Quasiconformal Folding
(or dessins d'adolescents)

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lecture slides available at
`www.math.sunysb.edu/~bishop/lectures`

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- every edge has equal harmonic measure from ∞
- edge subsets have same measure from both sides

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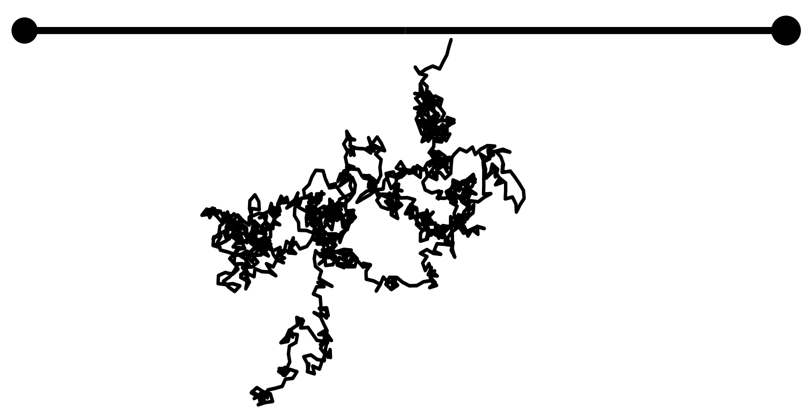
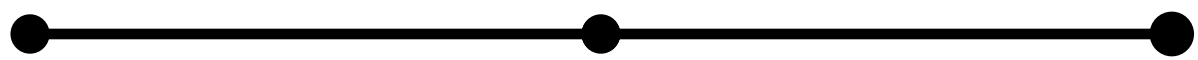
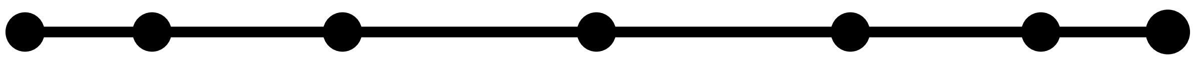
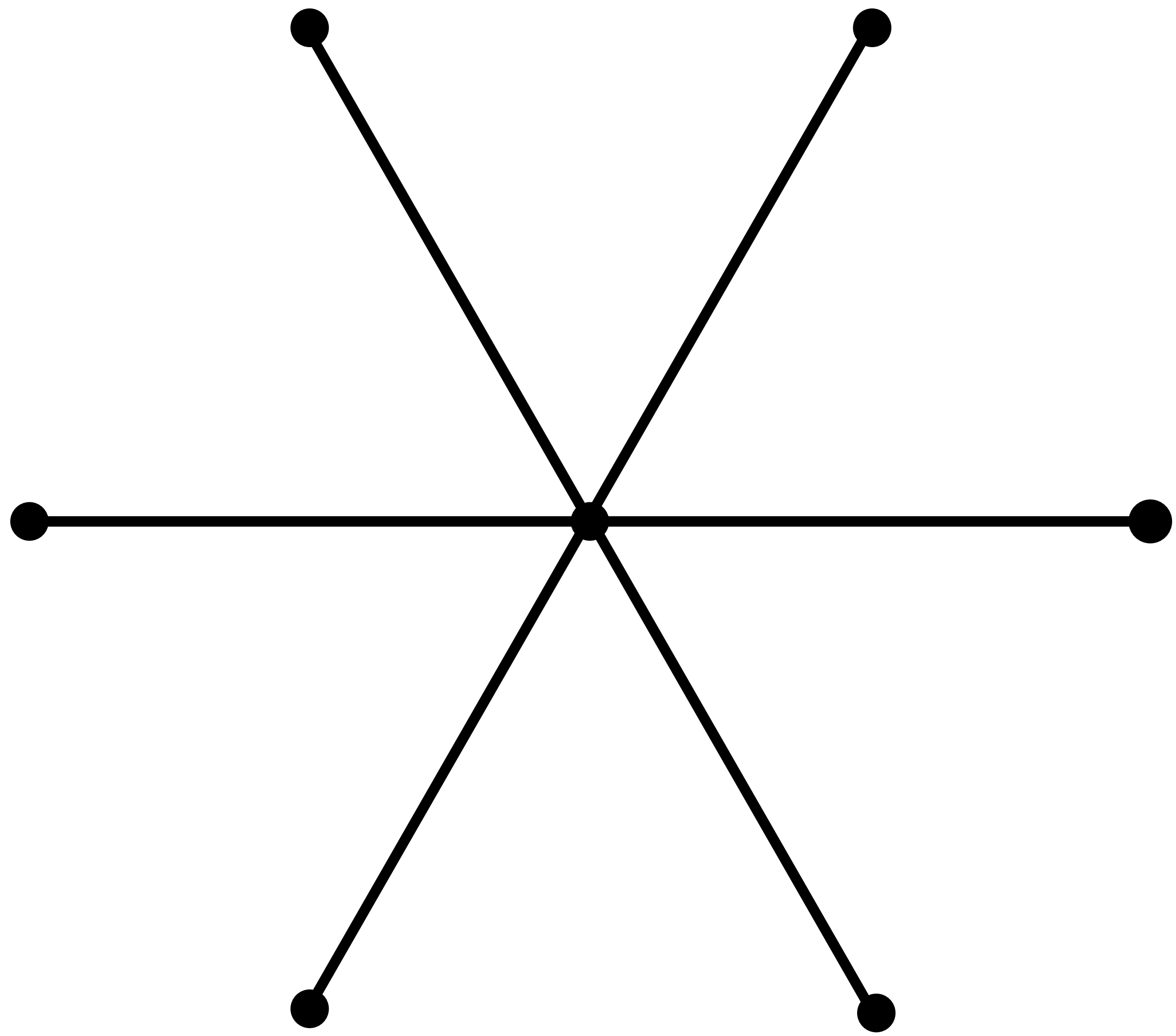
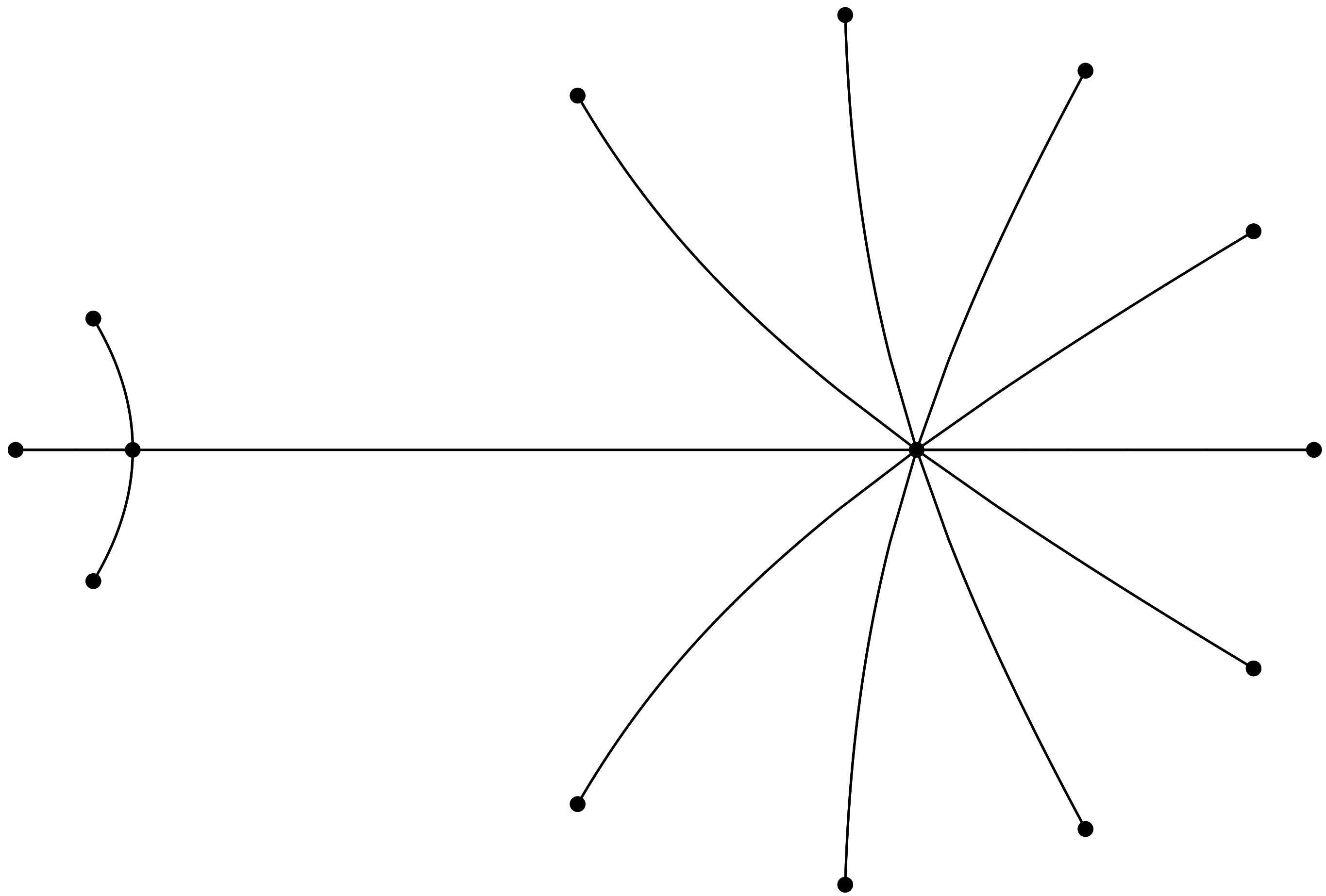
A line segment is an example. Are there others?

