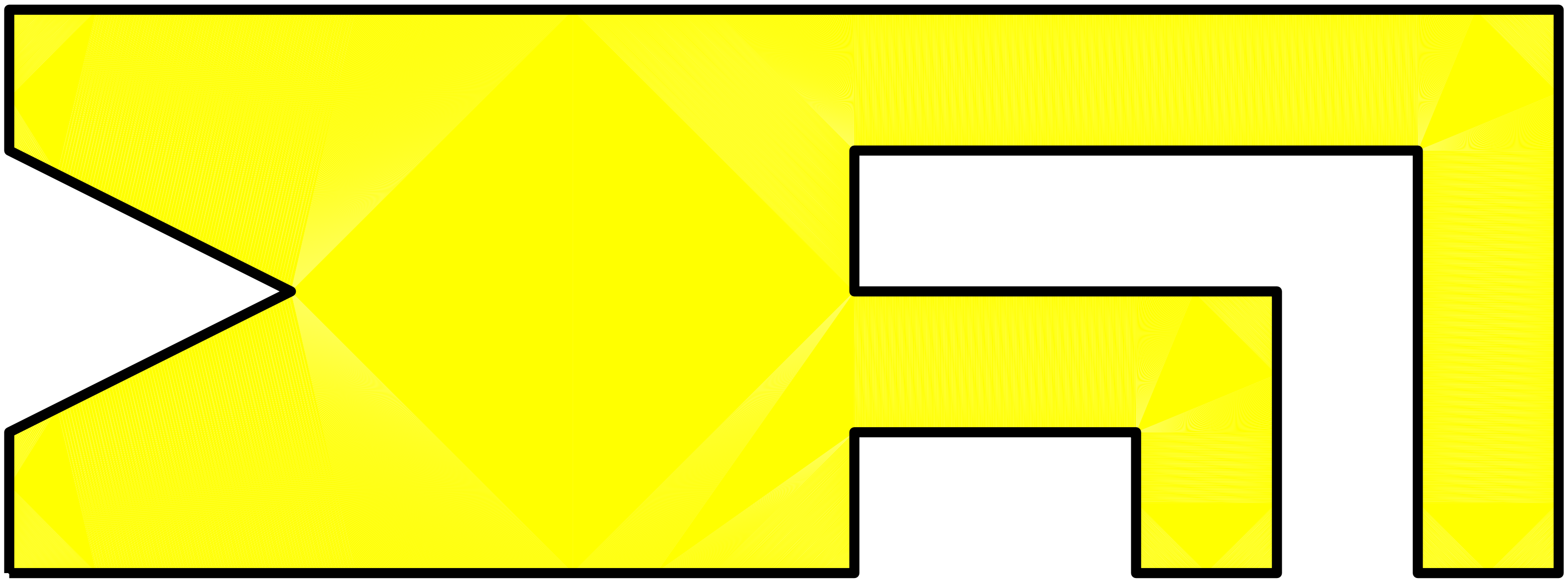


10,000 step random walk.



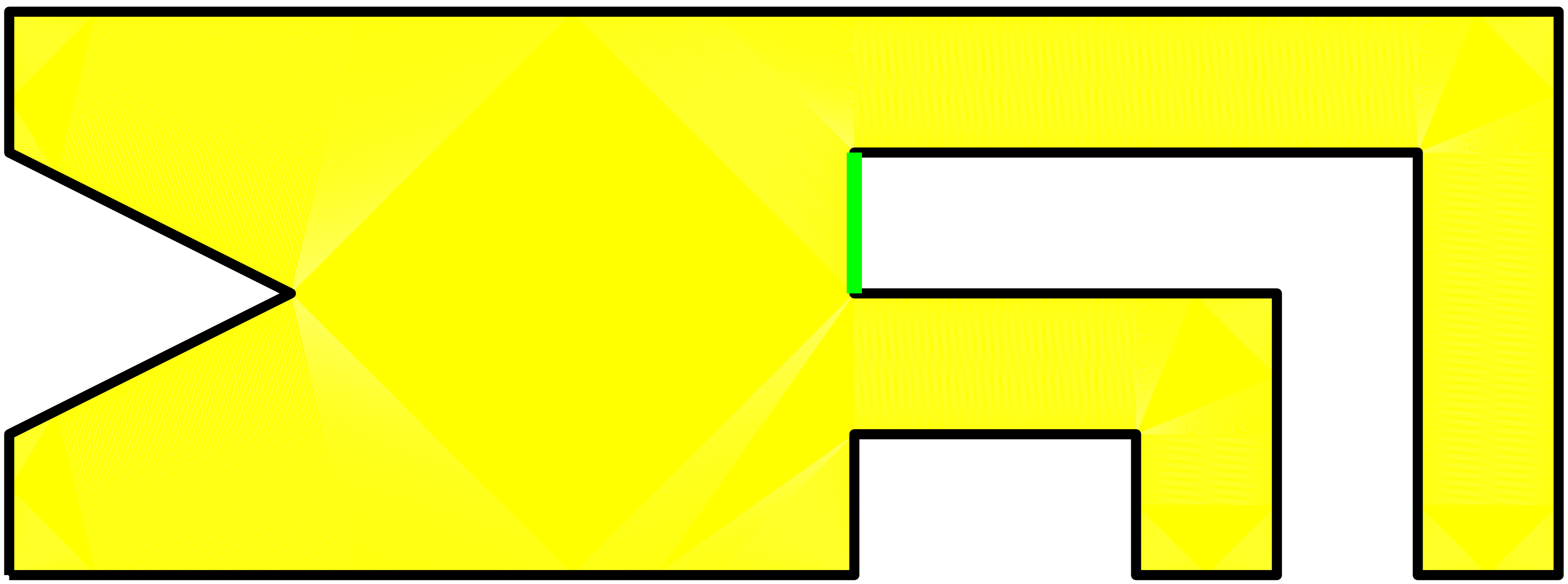
100,000 step random walk.

Harmonic measure = hitting distribution of Brownian motion



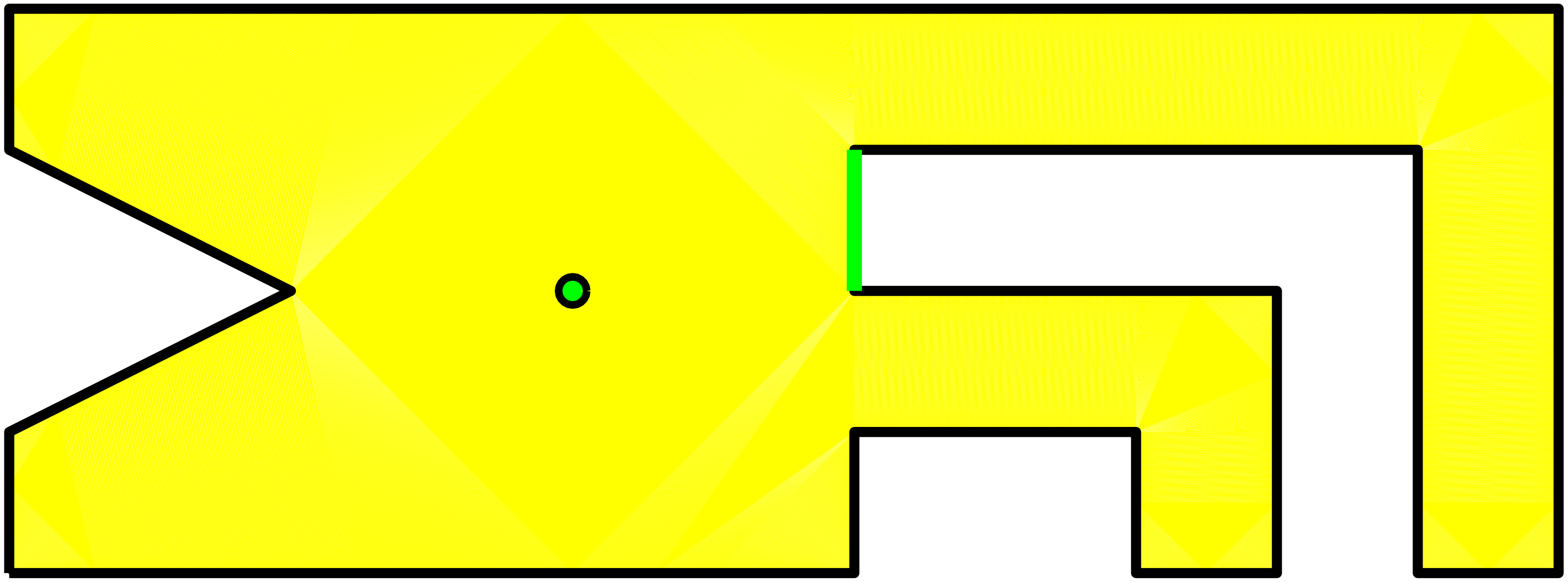
Suppose Ω is a planar Jordan domain.

Harmonic measure = hitting distribution of Brownian motion



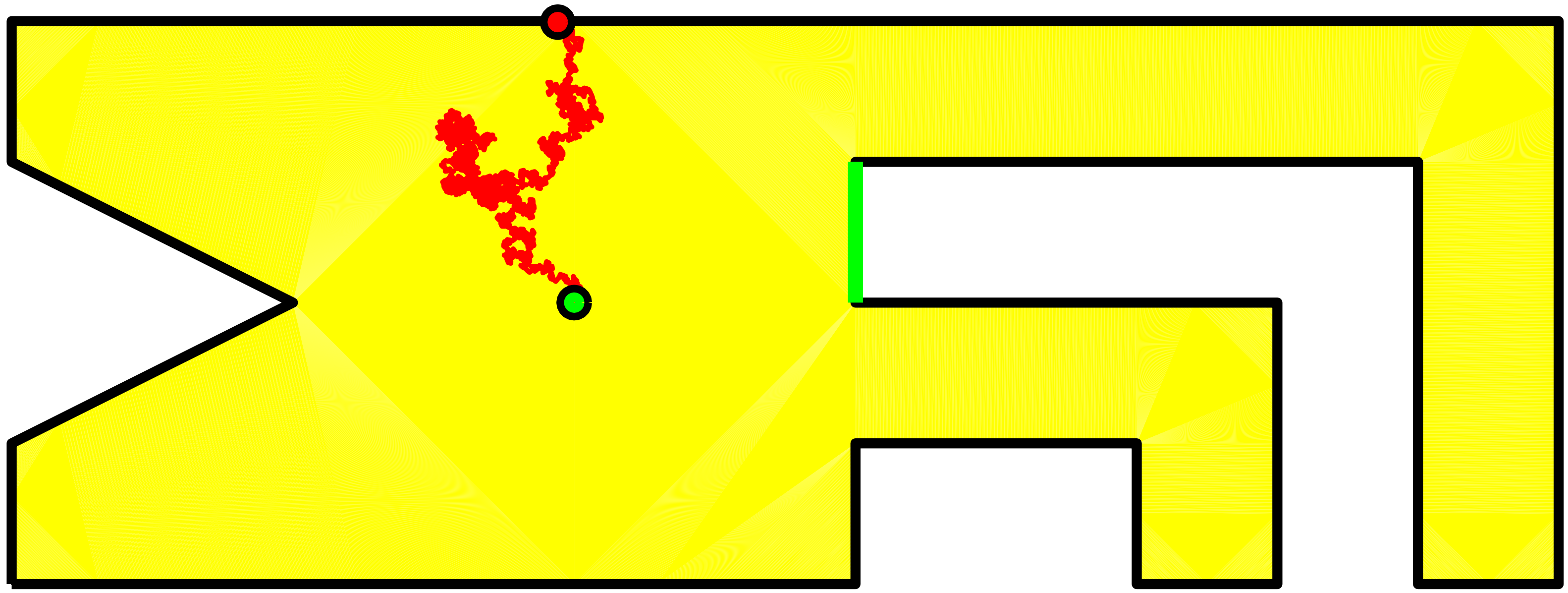
Let E be a subset of the boundary, $\partial\Omega$.

Harmonic measure = hitting distribution of Brownian motion



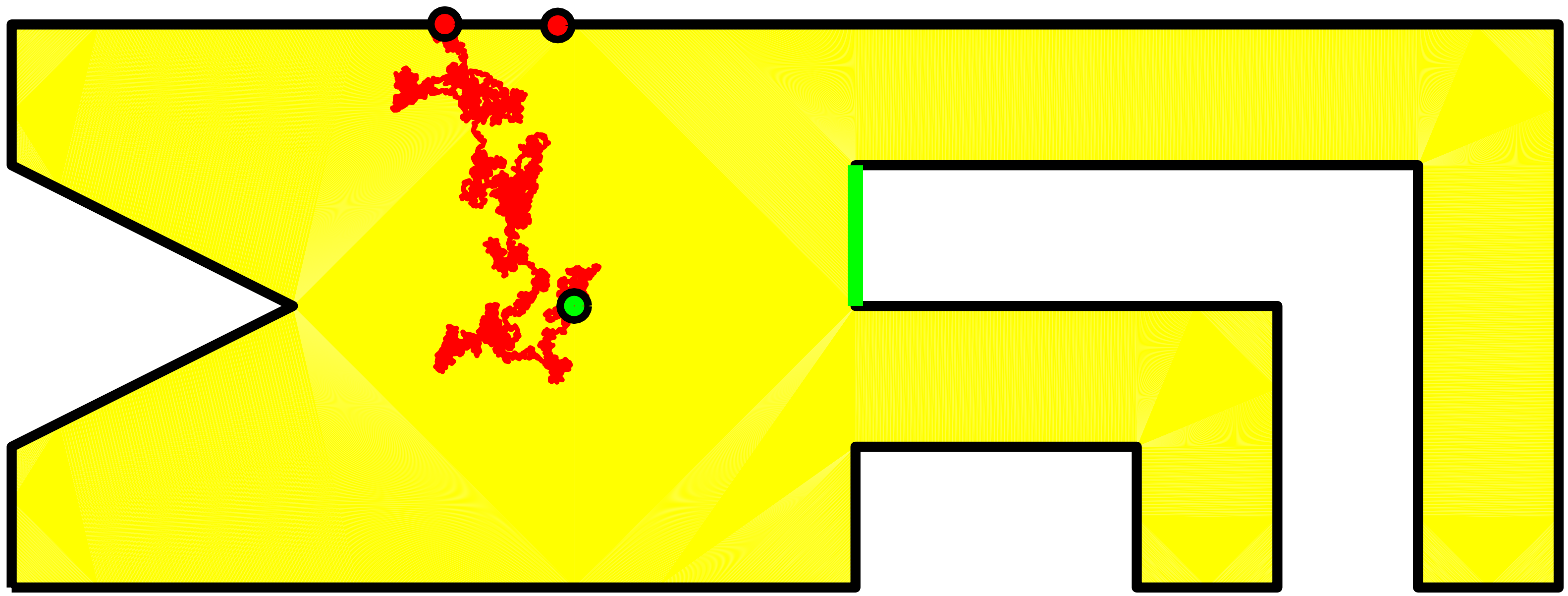
Choose an interior point $z \in \Omega$.

Harmonic measure = hitting distribution of Brownian motion



$\omega(z, E, \Omega)$ = probability a particle started at z first hits $\partial\Omega$ in E.

Harmonic measure = hitting distribution of Brownian motion



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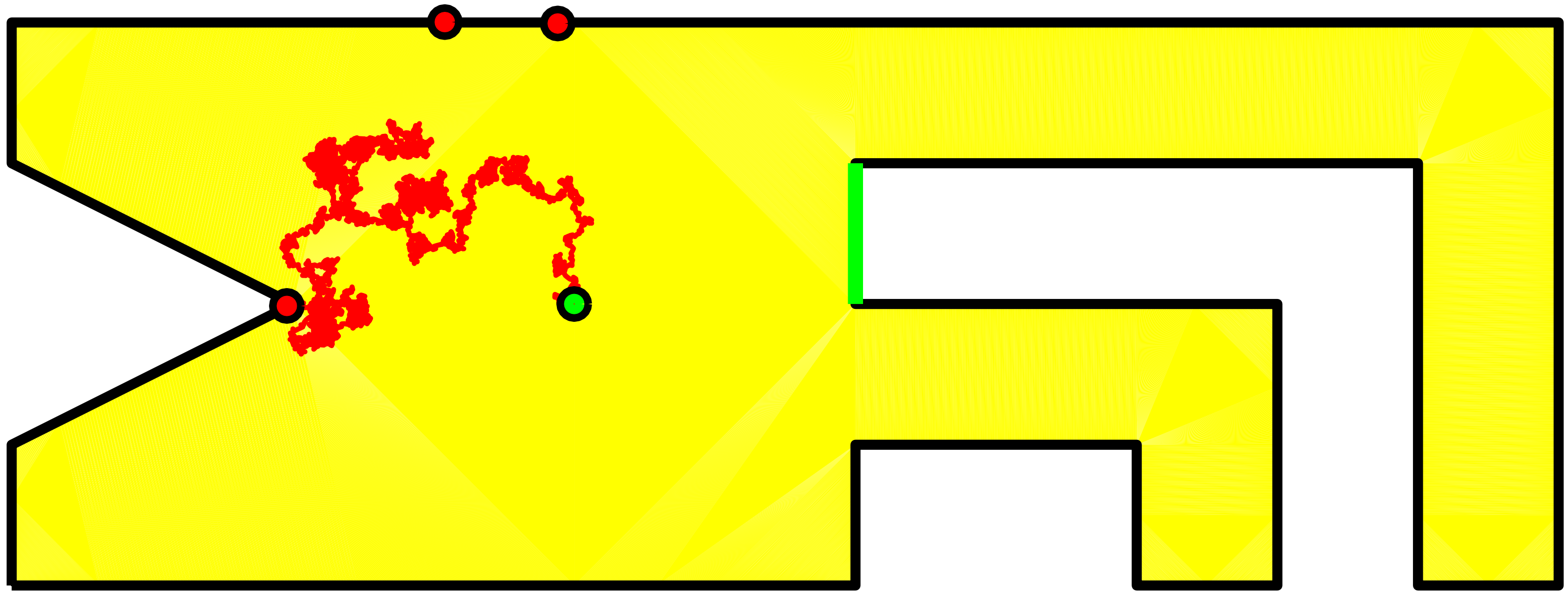

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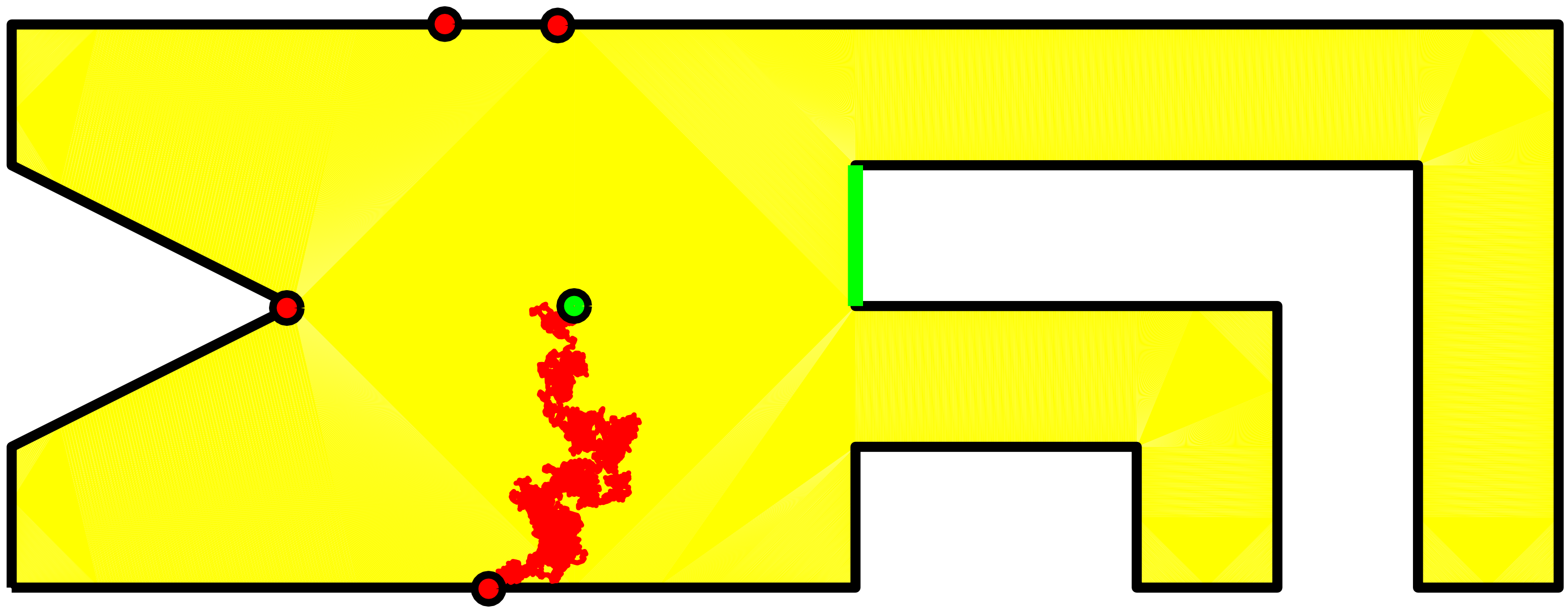

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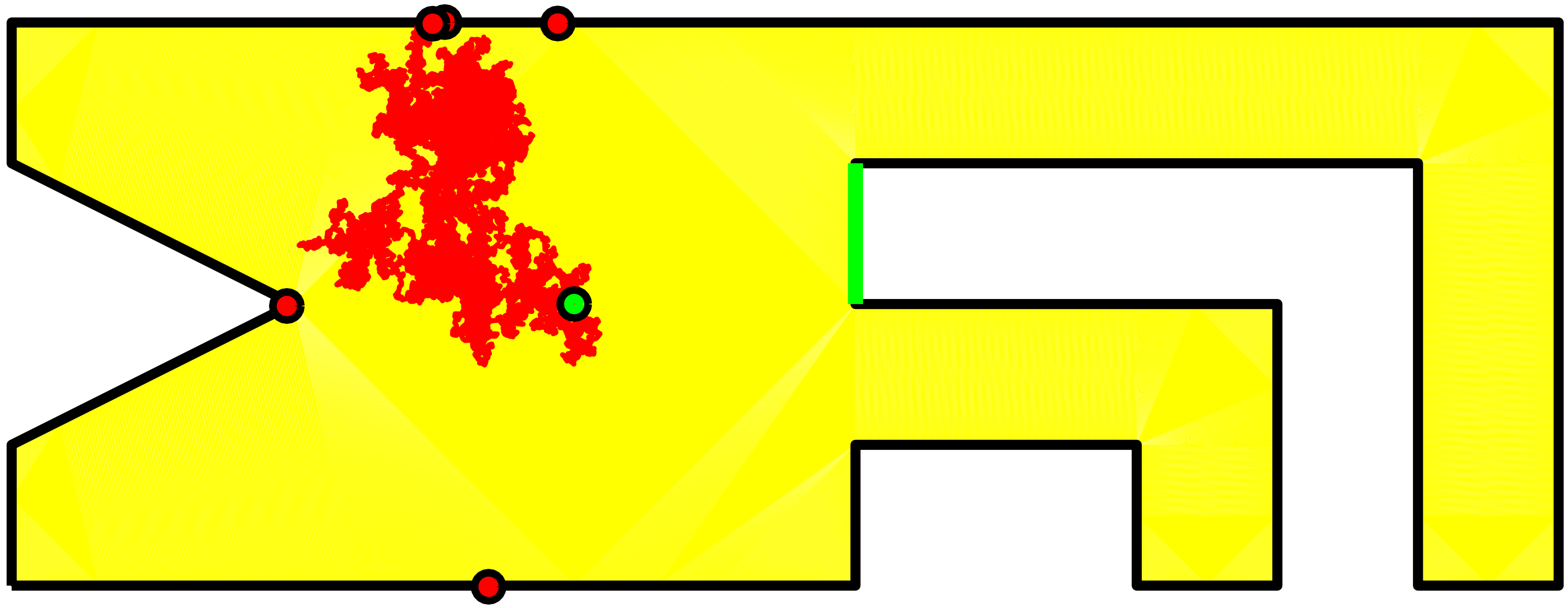

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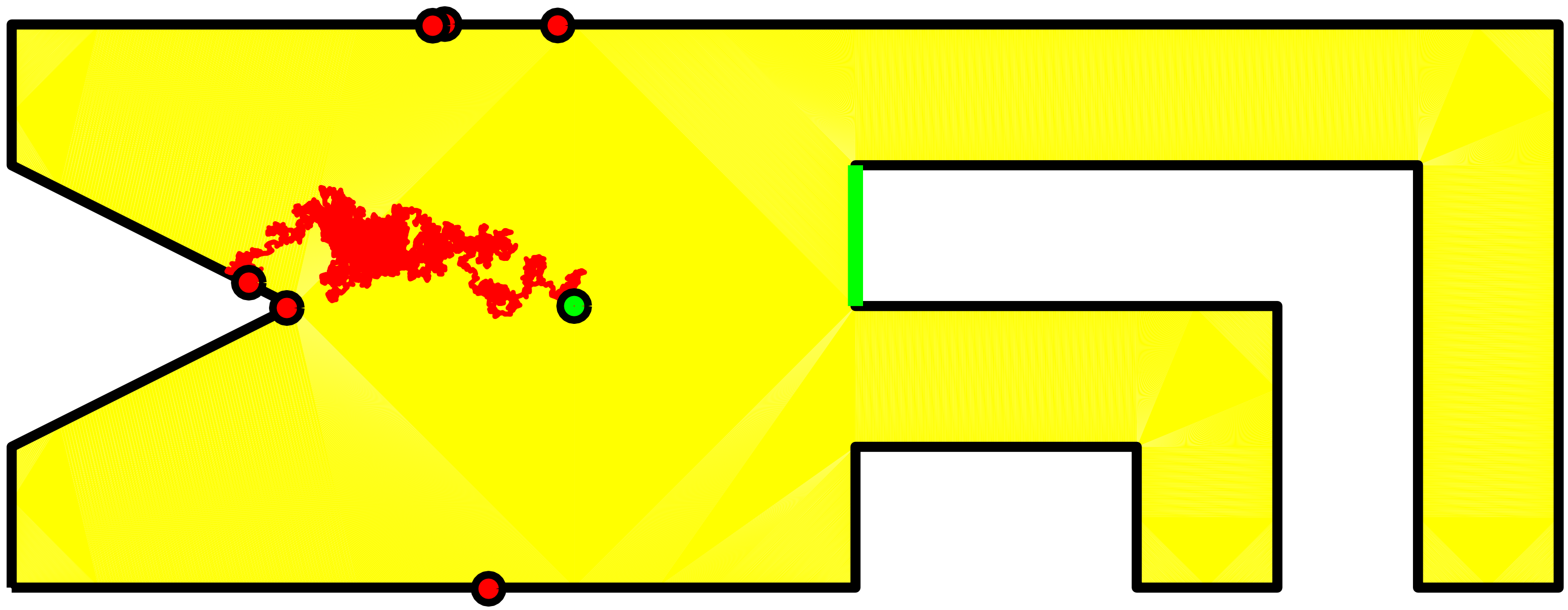

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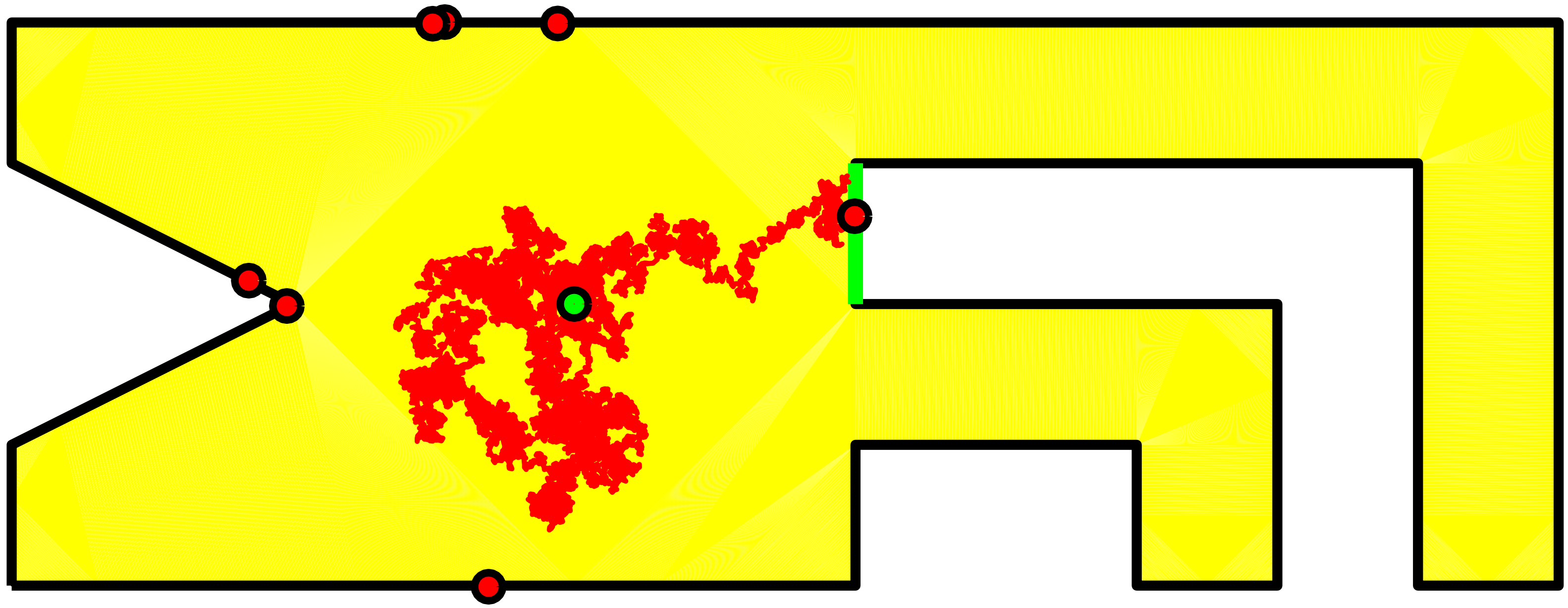

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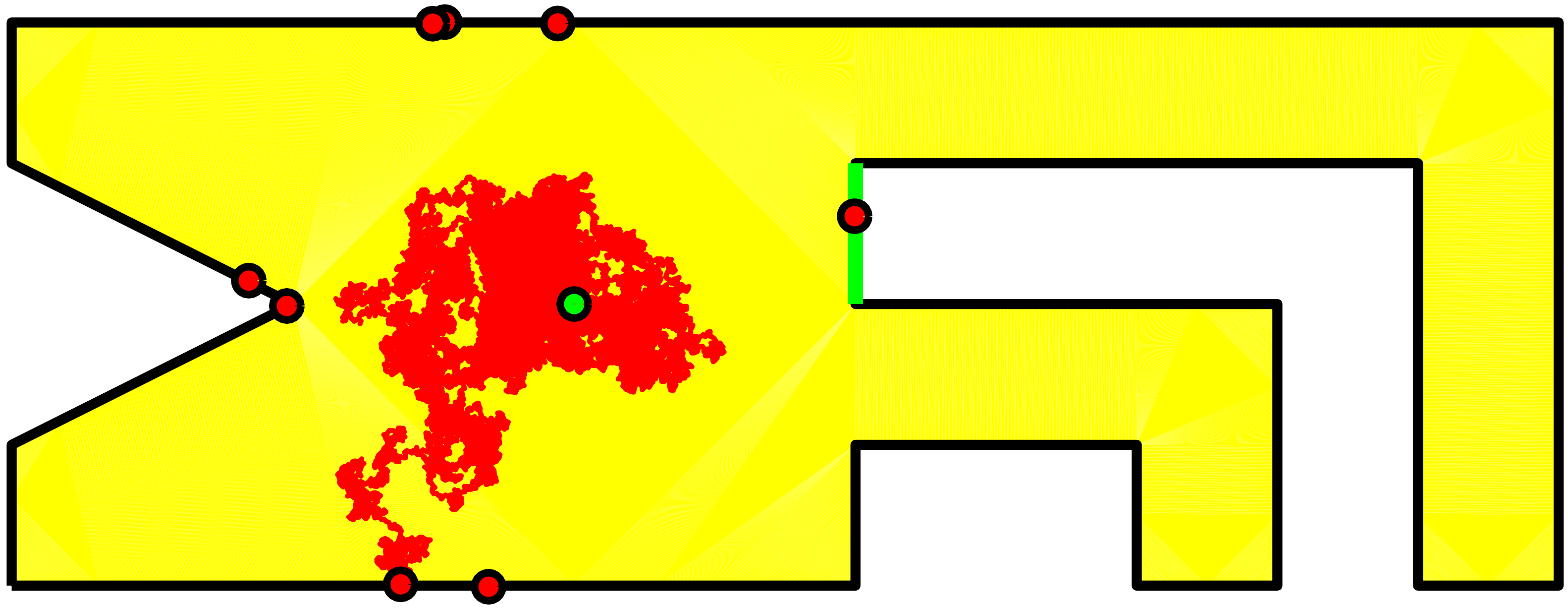

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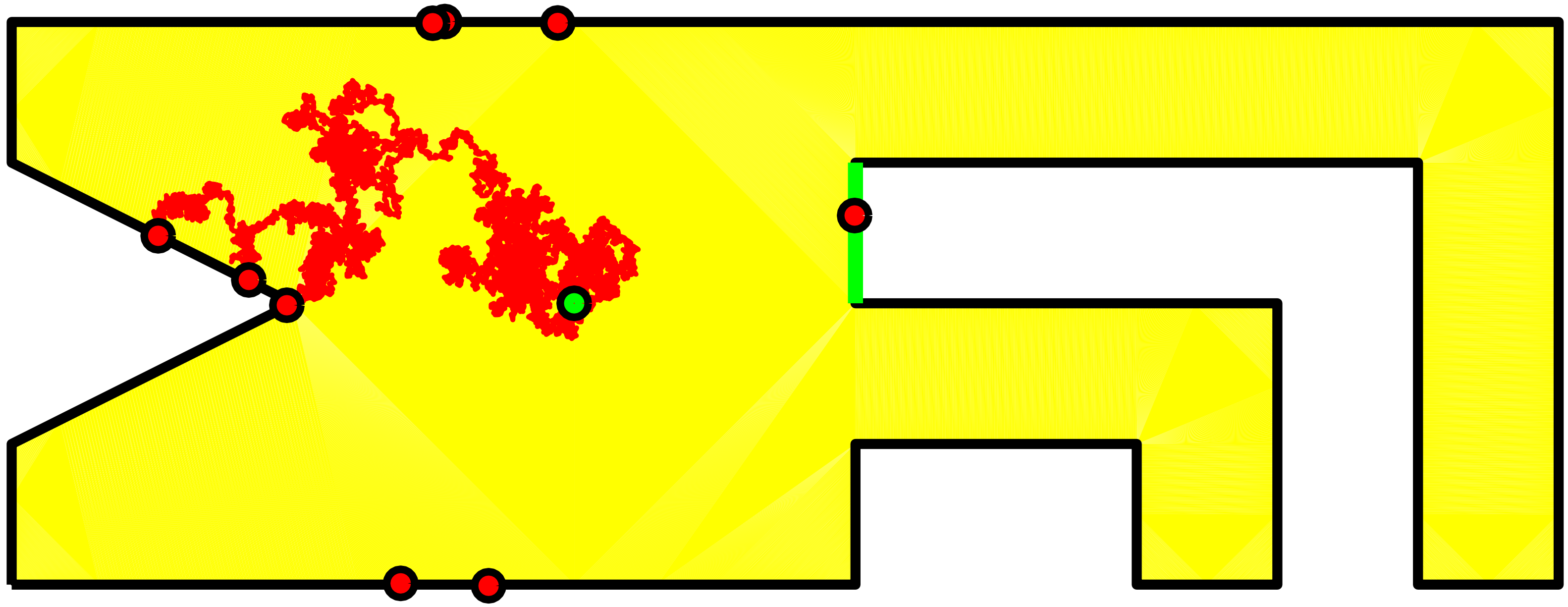

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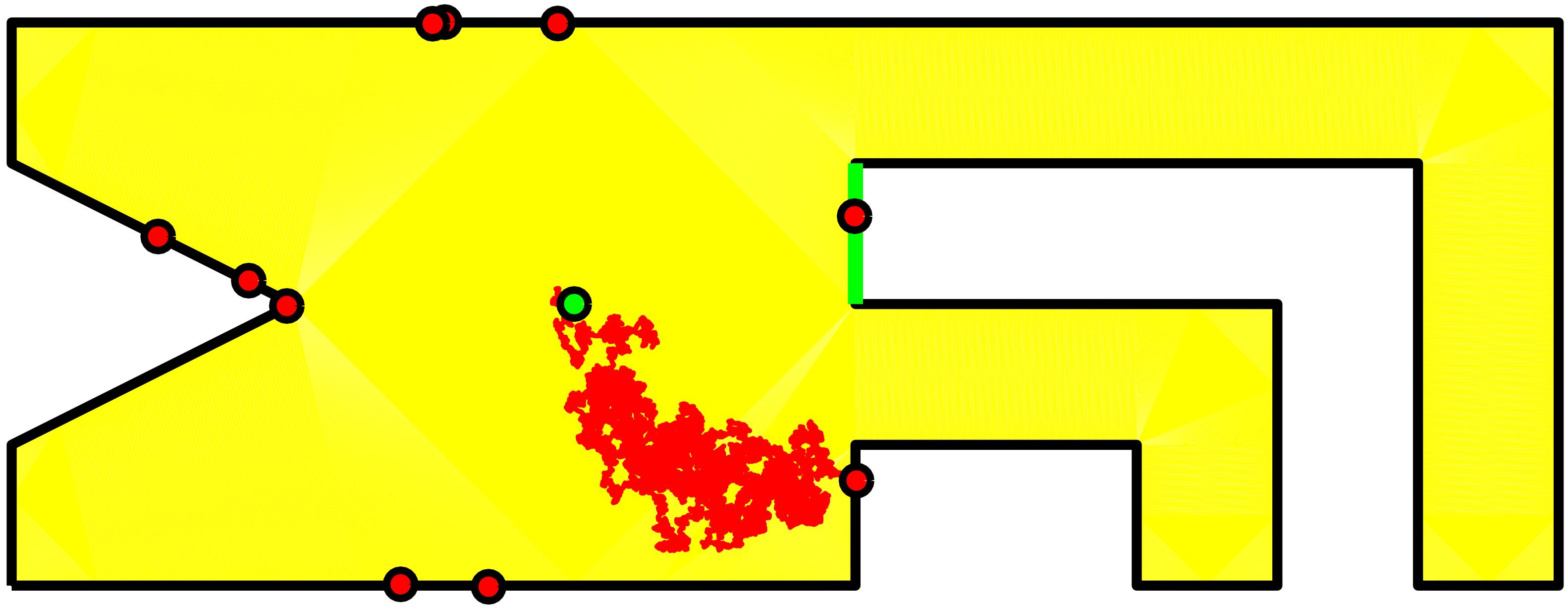
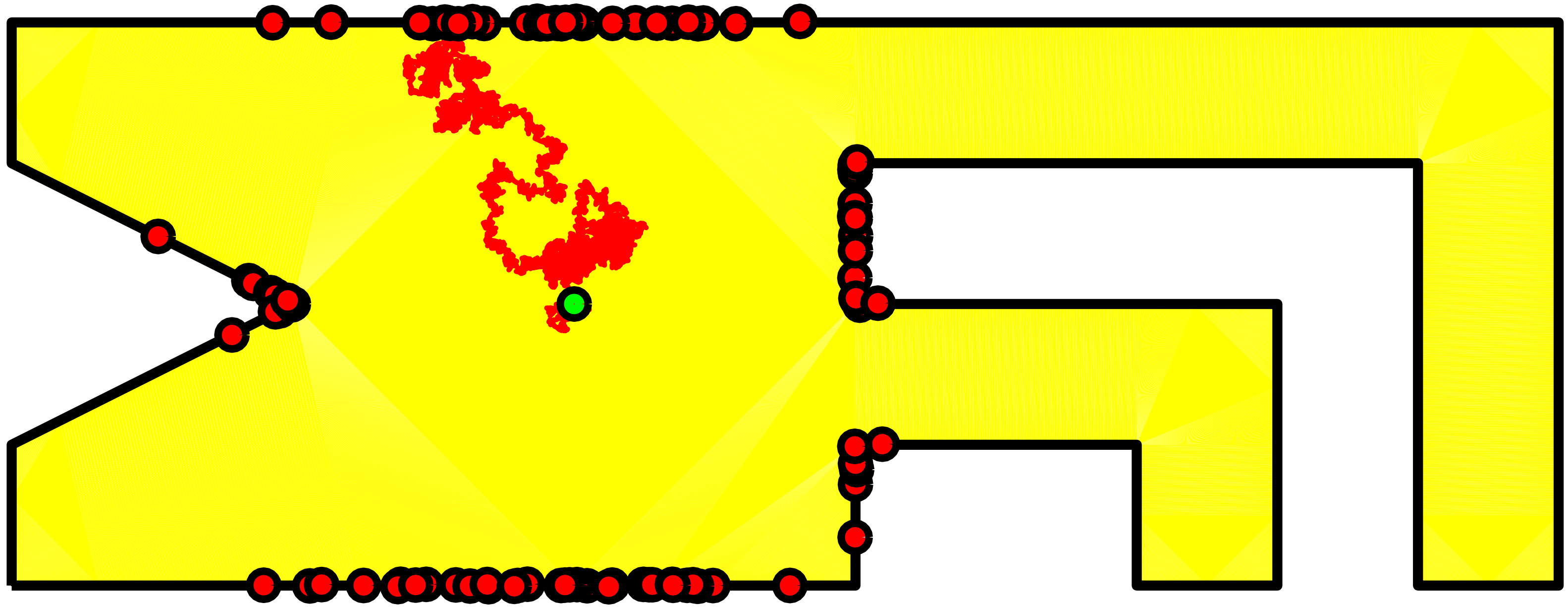


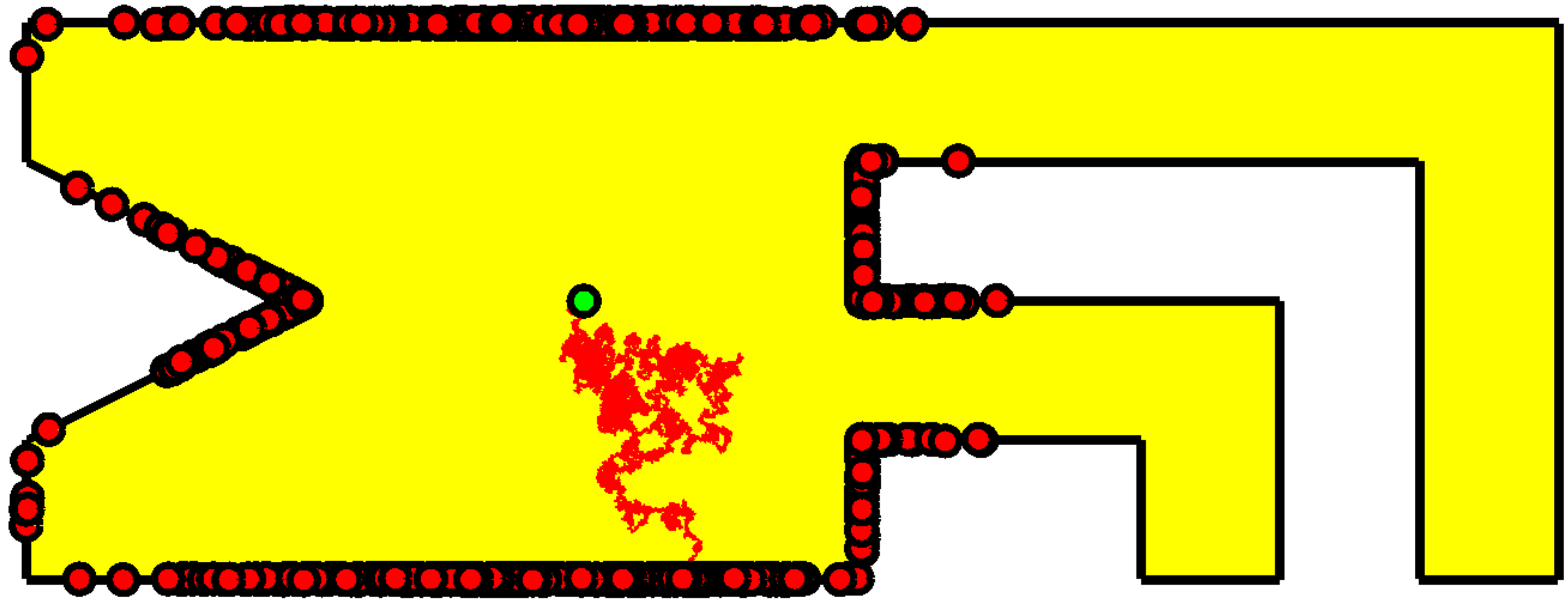
$$\omega(z, E, \Omega) \approx 1/10.$$

Harmonic measure = hitting distribution of Brownian motion



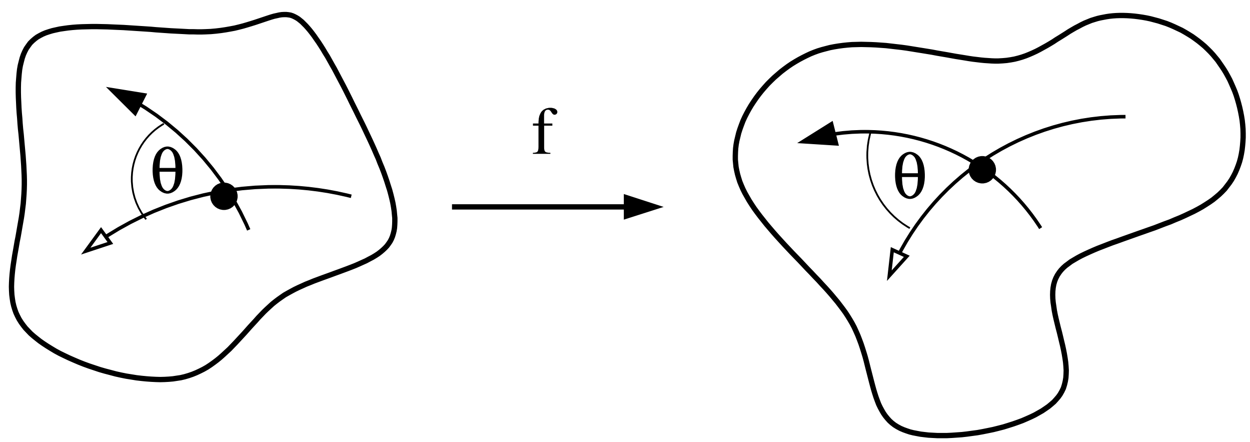
$\omega(z, E, \Omega) \approx 13/100.$

Harmonic measure = hitting distribution of Brownian motion

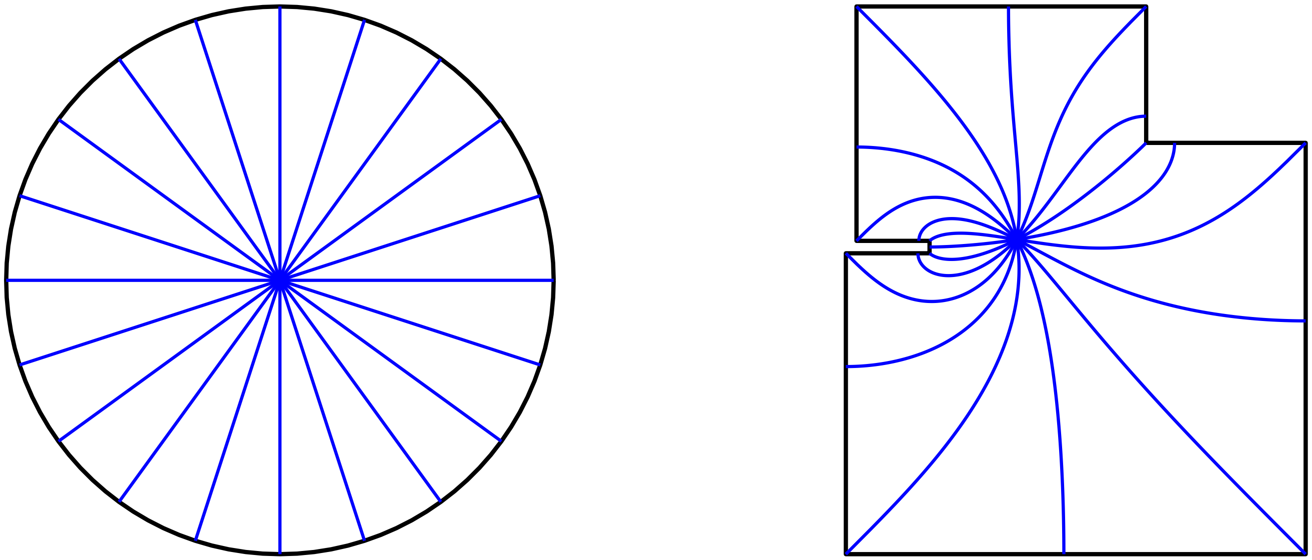


$$\omega(z, E, \Omega) \approx 126/1000.$$

Riemann Mapping Theorem: If $\Omega \subsetneq \mathbb{R}^2$ is simply connected, then there is a conformal map $f : \mathbb{D} \to \Omega$. (conformal = angle preserving)



Brownian motion is conformally invariant, so normalized length measure maps to harmonic measure. Fastest way to compute harmonic measure.



