Counting on Coincidences

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It's easier to compute the probability that everyone has a different birthday.

Let $P(N)$ be the probability that N random people have different birthdays.

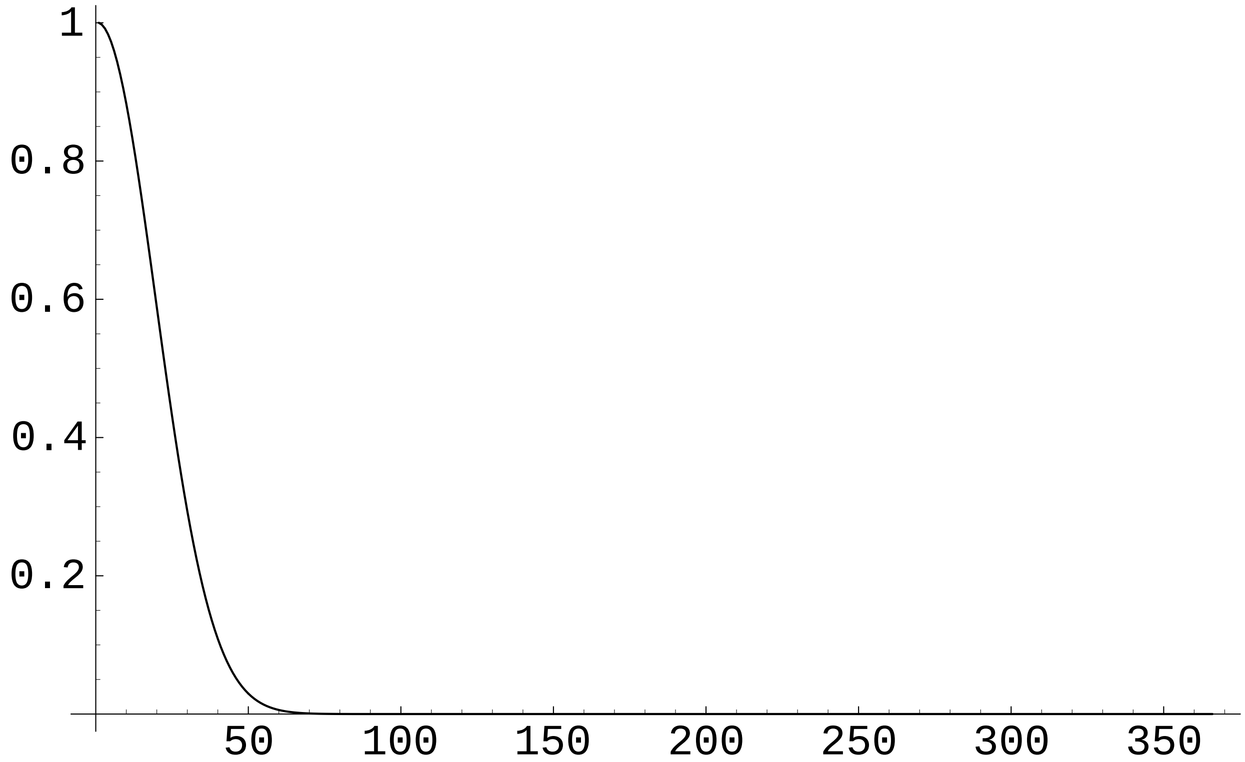
$$P(1) = 1$$

$$P(2) = 1 \cdot \frac{364}{365} \approx .99726$$

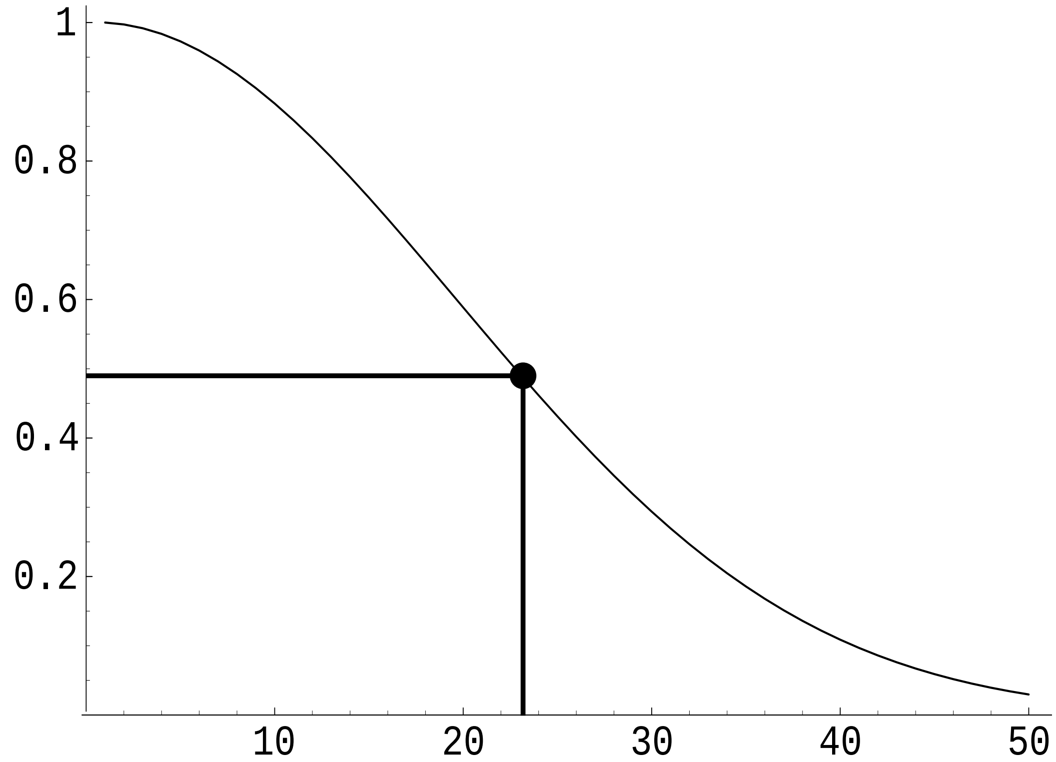
$$P(3) = 1 \cdot \frac{364}{365} \cdot \frac{363}{365} \approx .991796$$

$$P(4) = 1 \cdot \frac{364}{365} \cdot \frac{363}{365} \cdot \frac{362}{365} \approx .983644$$

$$P(N+1) = P(N) \cdot \frac{365-N}{365}$$



Probability of different birthdays for N people



Probability of different birthdays, $N \leq 50$

$P(22)$	≈ 0.524	$P(50)$	≈ 0.0296
$P(23)$	≈ 0.492	$P(60)$	≈ 0.00587
$P(30)$	≈ 0.293	$P(70)$	≈ 0.000840
$P(40)$	≈ 0.108	$P(100)$	≈ 0.000000307

$$
\begin{aligned}
P(365) &\approx 1.45495521563 \times 10^{-157} \\
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\end{aligned}
$$

A lottery sells a million tickets each day and chooses one winner every day.

What is the chance that someone wins twice in a year?

(a) about 1 in a trillion
(b) about 1 in a billion
(c) about 1 in a million
(d) about 1 in a thousand
(e) about 1 in a hundred
(f) about 1 in ten
(g) about 50-50

(Assume same million players every day.)

Answer: Probability of 365 different winners is

$$1 \cdot \frac{999,999}{1,000,000} \cdots \frac{999635}{1000000} \approx .9353.$$

6.5% chance of a double winner in one year.

