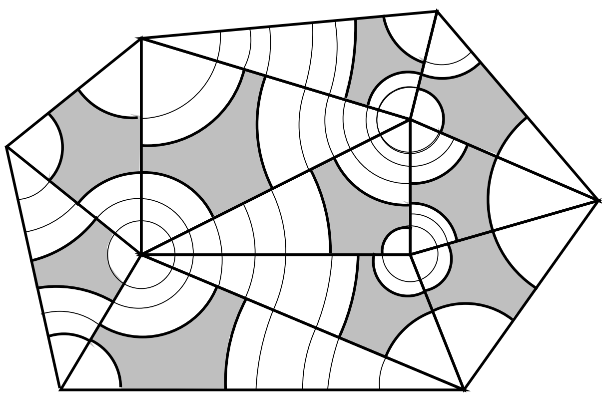
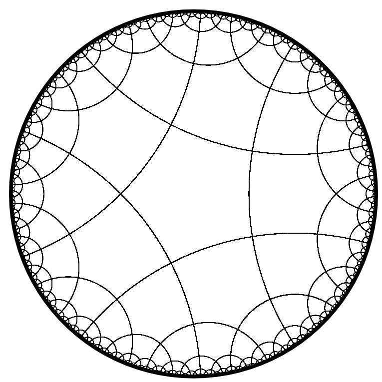
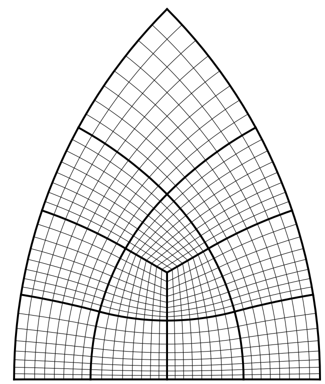
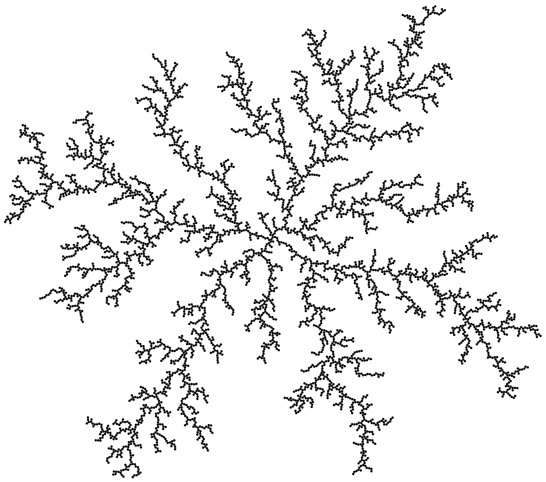
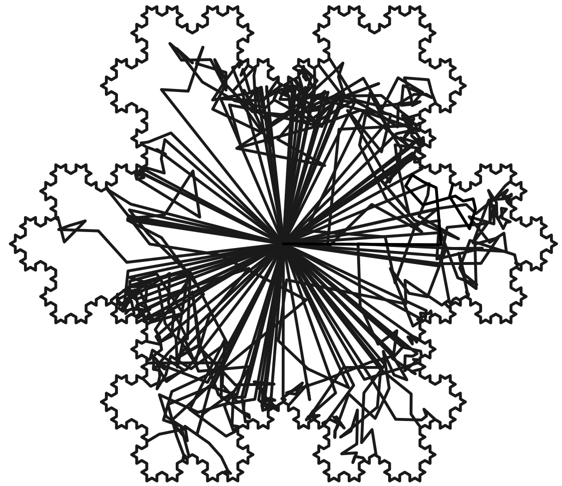
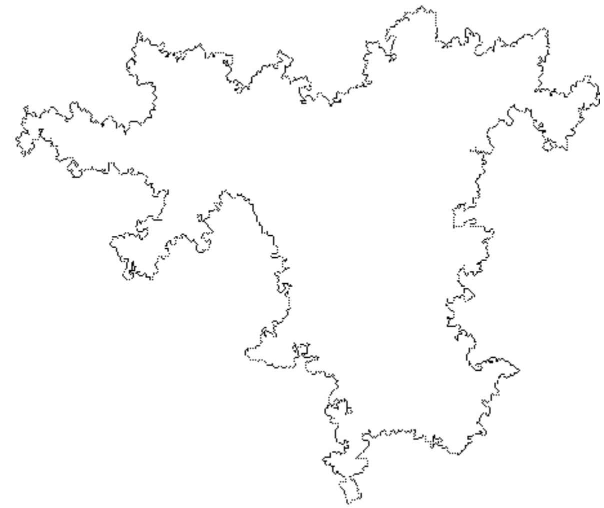
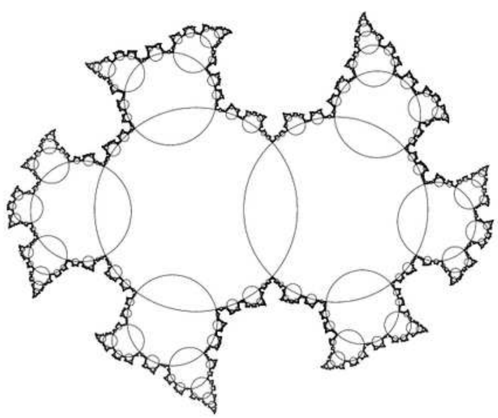
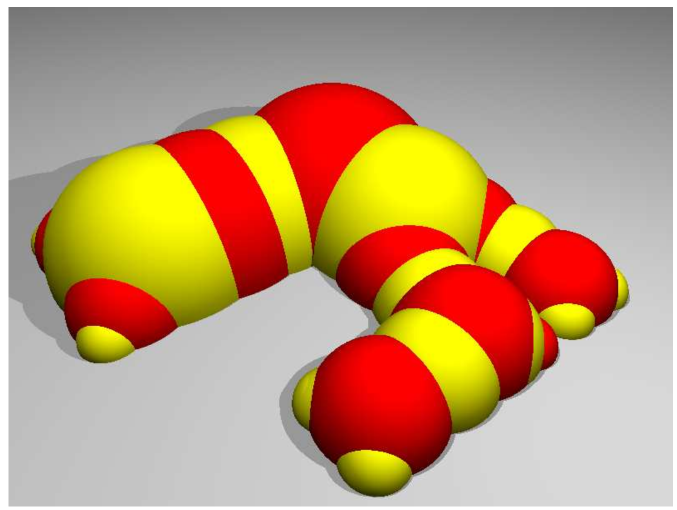
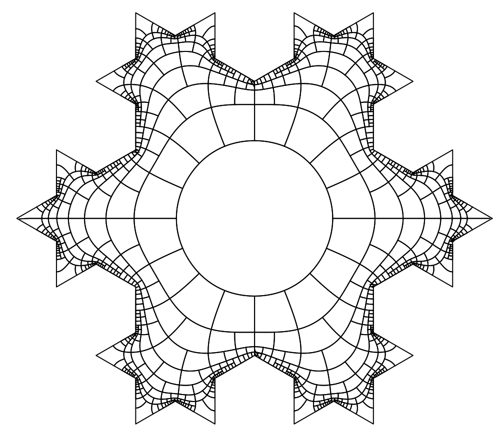
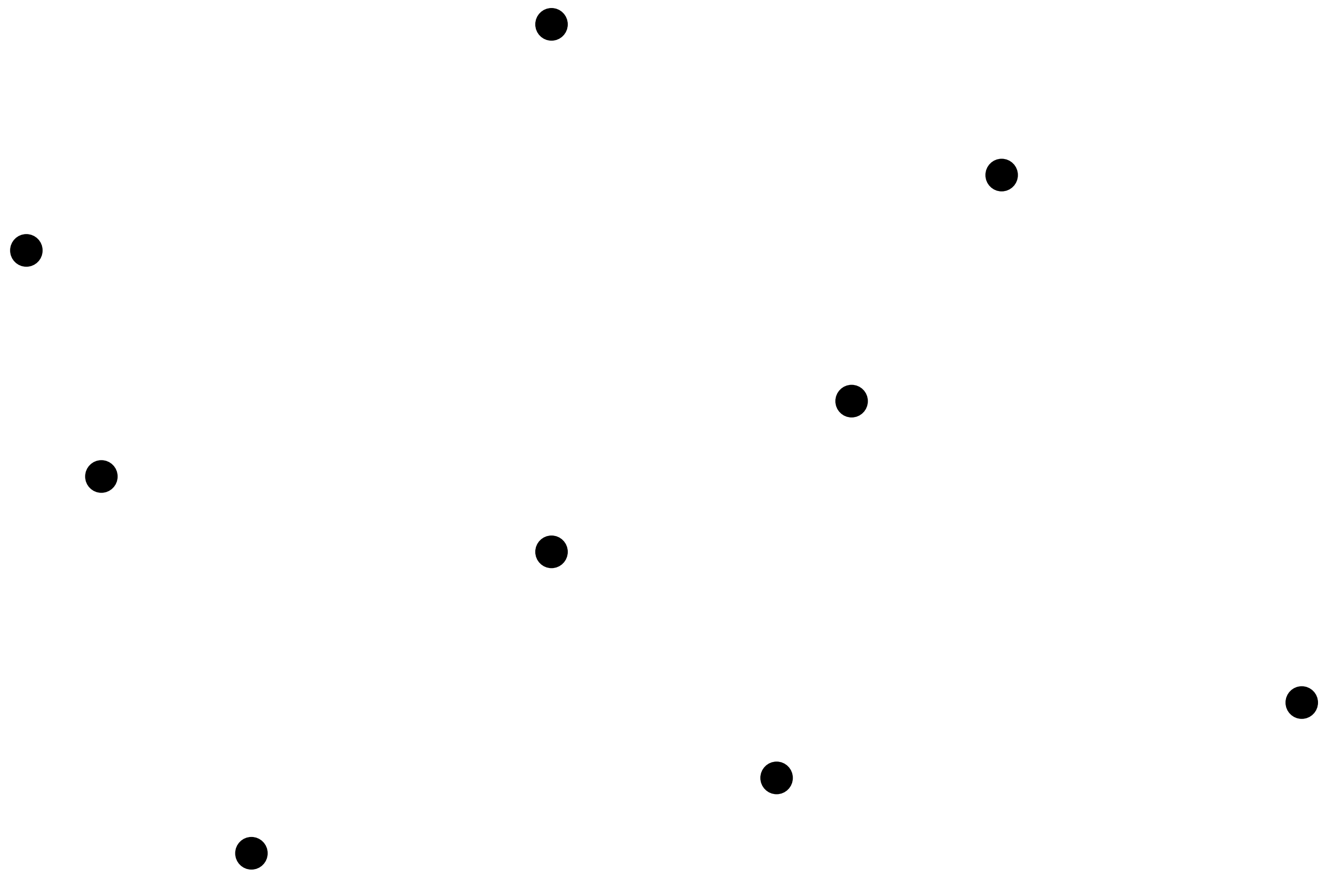
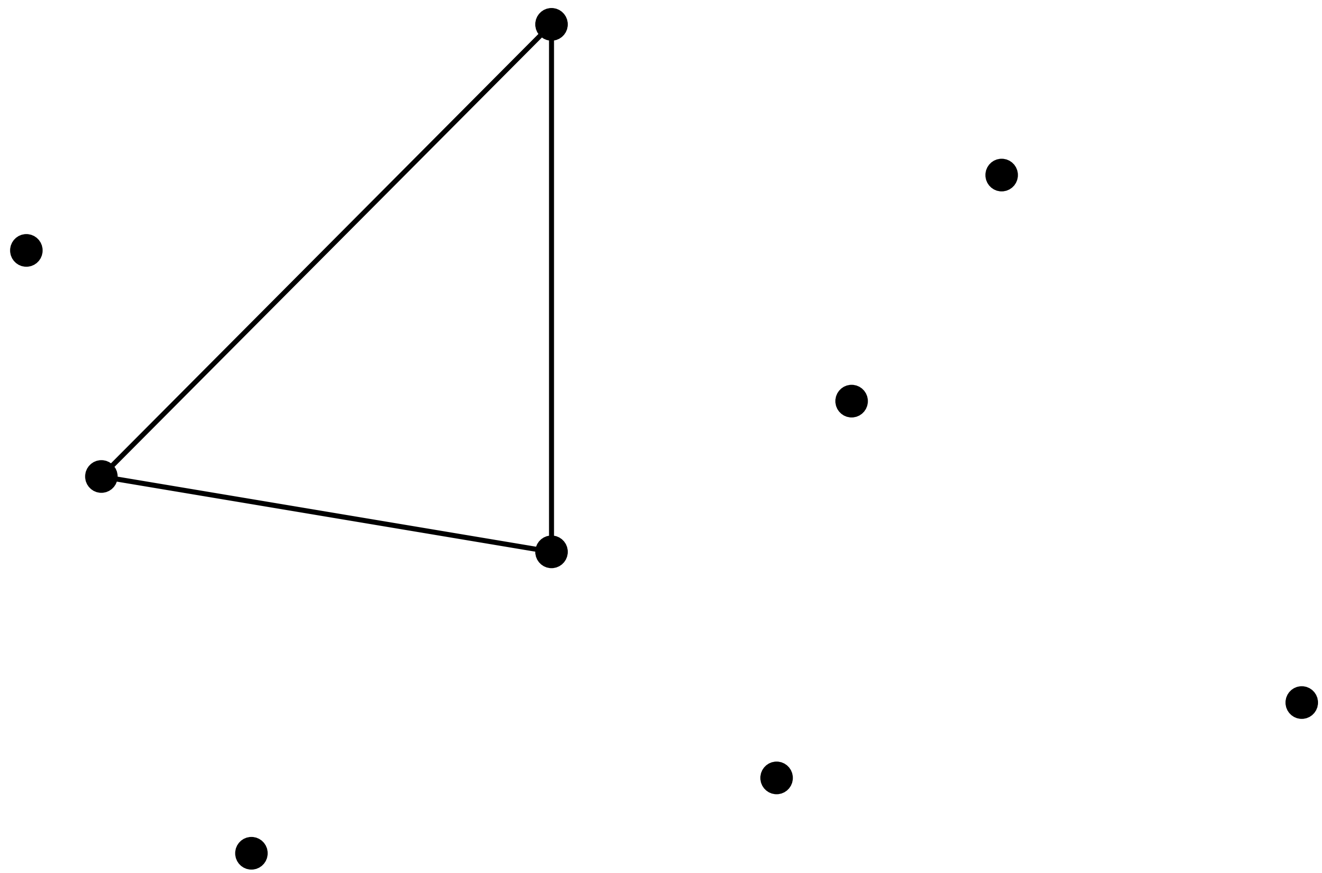
Optimal Meshing

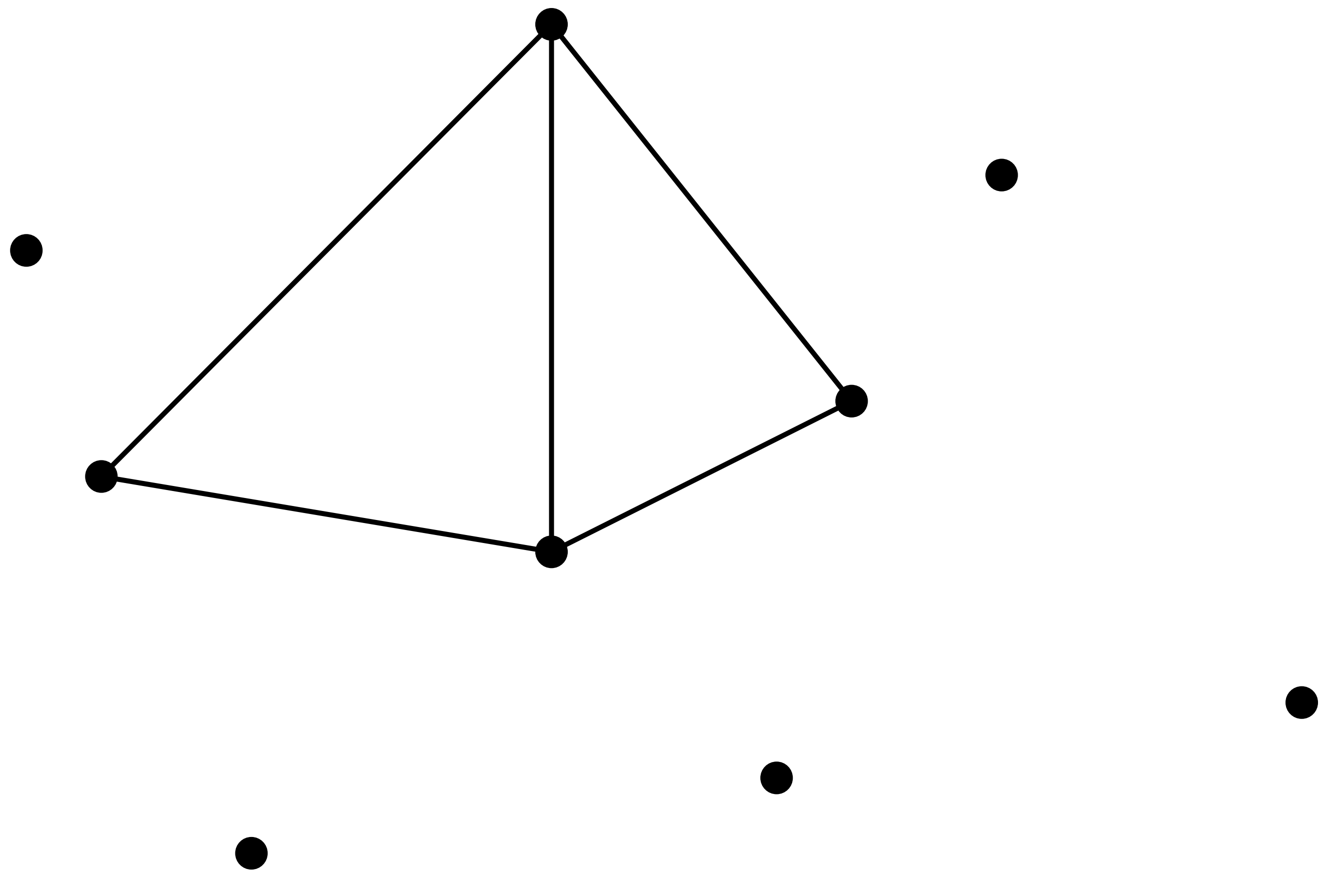
Christopher J. Bishop
SUNY Stony Brook

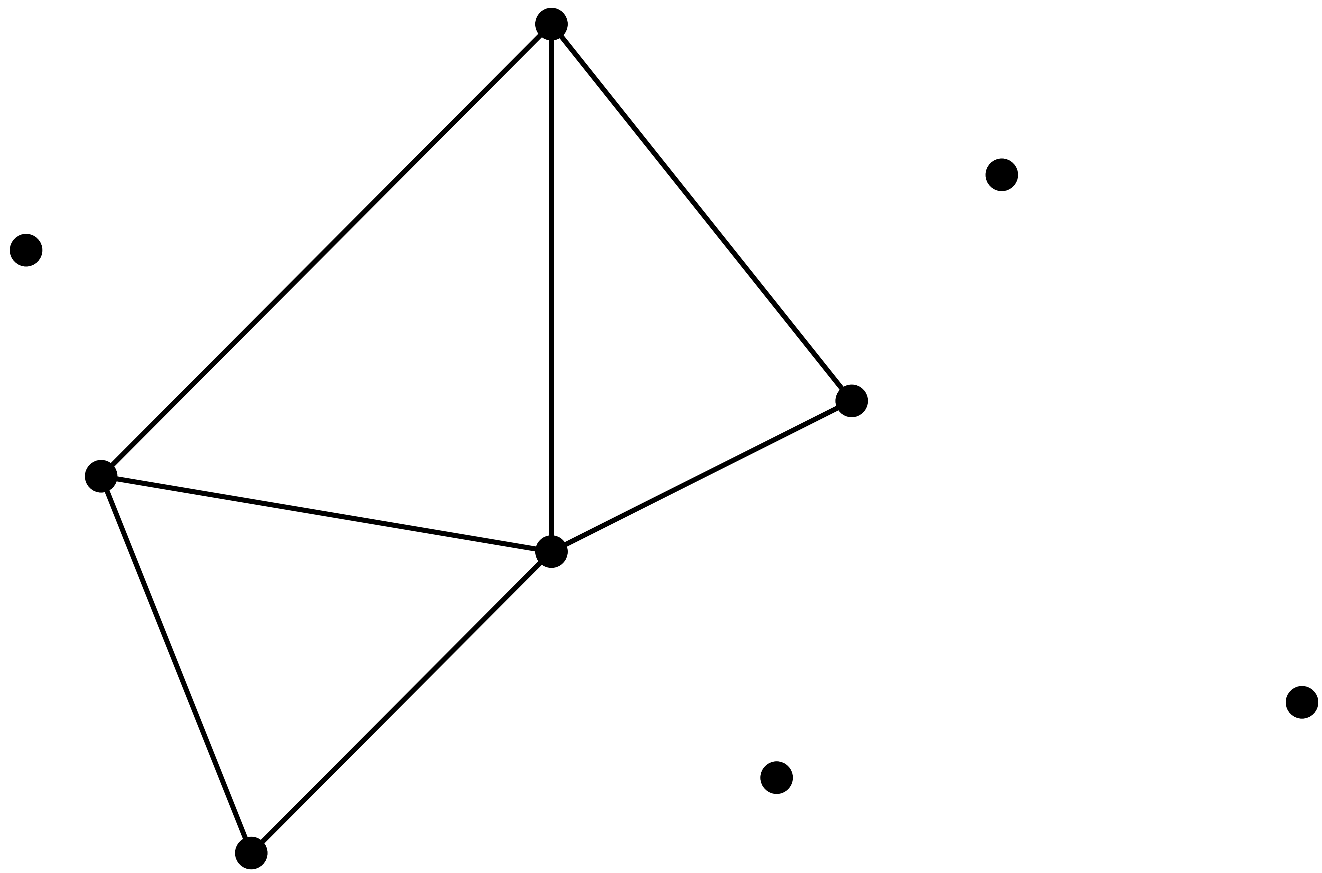
  

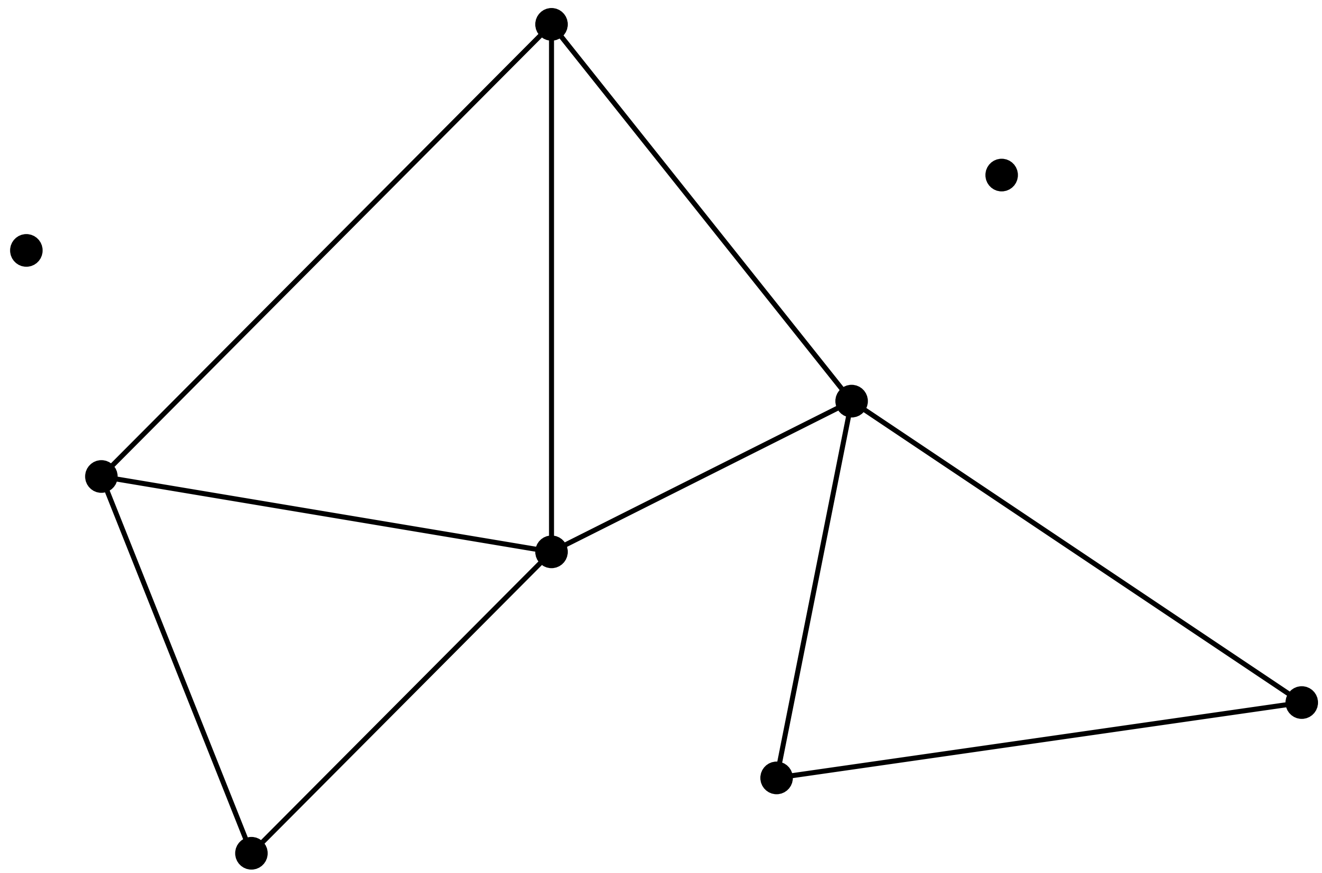







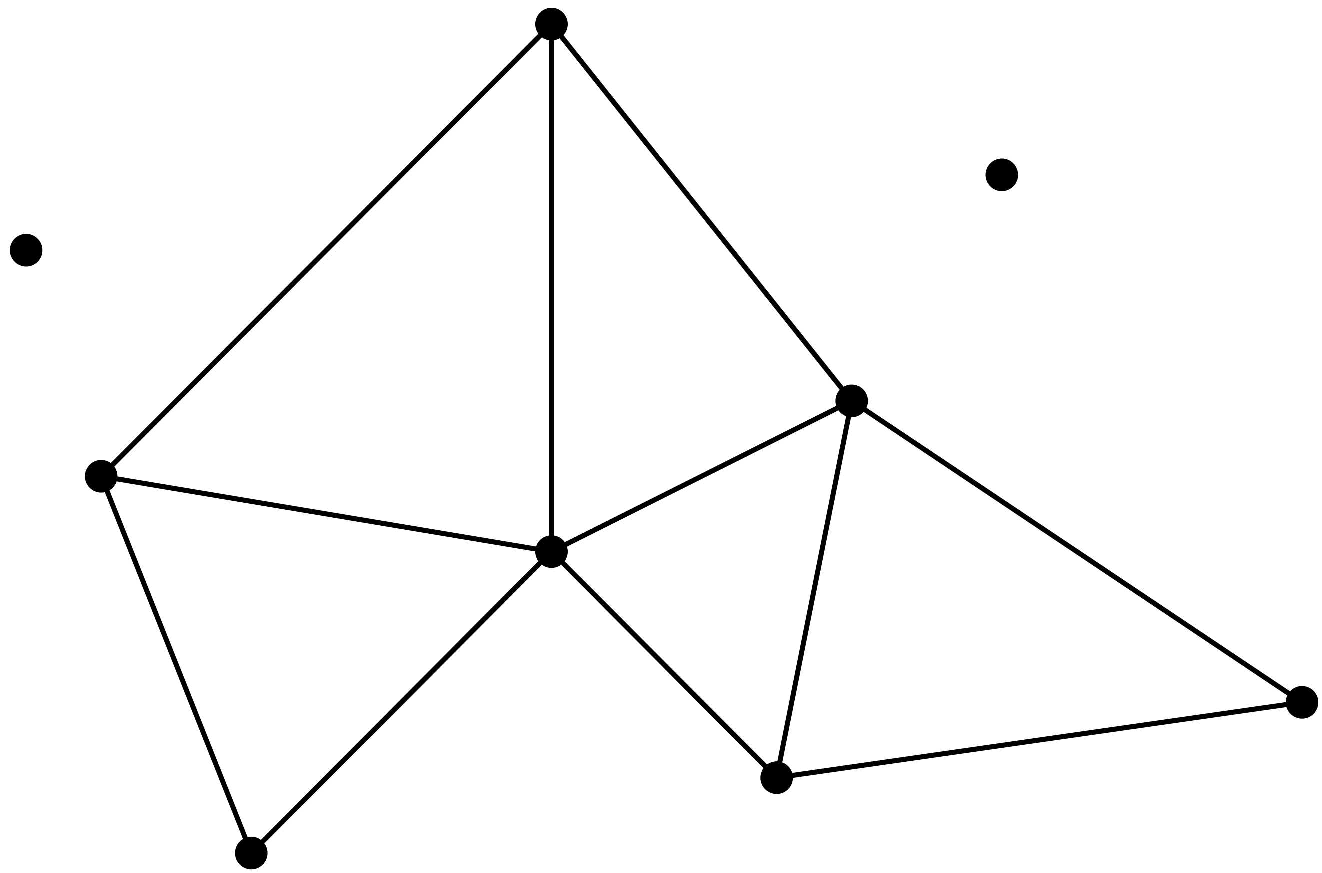


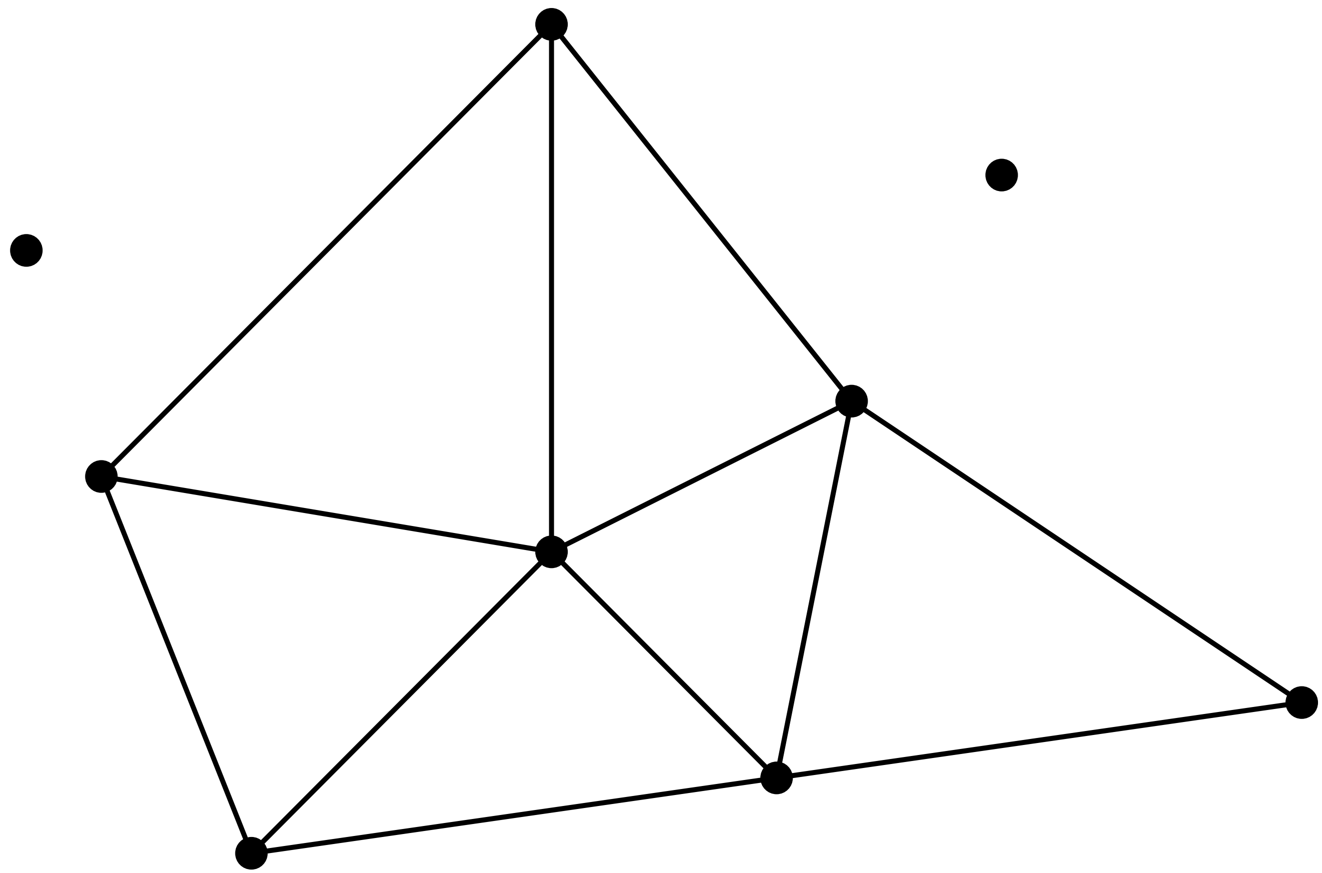


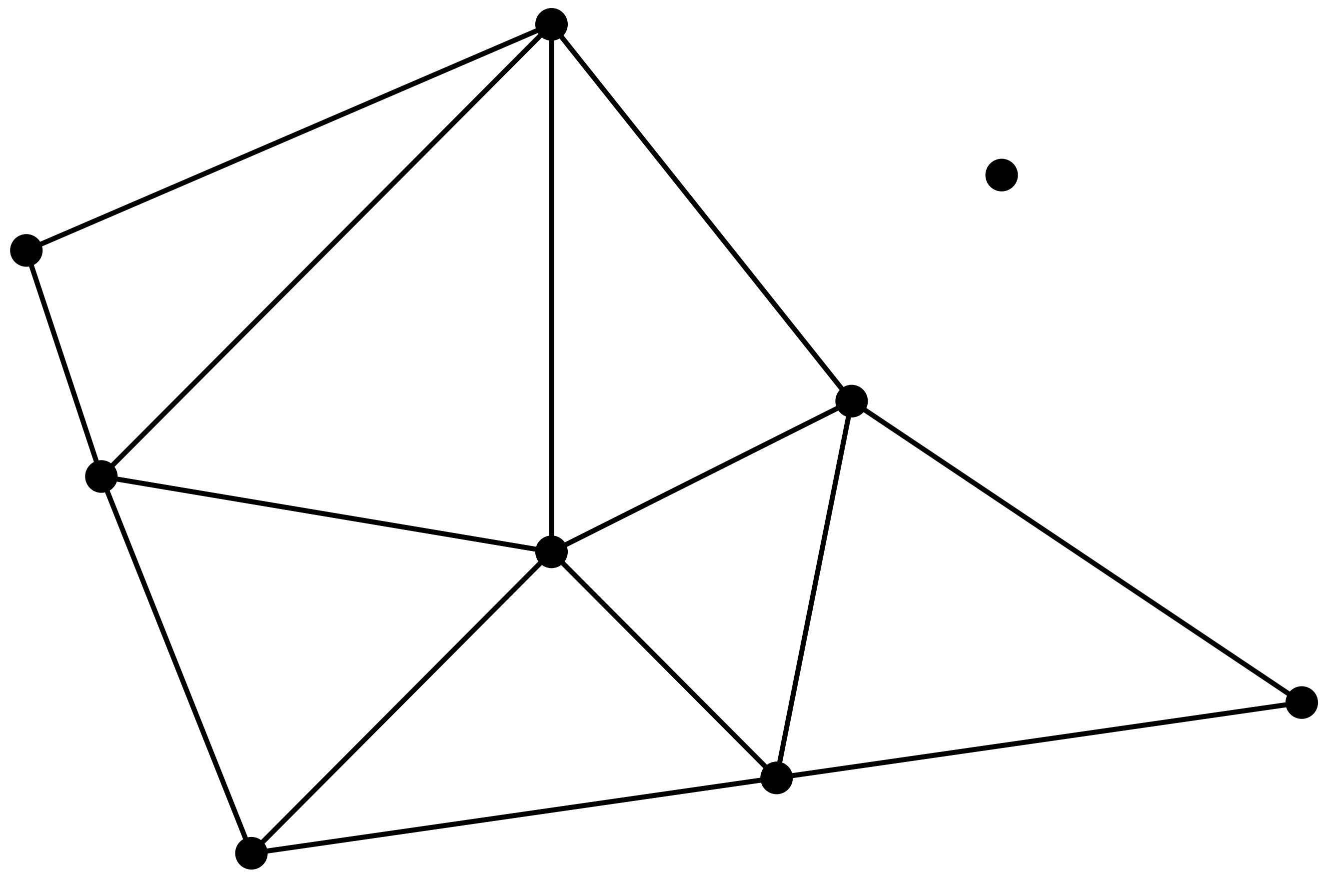


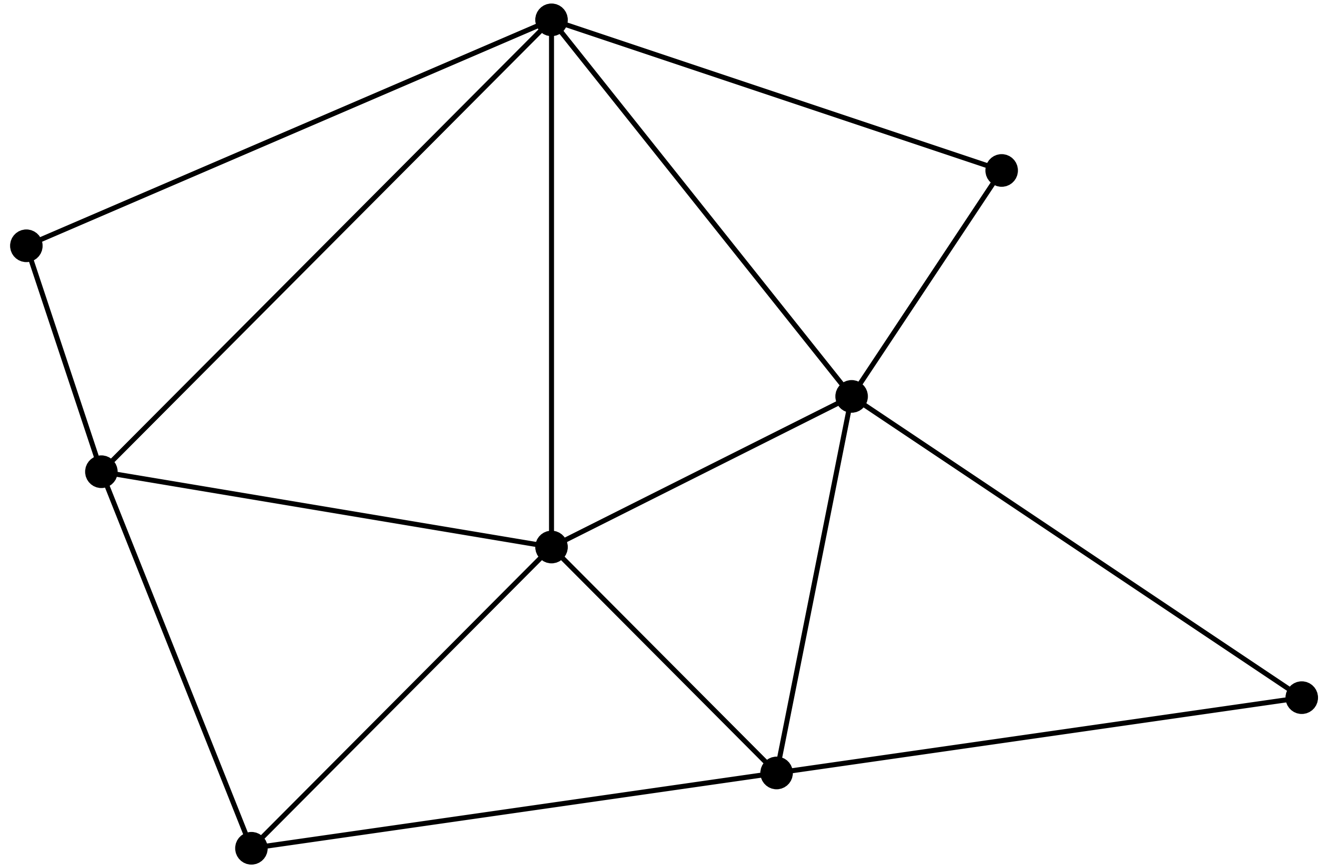


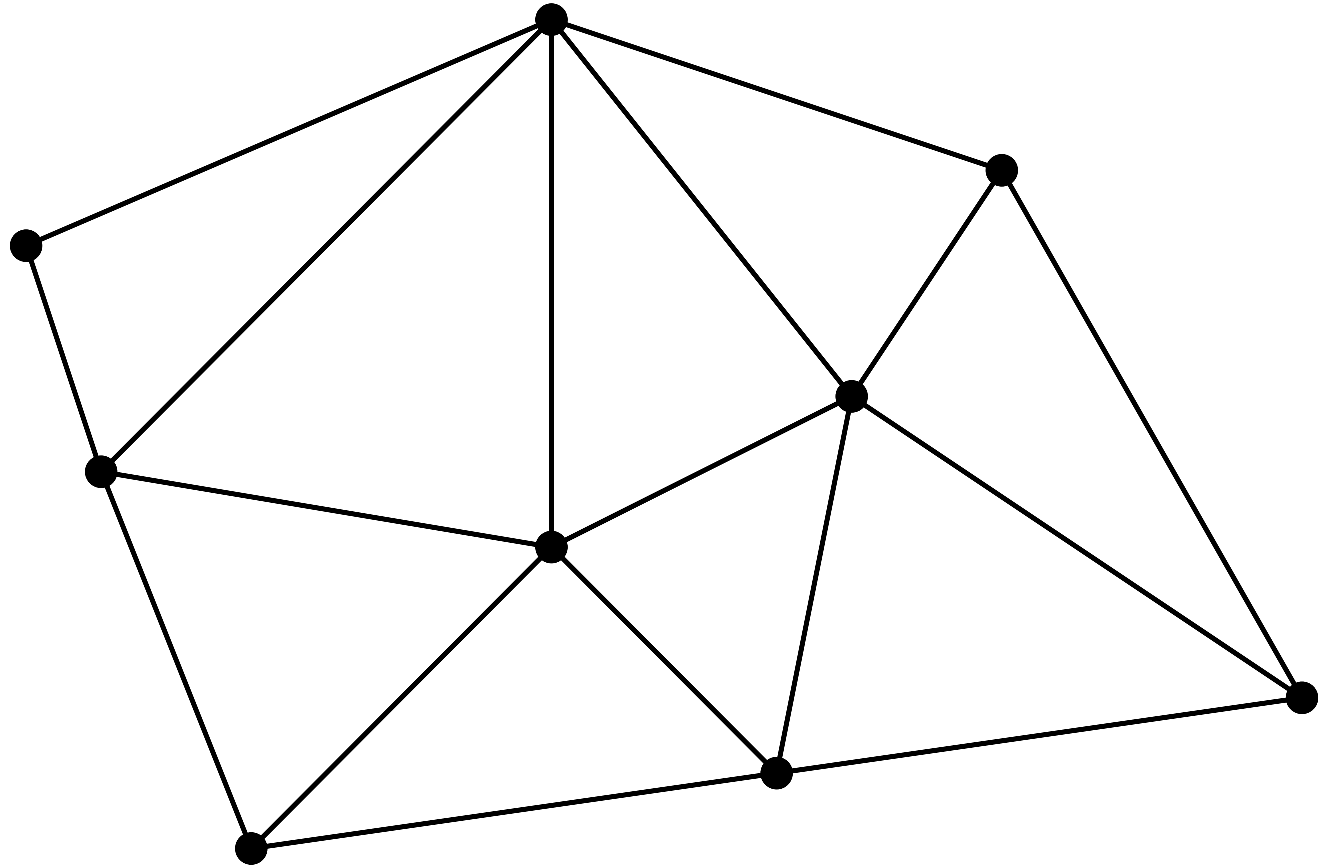






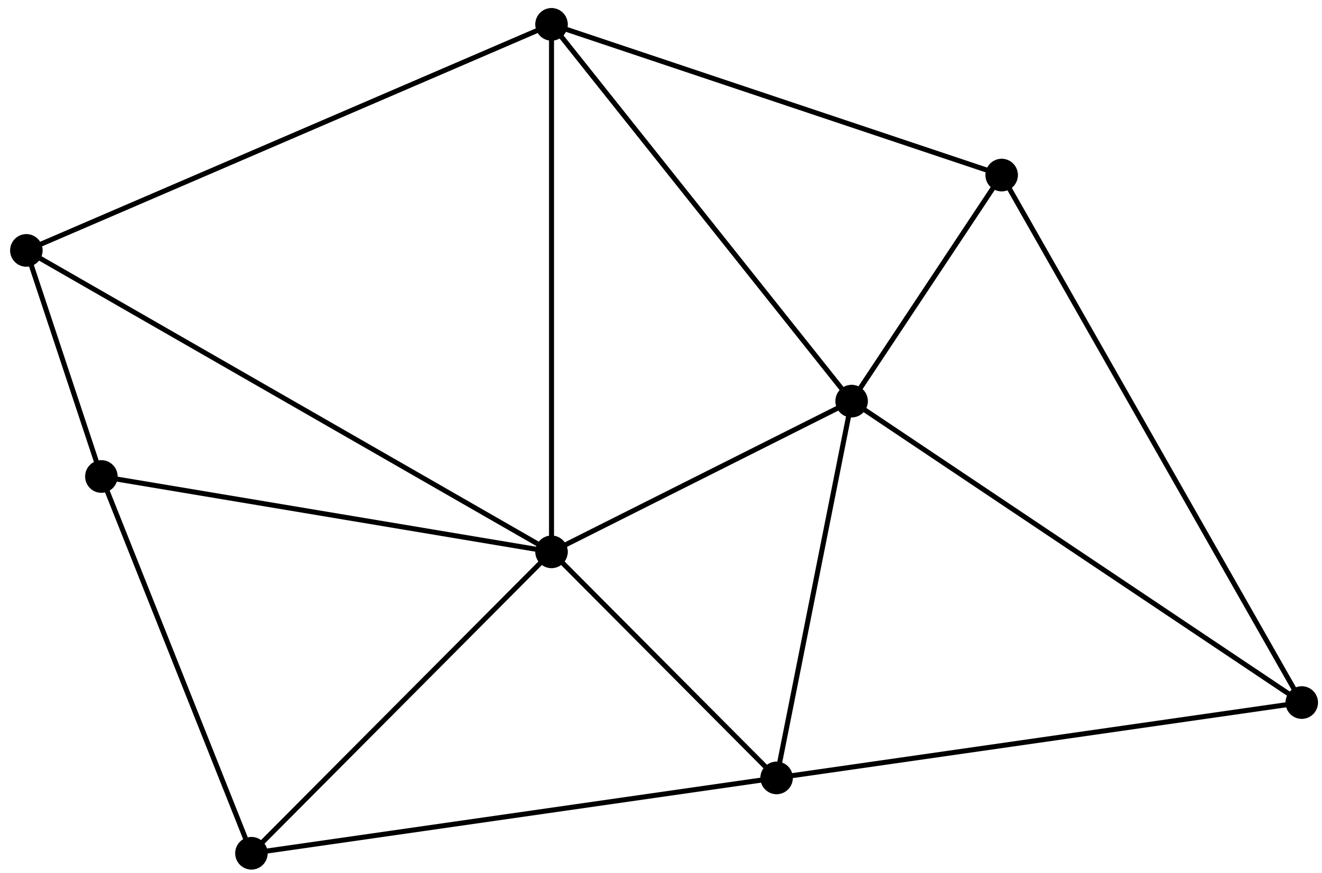




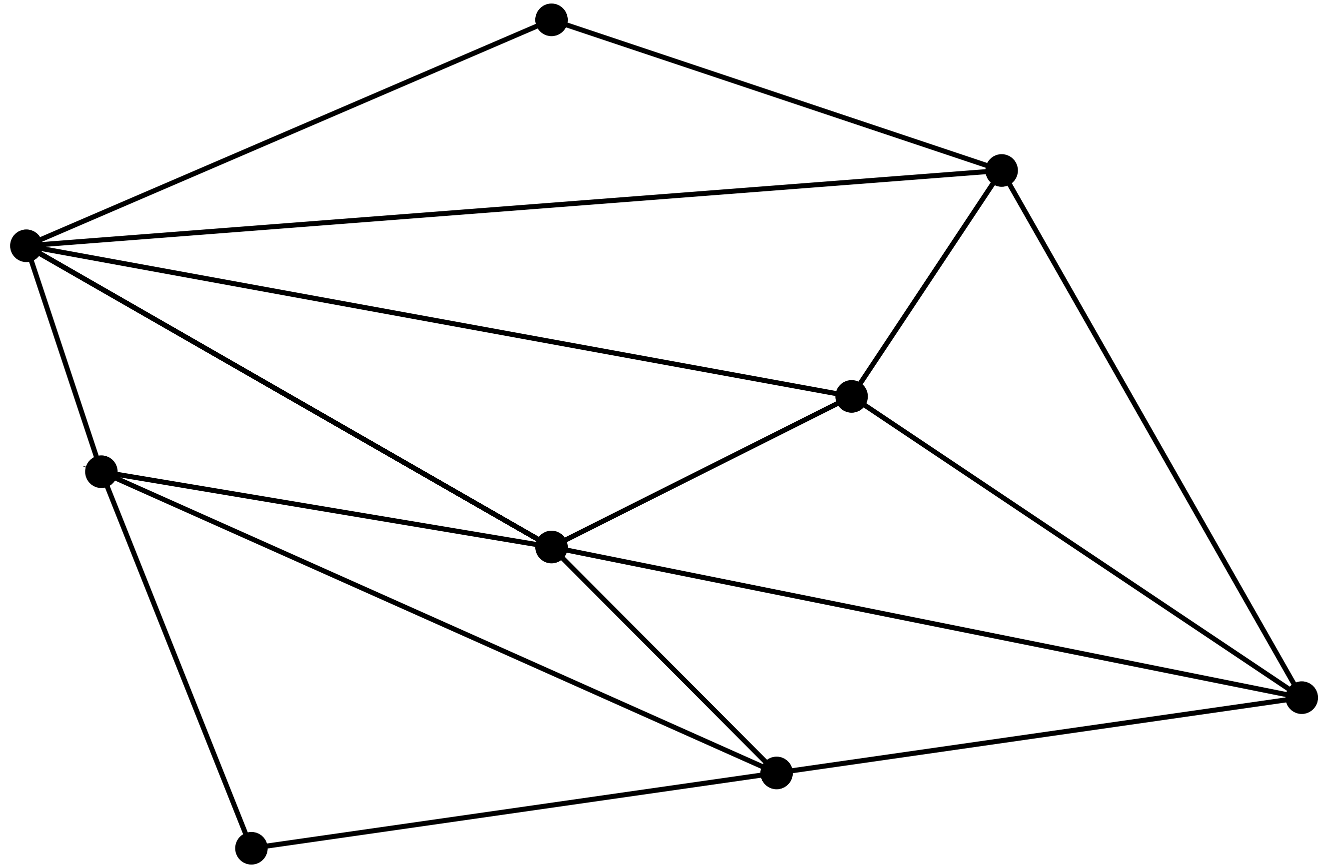


Triangulation of a point set

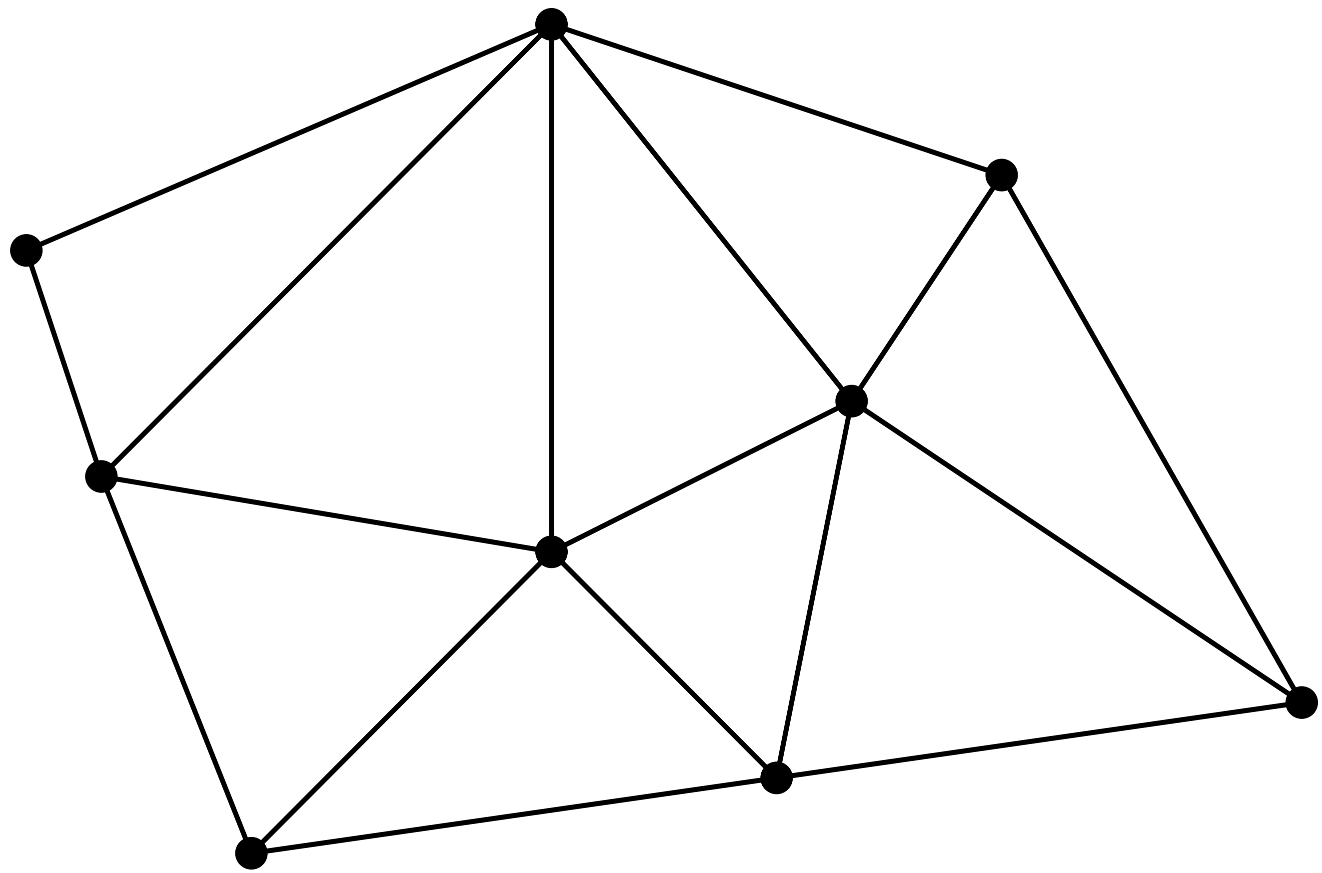
By convention, triangulation covers convex hull of points.



