Counting on Coincidences

Christopher J. Bishop
SUNY Stony Brook

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It's easier to compute the probability that everyone has a different birthday.

Let $P(N)$ be the probability that N random people have different birthdays.

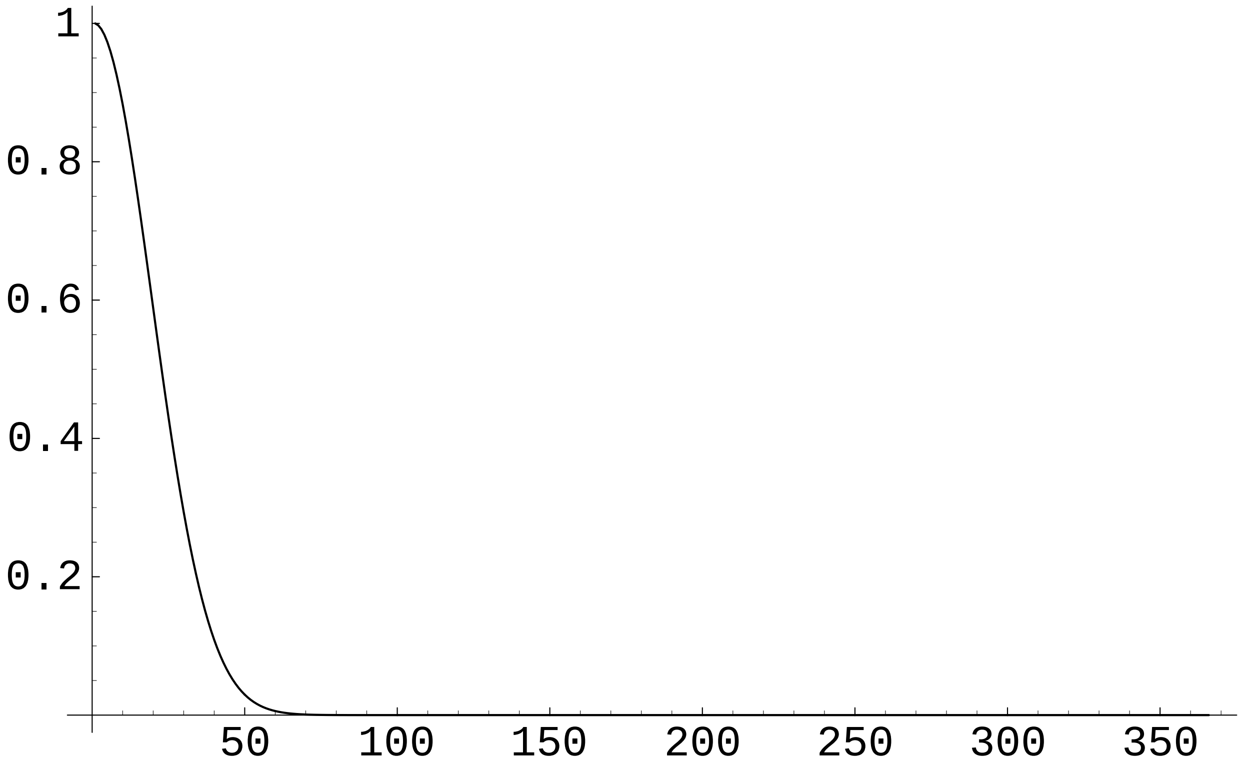
$$P(1) = 1$$

$$P(2) = 1 \cdot \frac{364}{365} \approx .99726$$

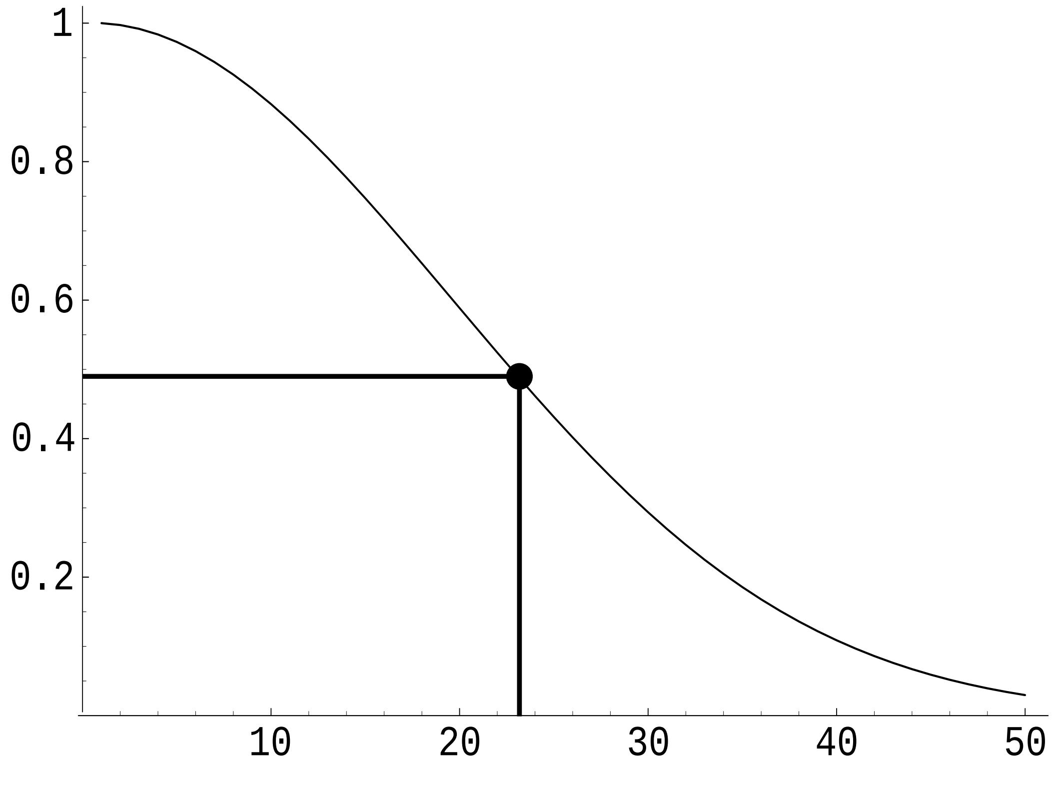
$$P(3) = 1 \cdot \frac{364}{365} \cdot \frac{363}{365} \approx .991796$$

$$P(4) = 1 \cdot \frac{364}{365} \cdot \frac{363}{365} \cdot \frac{362}{365} \approx .983644$$

$$P(N+1) = P(N) \cdot \frac{365-N}{365}$$



Probability of different birthdays for N people



Probability of different birthdays, $N \leq 50$

$P(22)$	≈ 0.524	$P(50)$	≈ 0.0296
$P(23)$	≈ 0.492	$P(60)$	≈ 0.00587
$P(30)$	≈ 0.293	$P(70)$	≈ 0.000840
$P(40)$	≈ 0.108	$P(100)$	≈ 0.000000307

$$
\begin{aligned}
\mathrm{P}(365) &\approx 1.45495521563 \times 10^{-157} \\
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\end{aligned}
$$

How many people do we need to have a 50-50 chance of having k people with same birthday?

number with same birthday	size of group needed
2	23
3	88
4	187
5	313
6	460
7	623
8	798
9	985
10	1181
11	1385
12	1596
13	1813

A lottery sells a million tickets and chooses one winner every day. What is the chance that someone wins the lottery twice in a year?

(a) about 1 in a trillion
(b) about 1 in a billion
(c) about 1 in a million
(d) about 1 in a thousand
(e) about 1 in a hundred
(f) about 1 in ten
(g) about 50-50

(Assume same million players every day.)

Answer: Probability of 365 different winners is

$$1 \cdot \frac{999,999}{1,000,000} \cdots \frac{999635}{1000000} \approx .9353.$$

About a 6.5% chance of a double winner.

45% chance of double winner in 3 year period.

99.87% chance of a double winner in 10 years.

Suppose we randomly put N balls into K boxes. What is the chance that no box has more than M balls in it?

Call this probability $P(N, K, M)$.

The Birthday Problem is computing $P(N, 365, 1)$.

MIDTERM: What is $P(14400, 9000, 7)$?