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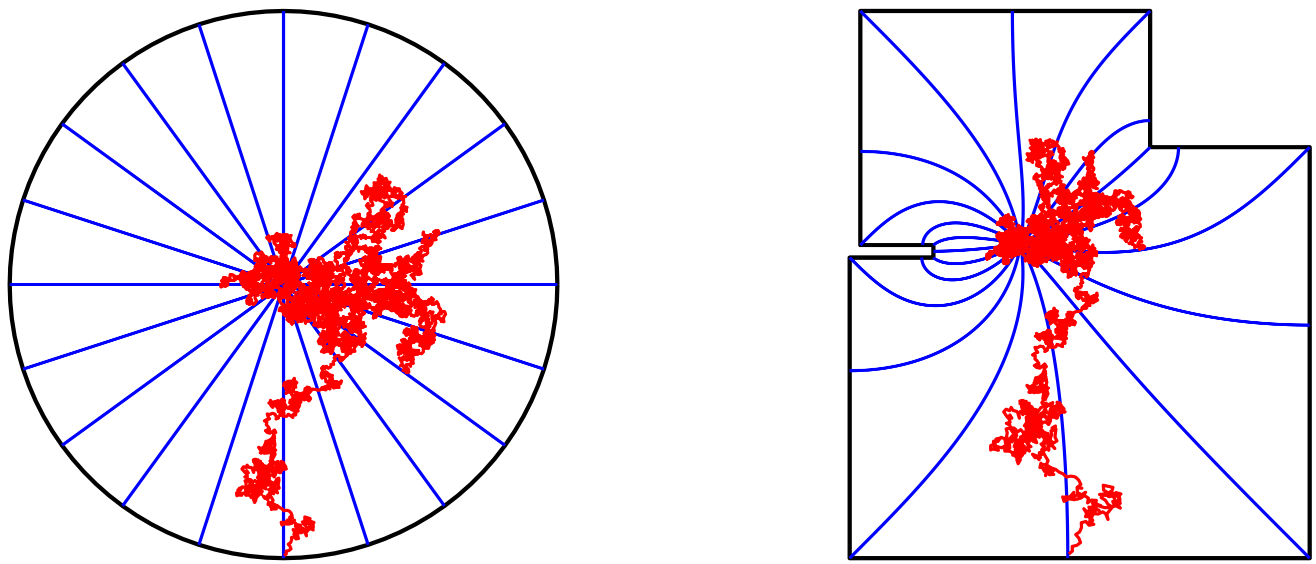

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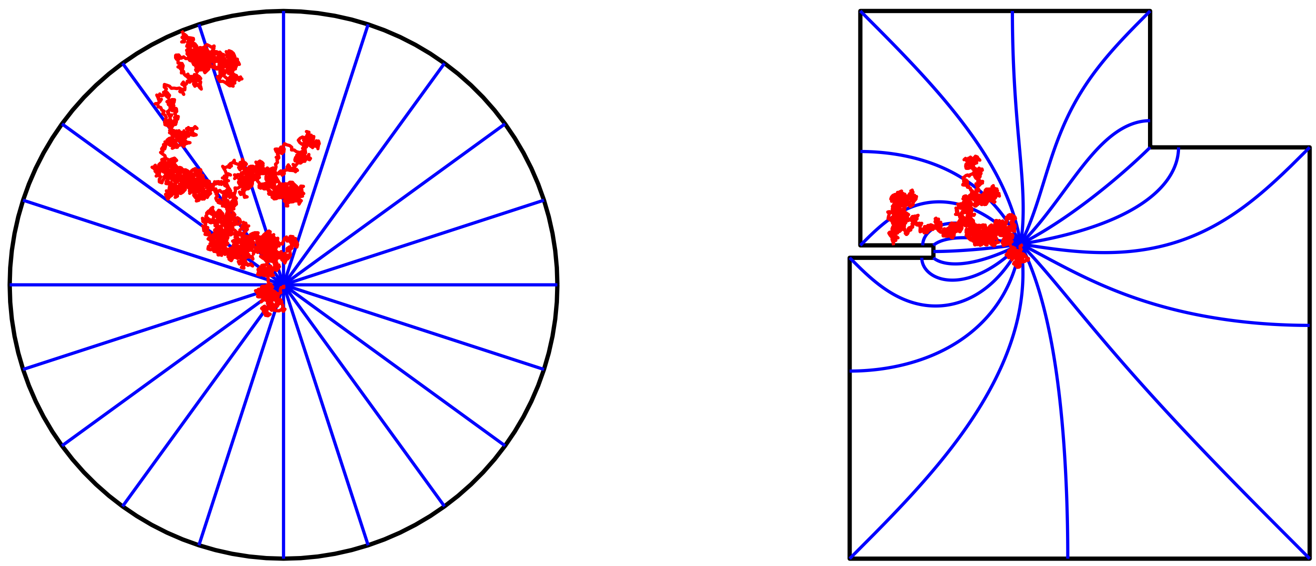

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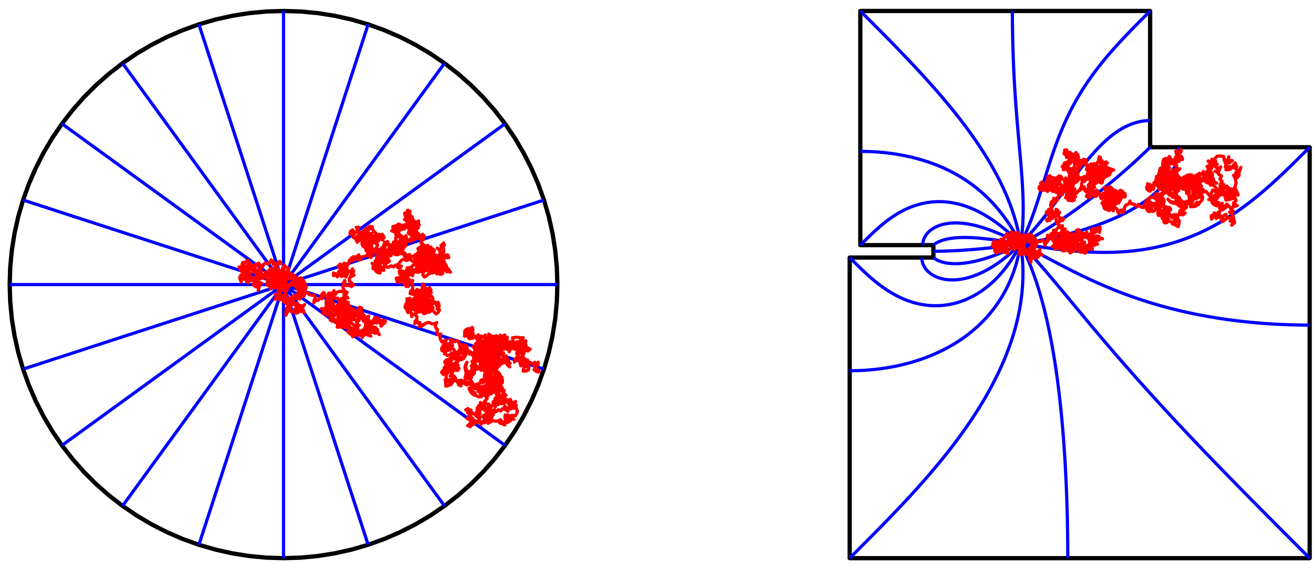
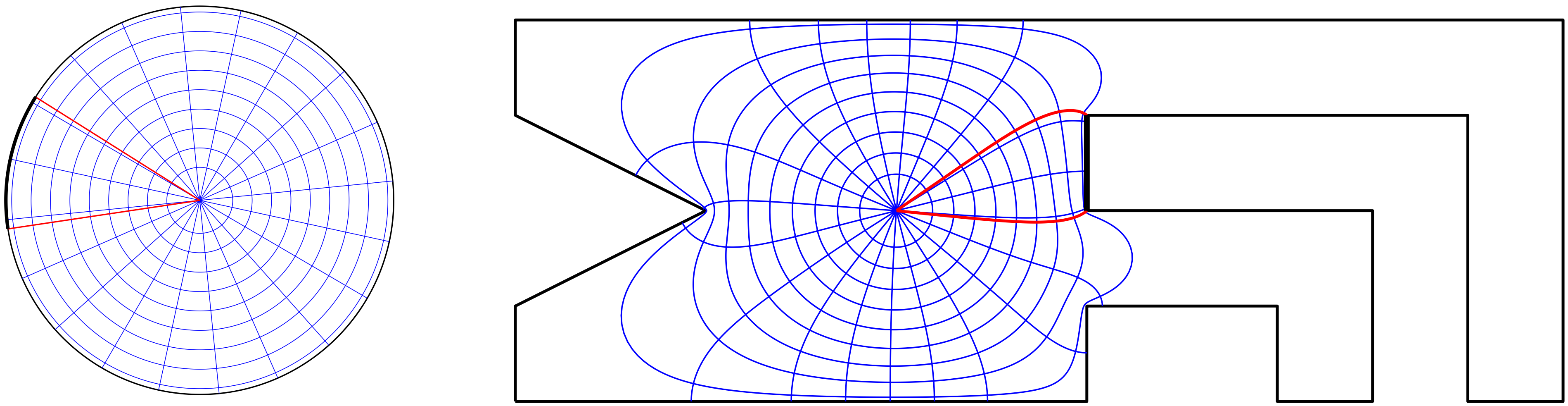
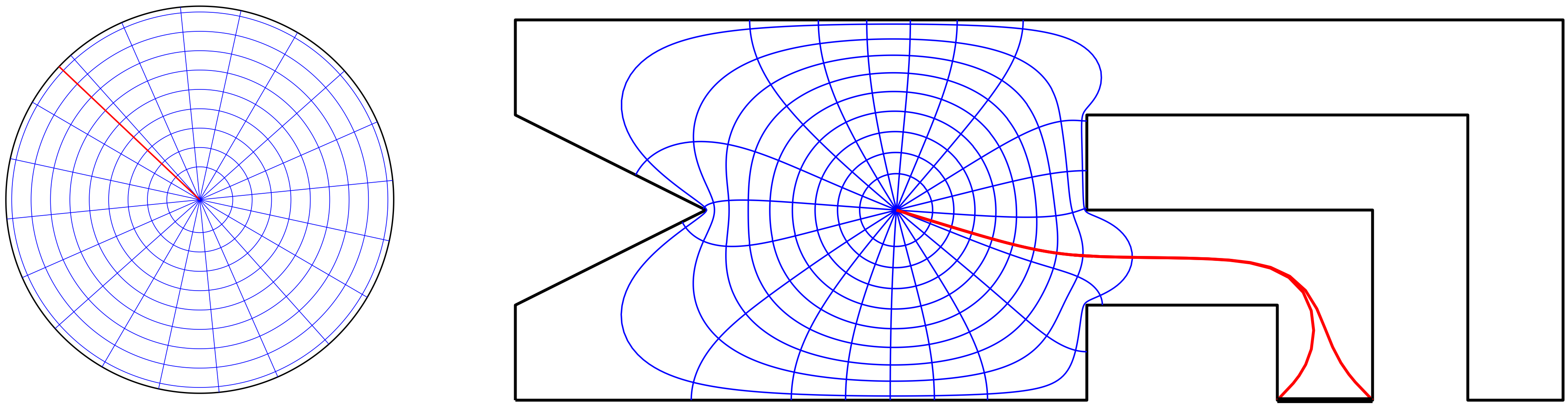




harmonic measure ≈ 0.1128027



harmonic measure $\approx 1.22155 \times 10^{-6}$

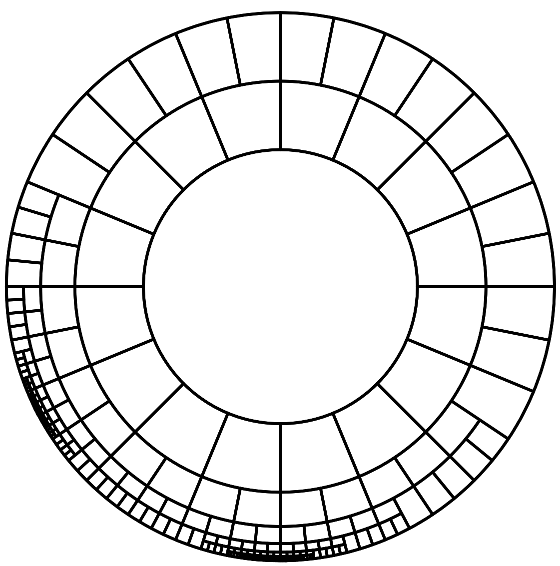
Some basic properties of harmonic measure:

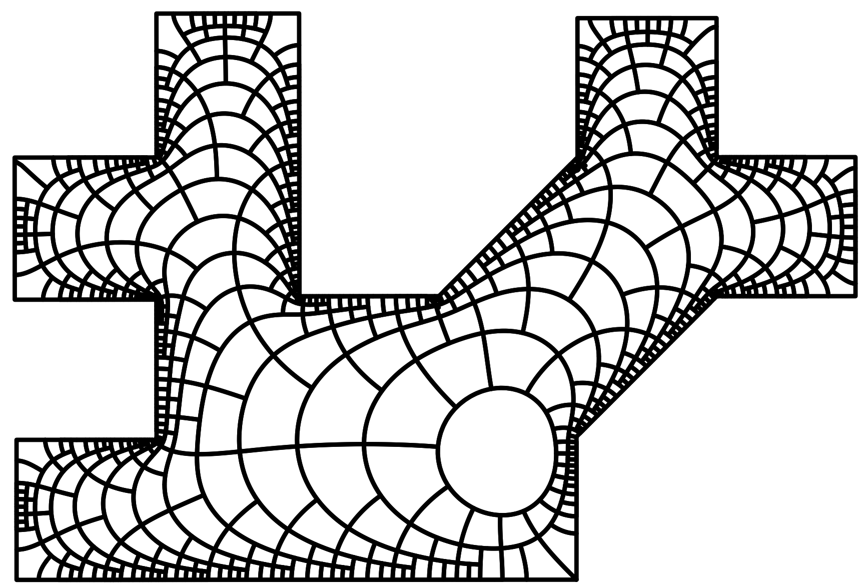
$\omega(z, E, \Omega)$ is the harmonic measure of $E \subset \partial\Omega$.

ω is harmonic in z and $0 \leq \omega \leq 1$

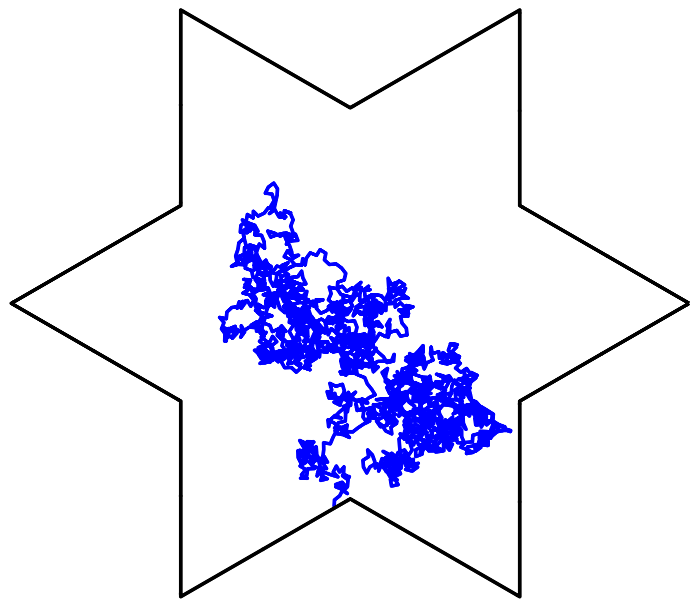
A harmonic function attaining its min or max is constant.

So a set E has positive measure for one point iff for all points.





Thm (F & M Riesz 1916):
For rectifiable boundaries, $\omega(E) = 0$ iff E has zero length.



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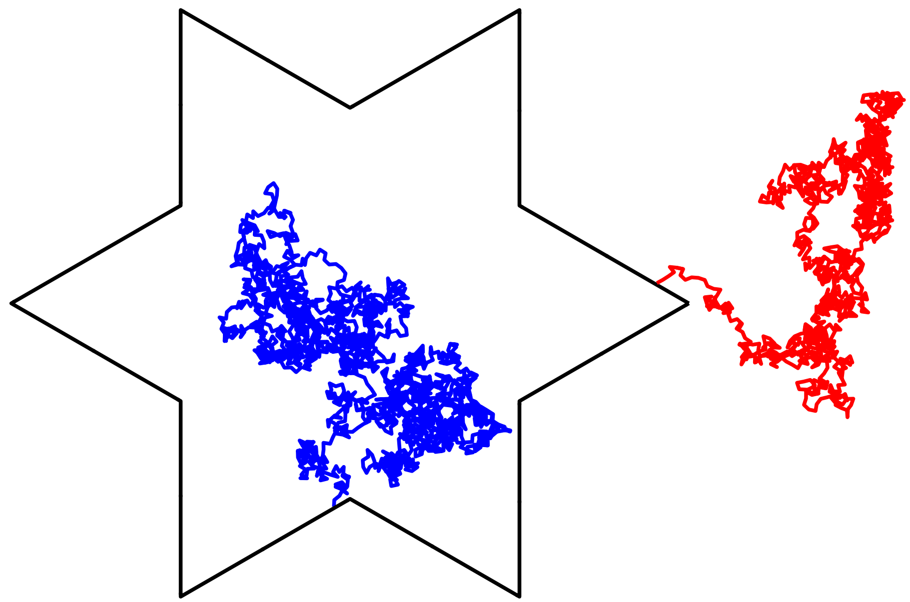
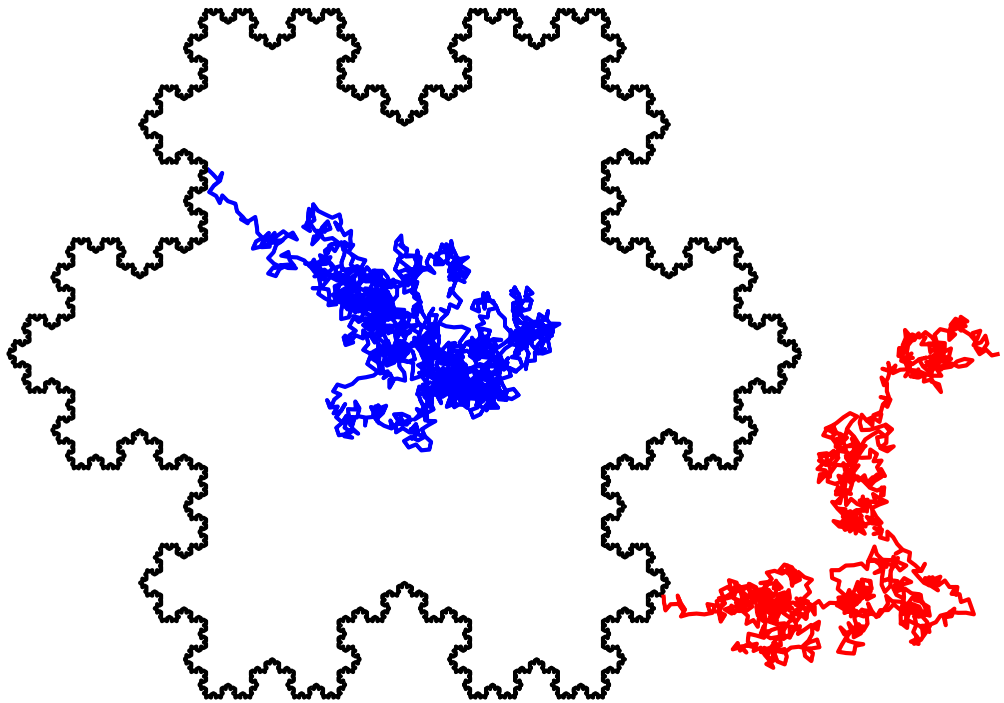


"Inside" and "outside" harmonic measures have same null sets.

Measures are mutually absolutely continuous. Same measure class.

For a fractal curve, inside and outside harmonic measures are singular.

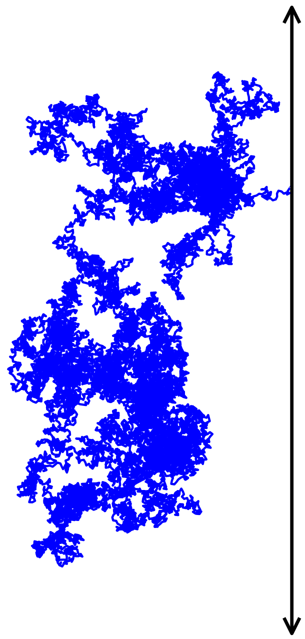


$\omega_1 \perp \omega_2$ iff tangents points have zero length.

(B. 1987, based on results of N.G. Makarov)

For which curves is $\omega_1 = \omega_2$?

For which curves is $\omega_1 = \omega_2$? True for lines:



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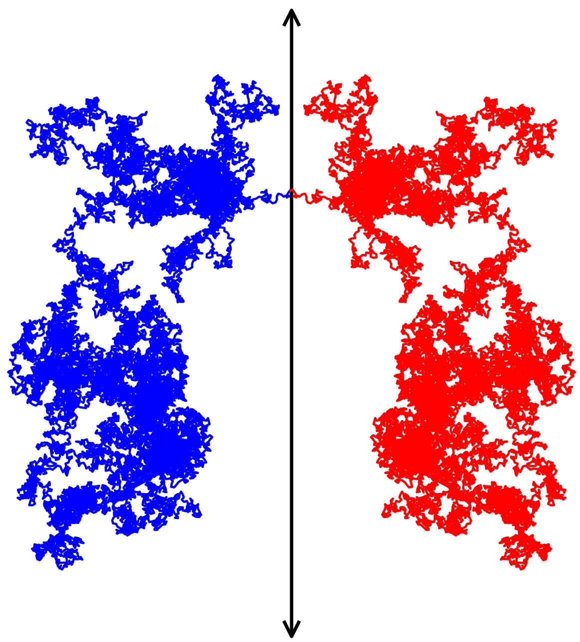

Also for circles (= lines conformally).

Converse? If $\omega_1 = \omega_2$ must Γ be a circle/line?

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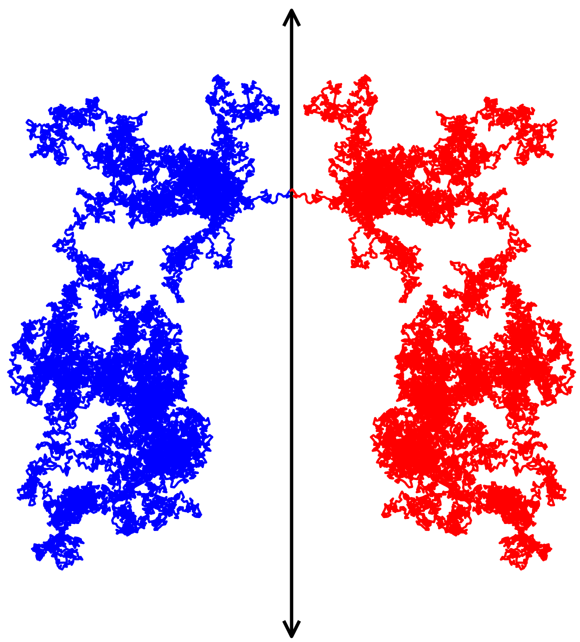
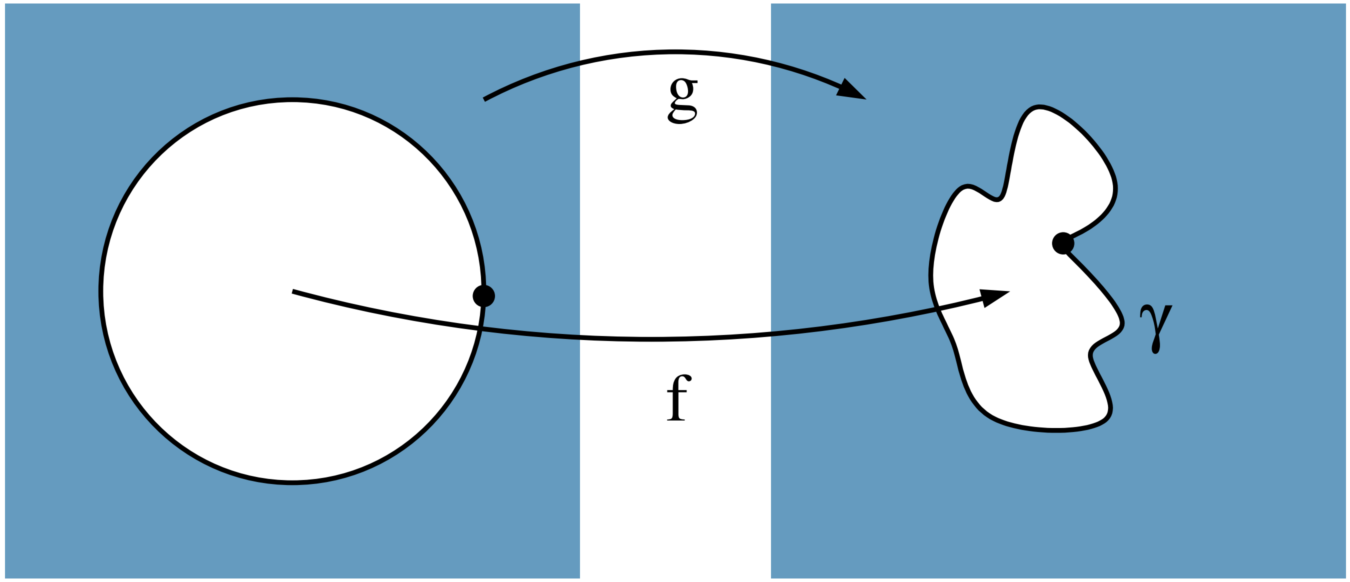


Also for circles (= lines conformally).

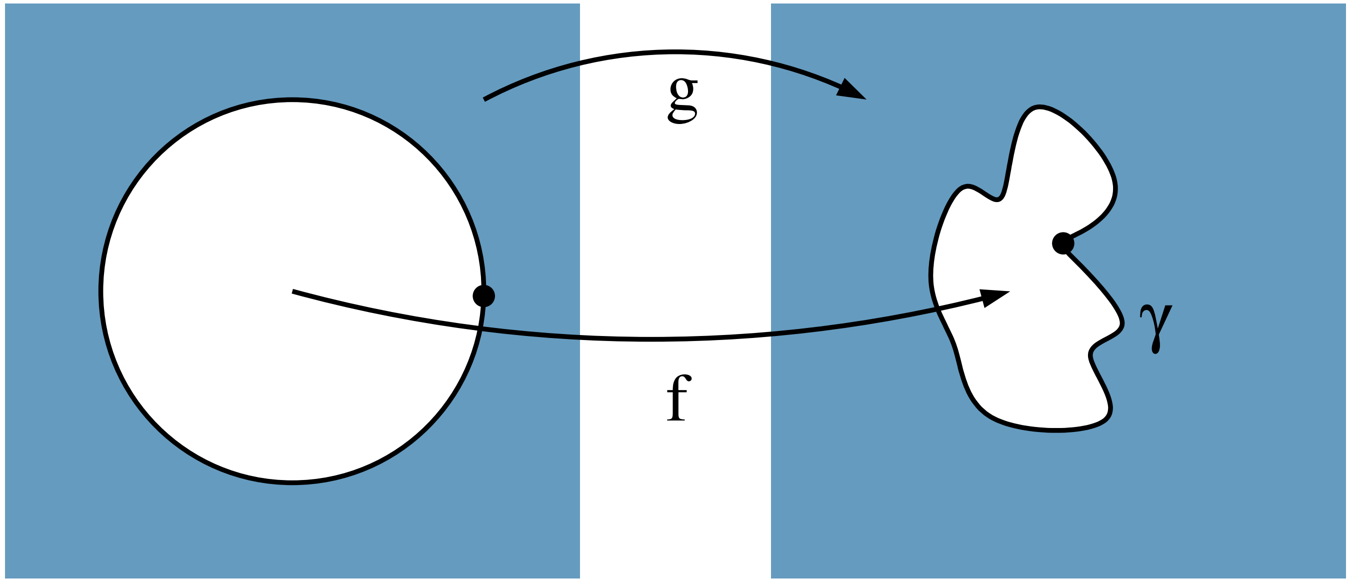
Converse? If $\omega_1 = \omega_2$ must Γ be a circle/line? **Yes**



Suppose $\omega_1 = \omega_2$ for a curve γ.

Conformally map two sides of circle to two sides of γ so $f(1) = g(1)$.

$\omega_1 = \omega_2$ implies maps agree on whole boundary.



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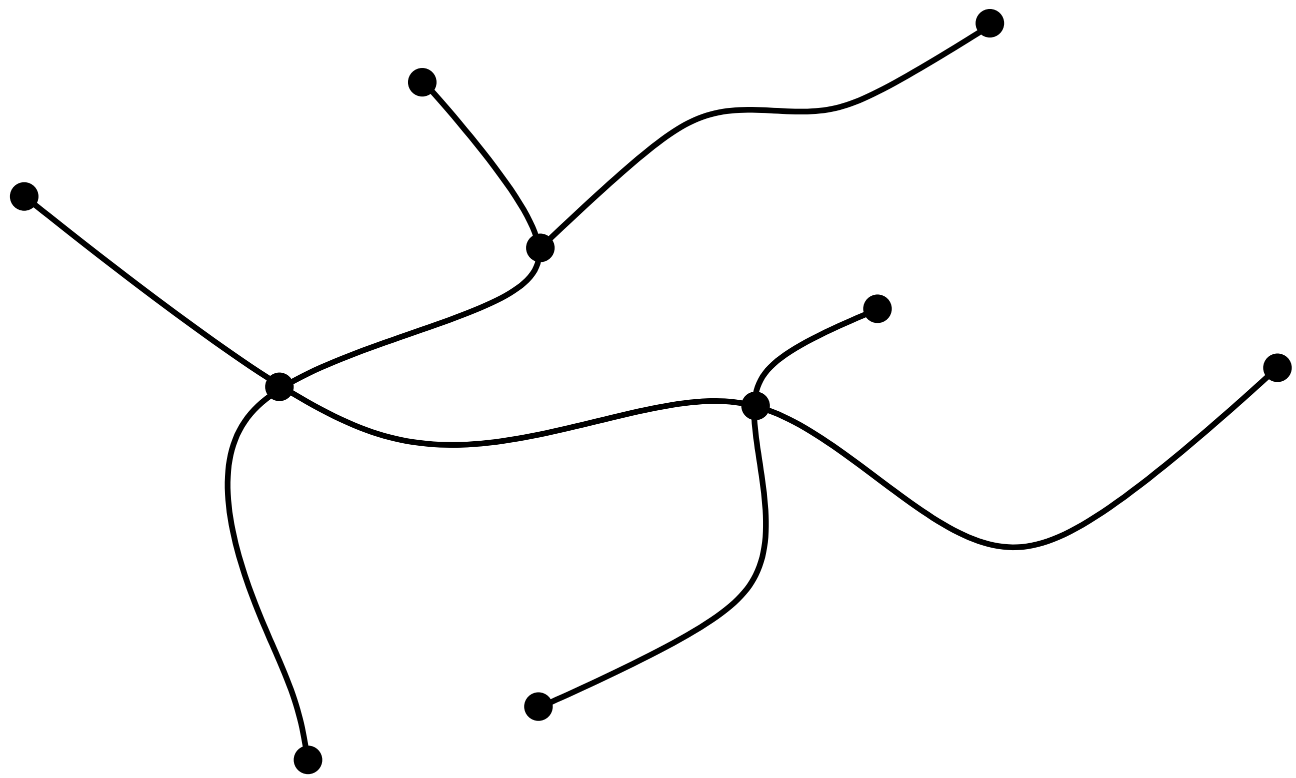
So f, g define homeomorphism h of plane holomorphic off circle.

Then h is entire by Morera's theorem.

Entire and 1-1 implies h is linear (Liouville's thm), so γ is a circle.

Circles are only closed curves with equal harmonic measures on both sides.

What about other kinds of 2-sided objects?



A planar graph is a finite set of points connected by non-crossing edges.

It is a tree if there are no closed loops.

A planar tree is **conformally balanced** if

- every edge has equal harmonic measure from ∞
- edge subsets have same measure from both sides

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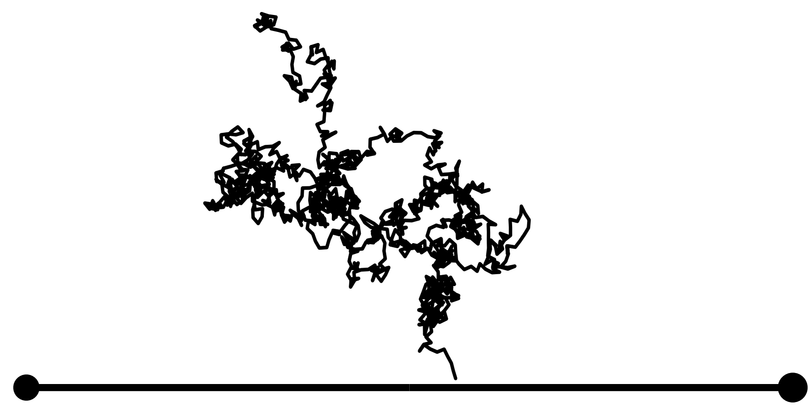

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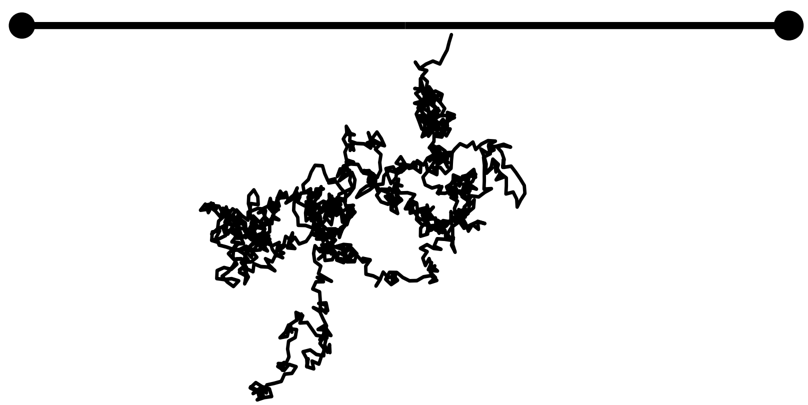
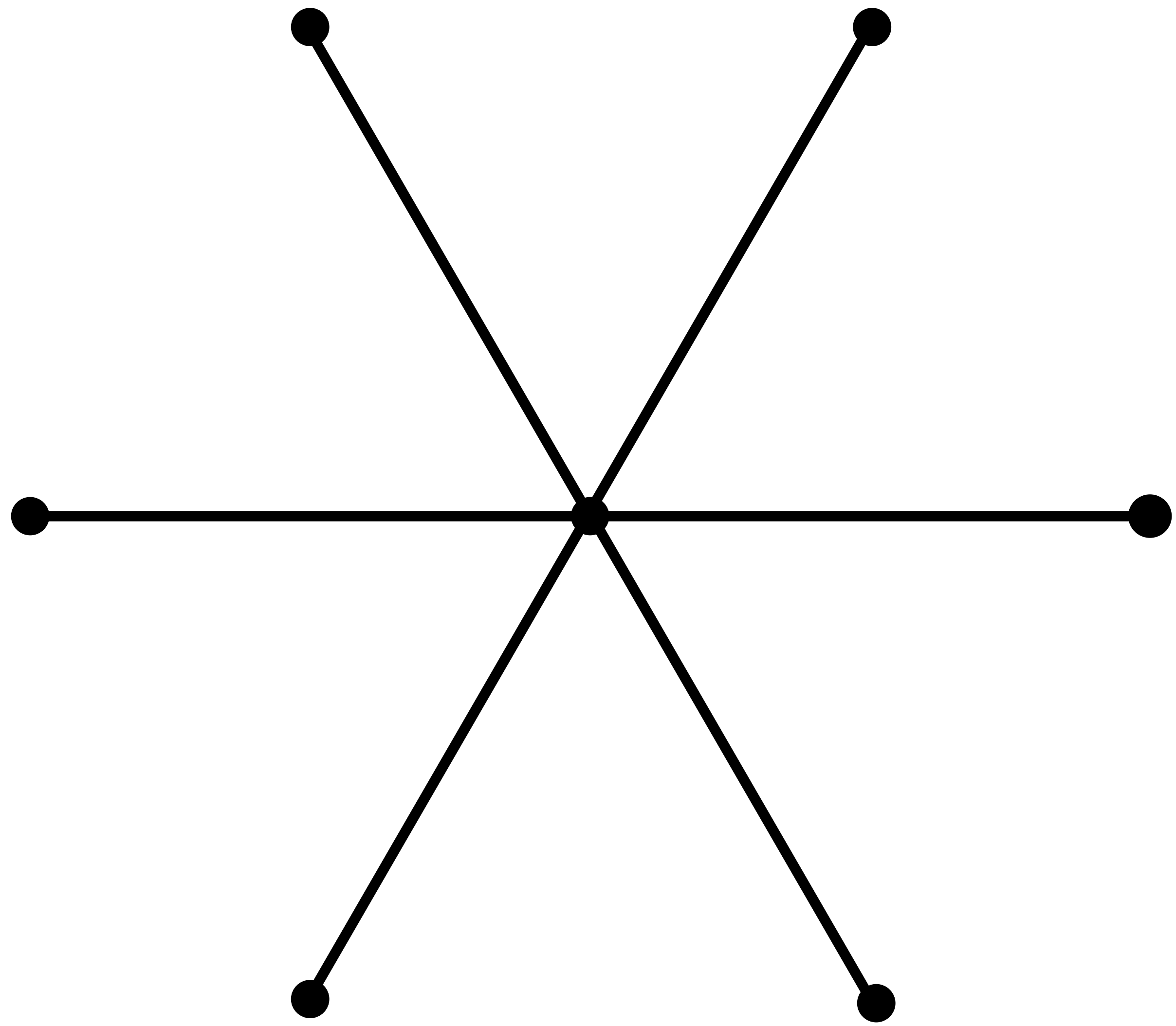
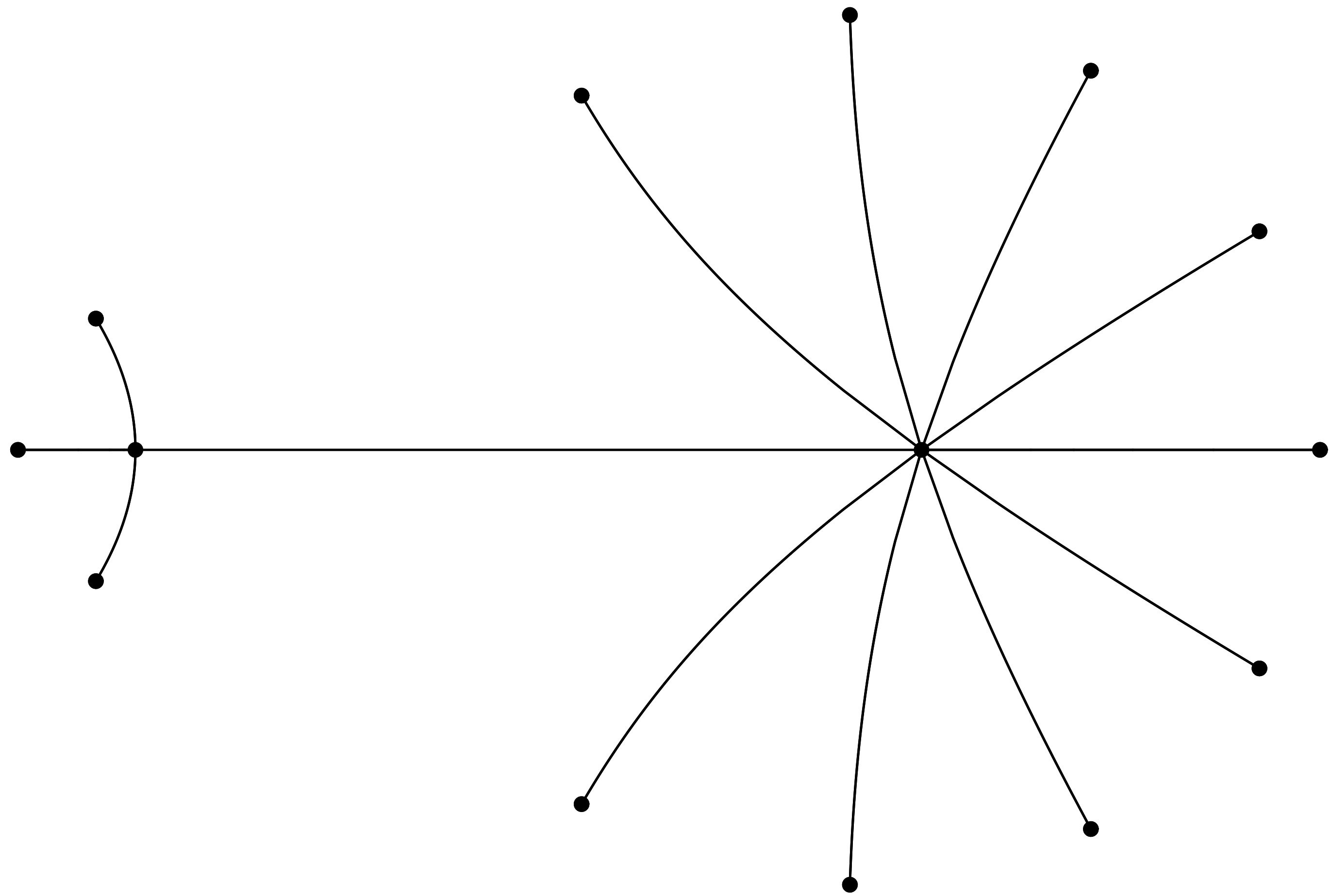




Trivially true by symmetry

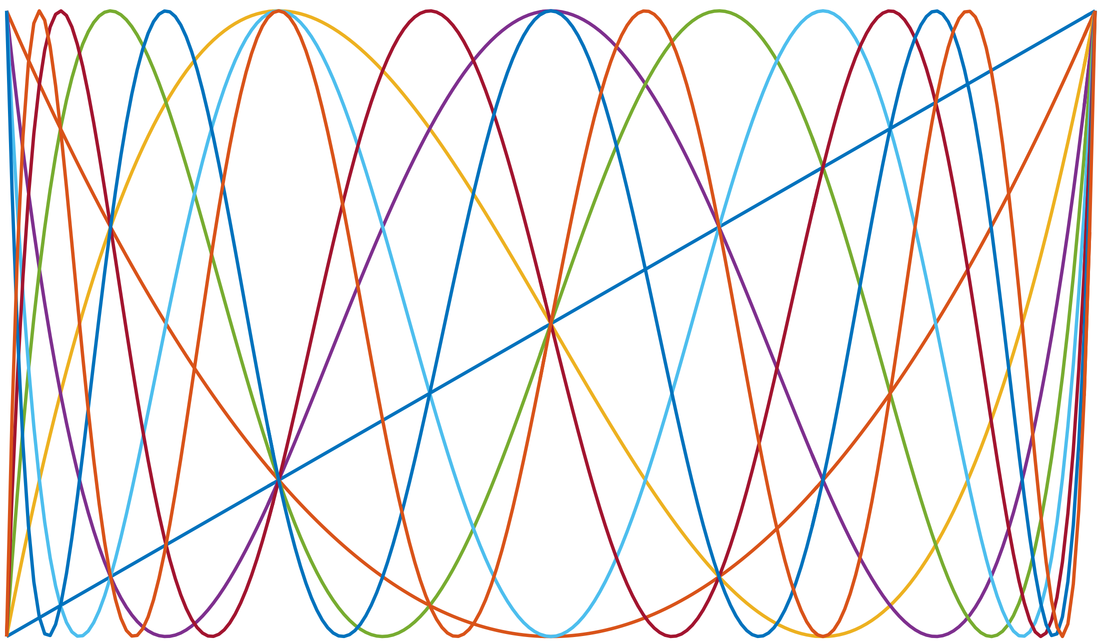


Non-obvious true tree

Definition of critical value: if p = polynomial, then

$$\mathrm{CV}(p) = \{p(z) : p'(z) = 0\} = \text{critical values}$$

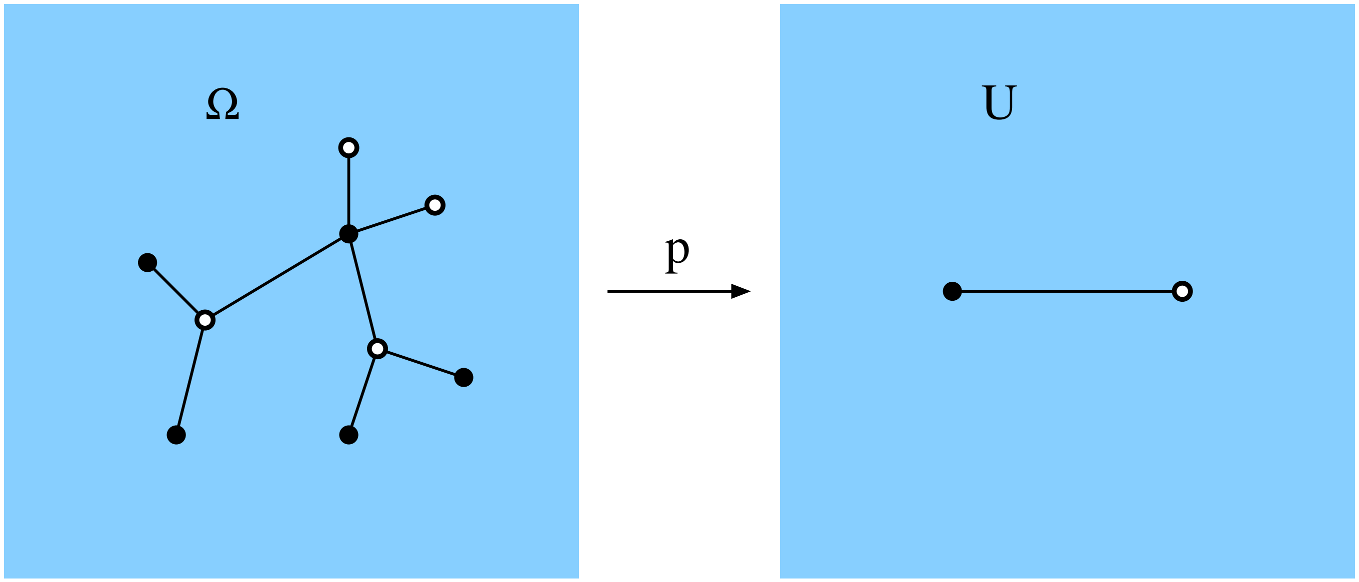
If $\mathrm{CV}(p) = \pm 1$, p is called **generalized Chebyshev** or **Shabat**.