Another triangulation (flipped a diagonal)

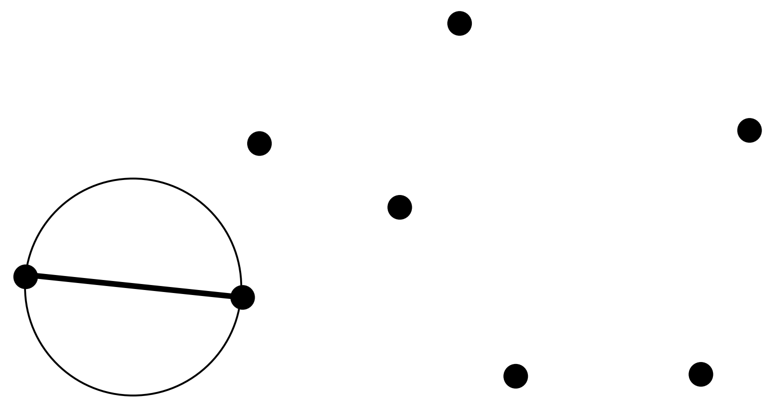


Yet another



This one is "best" = smallest maximum angle.

The segment $[v, w]$ is a **Gabriel** edge if it is the diameter of a disk containing no vertices.



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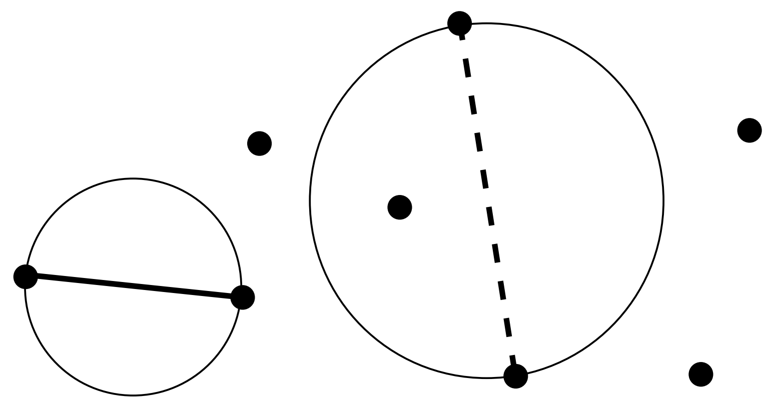

Not a Gabriel edge.

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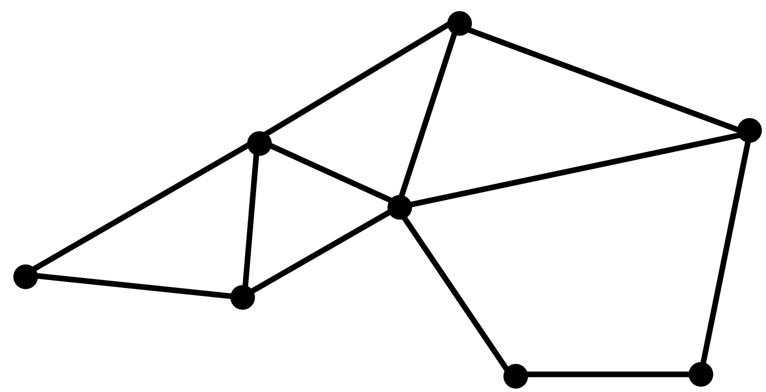
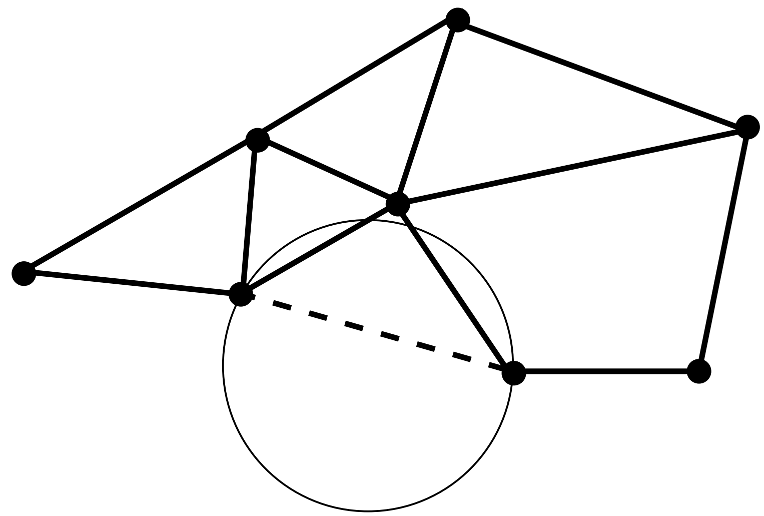


Gabriel graph contains the minimal spanning tree.

Gabriel and Sokol, *A new statistical approach to geographic variation analysis*, Systematic Zoology, 1969.

Gabriel edge is also a **Delaunay** edge: $[v, w]$ is a **chord** of an open disk not hitting vertex set.



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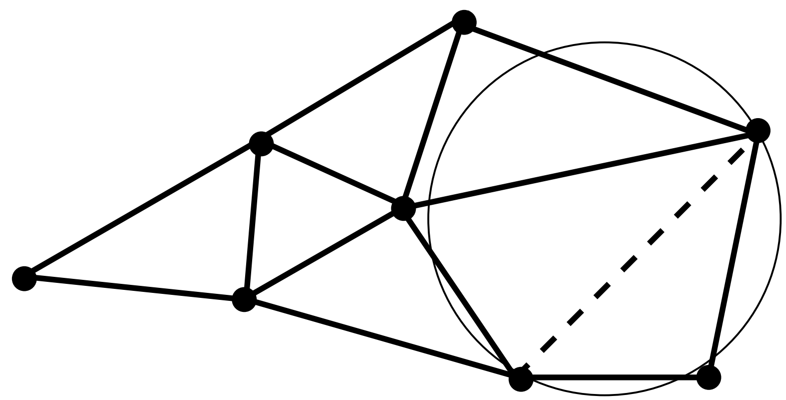

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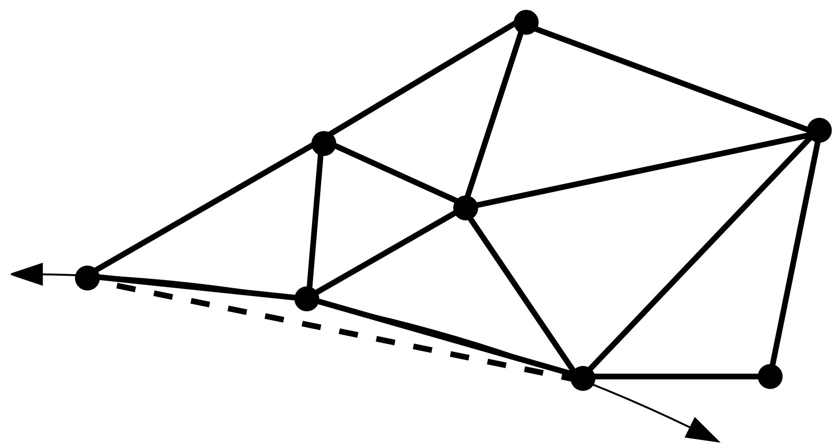

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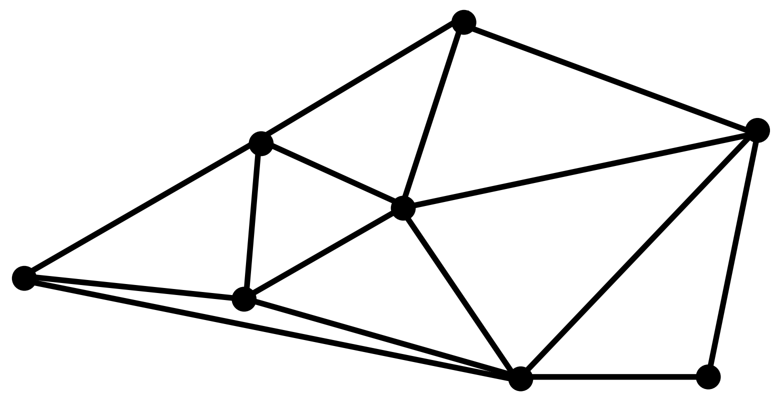
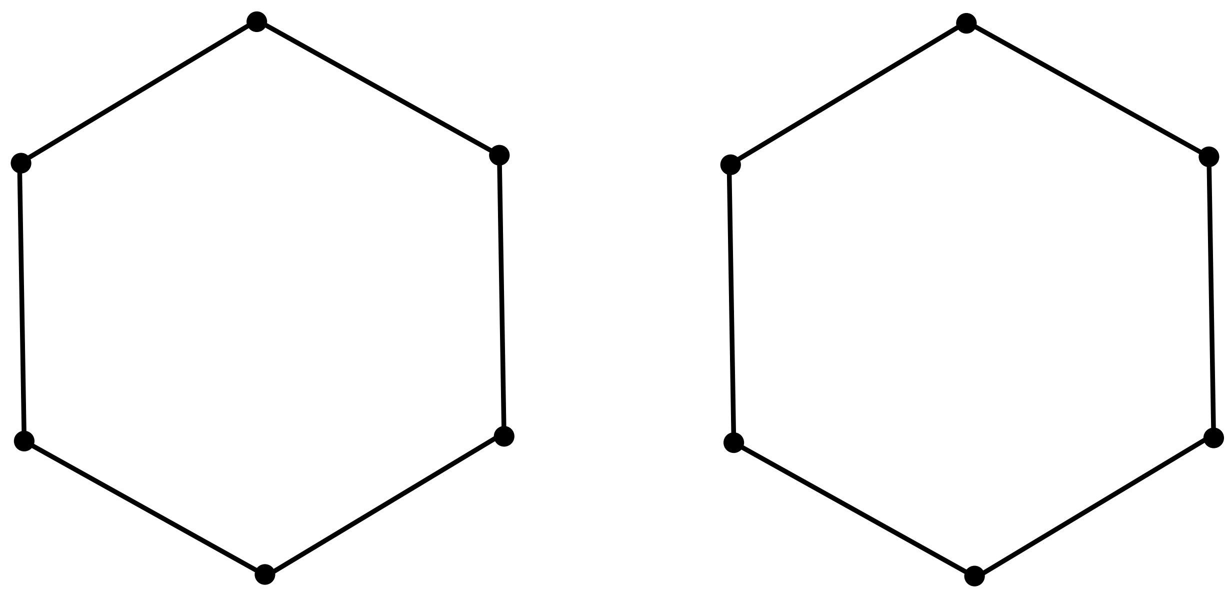


Delaunay edges triangulate and minimize the max angle.

Has the "best" geometry if only original points are used.

Add Steiner points to get better shaped triangles.



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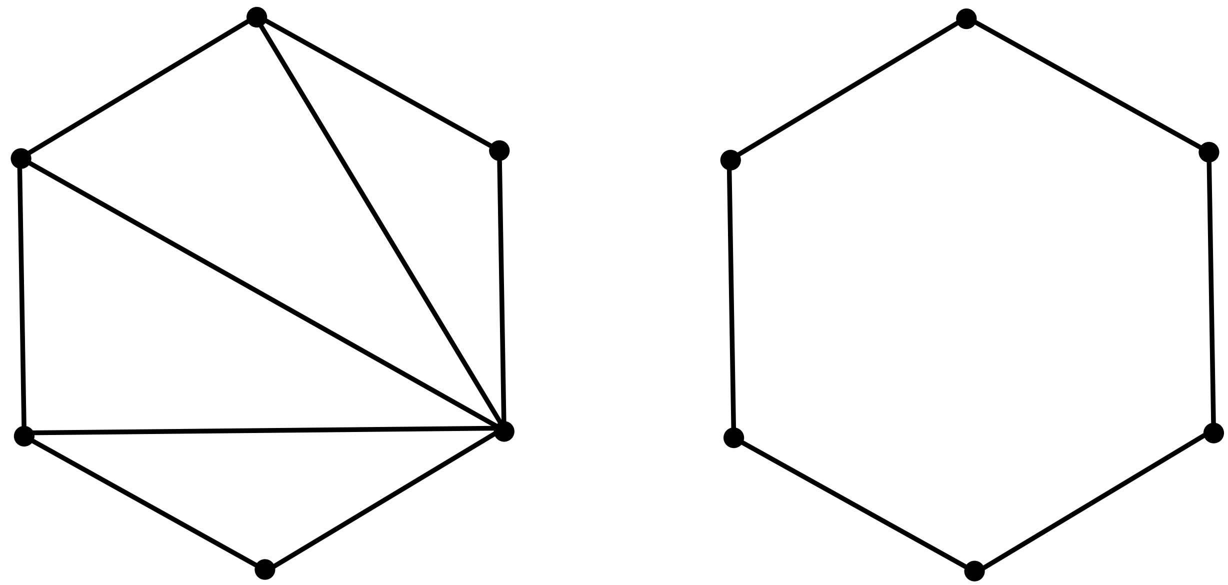

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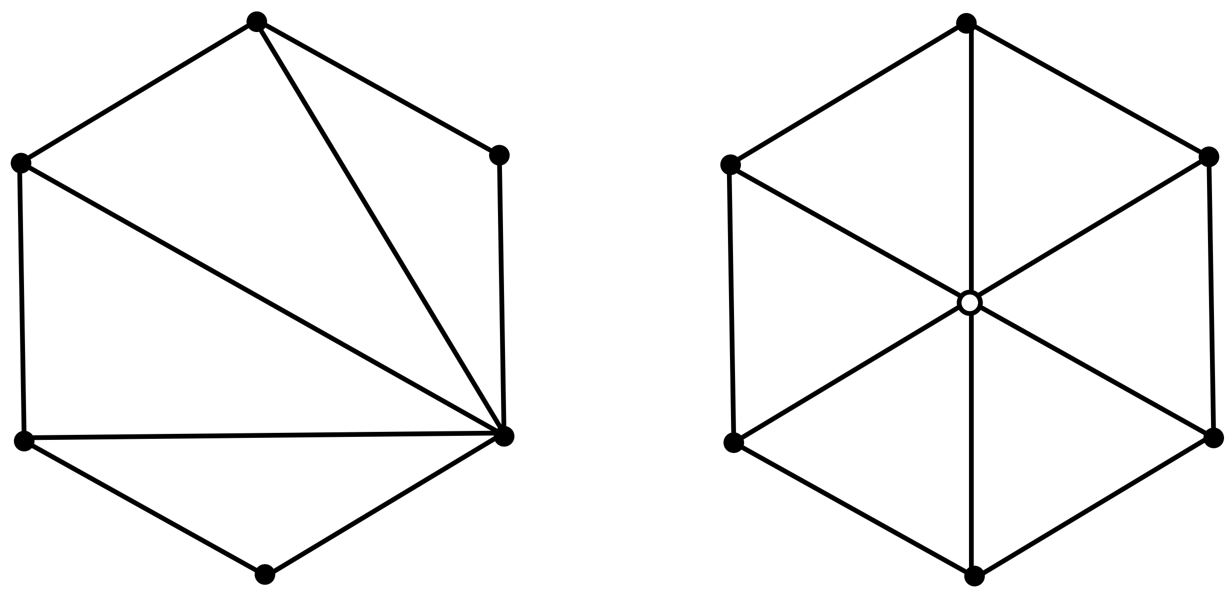
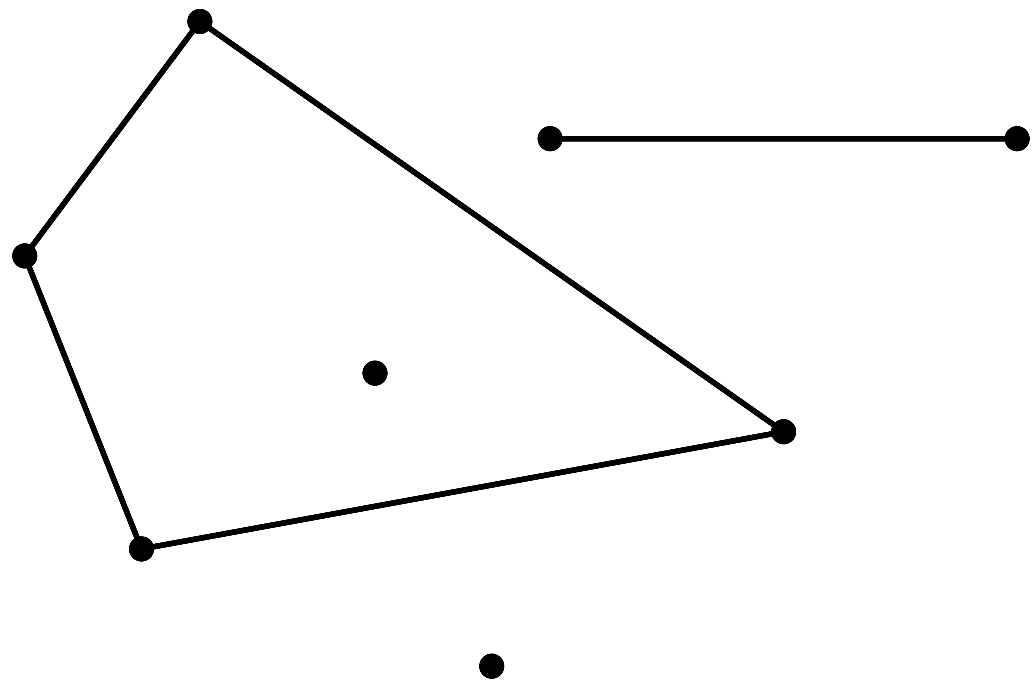


A **Planar Straight Line Graph** (PSLG) is a finite point set plus a set of disjoint edges between them.



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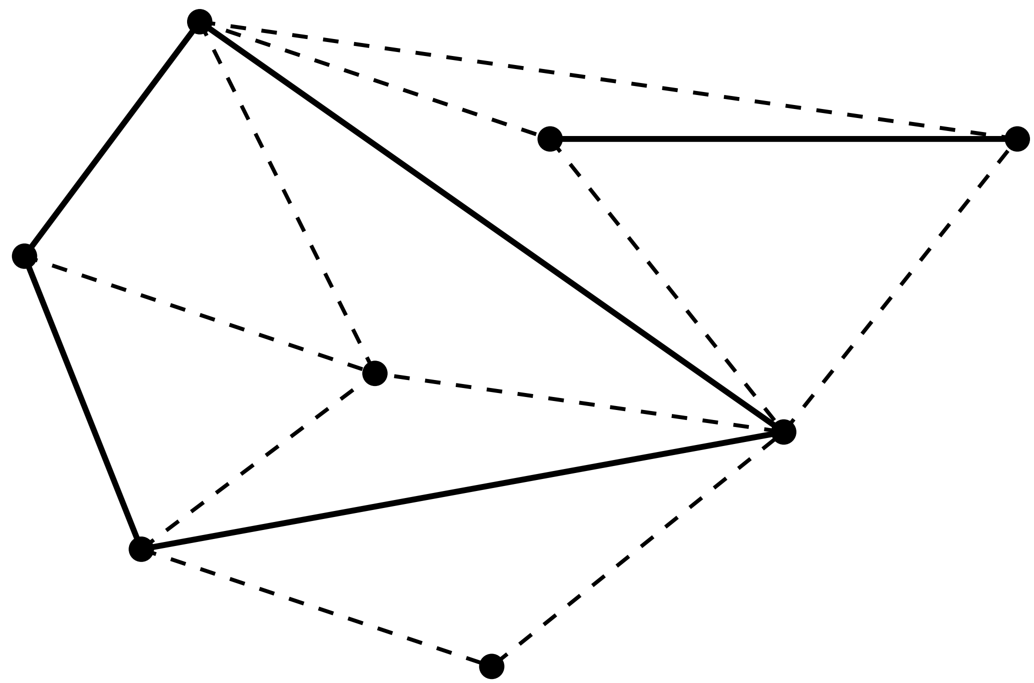

Triangulation must **use** the edges of the PSLG.

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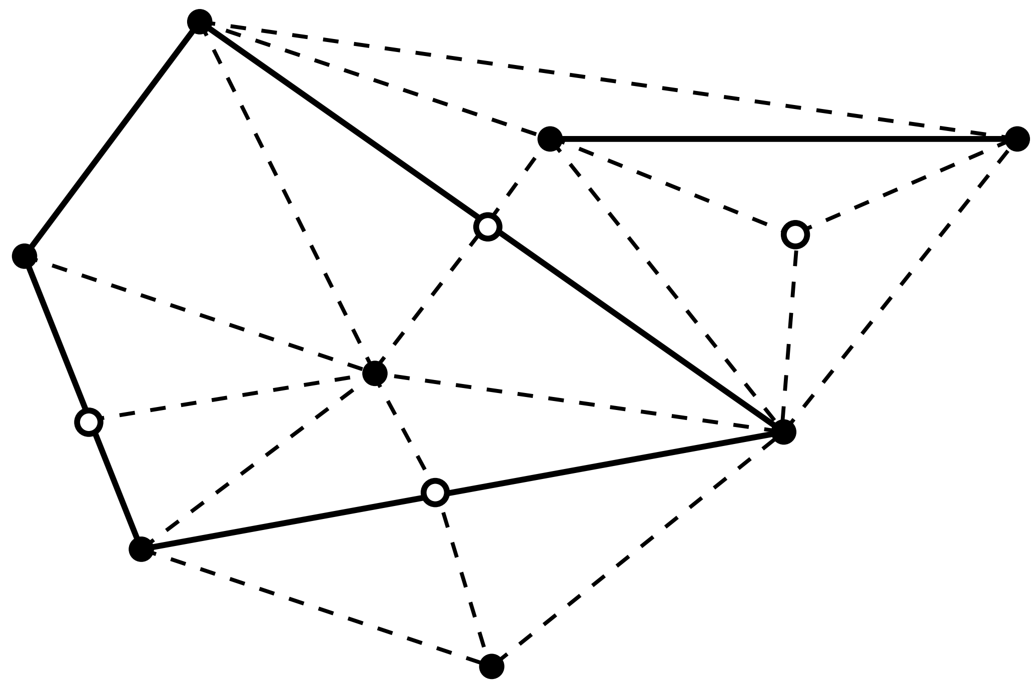
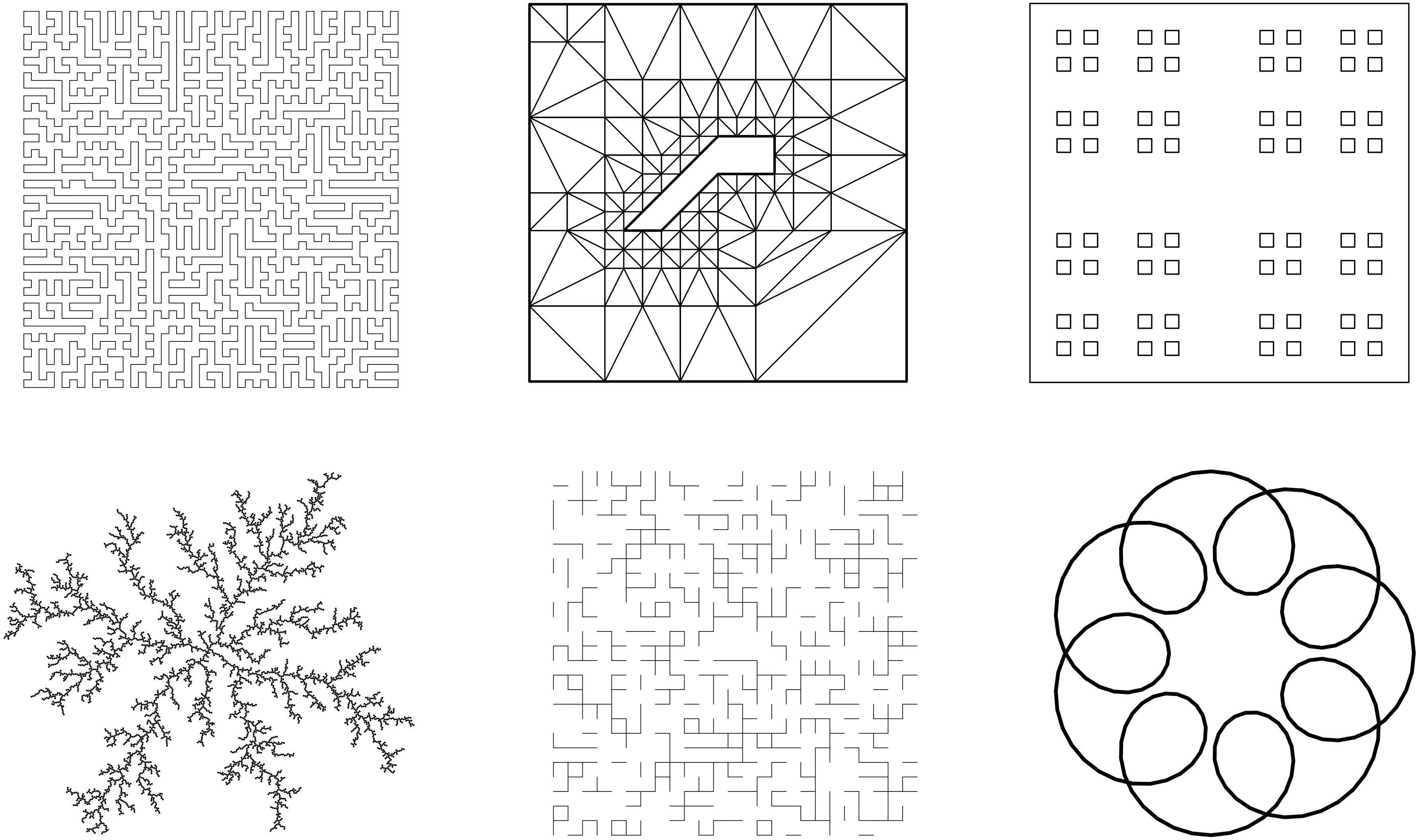
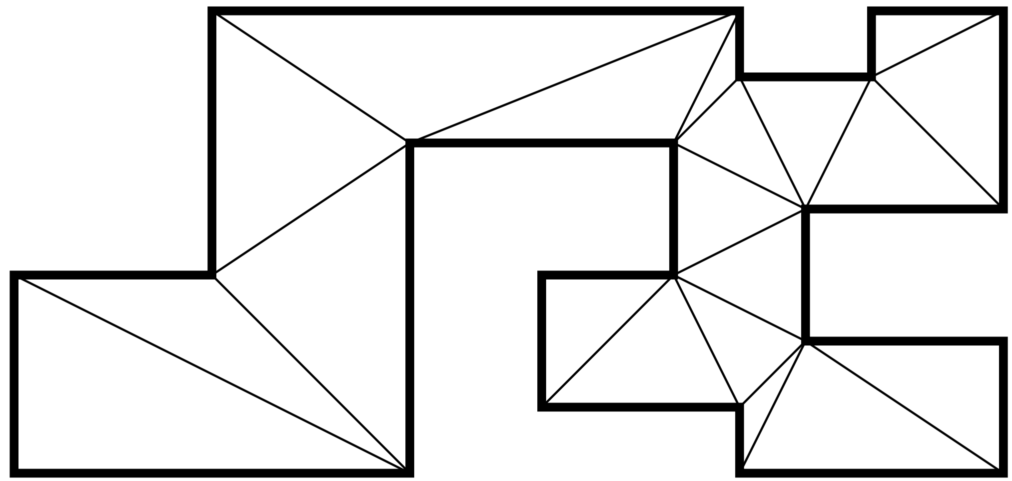


Triangulation must **cover** the edges of the PSLG.



More PSLGs

Special case of PSLG is a polygon.



A triangulation of a polygon only covers the interior.

A triangulation of a polygon has a tree structure.

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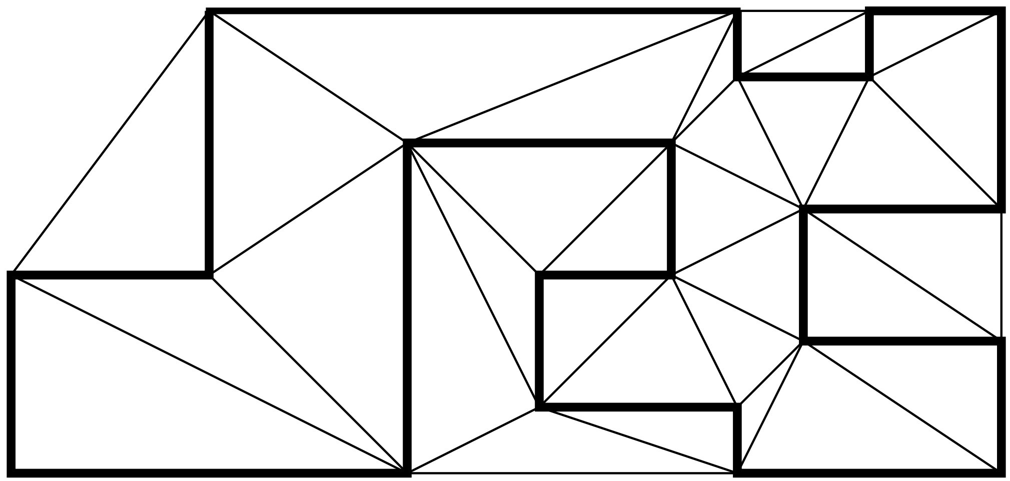

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Two conflicting goals: add Steiner points so we

- Triangulate with best geometry (angles bounded)
- Triangulate with least complexity (fewest points).

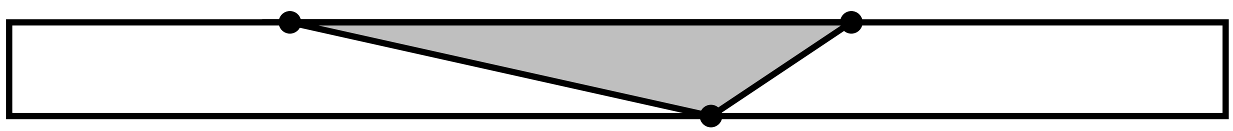
Compromise: find best angle bounds that allow complexity bounds depending only on n.

Angles $\leq 90^\circ$ is best we can do.

Why?

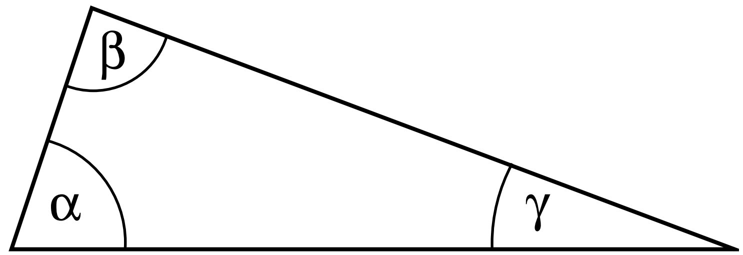
For $1 \times R$ rectangle

number of triangles $\gtrsim R \times$ (smallest angle)



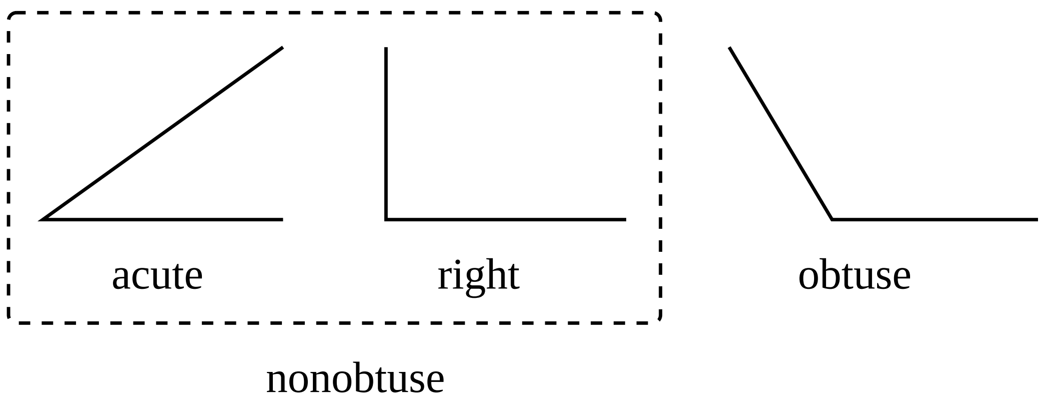
So uniform complexity $\Rightarrow$ no lower angle bound.

If all angles are $\leq 90^\circ - \epsilon$ then all angles are $\geq 2\epsilon$.



$$\alpha, \beta, \gamma \leq 90 - \epsilon,$$
$$\gamma = 180 - \alpha - \beta \geq 180 - (90 - \epsilon) - (90 - \epsilon) \geq 2\epsilon.$$

So nonobtuse triangulation is best we can hope for.



Brief history of nonobtuse triangulation:

- Always possible: Burago, Zalgaller 1960.
- Rediscovered: Baker, Grosse, Rafferty, 1988.
- $O(n)$ for points sets: Bern, Eppstein, Gilbert 1990
- $O(n^2)$ for polygons: Bern, Eppstein 1991
- $O(n)$ for polygons: Bern, Mitchell, Ruppert 1994

No known bounds for PSLGs.

Many heuristics.

What if we replace " $\leq 90^\circ$ " with " $\leq \theta < 180^\circ$ "?

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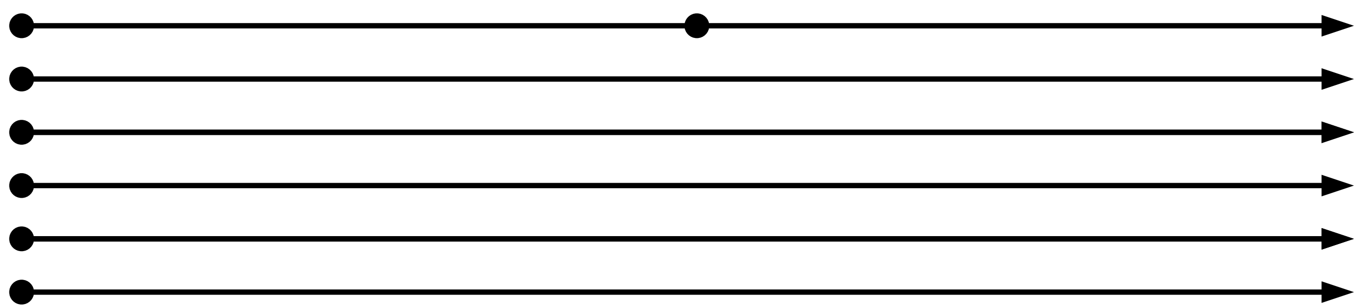
Any bound $< 180^\circ$ sometimes requires n^2 vertices.



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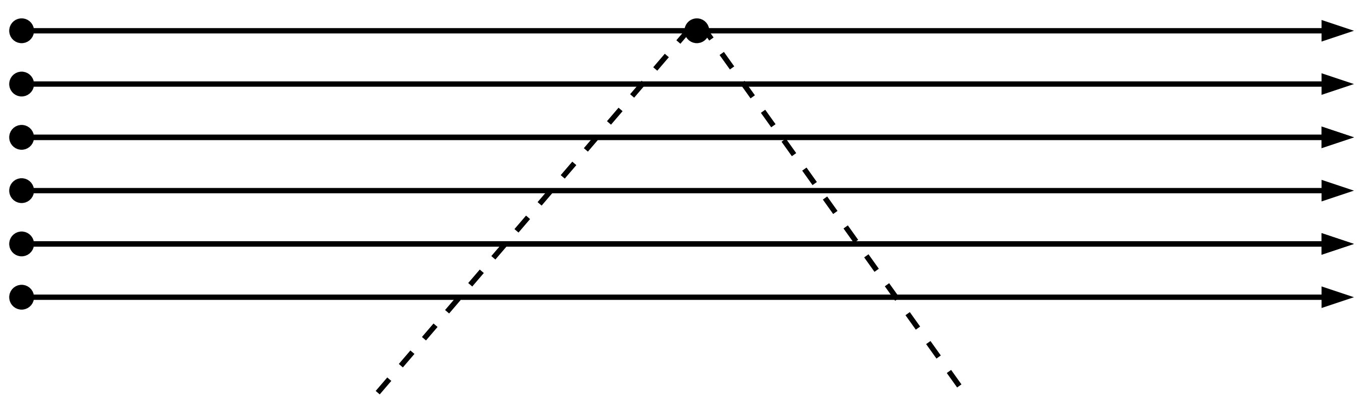

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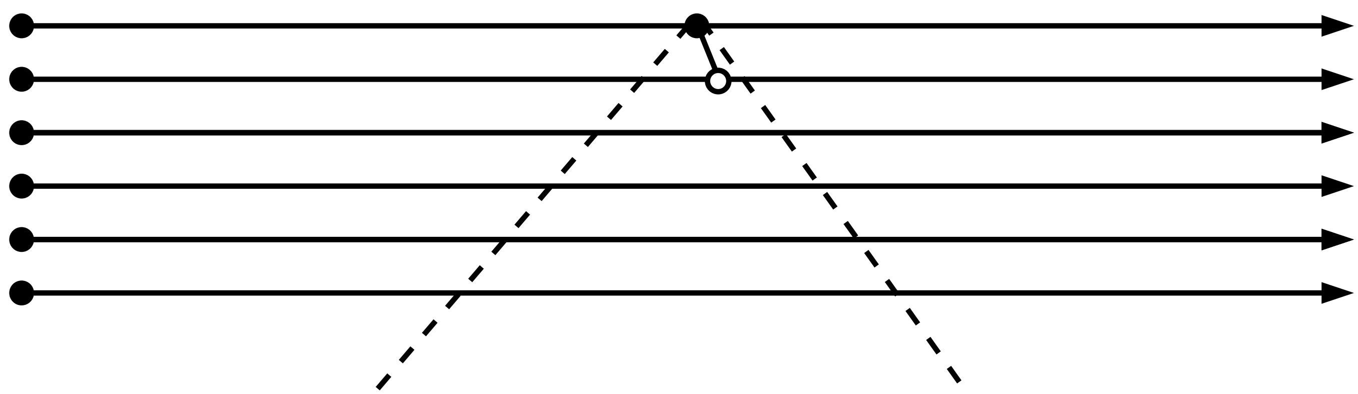

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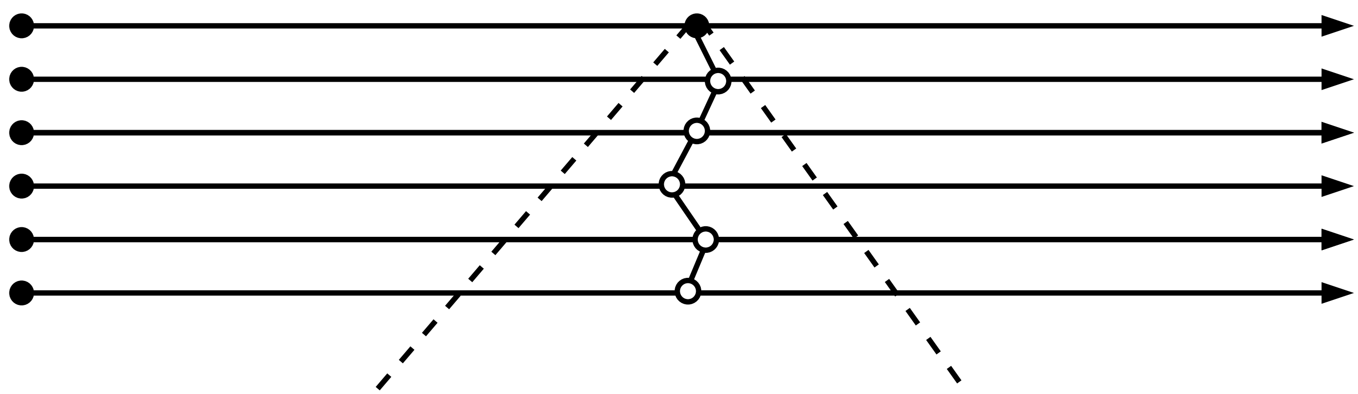

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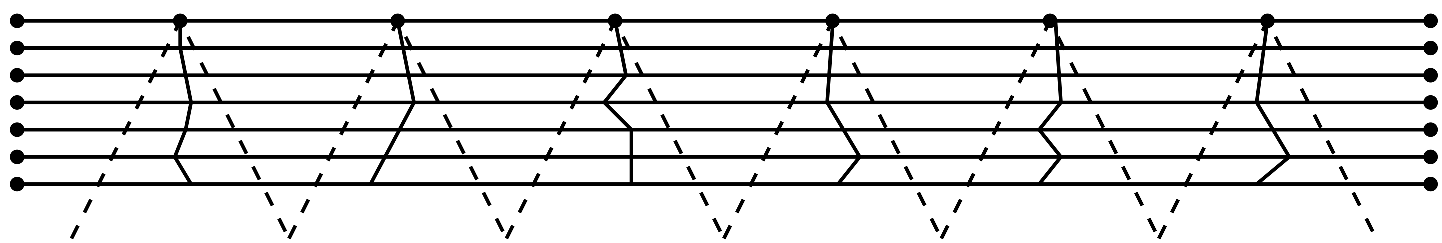


Applications of nonobtuse triangulations:

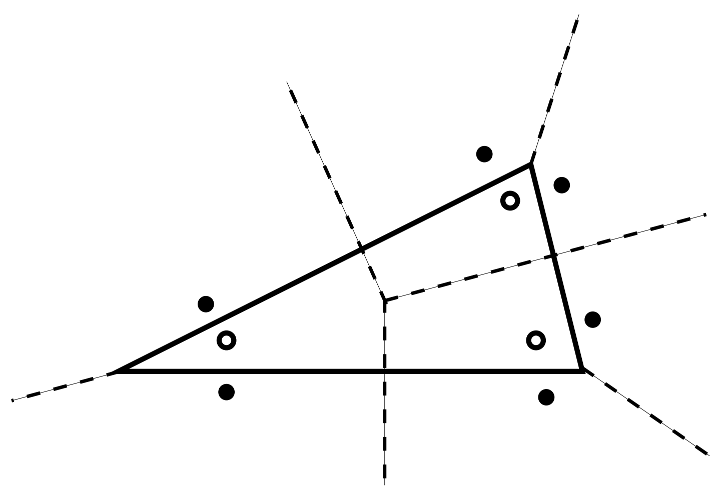
- Discrete maximum principle (Ciarlet, Raviart, 1973)
- Convergence of finite element methods (Vavasis, 1996)
- Fast marching method (Sethian, 1999)
- Meshing space-time (Ungör, Sheffer, 2002)
- Machine learning

Salzberg, Delcher, Heath, Kasif, 1995, *Best-case results for nearest-neighbor learning*.

Given polygon Γ find sets I, O so that

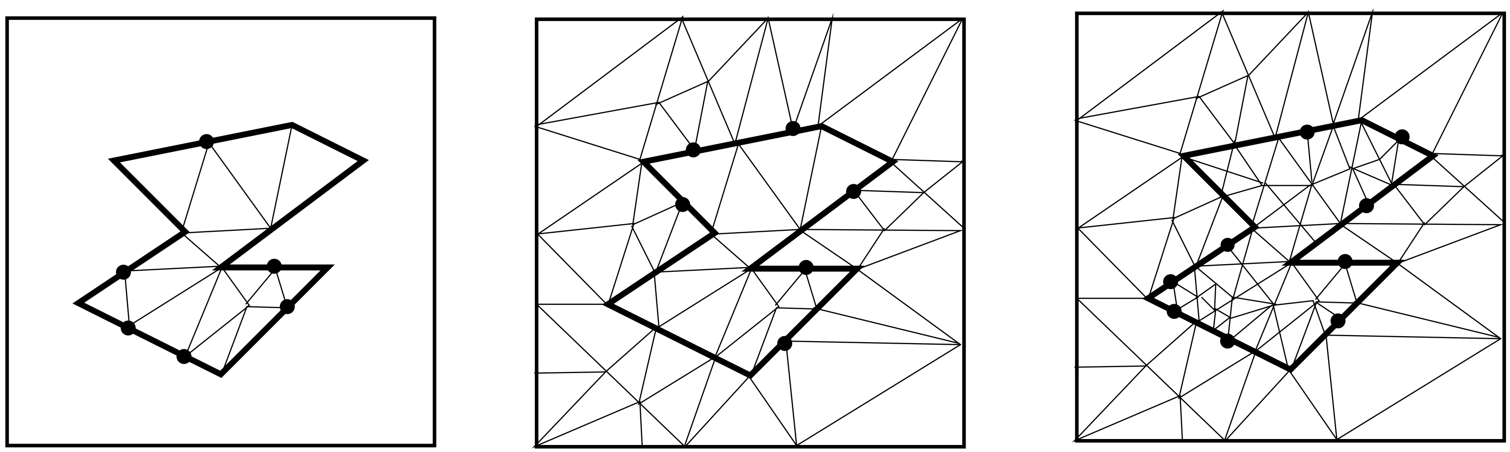
$$\text{int}(\Gamma) = \{z : \text{dist}(z, I) < \text{dist}(z, O)\},$$

Easy for nonobtuse triangles.



SDHK reduce learning Γ to nonobtuse triangulation of both sides of Γ.

Obvious strategy: mesh one side then other. But must re-mesh to include new vertices, ...



They ask if a polynomial number of points suffice.

Theorem (B, 2010): Every PSLG has a nonobtuse triangulation with $O(n^{2.5})$ elements.

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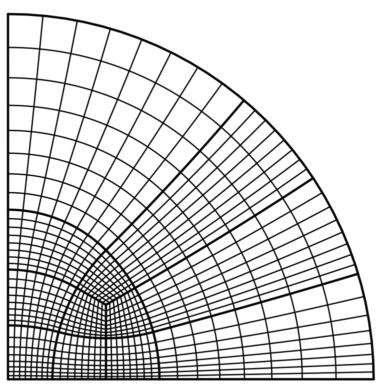
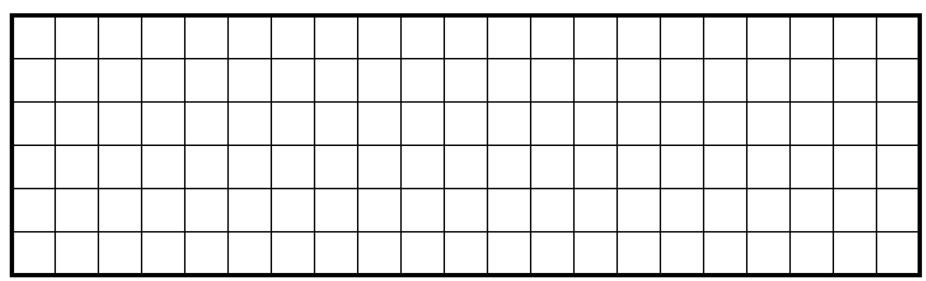
Theorem (B, 2010): Every PSLG has a triangulation with all angles $\leq 90^\circ + \epsilon$ and $O(n^2/\epsilon^2)$ elements.

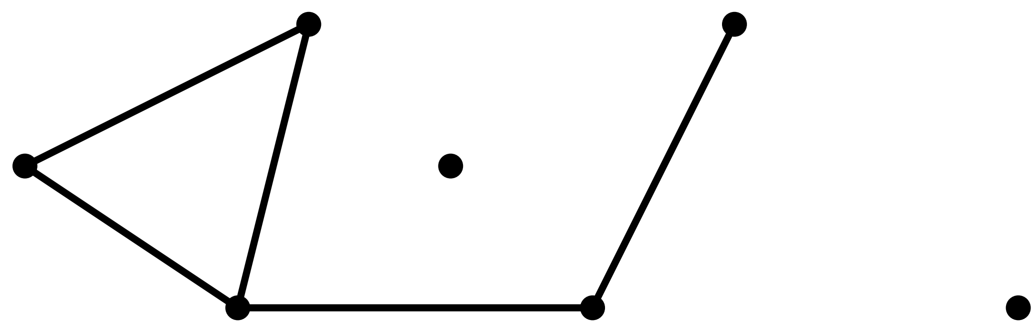
Theorem (B, 2010): Every PSLG has a quadrilateral mesh with $O(n^2)$ elements, all angles less than 120° and all new angles greater than 60°.





The nonobtuse triangulation problem reduces to:

Theorem: For any PSLG Γ of size n there is set of size $O(n^{2.5})$ whose Gabriel graph covers Γ.



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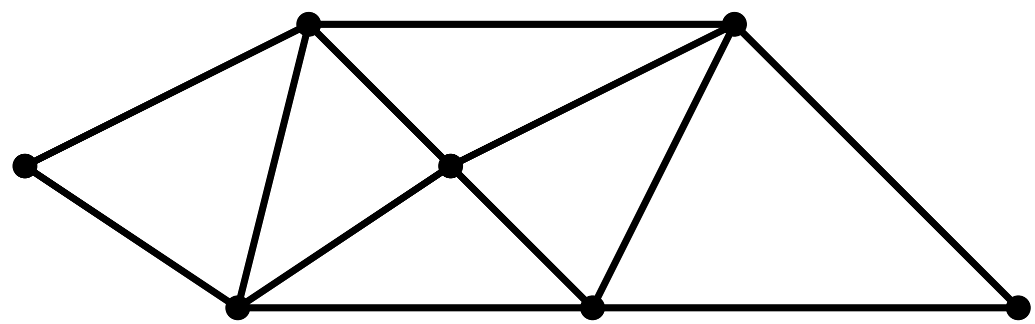

Suffices to consider Γ a triangulation.

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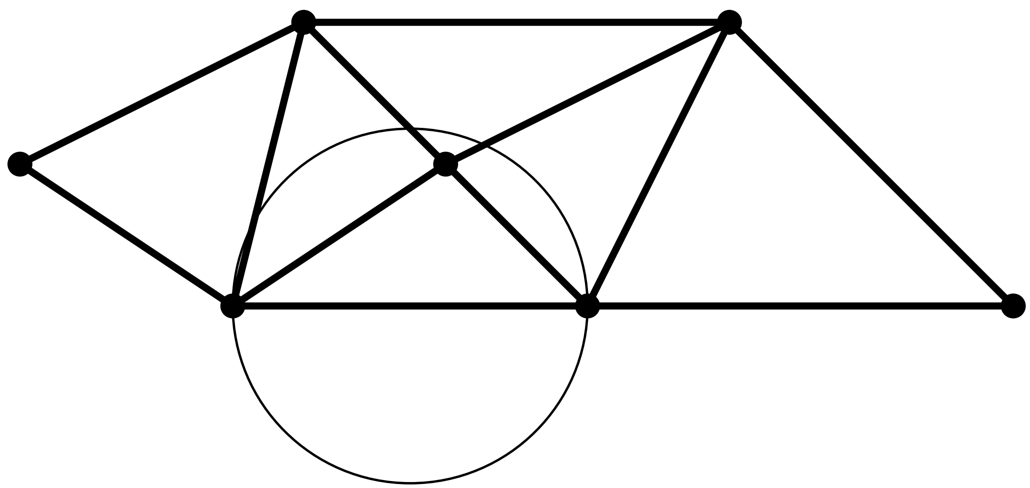

This triangulation is not Gabriel.

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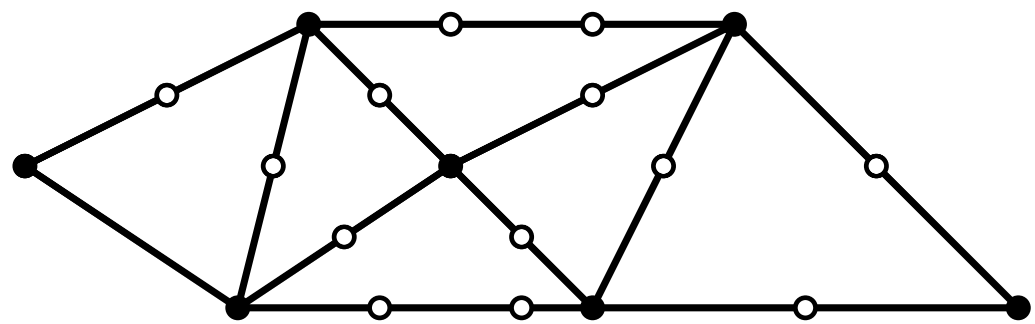

Add $O(n^{2.5})$ points to the PSLG.

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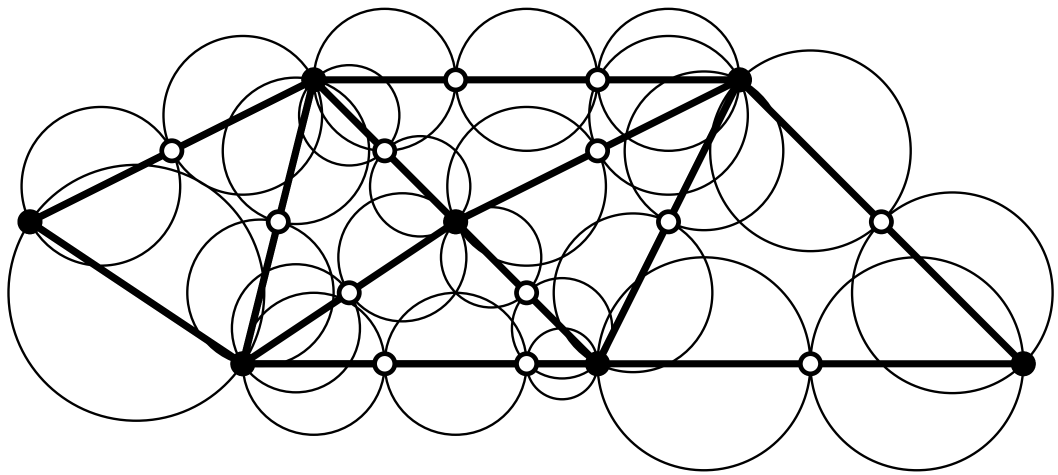
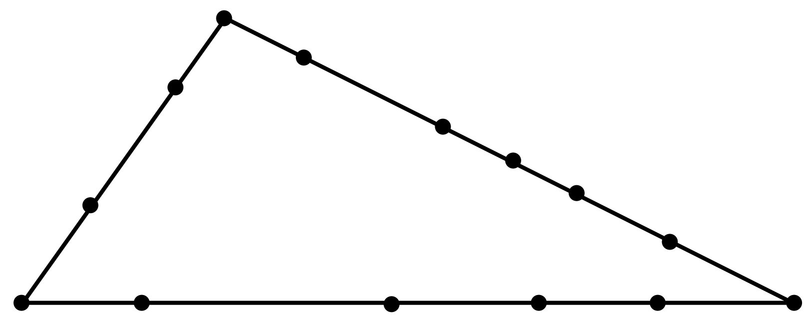


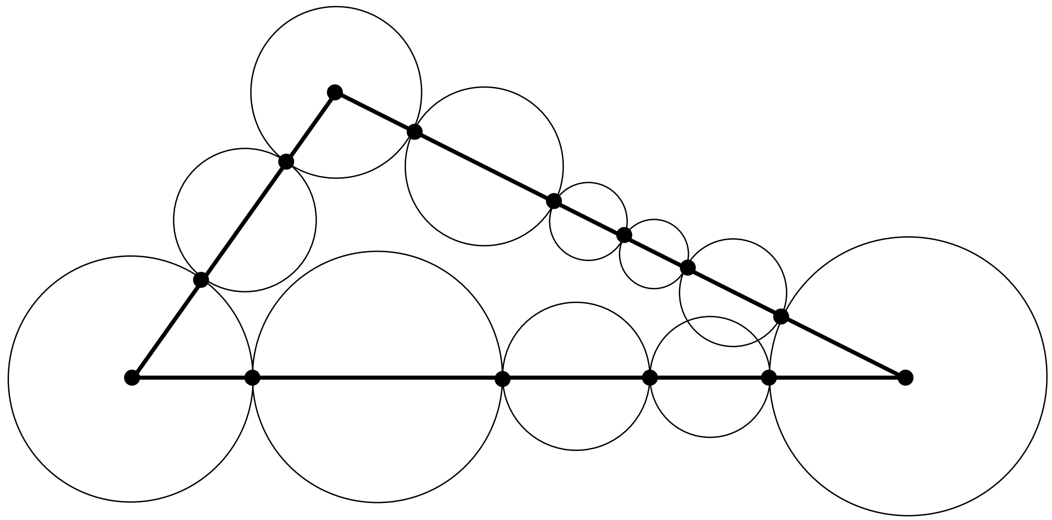
Gabriel graph covers PSLG.