(a) .0000000000000000000000000132
(b) .095395
(c) .664954
(d) .999323
(e) .999999999999999999999999845

If we drop 14,000 balls into 9000 boxes, what is the chance no box has more than 7 balls in it?

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Why is this an important example?

In 1960 there were 14,400 cases of leukemia in US and 8 cases in Niles, IL, population 20,000. The average for a town this size would be 1.6 cases.

Is the cluster random?

Population of US in 1960 was

$$180,000,000 = 9,000 \times 20,000.$$

Divide US population into 9,000 “boxes” of 20,000 people each. Drop in 14,400 “cases”. What is the chance that one box has 8 cases?

Answer $= 1 - P(14400, 9000, 7)$.

Calculation gives $P(14400, 9000, 7) = .095395$

N	Probability biggest cluster $\leq N$	Probability biggest cluster $> N$
6	.000005	.999995
7	.095395	.904605
8	.664954	.335046
9	.937864	.062137
10	.990843	.009157
11	.998788	.001212
12	.999852	.000148

A town of 20,000 with 8 or 9 cases is expected. More cases is very unlikely.

A bag contains N numbered balls. You pull one out, look at it and put it back. Continue until you know how big N is. How long does this take?

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Answer: $\approx \sqrt{N}$

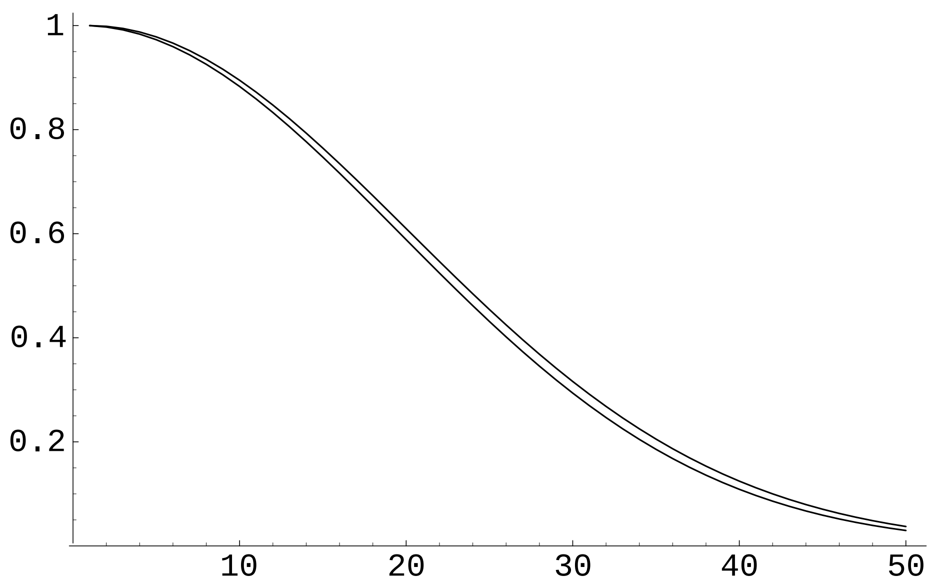
N balls into K boxes. Chance of "no repeats":

Exact Formula

$$P(N) = 1 \cdot \frac{K-1}{K} \cdots \frac{K-N+1}{K}$$

Approximate Formula

$$P(N) \approx e^{-N^2/2 \cdot K}$$



Comparing exact and approximate formulas

How many balls to get a 50% chance of a repeat?

Must solve

$$e^{-N^2/2K} = .5$$

$$\frac{-N^2}{2K} = \log .5$$

$$N \quad = \quad \sqrt{2K \log 2} \quad \approx \quad 1.17741\sqrt{K}$$

If we throw $N \approx \sqrt{K}$ balls into K boxes we have a good chance to get two in the same box.

Note $1.177\sqrt{365} \approx 22.49$.

$1.177\sqrt{1000000} = 1177\text{days} = 3.22\text{years}$.

Counting balls in a bag problem

Rough guess:

If first repeat in on the nth draw from the bag, guess $K = (n/1.1774)^2$ as number of balls in bag.