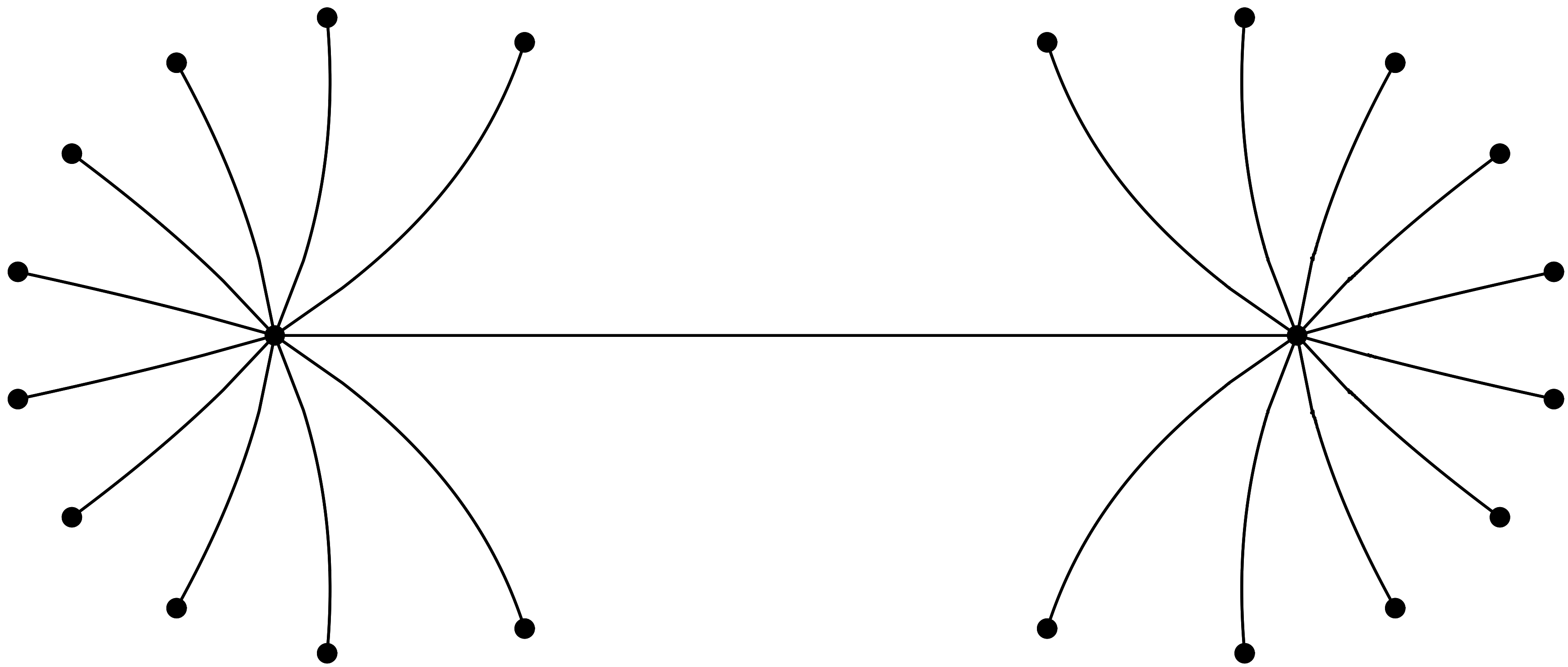










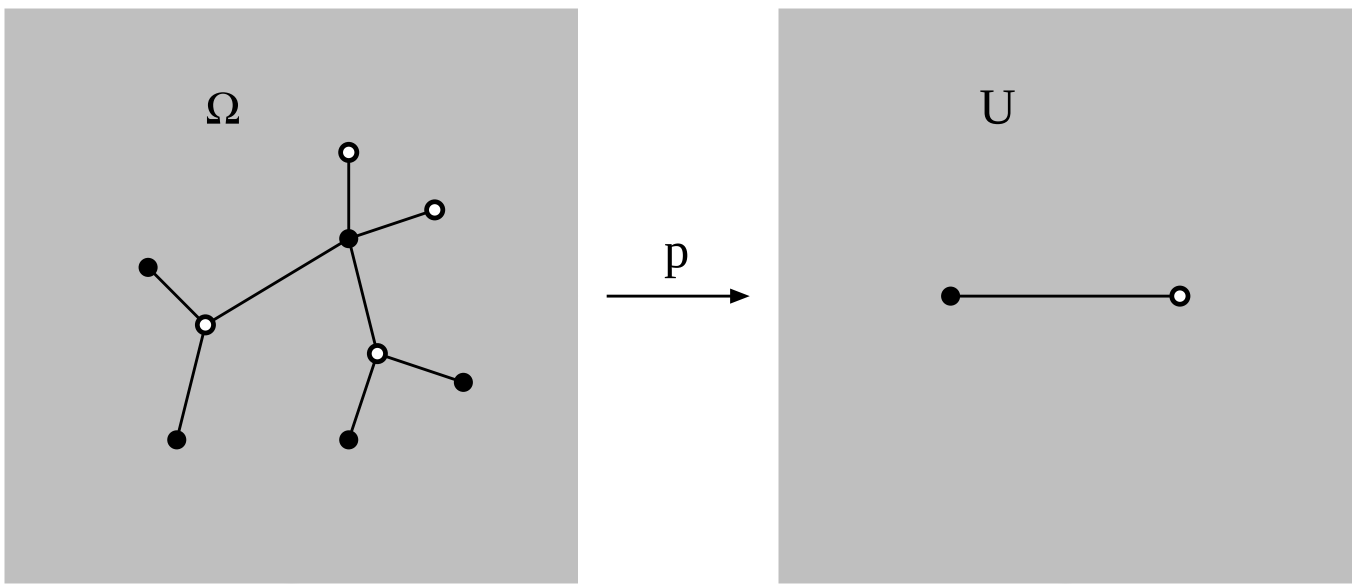