45% chance of double winner in 3 year period.

99.87% chance of a double winner in 10 years.

Suppose we randomly put K balls into N boxes. What is the chance that no box has more than M balls in it?

Call this probability $P(K, N, M)$.

The Birthday Problem is computing $P(K, 365, 1)$.

MIDTERM: What is $P(14400, 9000, 7)$?

(a) .0000000000000000000000000132
(b) .095395
(c) .664954
(d) .999323
(e) .999999999999999999999999845

If we drop 14,400 balls into 9000 boxes, what is the chance no box has more than 7 balls in it?

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Why is this an important example?

In 1960 there were 14,400 cases of leukemia in US and 8 cases in Niles, IL, population 20,000.

The average for a town this size would be 1.6 cases.

Is the cluster random?

Population of US in 1960 was

$$180,000,000 = 9,000 \times 20,000.$$

Divide population into 9,000 “boxes” of 20,000 each.

Drop in 14,400 “cases”.

What is the chance that some box has 8 cases?

Answer $= 1 - P(14400, 9000, 7)$.

Calculation gives $P(14400, 9000, 7) = .095395$

The probability of some town of size 20,000 having 8 cases at random is about 90%.

N	Probability biggest cluster $\leq N$	Probability biggest cluster $> N$
6	.000005	.999995
7	.095395	.904605
8	.664954	.335046
9	.937864	.062137
10	.990843	.009157
11	.998788	.001212
12	.999852	.000148

A town of 20,000 with 8 or 9 cases is highly likely.

Probability of 11 cases at random is about 1%.