10 classical Chebyshev polynomials

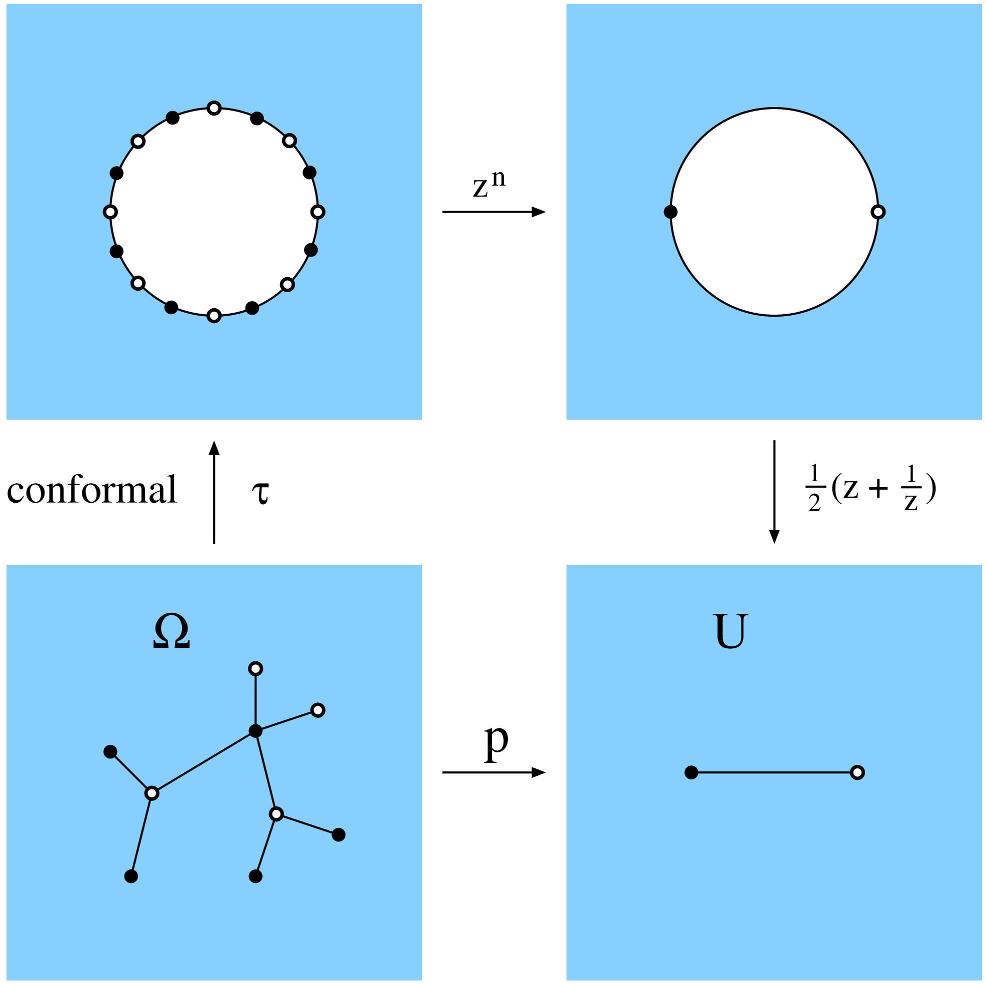
Balanced trees ↔ Shabat polynomials

Fact: T is balanced iff $T = p^{-1}([-1, 1])$, $p =$ Shabat.



$\Omega = \mathbb{C} \setminus T$ $\qquad$ $U = \mathbb{C} \setminus [-1, 1]$

T conformally balanced $\Leftrightarrow$ p Shabat.



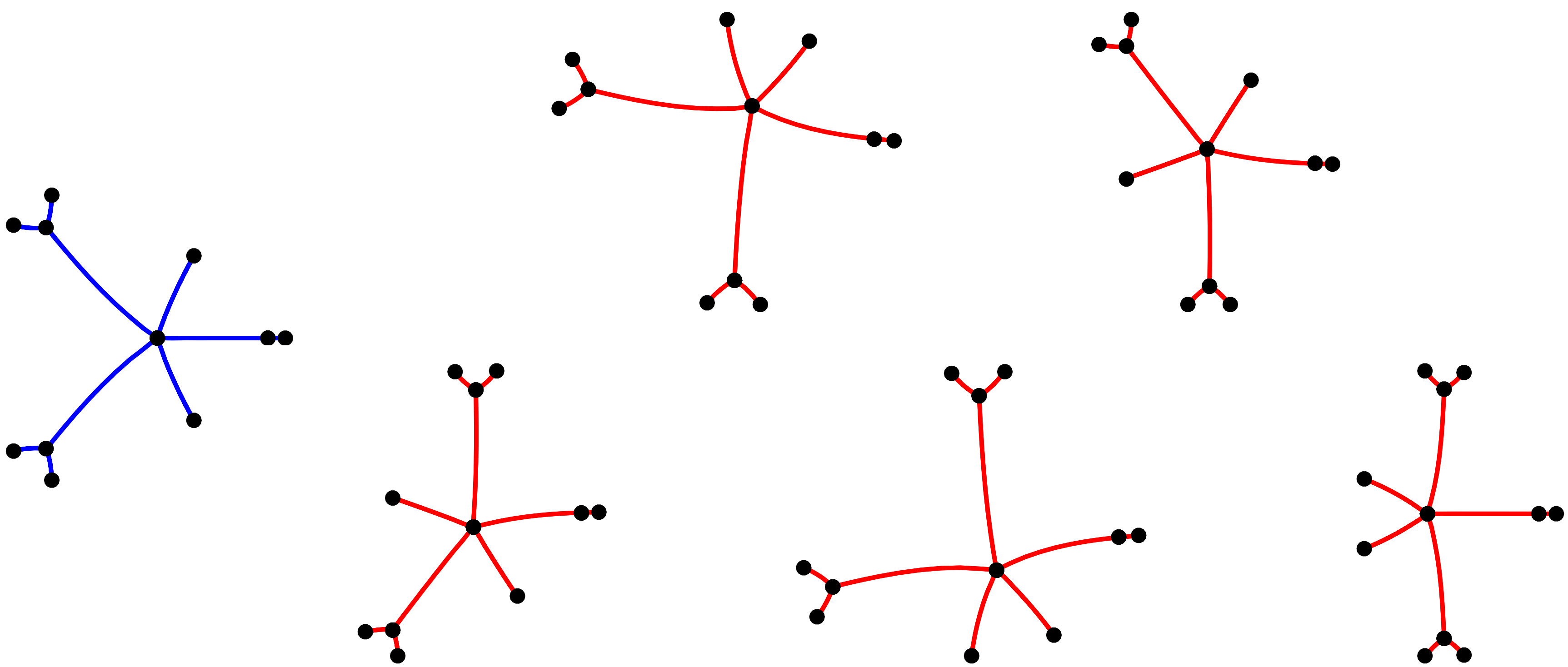
p is entire and n-to-1 $\Leftrightarrow$ p = polynomial.

$\mathrm{CV}(p) \not\in U \leftrightarrow p : \Omega \to U$ is covering map.

Algebraic aside:

True trees are examples of Grothendieck's *dessins d'enfants* on sphere.

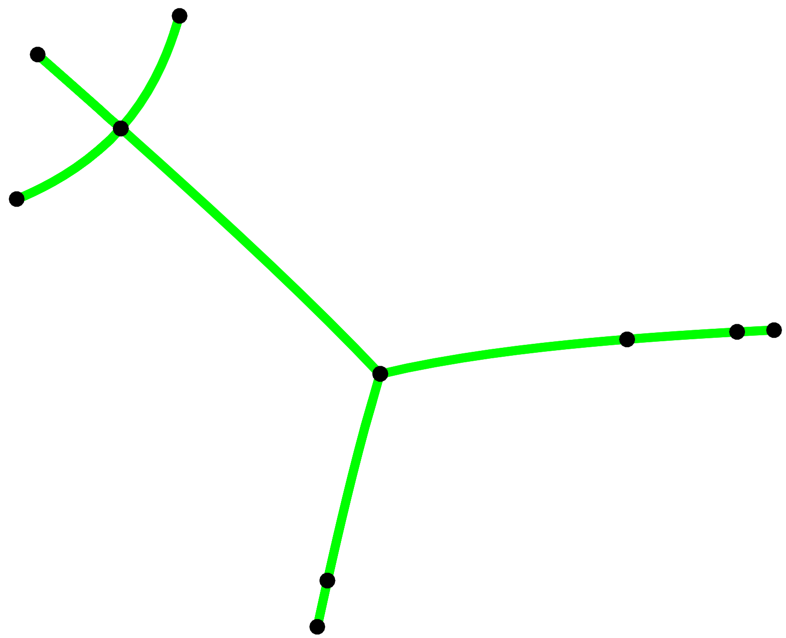
Normalized polynomials are algebraic, so planar trees correspond to number fields. Absolute Galois group acts on trees, but orbits unknown.



Six graphs of type 5 1 1 1 1 1 - 3 3 2 1 1, two orbits.

Even computing number field from tree is difficult.

Kochetkov (2009, 2014): did all trees with 9 and 10 edges.



For example, the polynomial for this 9-edge tree is

$$p(z) = z^4(z^2 + az + b)^2(z - 1),$$

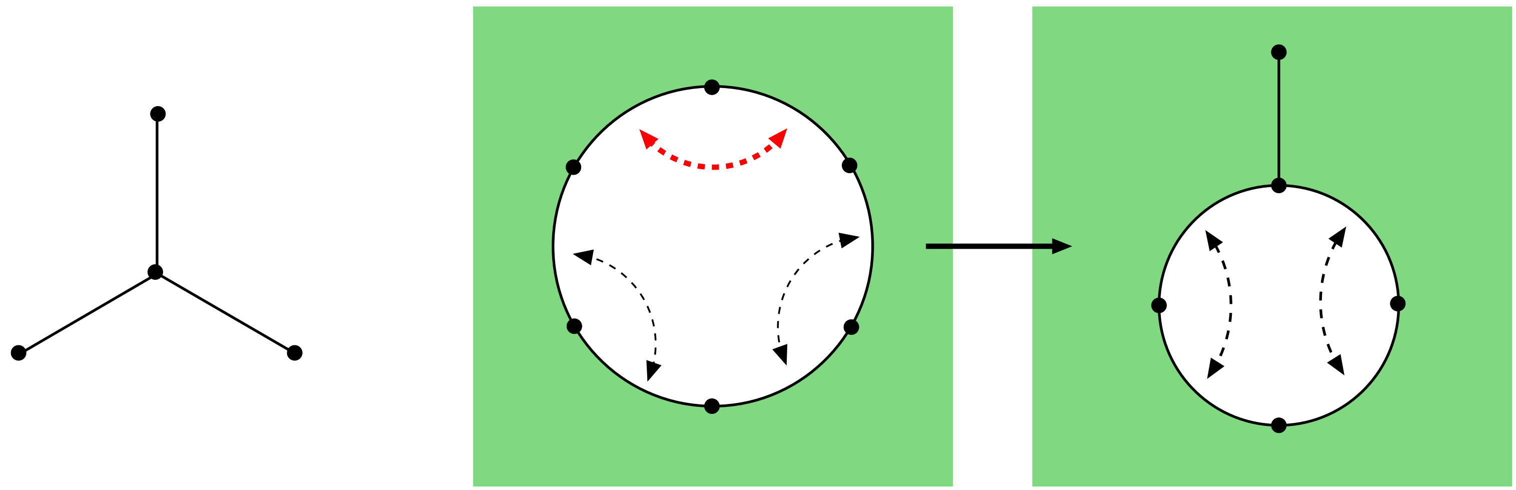
where a is a root of ...

$$
\begin{aligned}
0 = \quad & 126105021875\, a^{15} + 873367351500\, a^{14} \\
& +2340460381665\, a^{13} + 2877817869766\, a^{12} \\
& +3181427453757\, a^{11} - 68622755391456\, a^{10} \\
& -680918281137097\, a^{9} - 2851406436711330\, a^{8} \\
& -7139130404618520\, a^{7} - 12051656256571792\, a^{6} \\
& -14350515598839120\, a^{5} - 12058311779508768\, a^{4} \\
& -6916678783373312\, a^{3} - 2556853615656960\, a^{2} \\
& -561846360735744\, a - 65703906377728
\end{aligned}
$$

This is **not** the most complicated formula in Kochetkov's paper.

However, true form can be drawn without knowing the polynomial.

Don Marshall's `ZIPPER` uses conformal mapping to draw true trees.



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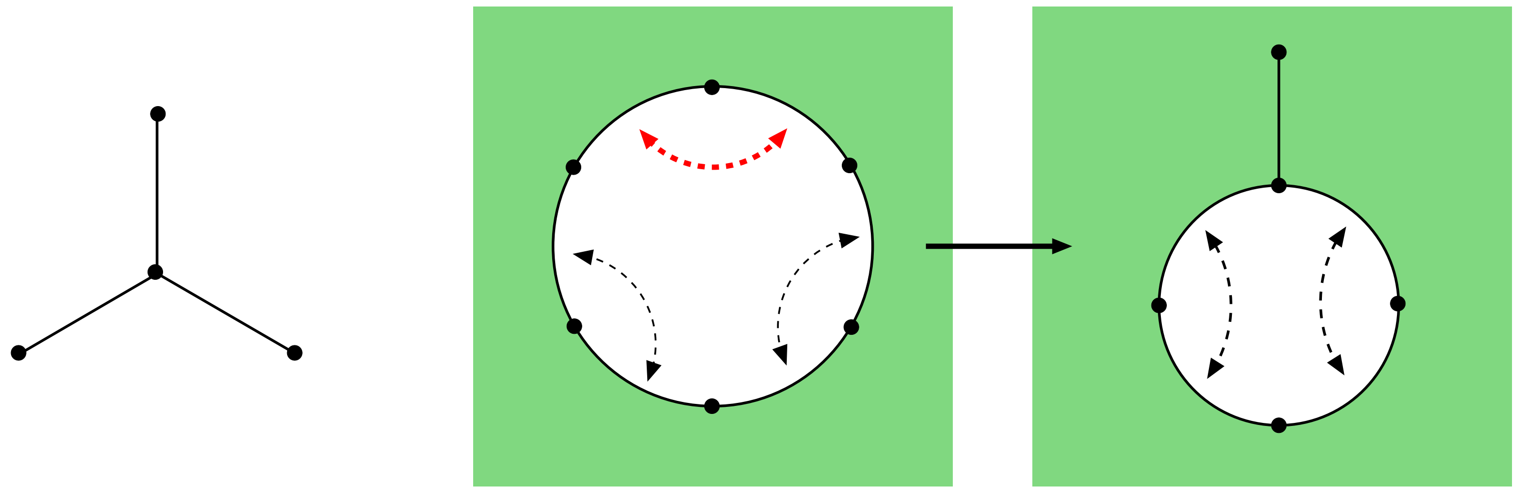
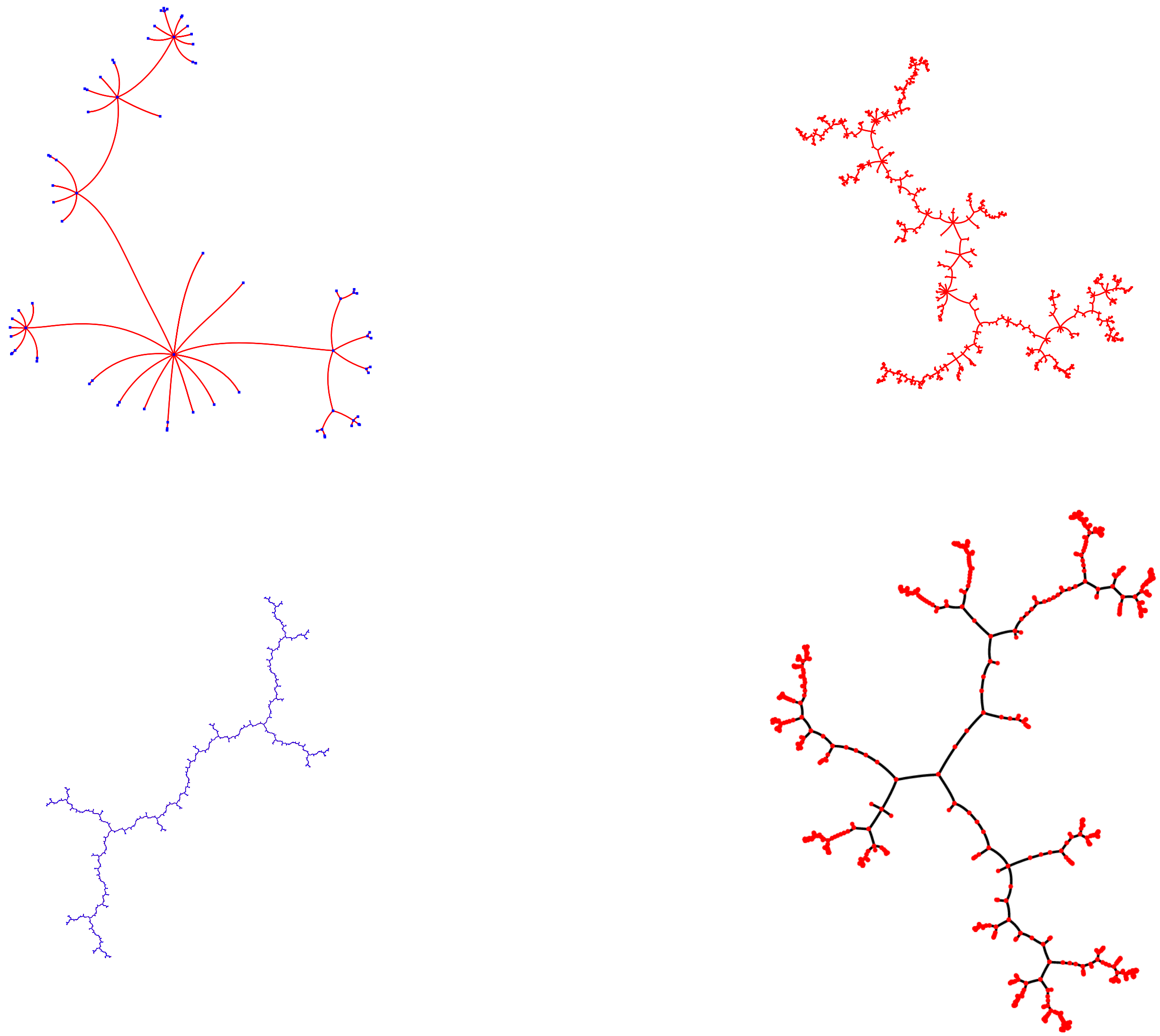


Marshall and Rohde approximated all true trees with ≤ 14 edges.

They can compute vertices to 1000's of digits of accuracy.

Test if $\alpha \in \mathbb{C}$ is algebraic by seeking integer relationships between $1, \alpha, \alpha^2, \ldots$ using lattice reduction or PSLQ algorithm.

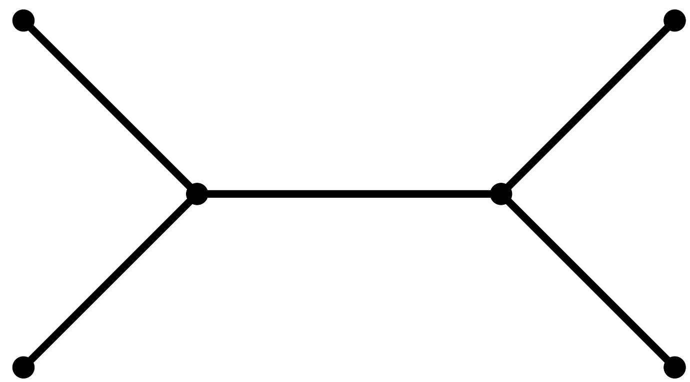


Some true trees, courtesy of Marshall and Rohde

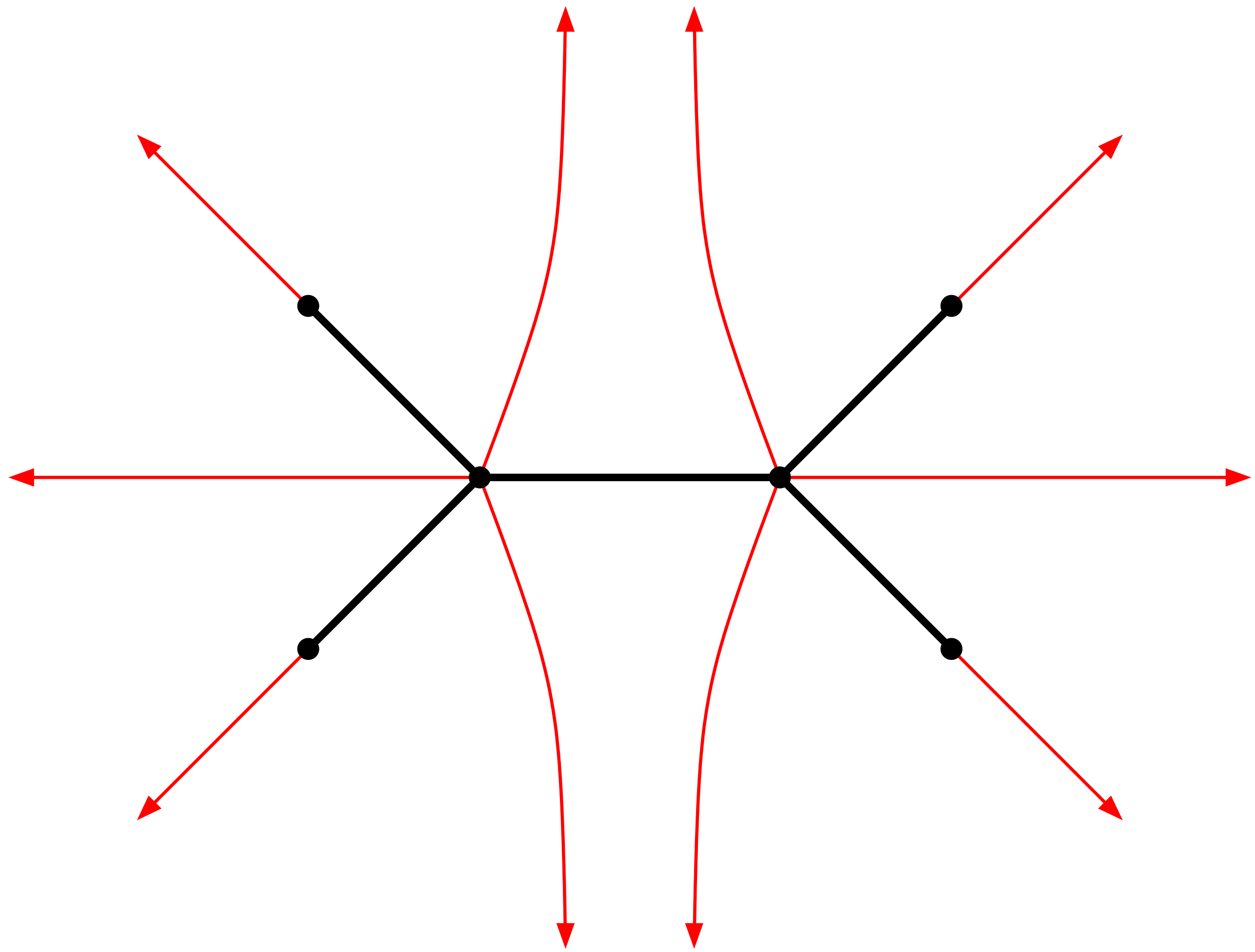
Possible to draw true trees with thousands of edges.

Theorem: Every finite tree has a true form.

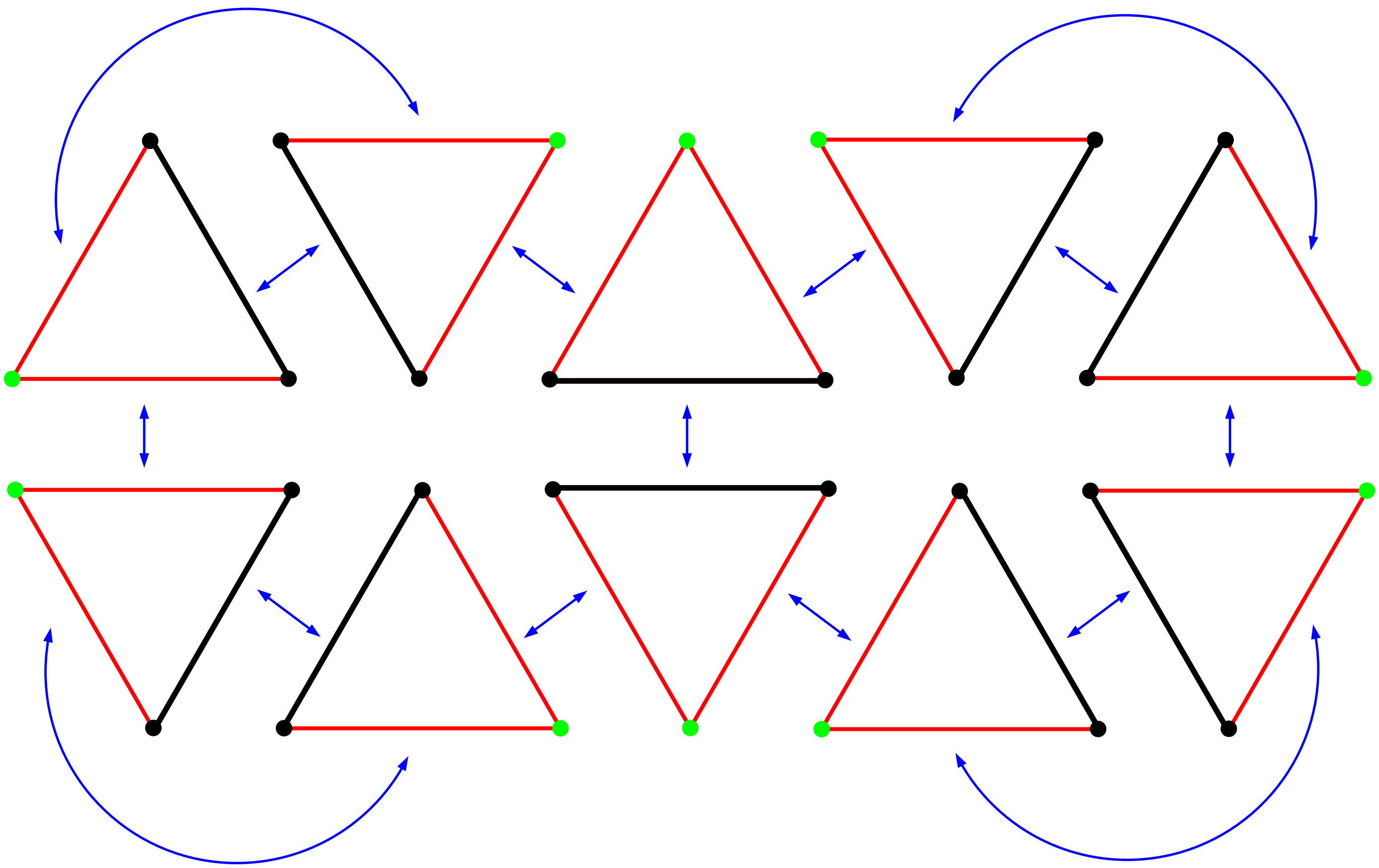
Standard proof uses the uniformization theorem.



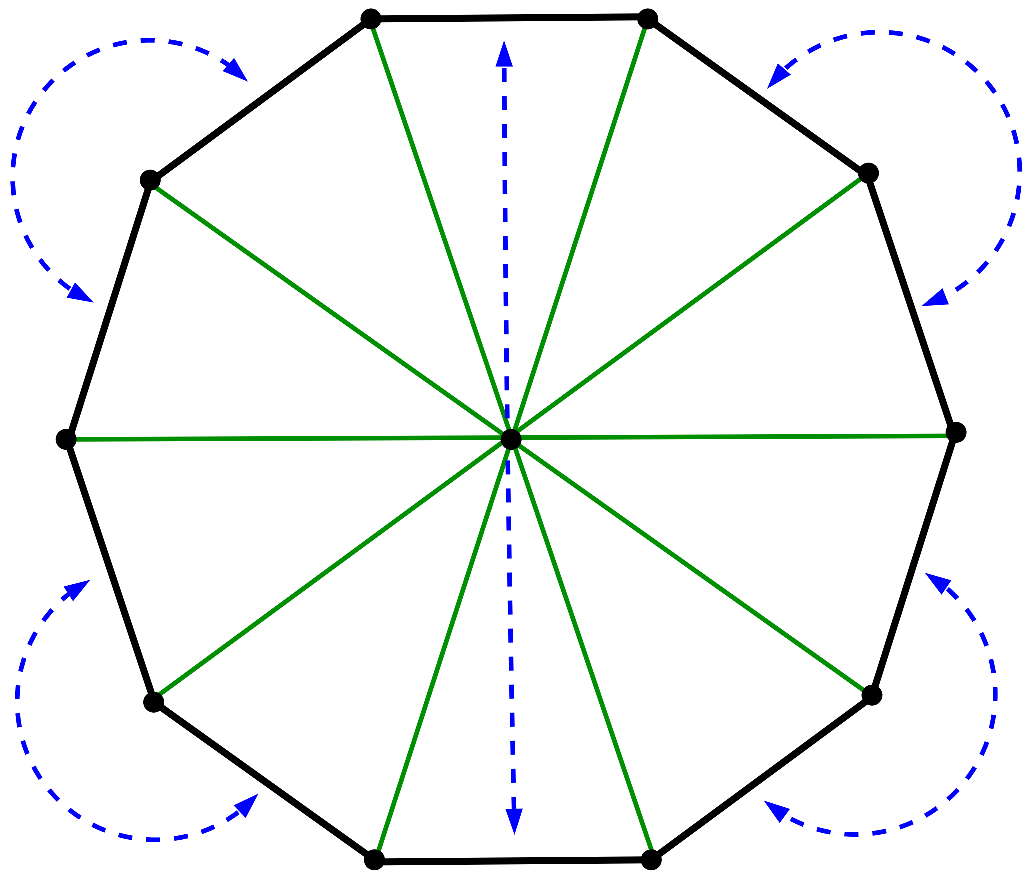
- start with a finite tree.



- connect vertices of T to infinity; gives finite triangulation of sphere.
- Defines adjacencies between triangles.



- Glue equilateral triangles using adjacencies: get a conformal 2-sphere.
- By uniformization theorem, conformal maps to Riemann sphere.
- Can check that tree maps to balanced tree.



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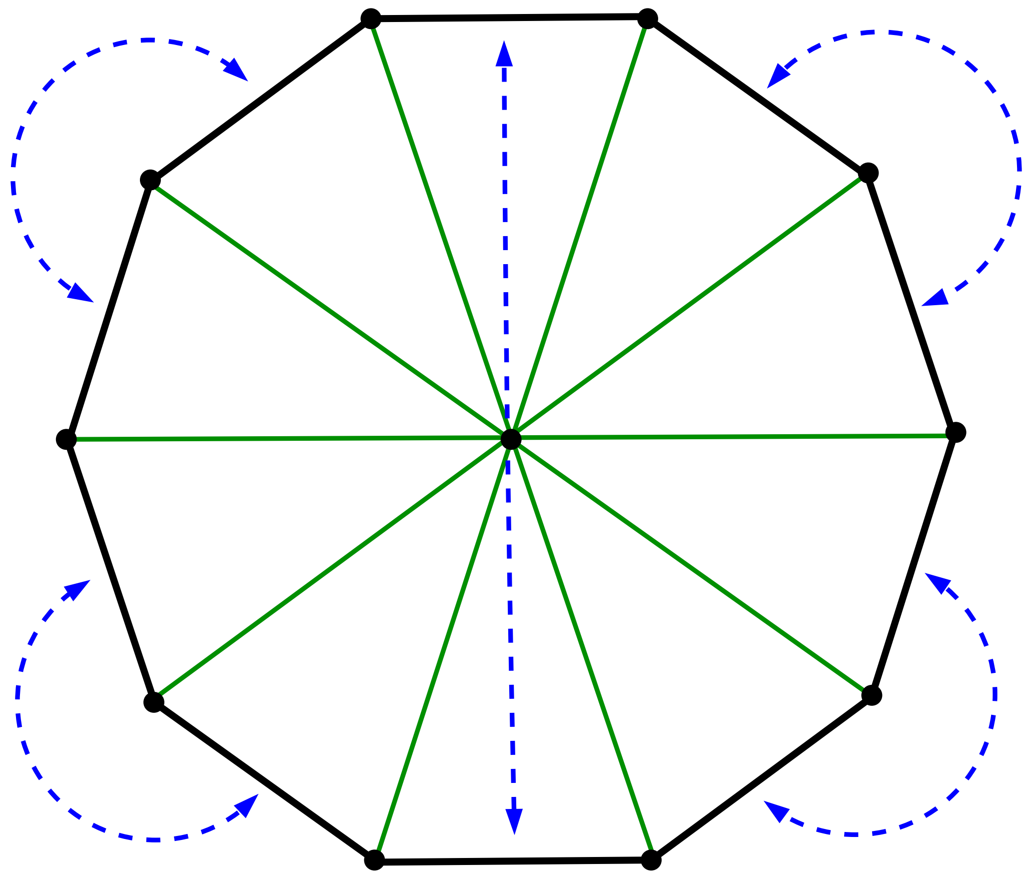
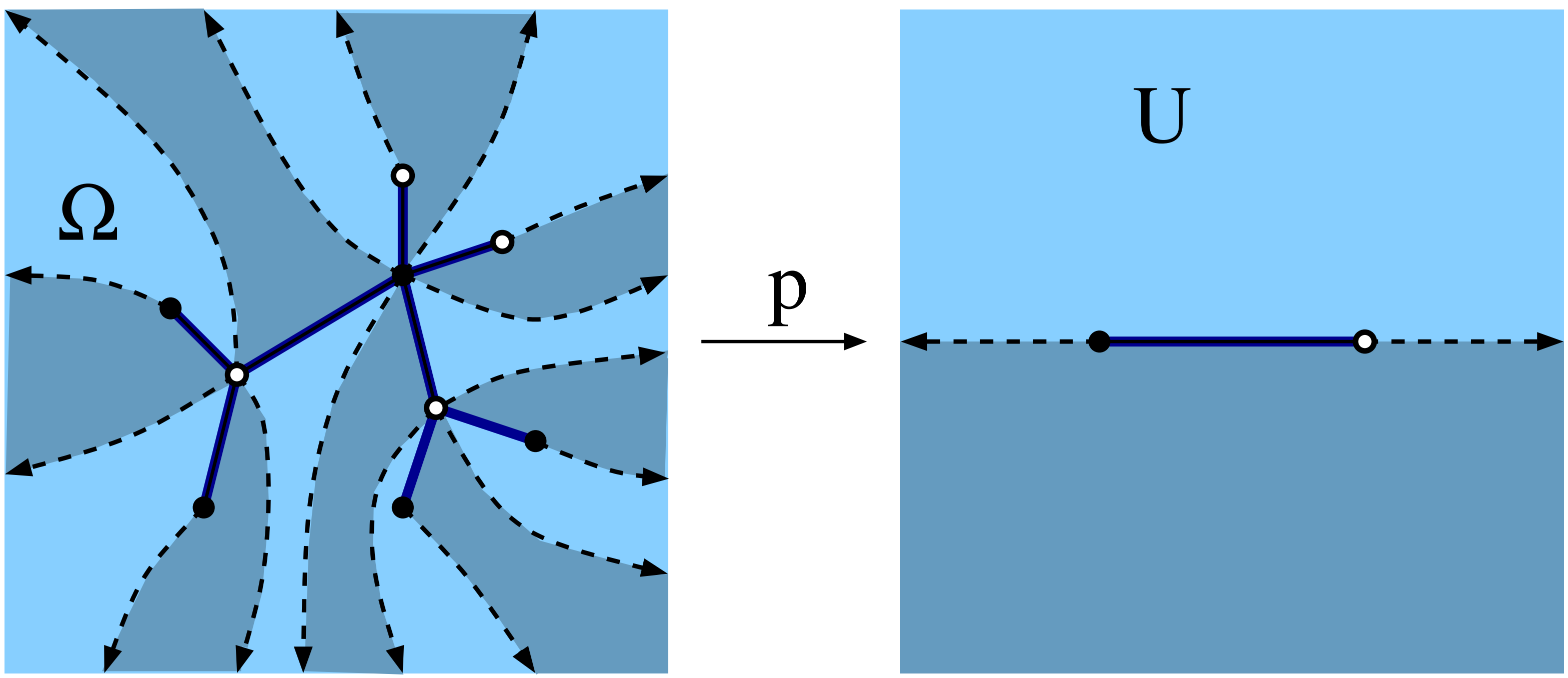


- Note: conformal 2-sphere has many “equilateral triangulations”.
- Which other Riemann surfaces can be obtained in this way?

Belyi function = holomorphic map from Riemann surface with 3 critical values (usually $0, 1, \infty$).

Theorem: A Riemann surface can constructed from equilateral triangles iff it has a Belyi function.



Triangles are inverse images of upper and lower half-planes.

Belyi function $=$ holomorphic map from Riemann surface with 3 critical values (usually $0, 1, \infty$).

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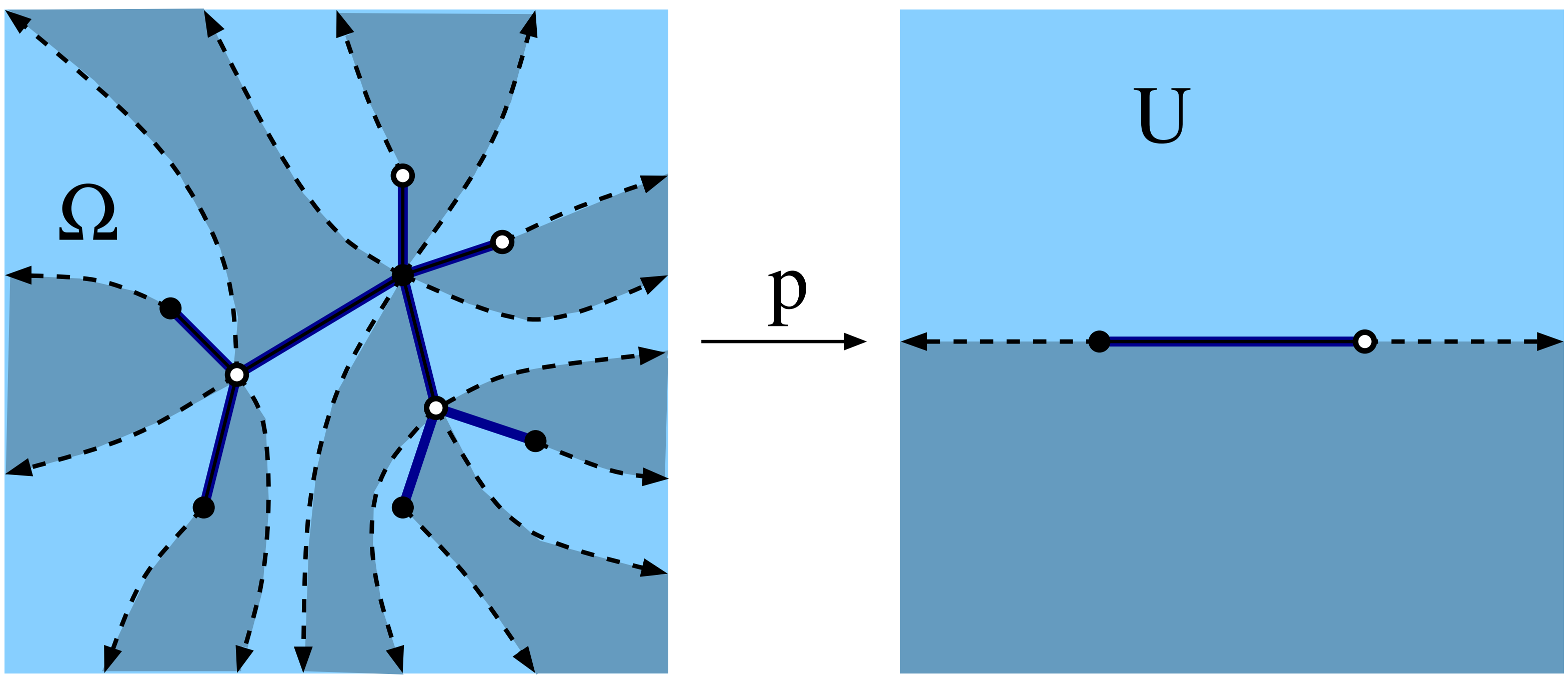

A triangulation is equilateral iff every edge has an anti-holomorphic reflection fixing it pointwise and swapping the two adjacent triangles.