Reducing nonobtuse triangulation to Gabriel covers follows ideas of Bern, Mitchell and Ruppert (1994).

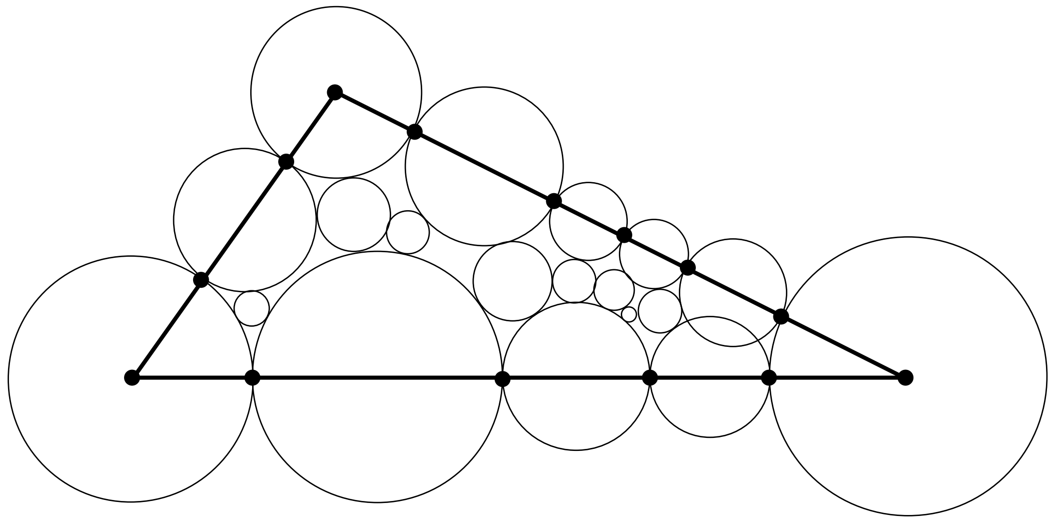
Proof of reduction:



Proof of reduction:

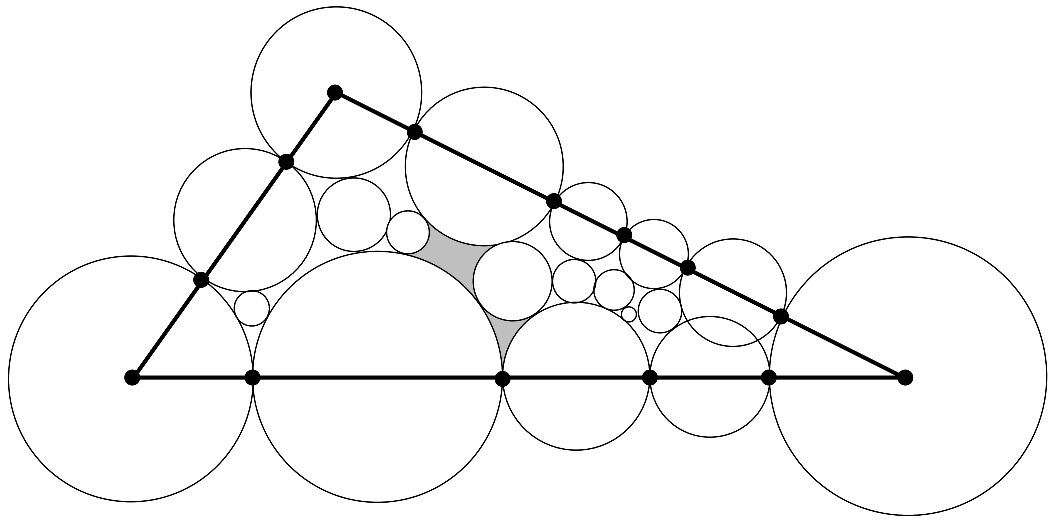


Proof of reduction:



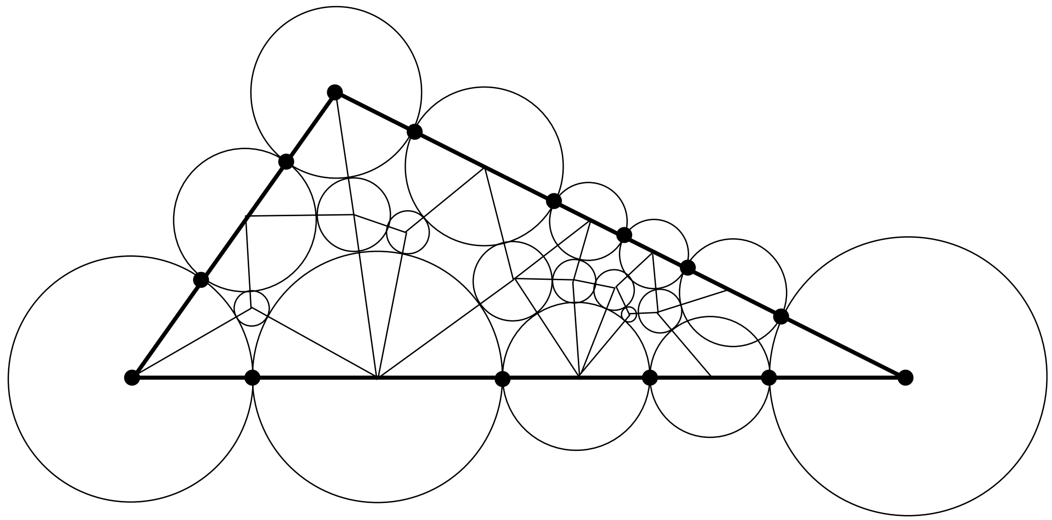
• Pack interior with disjoint disks so only 3-sided and 4-sided regions remain.

Proof of reduction:



• Pack interior with disjoint disks so only 3-sided and 4-sided regions remain.

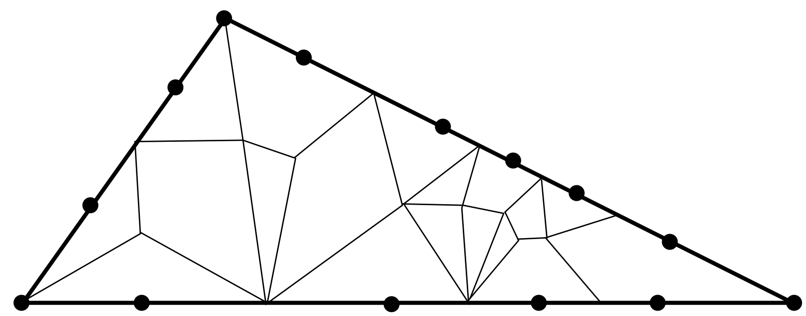
Proof of reduction:



• Pack interior with disjoint disks so only 3-sided and 4-sided regions remain.

• Connect centers.

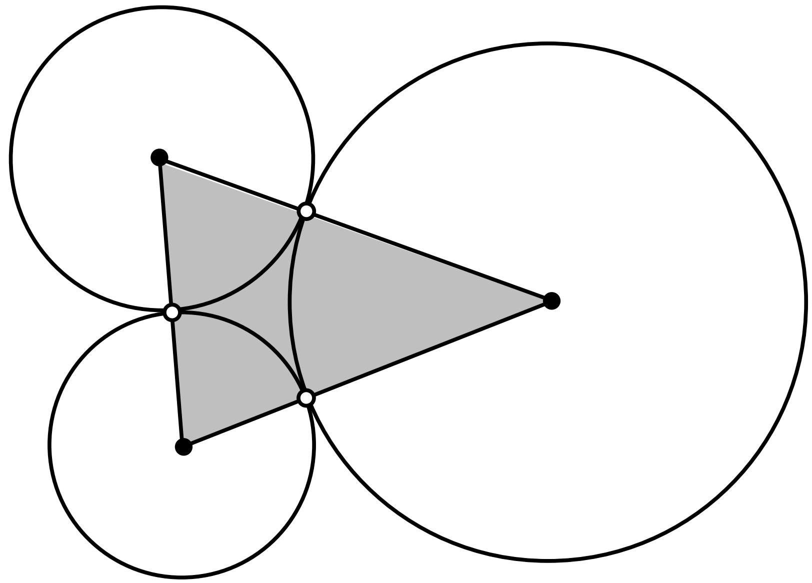
Proof of reduction:



• Pack interior with disjoint disks so only 3-sided and 4-sided regions remain.
• Connect centers.
• Divides triangle into triangles and quadrilaterals.

Must nonobtusely triangulate each region without adding new vertices along boundary. Several cases.

First case: decompose 3-region into right triangles



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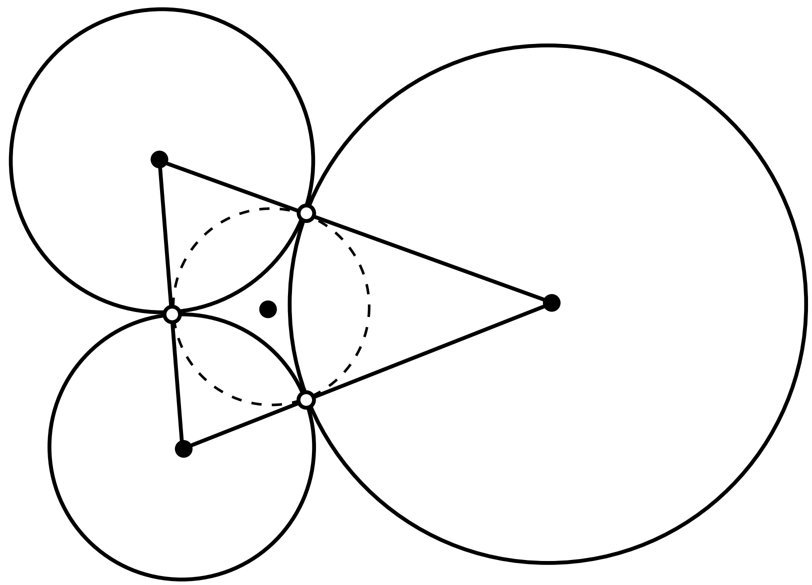
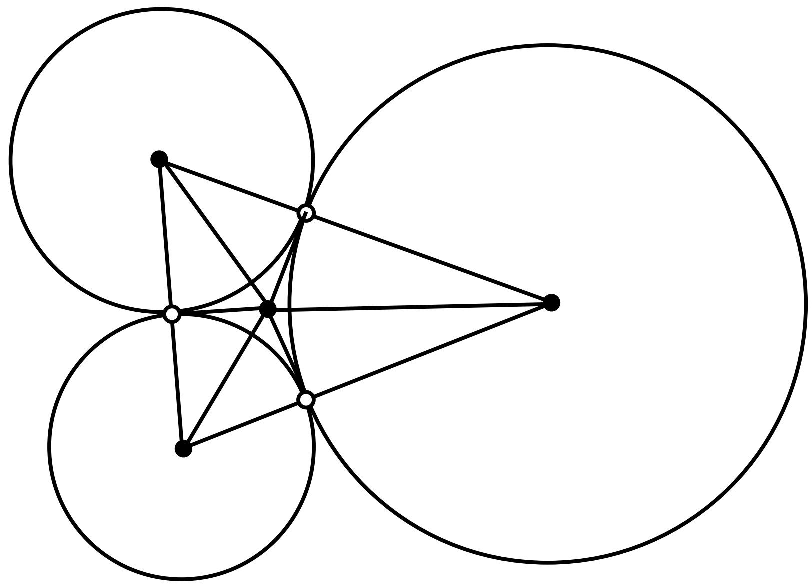


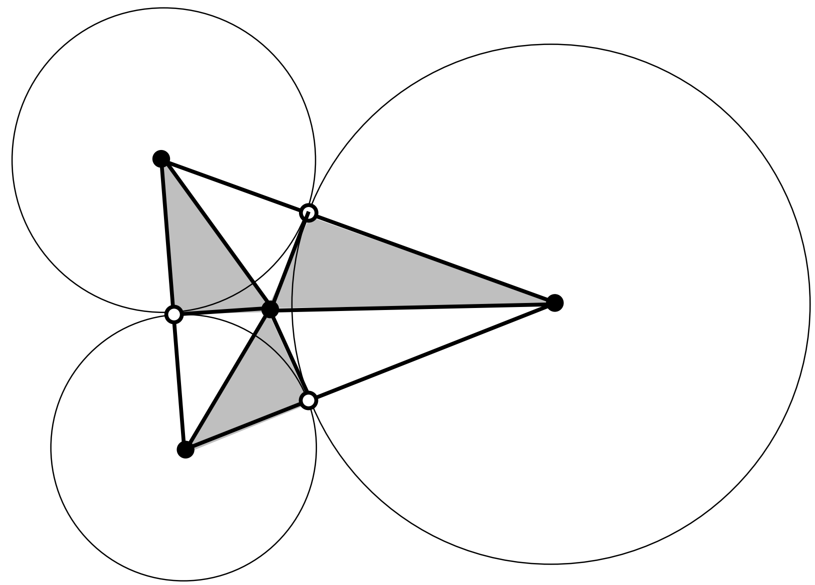
- Add center of circle through the three tangent points.

First case: decompose 3-region into right triangles



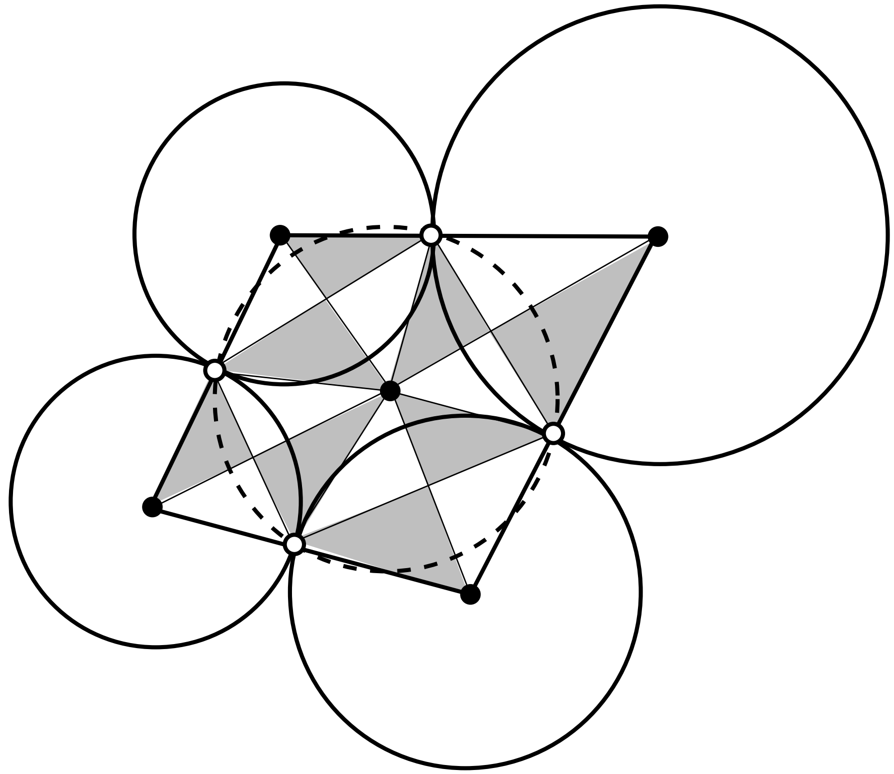
- Add center of circle through the three tangent points.
- Connect center to tangent points and centers of circles

First case: decompose 3-region into right triangles



- Add center of circle through the three tangent points.
- Connect center to tangent points and centers of circles

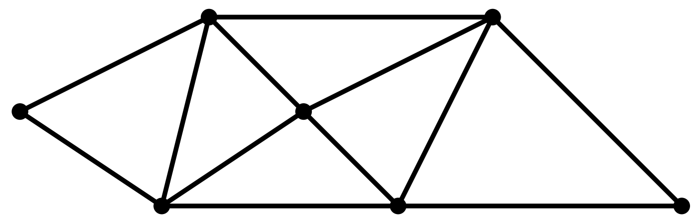
The 4-regions are similar (but several cases arise).



So Gabriel covering ⇒ nonobtuse triangulation.

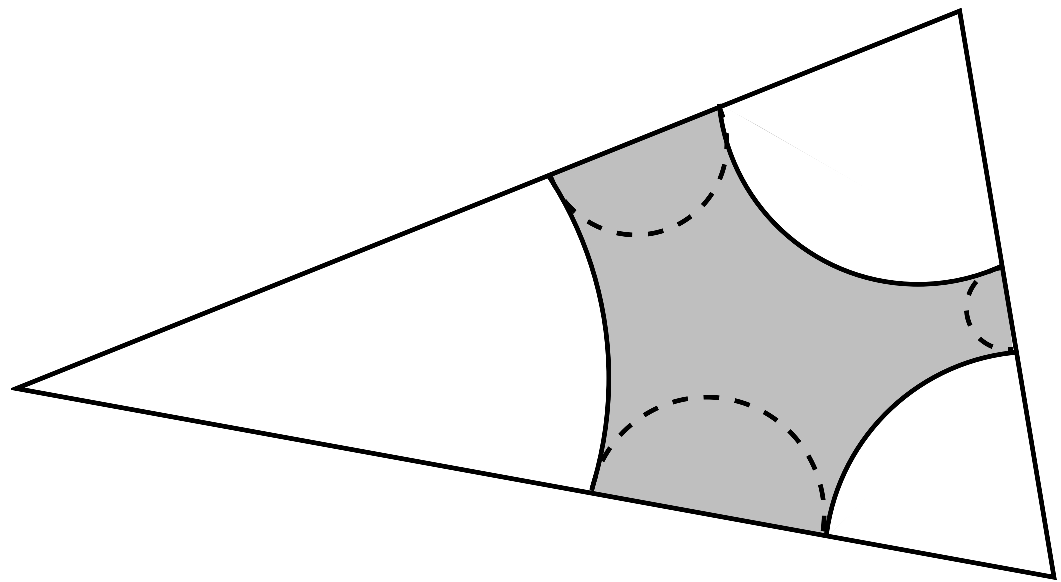
How to prove:

Theorem: For any PSLG Γ of size n there is set of size $O(n^{2.5})$ whose Gabriel graph covers Γ.



- Partition triangles into thick and thin pieces
- Define foliation in thin parts
- Flow vertices of thick parts along foliation
- Intersections of flow paths and Γ are the Gabriel points
- Bound number of points by bending flow lines

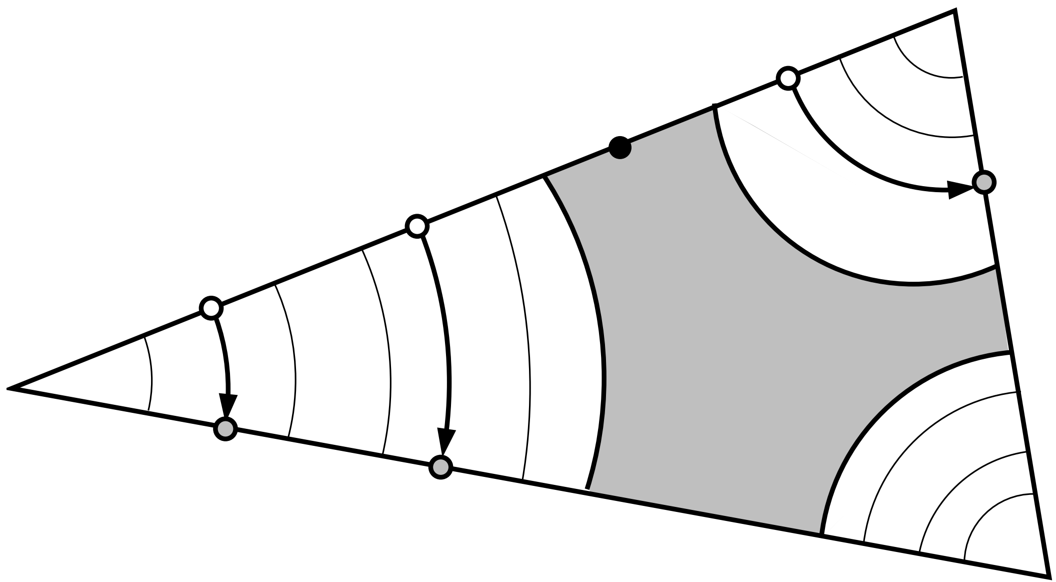
Partition triangles:



Thin parts = sectors, Thick part = central region

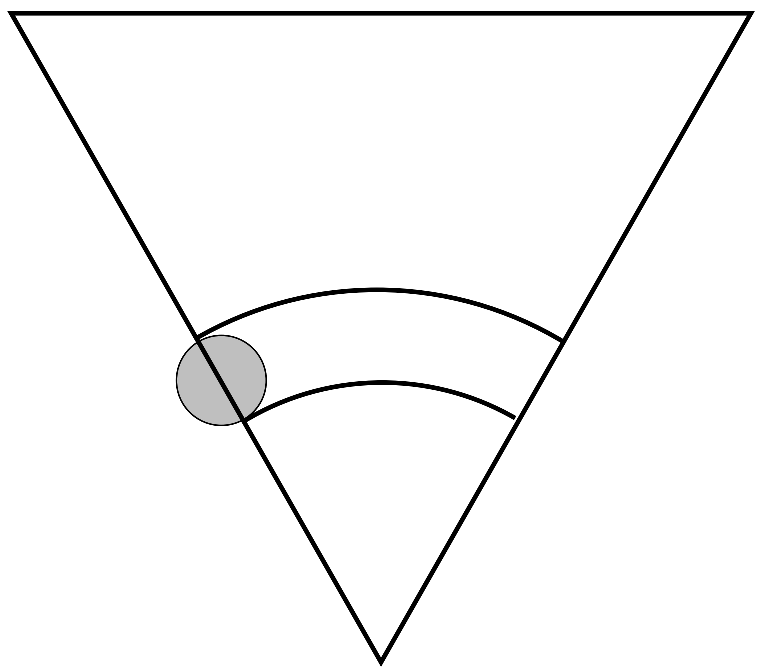
Thick sides are diameters of disjoint disks.

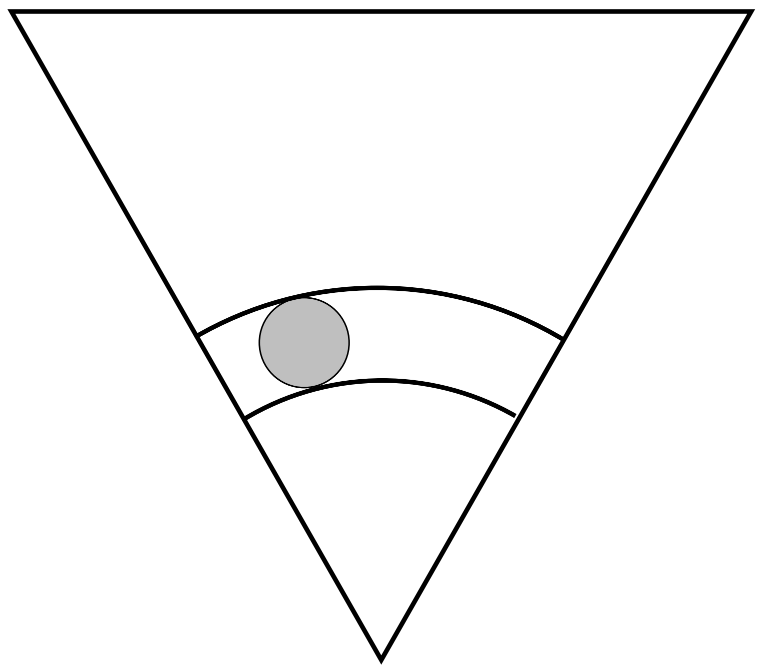
Foliate thin parts by circular arcs concentric with vertices:

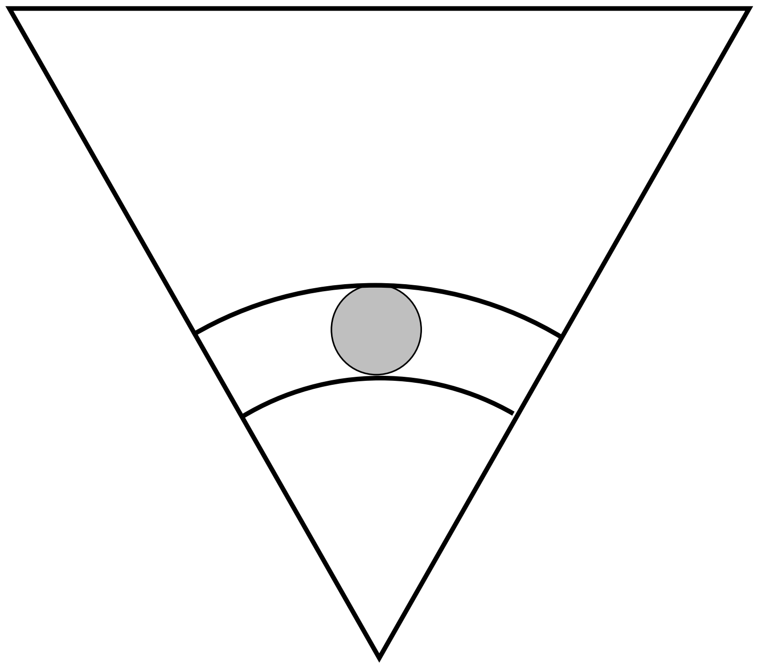


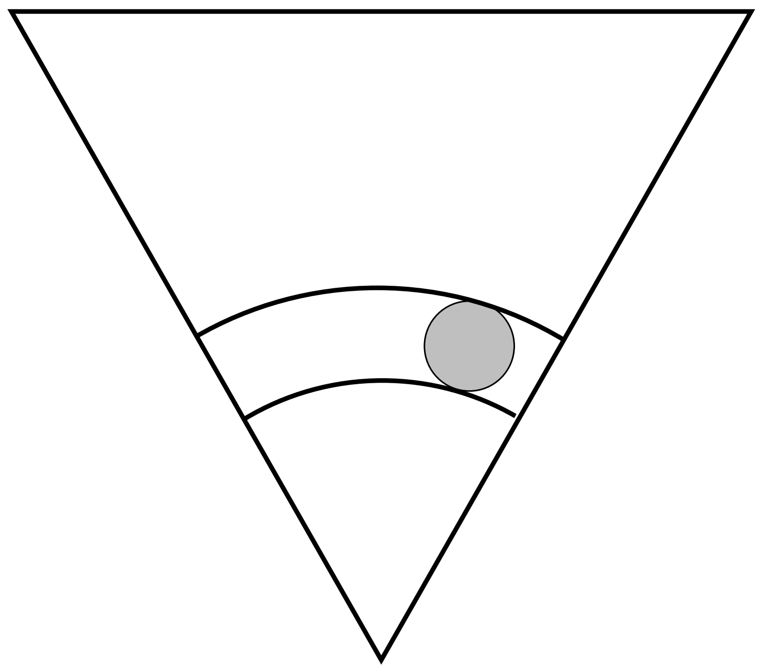
Parallel arcs form tubes of fixed width.

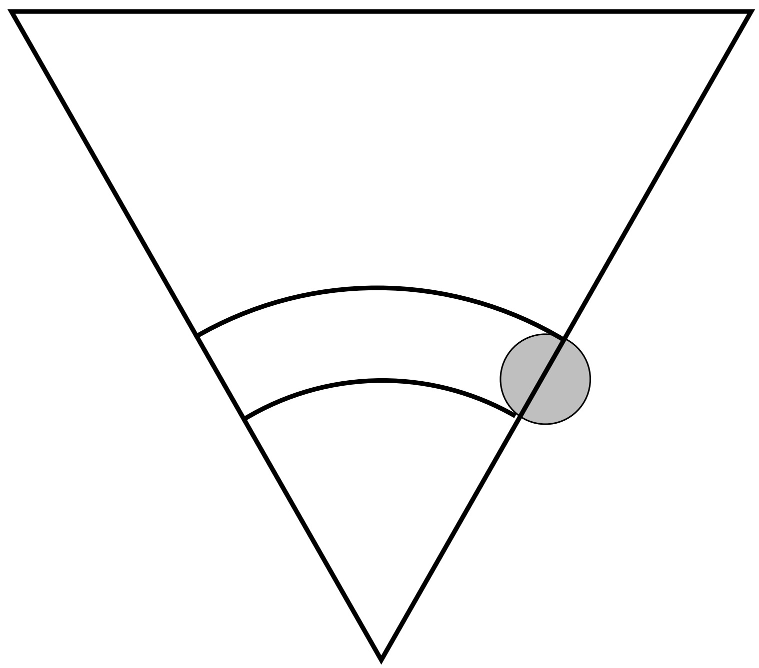
The tube is swept out by a disk of fixed size.