A counting problem:

A bag contains N numbered balls. You pull one out, look at it and put it back. Continue until you know how big N is. How long does this take?

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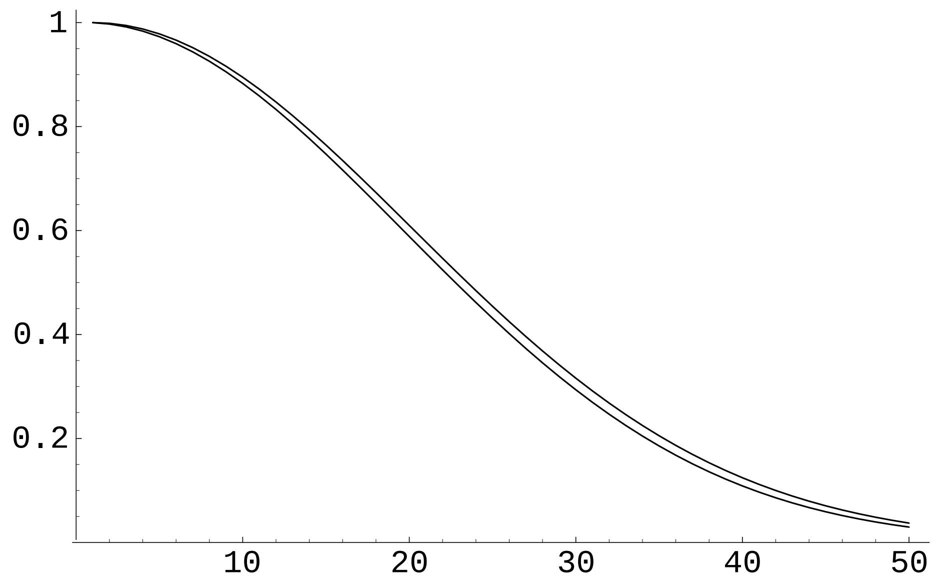
Answer: $\approx \sqrt{N}$

N balls into K boxes. Chance of "no repeats":

Exact Formula:

$$P(N) = 1 \cdot \frac{K-1}{K} \cdots \frac{K-N+1}{K}$$

Approximate Formula: $P(N) \approx e^{-N^2/2 \cdot K}$



Comparing exact and approximate formulas

How long before a 50% chance of a repeat?

Must solve

$$e^{-N^2/2K} = .5$$

$$\frac{-N^2}{2K} = \log .5$$

$$N \quad = \quad \sqrt{2K \log 2} \quad \approx \quad 1.17741\sqrt{K}$$

If we draw $N \approx \sqrt{K}$ random samples (with repetition) from a bag with K items, we have a good chance to get a repeat.

Counting balls in a bag problem

Rough guess:

If first repeat in on the nth draw from the bag, guess $K = (n/1.1774)^2$ as number of balls in bag.

Repeat and take average for better estimate.

Better solution:

Examine and return m samples.

Let t be total number of repeats.

Estimate $K \approx \frac{m(m-1)}{2t}$

Estimate is probably accurate if $m \gg \sqrt{K}$.

Example: I picked K distinct 3 digit numbers, then drew 30 random samples:

231 903 979 705 153 231 402 231 540 716

930 836 386 836 659 671 284 114 814 716

317 588 386 358 143 660 392 588 979 553

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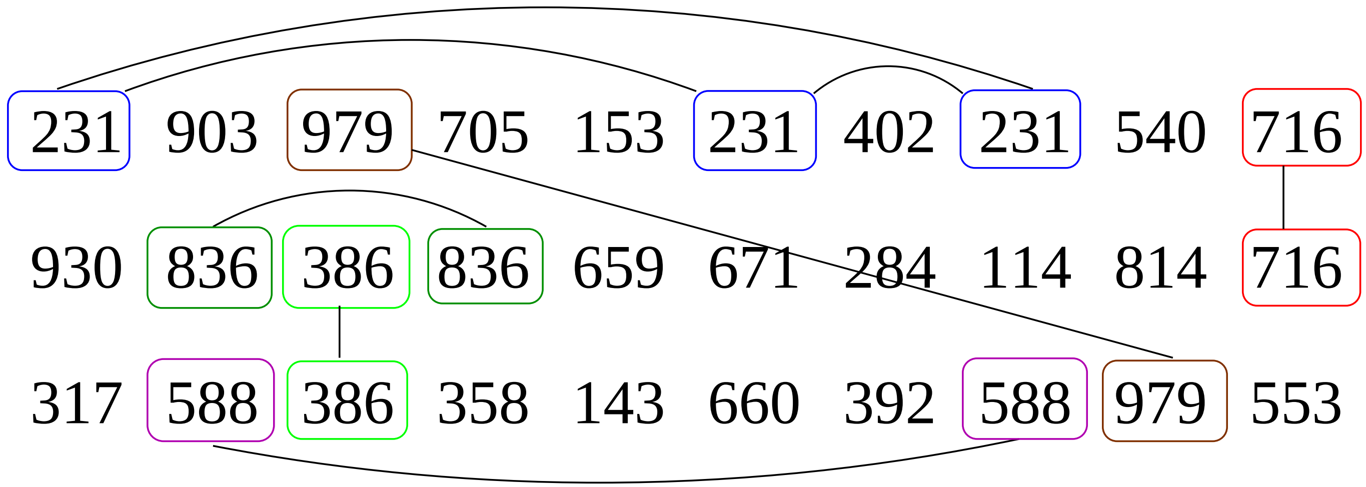