Belyi function = holomorphic map from Riemann surface with 3 critical values (usually $0, 1, \infty$).

Theorem: A Riemann surface can constructed from equilateral triangles iff it has a Belyi function.

There are only finitely many ways to glue together n triangles.

$\Rightarrow$ only countably many compact surfaces have a Belyi function.

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Theorem: A Riemann surface can constructed from equilateral triangles iff it has a Belyi function.

There are only finitely many ways to glue together n triangles.

$\Rightarrow$ only countably many compact surfaces have a Belyi function.

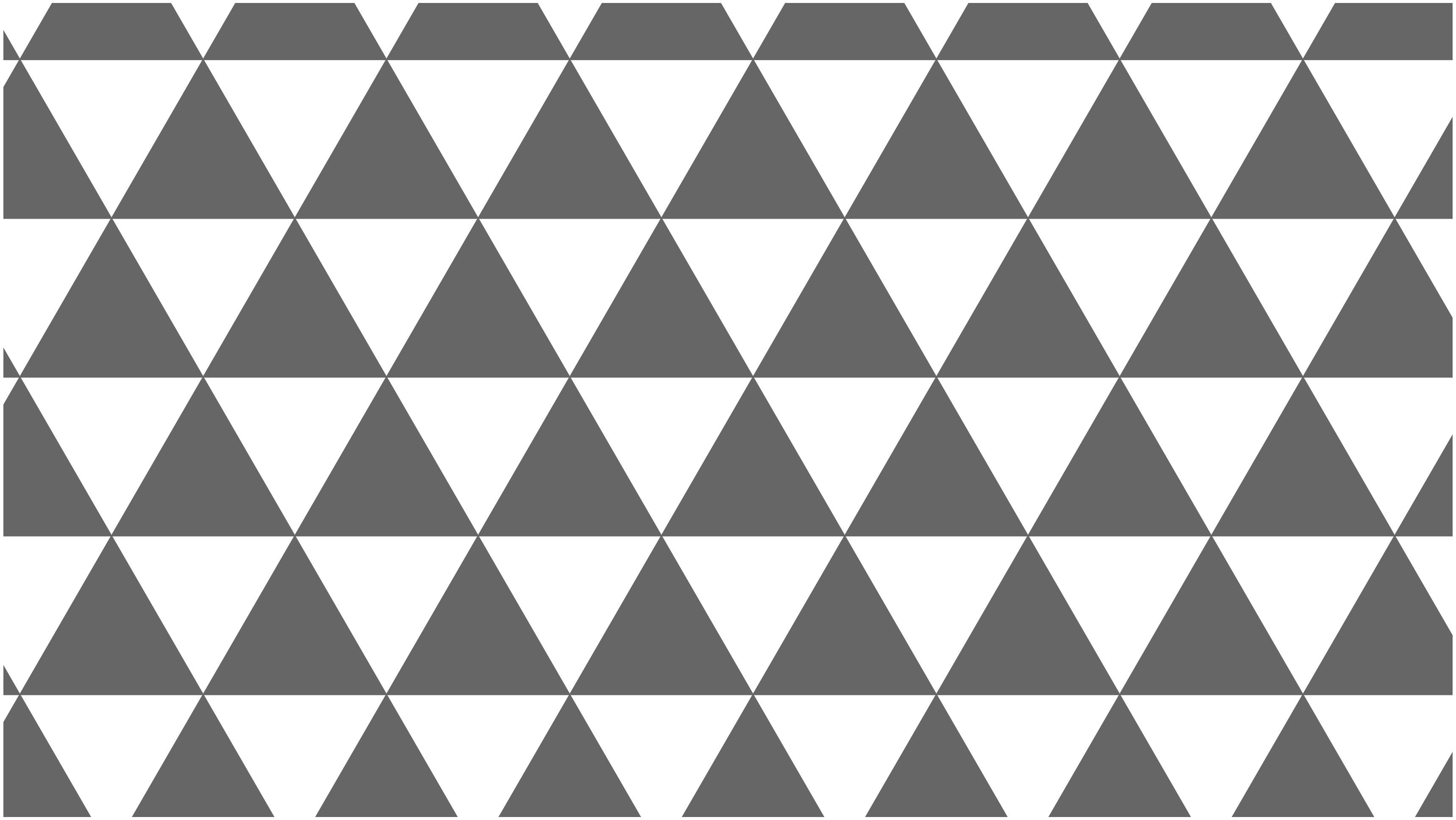
Belyi's Thm: A compact surface has a Belyi function iff it is algebraic.

(Zero set of $P(z, w)$ with algebraic integers as coefficients).

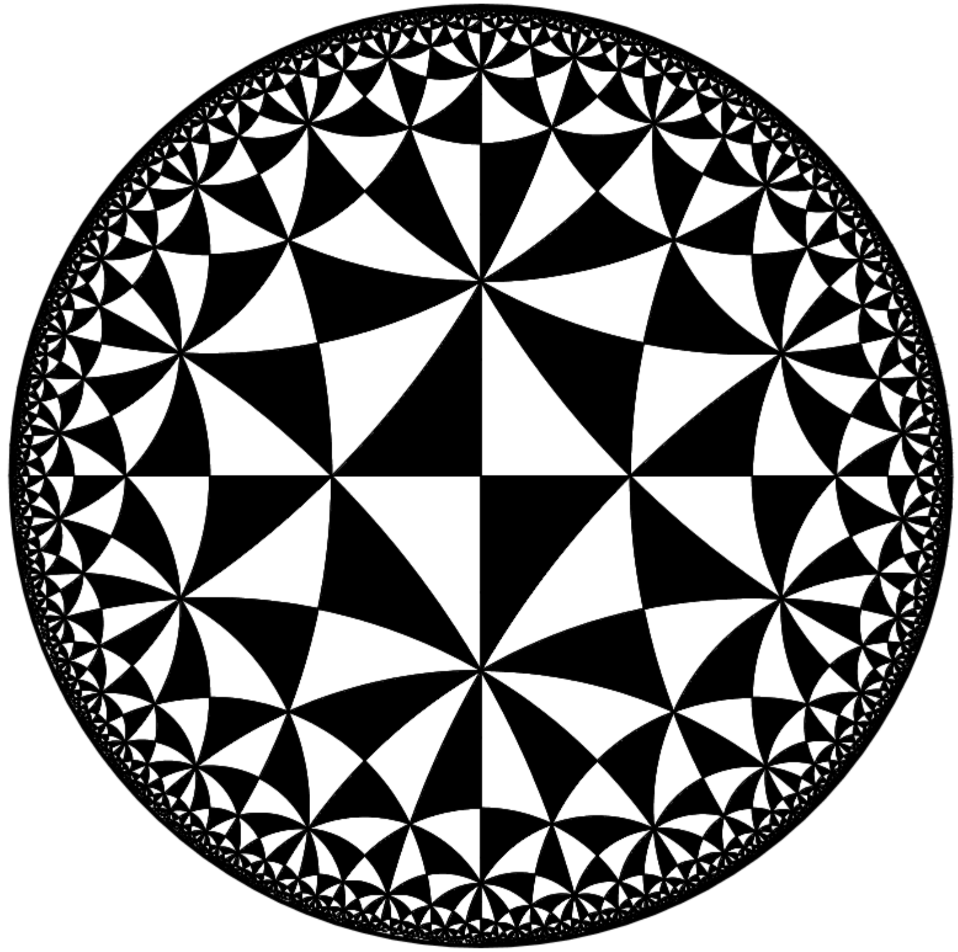
Which non-compact Riemann surfaces have a Belyi function?

Countable many triangles can be used to create non-compact surfaces.

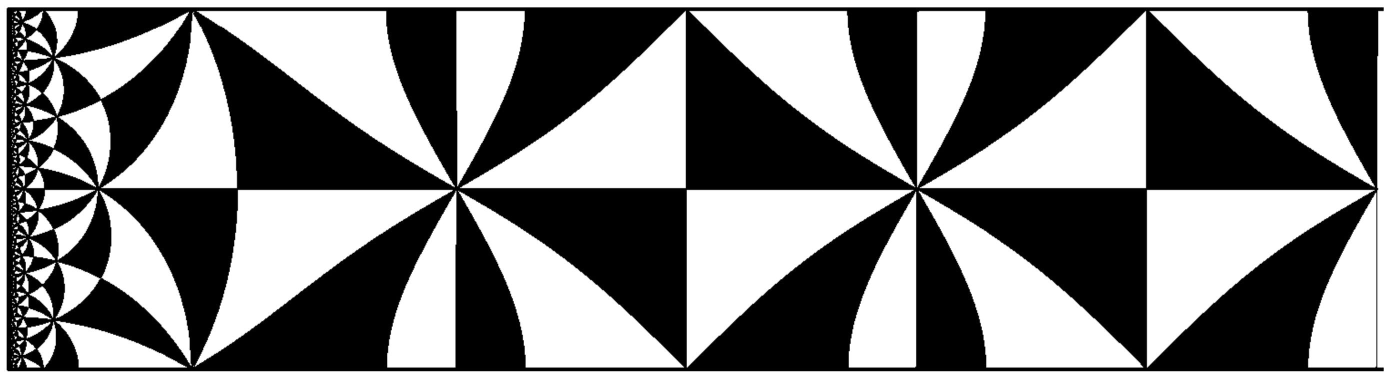
$\Rightarrow$ potentially uncountably many non-compact surfaces arise in this way.



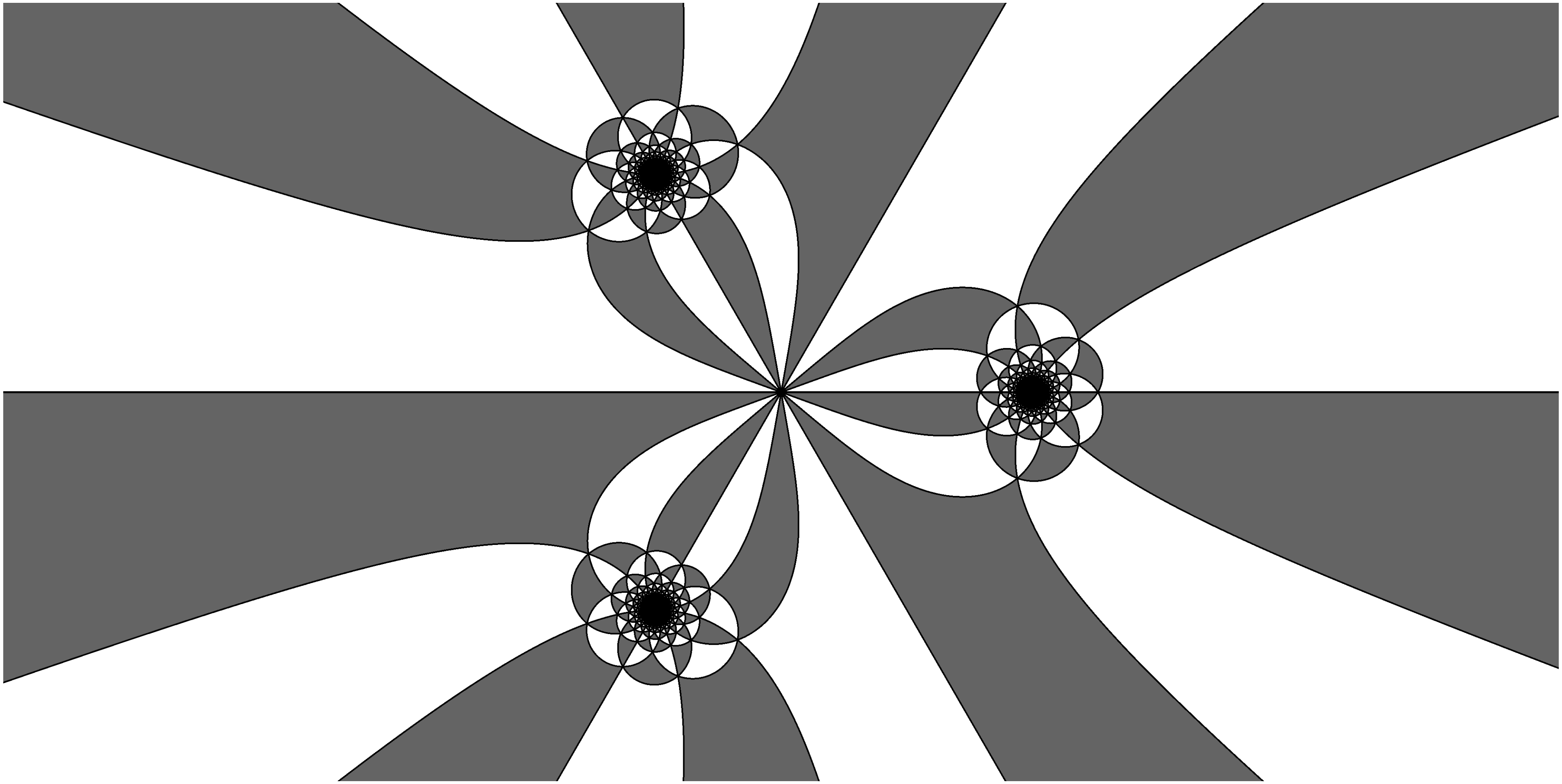
The plane



The disk



The punctured disk (identity top and bottom sides)

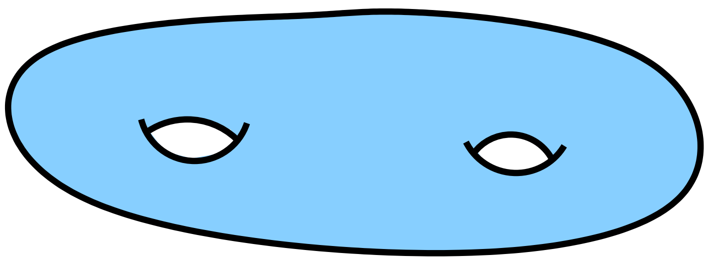


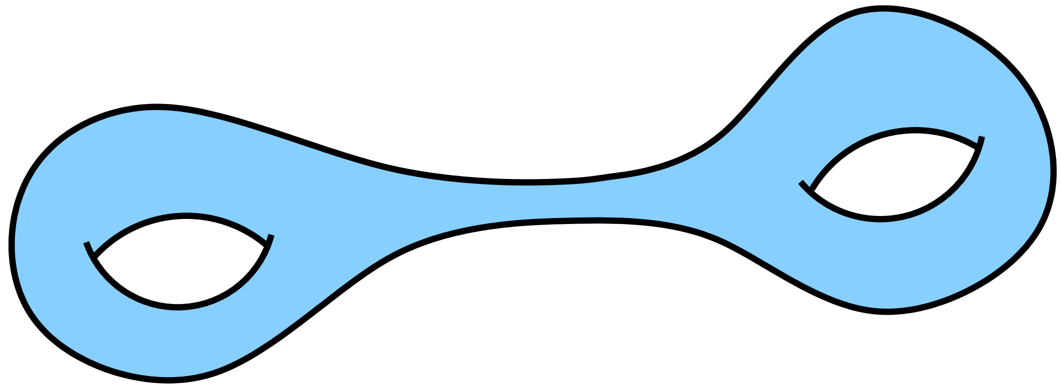
The trice punctured sphere

Most surfaces have many conformal structures = Moduli space.

E.g., plane = disk topologically, but not as Riemann surfaces

Higher genus surfaces have multi-dimensional moduli spaces.





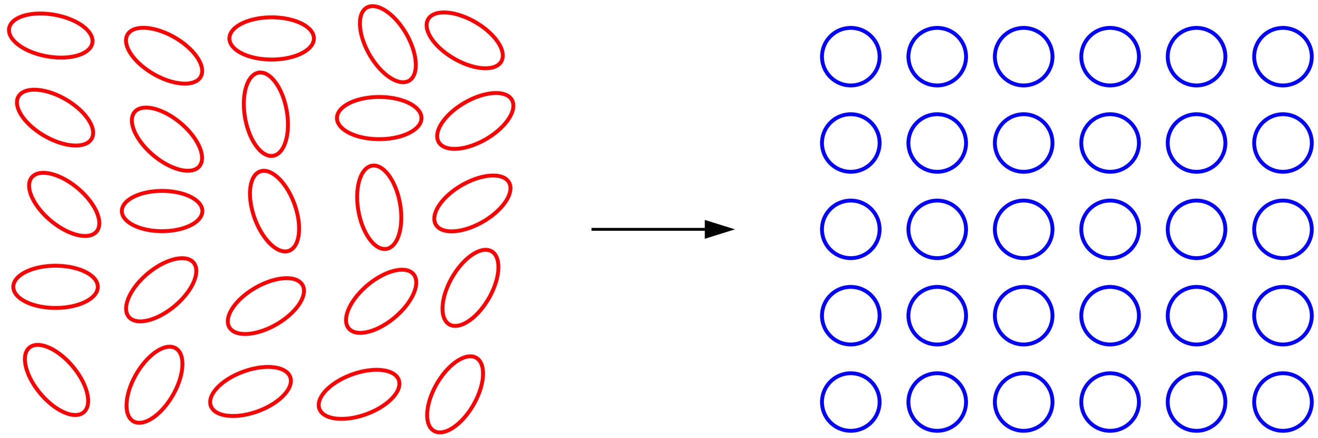
For a triangulation of R, glue equilateral triangles in same pattern.

Gives homomorphic surface R'. Possibly a different conformal structure.

For better control of conformal structures, we give an alternate approach.

We start with brief review of quasiconformal maps.

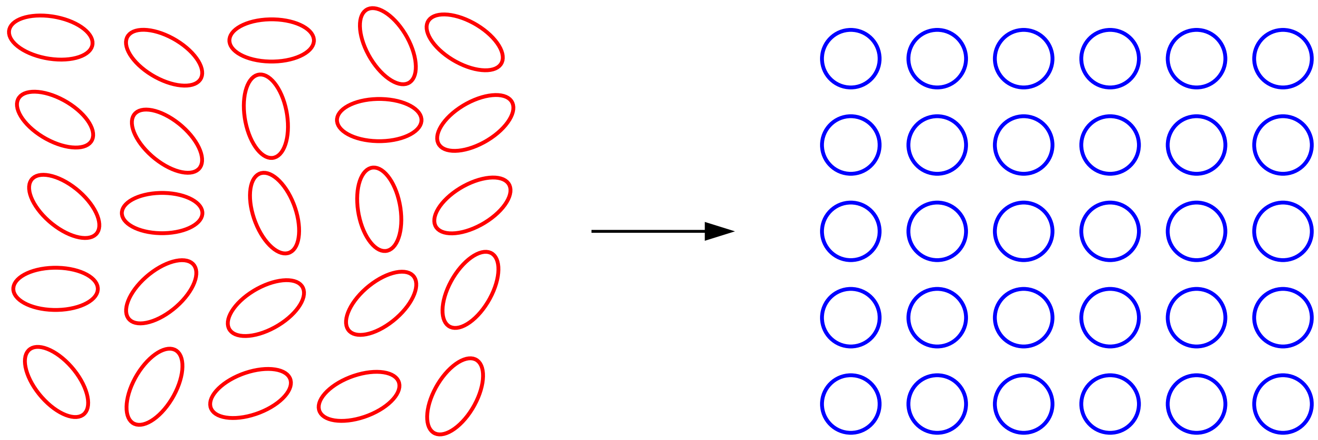
Quasiconformal (QC) maps send infinitesimal ellipses to circles.



Eccentricity = ratio of major to minor axis of ellipse.

For K-QC maps, ellipses have eccentricity $\leq K$

Quasiconformal (QC) maps send infinitesimal ellipses to circles.



Eccentricity = ratio of major to minor axis of ellipse.

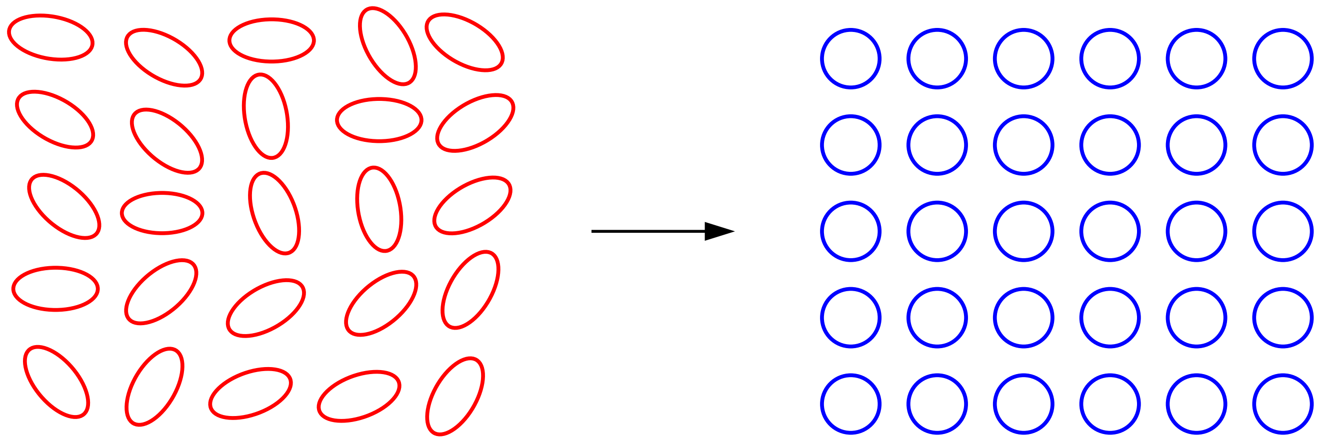
For K-QC maps, ellipses have eccentricity $\leq K$

Ellipses determined a.e. by measurable dilatation $\mu = f_{\overline{z}}/f_z$ with

$$|\mu| \leq \frac{K-1}{K+1} < 1.$$

Conversely, ...

Quasiconformal (QC) maps send infinitesimal ellipses to circles.



Mapping theorem: any such μ comes from some QC map f.

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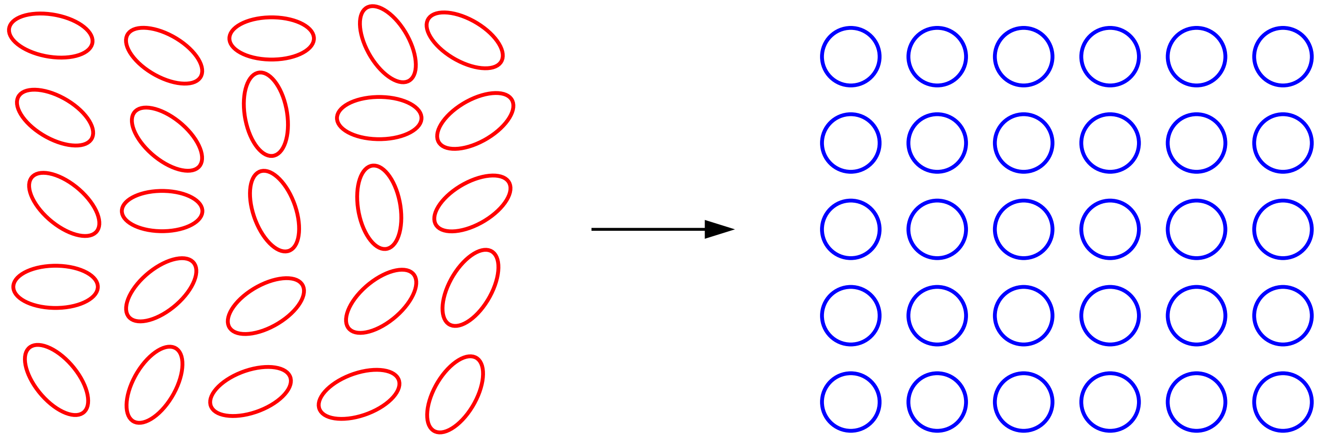
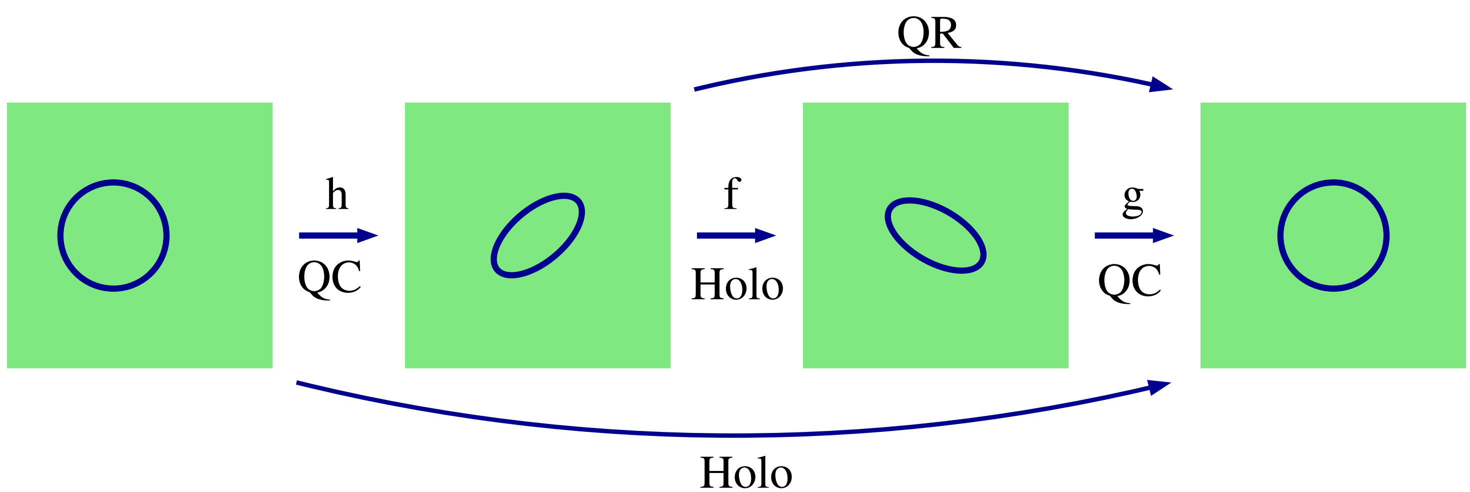


Mapping theorem: any such μ comes from some QC map f.

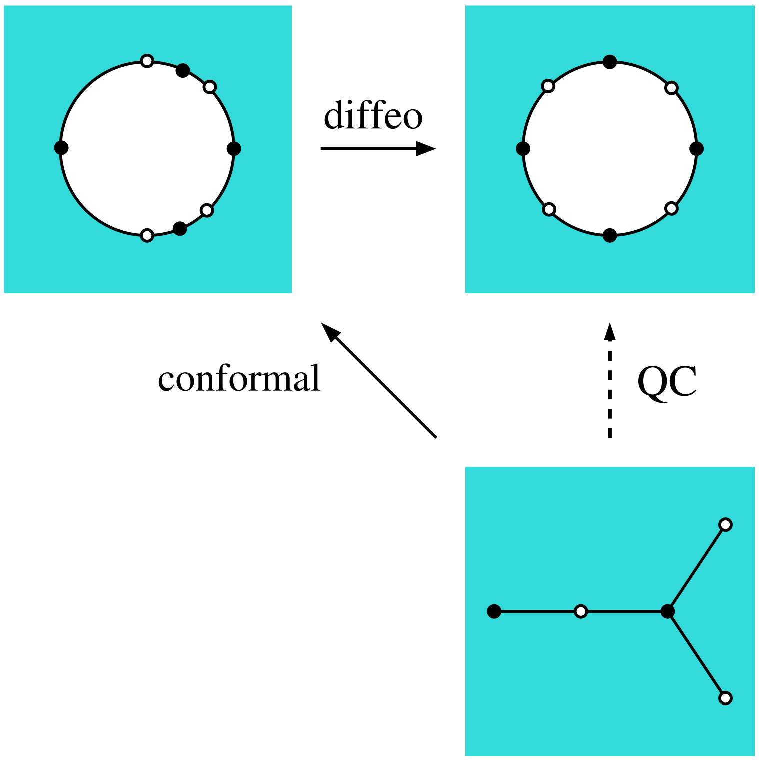
Cor: If f is holomorphic and g is QC, then there is a QC map h so that $F = g \circ f \circ h$ is also holomorphic. ($g \circ f$ is quasiregular.)



QC proof that every finite tree has a true form:

Map $\Omega = \mathbb{C} \setminus T$ to $\{|z| > 1\}$ conformally.

"Equalize intervals" by diffeomorphism. Composition is quasiconformal.



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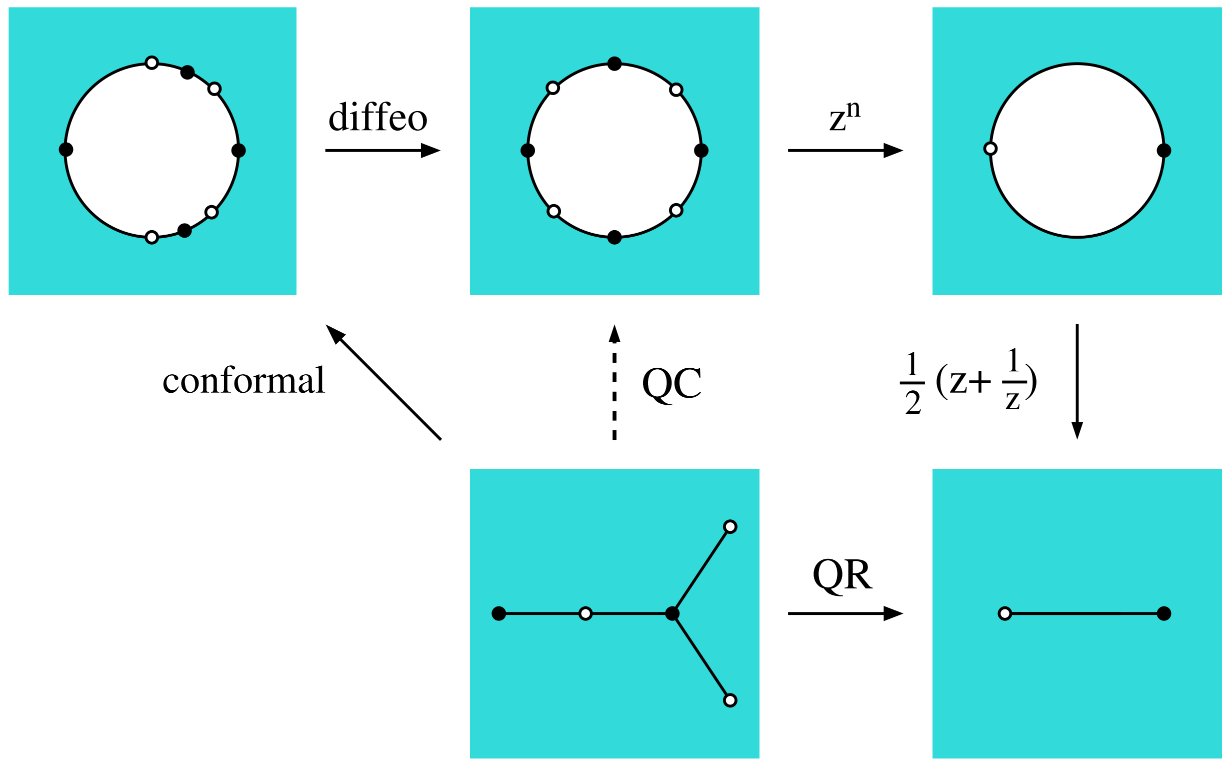
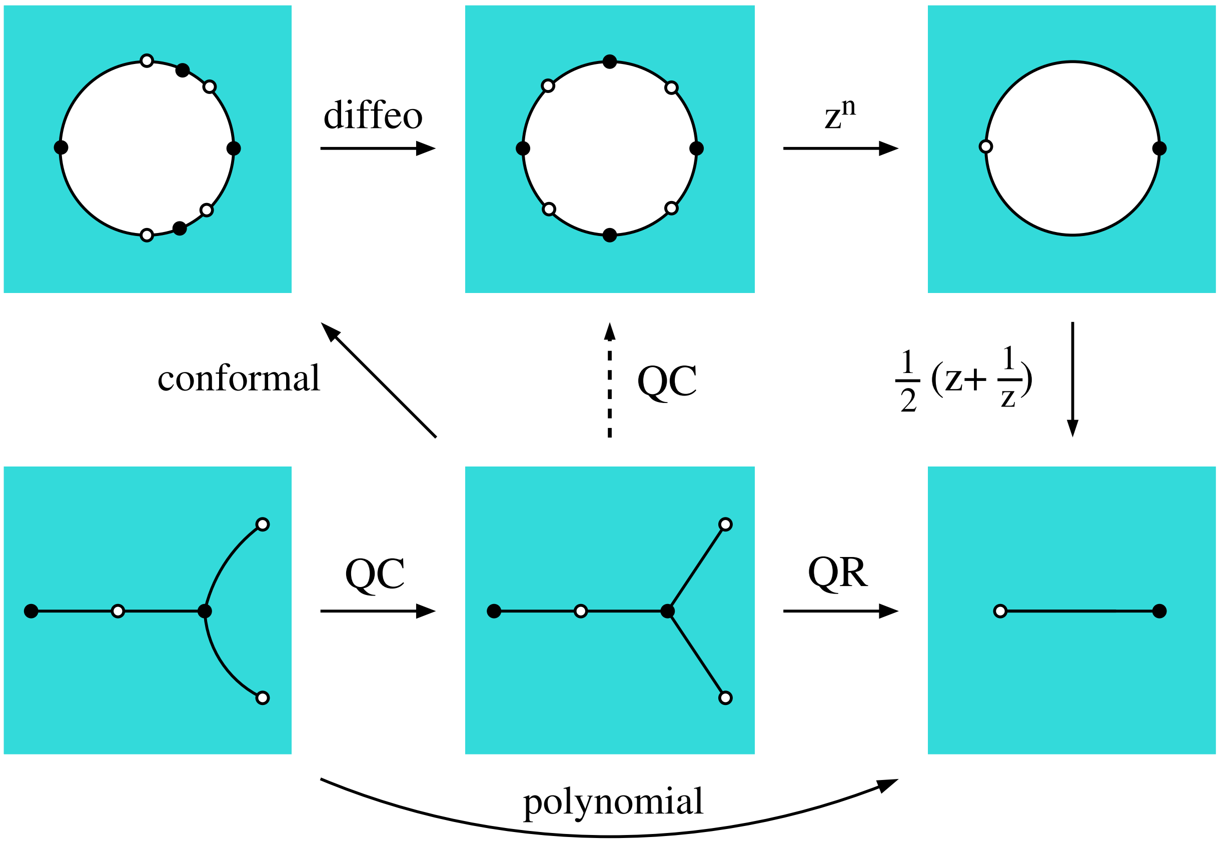


QC proof that every finite tree has a true form:

Map $\Omega = \mathbb{C} \setminus T$ to $\{|z| > 1\}$ conformally.

Mapping theorem implies there is a QC φ so $p = q \circ \varphi$ is a polynomial.



Only possible critical points are vertices of tree; these map to ± 1.

A. Eremenko: do true trees approximate all possible shapes? .

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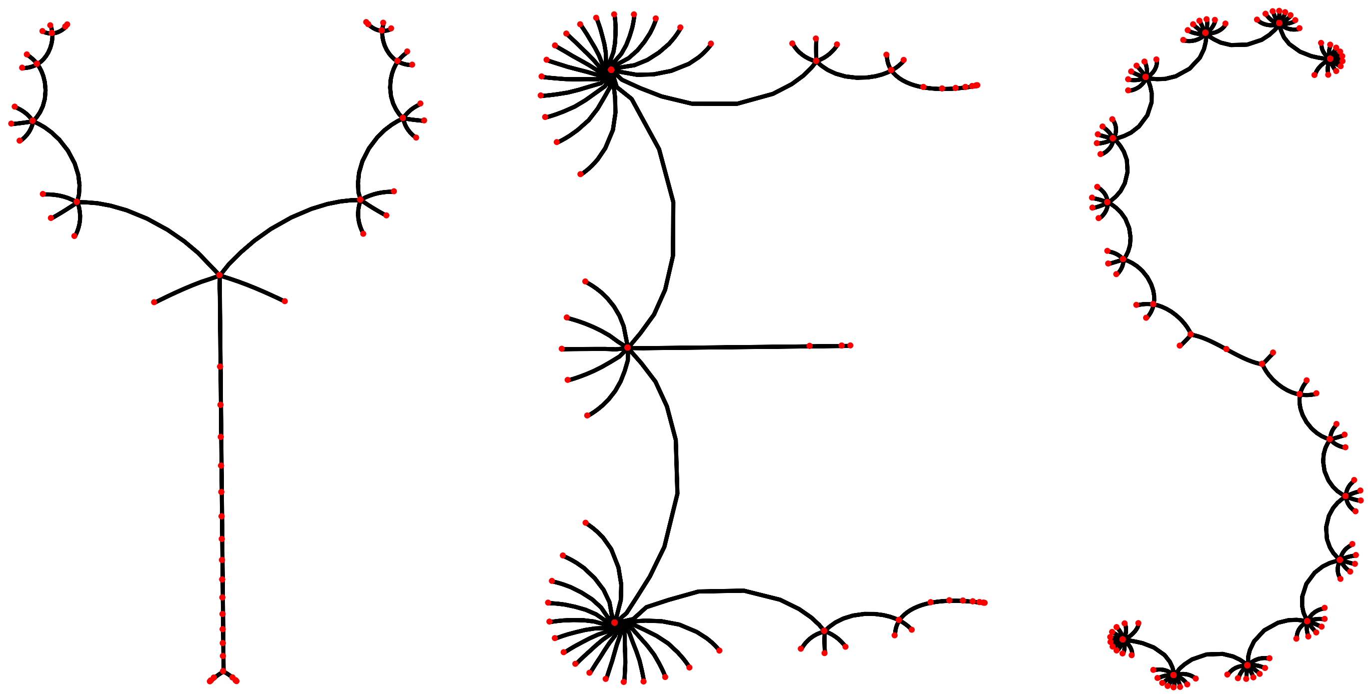

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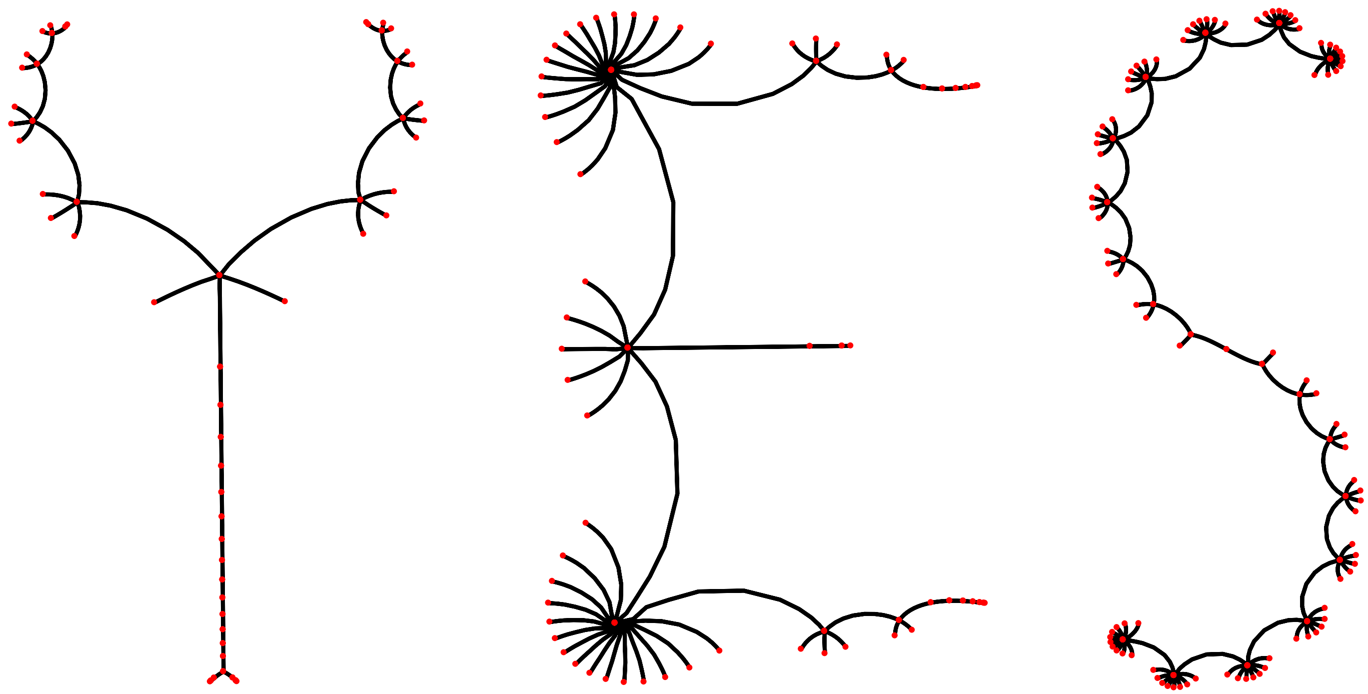


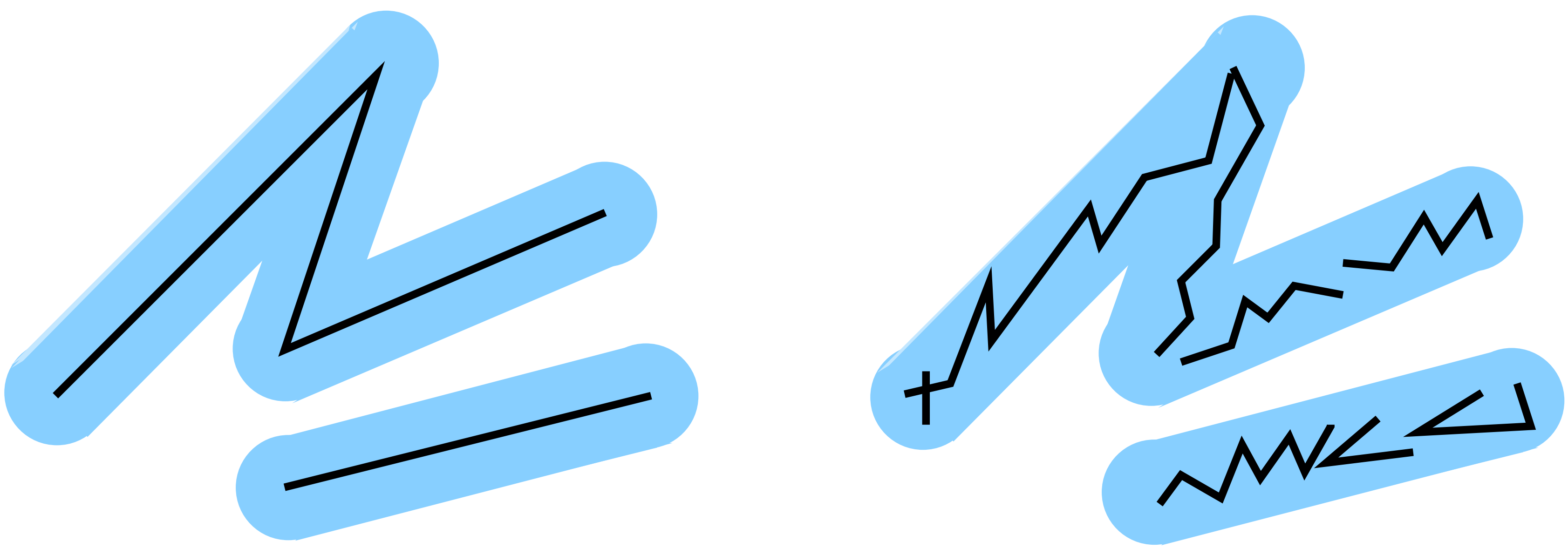
Thm: (B. 2013) Every planar continuum is Hausdorff limit of true trees.