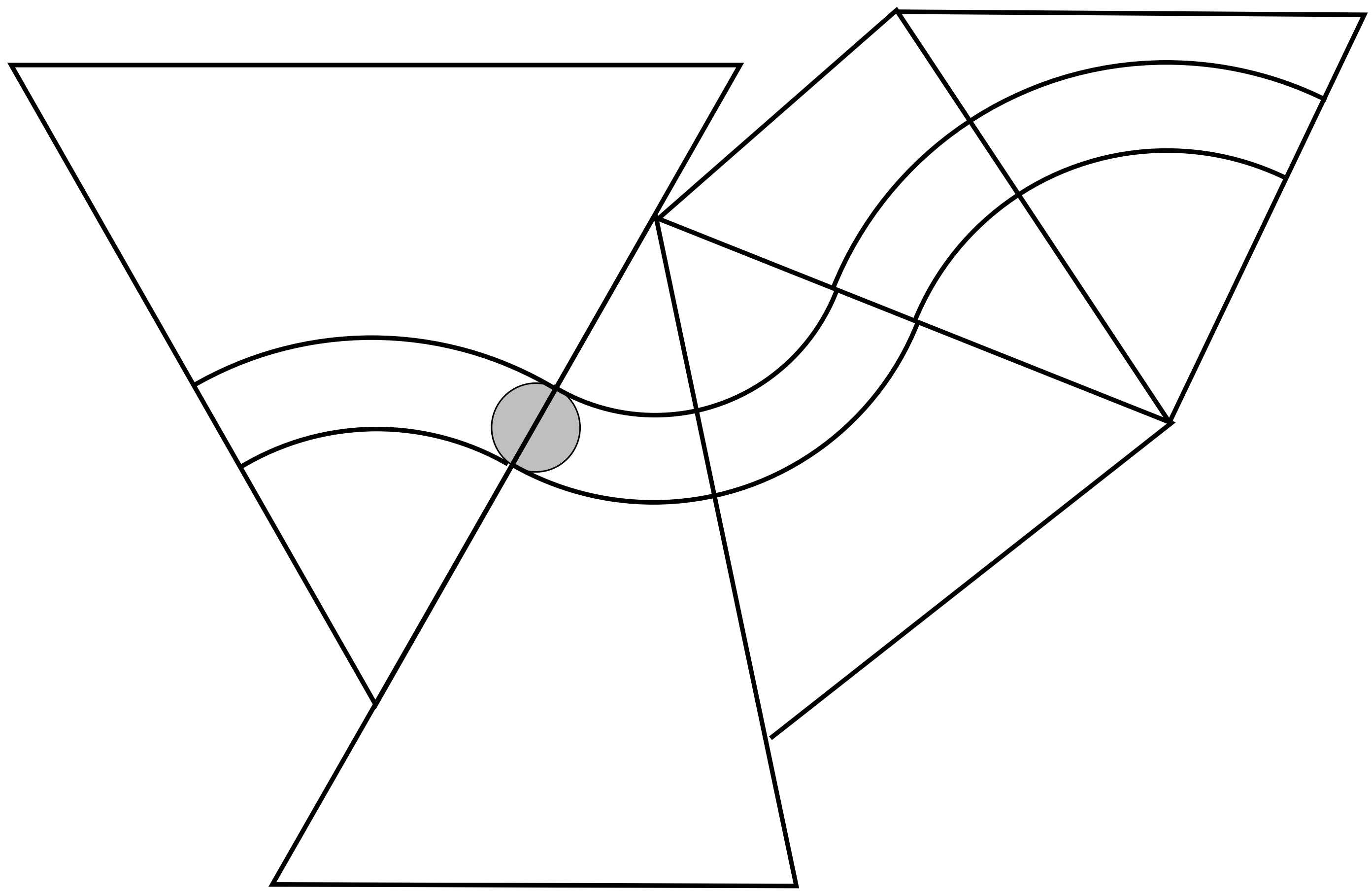


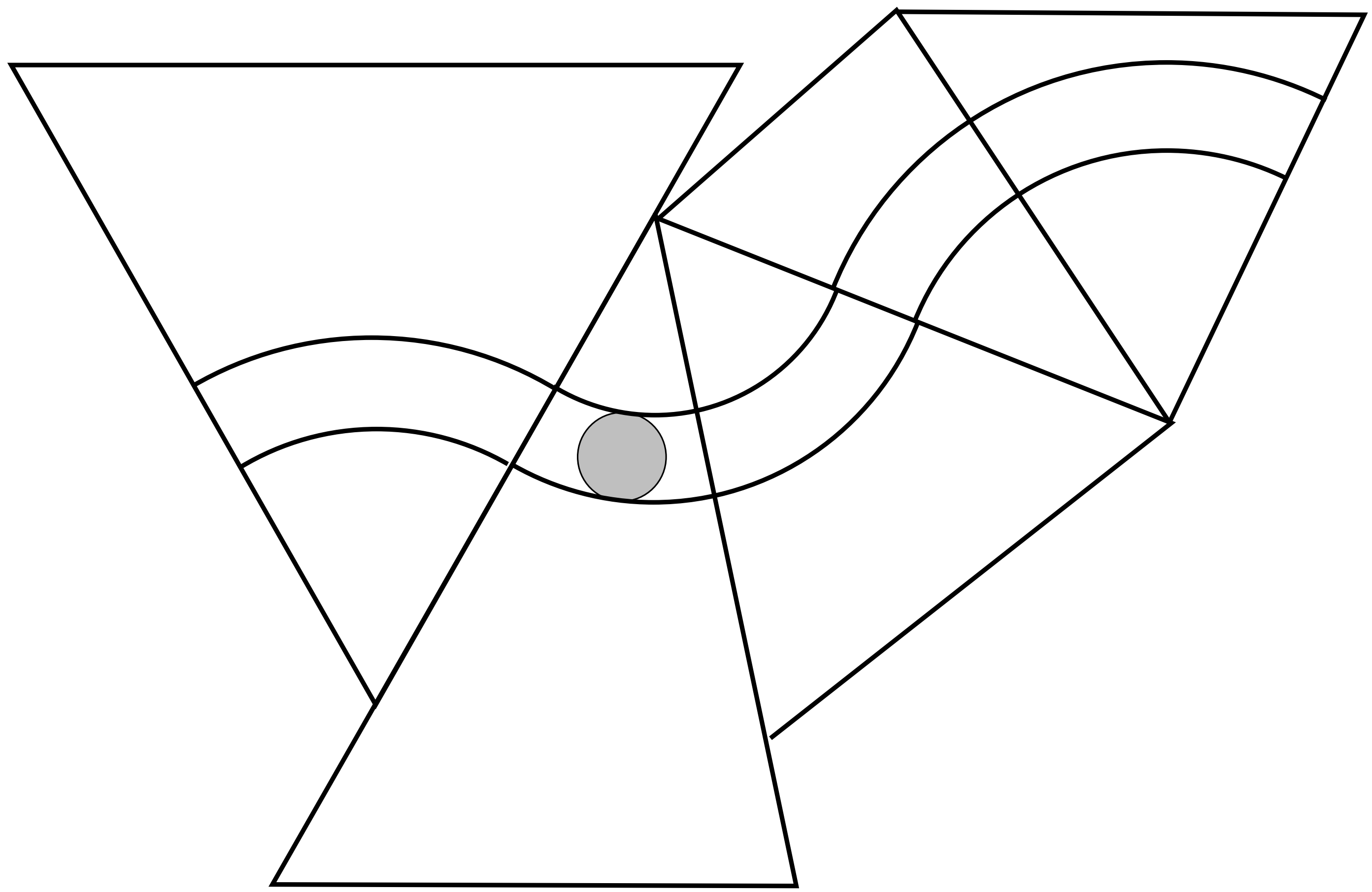


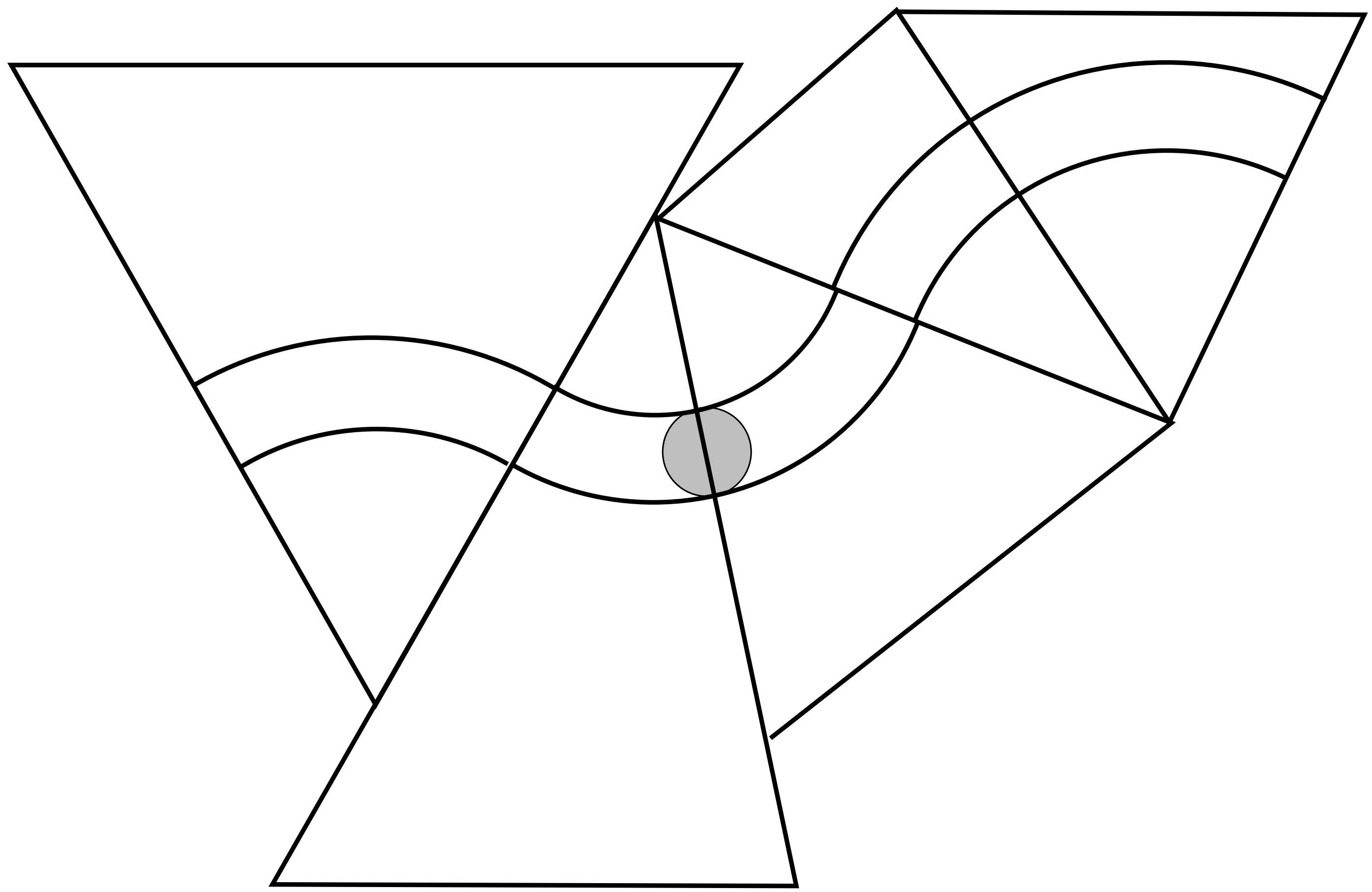


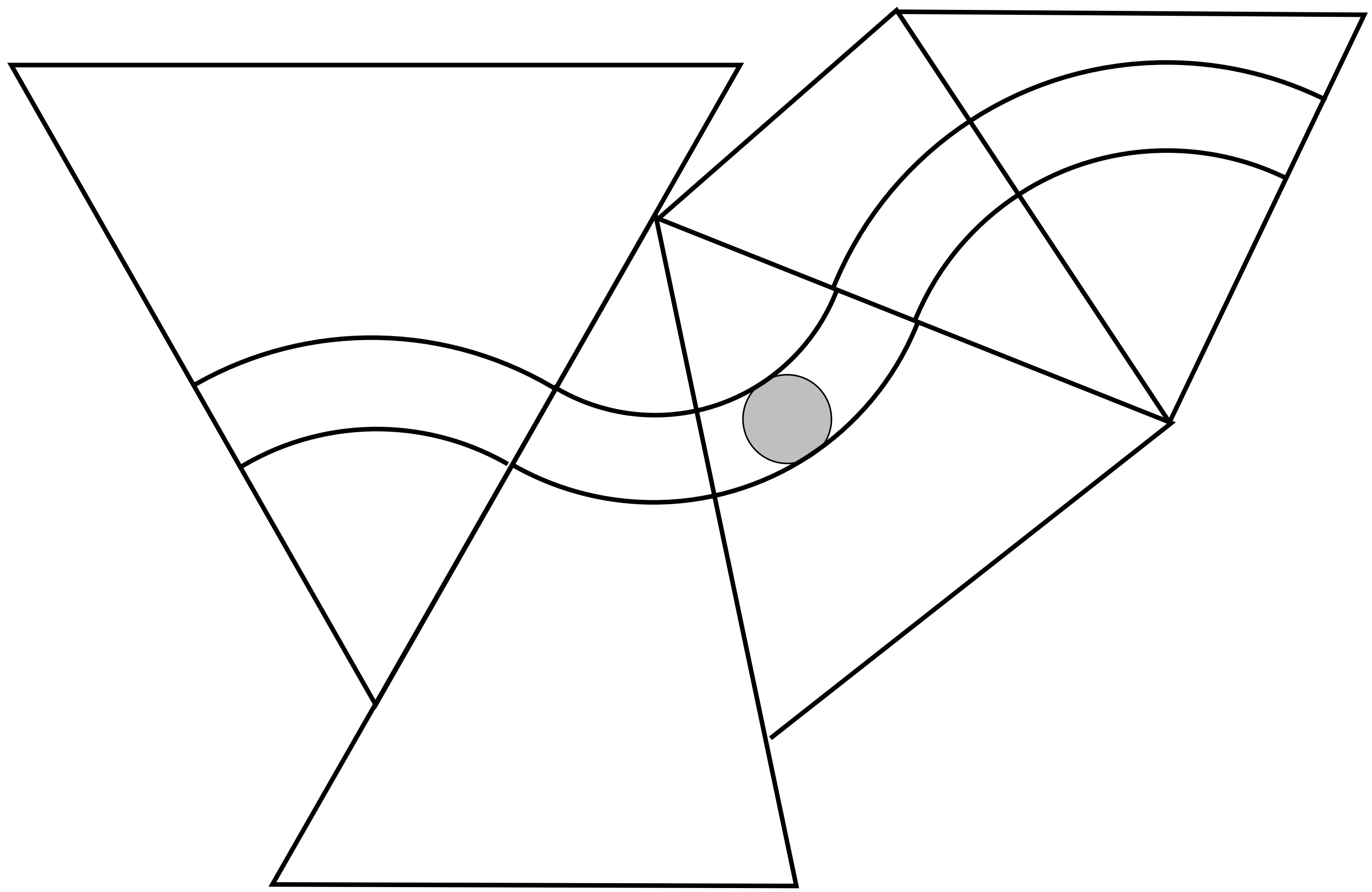


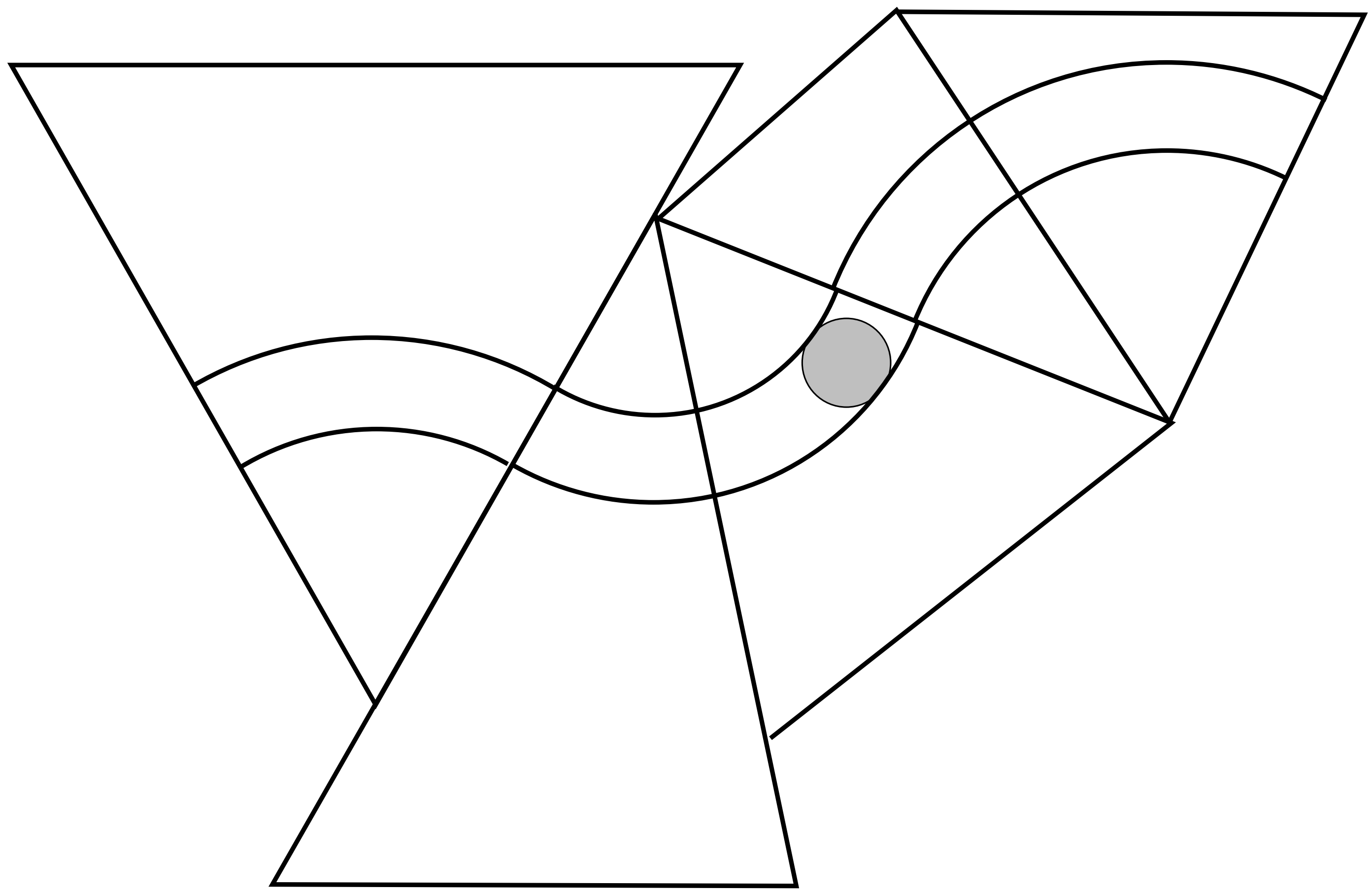


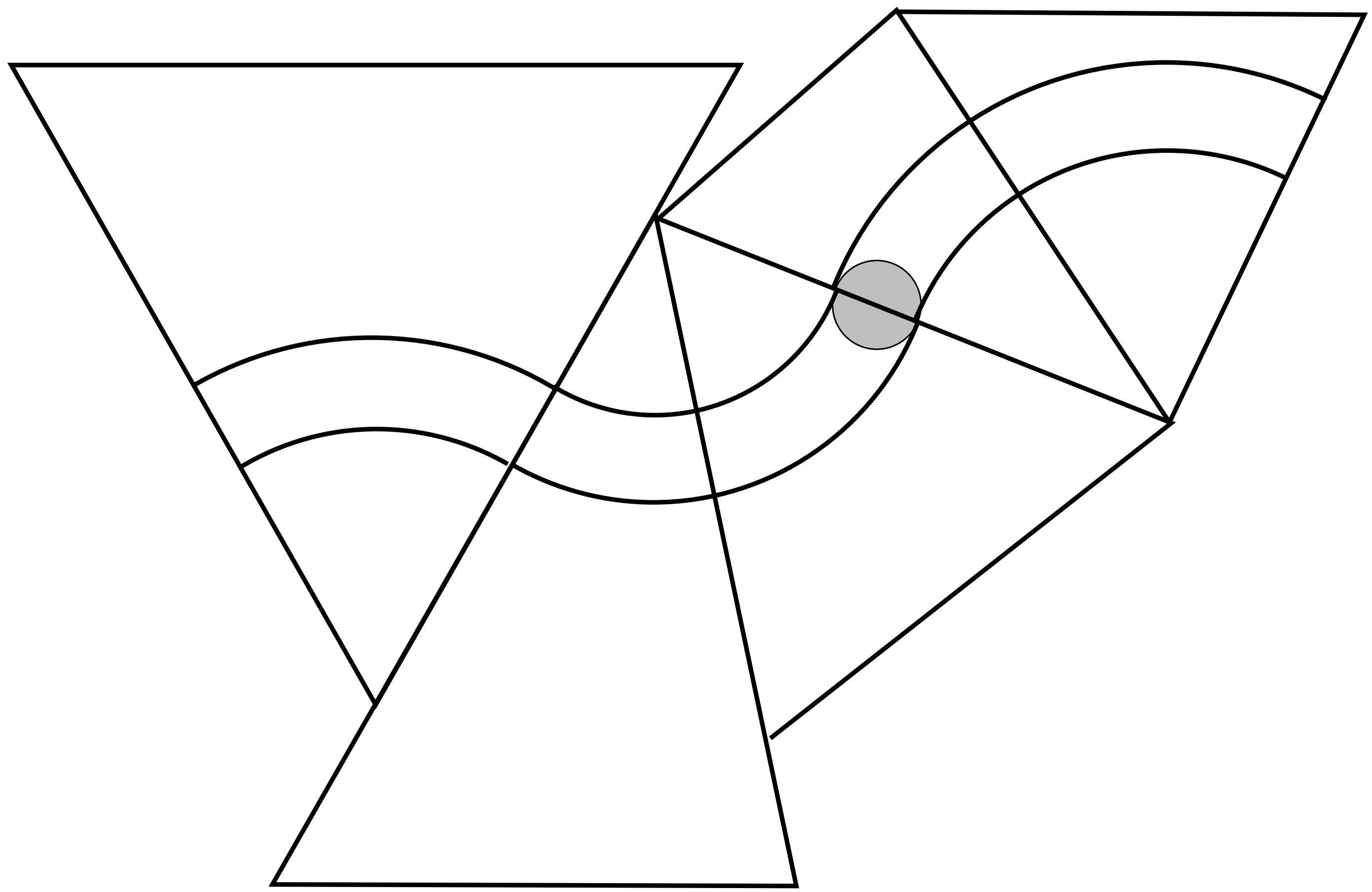


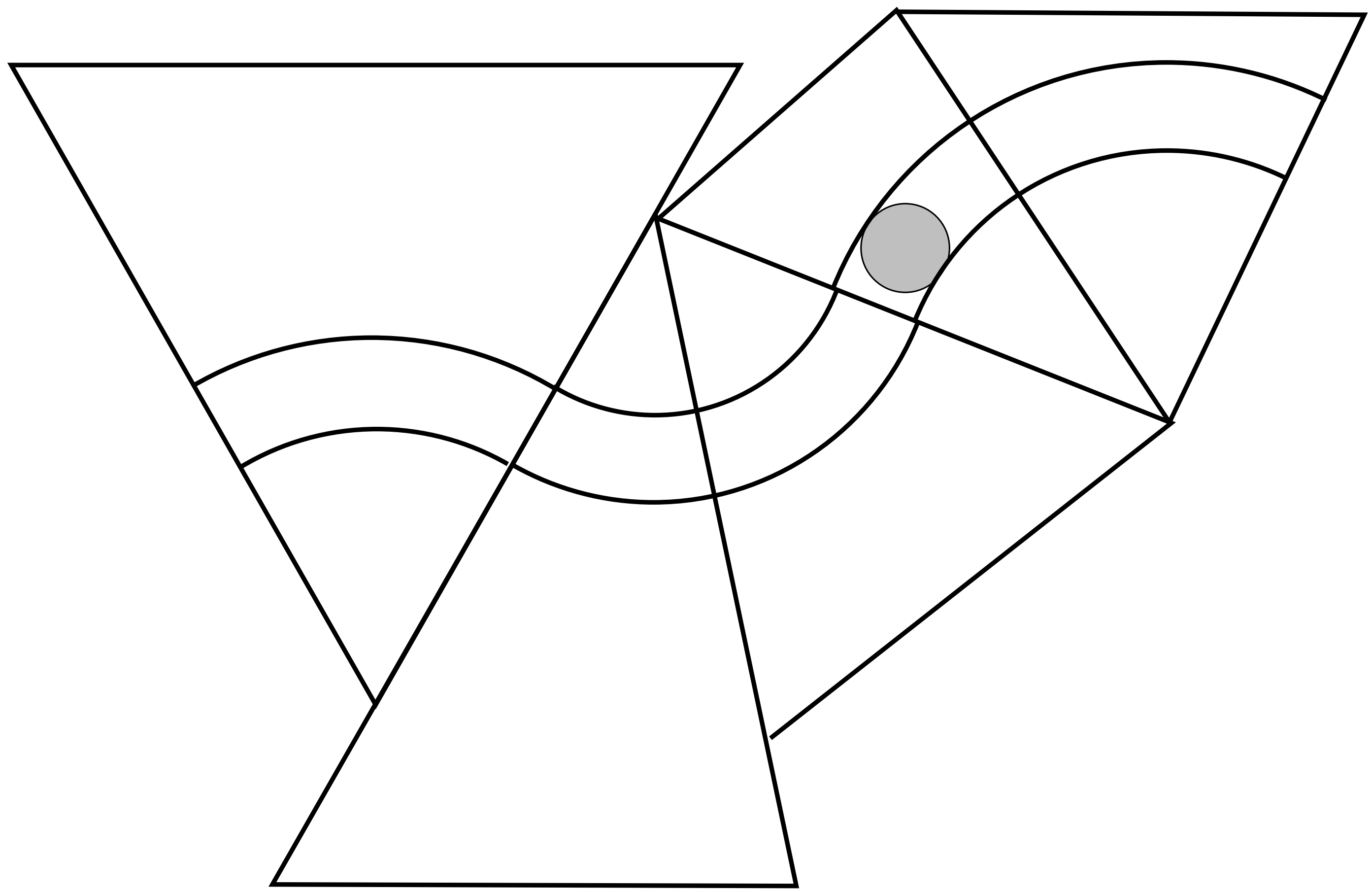


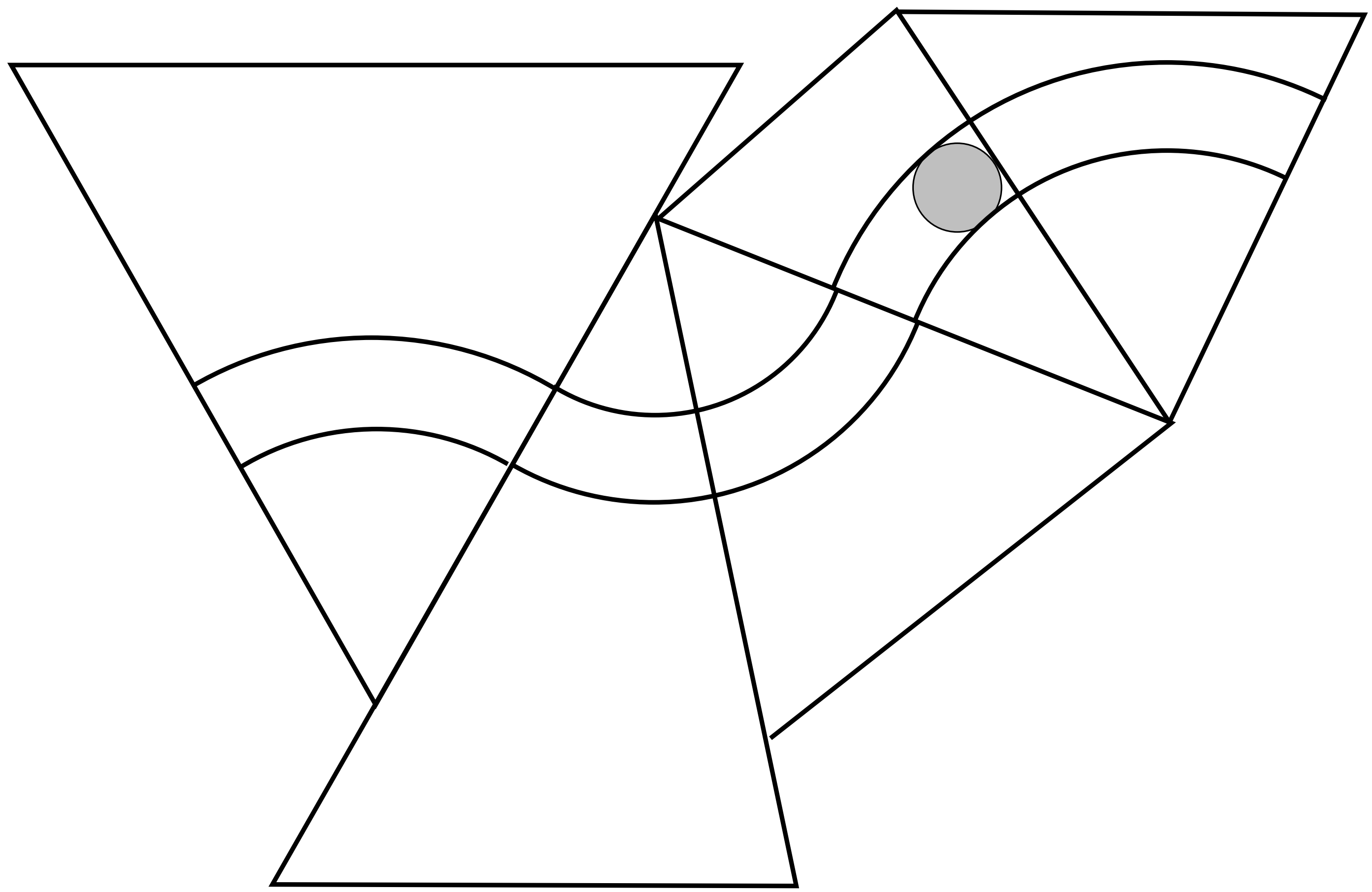


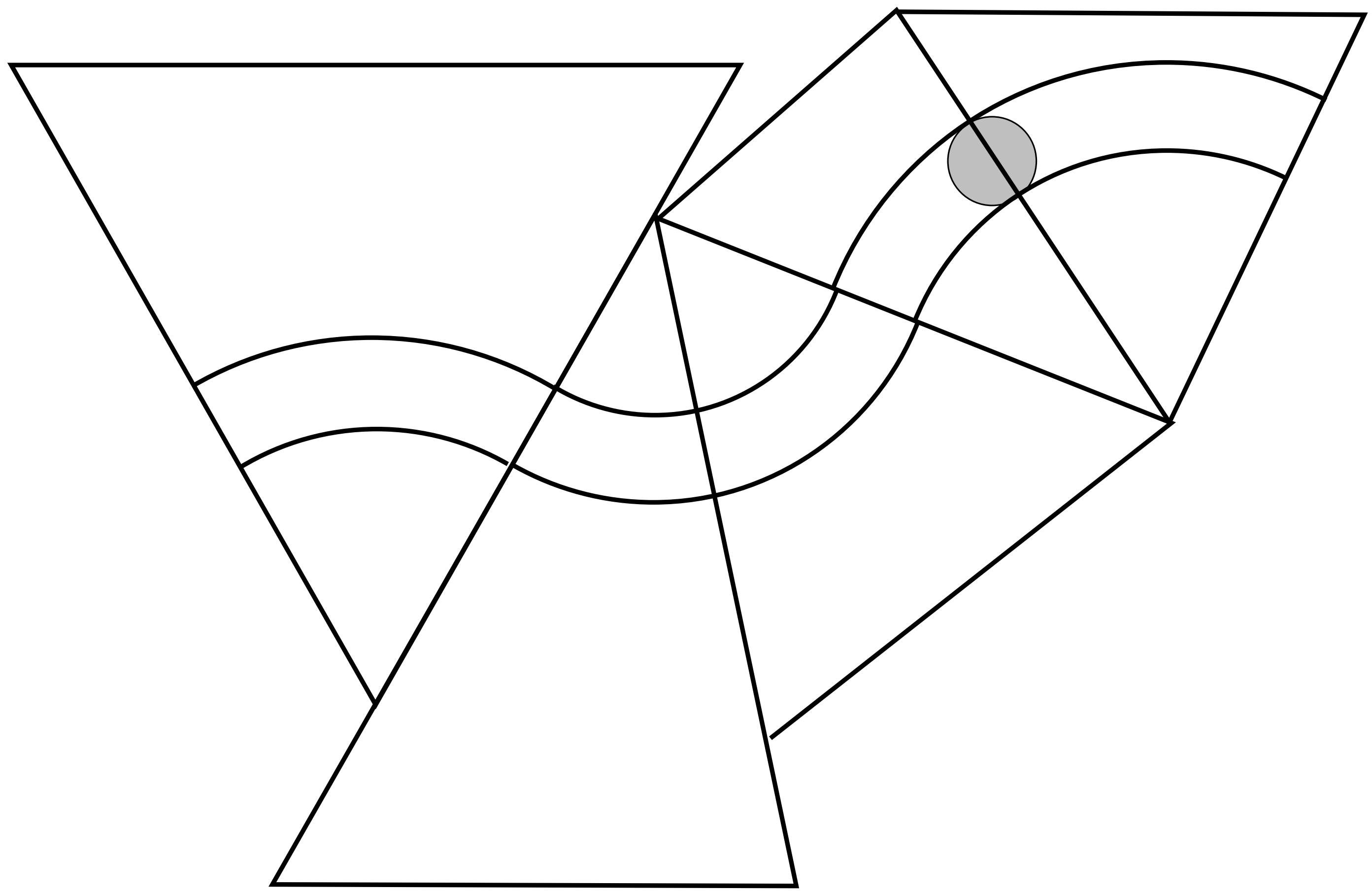


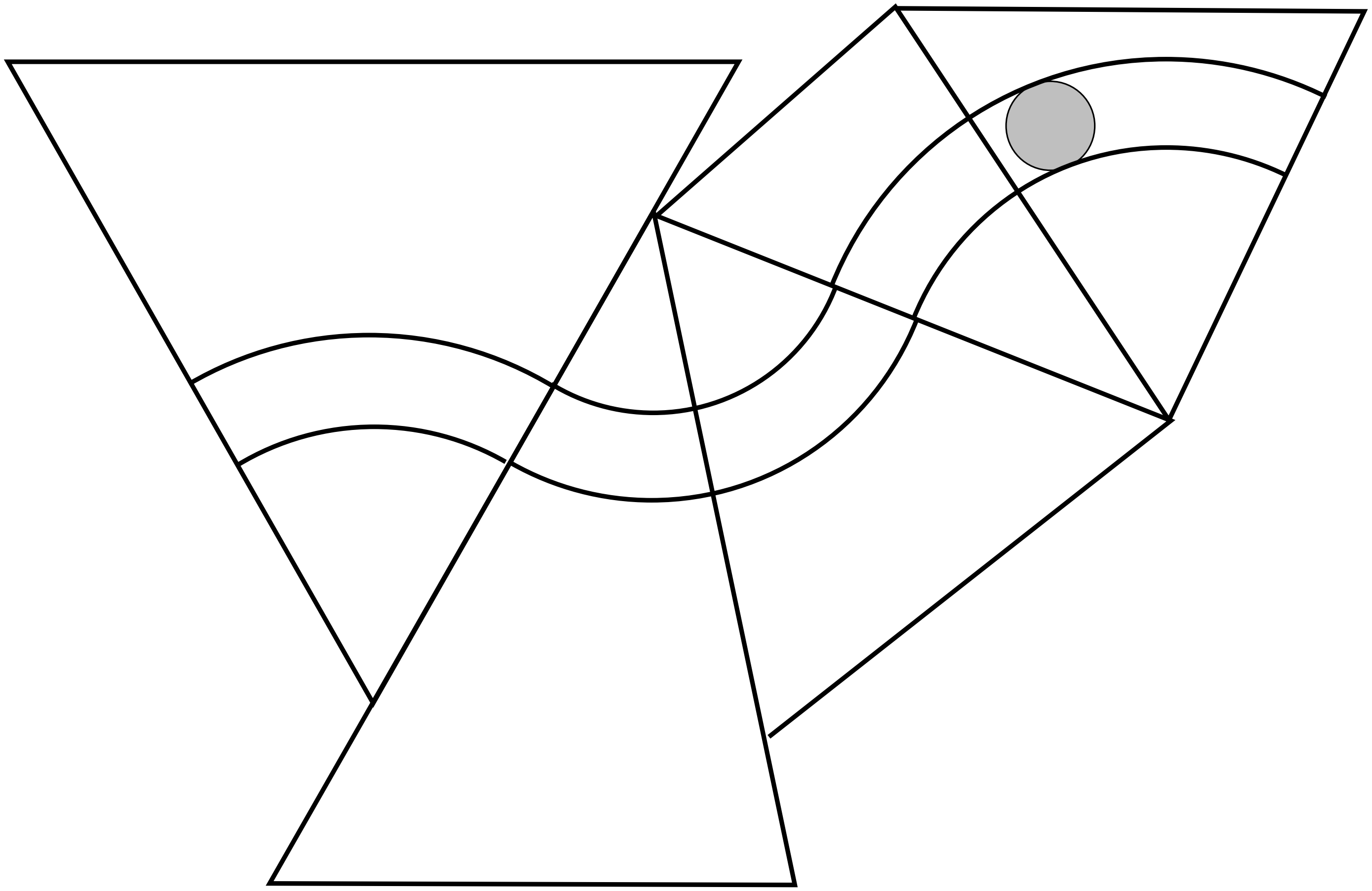


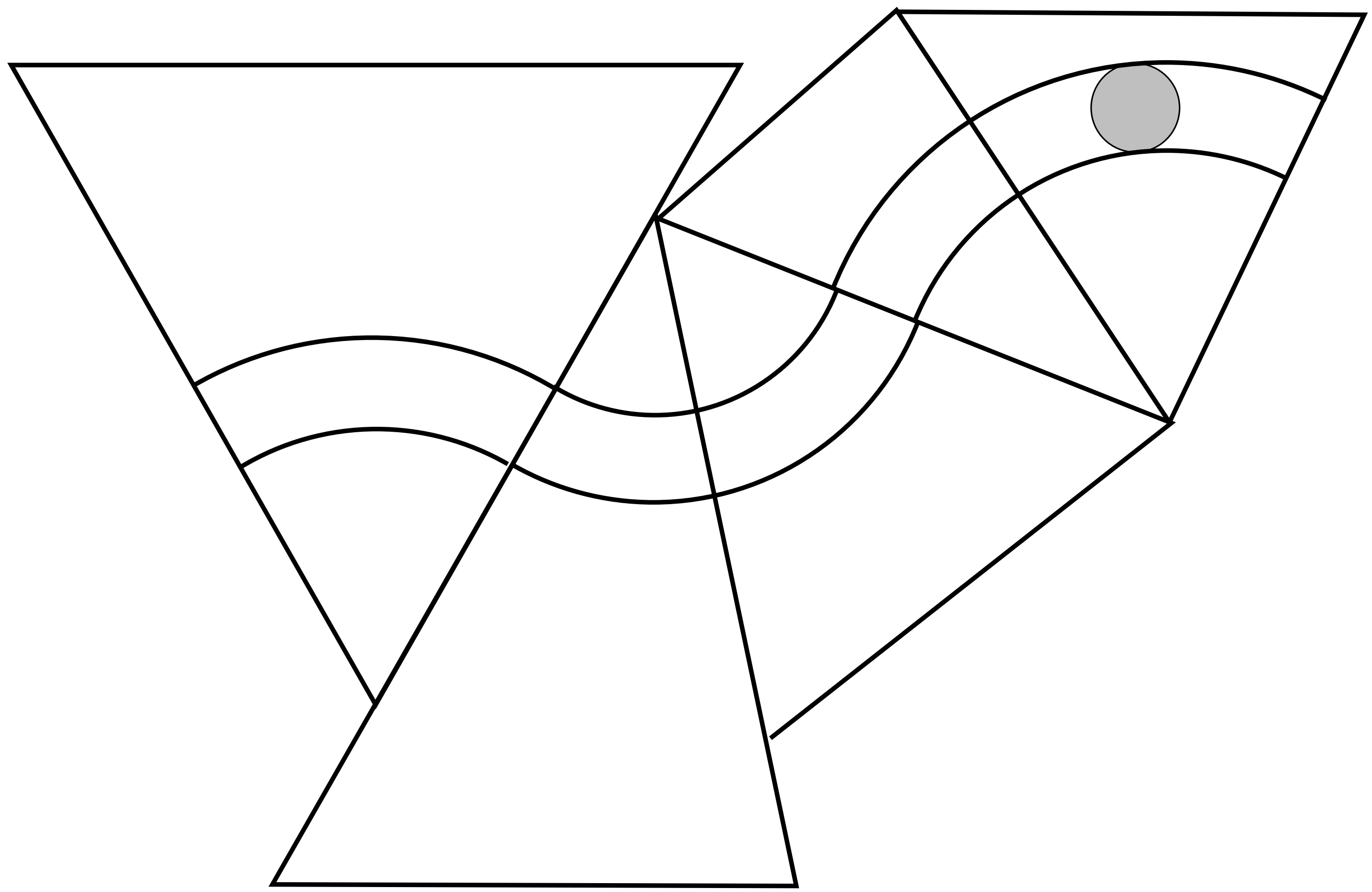


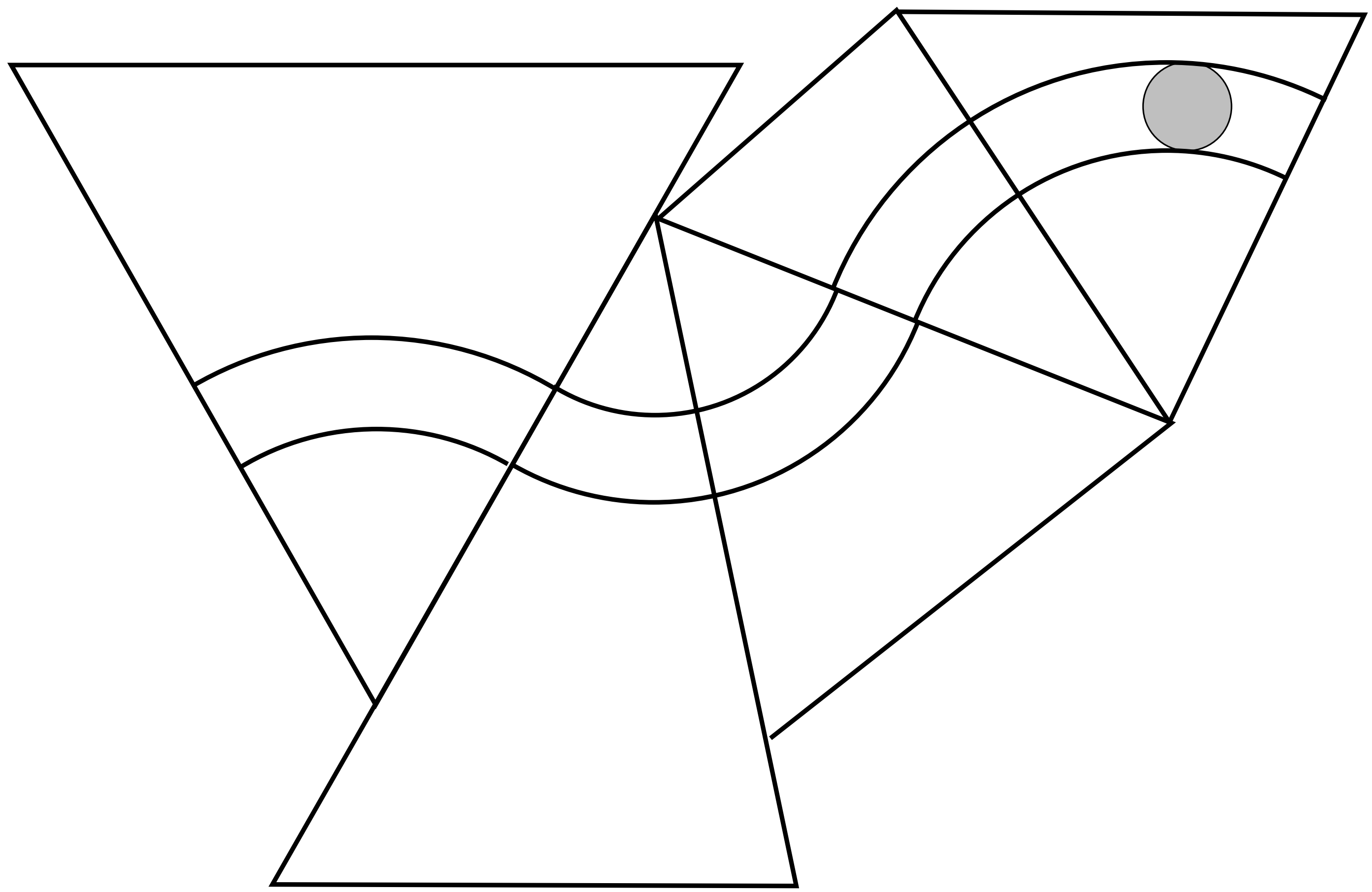


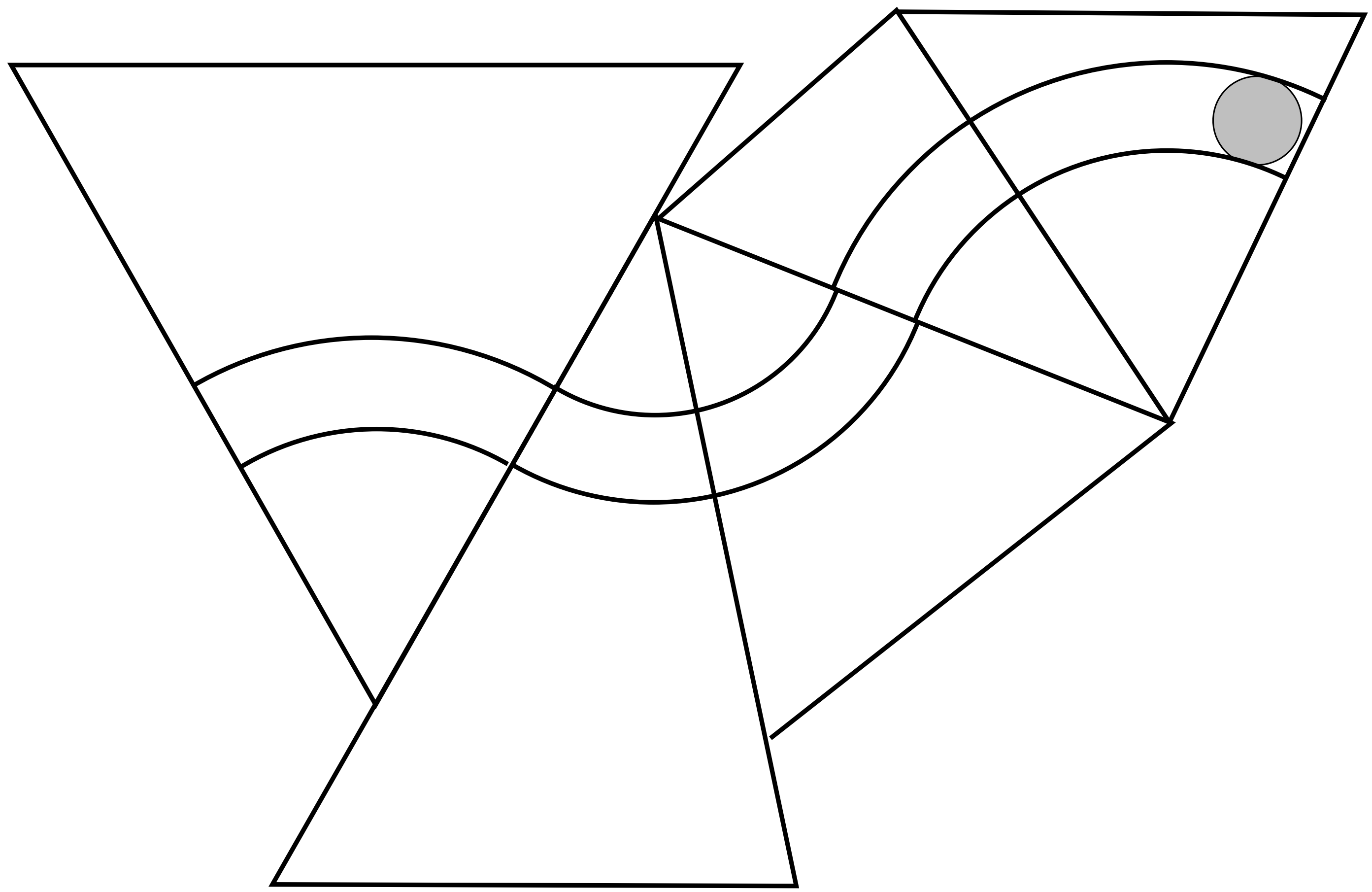


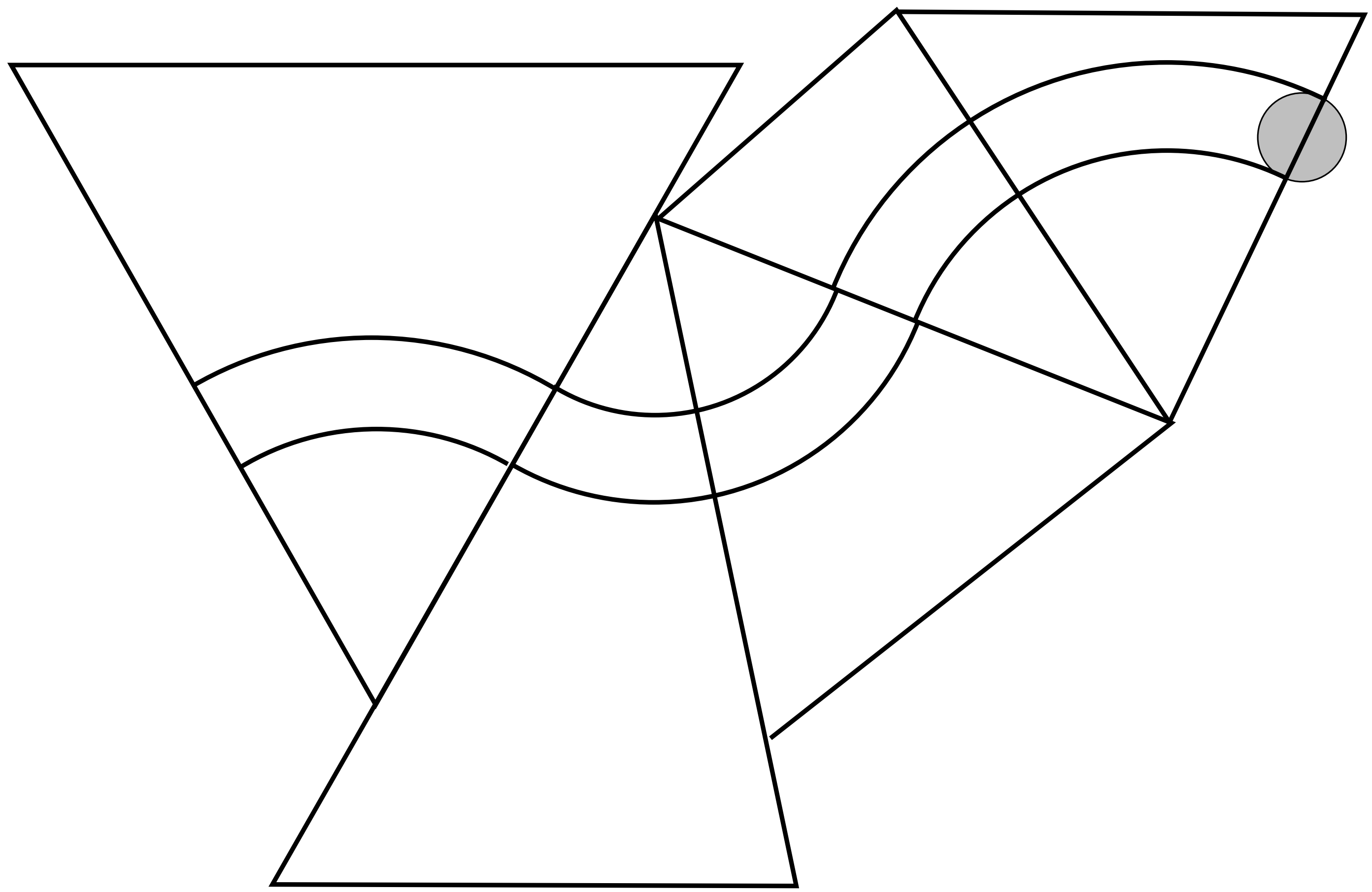


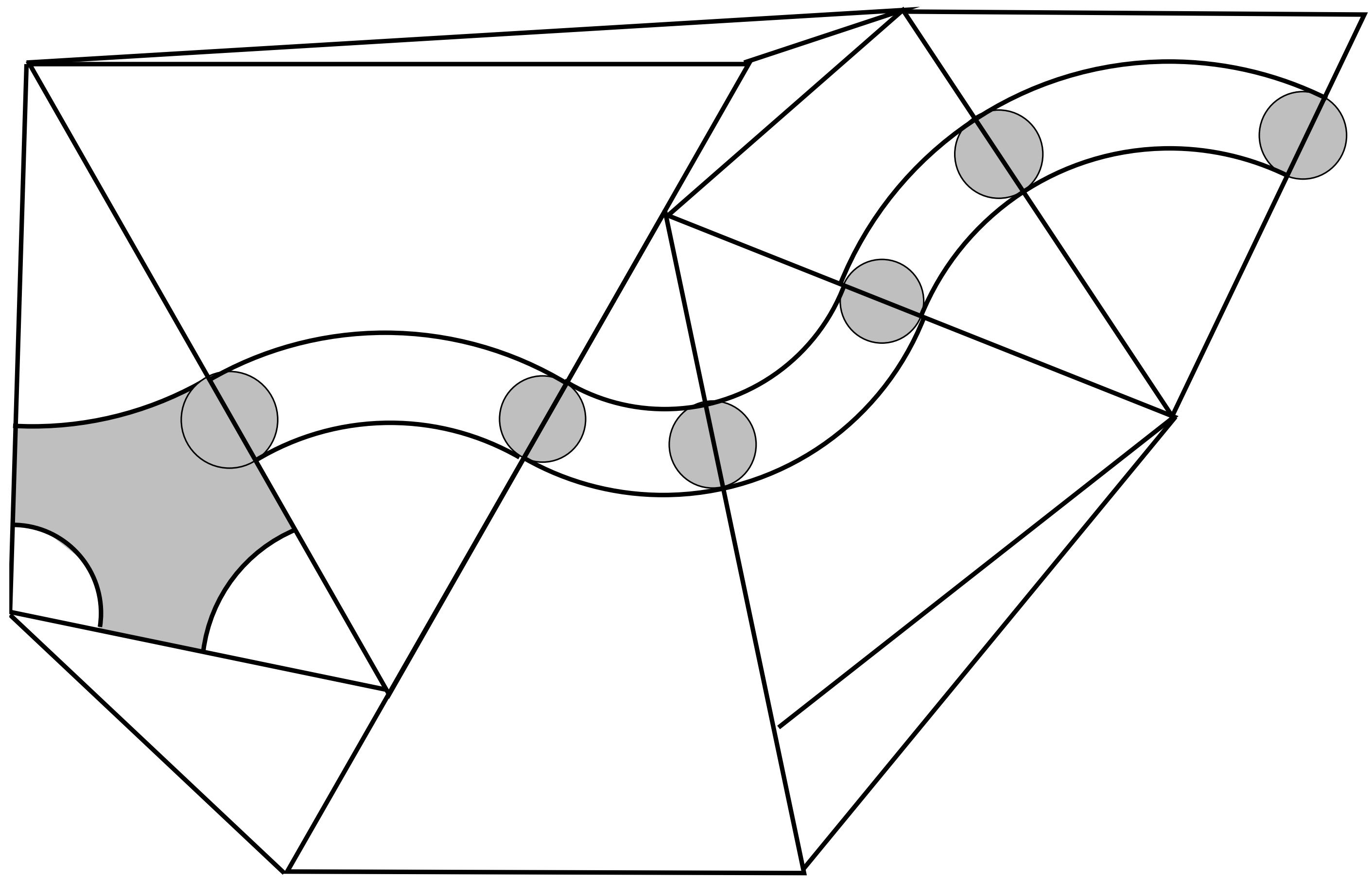






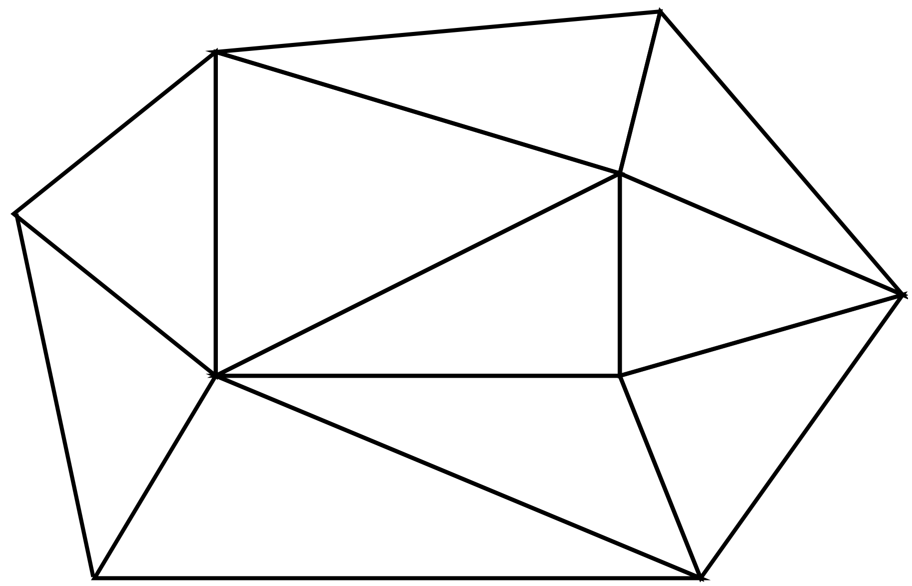




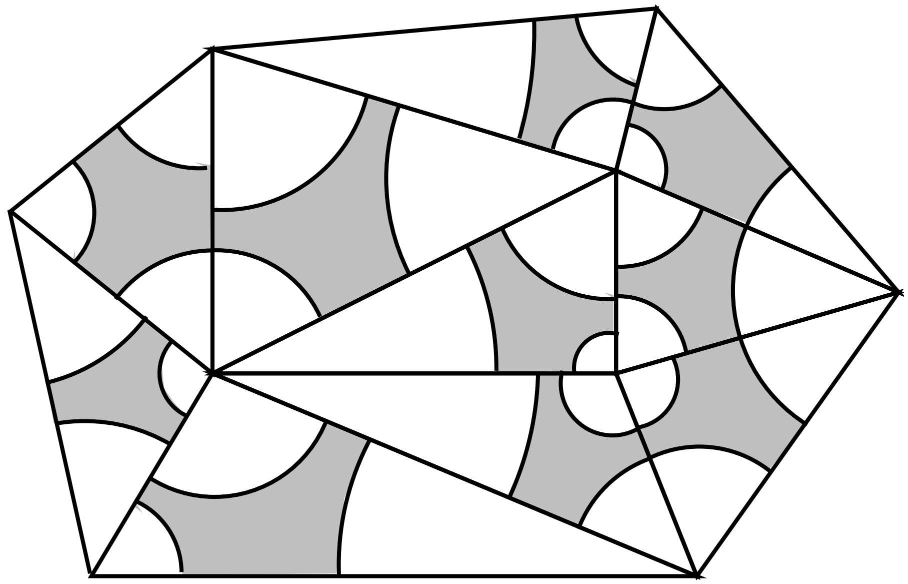


Intersection of tube and triangle edge is a Gabriel edge.

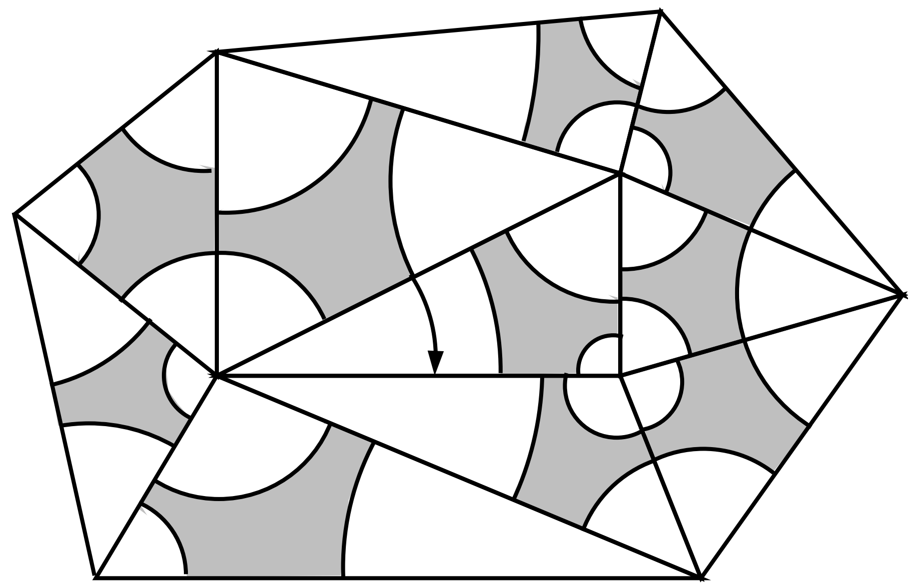
Disk lies inside tube or thick part or outside convex hull.



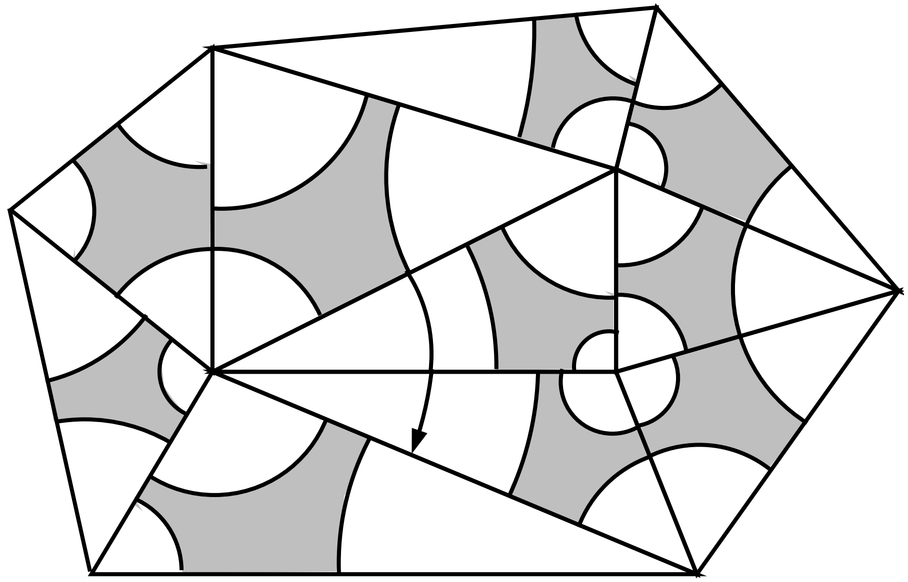
- Start with any triangulation.



- Start with any triangulation.
- Make thick/thin parts.



- Start with any triangulation.
- Make thick/thin parts.
- Propagate vertices until they leave thin parts.



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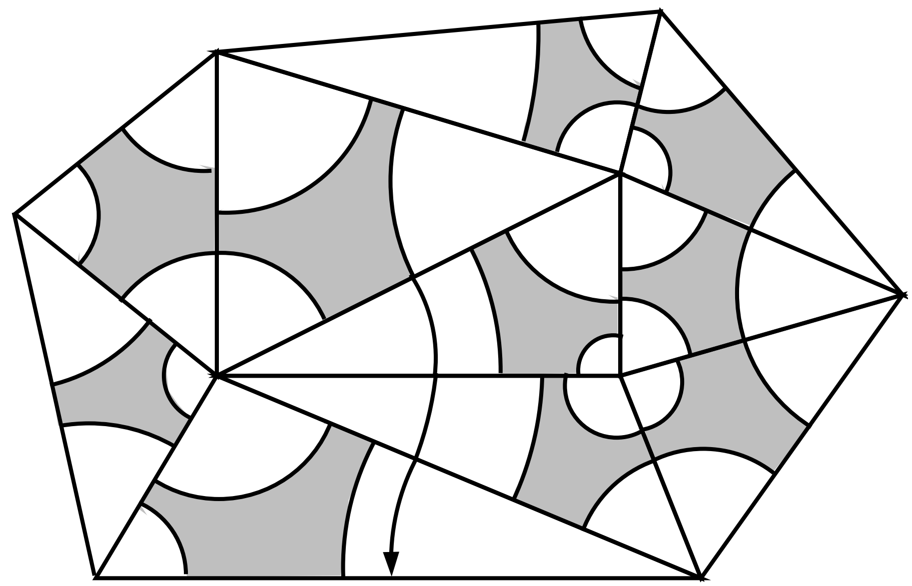

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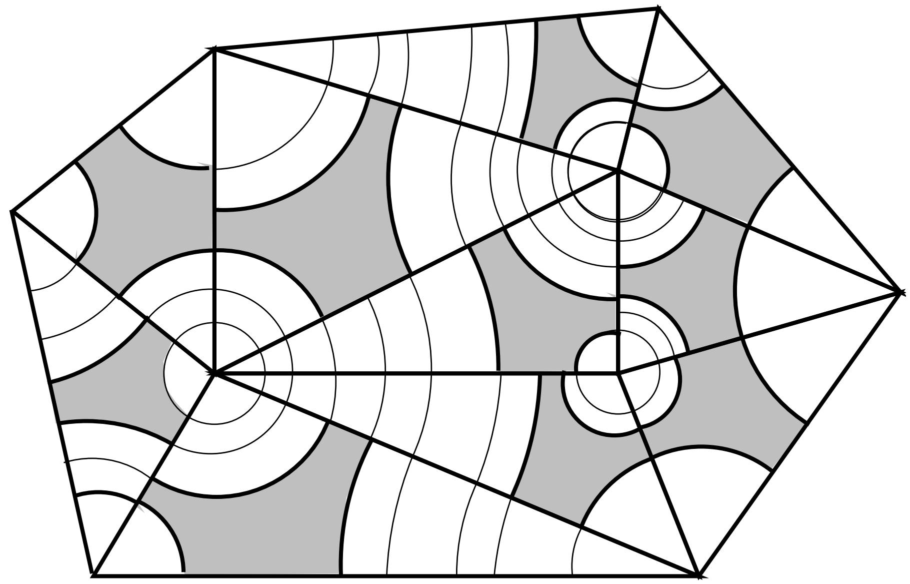
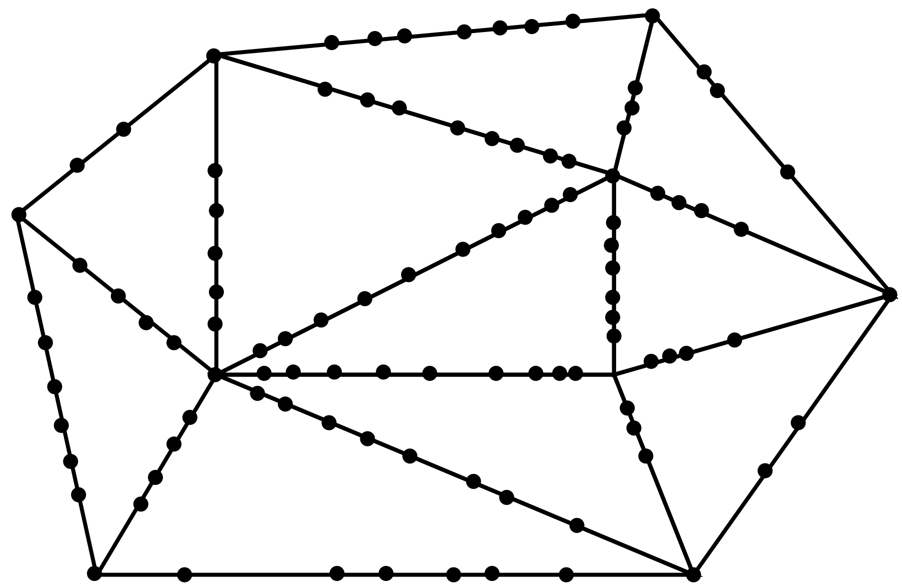


- Start with any triangulation.
- Make thick/thin parts.
- Propagate vertices until they leave thin parts.

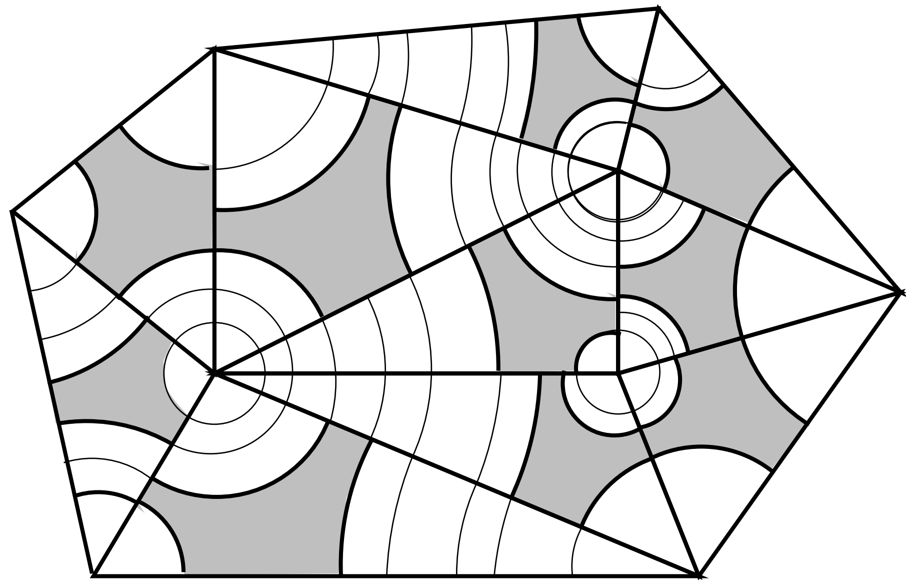
Gabriel graph of intersection points covers triangulation.



- Start with any triangulation.
- Make thick/thin parts.
- Propagate vertices until they leave thin parts.

Earlier argument gives nonobtuse triangulation.

But no bound on number of triangles.

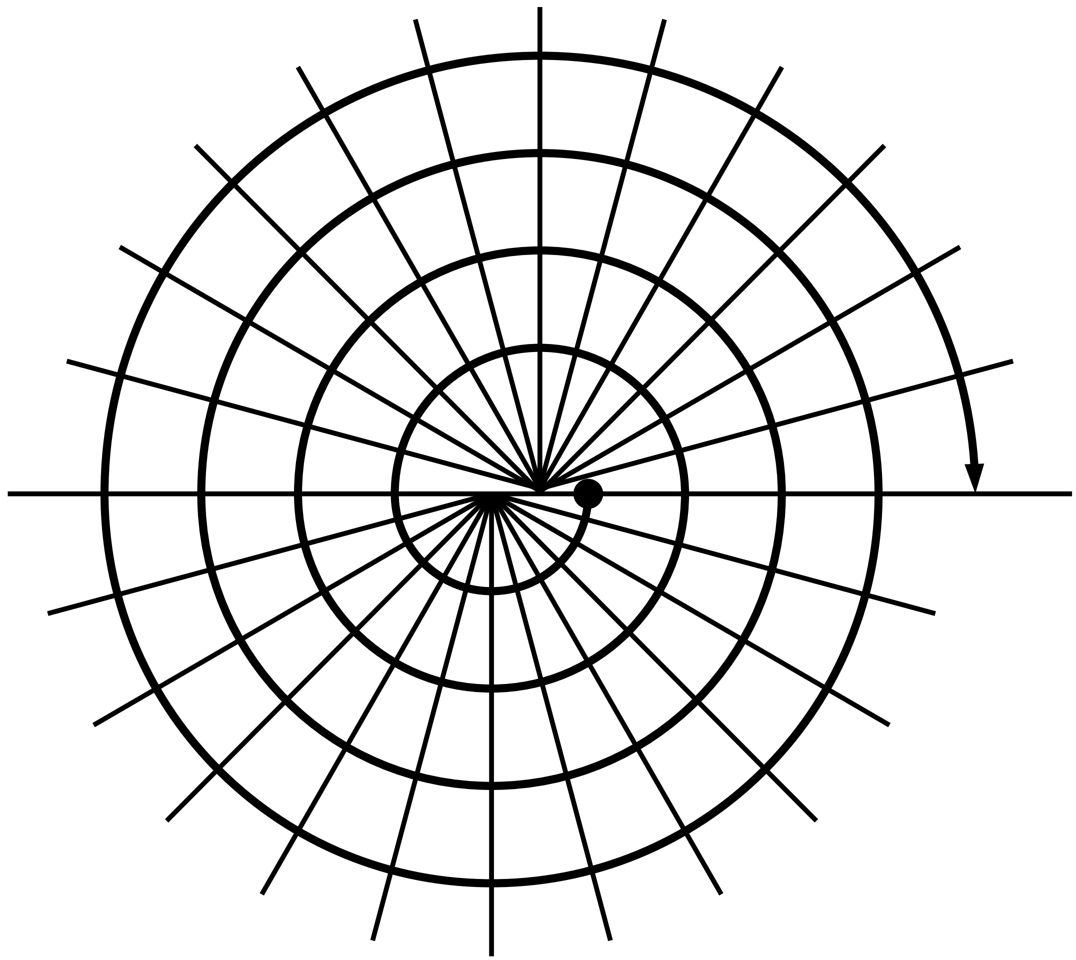


If paths never revisit a triangle, $O(n^2)$ points created.

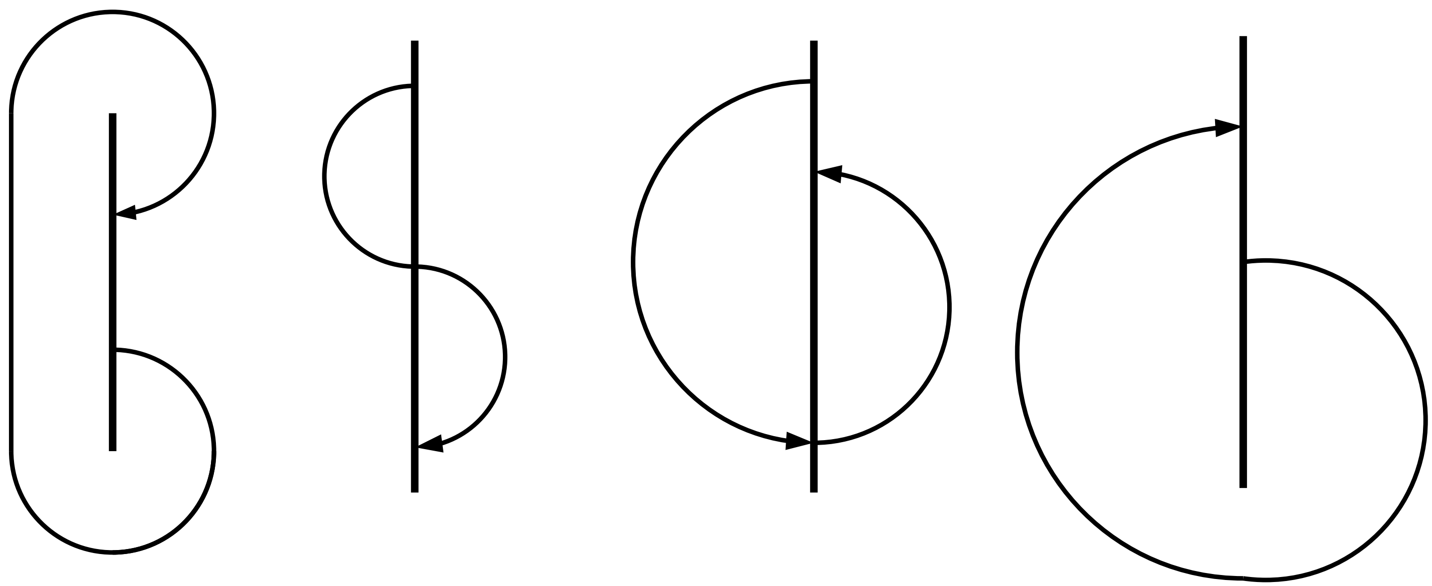
Corollary: Any triangulation of an n-gon has a refinement into $O(n^2)$ right triangles.

Improves $O(n^4)$ bound by Bern and Eppstein (1992).

In general, path can hit same thin part many times.

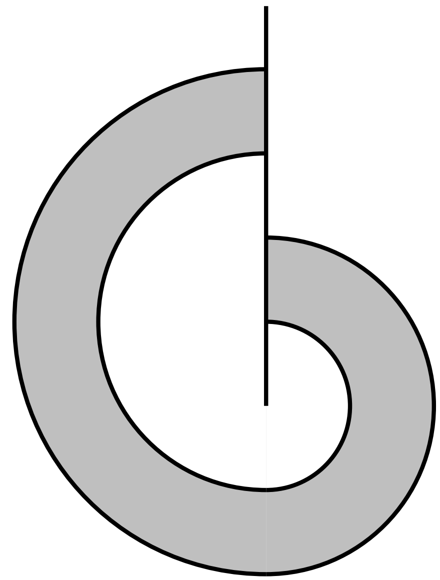


If a path returns to same edge 3 times it has a sub-path that looks like one of these:



C-curve S-curve G-curves

Return region consists of paths "parallel" to one of these.