Balanced trees $\leftrightarrow$ Shabat polynomials

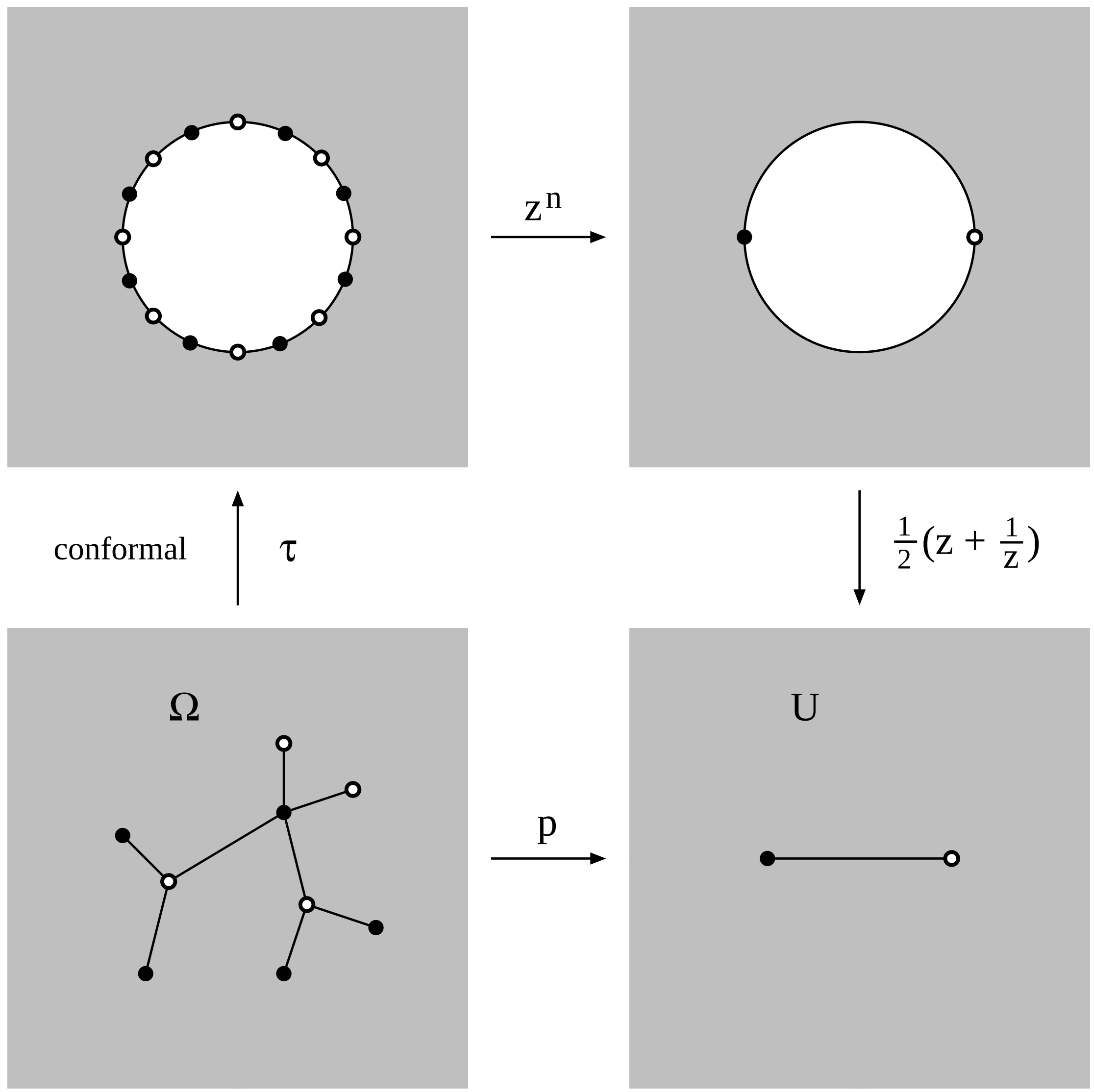
$\mathrm{CV}(p) = \{p(z) : p'(z) = 0\}$

T is balanced iff $T = p^{-1}([-1,1])$, $\mathrm{CV}(p) = \pm 1$.