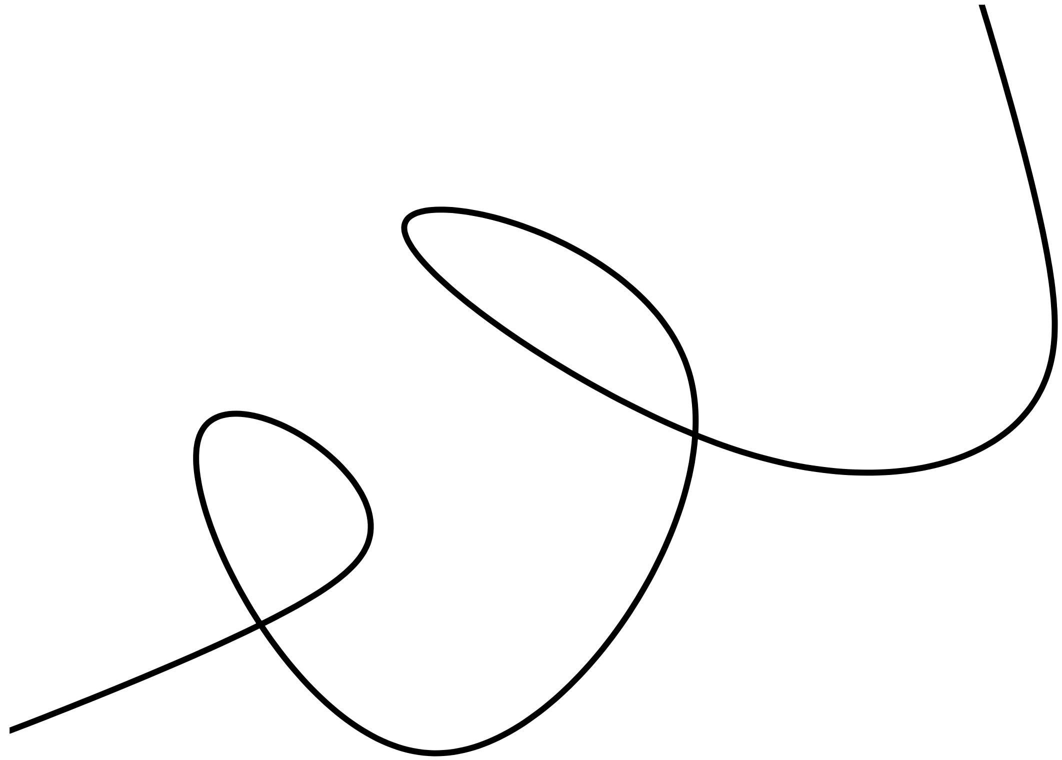
For a set E and $\epsilon > 0$ define ϵ-neighborhood of E:

$$E(\epsilon) = \{z : \text{dist}(z, E) < \epsilon\}.$$

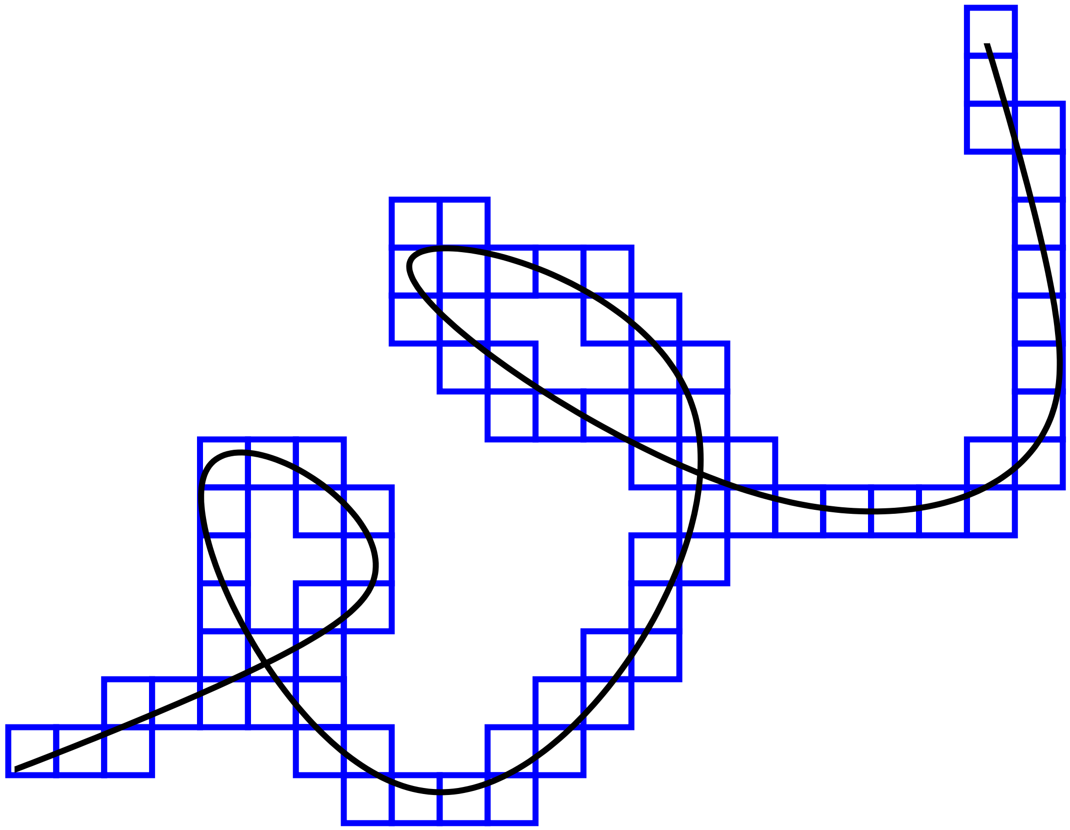
Defn: Hausdorff distance between sets is

$$d(E, F) = \inf\{\epsilon > 0 : E \subset F(\epsilon), F \subset E(\epsilon)\}.$$





Suffices to approximate subtrees of a grid.



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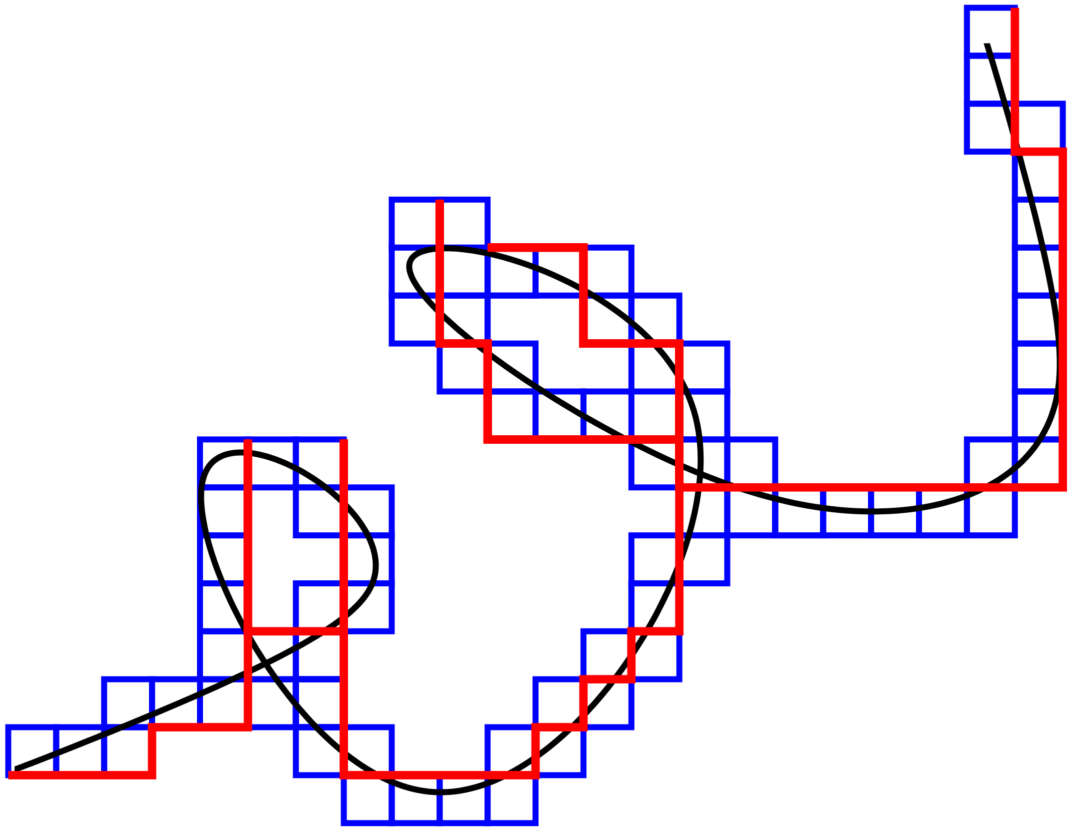

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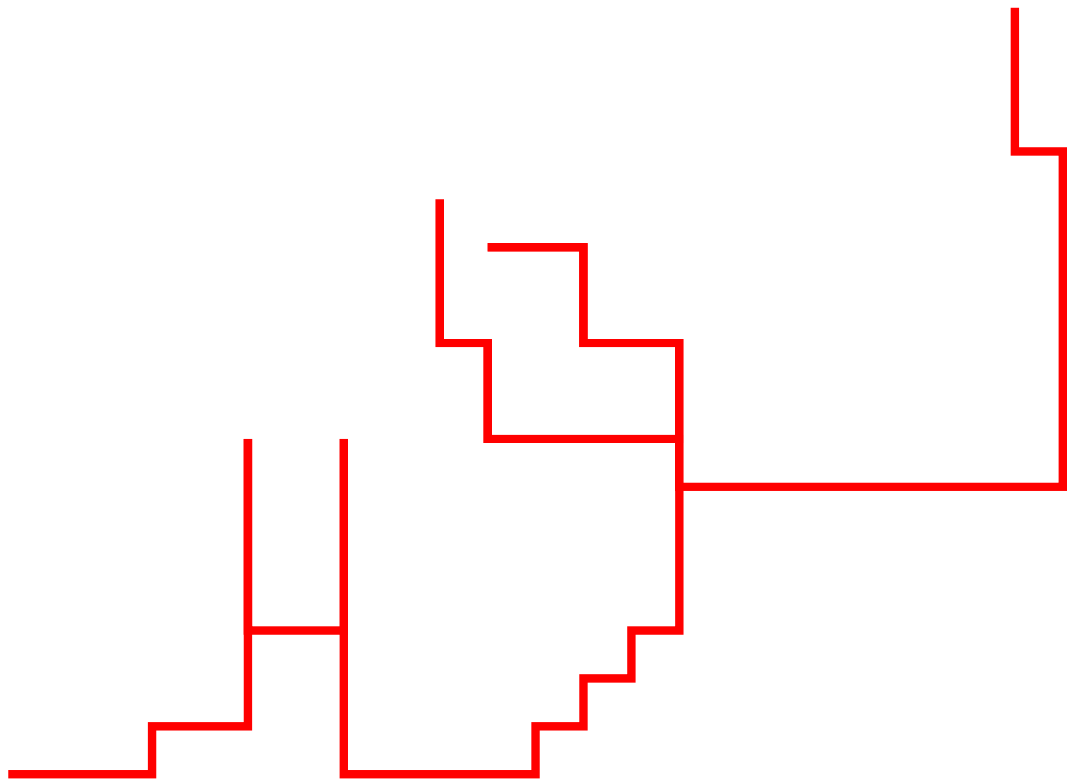
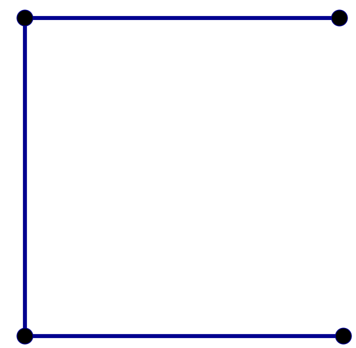


Suffices to approximate subtrees of a grid.

Theorem: Every planar continuum is a limit of true trees.

Idea of Proof: reduce harmonic measure ratio by adding edges.



Vertical side has much larger harmonic measure from left.

Theorem: Every planar continuum is a limit of true trees.

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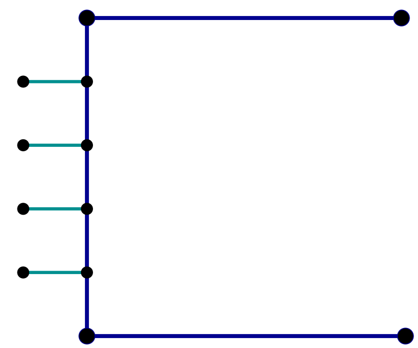

"Left" harmonic measure is reduced (roughly 3-to-1).

New edges are approximately balanced (universal constant).

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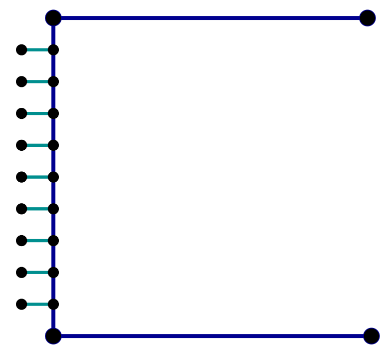


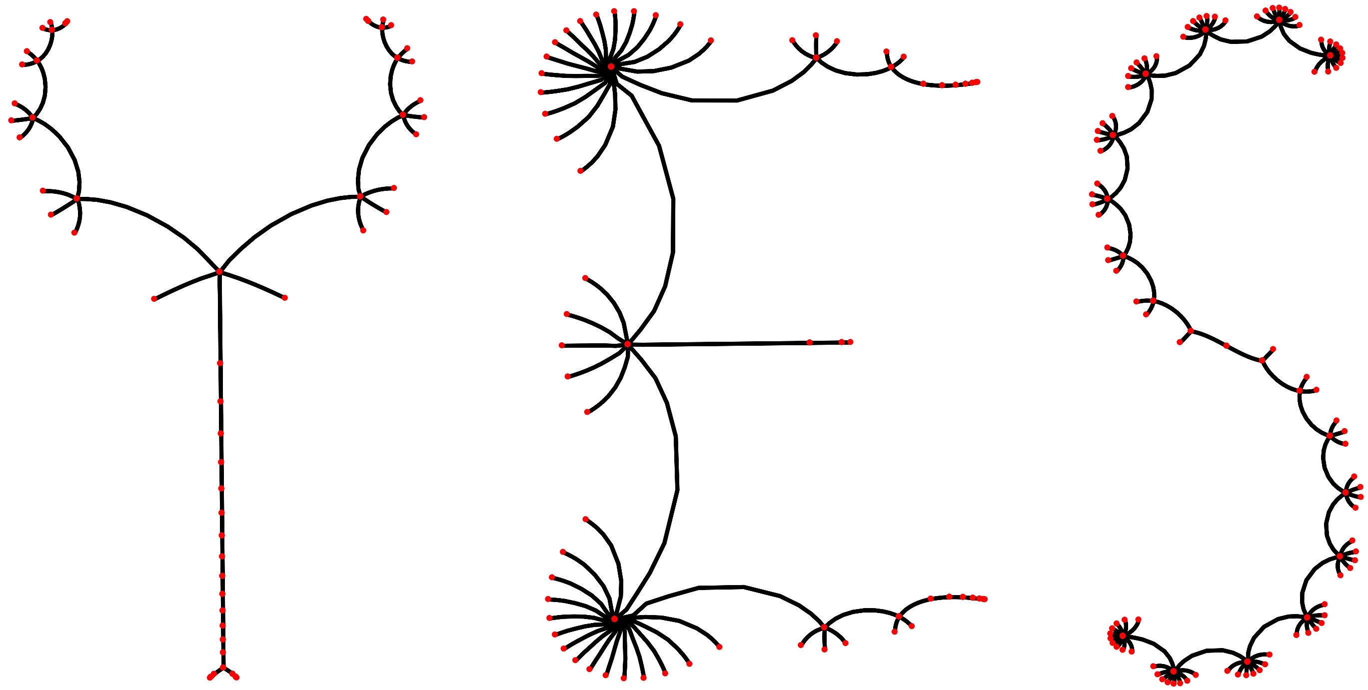
"Left" harmonic measure is reduced (roughly 3-to-1).

New edges are approximately balanced (universal constant).

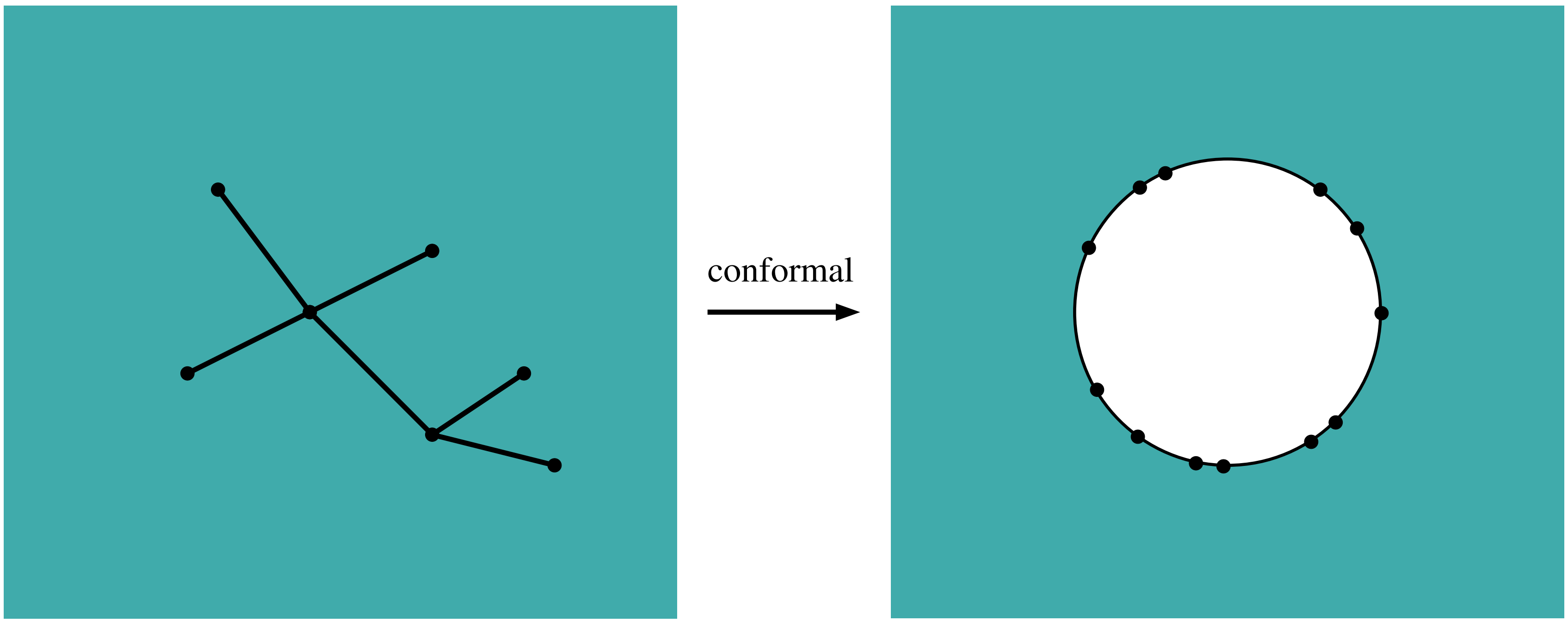
Mapping theorem gives exactly balanced.

QC correction map is near identity if "spikes" are short.

New tree approximates shape of old tree; different combinatorics.


YES

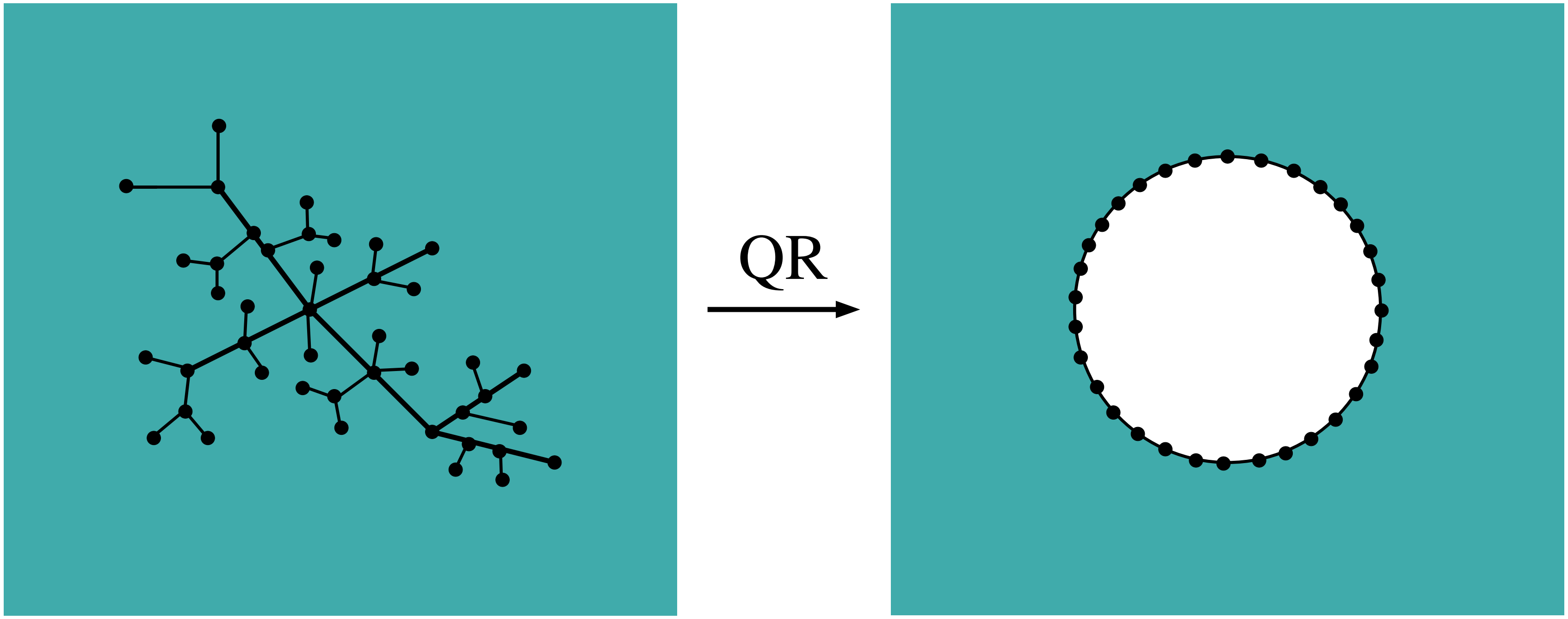
More precise statement.



Suppose we are given a “reasonable” tree (smooth edges, large angles).

Likely, the tree is not balanced. Edges have different harmonic measures.

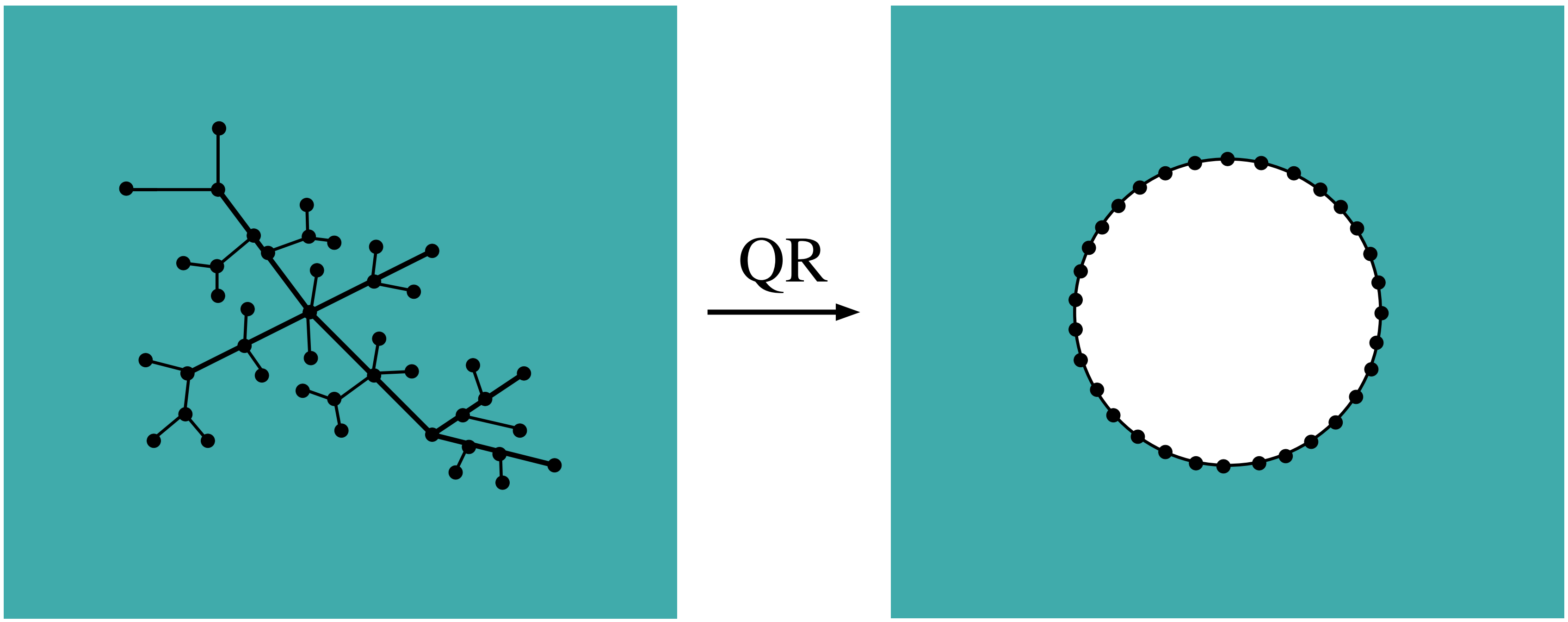
More precise statement.



We can add new branches to form an “almost balanced” tree.

There is a QR map to disk exterior mapping every side to equal length.

More precise statement.



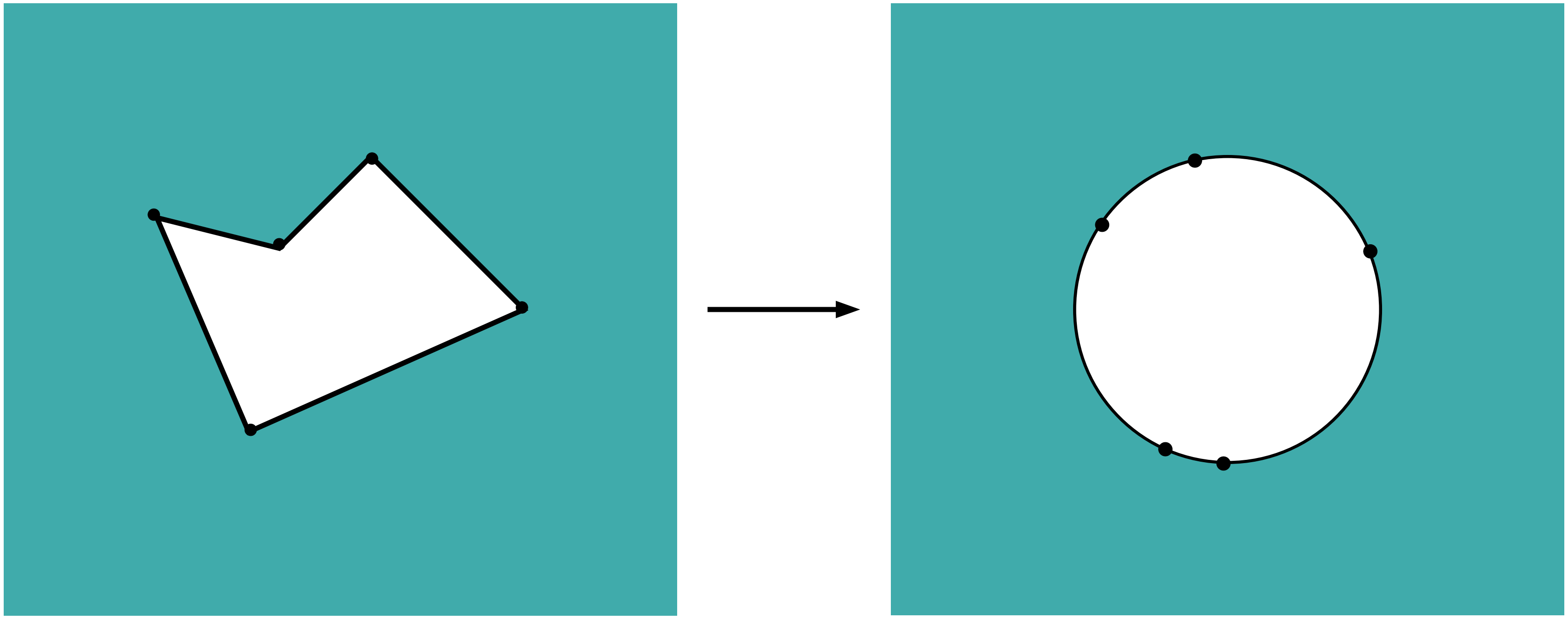
Distortion constant is uniformly bounded.

Can make all extra branches as small as we wish.

Map is conformal except on small area.

Quasiconformal correction is very close to identity.

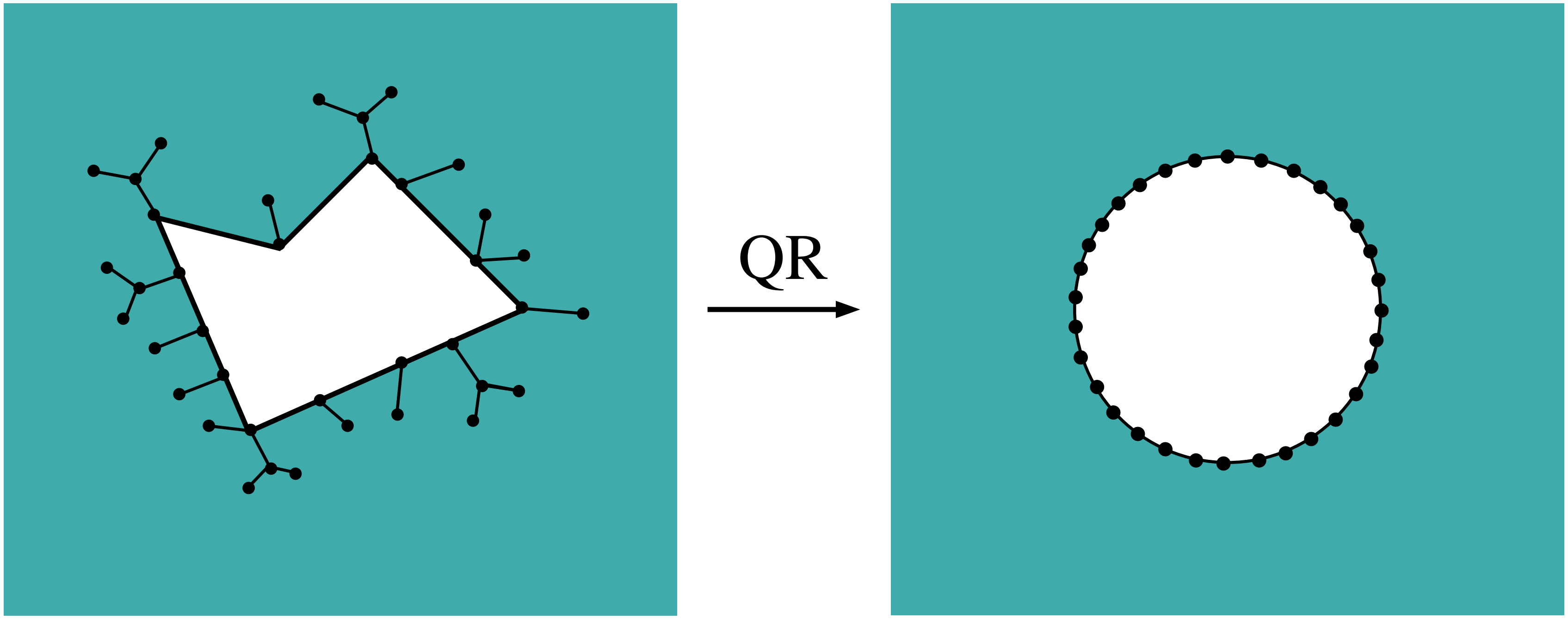
More precise statement.



This works on other shapes besides trees.

Suppose we are given “unbalanced polygon”.

More precise statement.

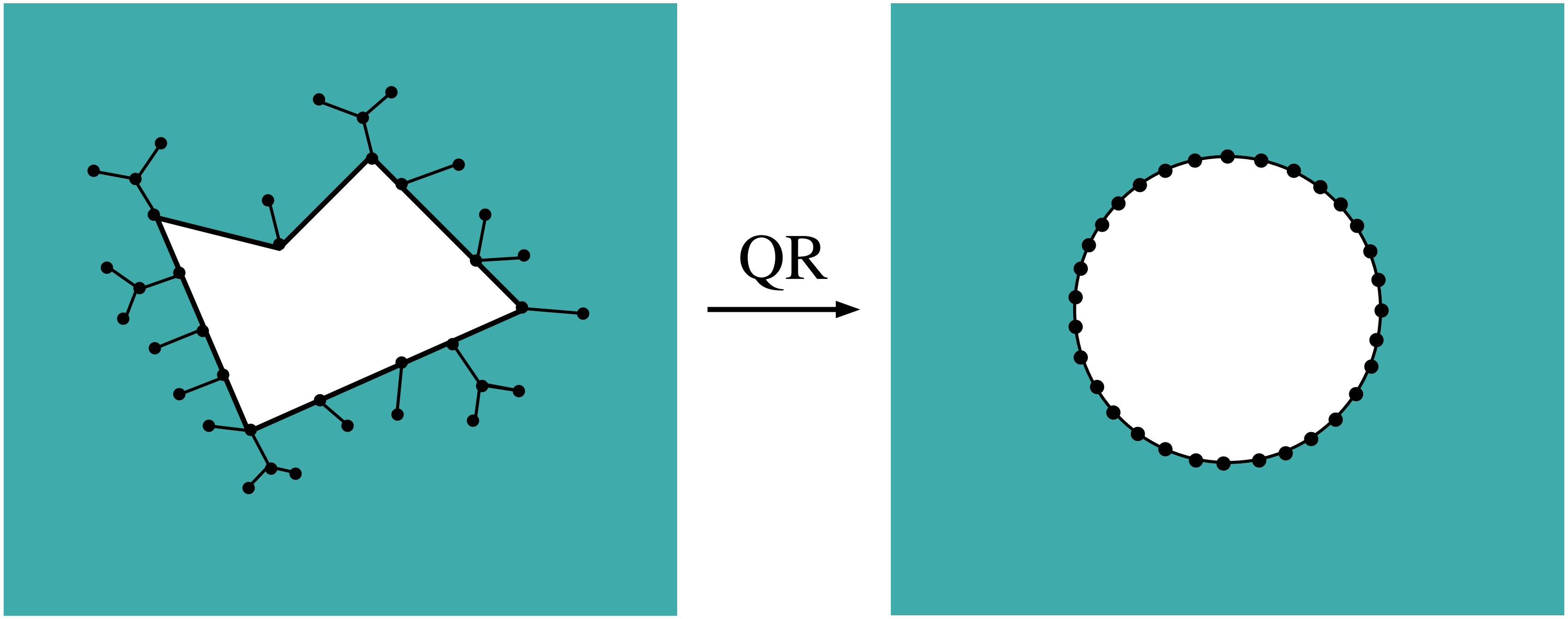


We can add short branches to make it almost balanced.

Exterior is QR mapped to evenly divided circle.

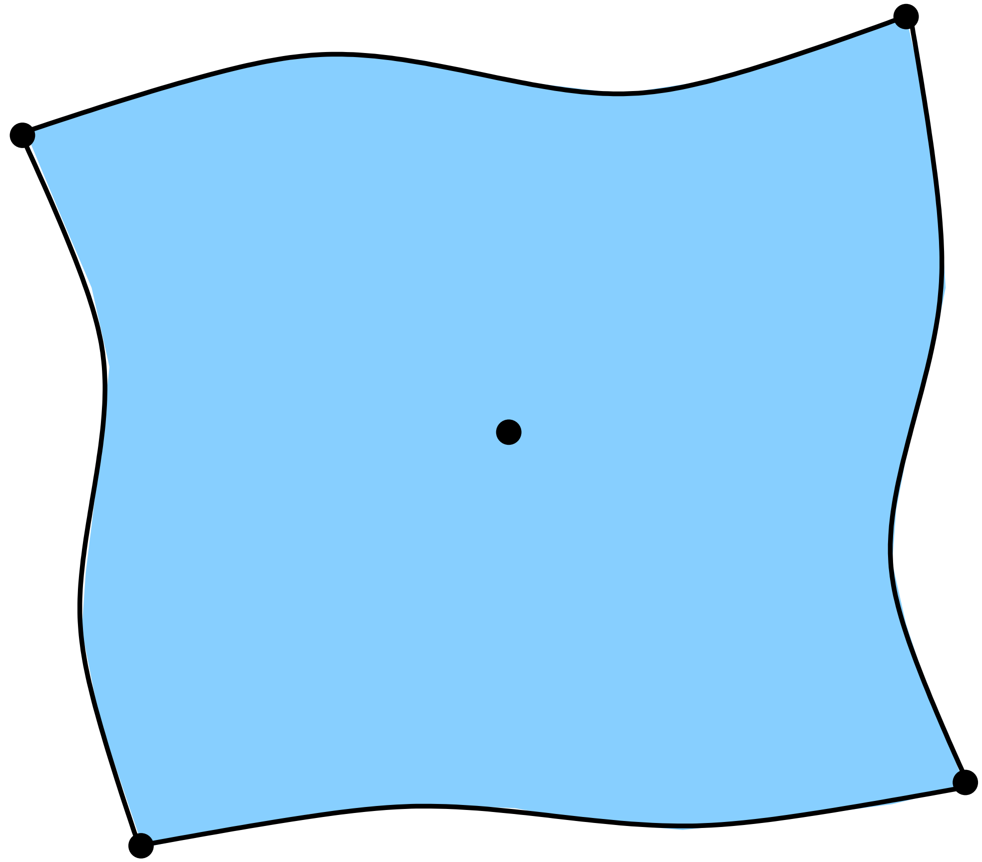
Again, QR distortion bounded, supported on tiny area.

More precise statement.

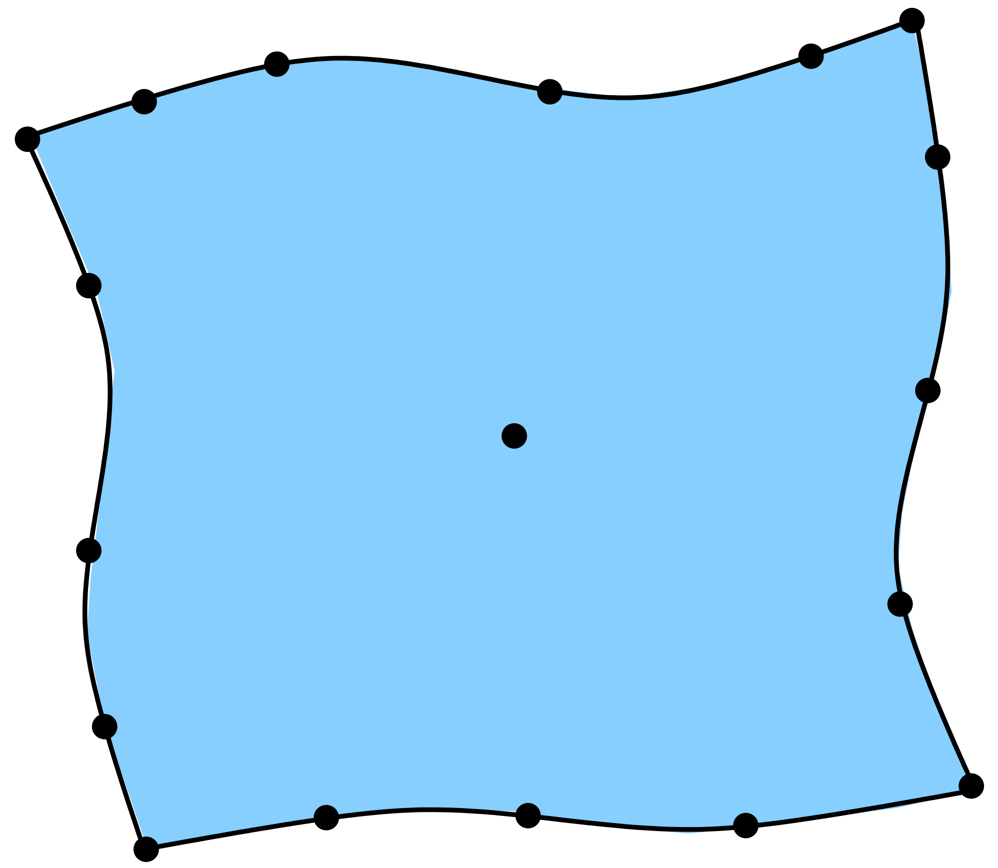


Important detail: map multiples arclength on each boundary segment.

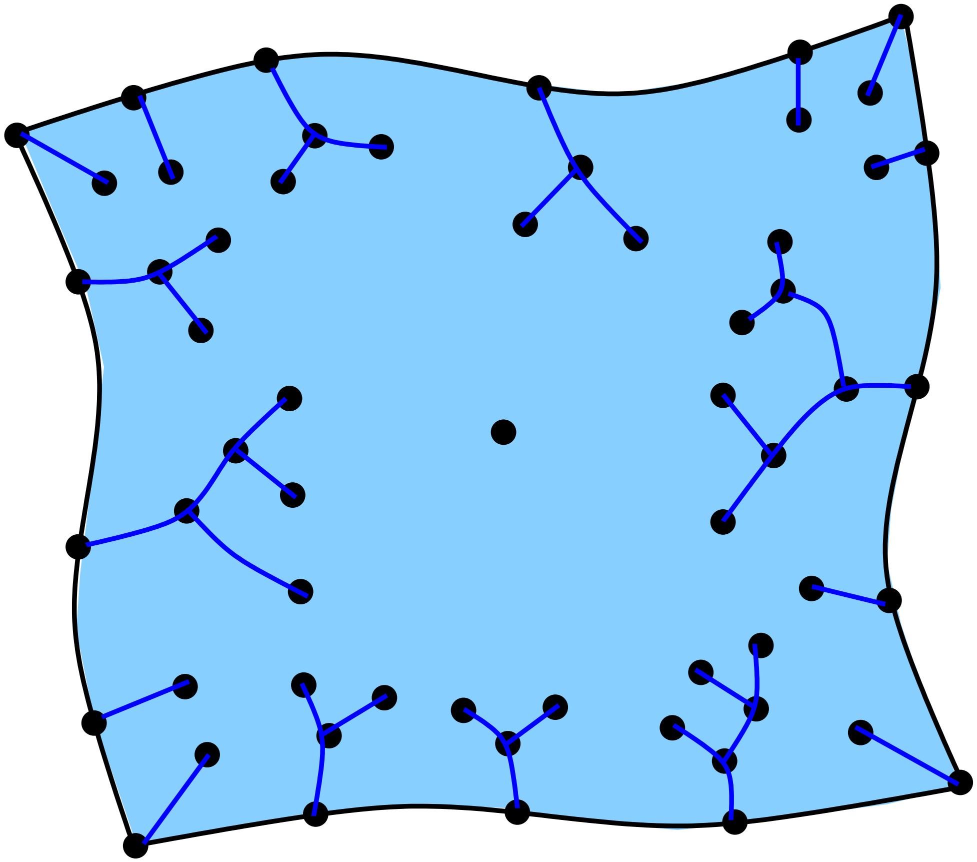
This allows us to "glue together" local maps to get a global QR map.



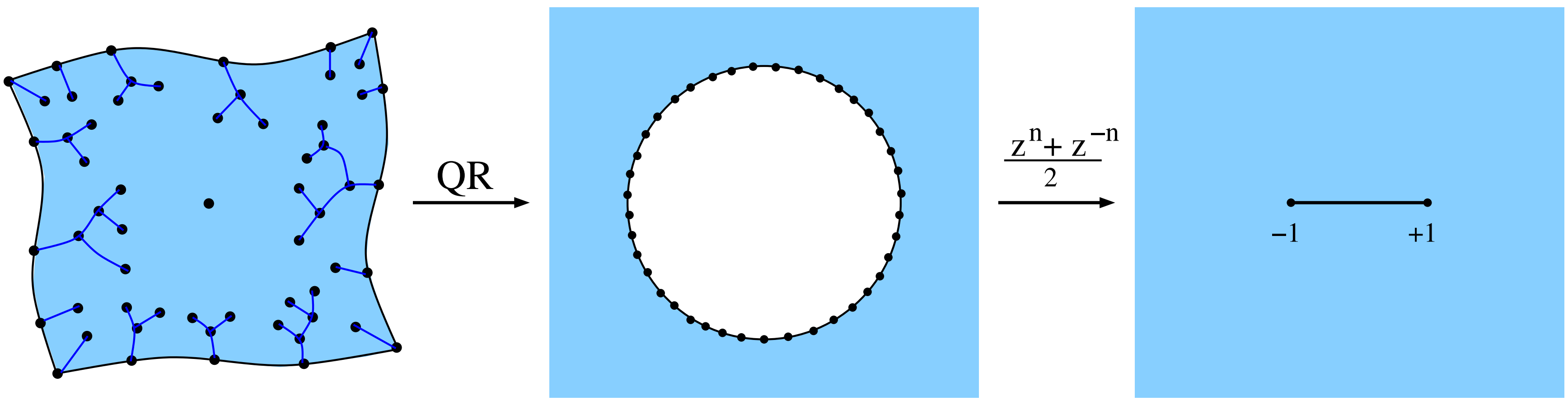
Start with a piecewise smooth domain.