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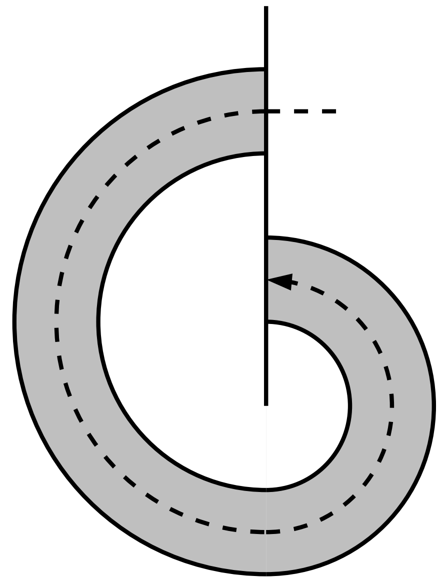
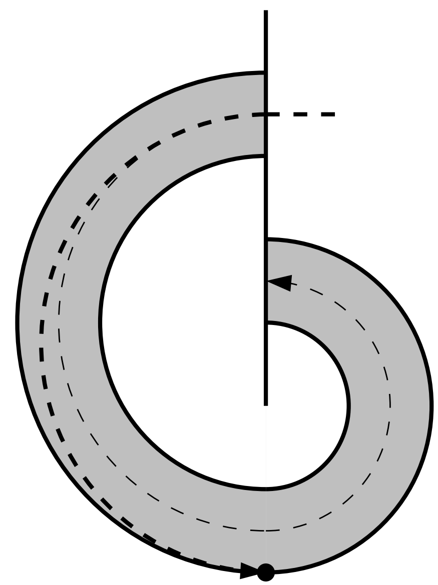


There are $O(n)$ return regions and every propagation path enters one after crossing at most $O(n)$ thin parts.

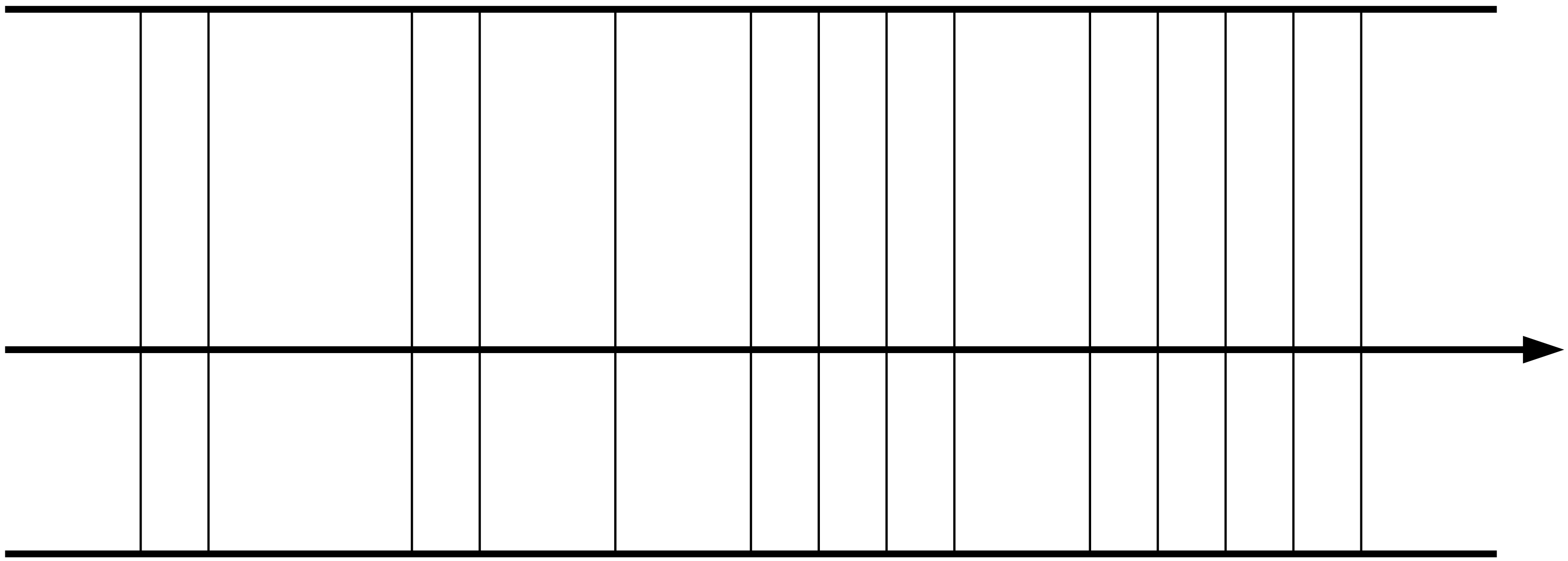
Return region consists of paths "parallel" to one of these.



There are $O(n)$ return regions and every propagation path enters one after crossing at most $O(n)$ thin parts.

We want to bend paths to terminate before they exit.

For simplicity, “straighten” region to rectangle.



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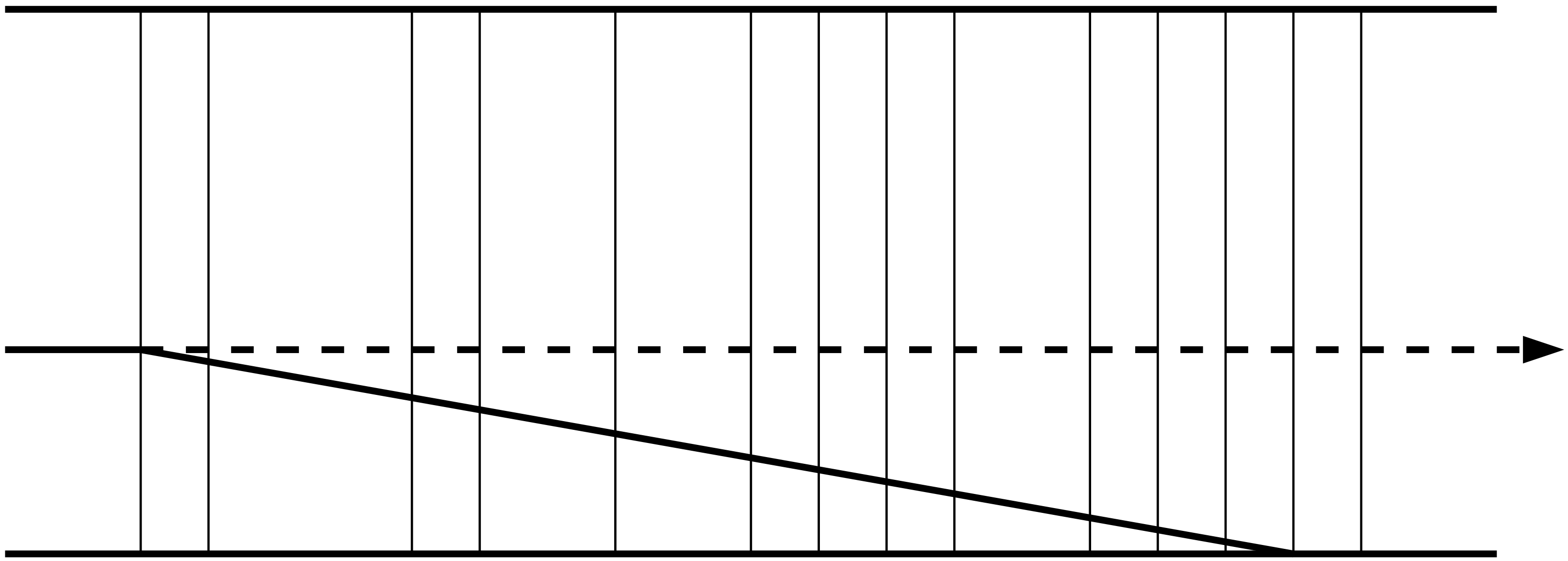
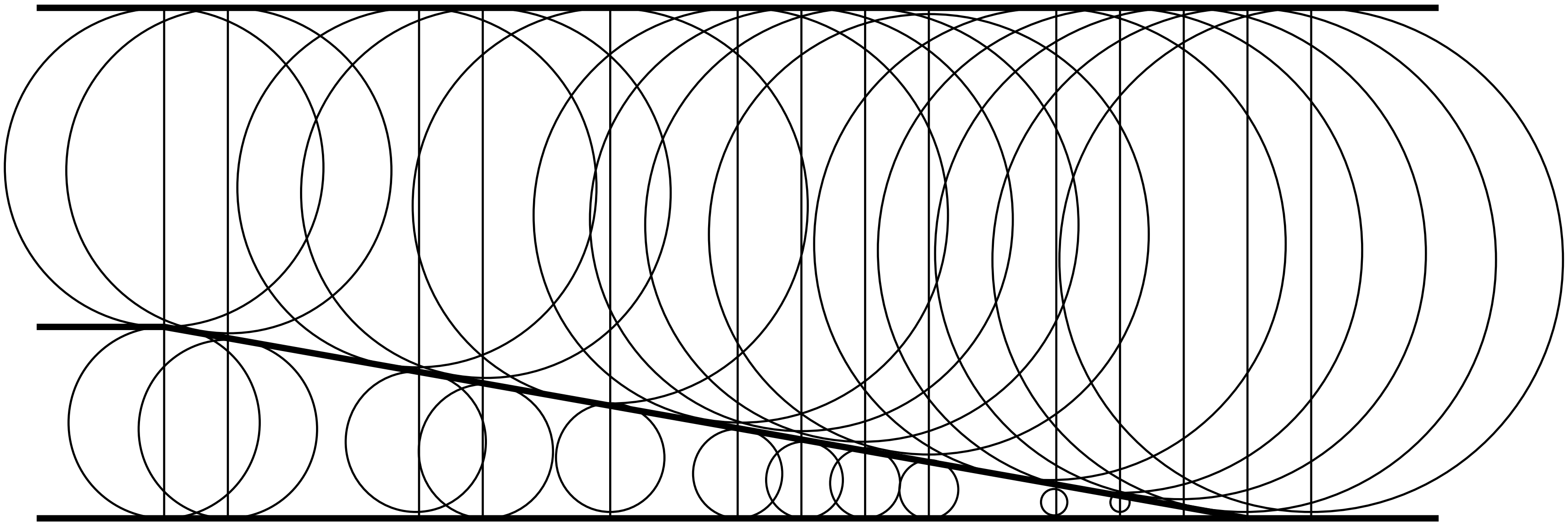


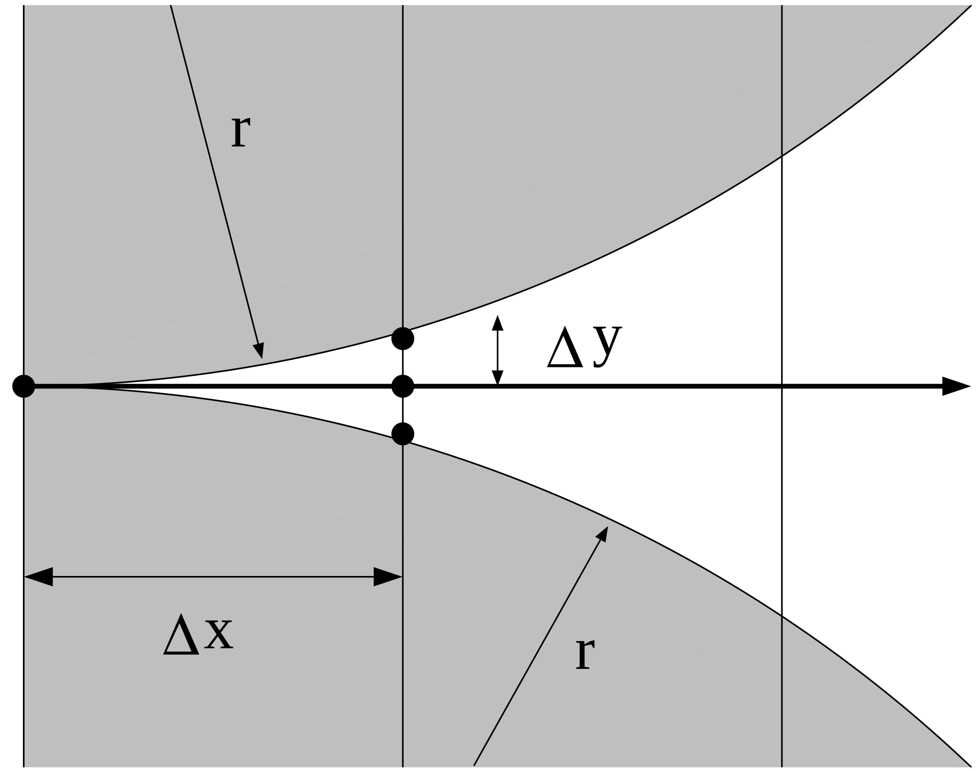
We want to bend path to hit side of tube. If it hits existing vertex, then path ends.

For simplicity, "straighten" region to rectangle.

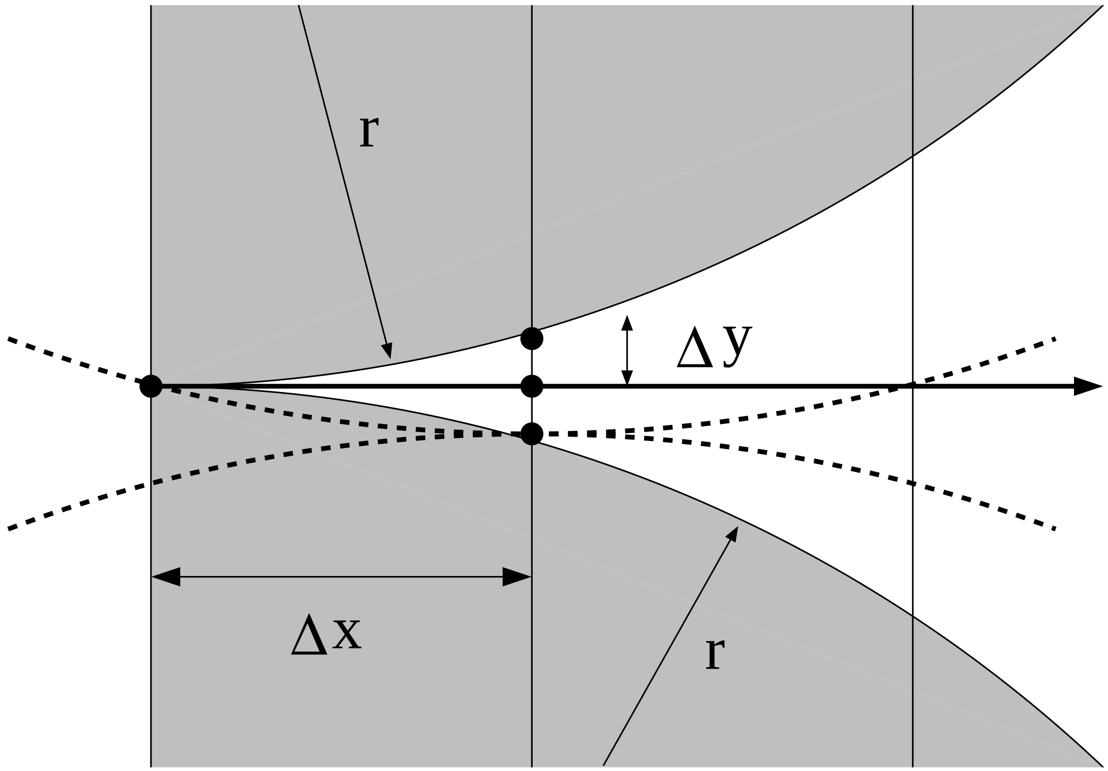


Bending must satisfy Gabriel condition. Diameter disks must not contain any intersection points.

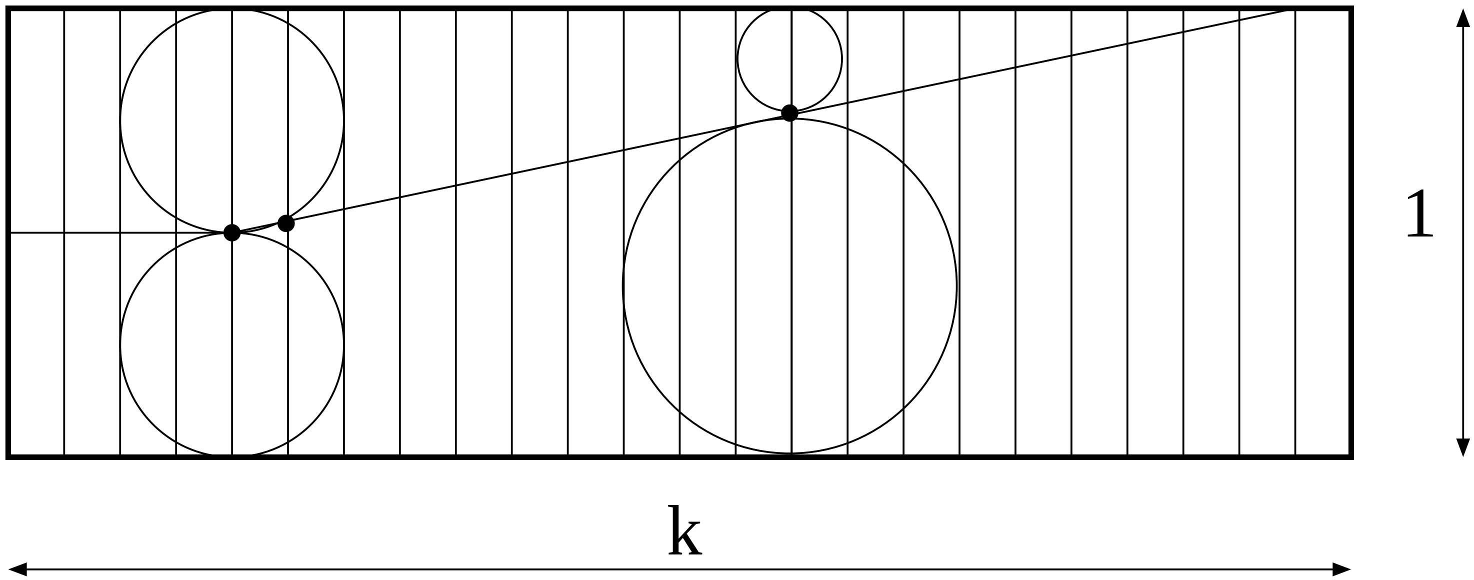
How far can we bend?



Answer: $\Delta y \approx (\Delta x/r)^2 r = (\Delta x)^2/r.$



Answer: $\Delta y \approx (\Delta x/r)^2 r = (\Delta x)^2/r$.



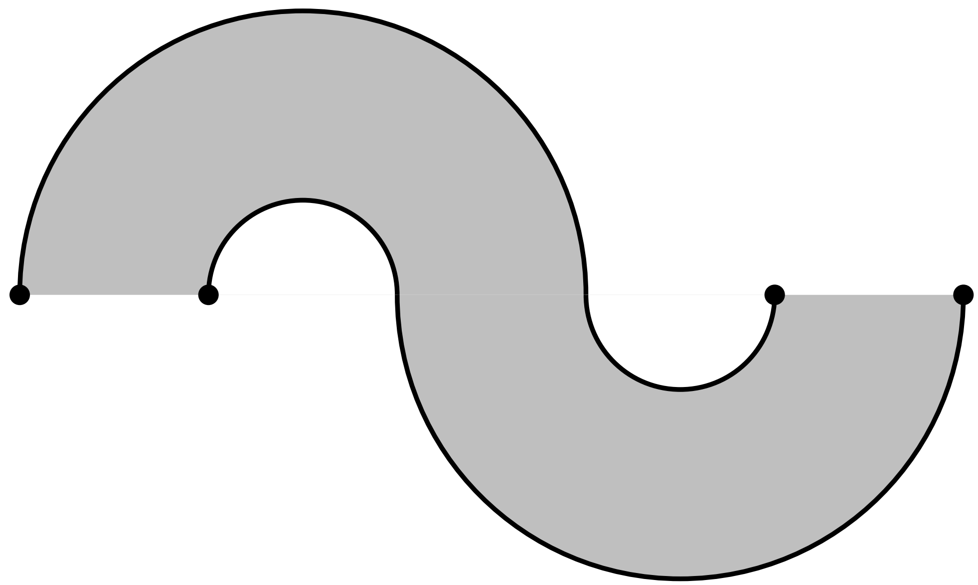
In $1 \times k$ tube crossing n (equally spaced) thin parts,

$$r \approx 1, \quad \Delta x \approx k/n, \quad \Delta y \approx k^2/n^2$$

Need $1 \approx \sum \Delta y = n\Delta y = n(k/n)^2 = k^2/n$.

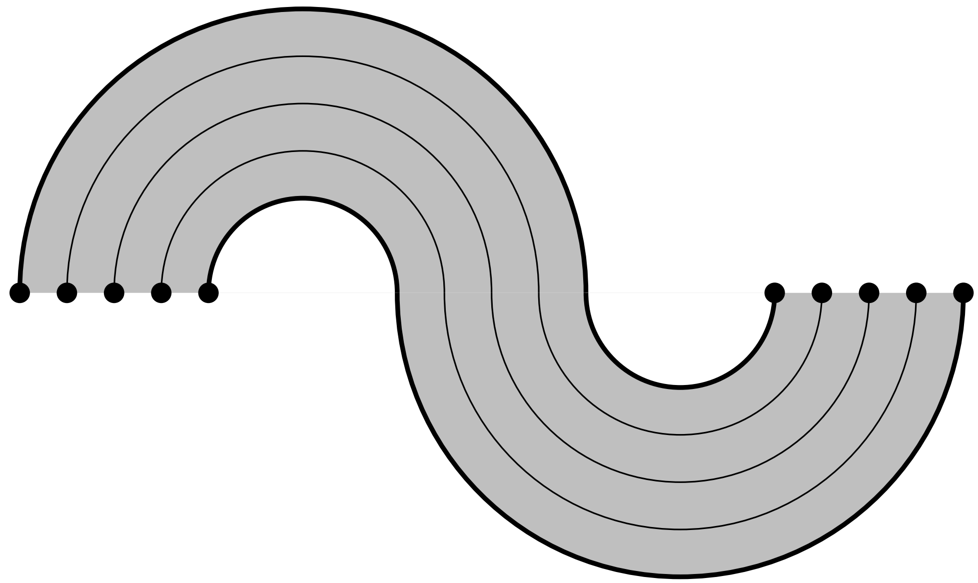
Can terminate path if $k \gg \sqrt{n}$.

Easy case: return region length > width.



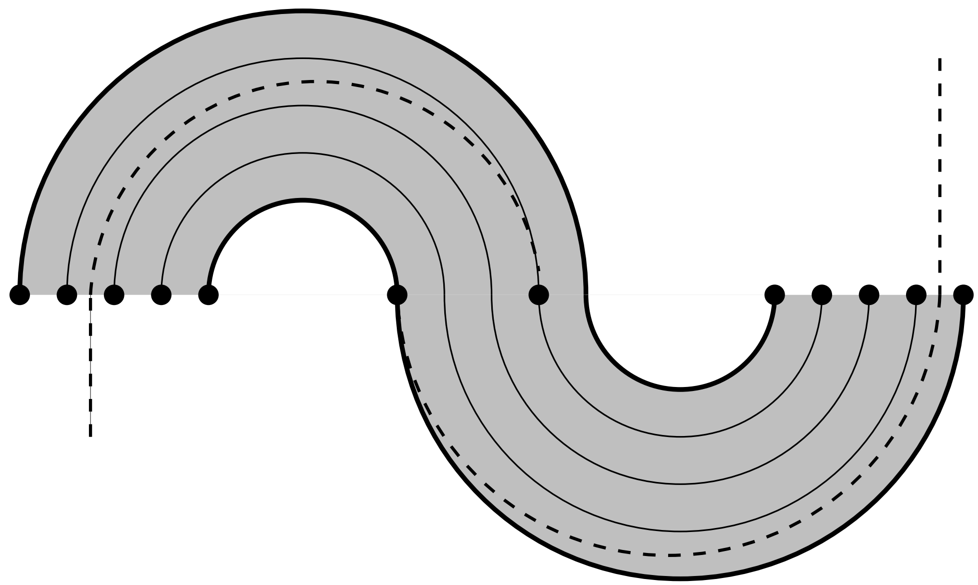
- Show there are $O(n)$ return regions.

Easy case: return region length > width.



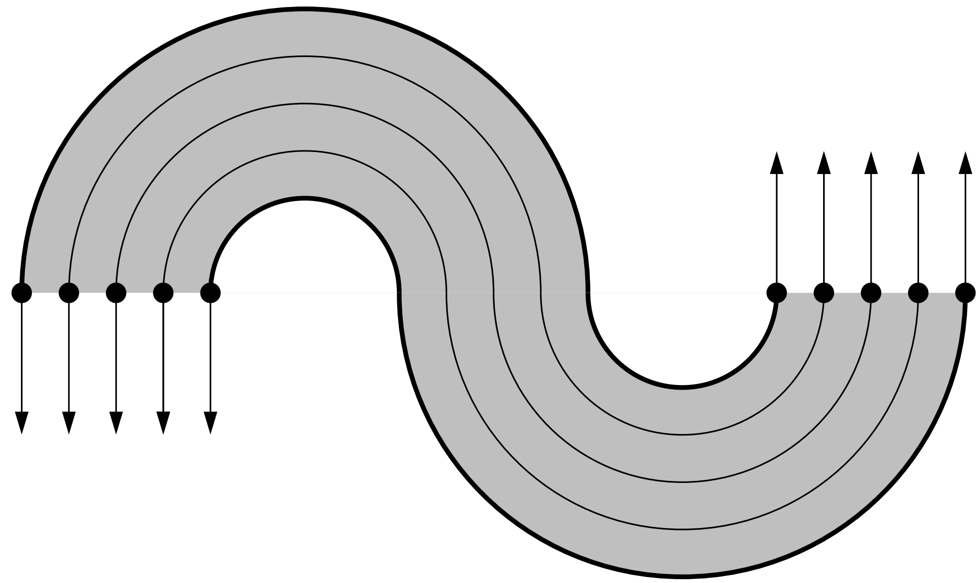
- Show there are $O(n)$ return regions.
- Divide each region into $O(\sqrt{n})$ long parallel tubes.

Easy case: return region length > width.



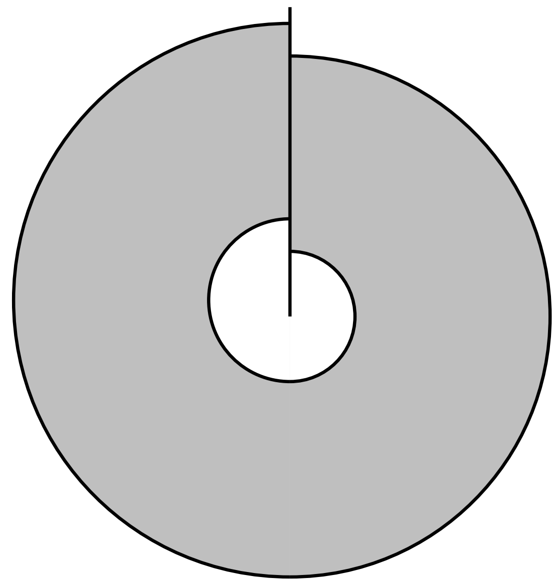
- Show there are $O(n)$ return regions.
- Divide each region into $O(\sqrt{n})$ long parallel tubes.
- Entering paths can be bent and terminated.
  Total vertices created $= O(n^2)$, but ...

Easy case: return region length > width.

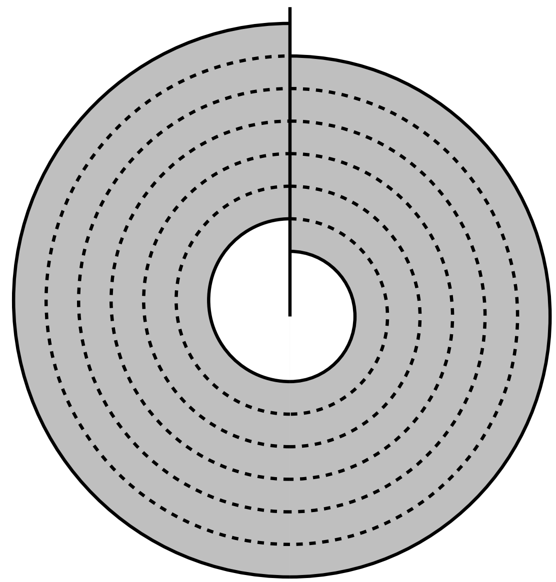


- Show there are $O(n)$ return regions.
- Divide each region into $O(\sqrt{n})$ long parallel tubes.
- Entering paths can be bent and terminated.
  Total vertices created $= O(n^2)$, but ...
- Each region has $O(\sqrt{n})$ new vertices to propagate.
  Vertices created is $O(n \cdot \sqrt{n} \cdot n) = O(n^{2.5})$.

Hard case is spirals: (length $\ll$ width)



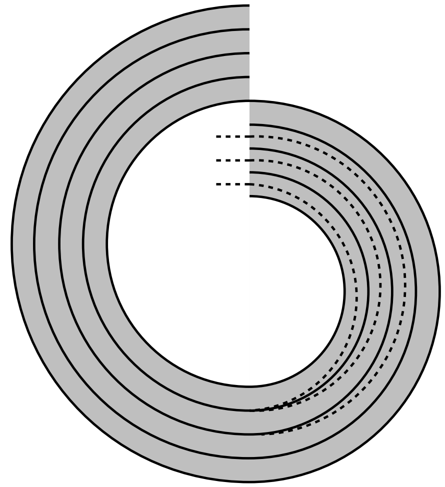
Hard case is spirals: (length $\ll$ width)



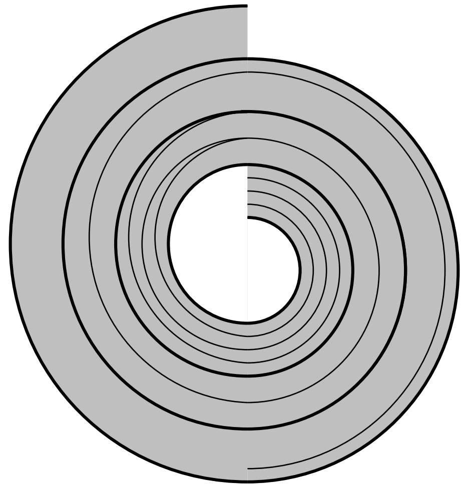
Curves may spiral arbitrarily often.

No curve can be allowed to pass all the way through the spiral. We stop them in a multi-stage construction.

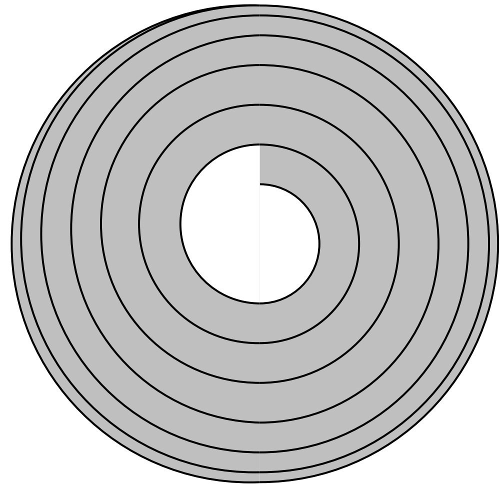
Normalize so "entrance" is unit width.



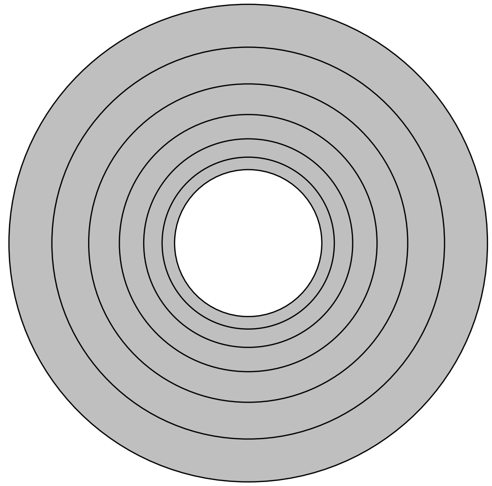
- Start with $\sqrt{n}$ parallel tubes at entrance of spiral. Terminate entering paths (1 spiral).



- Start with $\sqrt{n}$ parallel tubes at entrance of spiral. Terminate entering paths (1 spiral).
- Merge $\sqrt{n}$ tubes to single tube ($n^{1/3}$ spirals). (Spirals get longer as we move out.)



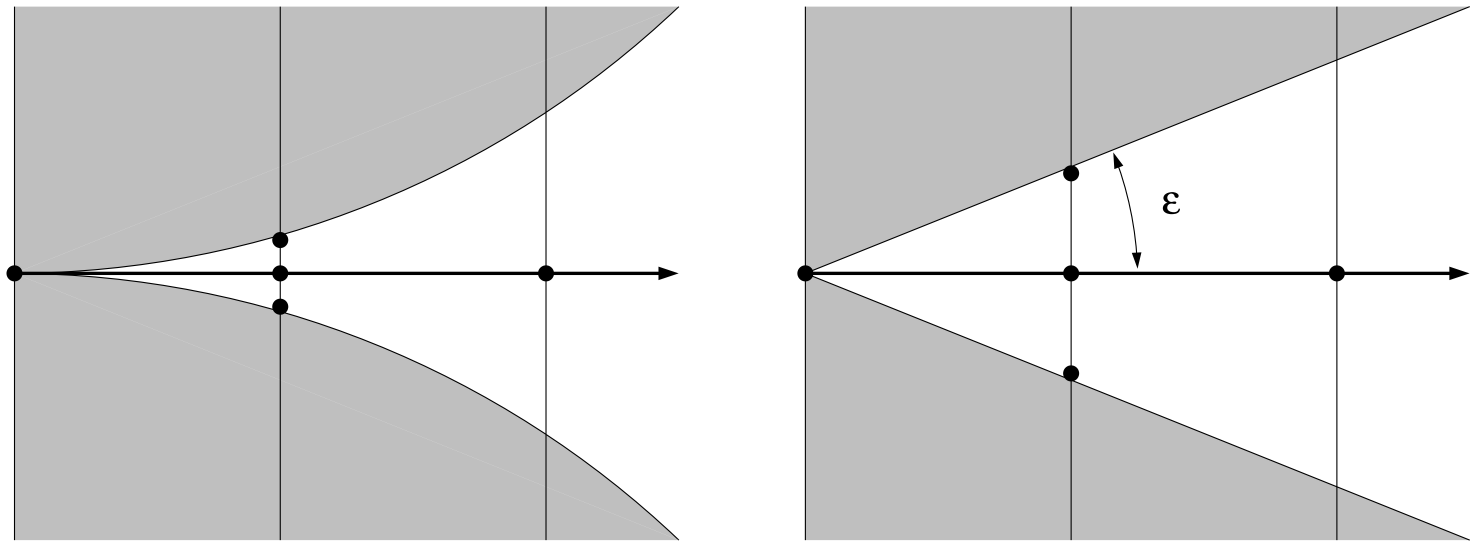
- Start with $\sqrt{n}$ parallel tubes at entrance of spiral. Terminate entering paths (1 spiral).
- Merge $\sqrt{n}$ tubes to single tube ($n^{1/3}$ spirals). (Spirals get longer as we move out.)
- Make tube edge self-intersect ($n^{1/2}$ spirals)



- Start with $\sqrt{n}$ parallel tubes at entrance of spiral. Terminate entering paths (1 spiral).
- Merge $\sqrt{n}$ tubes to single tube ($n^{1/3}$ spirals). (Spirals get longer as we move out.)
- Make tube edge self-intersect ($n^{1/2}$ spirals)
- Loops with increasing gaps ($n^{1/2}$ loops, n spirals)
- Beyond radius n spiral is empty.

Careful estimates needed to get $O(n^{2.5})$ vertices.

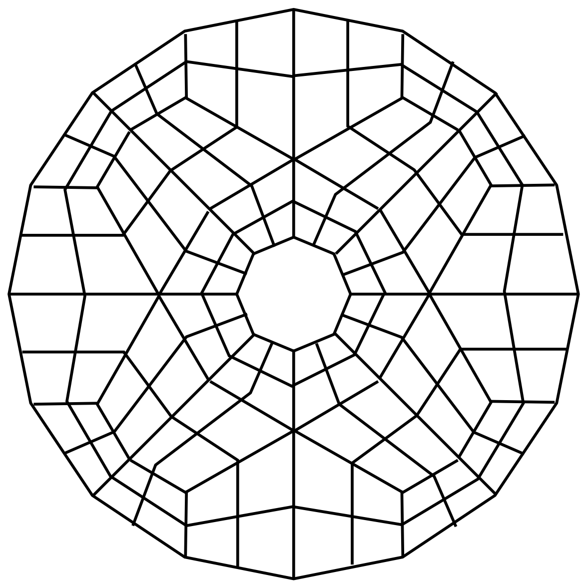
Almost Nonobtuse Triangulation: Replace cusps by cones of angle ϵ. Same construction in thick parts.

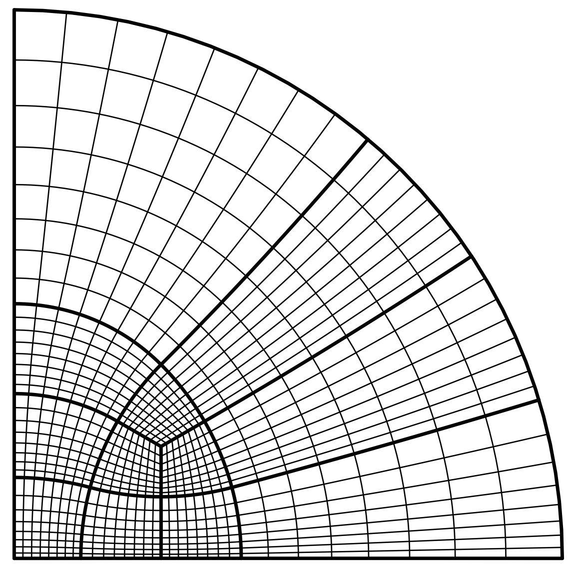


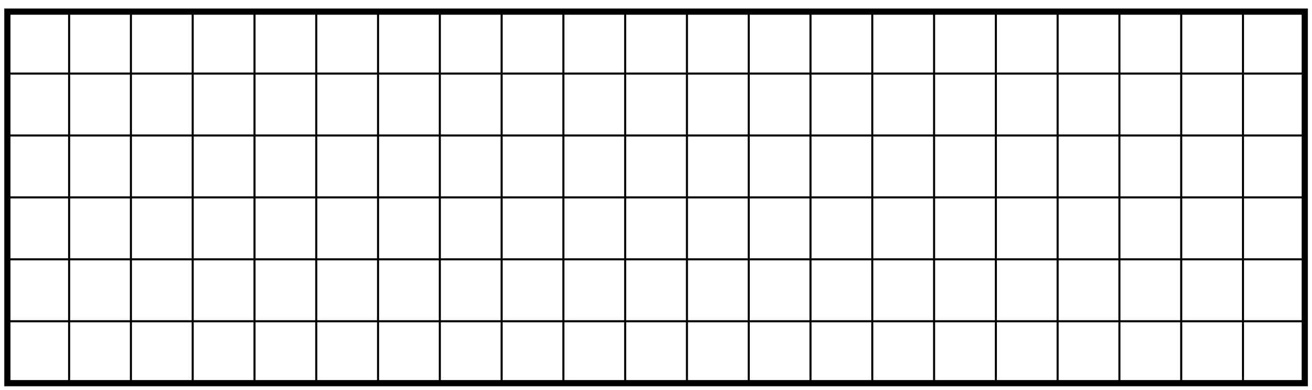
Paths can be terminated inside a $1 \times \frac{1}{\epsilon}$ tube.

Thm: Uses angles $\leq 90^\circ + \epsilon$ and $O(\frac{n^2}{\epsilon^2})$ triangles.

Quadrilateral meshes:







Some results

- Every simple n-gon has $O(n)$ quad mesh with angles $\leq 120°$. Bern and Eppstein, 2000. $O(n \log n)$ work.

- They showed any quad mesh of regular hexagon has at least one angle $\geq 120°$.

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Theorem: Every PSLG has a $O(n^2)$ quad mesh with all angles $\leq 120°$ and all new angles $\geq 60°$.

Angles bounds and complexity are sharp.

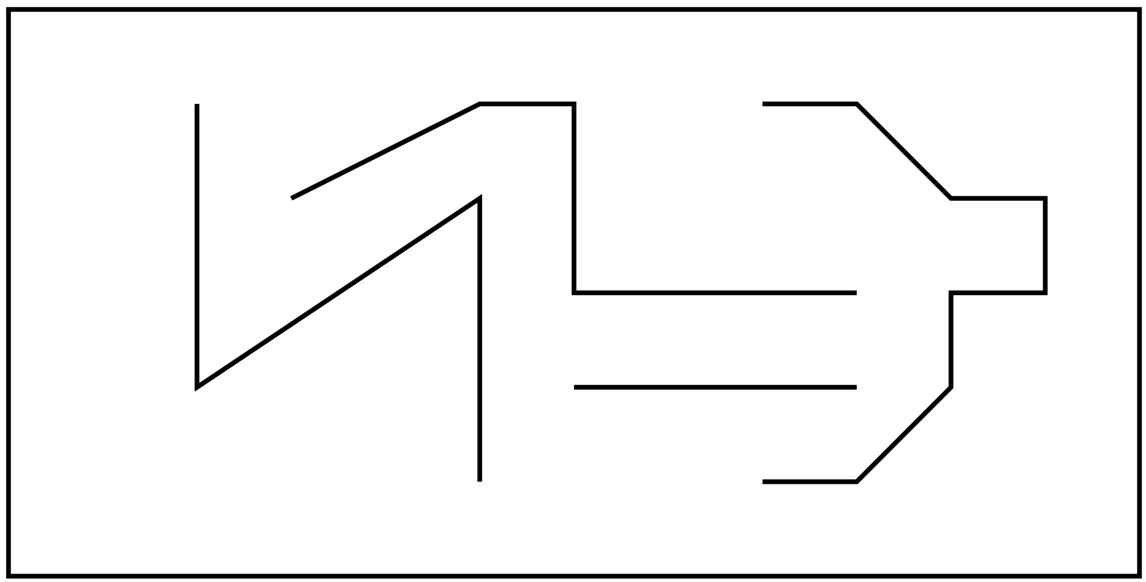
At most $O(\frac{n}{\epsilon})$ angles outside $[90° - \epsilon, 90° + \epsilon]$.

Mesh has partition into $O(n)$ "rectangular blocks".

Idea of proof:

- Connect Γ. Components now polygons, not triangles.
- Define thick/thin pieces.
- Mesh thin parts using propagation paths as before.
- Mesh thick parts using hyperbolic geometric in disk and conformal map to polygon.
- Connect meshes on thick and thin parts using "sinks".

Triangulating Γ may give small angles. Must connect without small angles.



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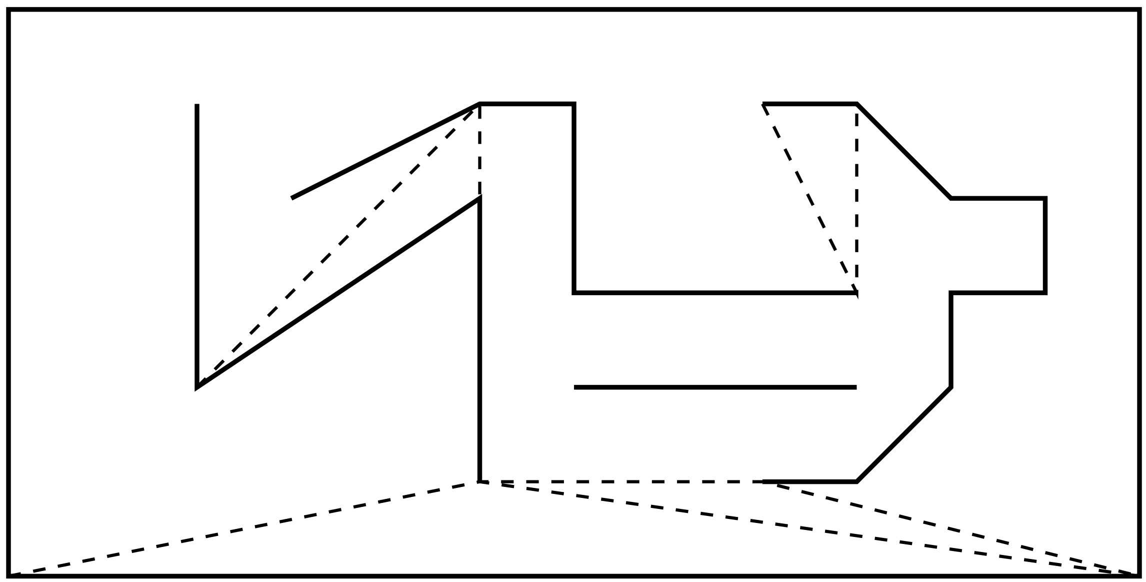

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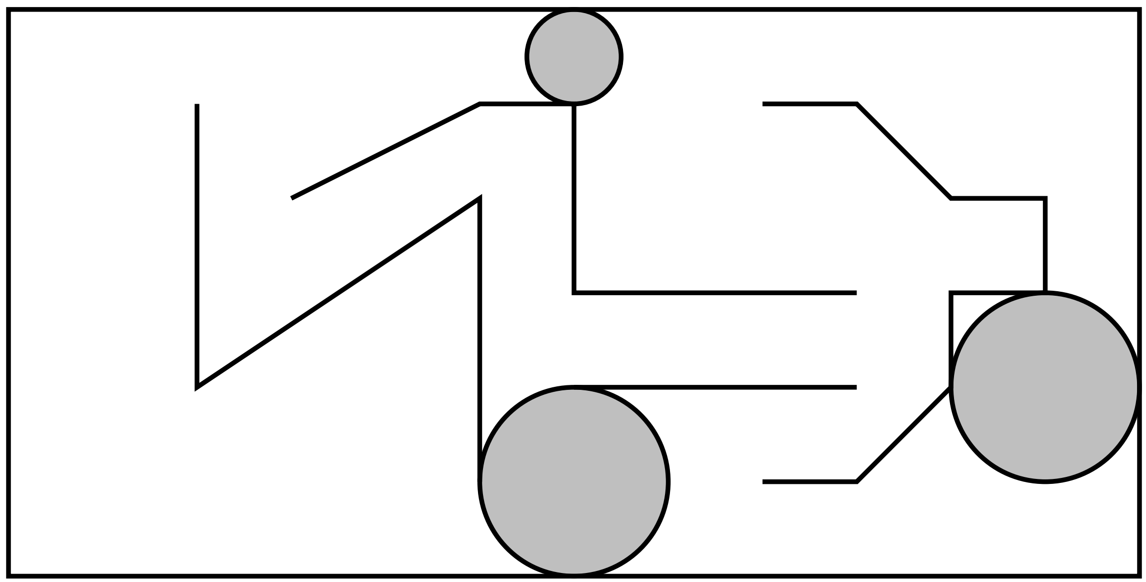
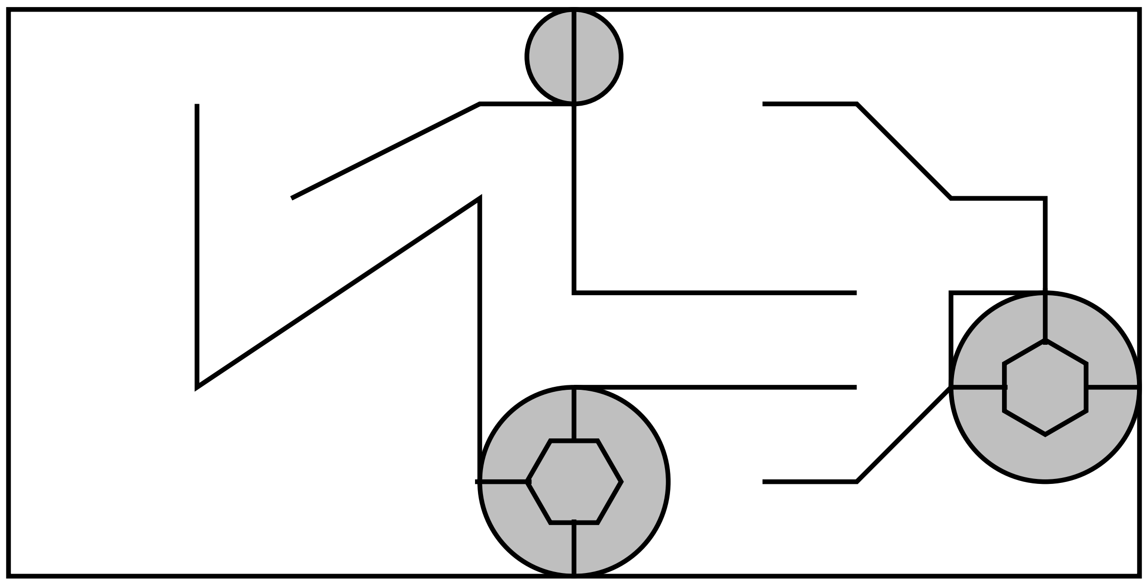


First connect Γ using disjoint disks.

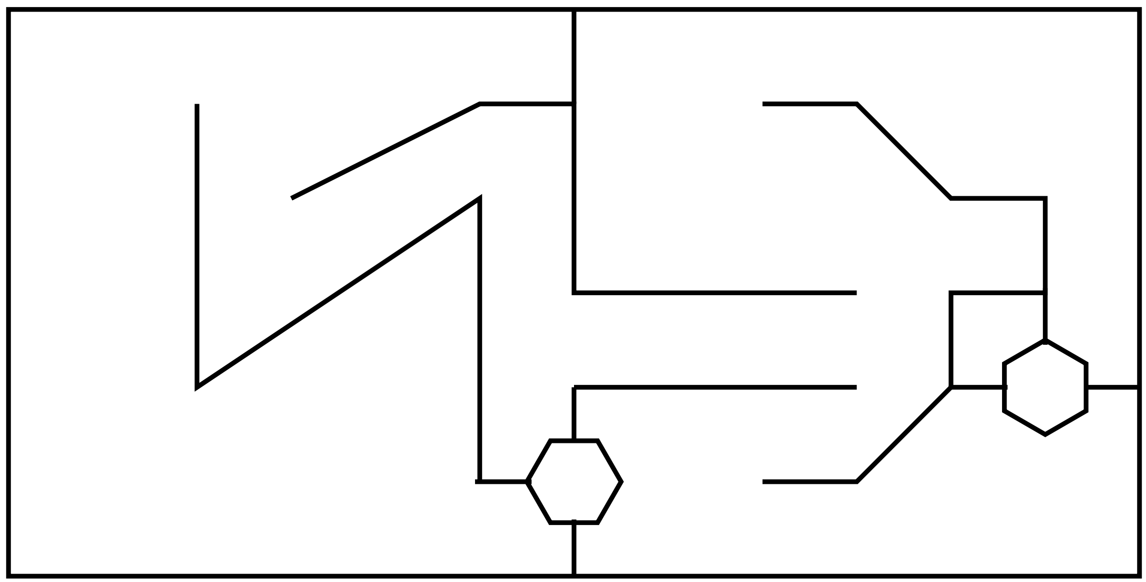
Triangulating Γ may give small angles. Must connect without small angles.



First connect Γ using disjoint disks.

Connect contact points using angles $\geq 60^\circ$

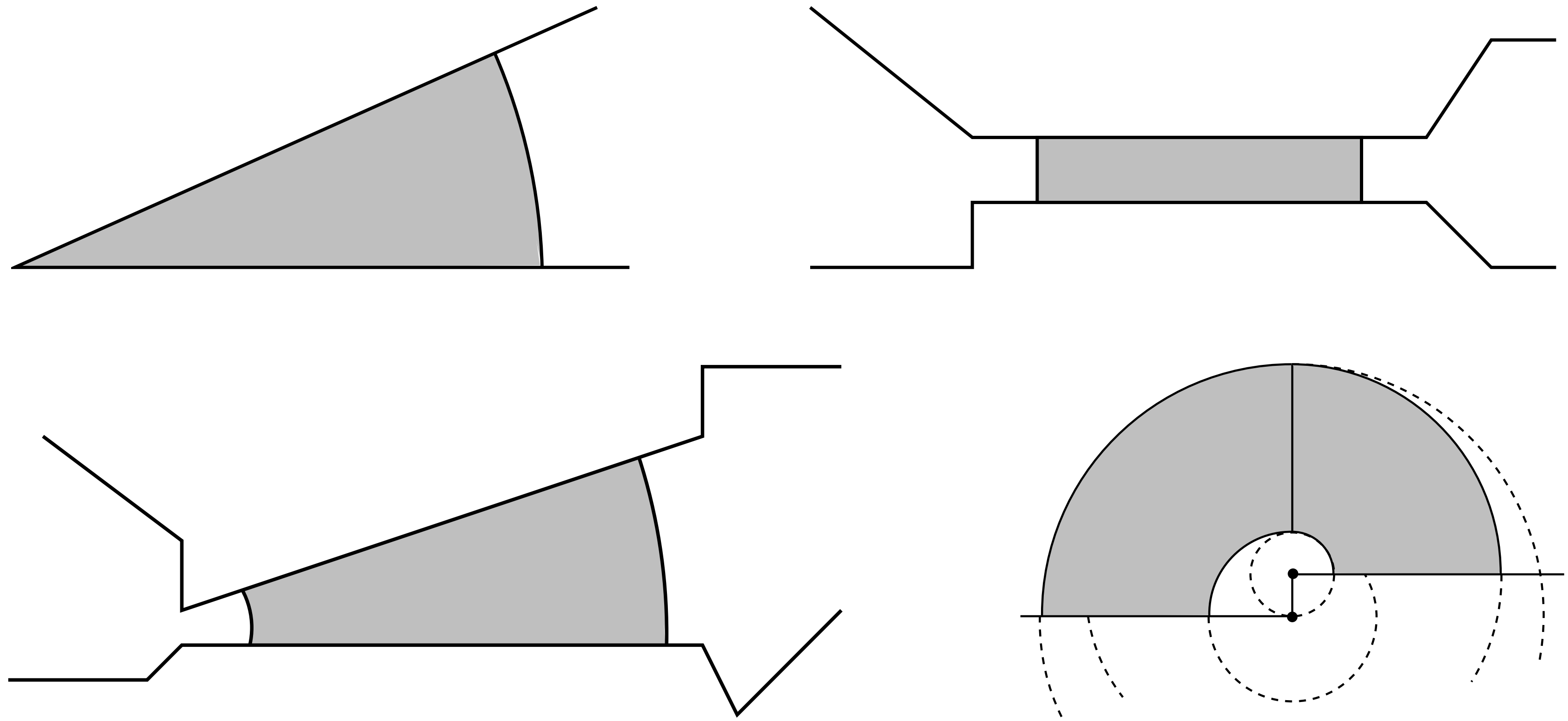
Triangulating Γ may give small angles. Must connect without small angles.



First connect Γ using disjoint disks.