Repeat and take average for better estimate.

n	$(n/1.177)^2$
2	2.92205
3	6.57462
4	1.6882
5	18.2628
6	26.2985
7	35.7952
8	46.7529
9	59.1716
10	73.0514
11	88.3921
12	105.194
13	123.457
14	143.181
15	164.366

Better solution:

Examine and return m samples.

Let t be total number of repeats.

Estimate $k = \frac{m(m-1)}{2t}$

Estimate is probably accurate if $m \gg \sqrt{K}$.

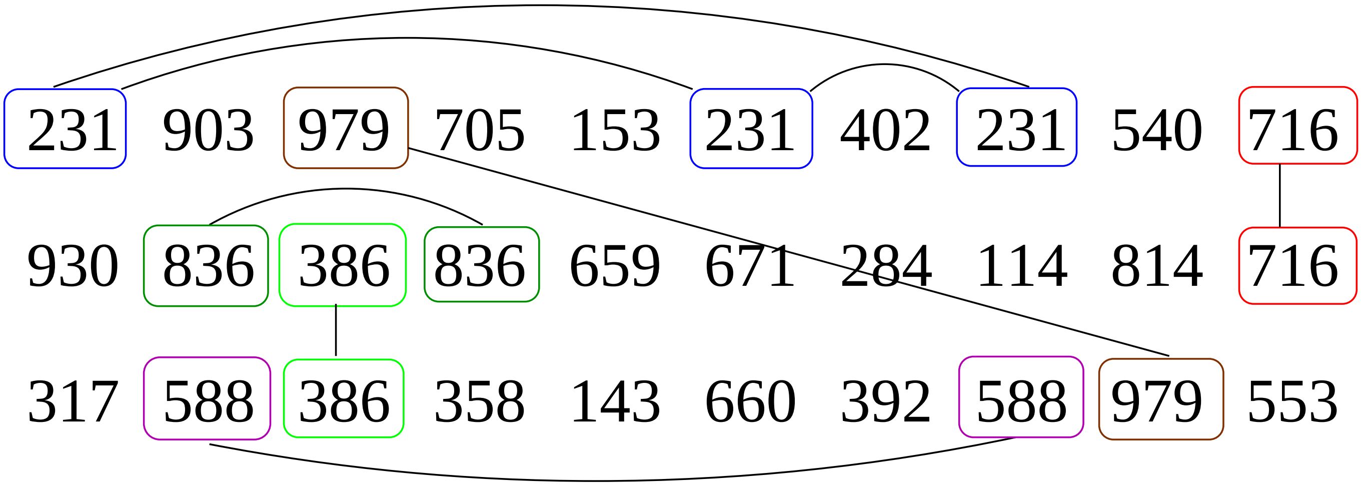
Example: I picked K distinct 3 digit numbers, then drew 30 random samples:

231 903 979 705 153 231 402 231 540 716

930 836 386 836 659 671 284 114 814 716

317 588 386 358 143 660 392 588 979 553

What is K?



There are $m = 30$ samples and $t = 8$ pairs of repeated numbers, so our guess is

$$\frac{m(m-1)}{2t} = \frac{30 \cdot 29}{16} \approx 54.375.$$

The true K is 57.

The large m is, the better the estimate.

Applications:

Counting fish in lake
Counting distinct users on internet
Cryptography (security of digital signatures)
Factoring integers (Pollard rho method)
Many others

FINAL EXAM

Two thieves steal N diamonds with random values between \$1 and \$1,000,000. Can they divide the loot into two piles of equal value?

Impossible if $N = 1$ and unlikely if $N = 2, 3, \ldots ?$.