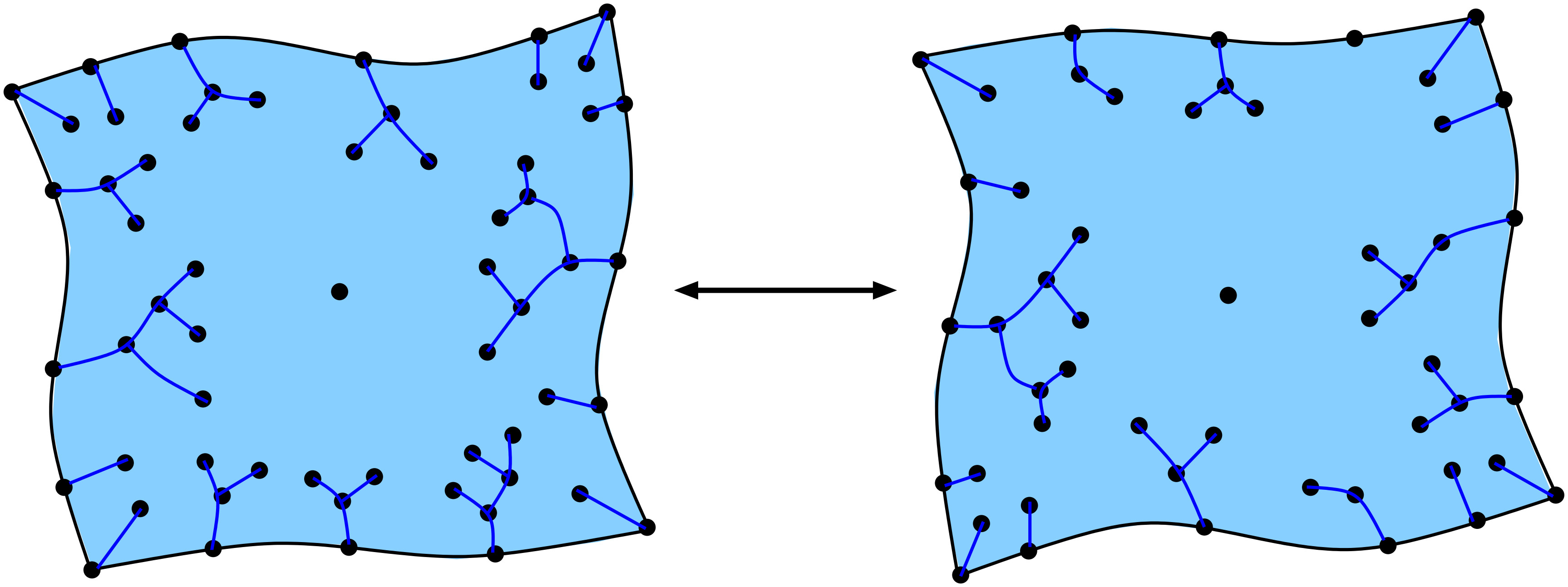
Add boundary points. Roughly evenly spaced.
Add interior point. Harmonic measures not all the same.



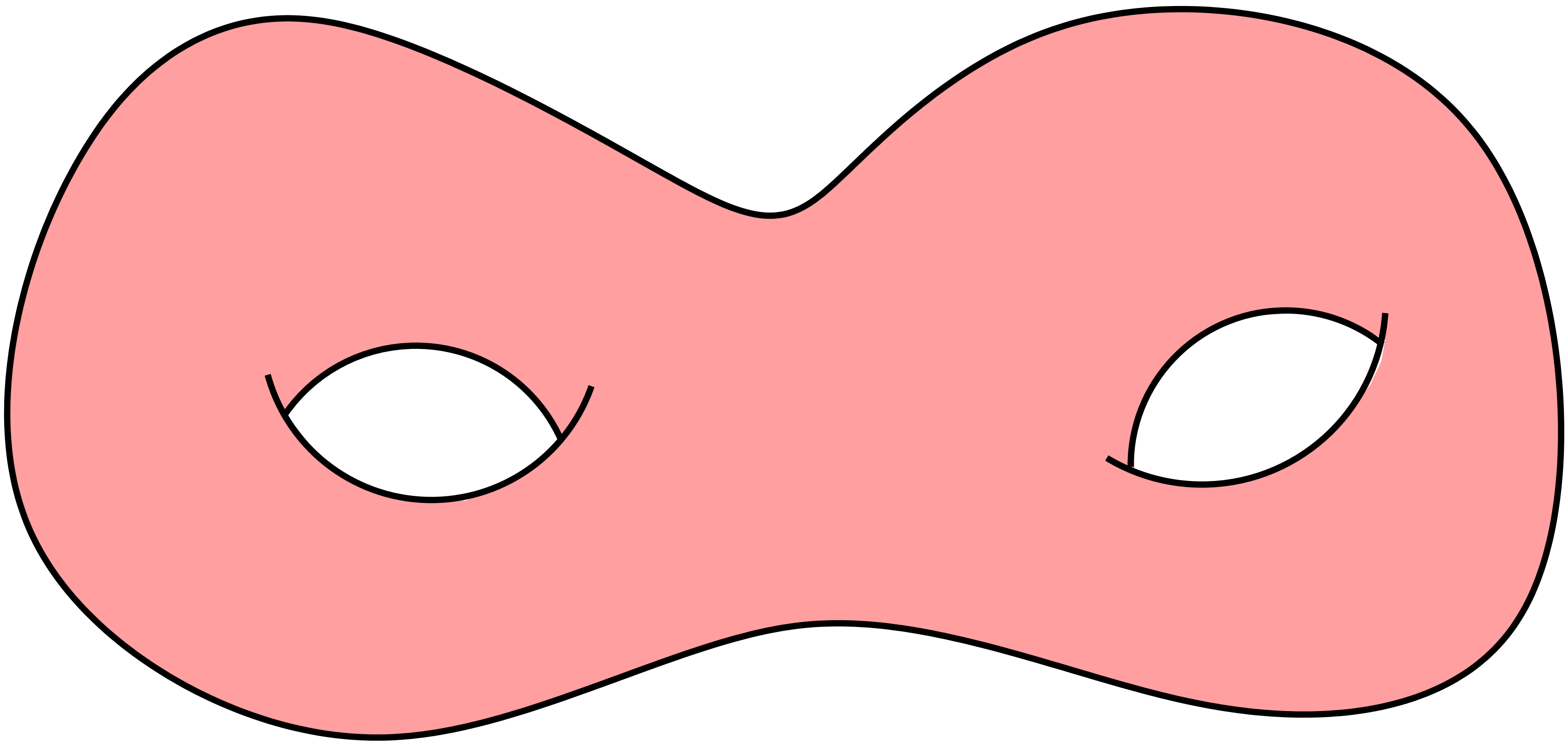
Add slits and trees so that all boundary arcs
have roughly the same harmonic measure.



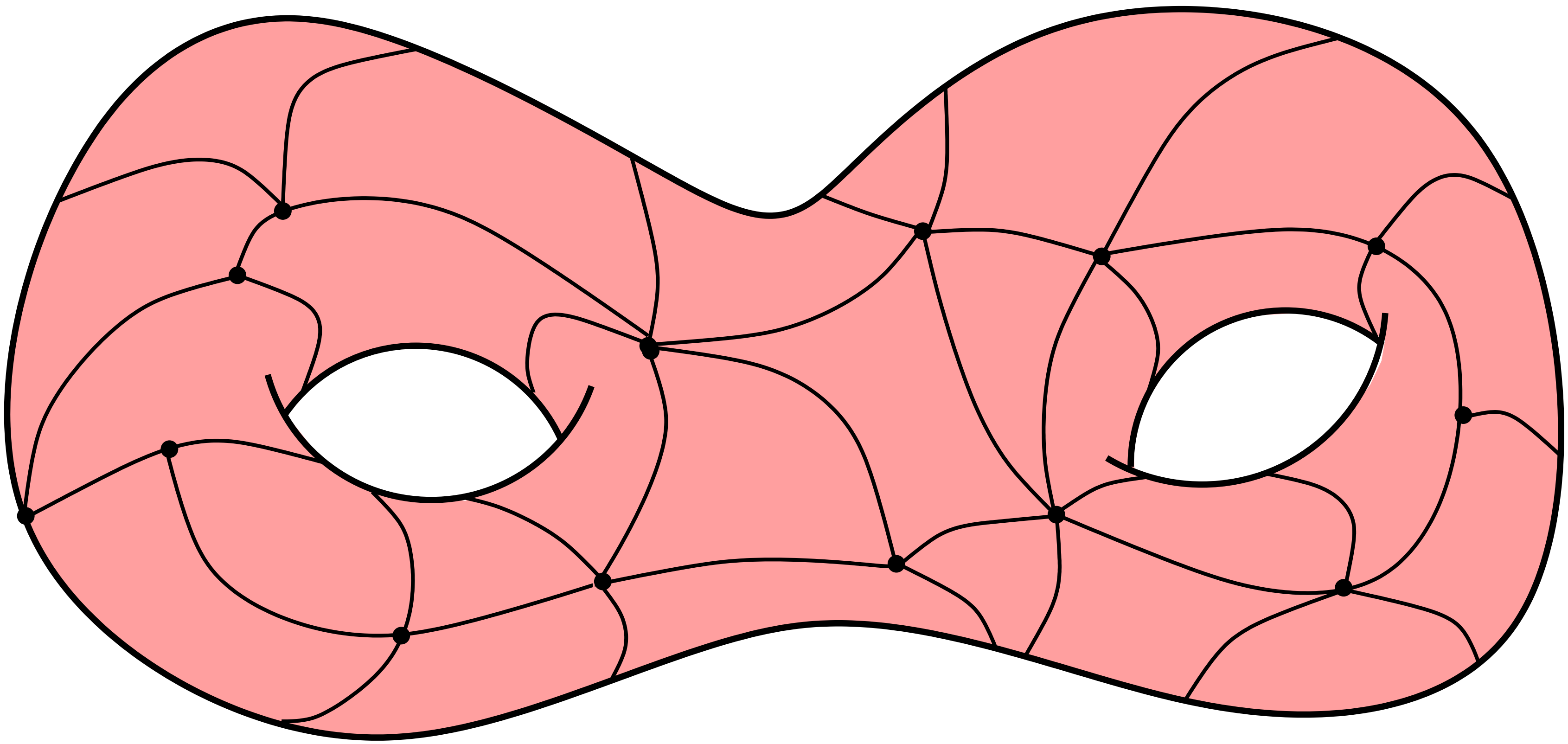
Find quasiregular map taking base point to ∞,
sending boundary arcs to equal intervals on circle.
Then each arc is mapped to $[-1, 1]$ via arclength.



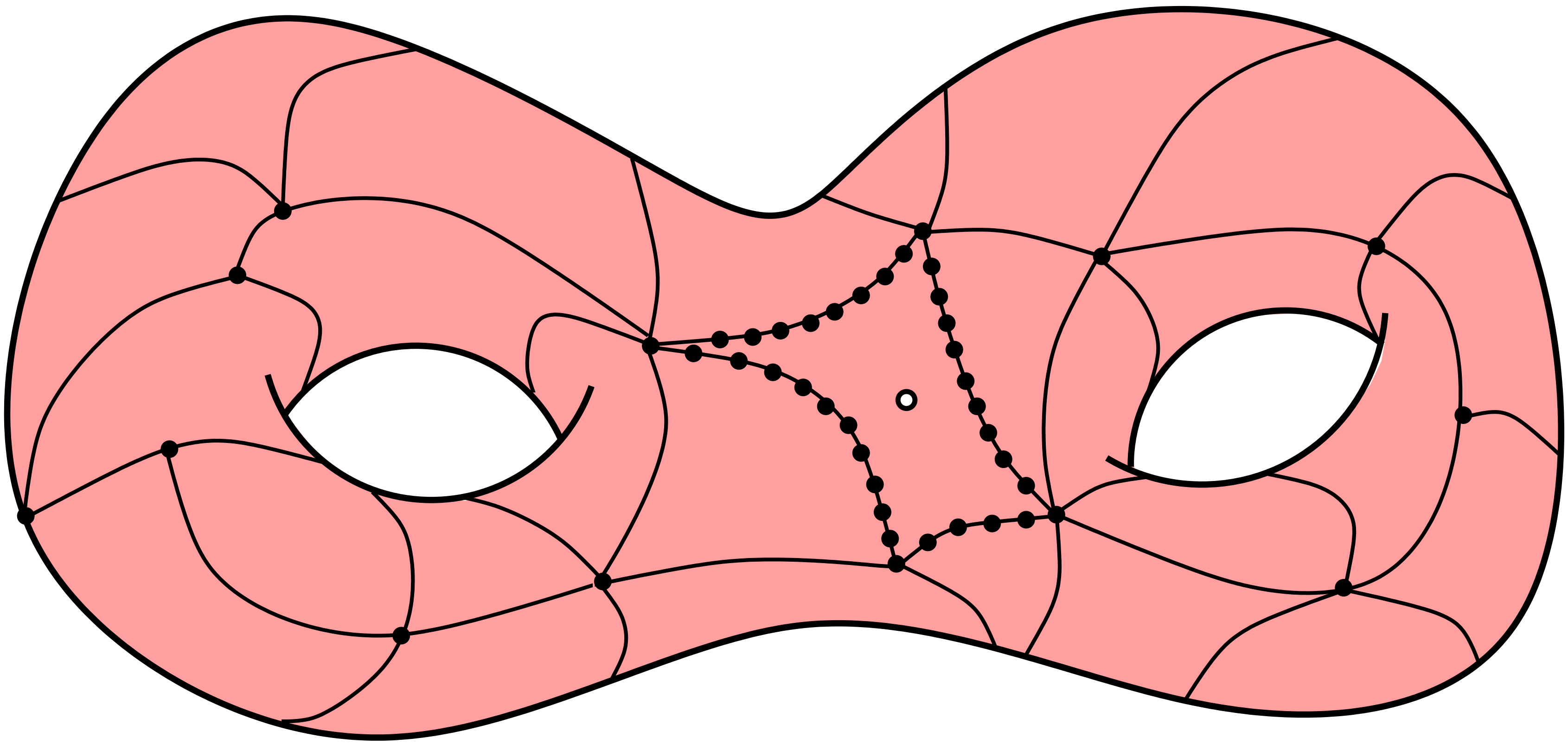
Arcs on both boundaries map to $[0, 1]$ via arclength.
We get a well defined, continuous function along boundary.
Uniformly quasiregular. Holomorphic except on small area.



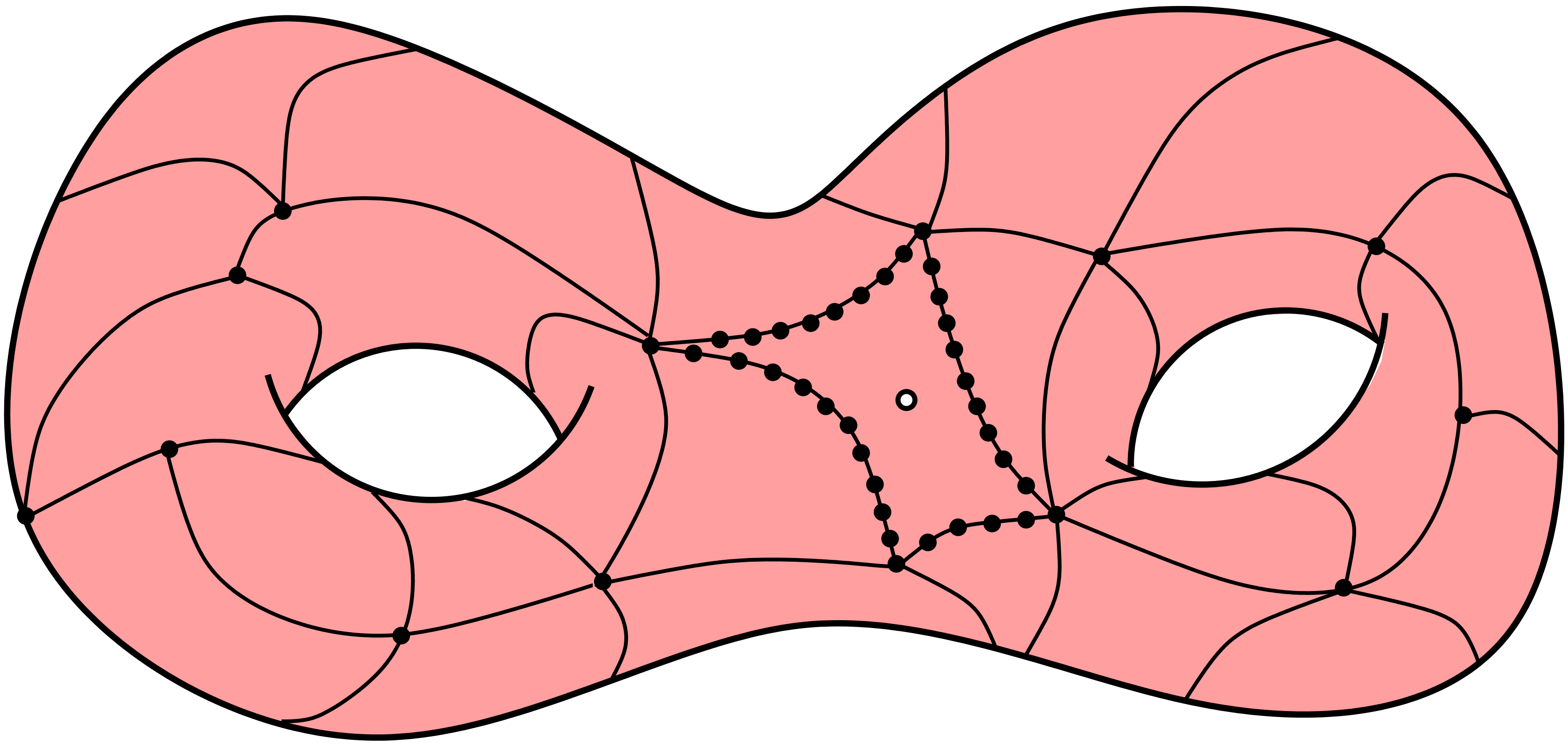
Now do this on a Riemann surface.



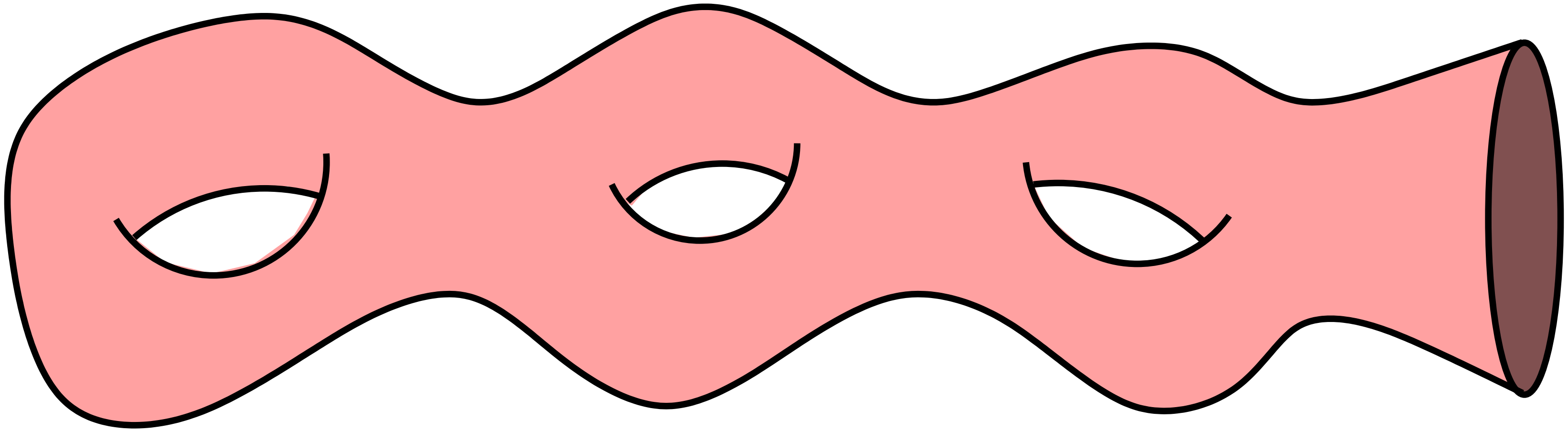
Cut the surface into nice pieces.



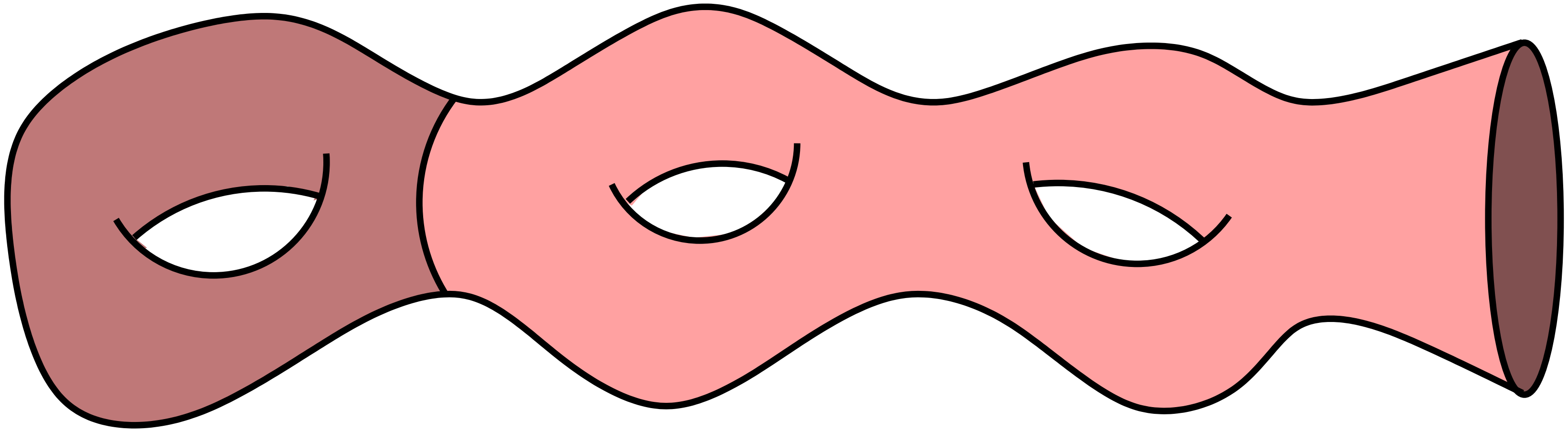
Subdivide edges into many small arcs.
Our construction gives QR maps on each piece.
Continuous across boundaries. QR on whole surface.



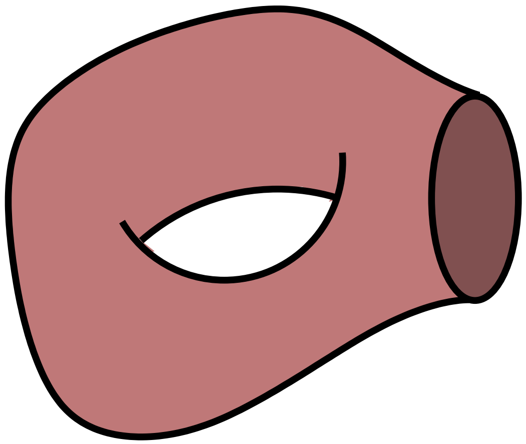
Change conformal structure to make QR map holomorphic.
Closely spaced dots $\Rightarrow$ small conformal change.
$\Rightarrow$ Belyi surfaces are dense in all surfaces.



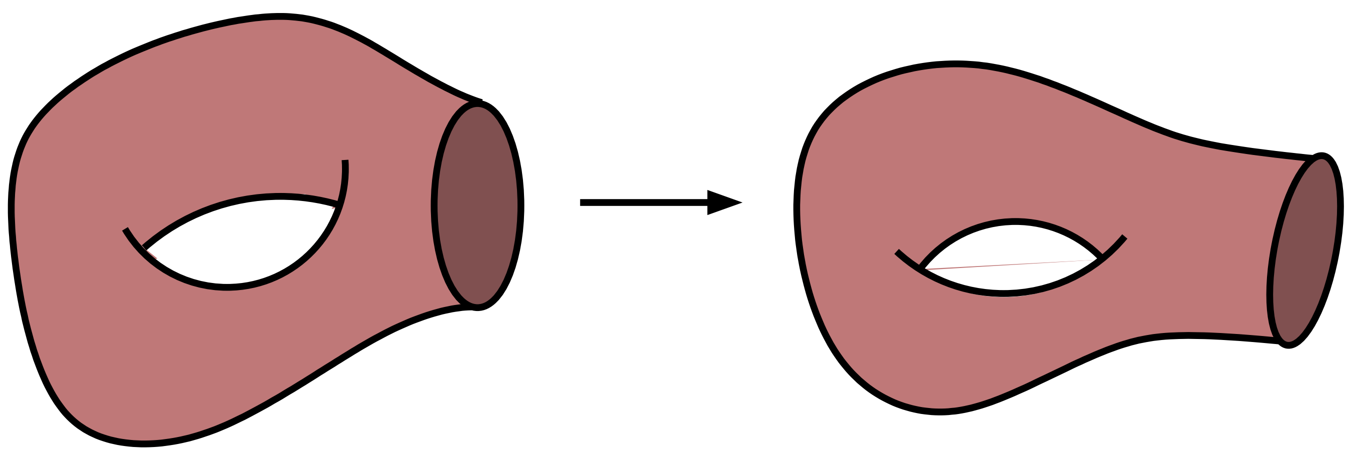
Consider a non-compact surface R.



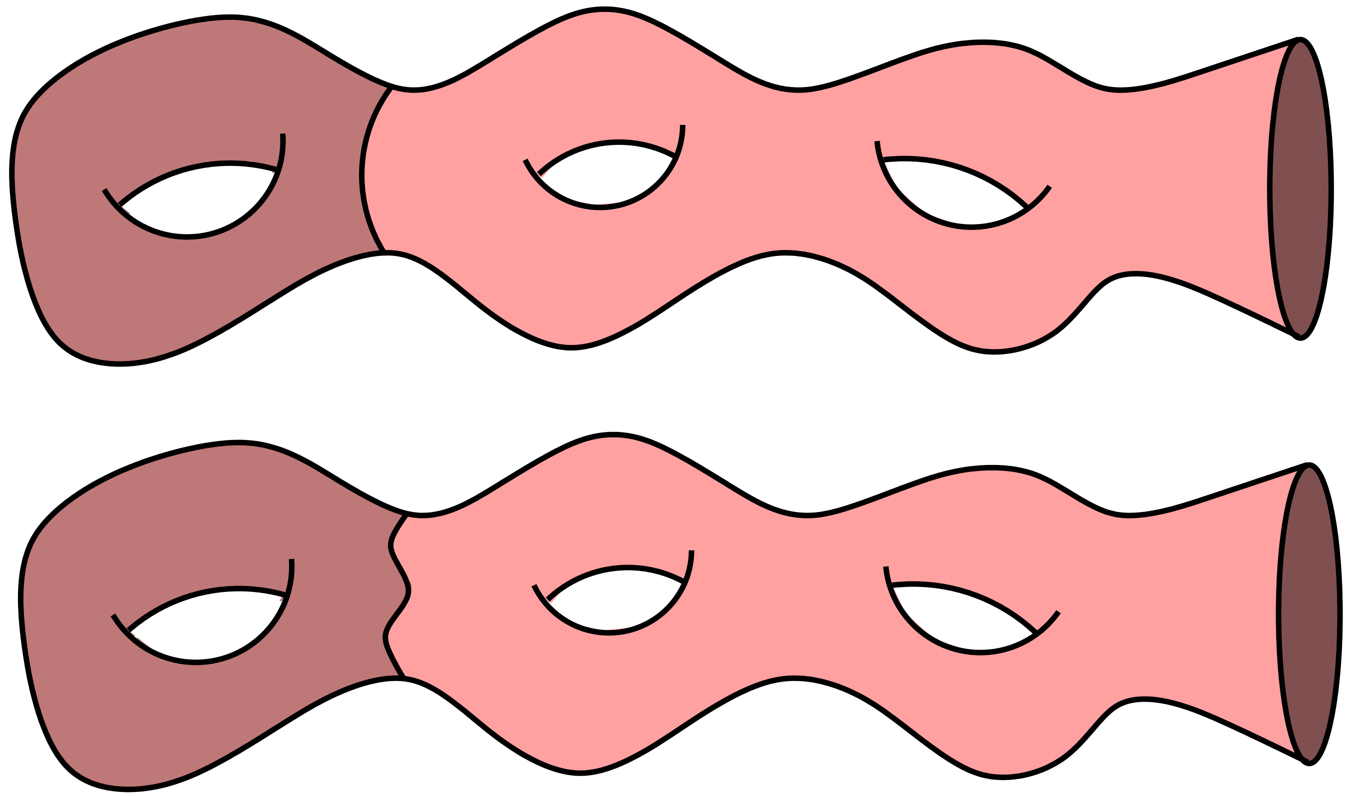
Take a compact, smoothly bordered piece Y_1.



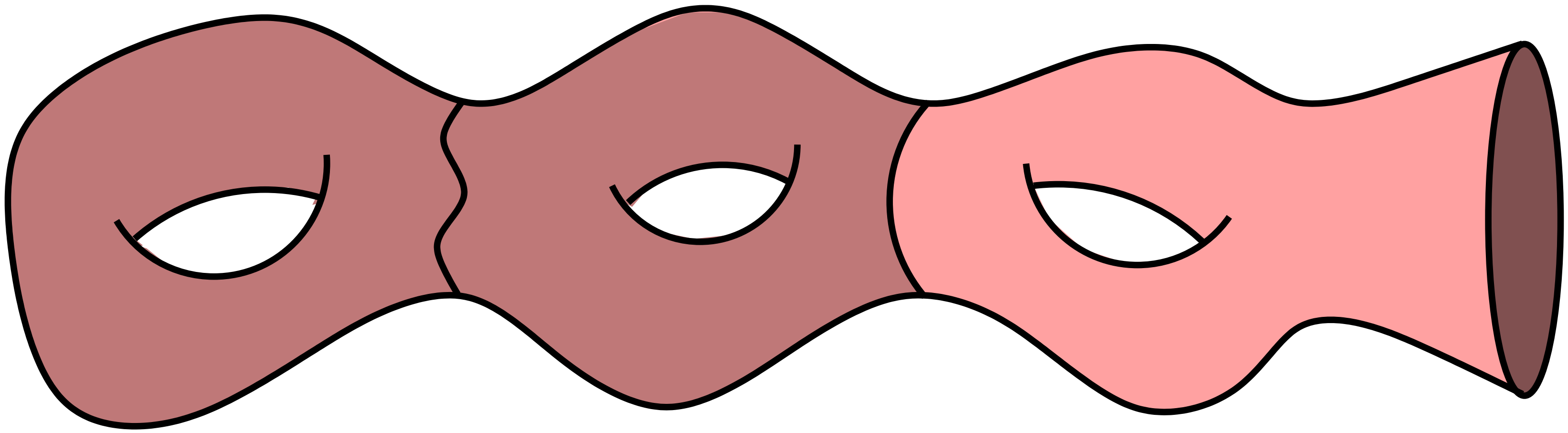
Take a compact, smoothly bordered piece Y_1.
Build a QR covering map on this piece as before.



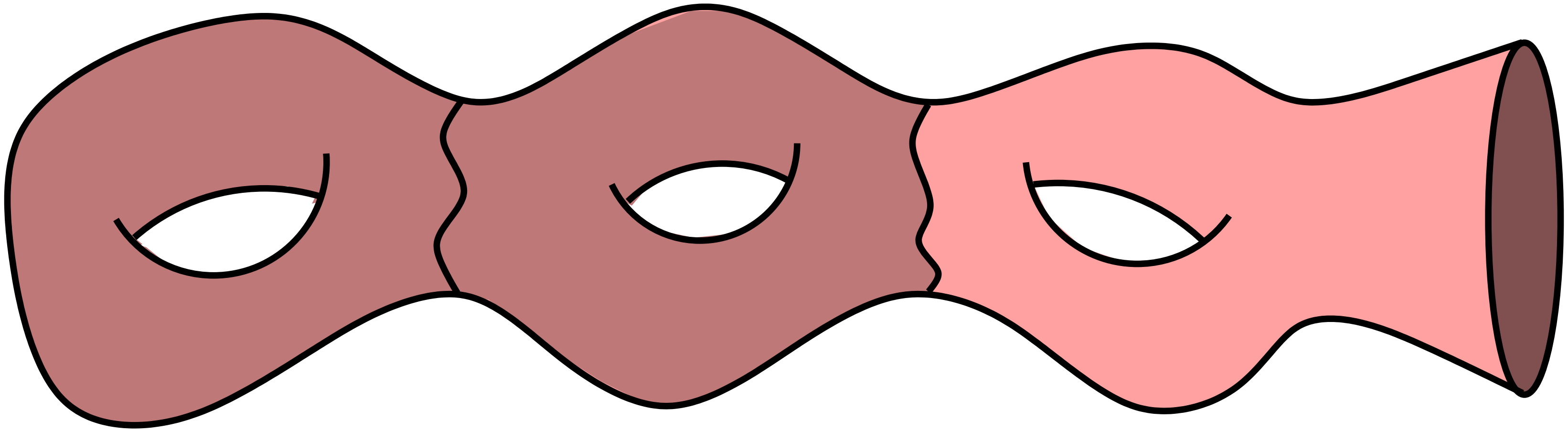
Take a compact, smoothly bordered piece Y_1.
Build a QR covering map on this piece as before.
Solve Beltrami on the compact piece. Get new surface $\widetilde{Y}_1 \neq Y_1$.



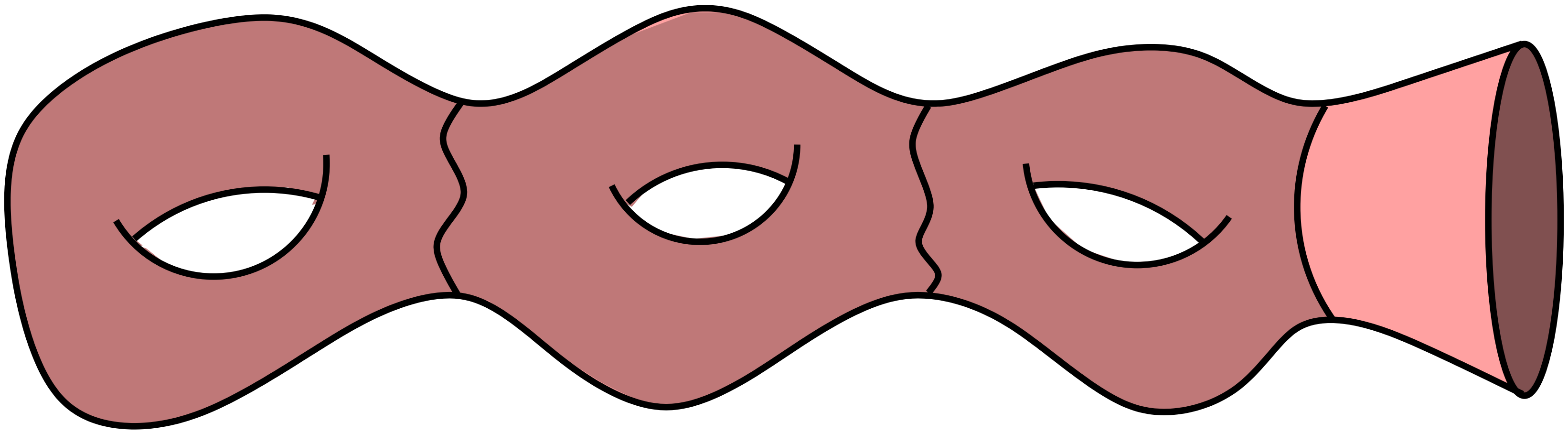
Key idea: small dilatation implies $\widetilde{Y}_1$ embeds in R.



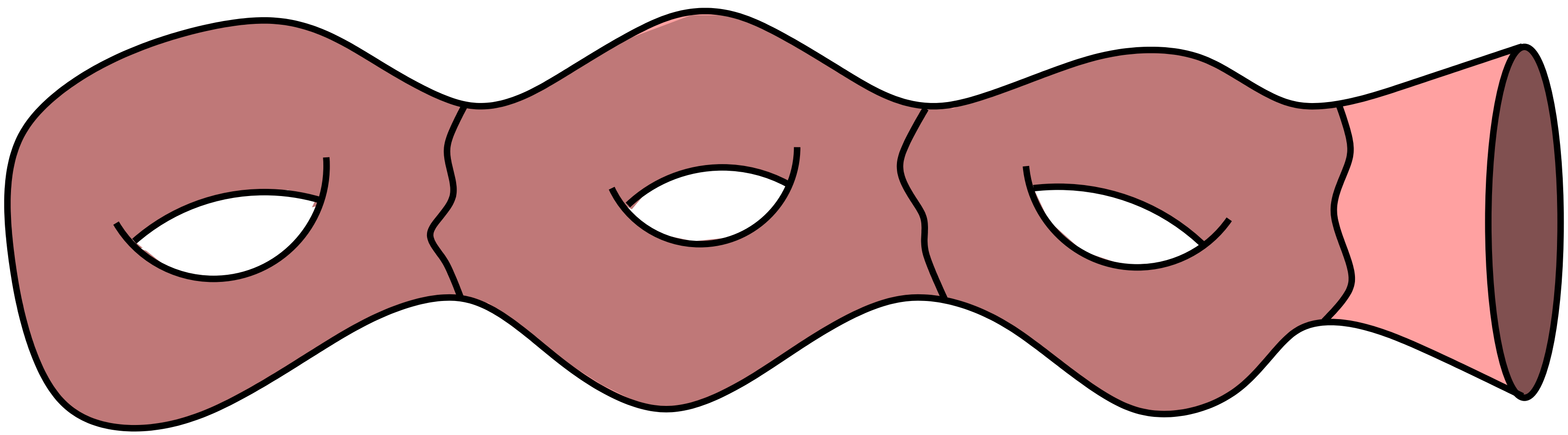
Take a larger compact, bordered piece Y_2.



Construct QR-Belyi, solve Beltrami equation, embed $\widetilde{Y}_2$ in R.



Induction: construct QR-Belyi, solve Beltrami equation, re-embed in R.



Induction: construct QR-Belyi, solve Beltrami equation, re-embed in R.

In limit, obtain a Belyi function on all of R.

Thm (B-Rempe): Every non-compact surface has a Belyi function.

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Corollary: Every non-compact surface can be obtained by gluing together countably many equilateral triangles.

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Corollary: A surface is non-compact iff it can be obtained in uncountably many combinatorially distinct ways.

Thm (B-Rempe): Every non-compact surface has a Belyi function.

Corollary: Every non-compact surface can be obtained by gluing together countably many equilateral triangles.

Corollary: A surface is non-compact iff it can be obtained in uncountably many combinatorially distinct ways.

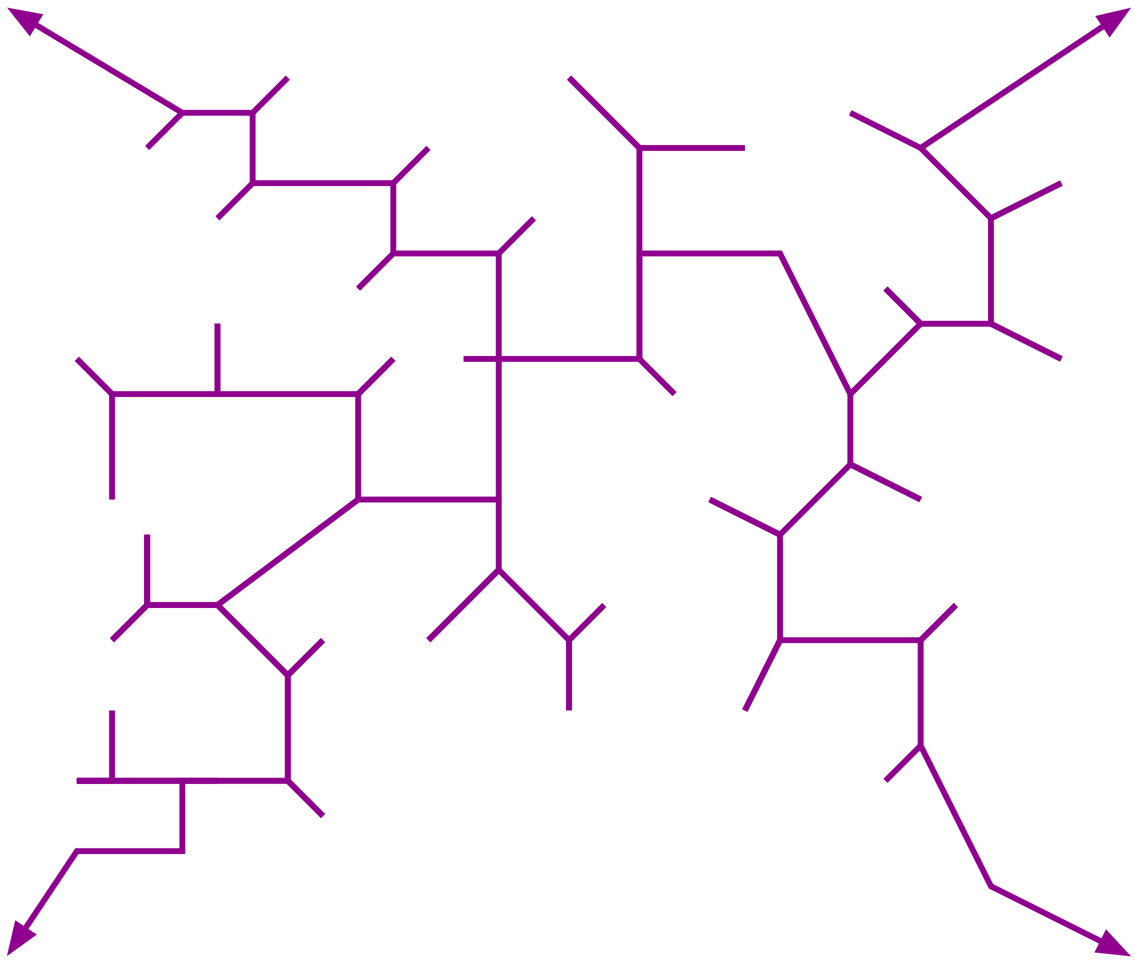
Corollary: Every Riemann surface is a branched cover of the sphere, branched over finitely many points.

For compact surfaces, this is Riemann-Roch.

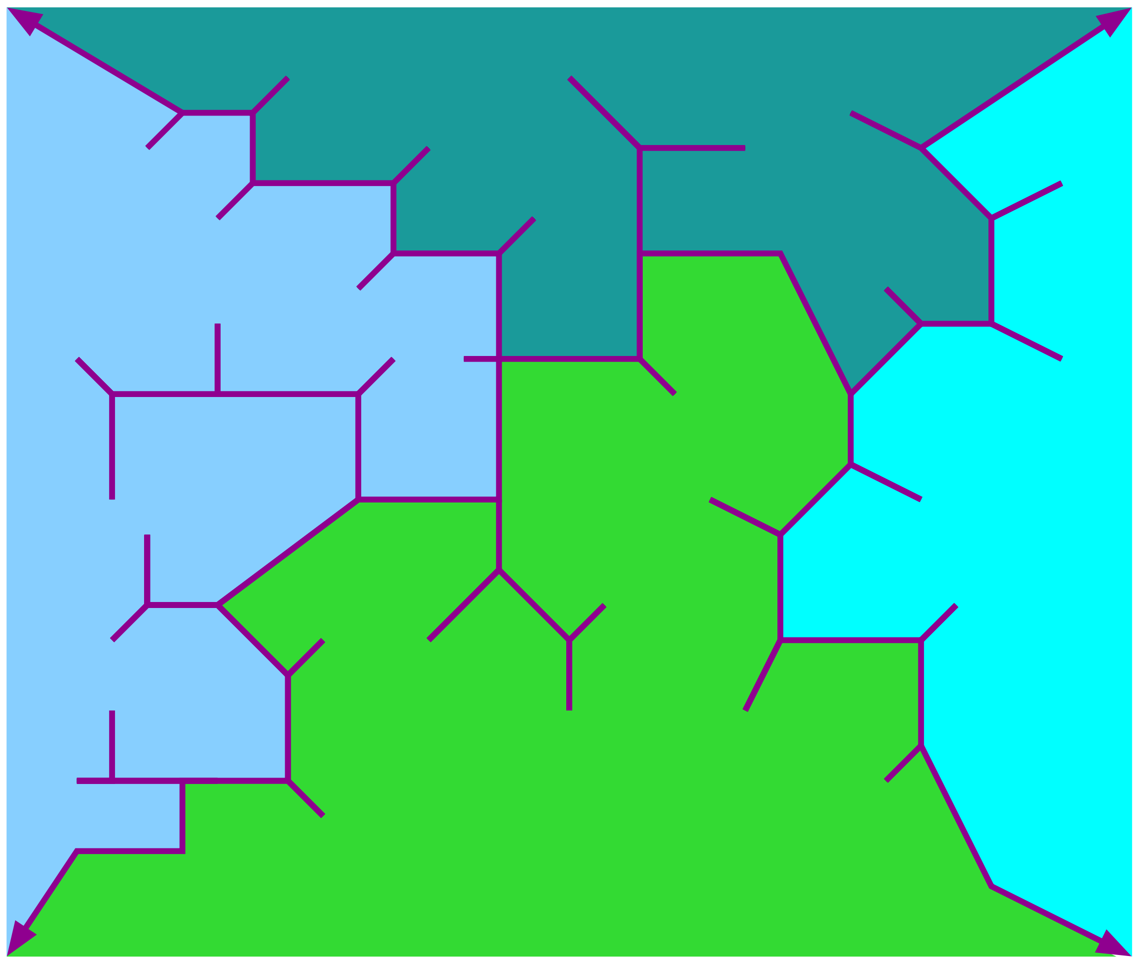
Compact, genus g sometimes needs $3g$ branch points.