Connect contact points using angles $\geq 60^\circ$

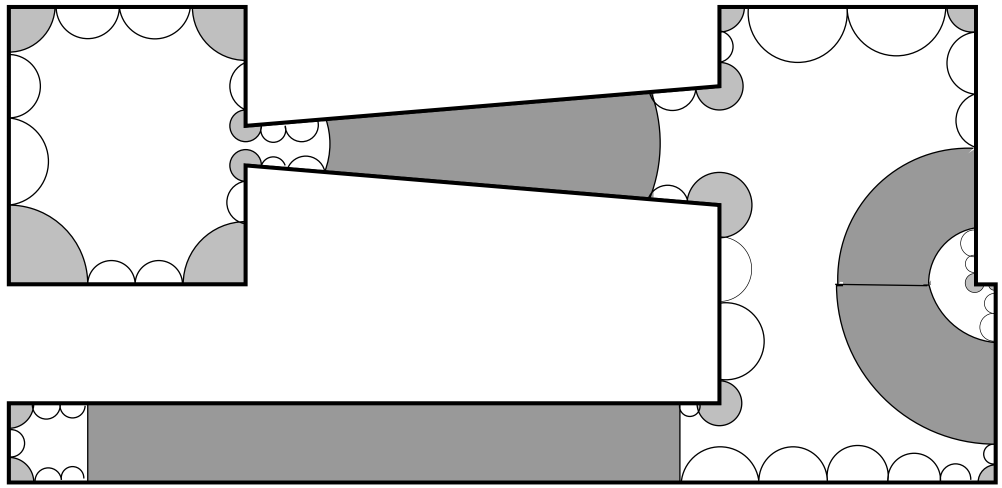
Thick/thin pieces of polygons.



Thin piece is a sector whose two straight sides satisfy

$$\mathrm{dist}(I, J) \ll \min(|I|, |J|).$$

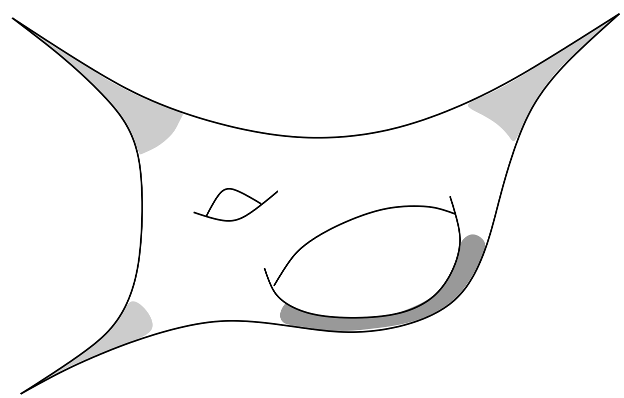
Conformally equivalent to $1 \times R$ rectangle, $R \gg 1$.



Parabolic = adjacent edges $\approx$ cusps

Hyperbolic = non-adjacent edges $\approx$ short geodesics.

Thick sides covered by $O(n)$ interior half-disks.



Thin parts computable in $O(n)$ using conformal map.

- Map polygon conformally to half-plane.
  Vertices map to points on line.

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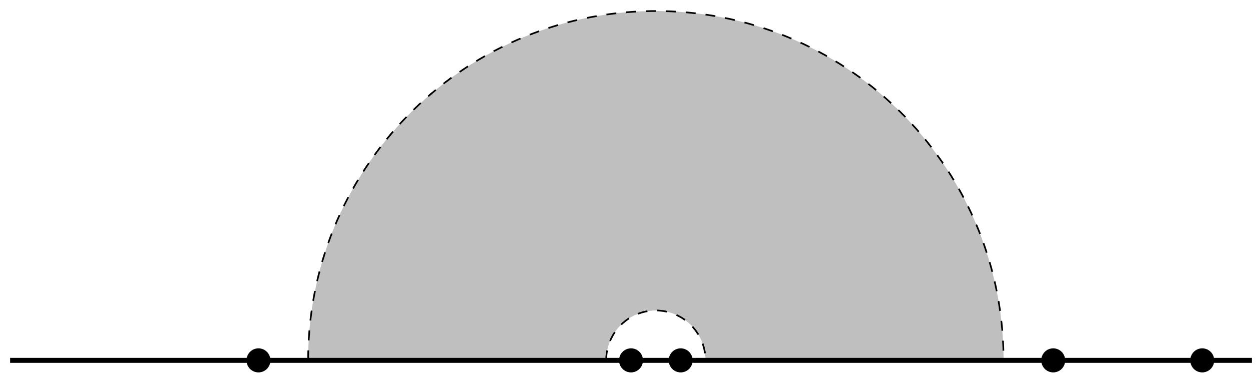
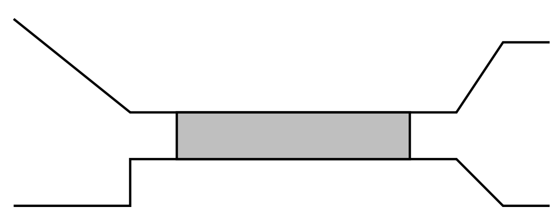
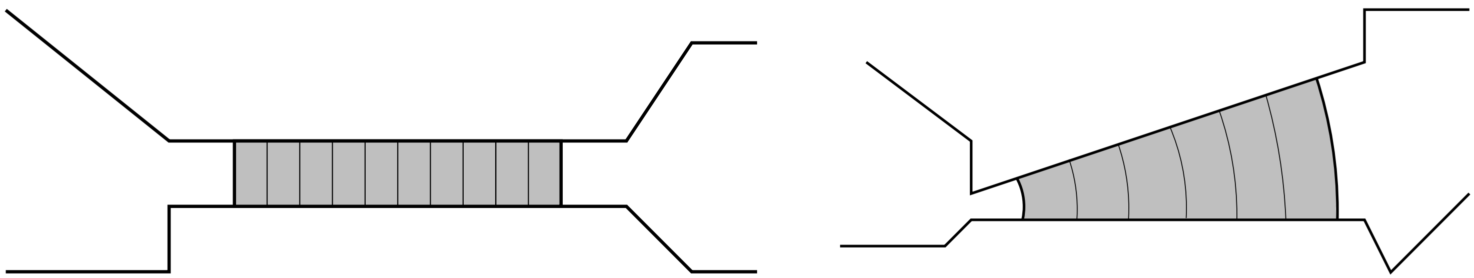


- Map polygon conformally to half-plane.
  Vertices map to points on line.
- Thin parts = wide annuli separating vertices.
- Can be found in $O(n)$ using known methods.



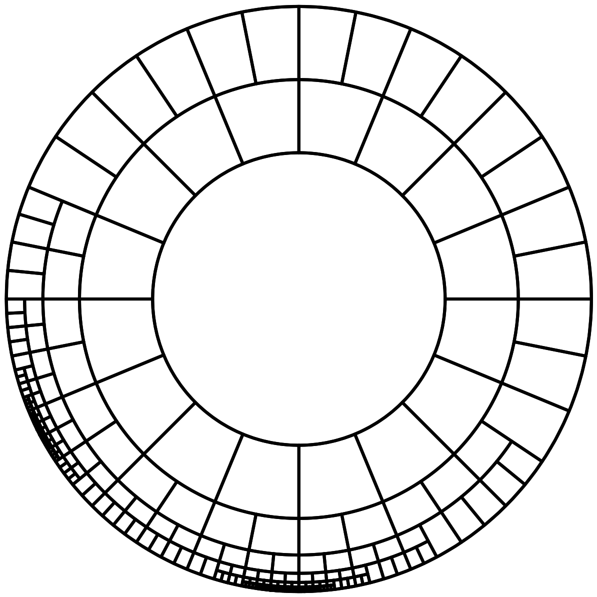
Thin parts have foliations.

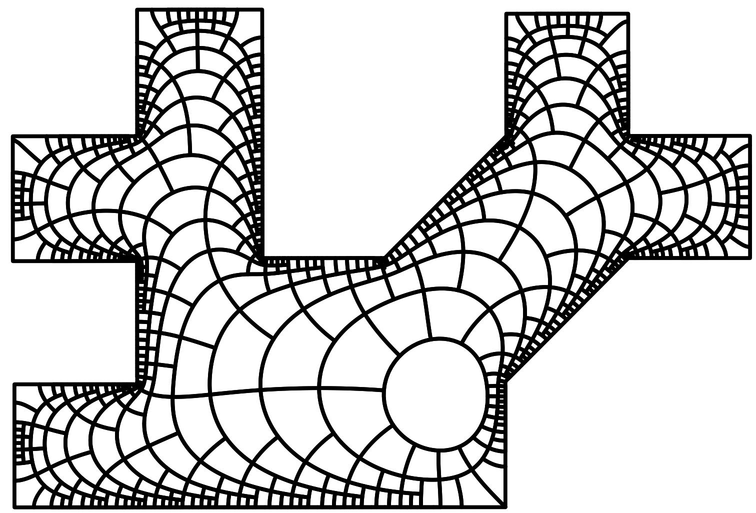


Quad mesh in thin parts is just like $90^\circ + \epsilon$ triangulation.

Harder part is to mesh the thick parts.

Basic idea for thick parts: Conformal map from disk preserves angles except near vertices.





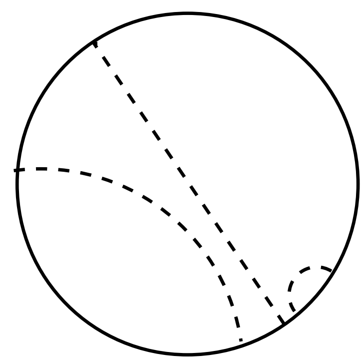
Transfer mesh on disk to mesh of polygon.

Need to be careful with tiles and timing.

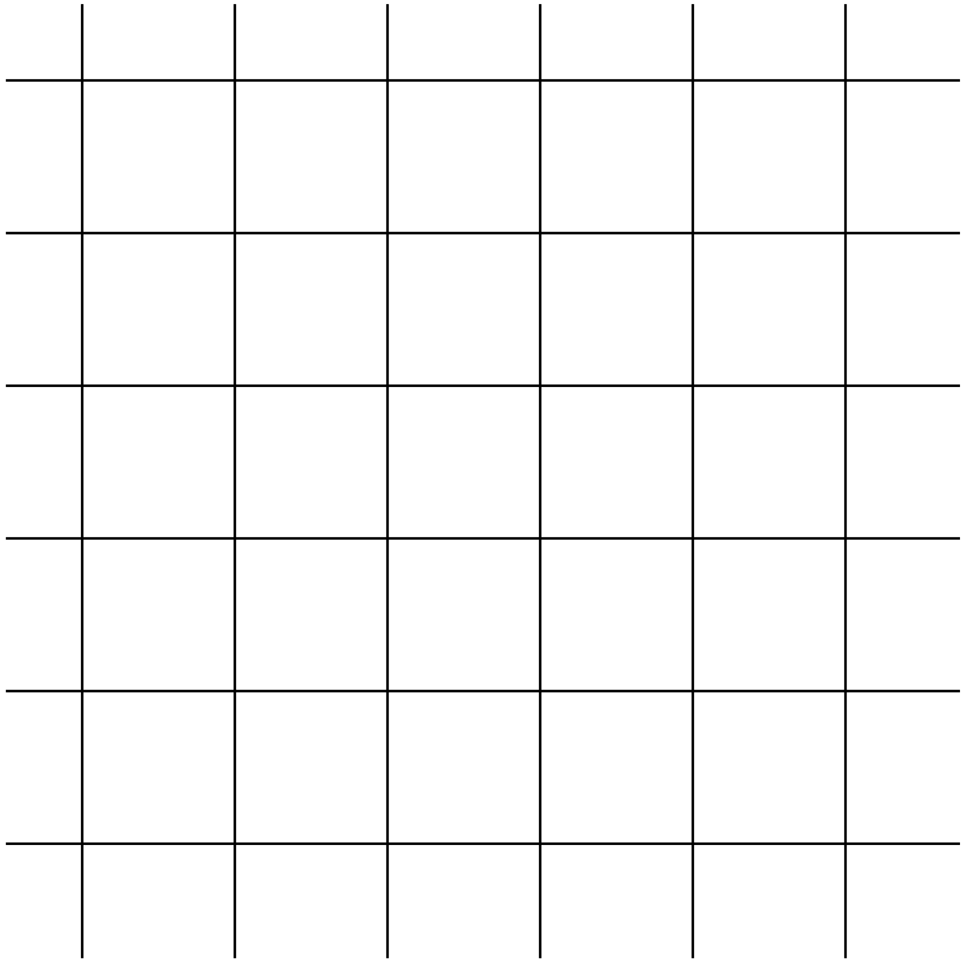
Hyperbolic metric on disk given by

$$d\rho = \frac{ds}{1-|z|^2} \simeq \frac{ds}{\operatorname{dist}(z, \partial D)}.$$

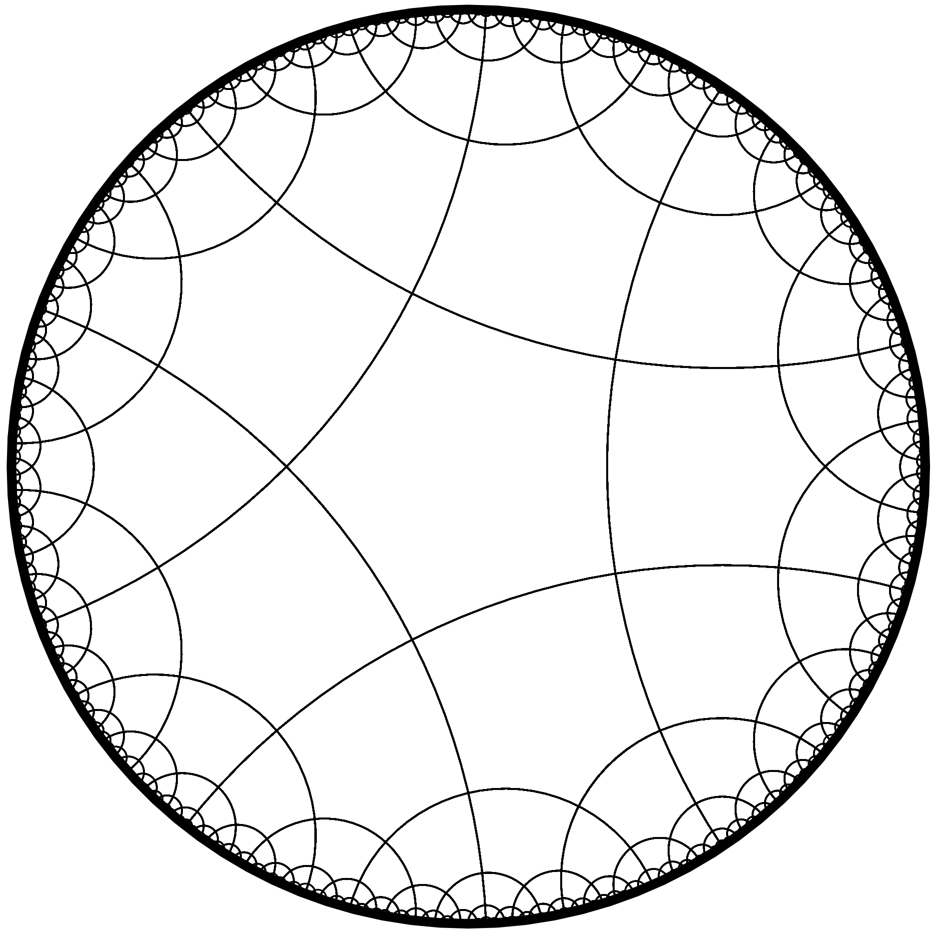
- Isometries are the Möbius transformations.
- Geodesics are circles perpendicular to boundary.



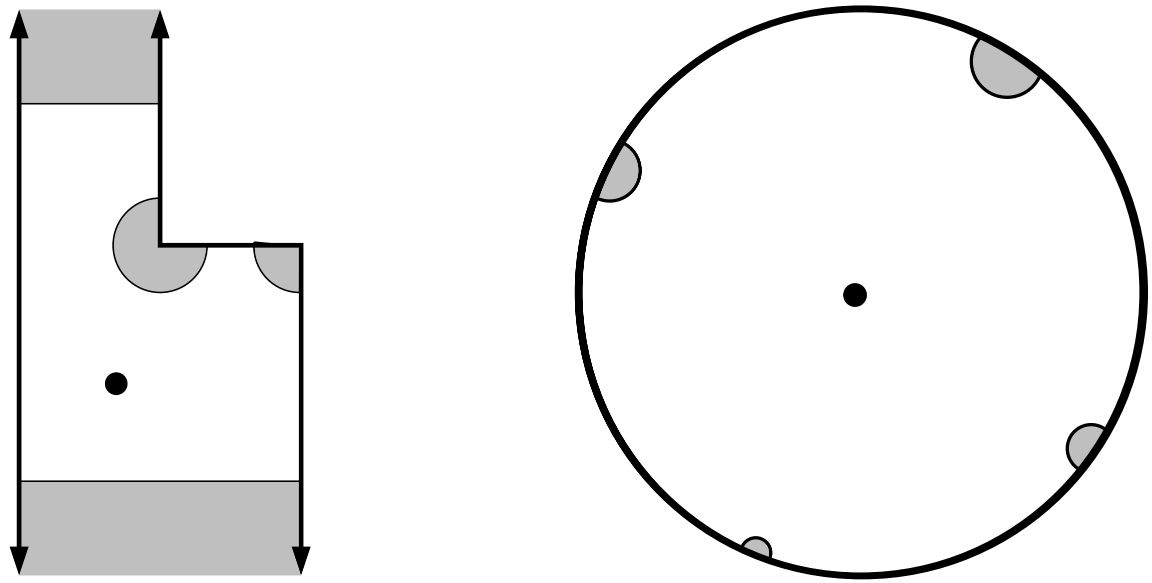
Euclidean space can be tesselated by squares



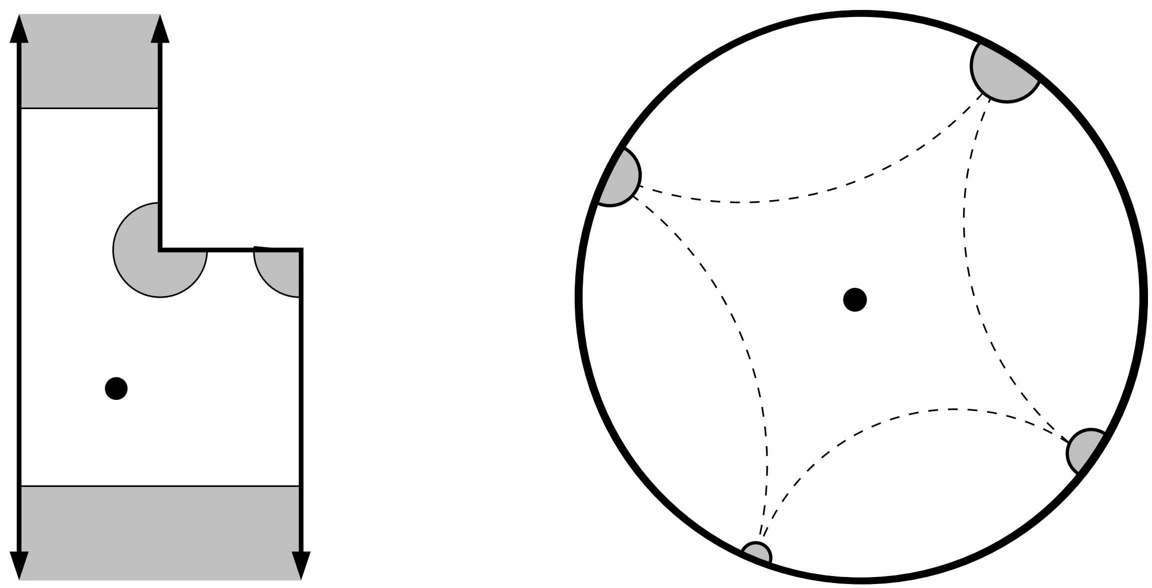
Hyperbolic space can be tesselated by right pentagons.



Conformal map from polygon to disk takes thick and thin parts to disk as shown.

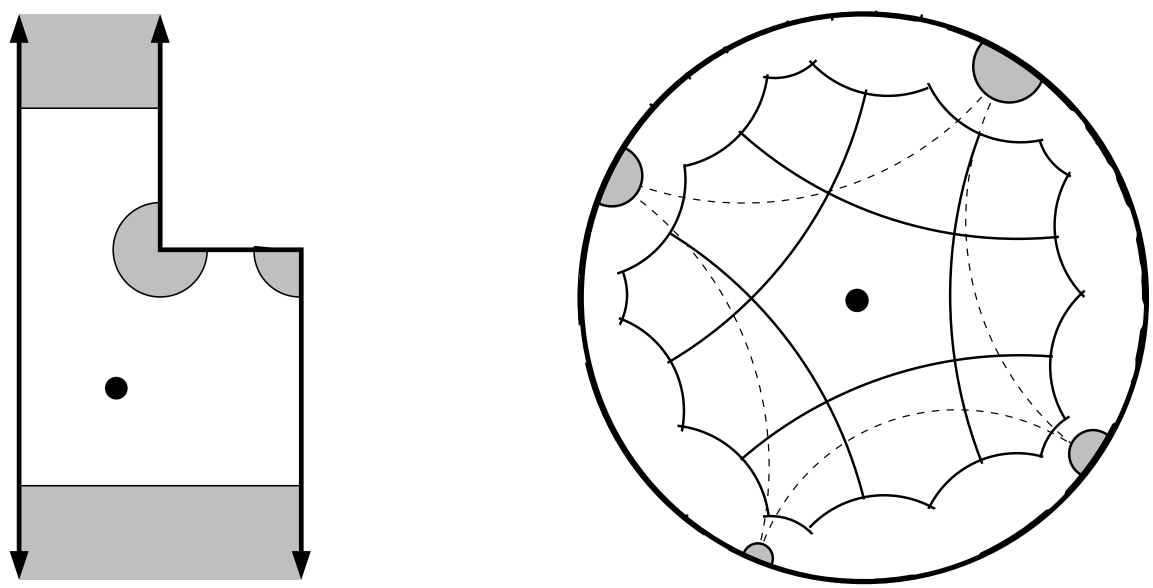


Conformal map from polygon to disk takes thick part to region as shown.



Draw (hyperbolic) convex hull of thin regions.

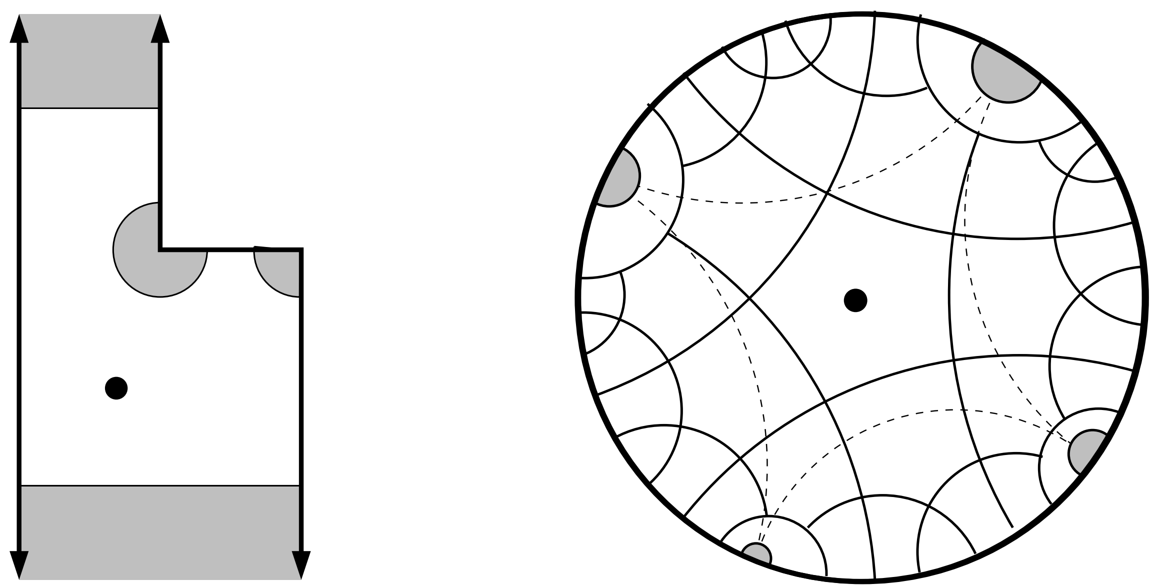
Conformal map from polygon to disk takes thick part to region as shown.



Draw (hyperbolic) convex hull of thin regions.

Take pentagons from tesselation hitting convex hull but missing thin parts.

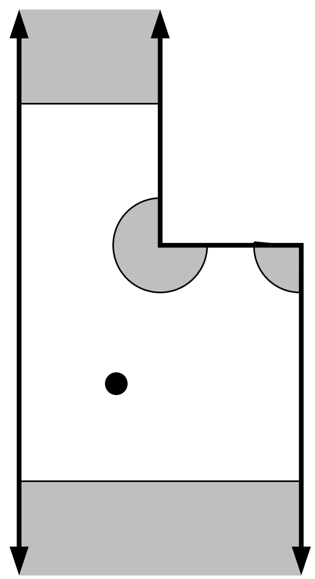
Conformal map from polygon to disk takes thick part to region as shown.

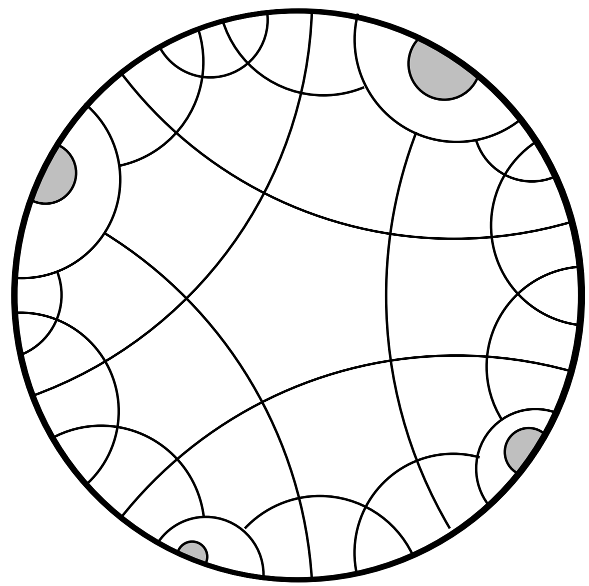


Draw (hyperbolic) convex hull of thin regions.

Take pentagons from tesselation hitting convex hull but missing thin parts. Extend pentagon edges to boundary.

Conformal map from polygon to disk takes thick part to region as shown.





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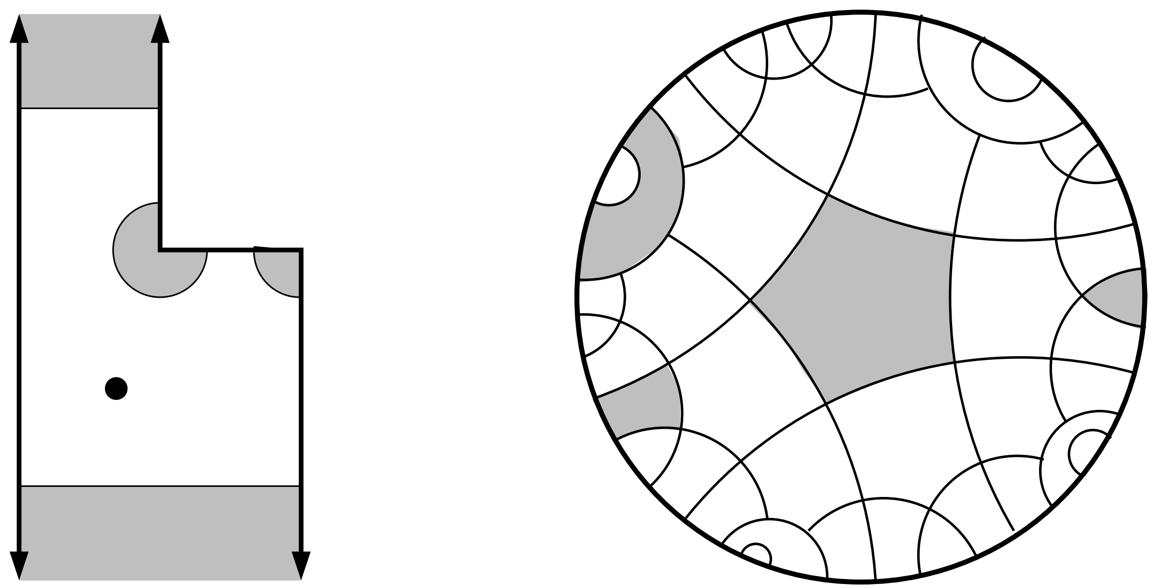
Analog of Whitney or quadtreee construction.

Conformal map from polygon to disk takes thick part to region as shown.



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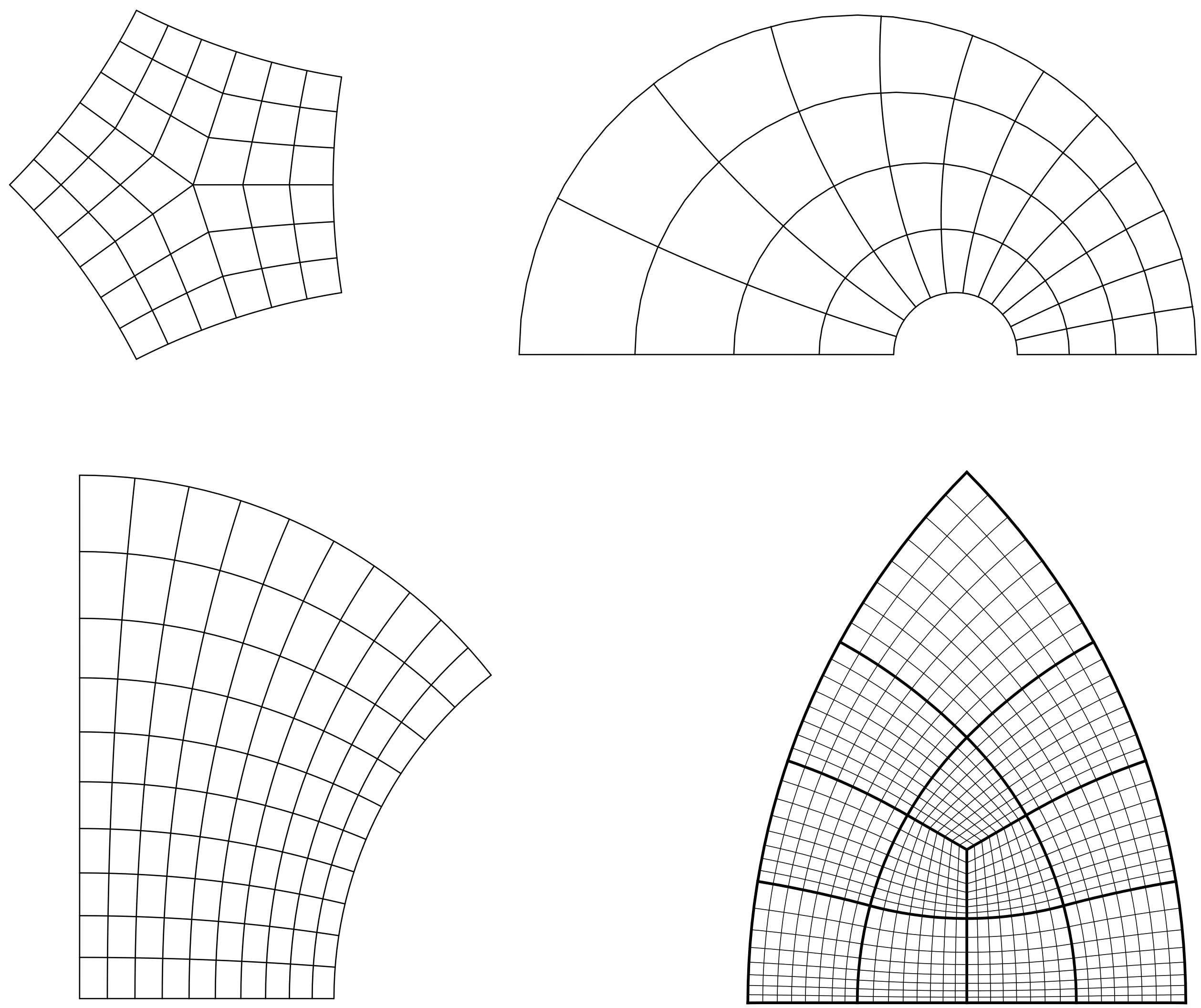
Take pentagons from tesselation hitting convex hull but missing thin parts. Extend pentagon edges to boundary.

Pentagons, quadrilaterals, triangles and half-annuli.



Shapes can be meshed to match along common edges.

How much time to create conformal map?

Theorem: We can compute a ϵ-conformal map onto n-gon in $O(n \log \frac{1}{\epsilon} \log \log \frac{1}{\epsilon})$.

ϵ-conformal $=$ ϵ-distortion of angles.

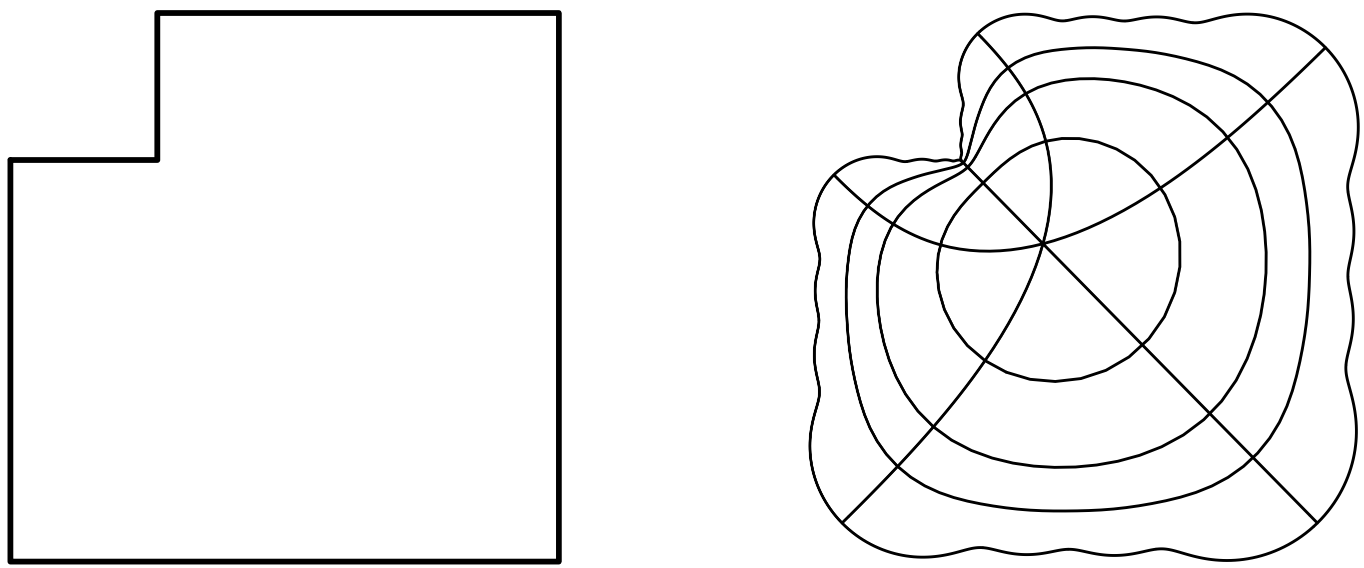
First method to give guaranteed success in linear time, e.g., CRDT algorithm of Driscoll and Vavasis is $O(n^2)$ and not proven to converge.

For our application we want to approximate conformal map onto n-gon at $O(n)$ points in $O(n)$ time.

Proof of the fast mapping theorem:

- Local representation of maps
- Newton's method for Beltrami's equation
- The iota map for initial guess

Conformal maps have power series, but corners of polygon create singularities on circle. Convergence is slow.



20 terms

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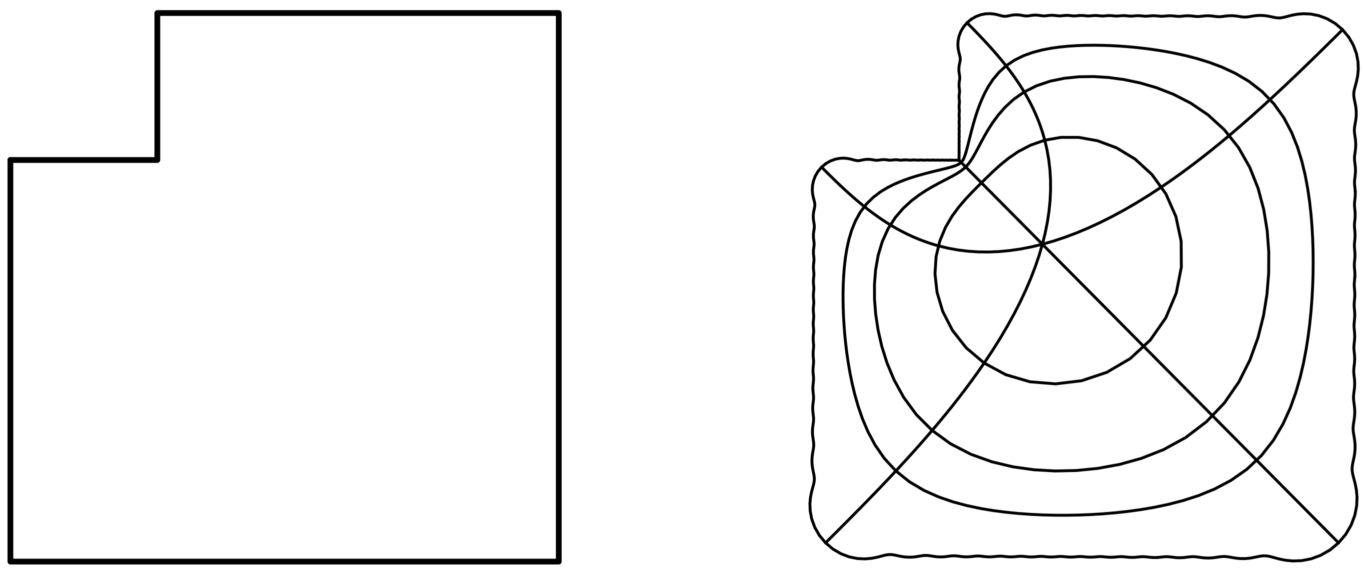
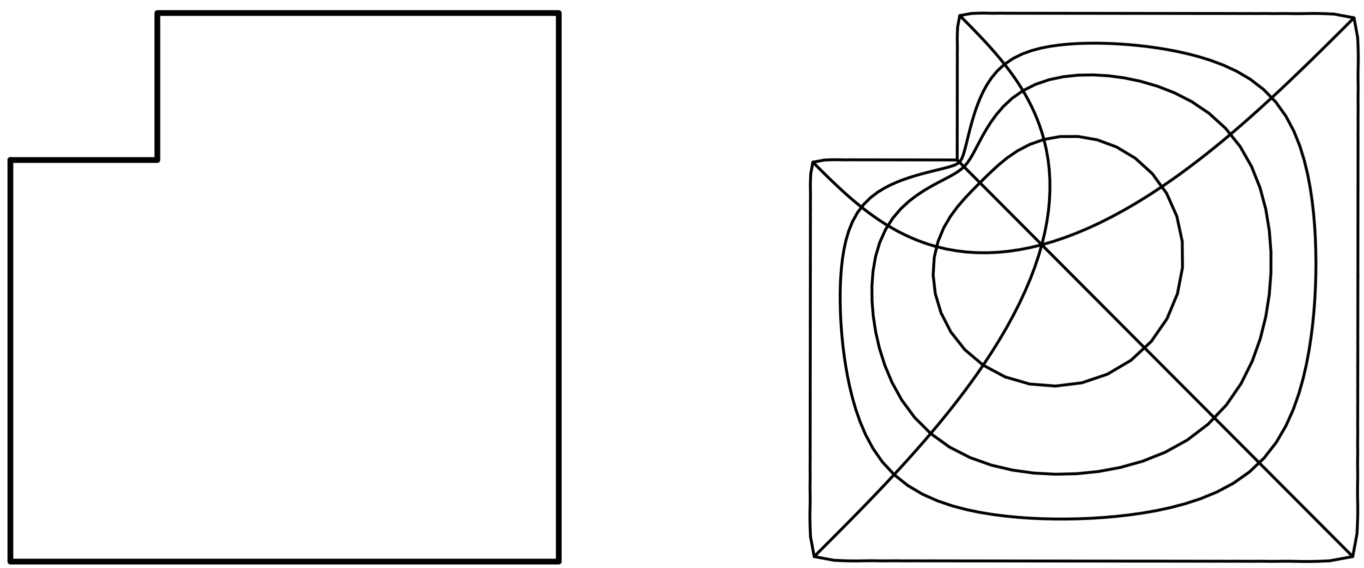


100 terms

Conformal maps have power series, but corners of polygon create singularities on circle. Convergence is slow.



1000 terms

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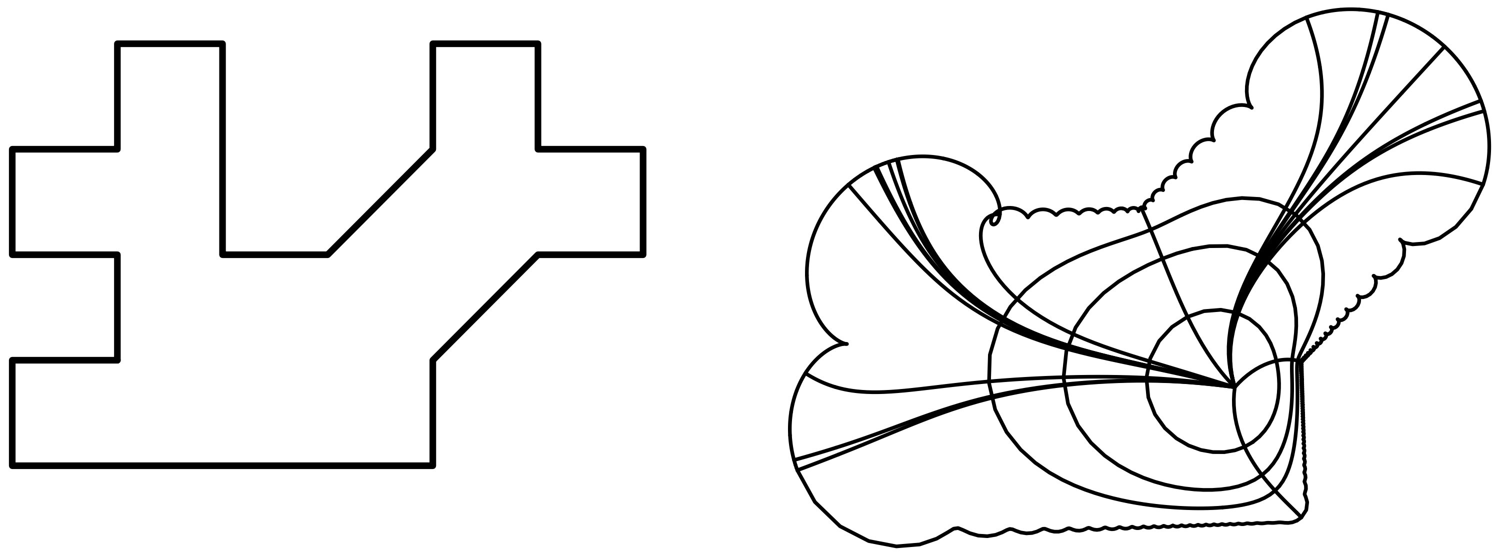

100 terms

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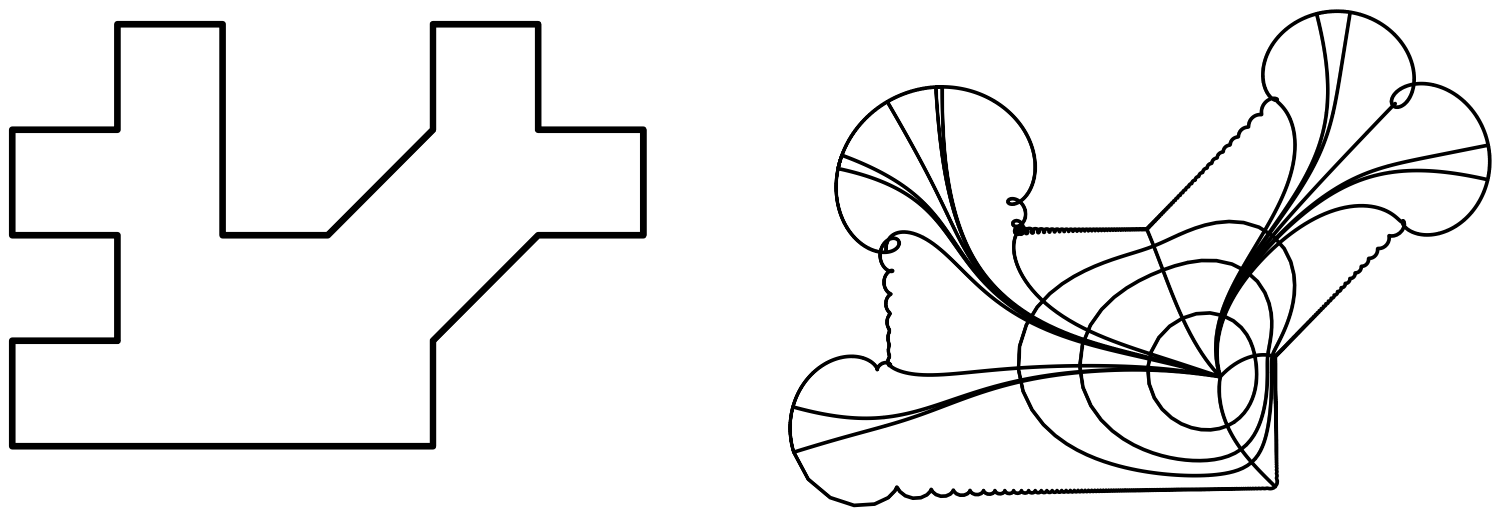
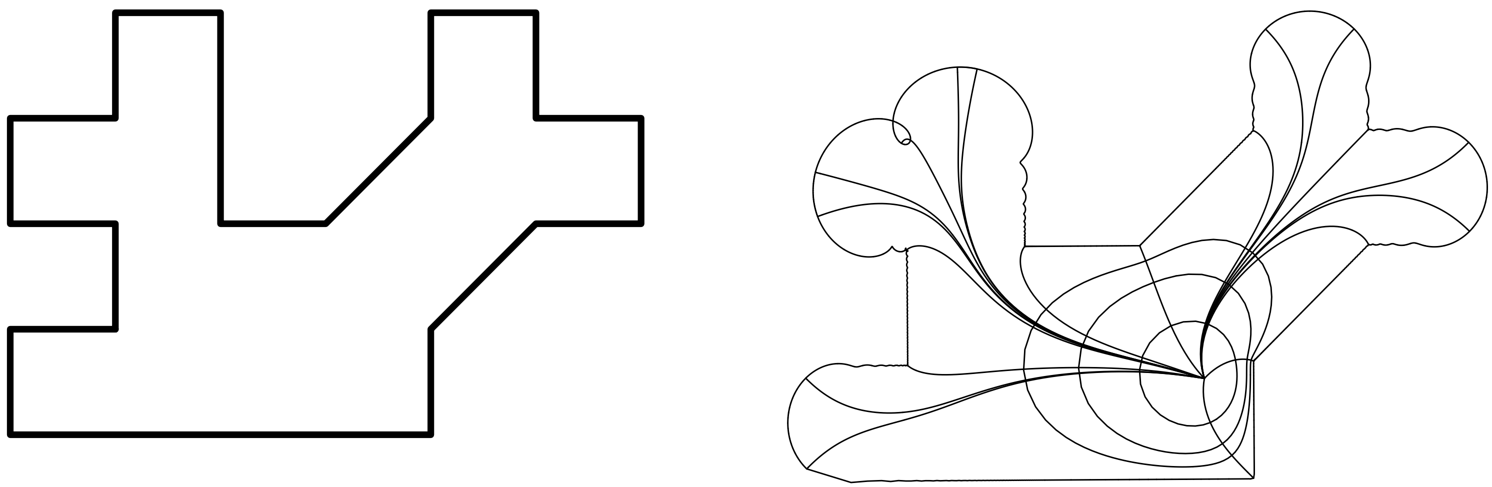


500 terms

Conformal maps have power series, but corners of polygon create singularities on circle. Convergence is slow.



2500 terms

1×20 rectangle would require about 10^{15} terms.

Schwarz-Christoffel representation:

$$f(z) = A + C \int^{z} \prod_{k=1}^{n} (1 - \frac{w}{z_k})^{\alpha_k - 1} dw,$$

$\{\alpha_1 \pi, \ldots, \alpha_n \pi\}$, are interior angles of polygon.
$\{z_1, \ldots, z_n\}$ are points on circle mapping to vertices.

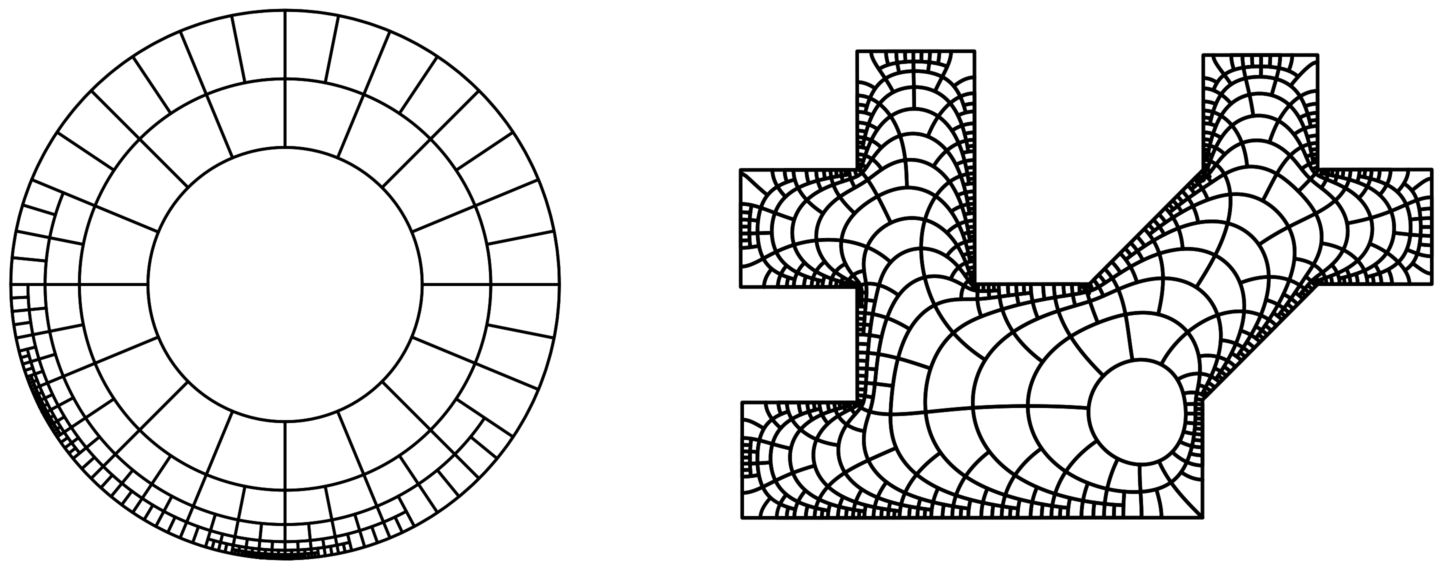
α's are known.

z's must be solved for.

Each evaluation of integrand is n-fold product. How many evaluations to compute integral accurately?

Recall we have a $O(n)$ time bound.

Local series representation: We cut disk into $O(n)$ regions and use a p-term series on each piece to approximate map with accuracy $\epsilon \approx 2^{-p}$ in hyperbolic metric.



Use partition of unity to get global map.

Easy to evaluate; just plug in.

Also more subtle advantage.

Schwarz-Christoffel always gives conformal map, but onto wrong polygon if z-parameters are only approximate.

Hard to understand relationship between parameters and image domain, so hard to update parameters in provably correct way (many unproven heuristics which seem to work in practice, e.g., Davis, CRDT.)

Local series give map onto correct domain, but are not conformal if series are approximate.

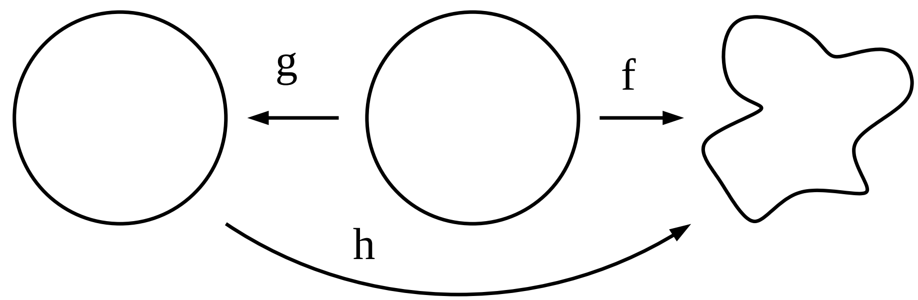
Easy to make a map conformal and preserve image. (Method has been known for 50 years.)

$$\partial f = \frac{1}{2}(f_x - i f_y), \quad \overline{\partial} f = \frac{1}{2i}(f_x + i f_y).$$

We want $f : \mathbb{D} \to \Omega$ with $\overline{\partial} f = 0$ (Cauchy-Riemann).
We measure distance to conformality by dilitation

$$\|f\| = \sup |\mu_f| = \sup |\overline{\partial} f / \partial f|.$$

Main point: If $f : \mathbb{D} \to \Omega$, $g : \mathbb{D} \to \mathbb{D}$ and $\mu_g = \mu_f$, then $h = f \circ g^{-1} : \mathbb{D} \to \Omega$ is conformal.



Given $\mu = \mu_f$ set $g = P[\mu(h+1)] + z$, where

$$h = T\mu + T\mu T\mu + T\mu T\mu T\mu + \dots,$$

T is the Beurling transform

$$T\varphi(w) = \lim_{r \to 0} \frac{1}{\pi} \iint_{|z-w|>r} \frac{\varphi(z)}{(z-w)^2} dxdy,$$

P is the Cauchy transform

$$P\varphi(w) = -\frac{1}{\pi} \iint \varphi(z)(\frac{1}{z-w} - \frac{1}{z}) dxdy.$$

Then $f \circ g^{-1} : \mathbb{D} \to \Omega$ is conformal.

If f has local representation by $O(n)$ p-term series, we can compute a g in time $O(np \log p)$ so that

$$\|f \circ g^{-1}\| = O(\|f\|^2).$$

Iteration gives quadratic convergence to conformal map.

Uses fast multipole method and FFTs on series.