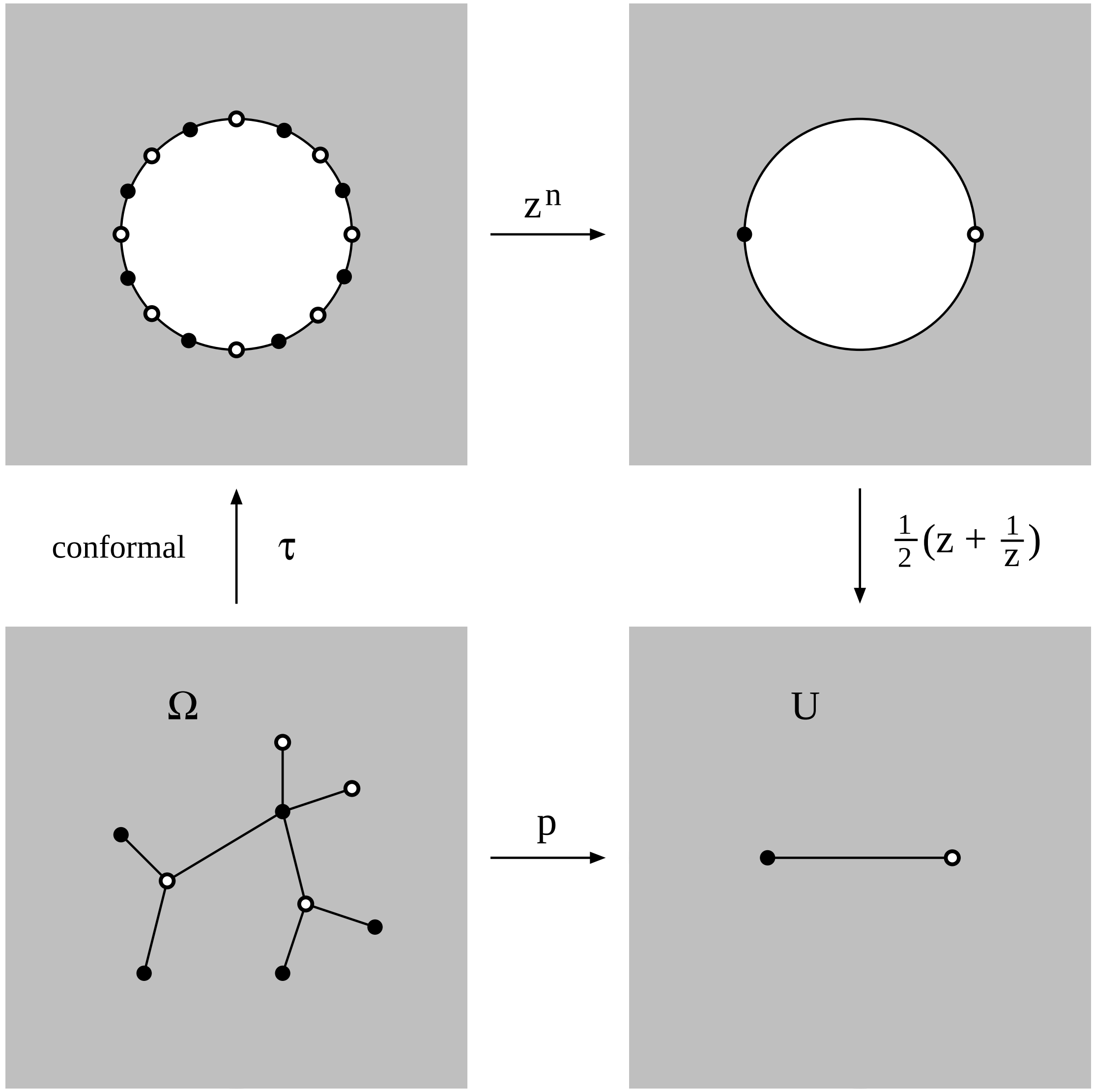
p is called **generalized Chebyshev** or **Shabat**.