There are $m = 30$ samples and $t = 8$ pairs of repeated numbers, so our guess is

$$\frac{m(m-1)}{2t} = \frac{30 \cdot 29}{16} \approx 54.375.$$

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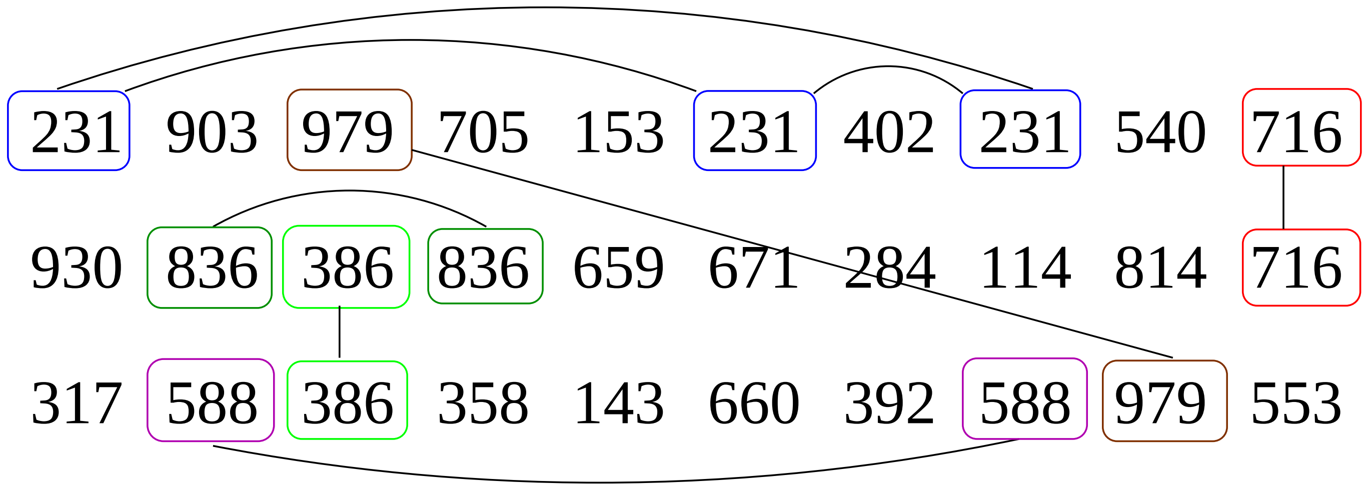

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The true K is 57.

Applications:

Counting fish in lake

Counting distinct users on internet

Cryptography (security of digital signatures)

Factoring integers (Pollard rho method)

Many others

FINAL EXAM

Two thieves steal N diamonds with random values between \$1 and \$1,000,000. Can they divide the loot into two piles of equal value?

Impossible if $N = 1$ and unlikely if $N = 2, 3, \ldots ?$.