For time bound iteration needs an starting map $\mathbb{D} \to \Omega$ that is close to conformal (independent of Ω).

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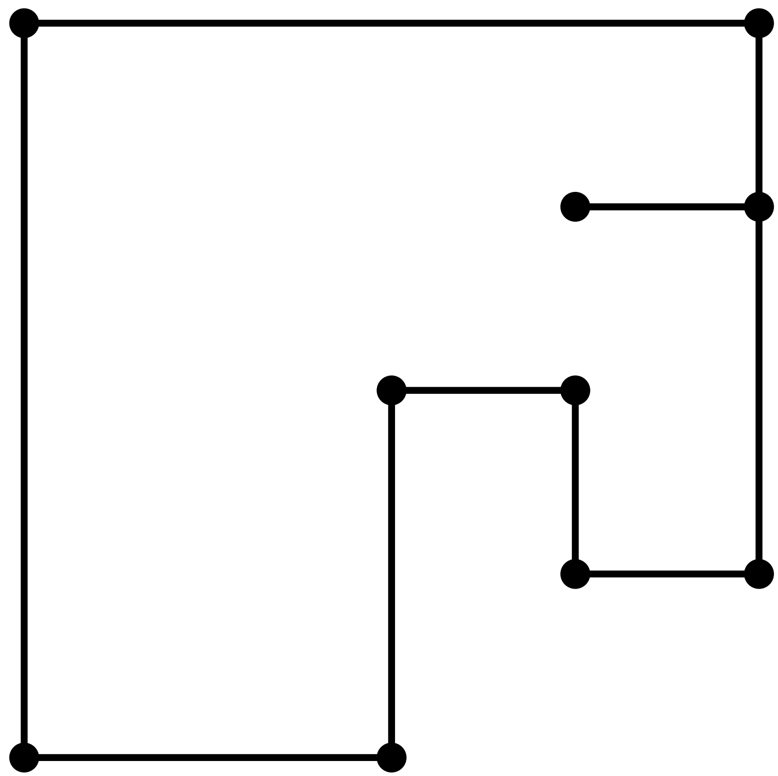
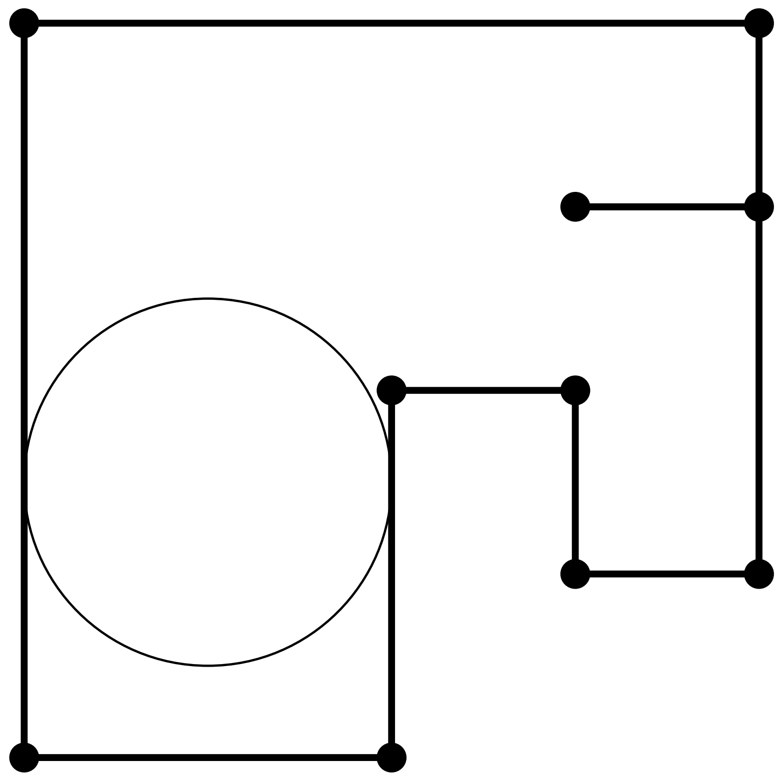
This is how I got involved with numerical mapping.

Initial guess comes from **iota map**.



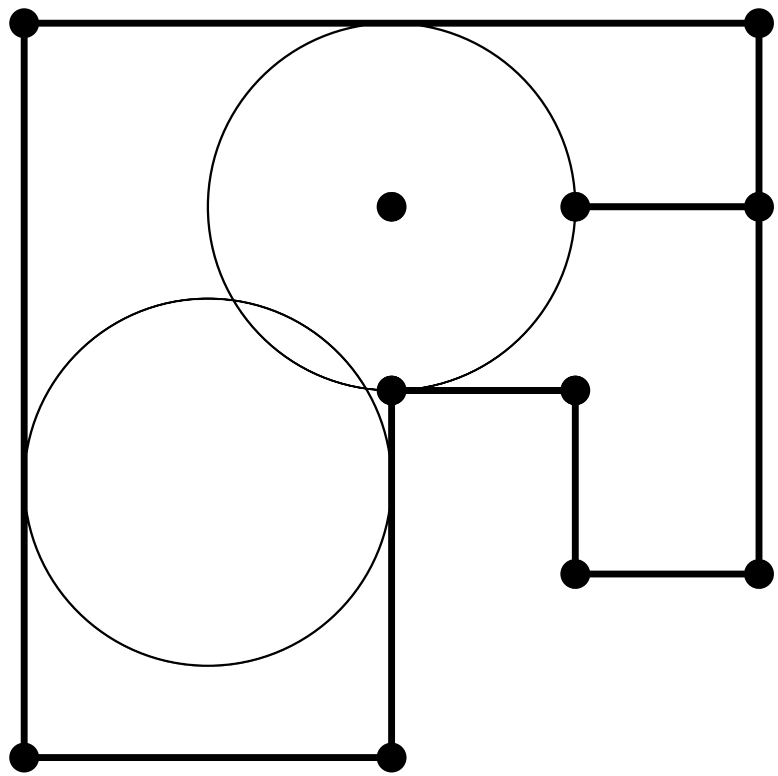
Start with a polygon.

Initial guess comes from **iota map**.



Consider interior disks with ≥ 2 contacts on boundary.

Initial guess comes from **iota map**.



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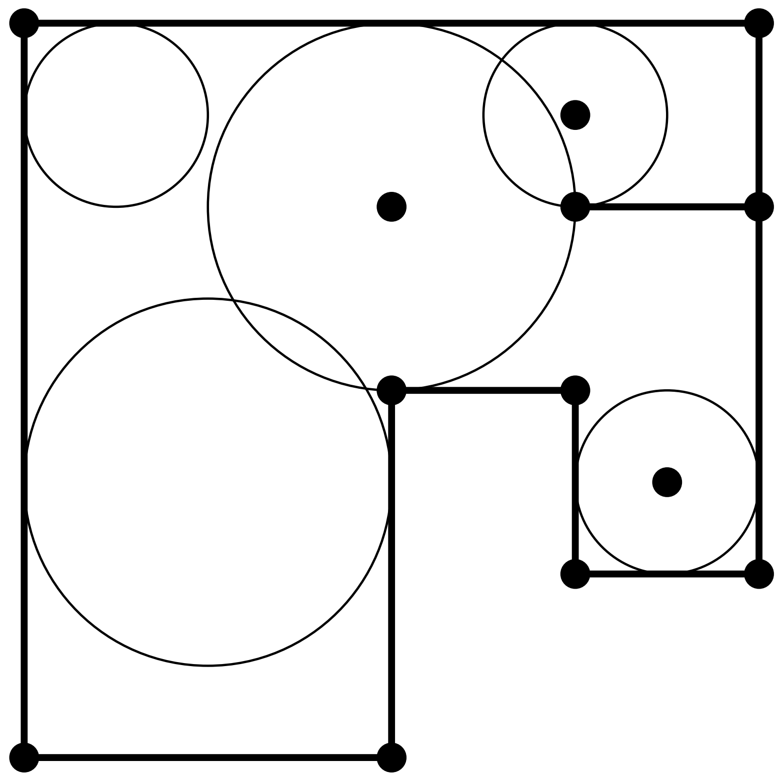

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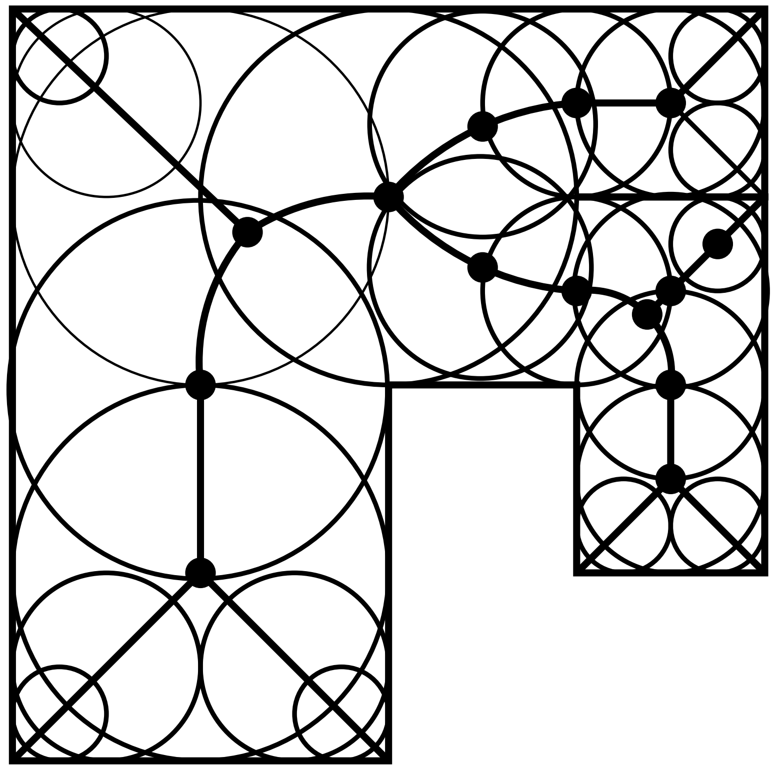
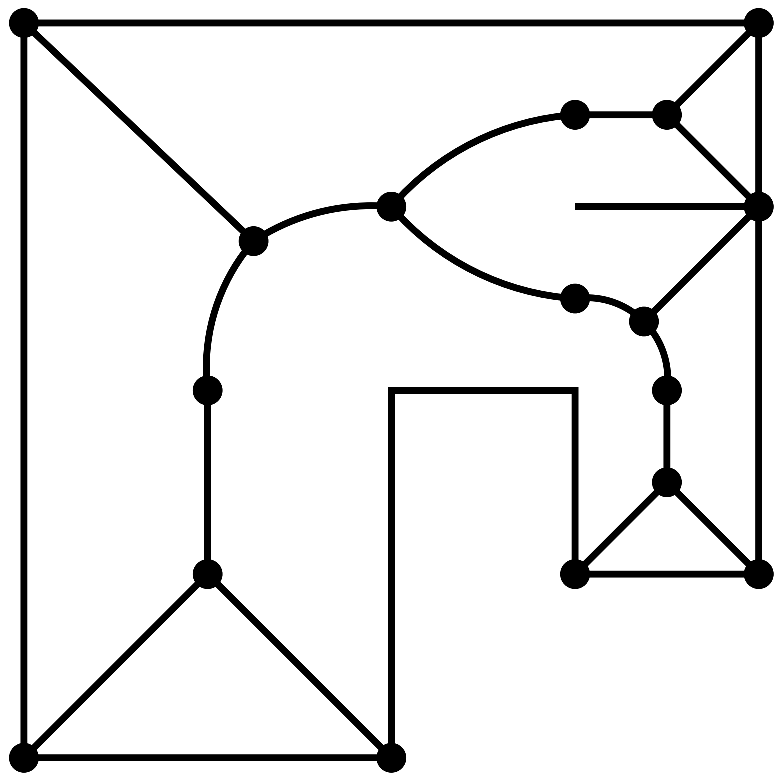


Centers of all such disks define **medial axis**.

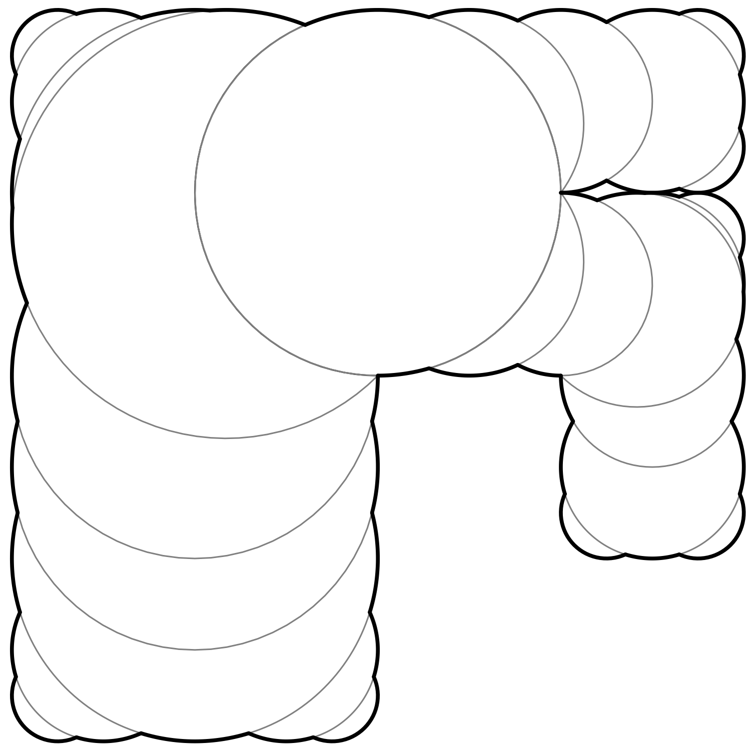
Initial guess comes from **iota map**.



Centers of all such disks define **medial axis**.

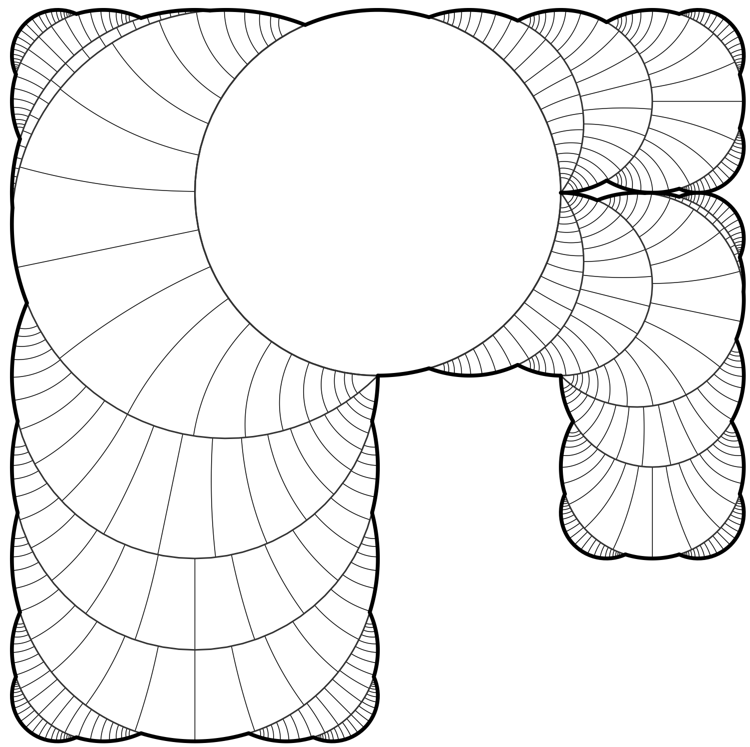
Computable in $O(n)$ (Chin, Snoeyink, Wang 1999).

Initial guess comes from **iota map**.



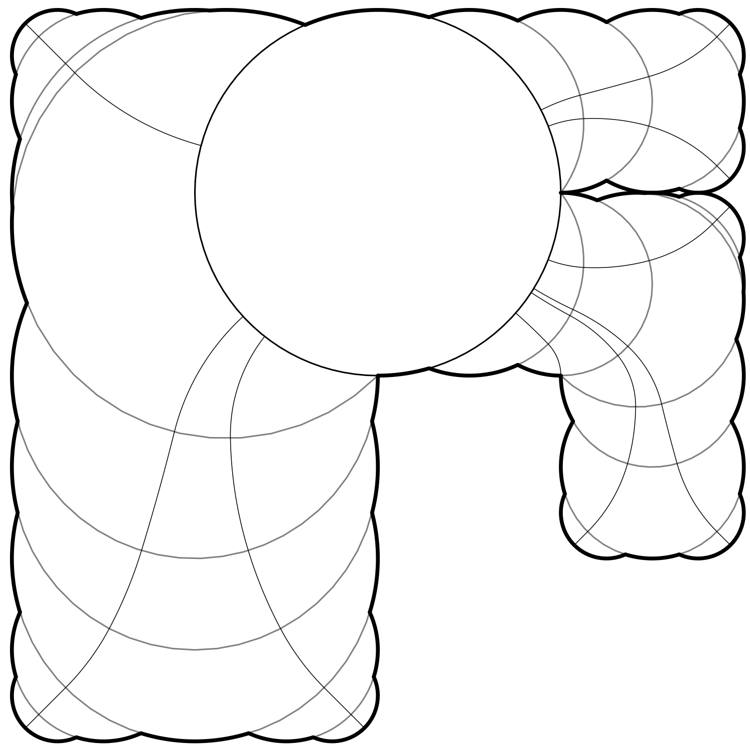
Take a finite set of medial axis disks. Choose a root.

Initial guess comes from **iota map**.



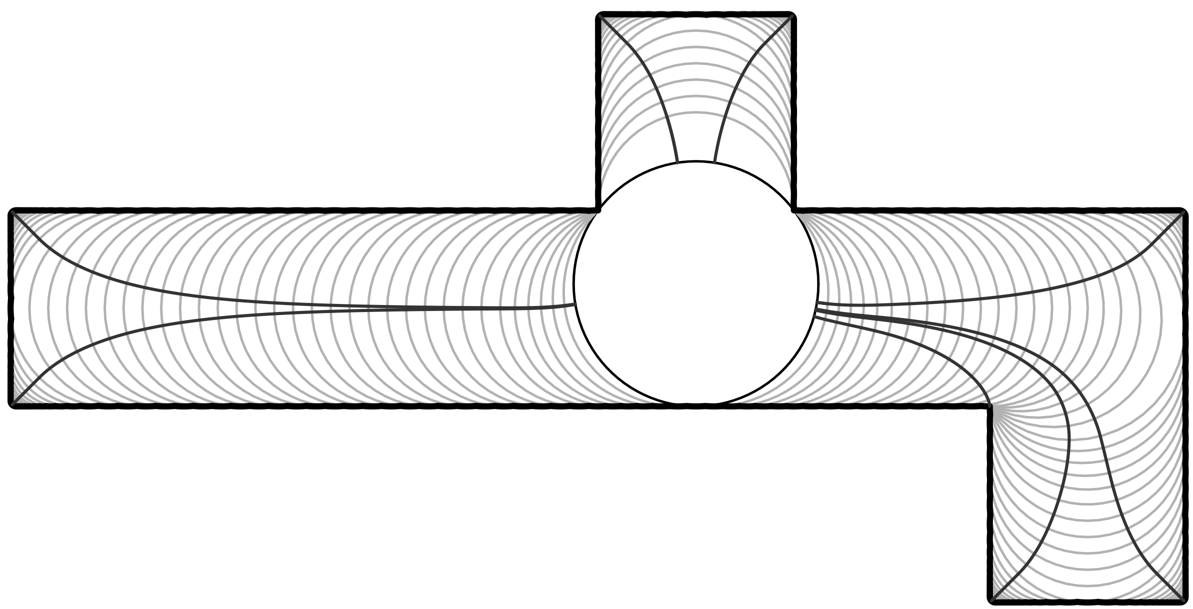
Foliate crescents by orthogonal arcs.

Initial guess comes from **iota map**.

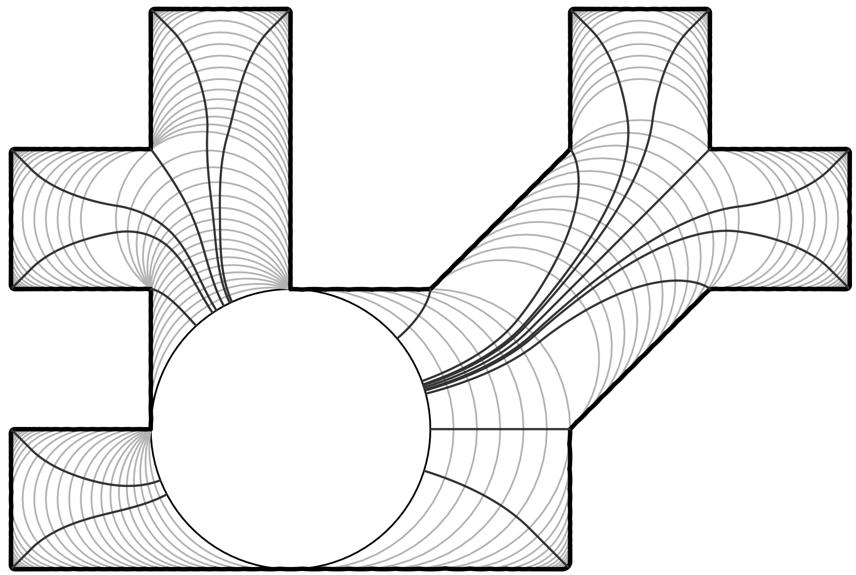


Follow arcs to define map of boundary to circle.

Medial Axis Flow = iota map: Take limit to get explicit formulas for n-gons. Images of vertices computable in $O(n)$.



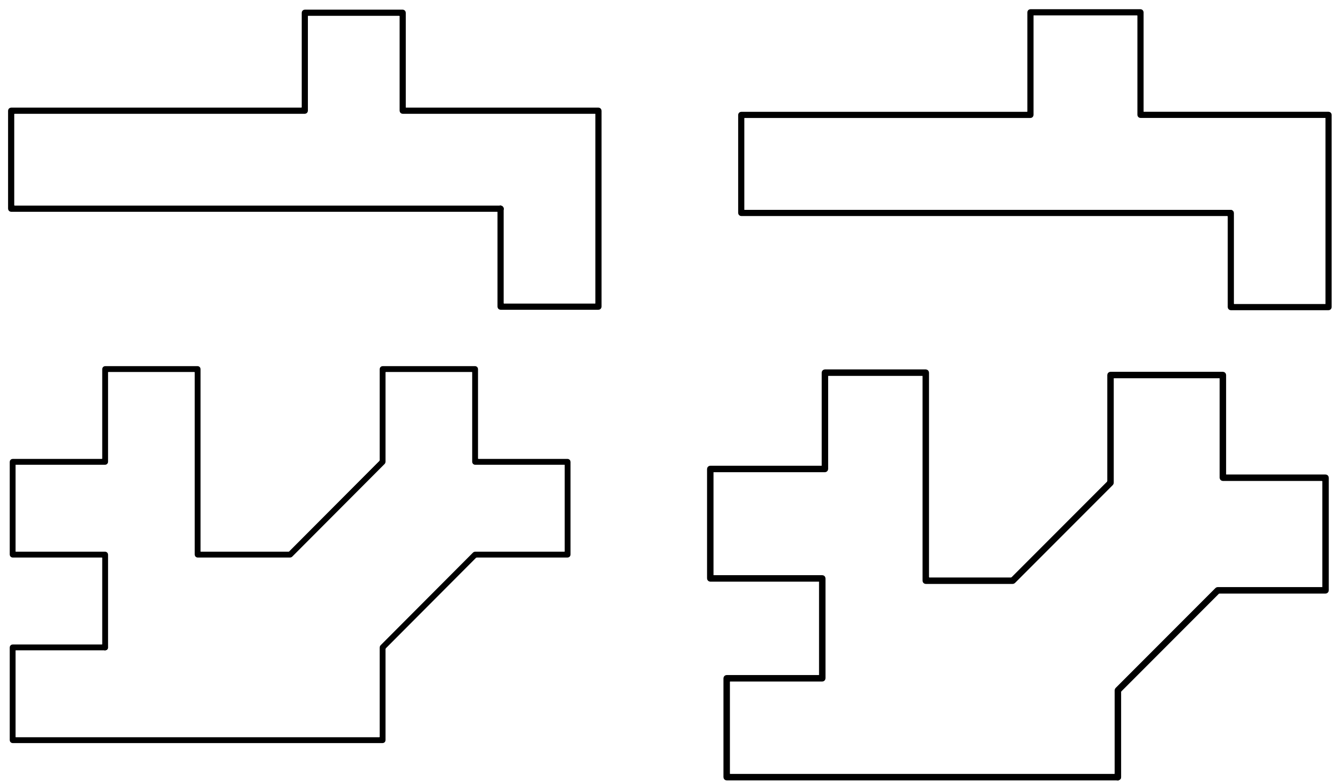
Theorem: Iota on boundary extends to interior map f with $|\mu_f| < k < 1$, universal k.



Iota gives good starting point for iteration. Only need $O(\log\log\frac{1}{\epsilon})$ iterations to attain accuracy ϵ.

Also "good enough" to compute thin parts in $O(n)$.

How close is iota map to conformal? Try using “iota parameters” in Schwarz-Christoffel formula.



Target Polygon

Iota Parameters

Suffices to show nearest point map onto certain convex sets in hyperbolic 3-space is bi-Lipschitz at large scales.

- short proof of special case, Sullivan, 1981
- long proof of general case, Epstein and Marden, 1985
- short proof of general case, B, 2001

I worked on this to compute fractal dimension of the set of directions in a hyperbolic manifold that corresponded to bounded geodesic rays.

Connection to conformal maps and meshing came later.

Conclusions:

- Any PSLG can be triangulated in time $O(n^{2.5})$ and all angles $\leq 90°$.

- Any PSLG has a quad mesh with $O(n^2)$ elements and angles in $[60°, 120°]$.

- An ϵ-conformal map onto an n-gon can be computed in time $O(n \log \epsilon \log \log \epsilon)$.

Why does iota approximate the Riemann Map?

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Because iota is the boundary map of a conformal map for a surface with the same boundary as Ω.

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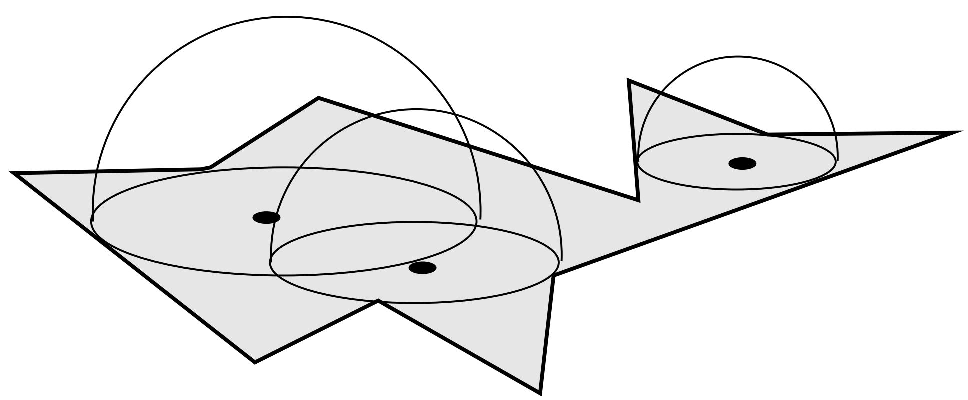
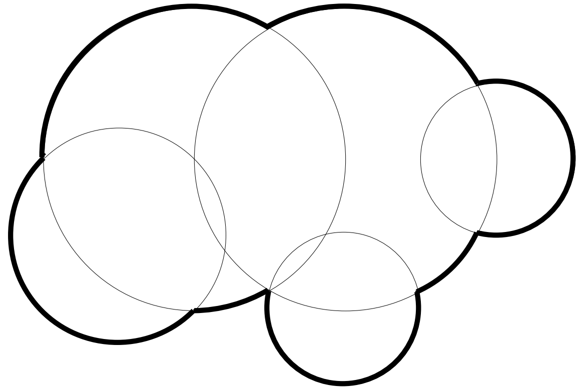
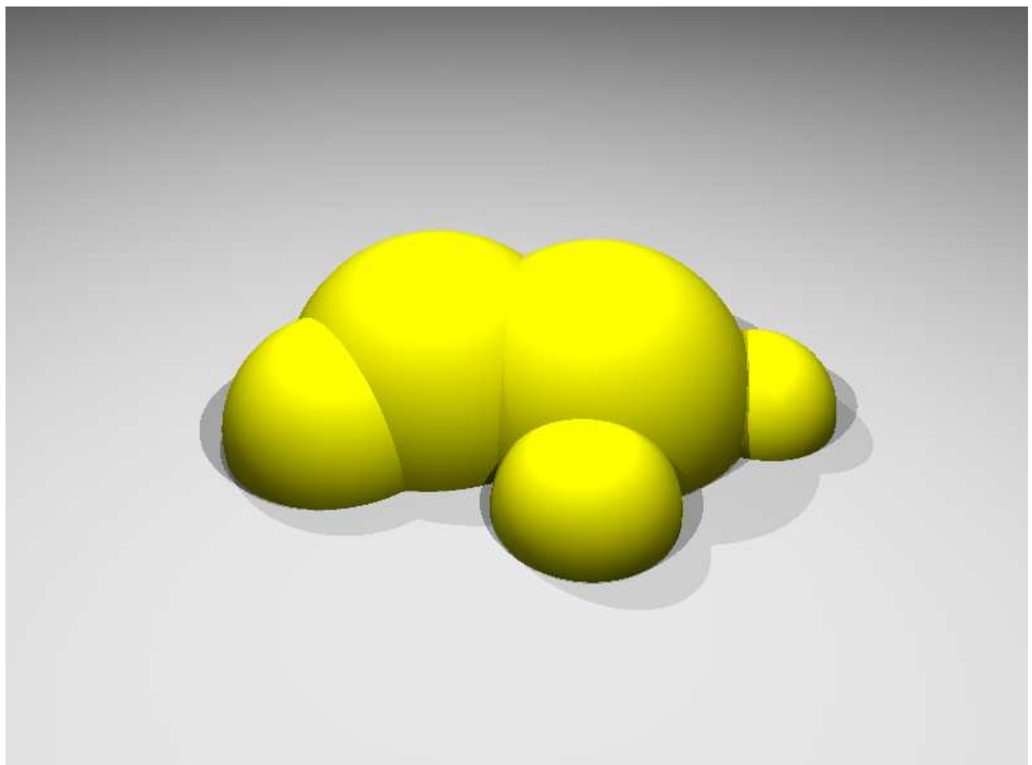
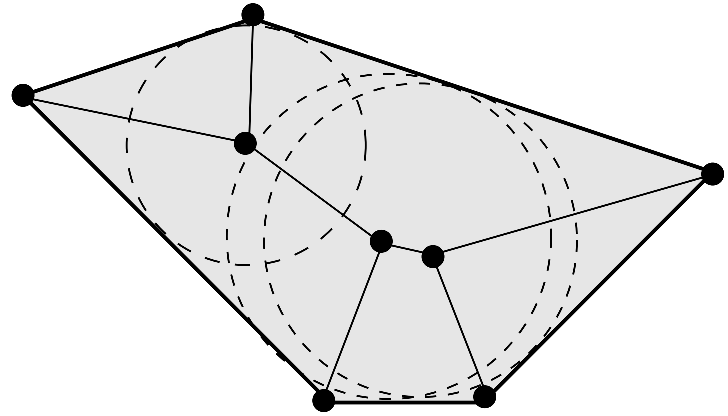
Take union of all hemispheres whose bases lie inside Ω. Upper envelope is the "dome" of Ω.

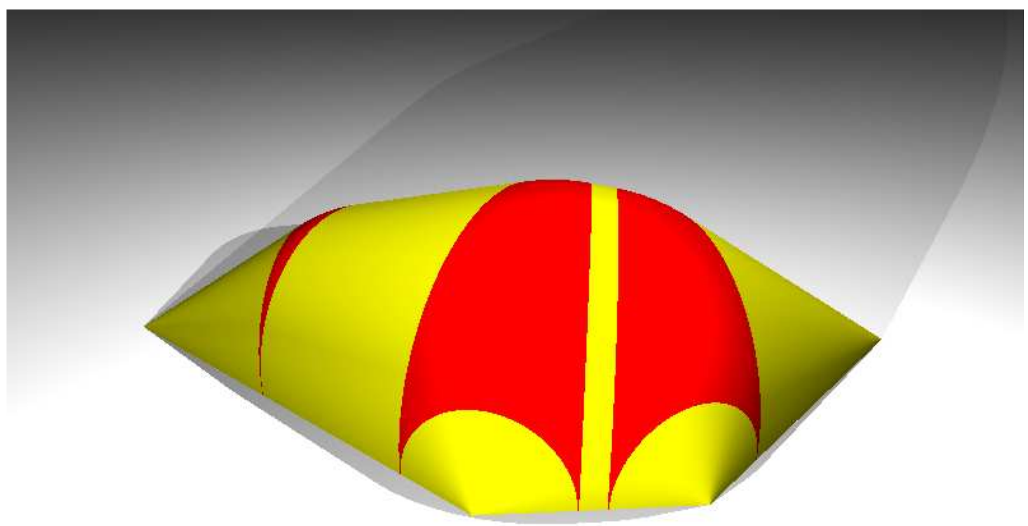


(This is hyperbolic convex hull of Ω^c.)

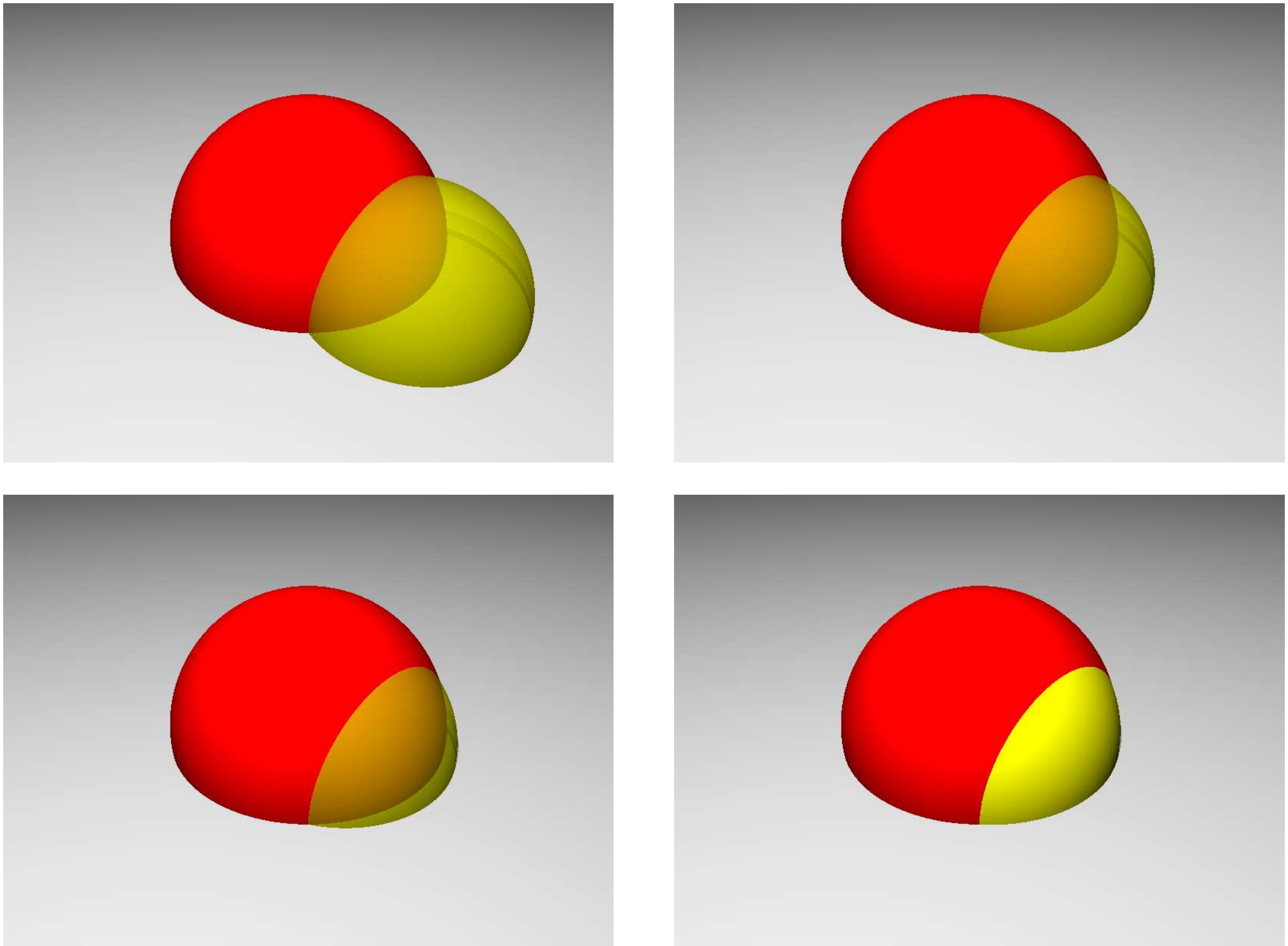




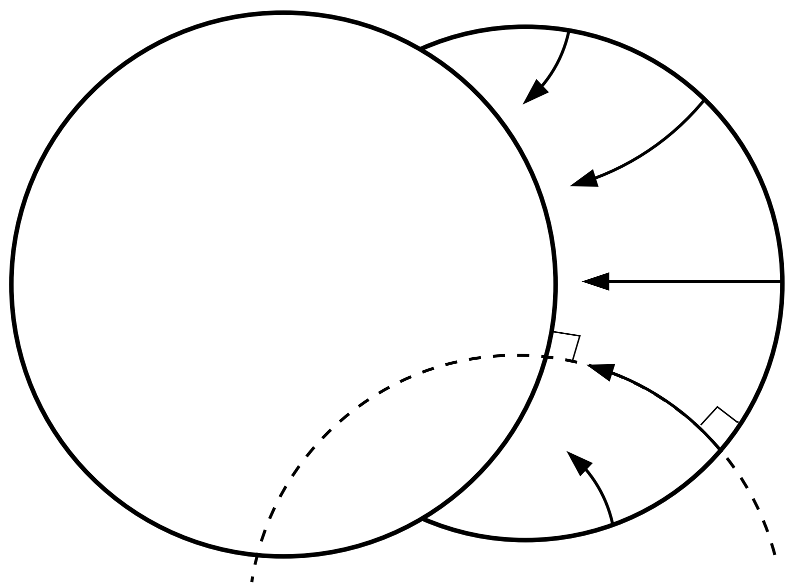


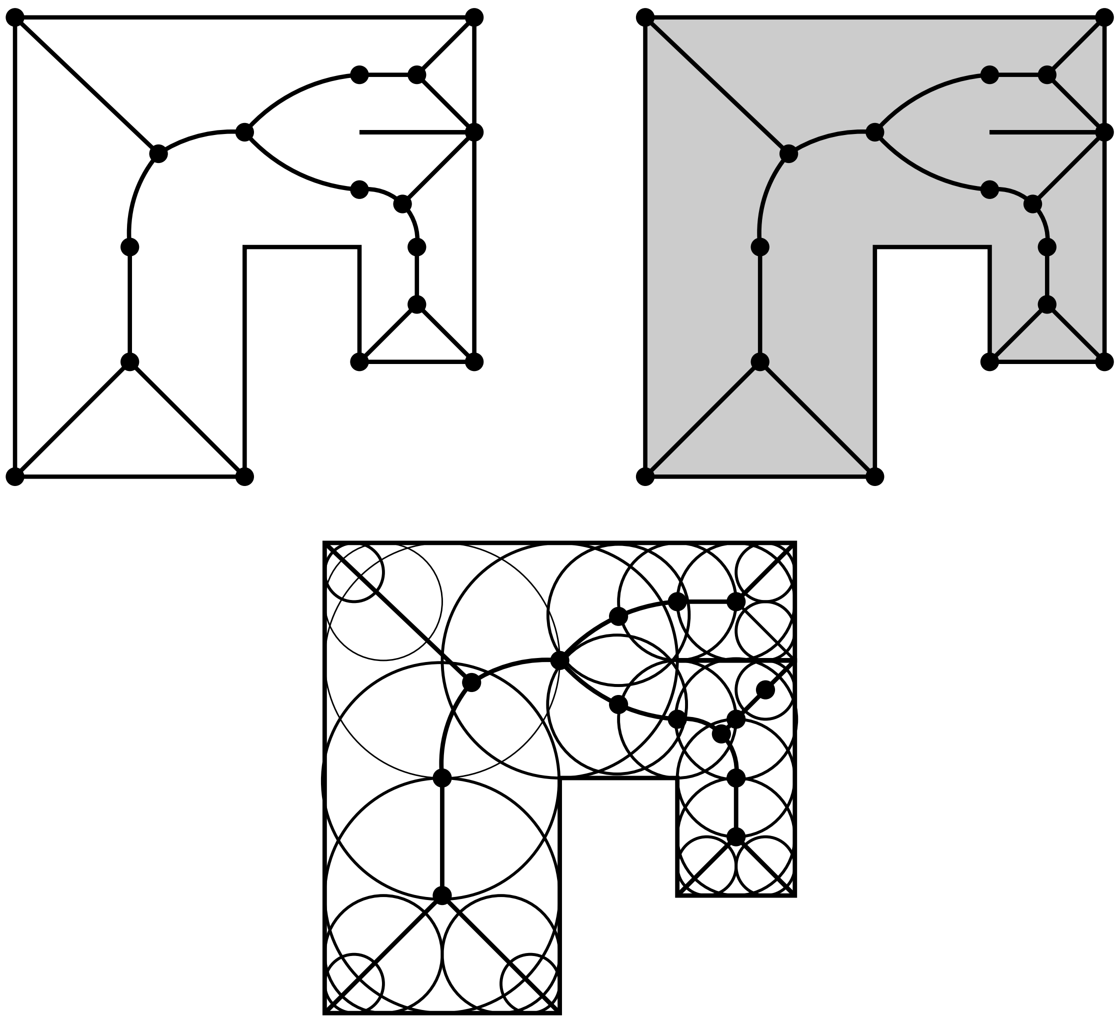


It is easy to map any dome conformally to a disk.

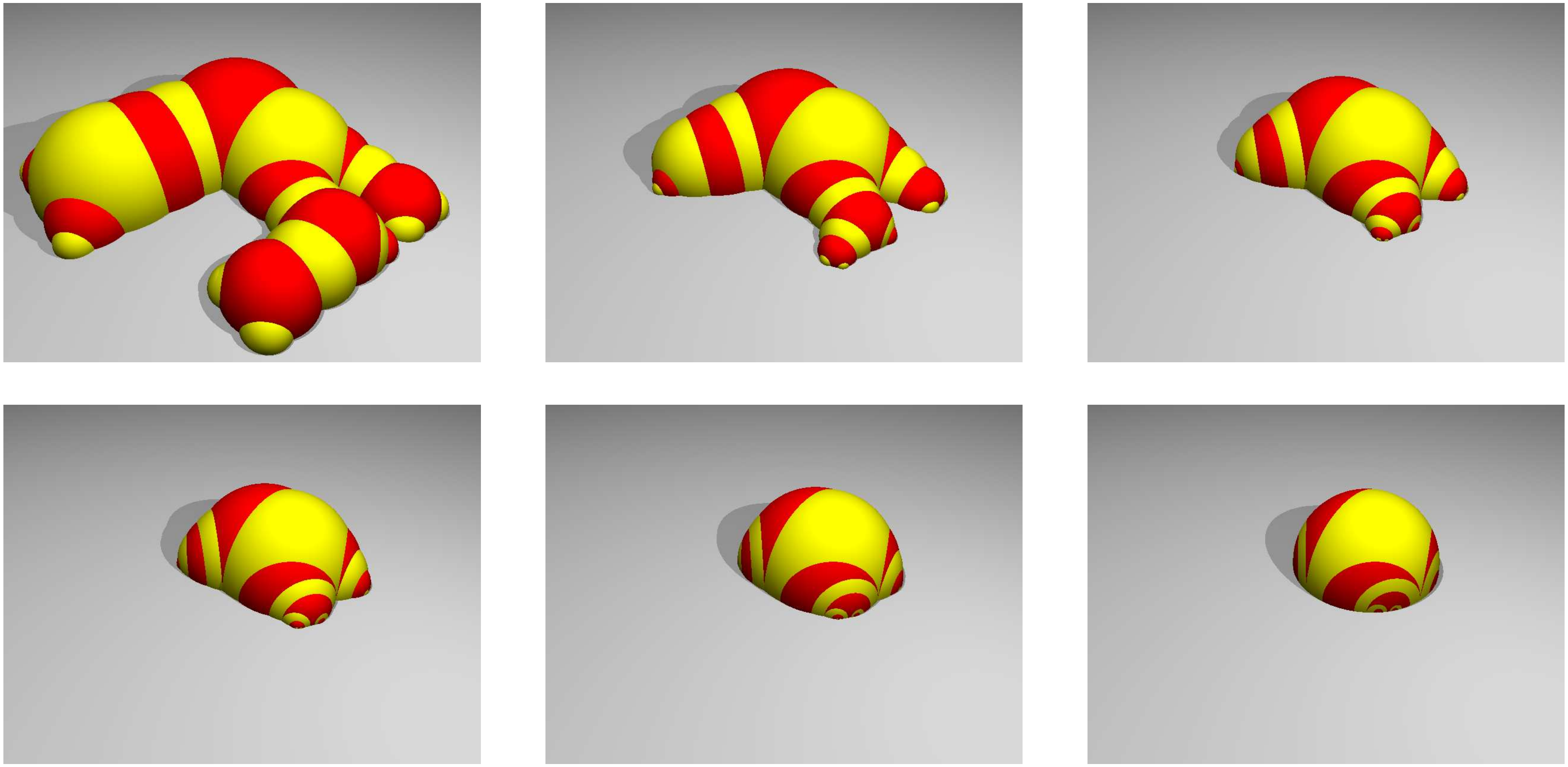


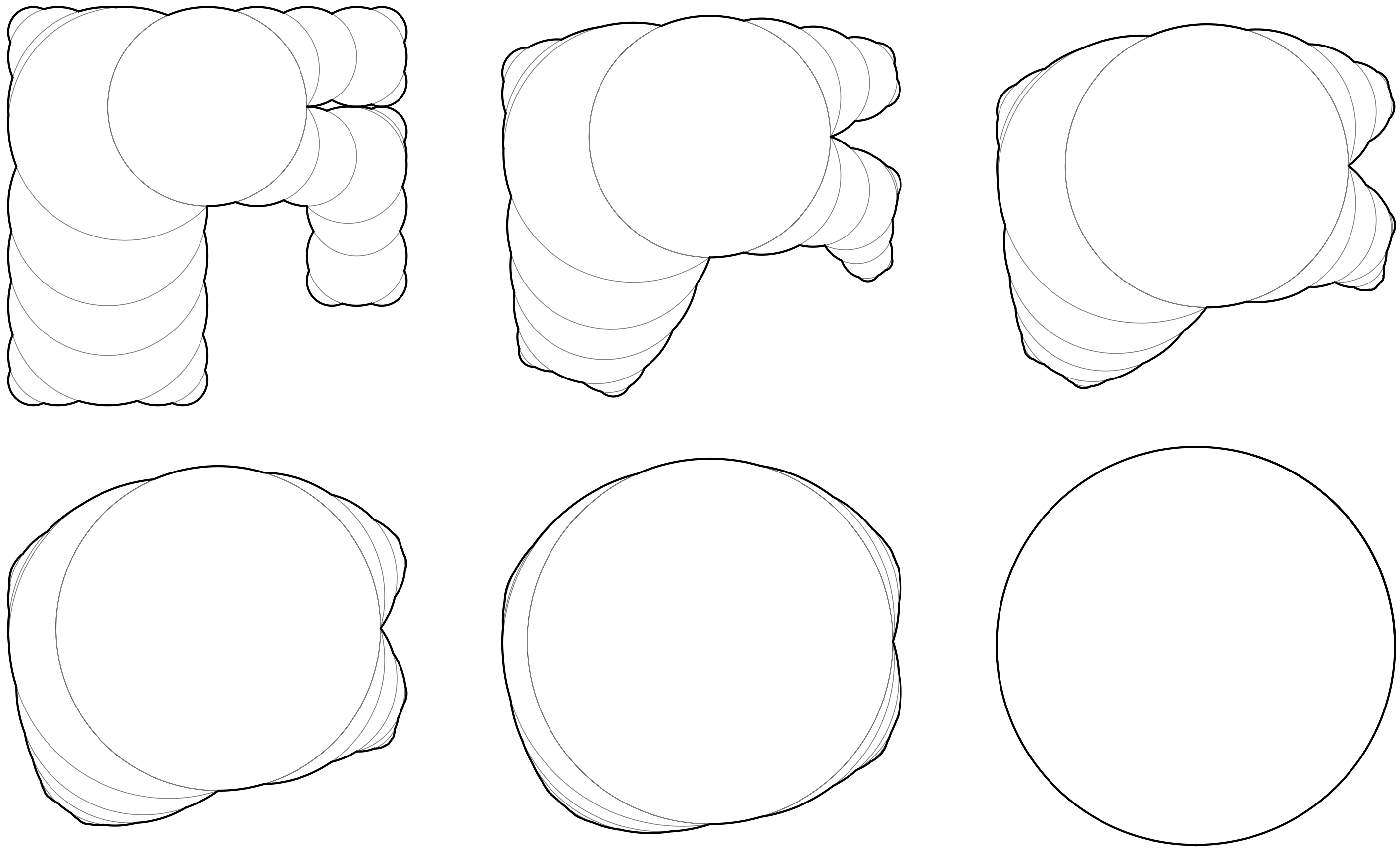
On base, foliate crescent by orthogonal arcs. Points in plane move along these arcs.





A polygon, medial axis, approximation by disks.



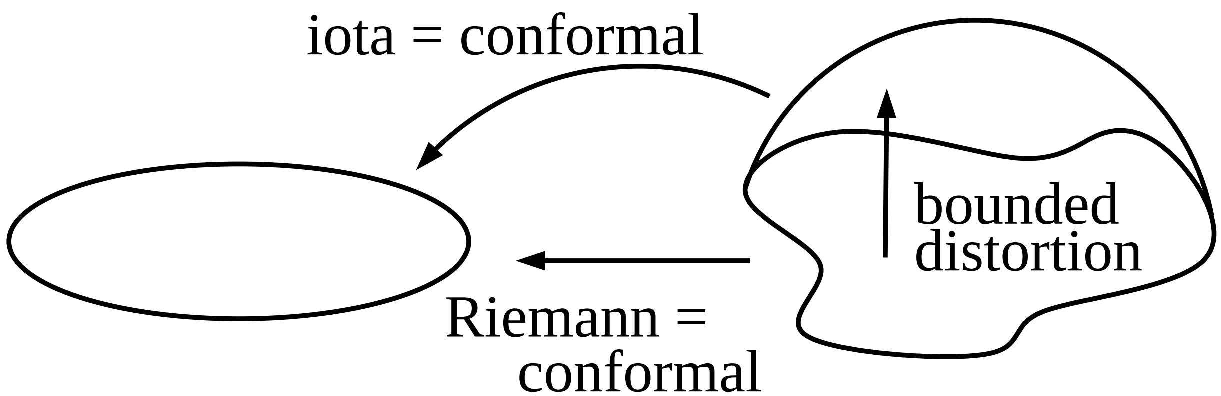


Angle scaling family

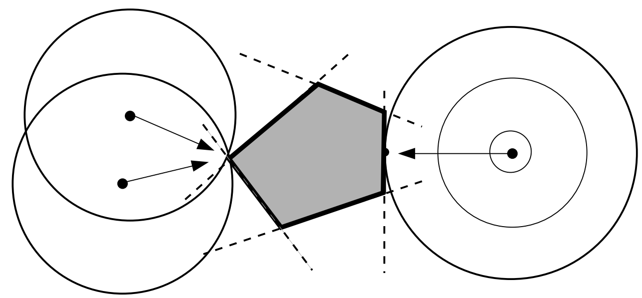
Iota = conformal map from dome to disk.

Riemann = conformal from base to disk

From base to dome = Sullivan's convex hull theorem

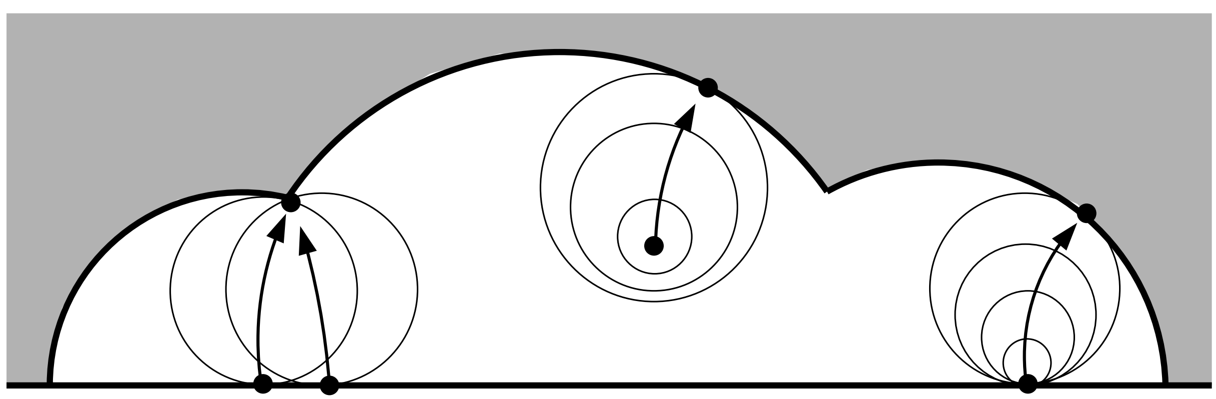


Nearest point map in $\mathbb{R}^n$ is Lipschitz.



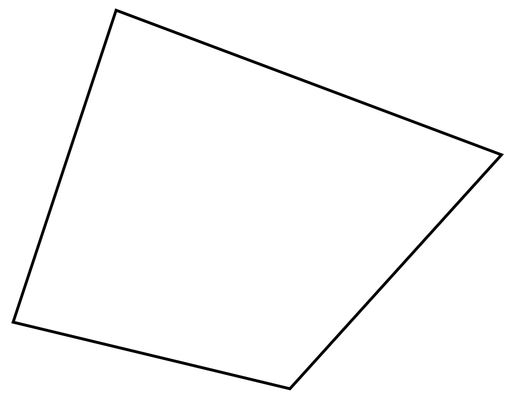
Nearest point retraction in hyperbolic space extends to map $R : \Omega \to S =$ Dome and is a quasi-isometry (Dennis Sullivan, David Epstein and Al Marden, C. Bishop)

$$\frac{1}{A}\rho_\Omega(x, y) - B \leq \rho_S(R(x), R(y)) \leq A\rho_\Omega(x, y).$$



Some special constructions needed to merge thick and thin meshes.