3 branch points suffice for all non-compact surfaces.

THANKS



Do infinite trees correspond to entire functions with 2 critical values?

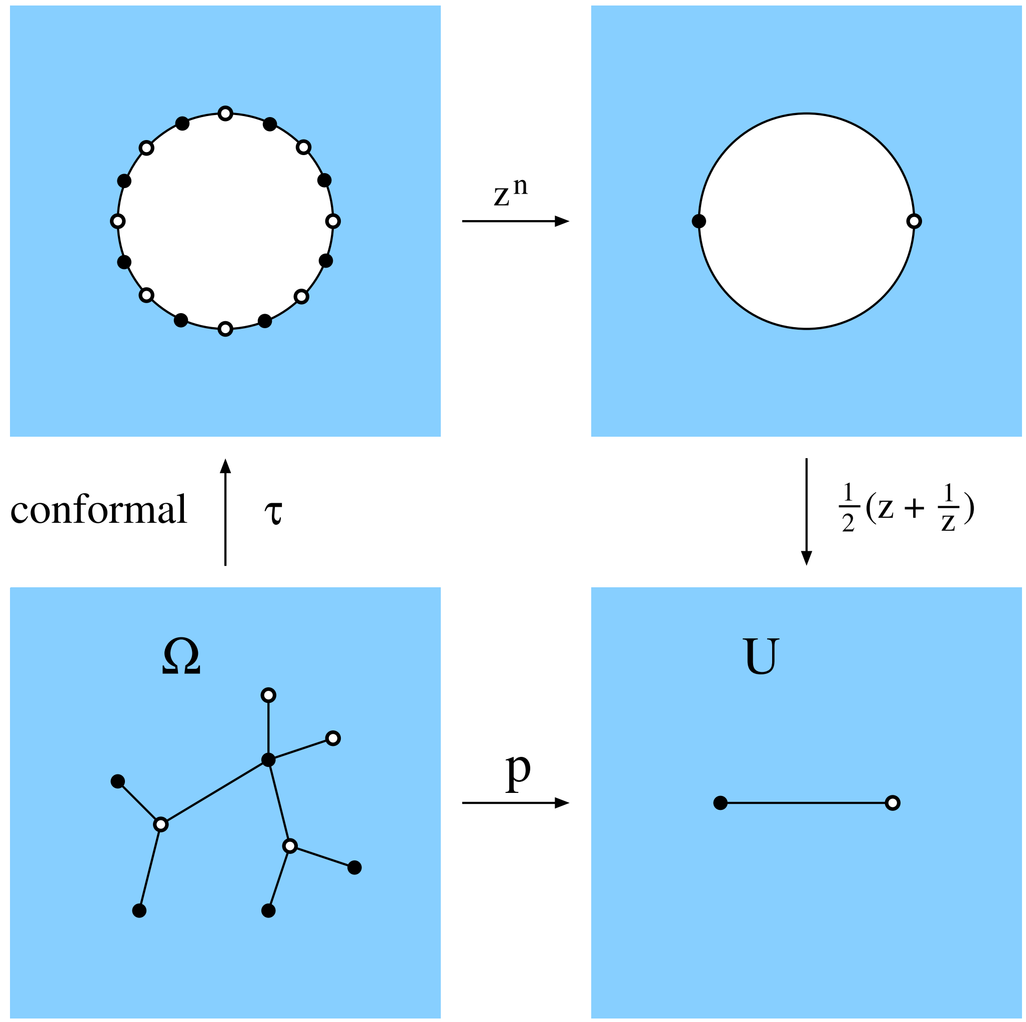


Main difference:

$\mathbb{C}\setminus$ finite tree = one topological annulus

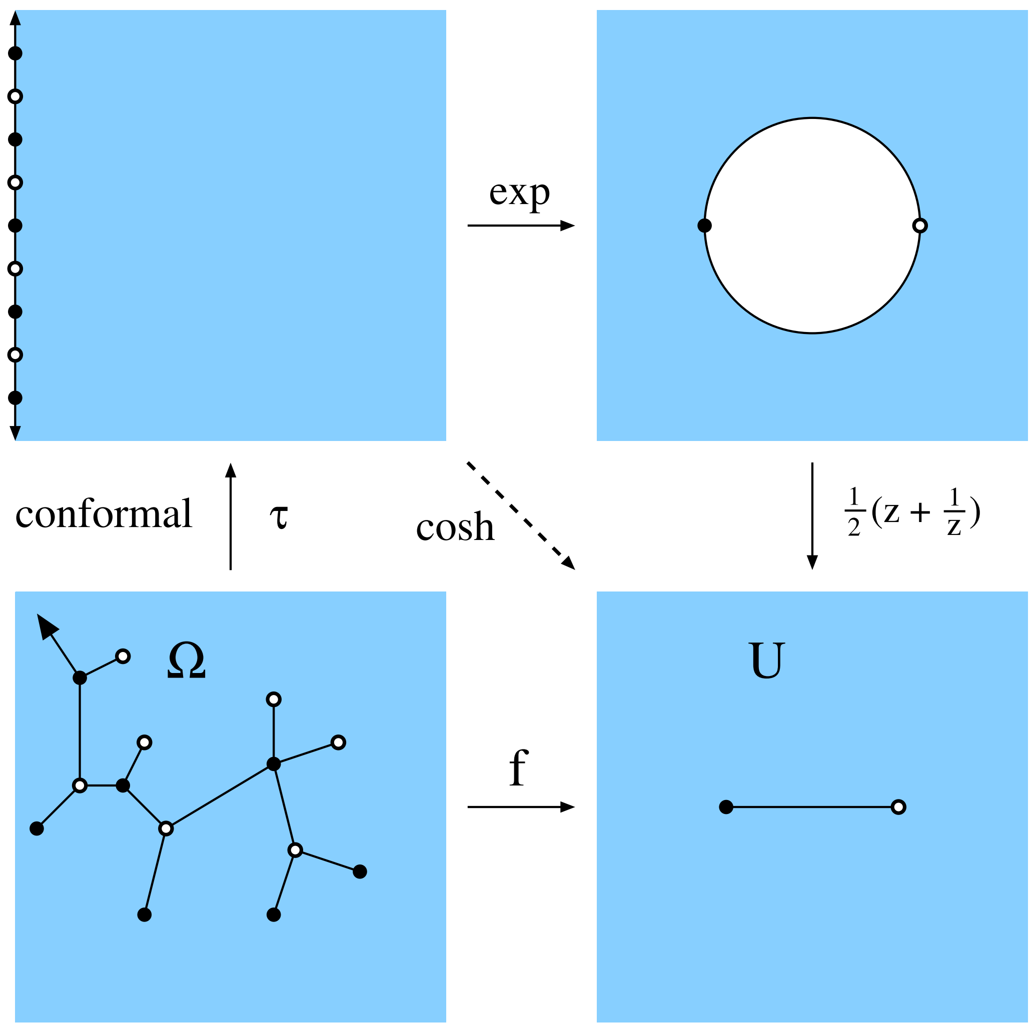
$\mathbb{C}\setminus$ infinite tree = many simply connected components

Recall finite case

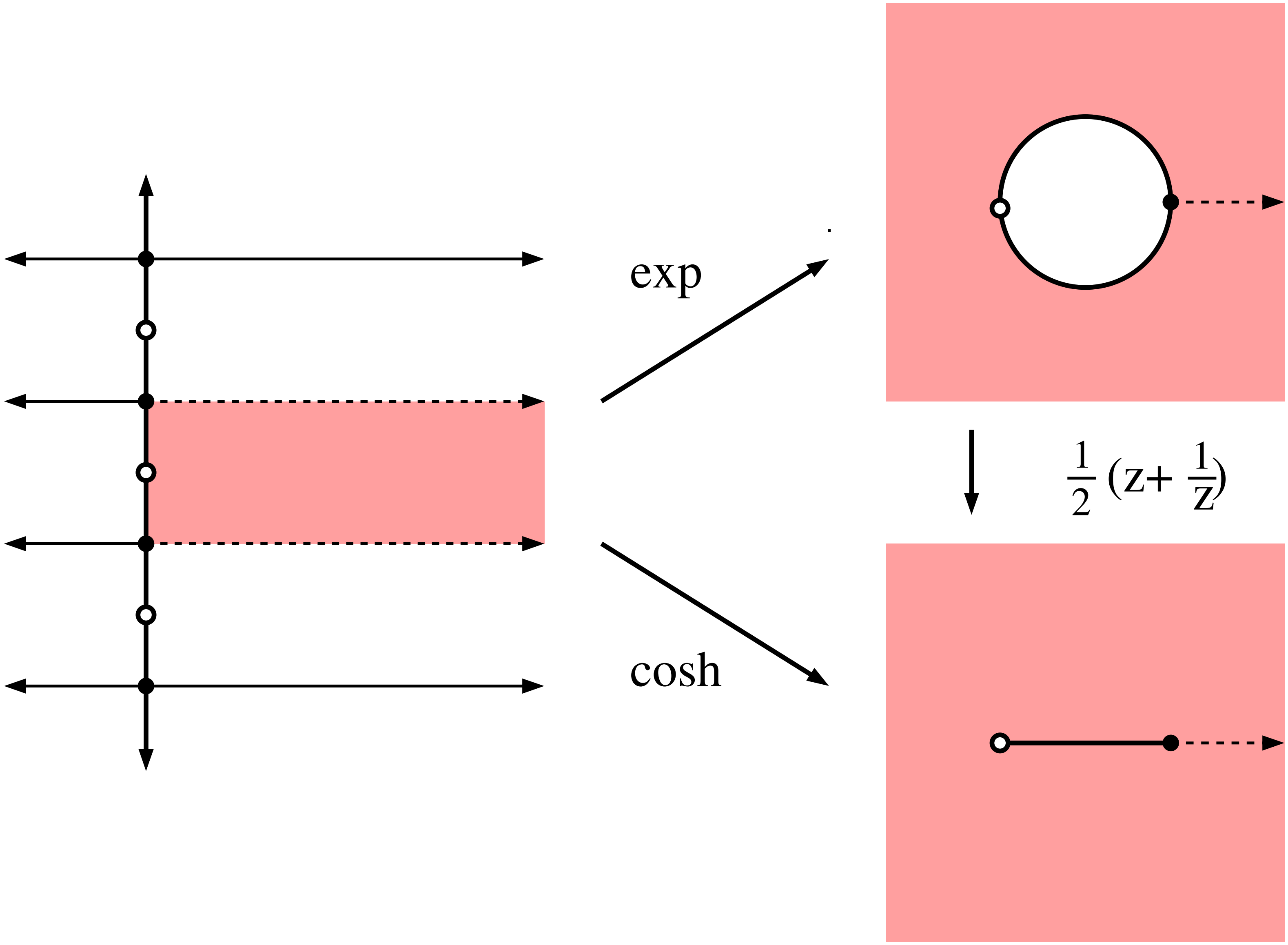


T is true tree $\Leftrightarrow p = \frac{1}{2}(\tau^n + 1/\tau^n)$ is continuous across T.

Infinite case



Infinite balanced tree $\Leftrightarrow$ $f = \cosh \circ \tau$ is continuous across T.



Definitions of exp and cosh.

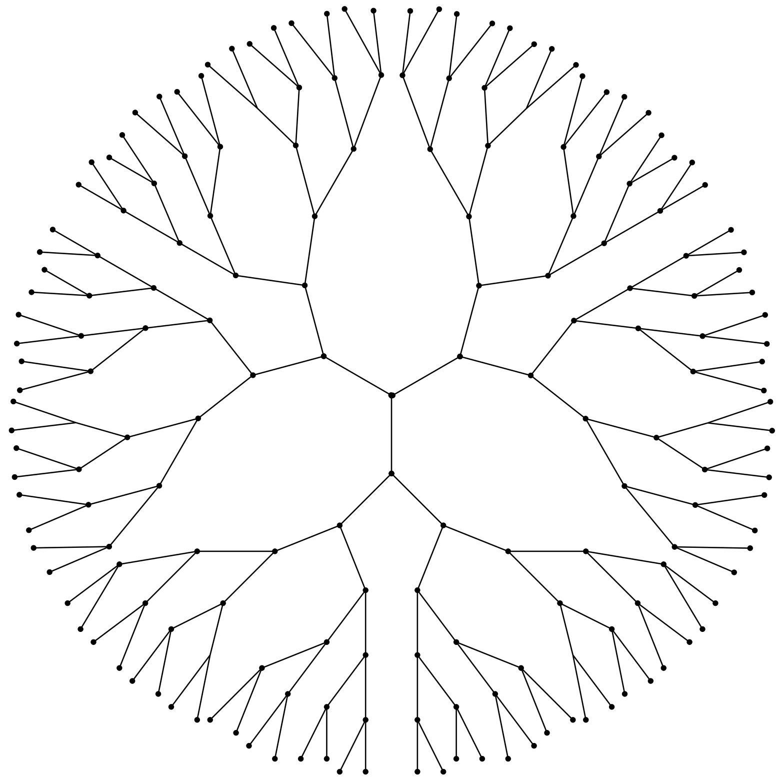
Do all infinite planar trees have true forms?

Do infinite true trees approximate any shape?

Do all infinite planar trees have true forms? **No.**

Do infinite true trees approximate any shape? **Yes (sort of).**

The infinite 3-regular tree has no true form in plane.



W. Cui, 2018: would contradict a theorem of Nevanlinna.

Now, the second question: which shapes can occur?

Vague theorem: every sufficiently nice infinite planar tree T is approximated by $f^{-1}([-1,1])$ for some entire function f with two singular values.

Both singular values are critical values at ± 1.

What does "nice" mean?

We will introduce two assumptions that substitute for finiteness, but first we define a certain neighborhood $T(r)$ of an infinite tree.

If e is an edge of T and $r > 0$ let

$$e(r) = \{z : \mathrm{dist}(z, e) \le r \cdot \mathrm{diam}(e)\}$$

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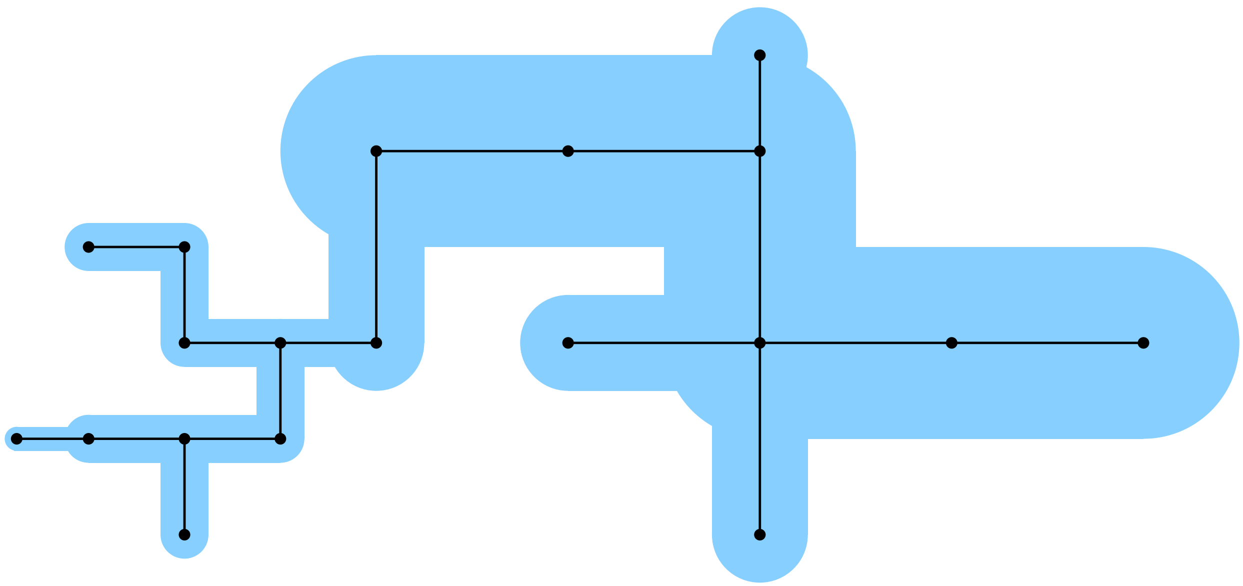
Define neighborhood of T: $T(r) = \cup\{e(r) : e \in T\}$.



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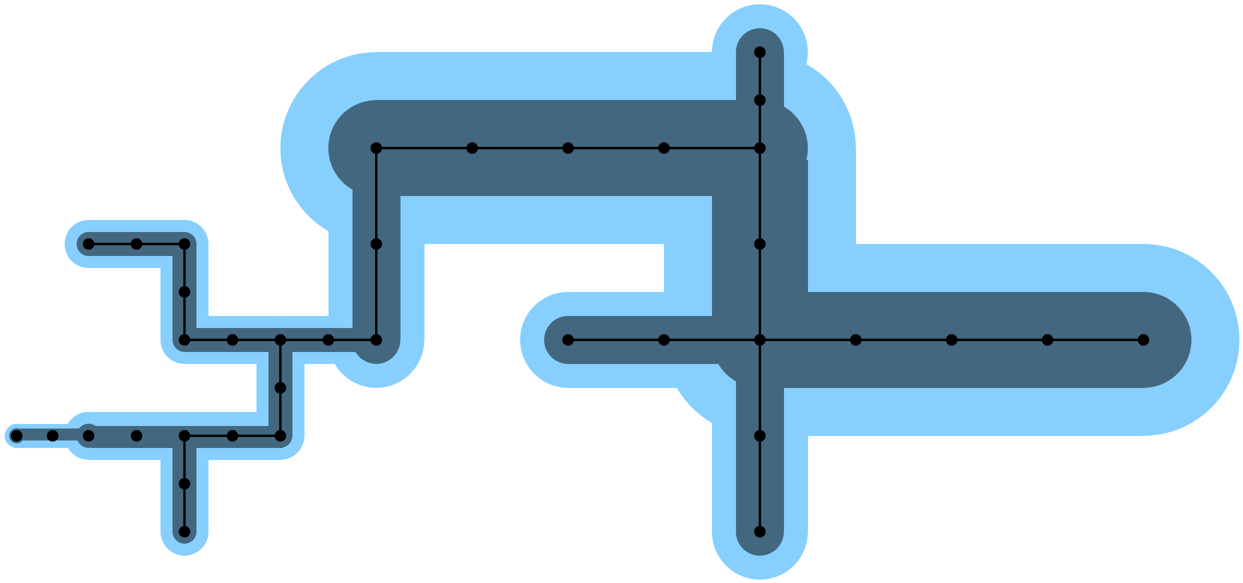
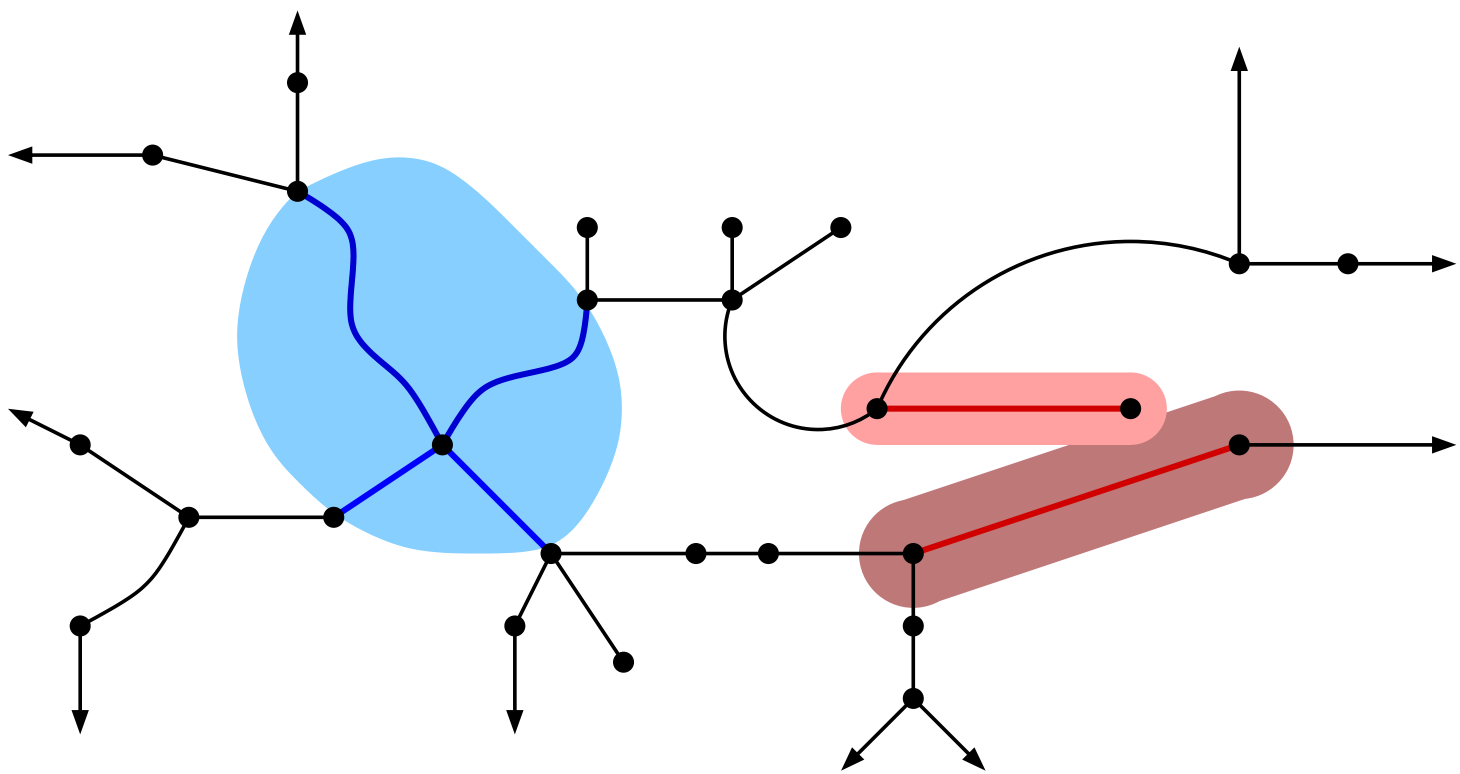


Adding vertices reduces $T(r)$. Useful scaling property.

(1) Bounded Geometry (local condition; easy to verify):

- edges are uniformly smooth.
- adjacent edges form bi-Lipschitz image of a star $= \{z^n \in [0, r]\}$
- non-adjacent edges are well separated,

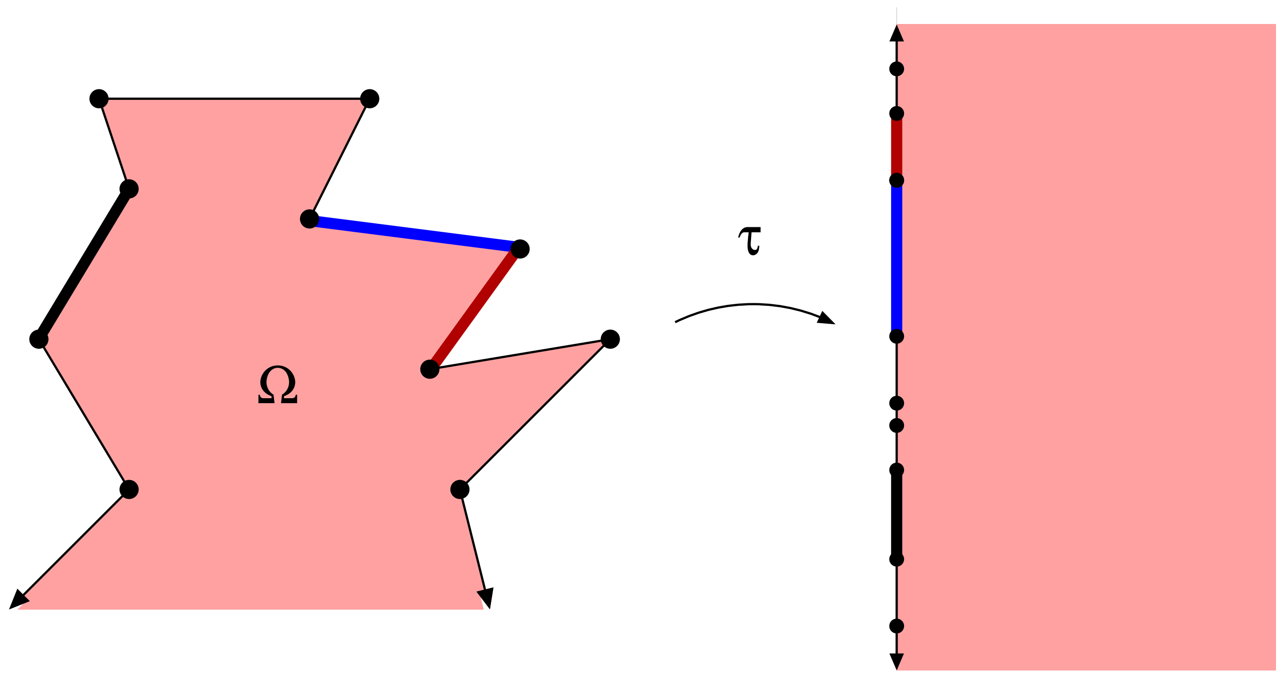
$$\mathrm{dist}(e, f) \geq \epsilon \cdot \min(\mathrm{diam}(e), \mathrm{diam}(f)).$$



(2) τ-Lower Bound (global condition; harder to check):

Complementary components of tree are simply connected.

Each can be conformally mapped to right half-plane. Call map τ.

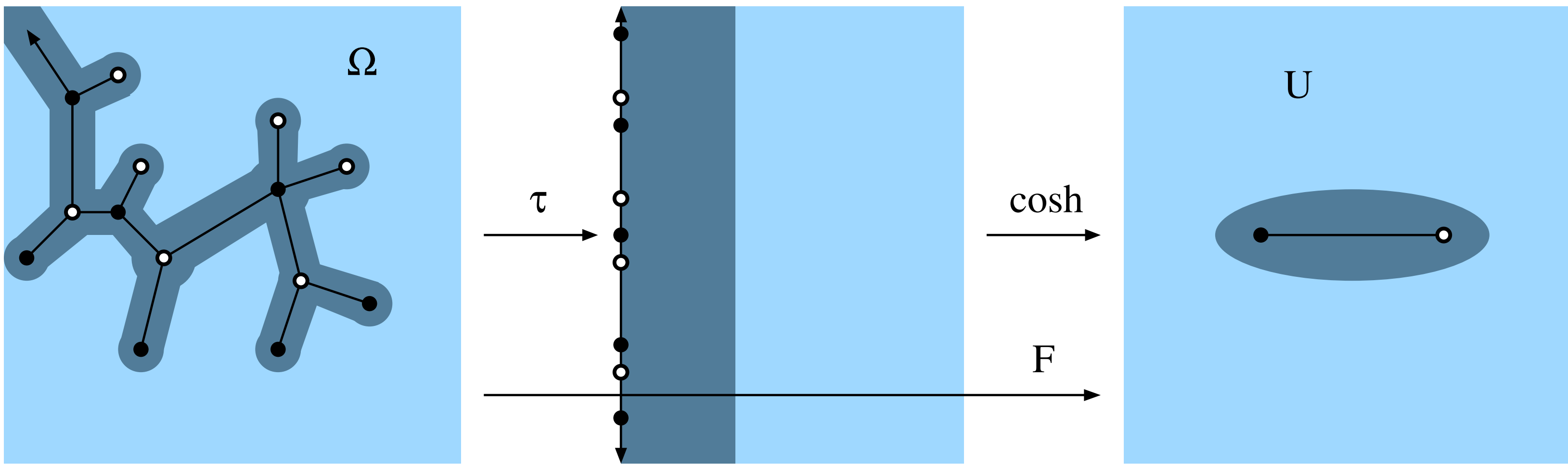


We assume all images have length $\geq \pi$.

Need positive lower bound; actual value usually not important.

Components are "thinner" than half-plane near ∞.

Folding Theorem: Suppose (1) T has bounded geometry and (2) T satisfies the τ-lower bound. Let $F = \cosh \circ \tau$. Then there is a K-QR g and $r > 0$ such that $g = F$ off $T(r)$ and $\mathrm{CV}(g) = \pm 1$.



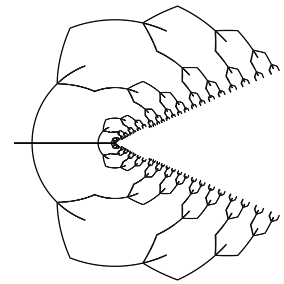
K and r depends only on the bounded geometry constants.

F may be discontinuous across T. g is continuous everywhere.

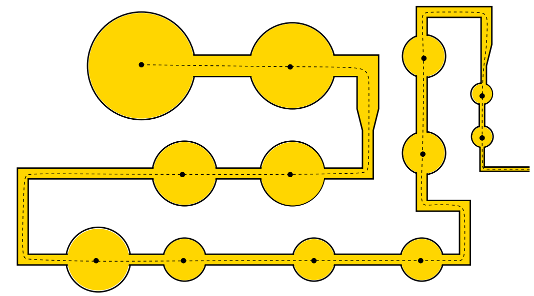
There is QC φ so $f = g \circ \varphi$ is entire, and $f^{-1}([-1, 1]) \approx T$.

APPLICATIONS OF QC-FOLDING

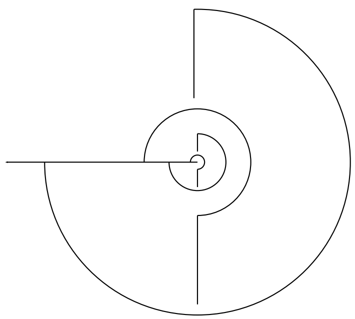
EL Models



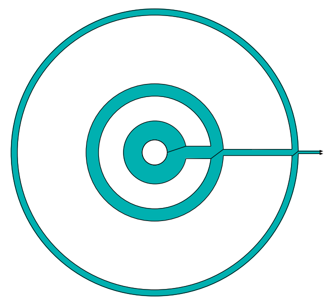
Order Conjecture



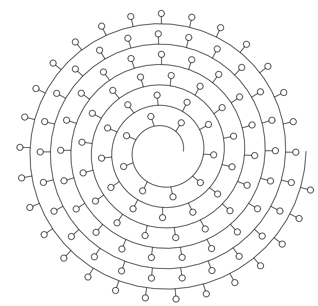
Post-singular dynamics



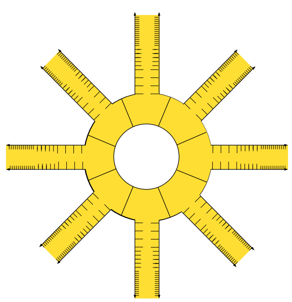
Eremenko Conjecture



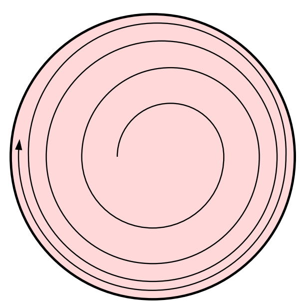
Near zero



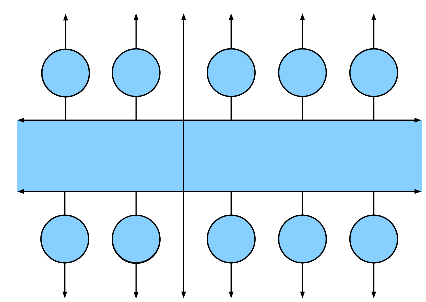
Wiman's Conjecture



Dimension near 1



Folding in disk

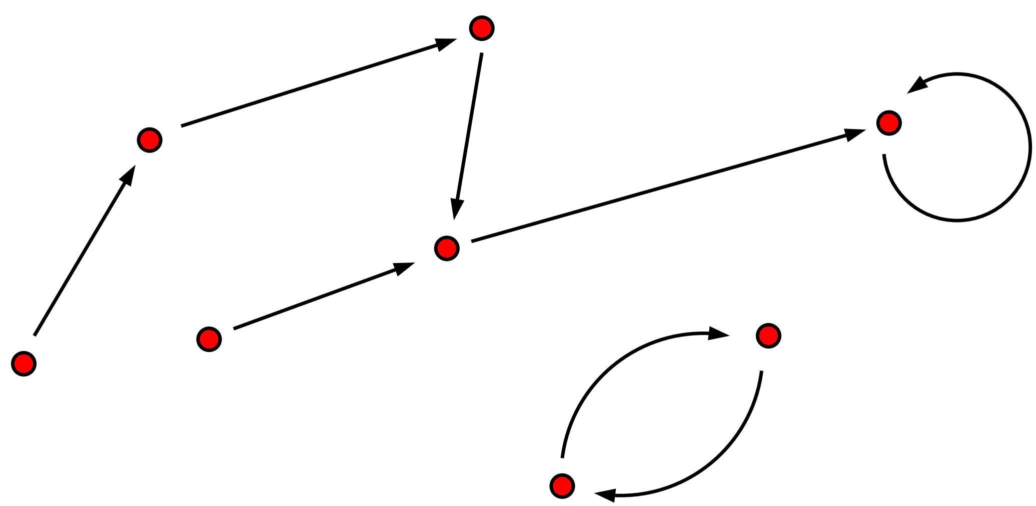


Wandering domain

Prescribing the post-critical dynamics:

DeMarco, Koch and McMullen proved that a post-critically finite rational map can have **any** given dynamics.

More precisely, given any $\epsilon > 0$, any finite set X and any map $h : X \to X$ there is a rational r whose post-singular set $P(r)$ is the same size as X, $P(r)$ ϵ-approximates X in Hausdorff metric, and $|r - h| < \epsilon$ on P.



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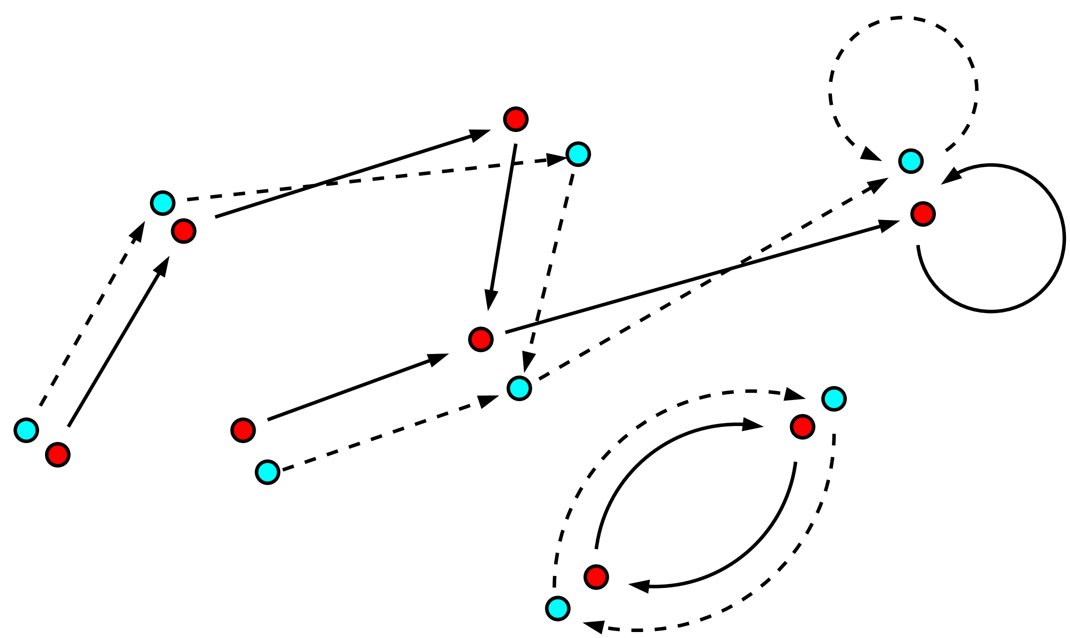
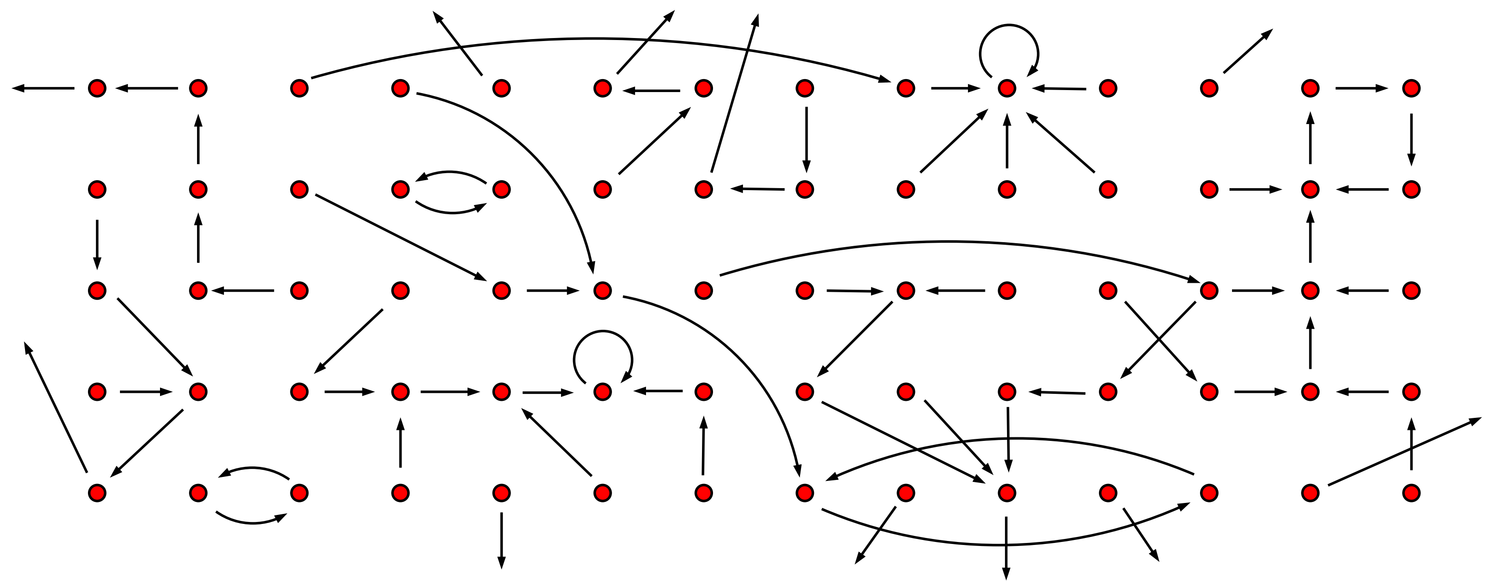




Theorem (Lazebnik, B.): Let X be discrete (≥ 4 points), let $h : X \to X$ be any map, and let $\epsilon > 0$. Then there is a transcendental meromorphic function f and a bijection $\psi : X \to P(f)$ so that

$$|\psi(z) - z| \leq \max(\epsilon, o(1)) \quad \text{and} \quad f = \psi \circ h \circ \psi^{-1} \text{ on } P(f).$$

Main argument (folding + fixed point thm) due to Kirill.

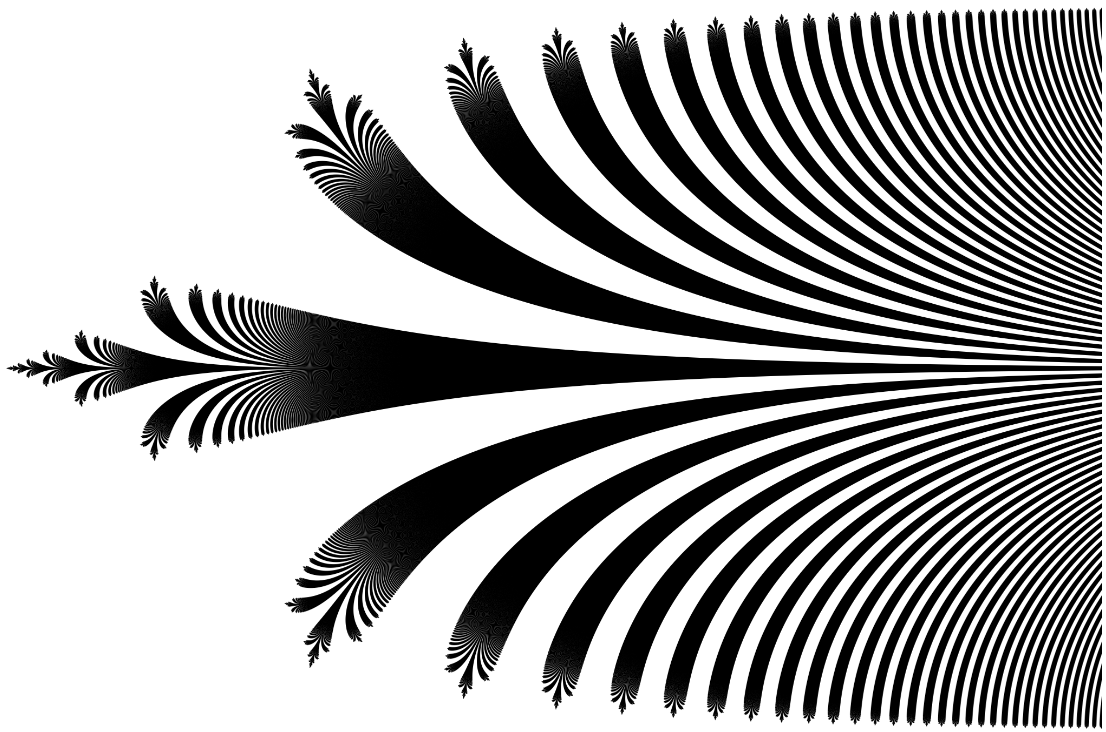
Dimensions of transcendental Julia sets:

Given an entire function f,

Fatou set $= \mathcal{F}(f) =$ open set where iterates are normal family.

Julia set $= \mathcal{J}(f) =$ complement of Fatou set.

Julia set is usually fractal. What is its (Hausdorff) dimension?



$\mathcal{J}((e^z - 1)/2)$, courtesy of Arnaud Chéritat

Dimensions of transcendental Julia sets:

Singular set $=$ closure of critical values and finite asymptotic values
$=$ smallest set so that f is a covering map onto $\mathbb{C} \setminus S$

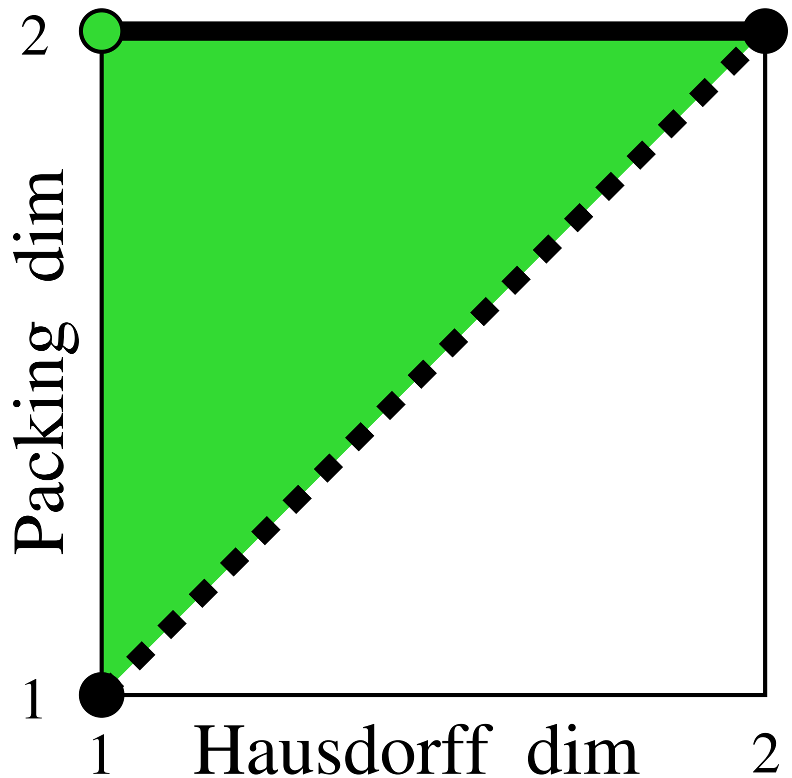
Eremenko-Lyubich class $=$ bounded singular set $= \mathcal{B}$

Speiser class $=$ finite singular set $= \mathcal{S} \subset \mathcal{B}$

Folding produces functions in the smaller Speiser class.