There is a 50% chance even splitting if $N >?$

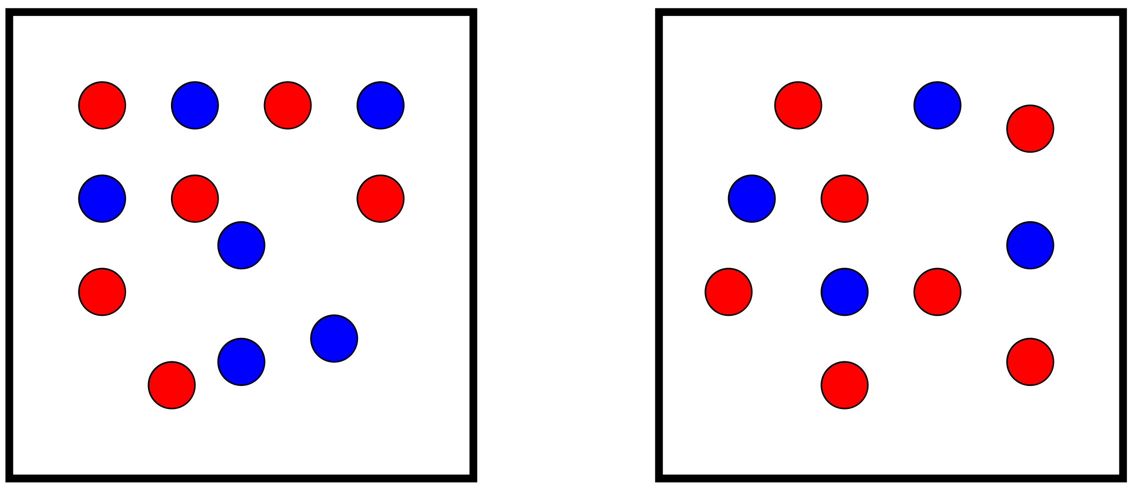
(a) 11
(b) 25
(c) 78
(d) 979
(e) 10,122

Divide diamonds into two groups of size $N/2$.

Randomly divide 1st group into red and blue subsets.

Let R_1, B_1 be the value of each subset.

Let $D_1 = R_1 - B_1$. Similarly $D_2 = R_2 - B_2$.



$D_1 = R_1 - B_1$ $\qquad$ $D_2 = R_2 - B_2$

There are $2^{N/2}$ ways to obtain D_1. Same for D_2.

If $D_1 = D_2$ then

$$R_1 - B_1 = R_2 - B_2,$$

$$R_1 + B_2 = R_2 + B_1,$$

so we get a division into two equal parts.

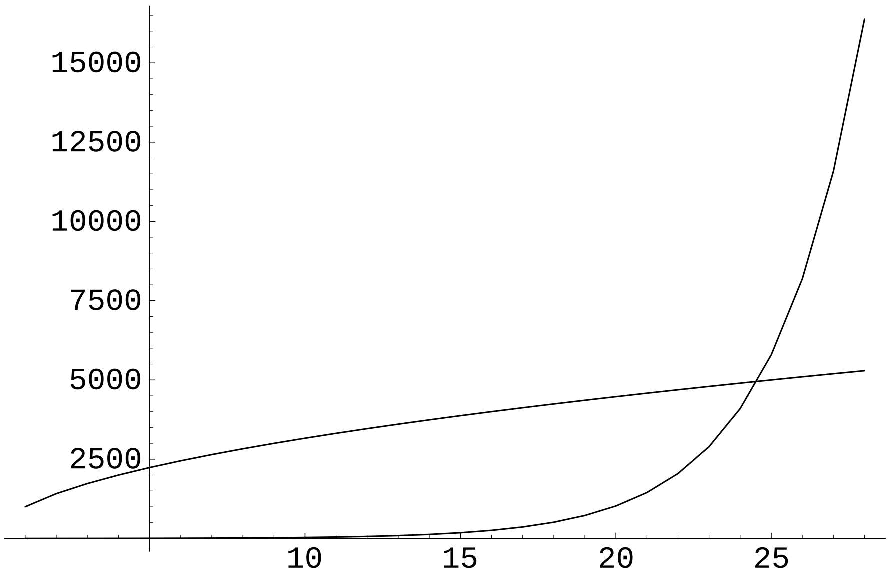
What is the chance $D_1 = D_2$?

What is chance of a repeat among $2^{N/2}$ random numbers of size between $-\frac{N}{2} \cdot 1,000,000$ and $\frac{N}{2} \cdot 1,000,000$?

By Birthday Problem the odds $\approx$ 50-50 if

$$\#\text{ random choices} \approx \sqrt{\#\text{ possible choices}}$$

$$2^{N/2} = \sqrt{N \times 1{,}000{,}000}$$



Answer: An equal division is likely if $N \geq 25$.

Number Partition Problem: given N integers can we divide them into two subsets with same sum?

This is an **NP-hard problem**. Roughly requires checking $\approx 2^N$ possible subsets in worst case.

Randomly choose N numbers in the range $[1, 2^{\kappa N}]$:

- there usually is an equal division if $\kappa < .96$
- there is usually not an equal division if $\kappa > .96$

The Number Partition Problem is only NP-hard problem that has such a precise analysis. Studied by computer scientists, mathematicians and physicists.

Nicknamed the "easiest" NP-hard problem.

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