There is a 50% chance even splitting if $N > ?$

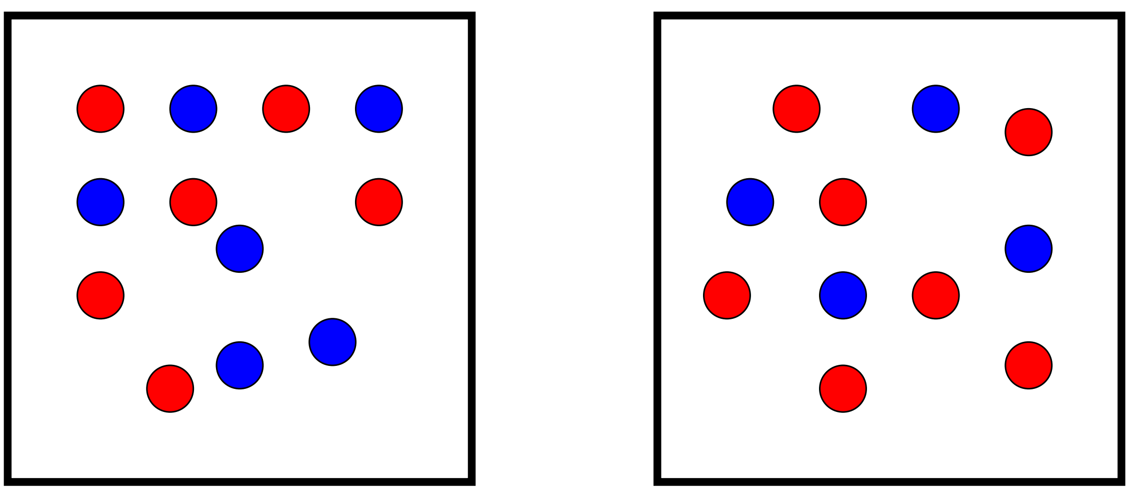
(a) 11
(b) 25
(c) 78
(d) 979
(e) 10,122

Divide diamonds into two piles of size $N/2$.

Randomly divide first pile into two smaller sets (red and blue). Let R_1, B_1 be the value of each subset. Let $D_1 = R_1 - B_1$. This is a random number between

$$-\frac{N}{2} \times 1,000,000 < D_1 < \frac{N}{2} \times 1,000,000$$

There are $2^{N/2}$ ways to choose D_1.



$D_1 = R_1 - B_1$ $\qquad$ $D_2 = R_2 - B_2$

Do the same for the second pile. Get a random number $D_2 = R_2 - B_2$

If $D_1 = D_2$ then

$$R_1 - B_1 = R_2 - B_2,$$
$$R_1 + B_2 = R_2 + B_1,$$

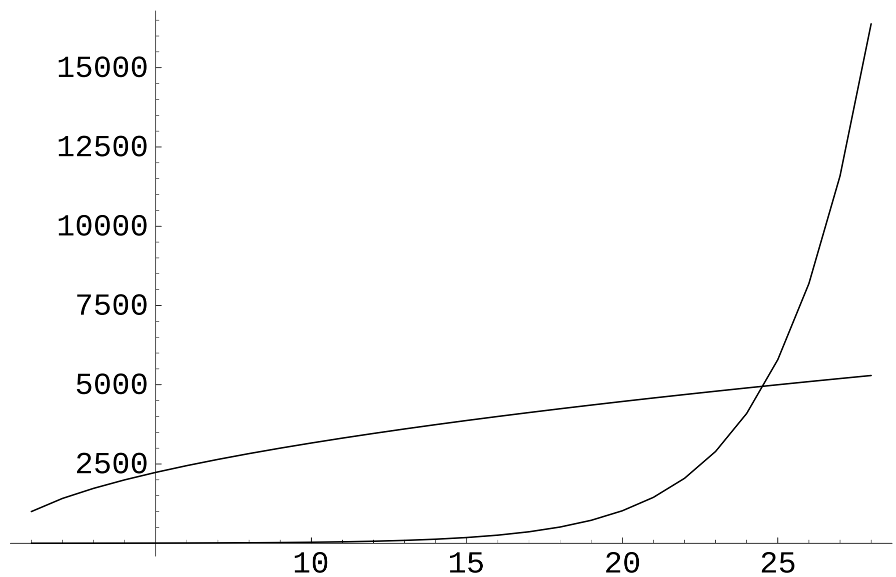
so we get a division into two equal parts.

What is the chance $D_1 = D_2$?

By Birthday Problem the odds ≈ 50-50 if

$$\# \text{ random choices} \approx \sqrt{\# \text{ possible choices}}$$

$$2^{N/2} = \sqrt{N \times 1,000,000}$$



Answer: An equal division is likely if $N \geq 25$.

(But finding the division can be very hard; what we call an NP-hard problem.)

References

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