Vertices propagate via linear interpolation.



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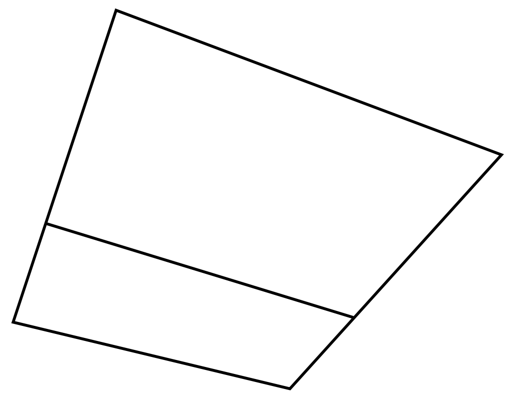
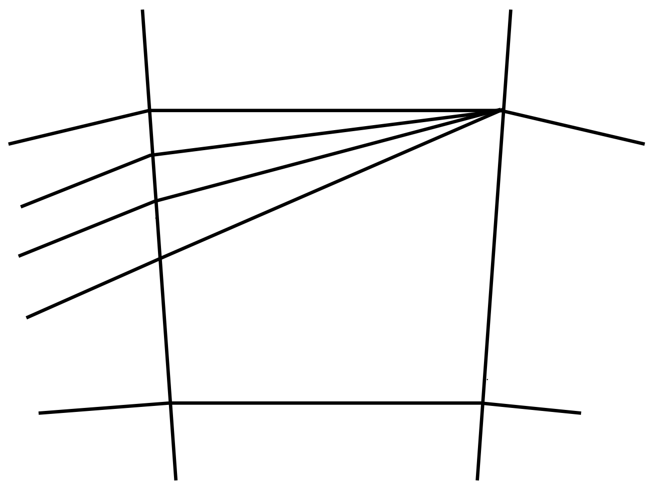
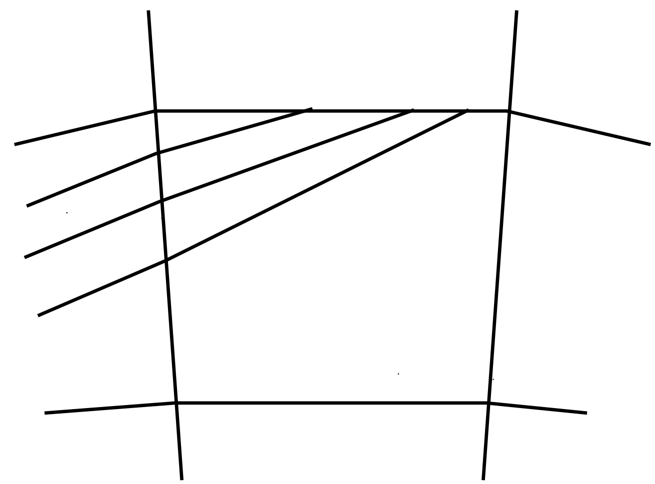


This preserves angle bounds.

In thin parts propagations paths are "bent" to hit tube wall instead of vertex.

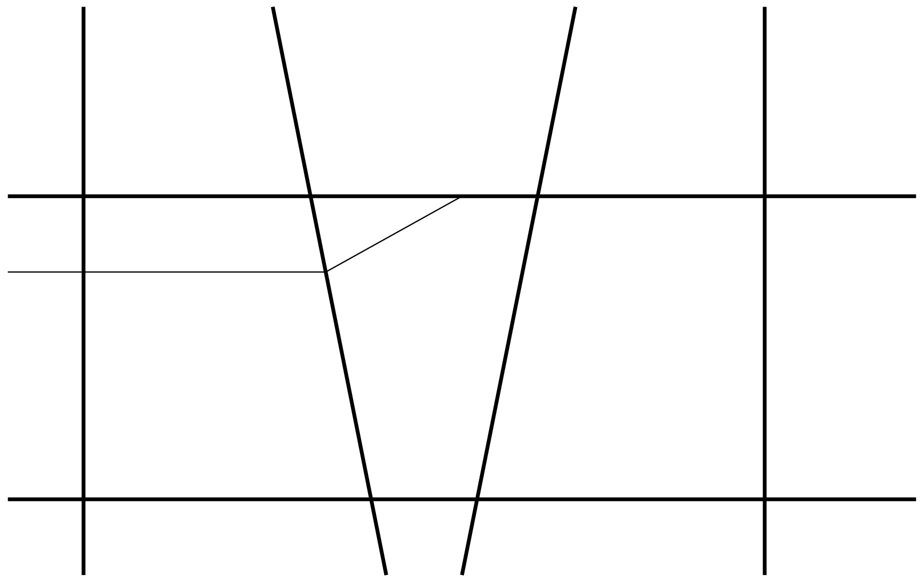


nonobtuse triangulation



quadrilateral meshing

Consider one such line.



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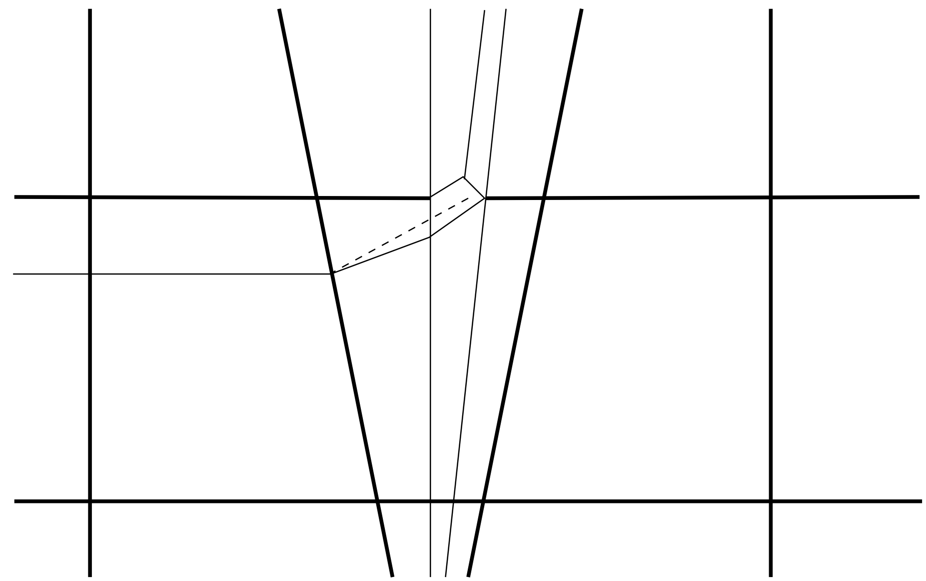

Construction "bends" line by about 90°. Path continues until it hits thick part.

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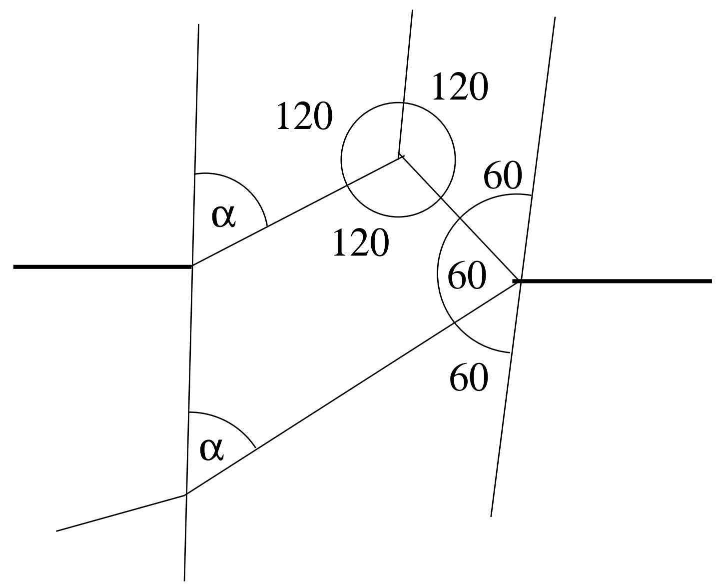

Enlargement of bending construction.

A **sink** is a polygon so that if we add an even number of vertices to boundary, it can quad meshed with angles in $[60°, 120°]$ and no new boundary vertices.

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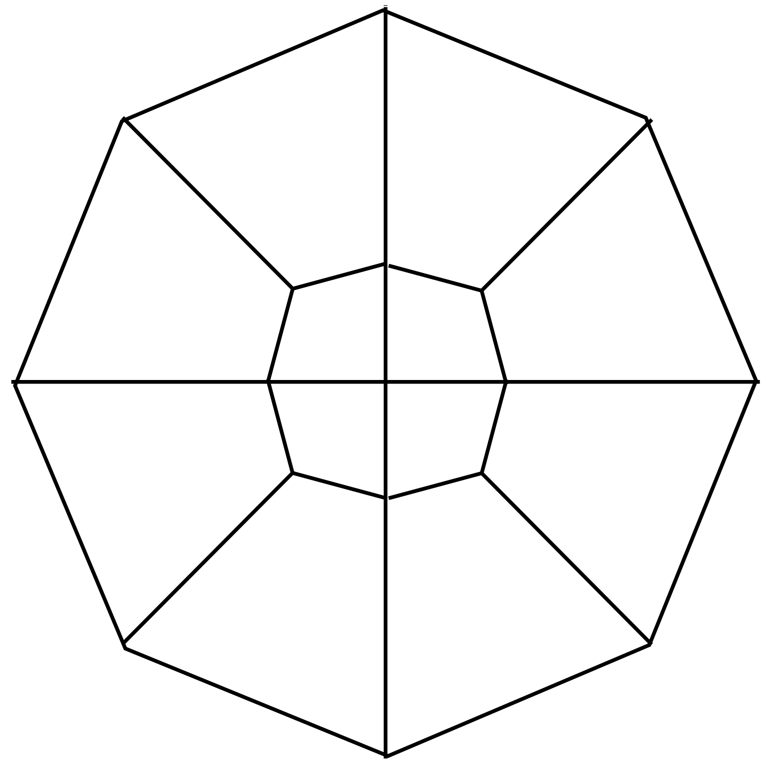
A simple polygon with a quad mesh must have an even number of boundary vertices.

Lemma: An octagon is a sink.



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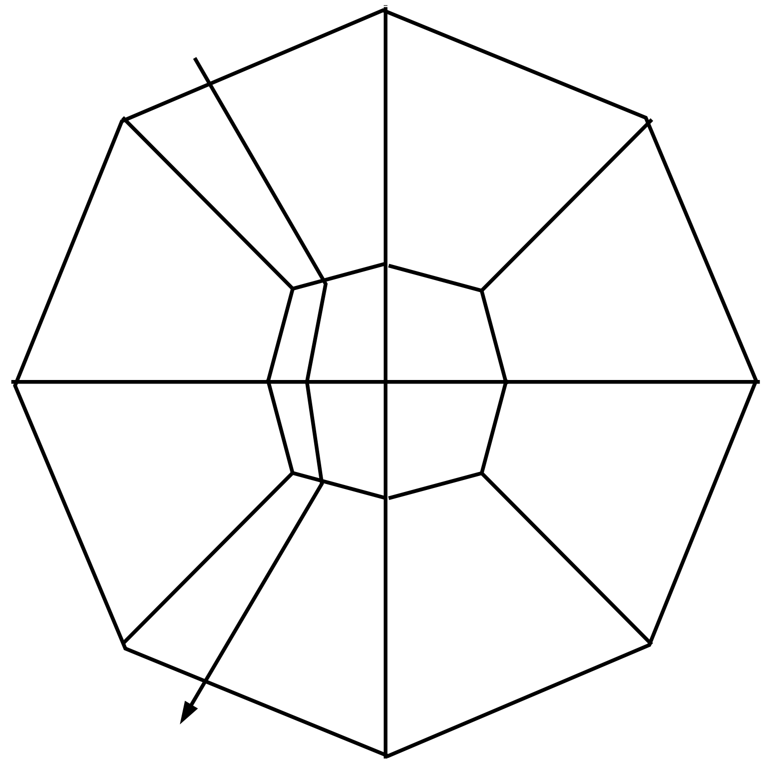

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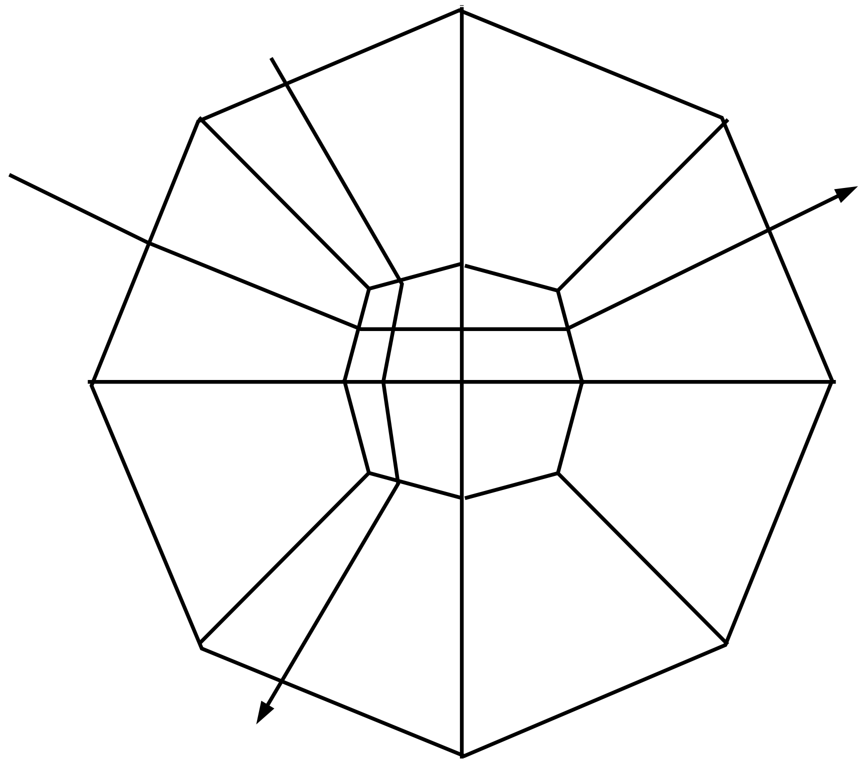

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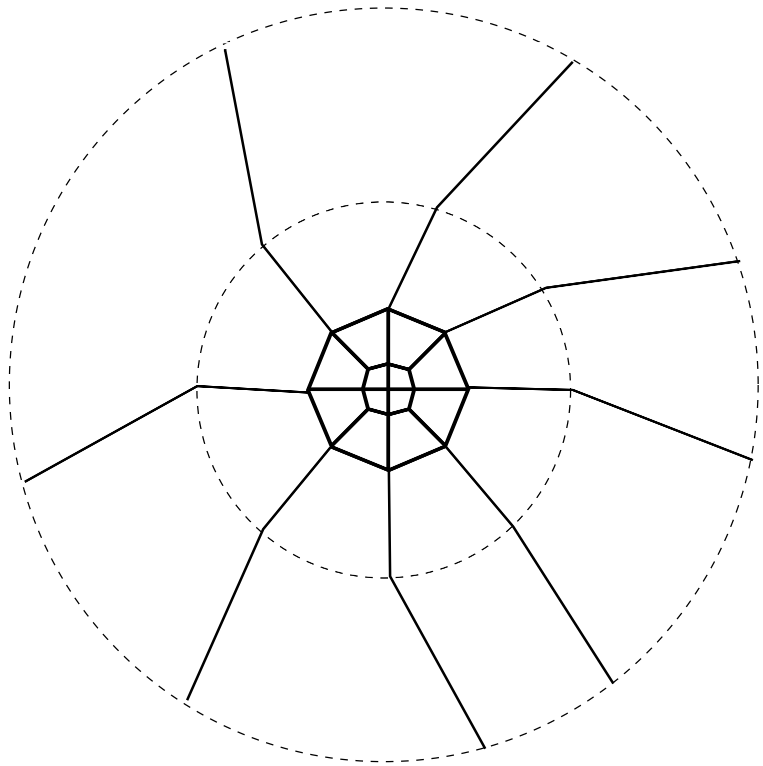

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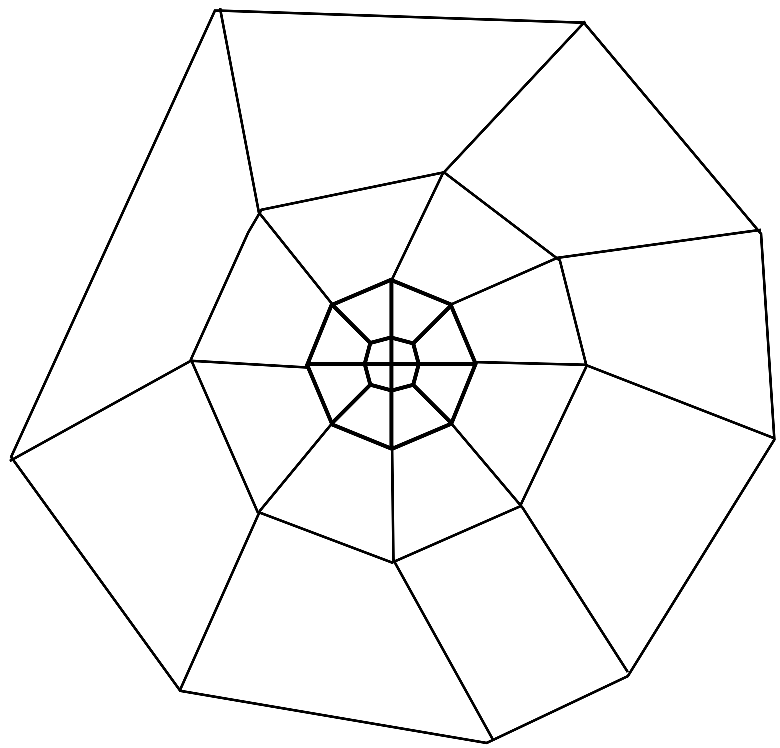

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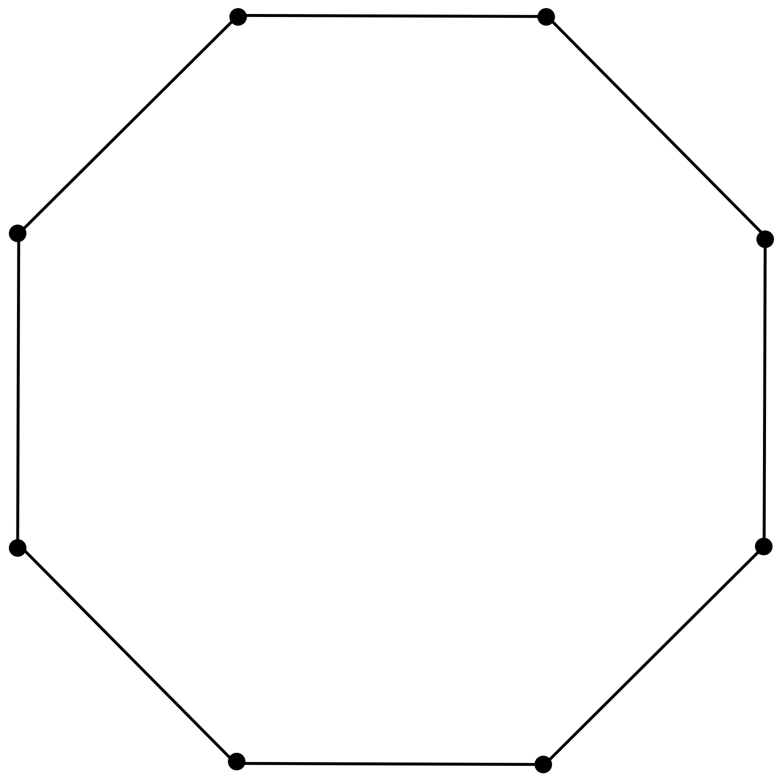

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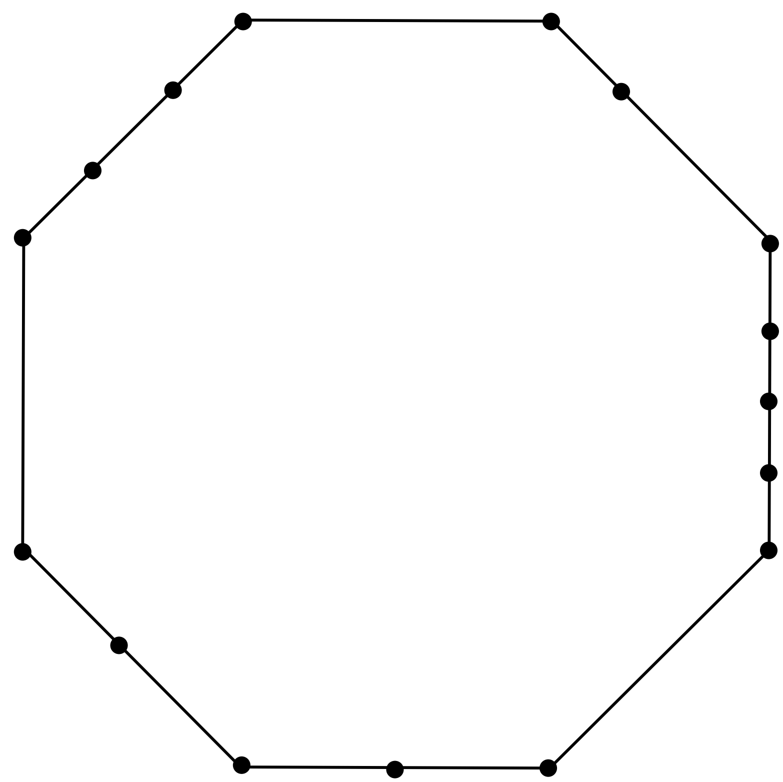

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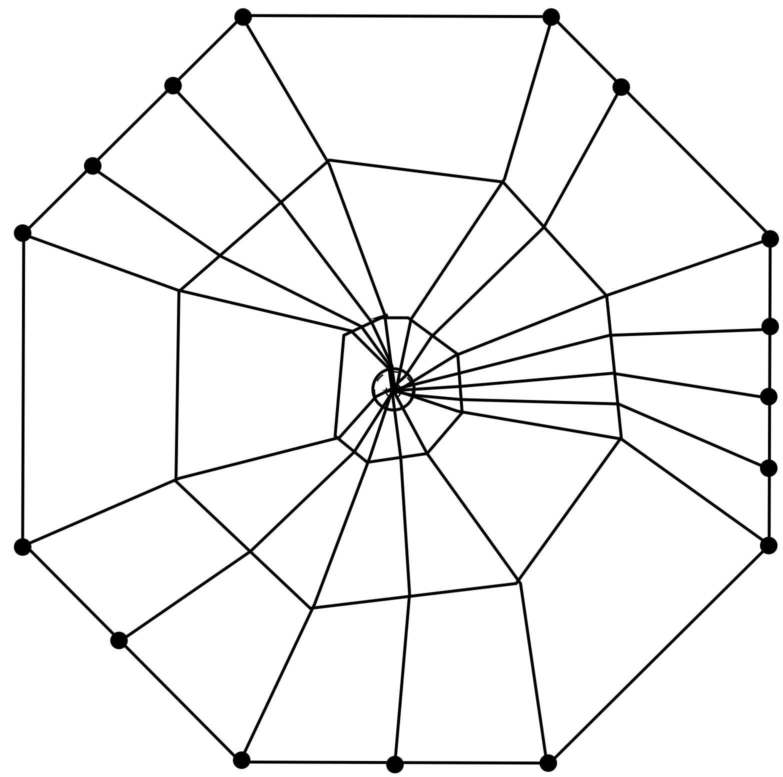

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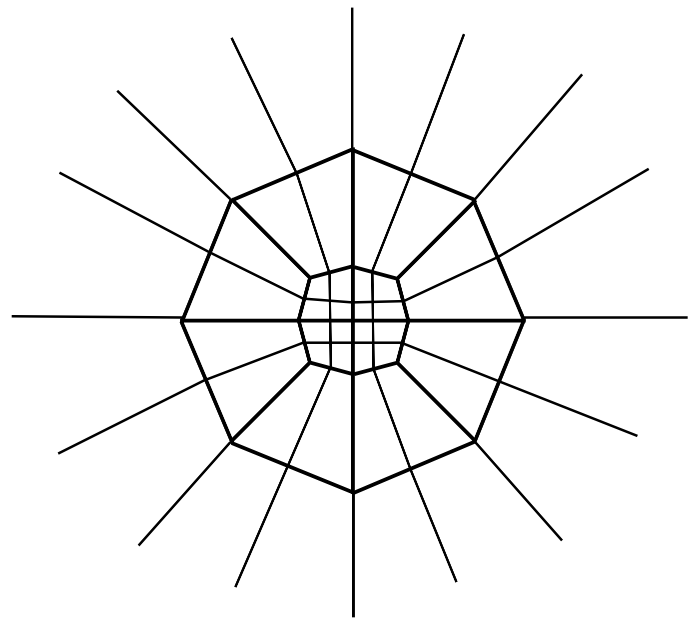

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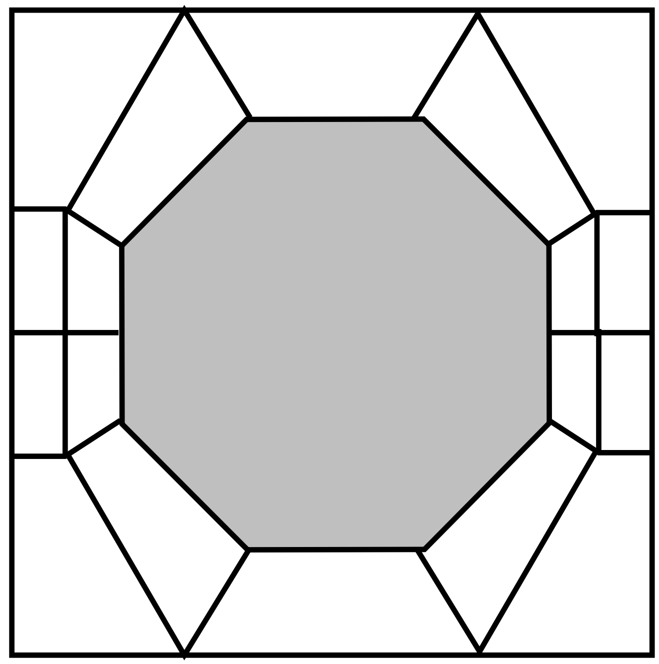

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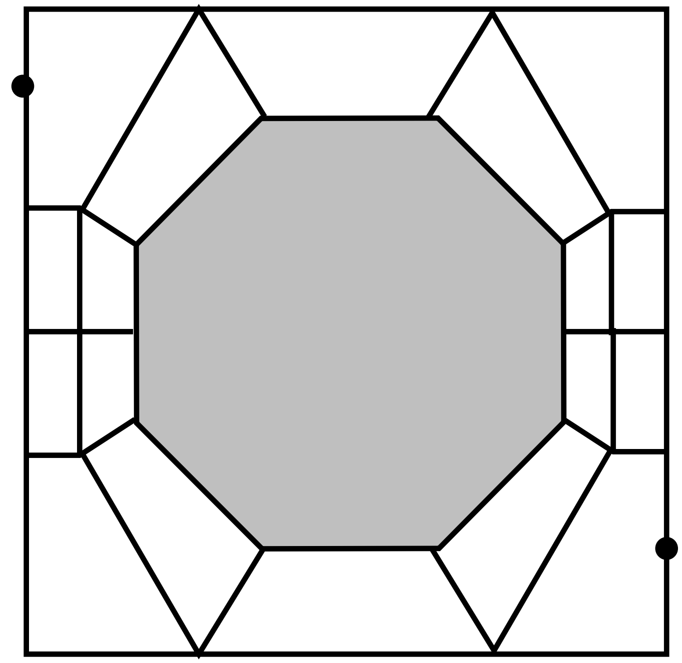

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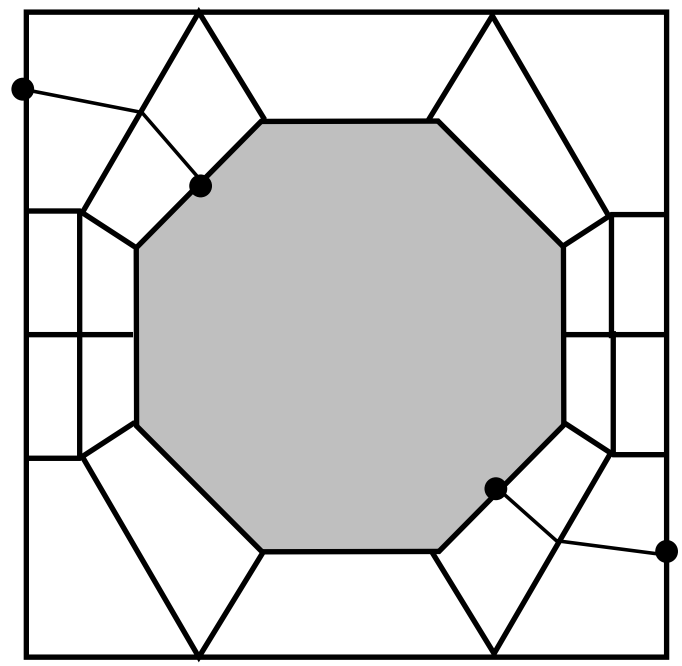

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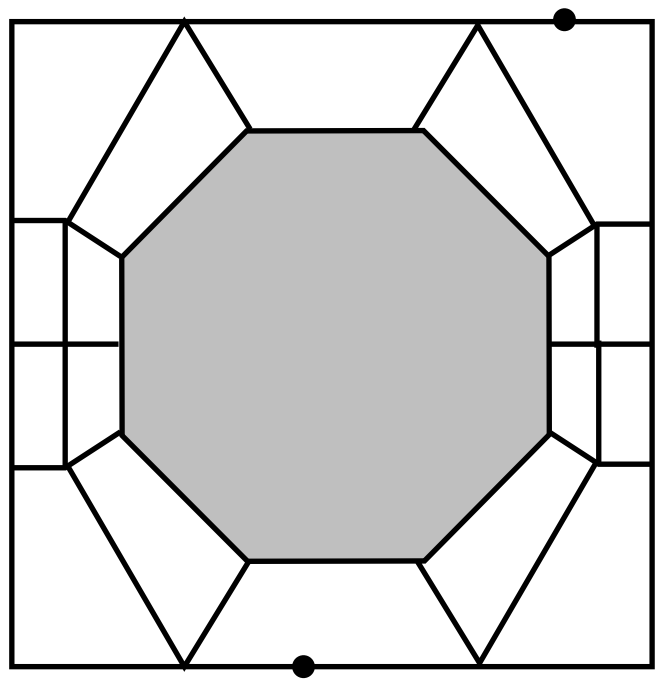

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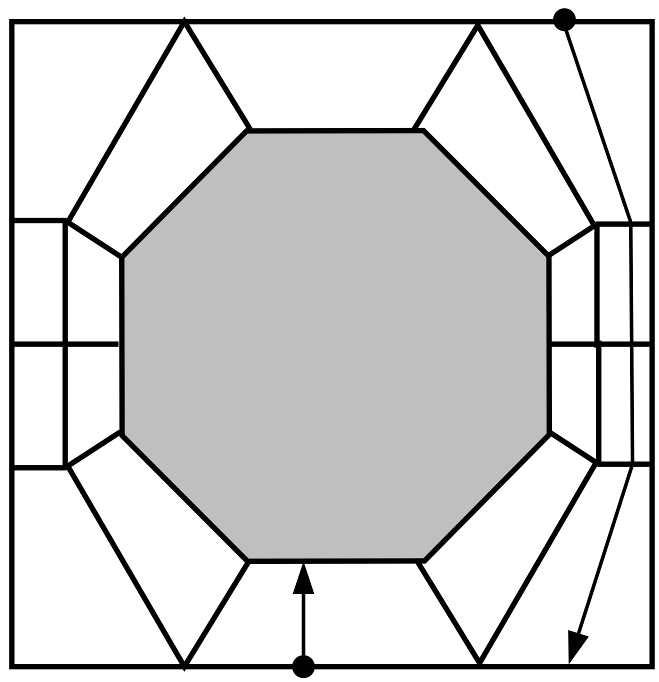

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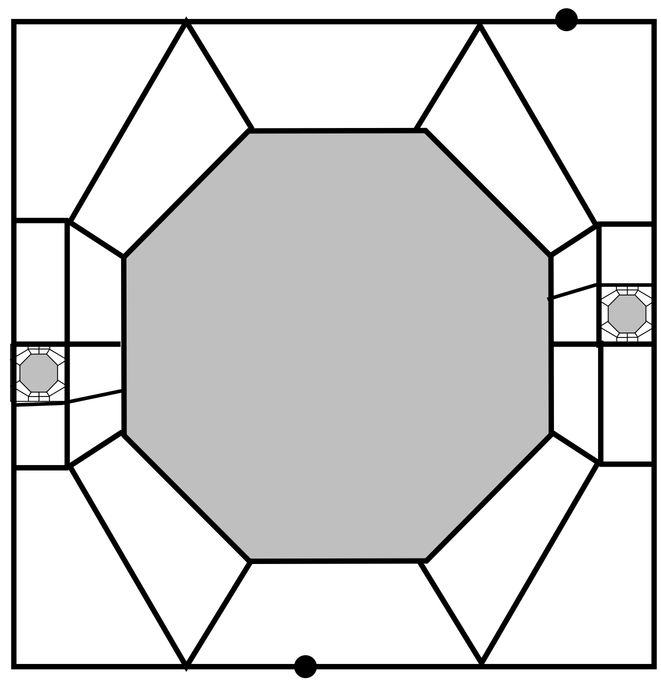

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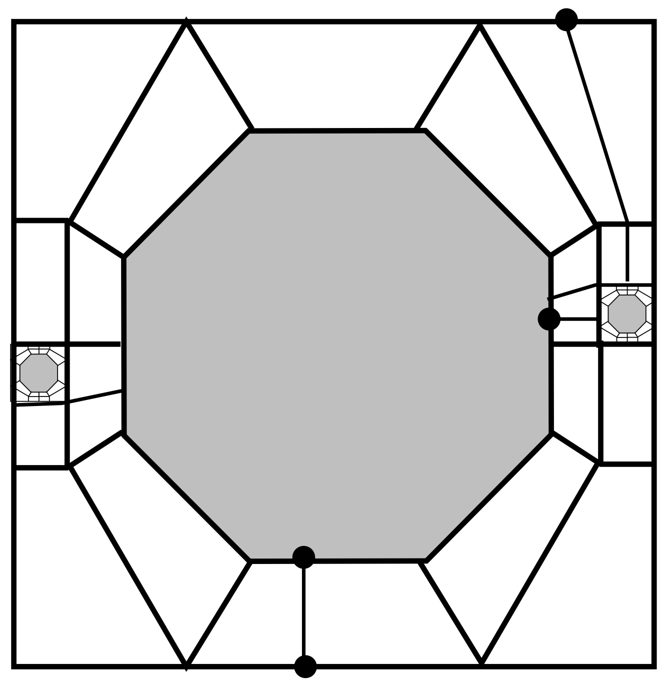
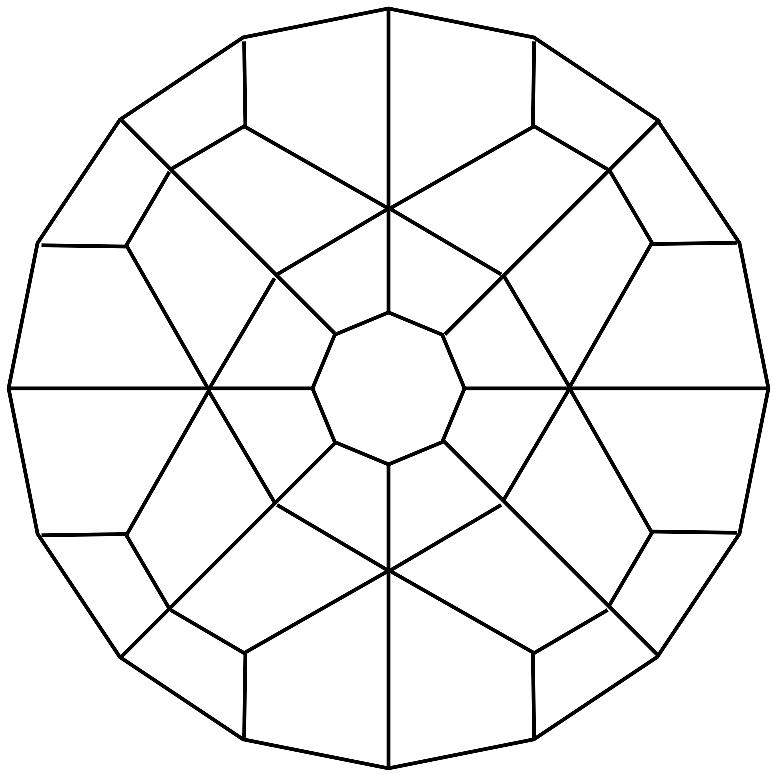


Every quad mesh can be doubled by connecting mid-points.



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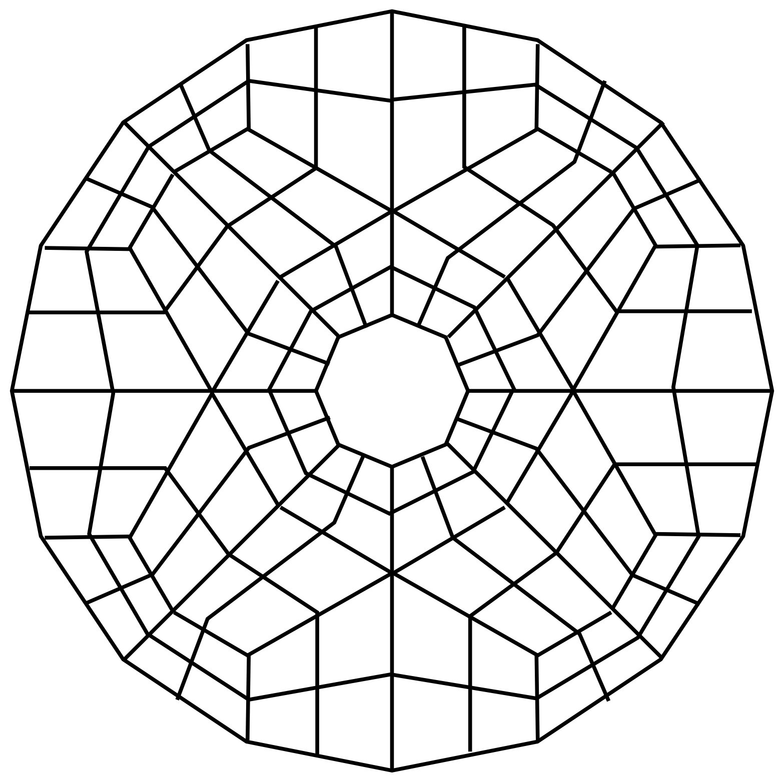
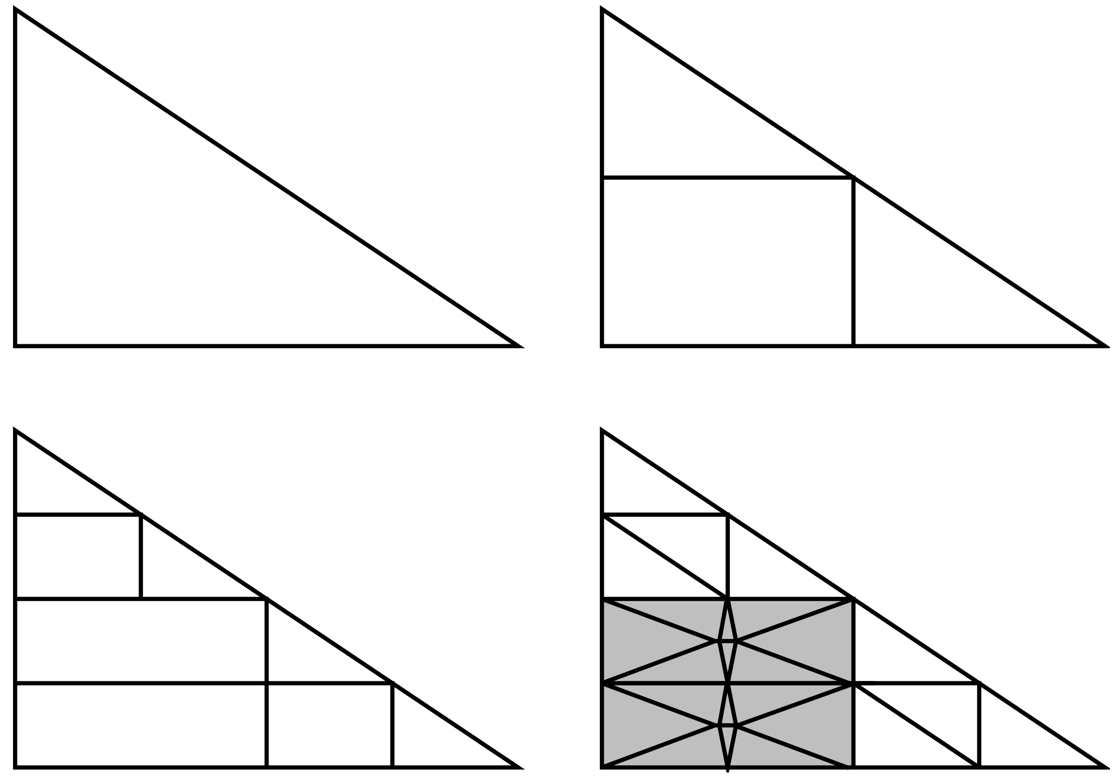
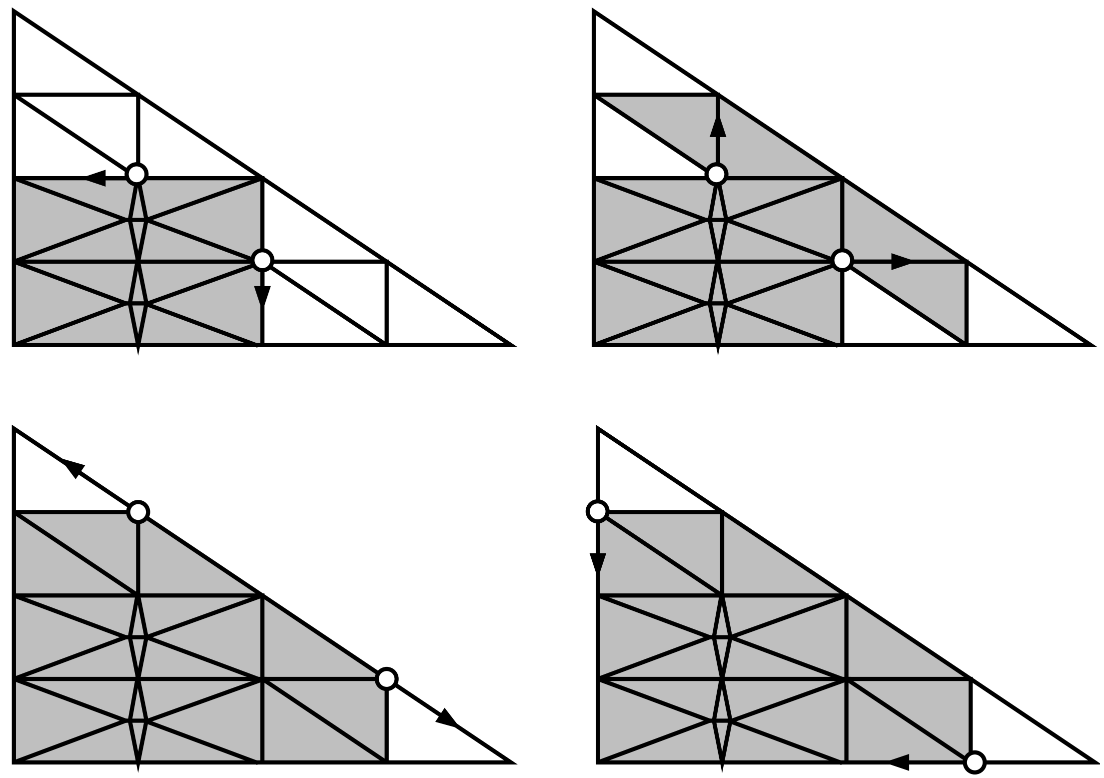
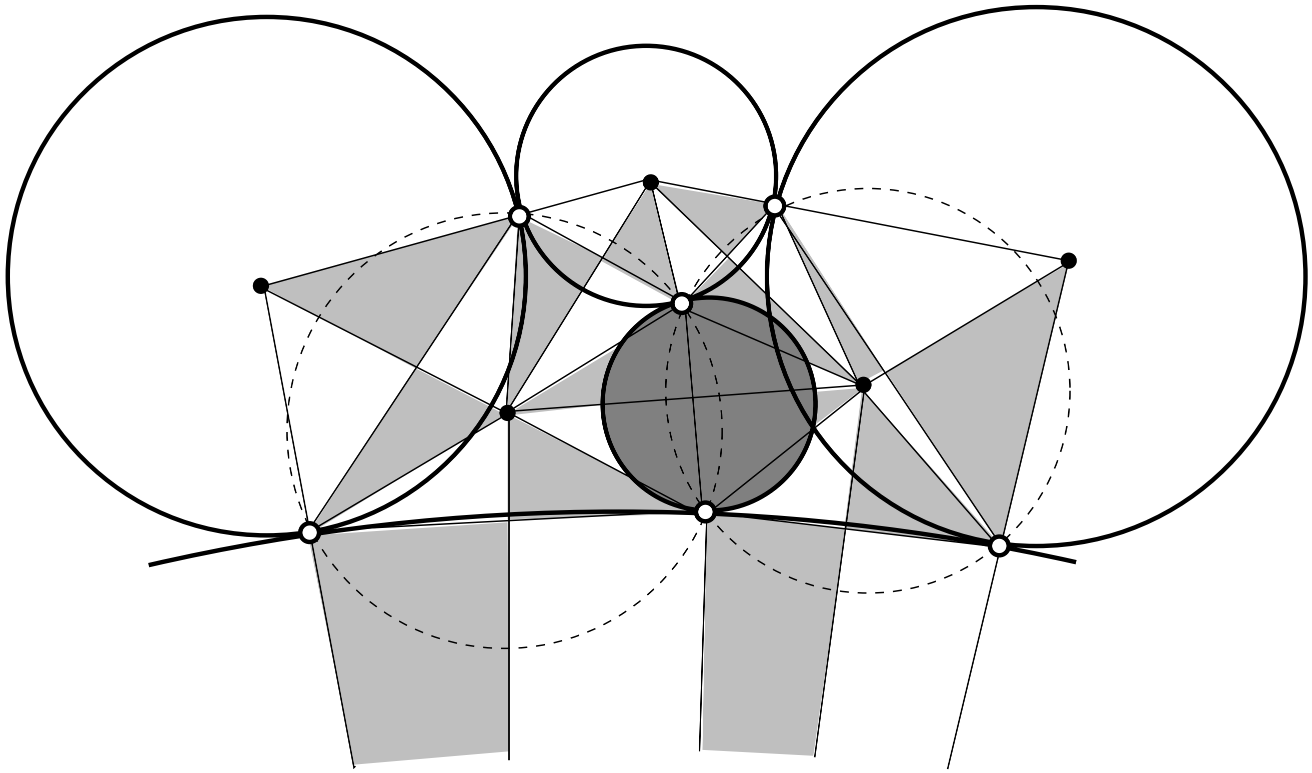


Doubling allows us to insure every sink has a even number of boundary vertices.





The 4-regions are similar (but several cases arise).



So Gabriel covering implies a nonobtuse triangulation.

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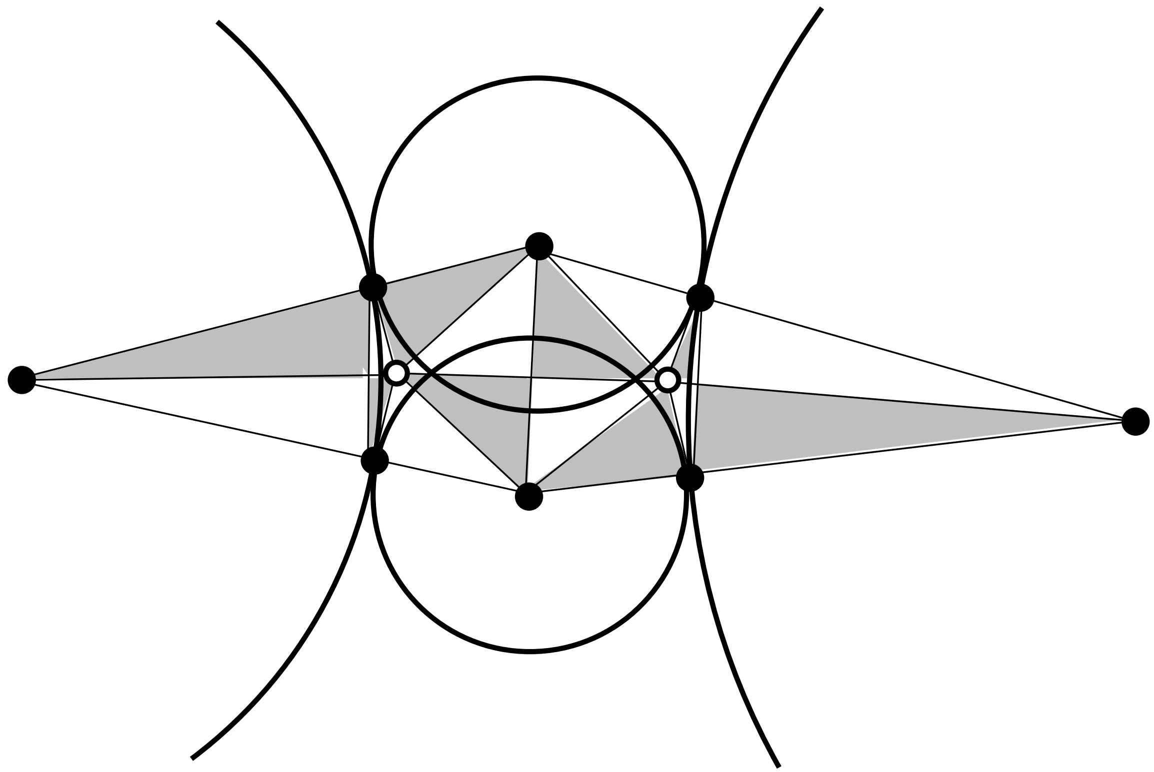
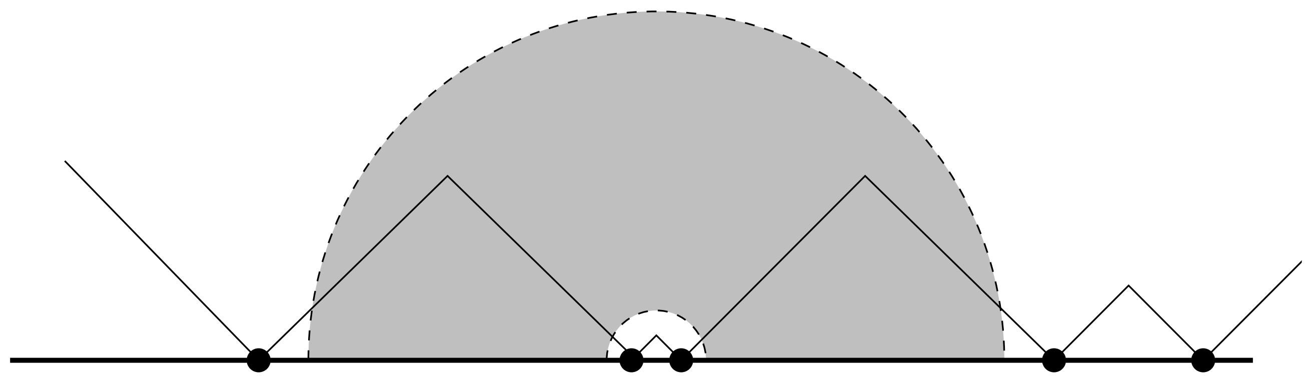


So Gabriel covering implies a nonobtuse triangulation.

Thin parts computable in $O(n)$ using conformal map.



- Map polygon conformally to half-plane.
  Vertices map to points on line.
- Thin parts = wide annuli separating vertices.
- Draw sawtooth domain, valleys at vertices.

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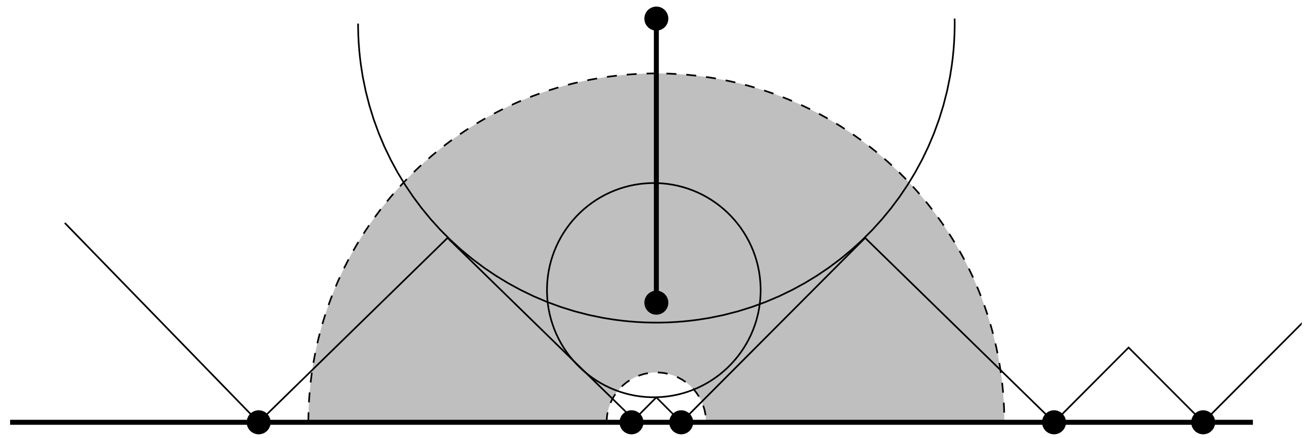

- Map polygon conformally to half-plane.
  Vertices map to points on line.
- Thin parts = wide annuli separating vertices.
- Draw sawtooth domain, valleys at vertices.
- Compute medial axis (internal maximal disks).
  Find long (hyperbolically) vertical edges.