A short (and incomplete) history:

- Baker (1975): f transcendental $\Rightarrow \mathcal{J}$ contains a continuum $\Rightarrow$ dim ≥ 1.
- Misiurewicz (1981), McMullen (1987) dim =2 occurs (is common)
- Stallard (1997, 2000): $\{\dim(\mathcal{J}(f)) : f \in \mathcal{B}\} = (1, 2]$.
- Rippon-Stallard (2005) Packing dim = 2 for $f \in \mathcal{B}$.
- B (2018) H-dim = P-dim = 1 can occur (example outside $\mathcal{B}$).
- Albrecht-B (2020). $1 <$ H-dim < 2 can also occur in Speiser class.
- Burkart (2021) $1 <$ P-dim < 2 can occur (outside $\mathcal{B}$).
- Many other results known, e.g., relating growth rate to dimension.

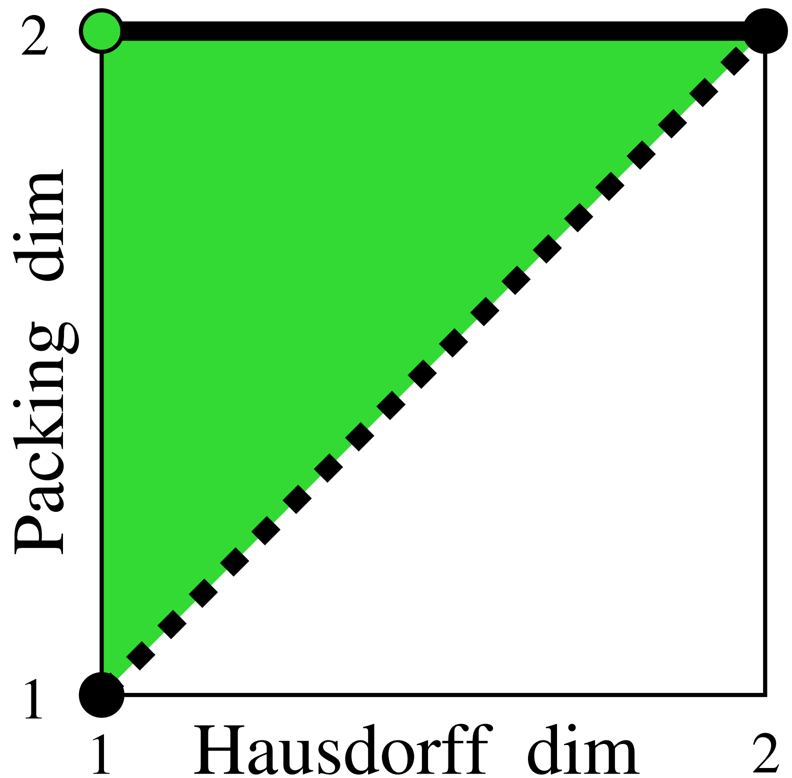


Hausdorff, upper Minkowski and packing dimension are defined as

$$\mathrm{Hdim}(K) = \inf\{s : \inf\{\textstyle\sum_j r_j^s : K \subset \cup_j D(x_j, r_j)\} = 0\},$$

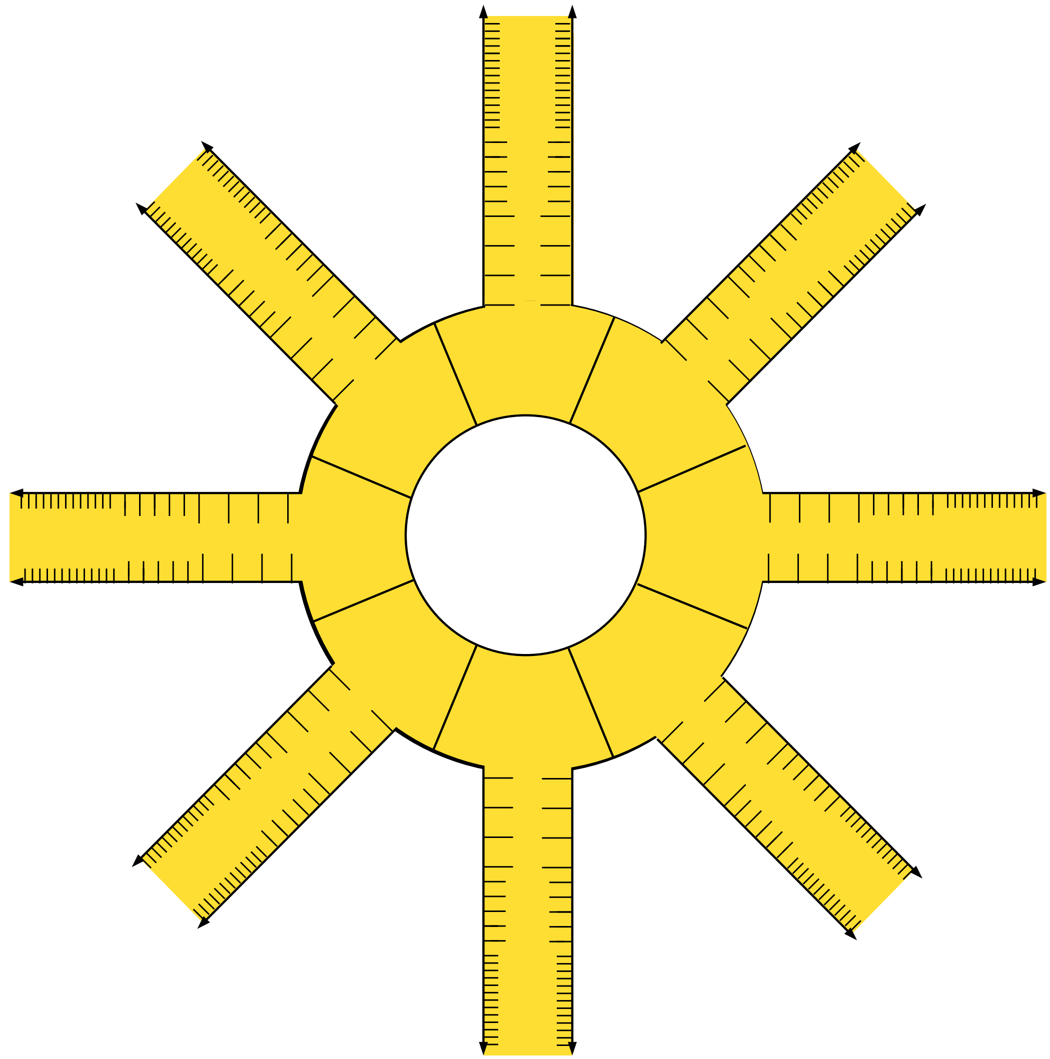
$$\overline{\mathrm{Mdim}}(K) = \inf\{s : \limsup_{r\to 0} \inf_N N r^s = 0 : K \subset \cup_{j=1}^N D(x_j, r)\},$$

$$\mathrm{Pdim}(K) = \inf\{s : K \subset \cup_{j=1}^\infty K_j : \overline{\mathrm{Mdim}}(K_j) \le s \text{ for all } j\},$$



In polynomial dynamics it is difficult to construct examples with large dimension or positive area (Shishikura, Buff, Cheritat).

For entire functions, it is harder to find small Julia sets (dimension 1, finite length, rectifiable).



$$\inf\{\dim(\mathcal{J}(f)) : f \in \mathcal{S}\} = 1 \qquad \text{(B.-Albrecht, 2018).}$$

Uses upper and lower τ-bounds to control dimension.

Wandering domains for entire functions:

Given an entire function f,

Fatou set $= \mathcal{F}(f) =$ open set where iterates are normal family.

Julia set $= \mathcal{J}(f) =$ complement of Fatou set.

f permutes components of its Fatou set.

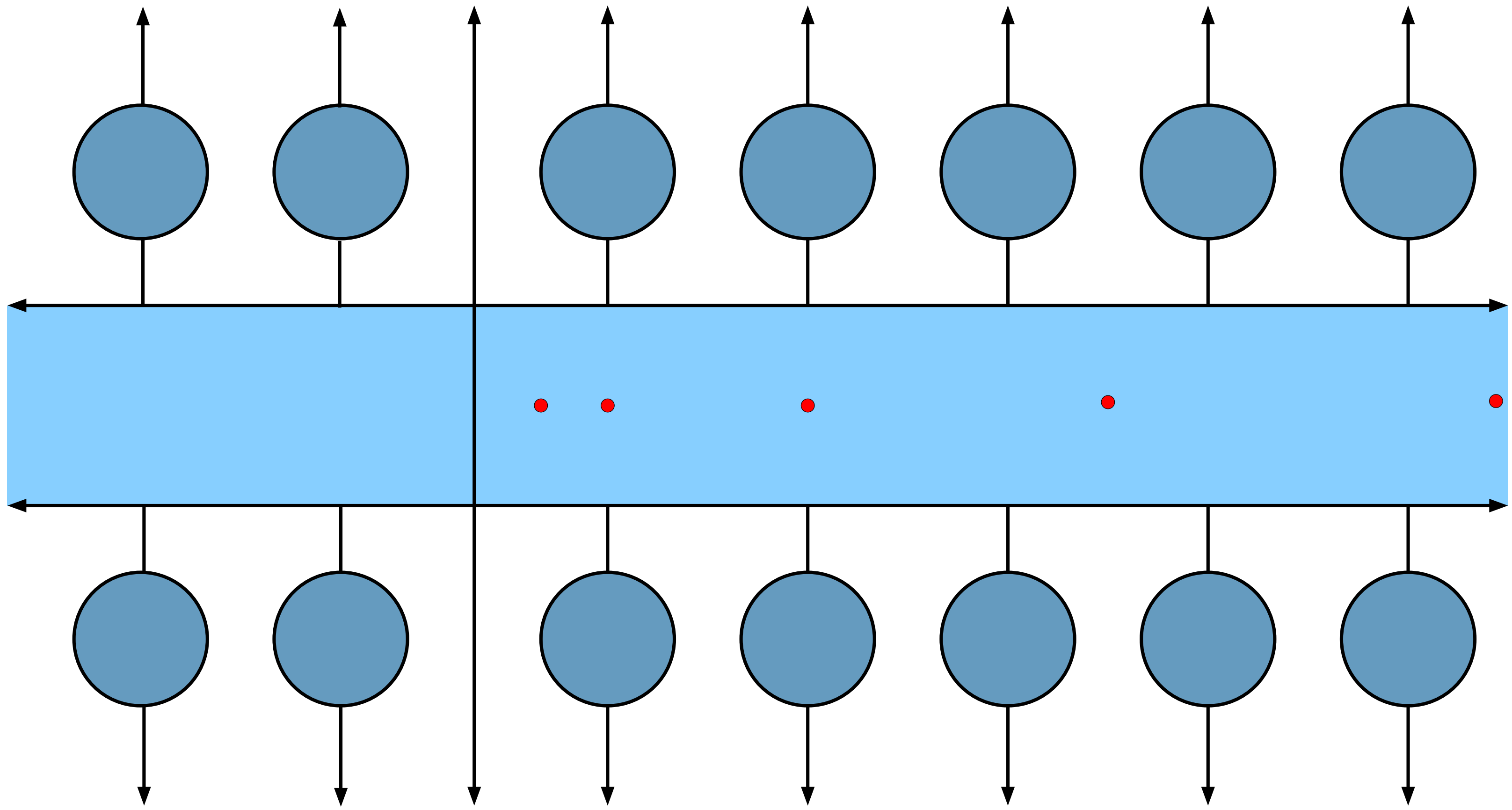
Wandering domain = Fatou component with infinite orbit.

- Entire functions can have wandering domains (Baker 1975).

- No wandering domains for rational functions (Sullivan 1985).

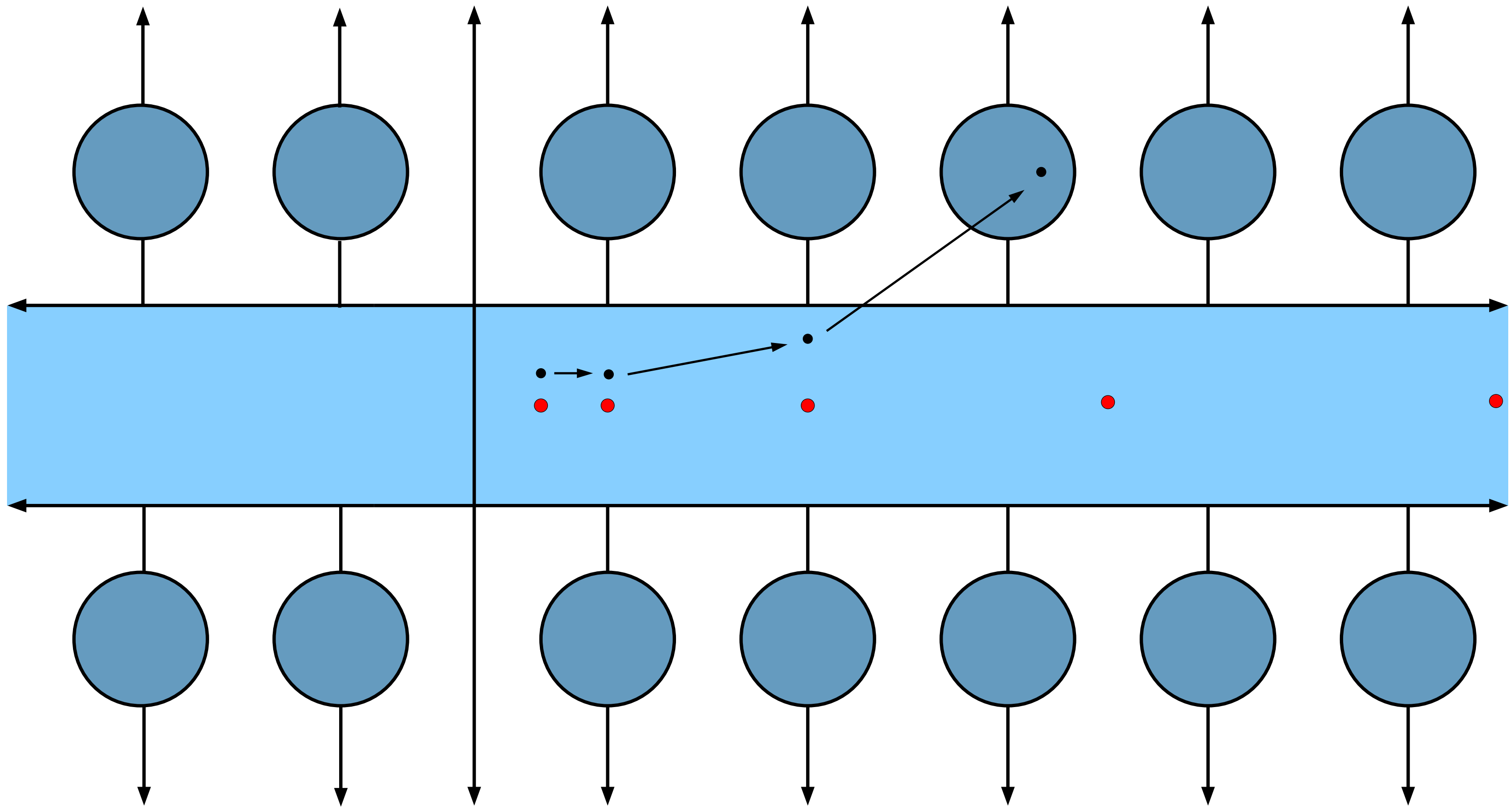
- Also none in Speiser class (Eremenko-Lyubich, Goldberg-Keen).

Are there wandering domains in Eremenko-Lyubich class?



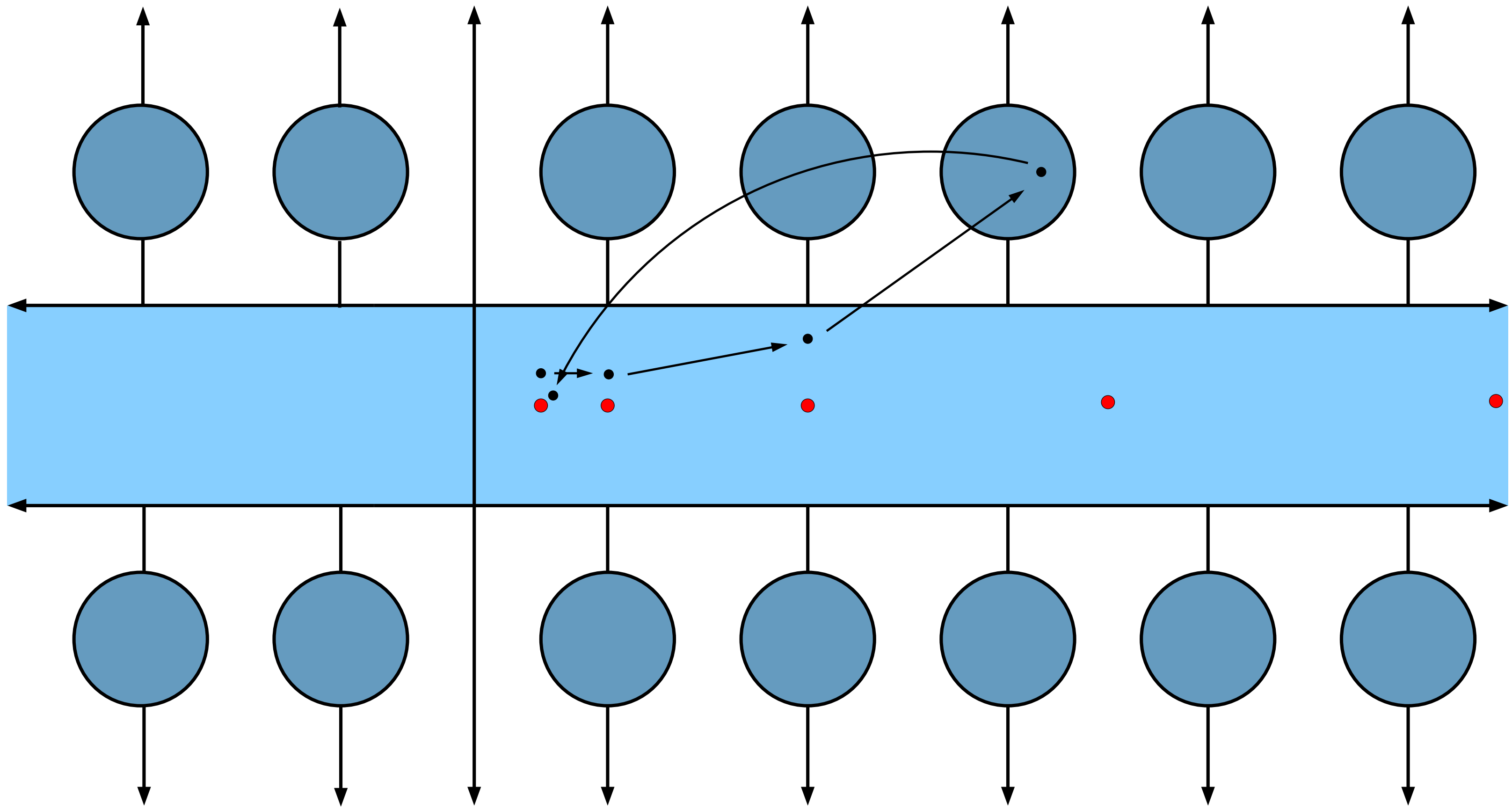
Graph giving wandering domain in Eremenko-Lyubich class.

Original proof corrected by Marti-Pete and Shishikura, who also give alternate construction.



Graph giving wandering domain in Eremenko-Lyubich class.

Variations by Lazebnik, Fagella-Godillon-Jarque, Osborne-Sixsmith.



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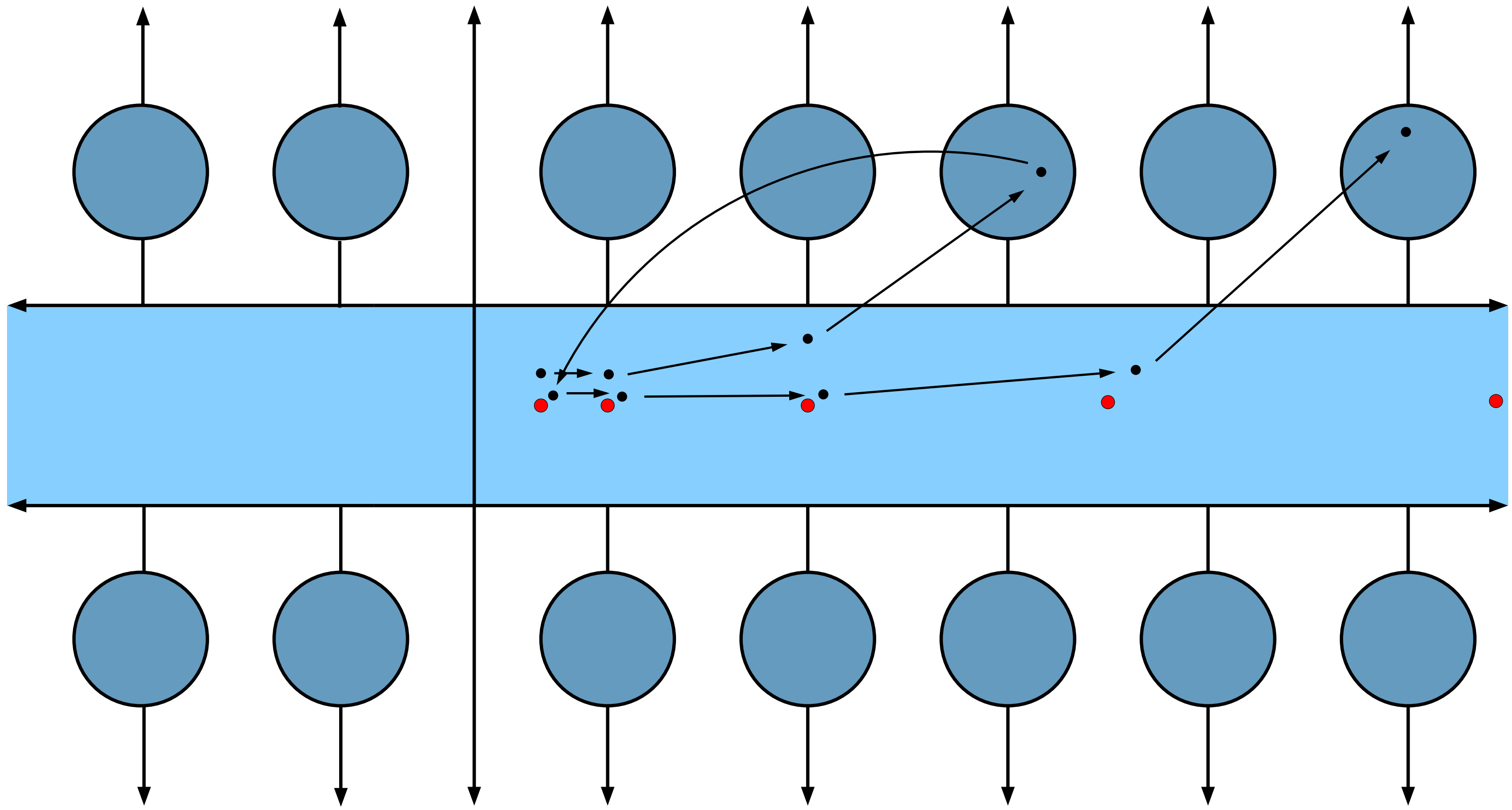

Graph giving wandering domain in Eremenko-Lyubich class.

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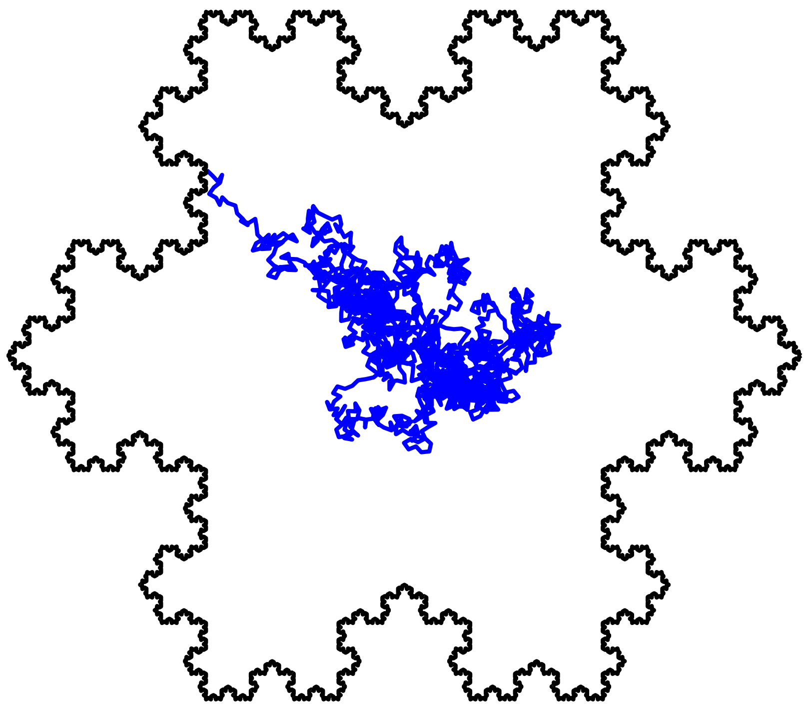
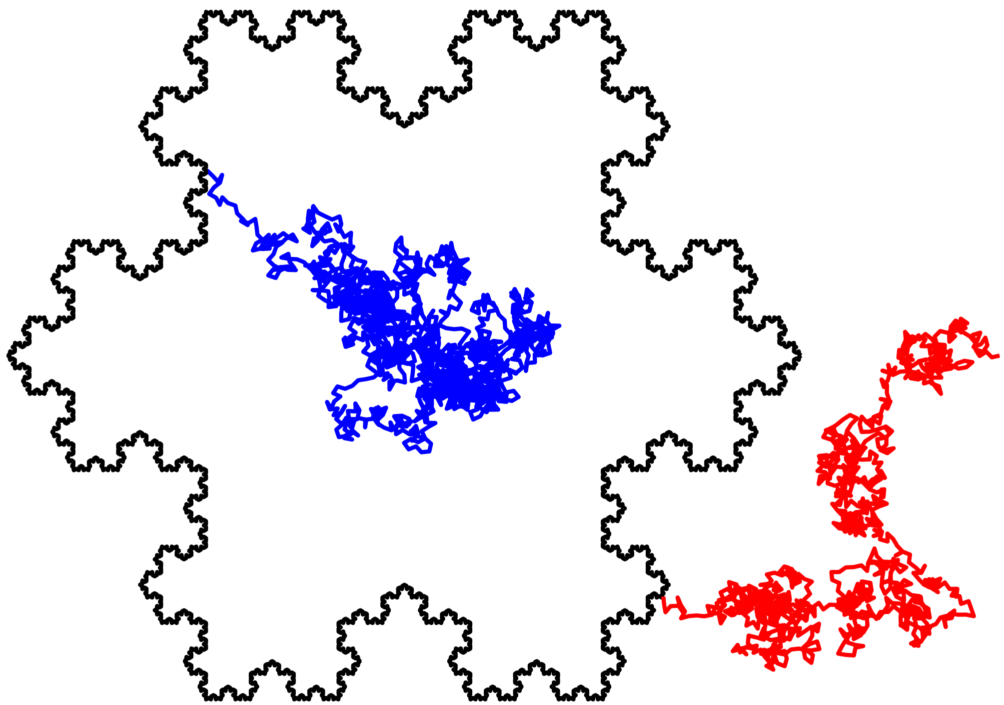
Thm (Makarov 1985):

For fractal domains, ω gives full measure to a set of zero length.



First such examples due to Lavrentiev (1936).

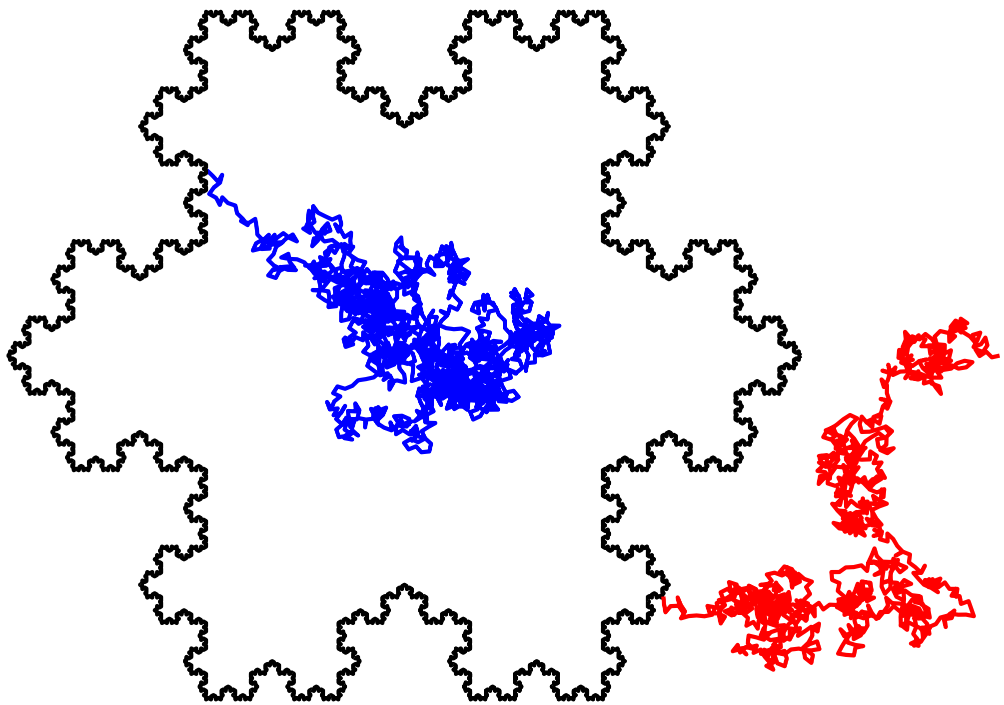
Thm (Makarov 1985):
For fractal domains, ω gives full measure to a set of zero length.



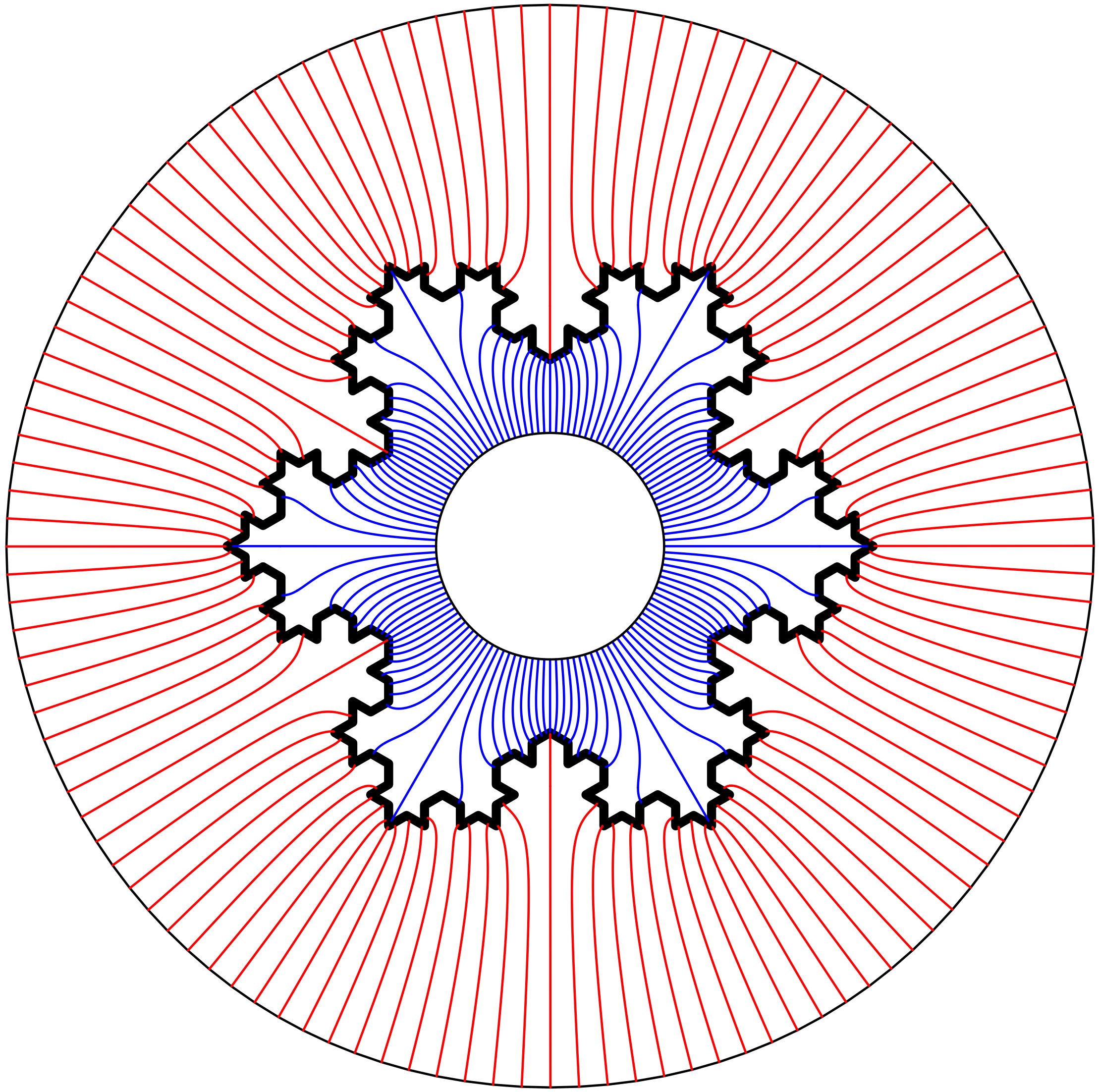
Outside is also fractal. Same set of length zero?

Theorem (B. 1987):

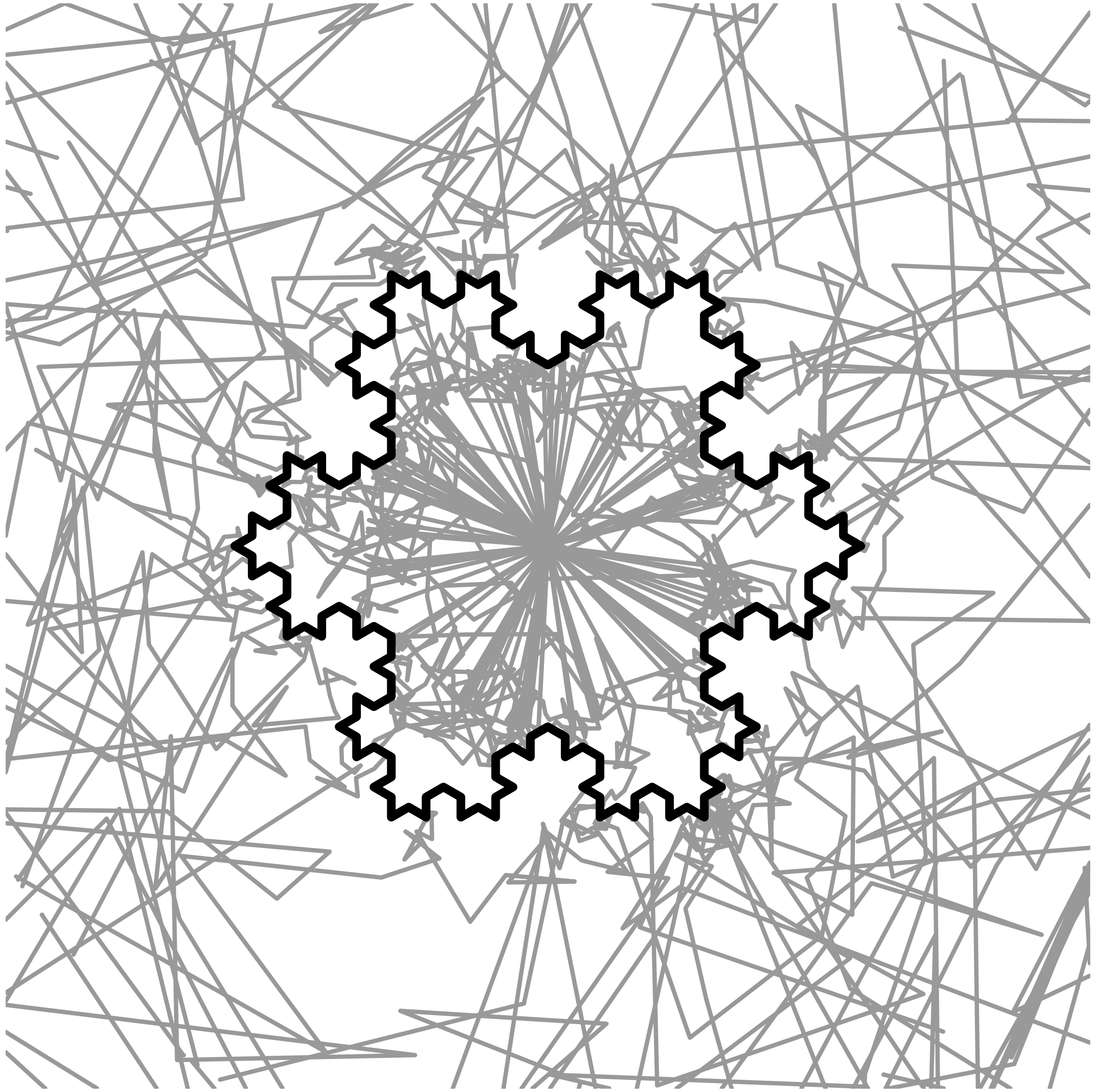
$\omega_1 \perp \omega_2$ iff tangents points have zero length.

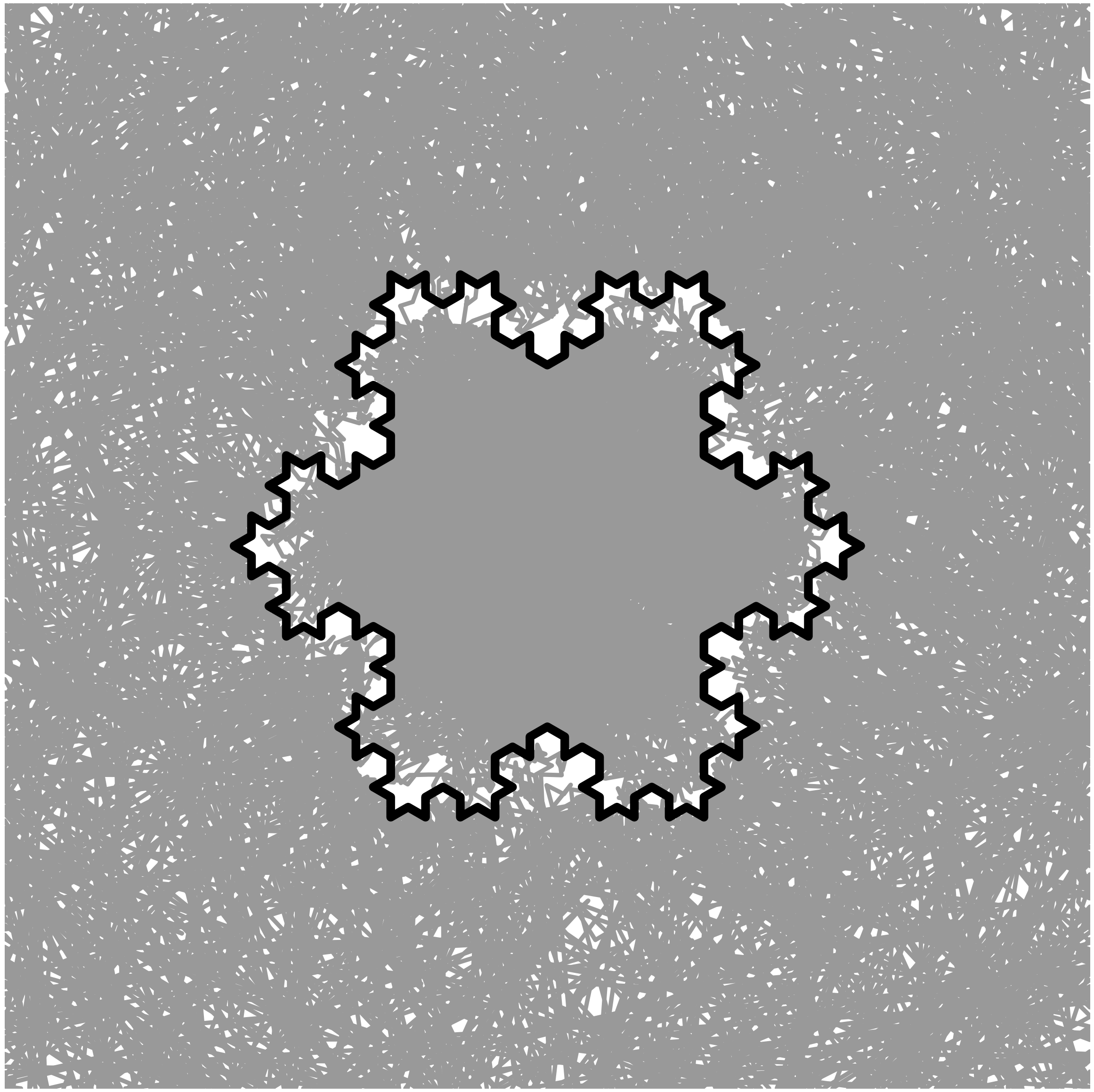


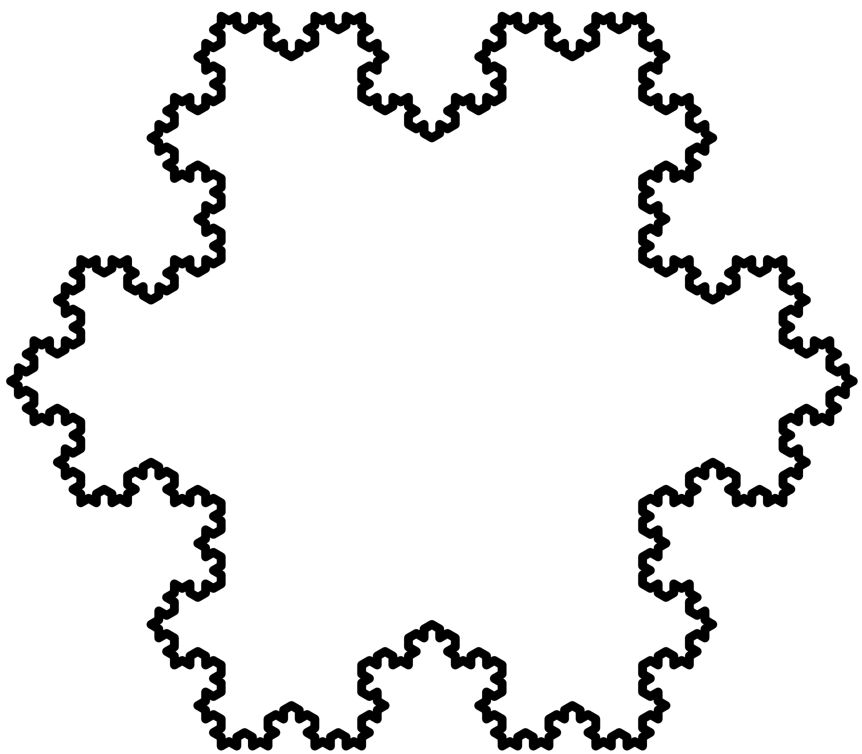
Inside and outside harmonic measures are singular

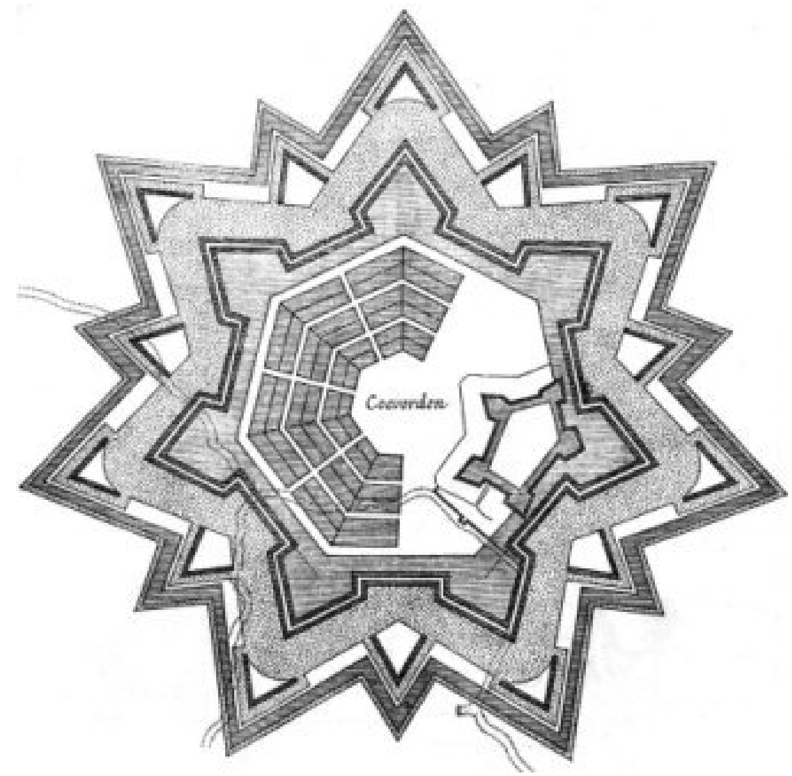


Images of radial lines for conformal maps to inside and outside.









Snowflake and star fort (“trace italienne”). Coincidence?