$\Omega = \mathbb{C} \setminus T$ $\qquad$ $U = \mathbb{C} \setminus [-1, 1]$



$$\tau = \text{conformal } \Omega \to \mathbb{D}^* = \{|z| > 1\}.$$



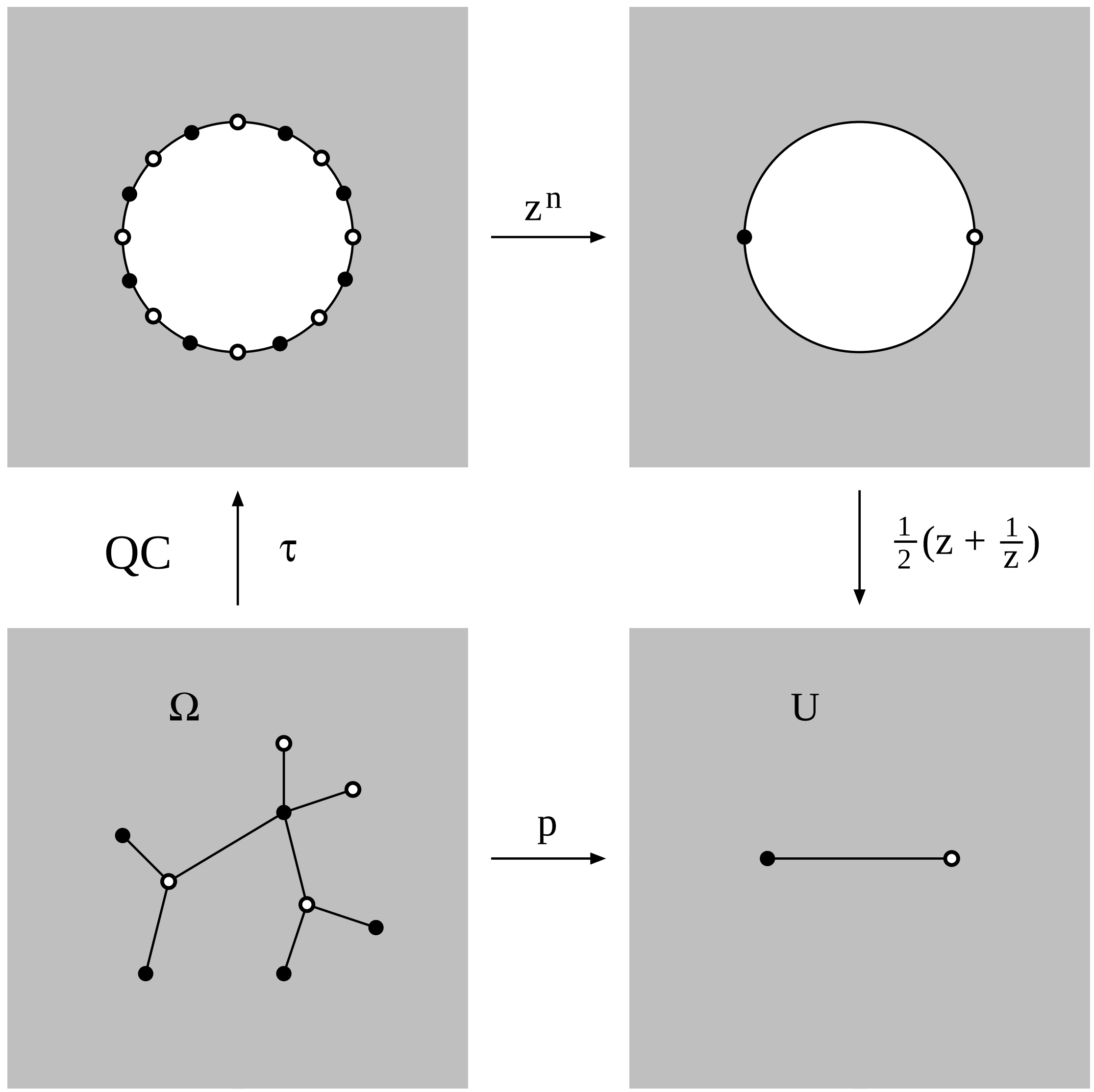
T is balanced iff $p = \frac{1}{2}(\tau^n + \tau^{-n})$ is continuous.

What if the tree is not balanced?

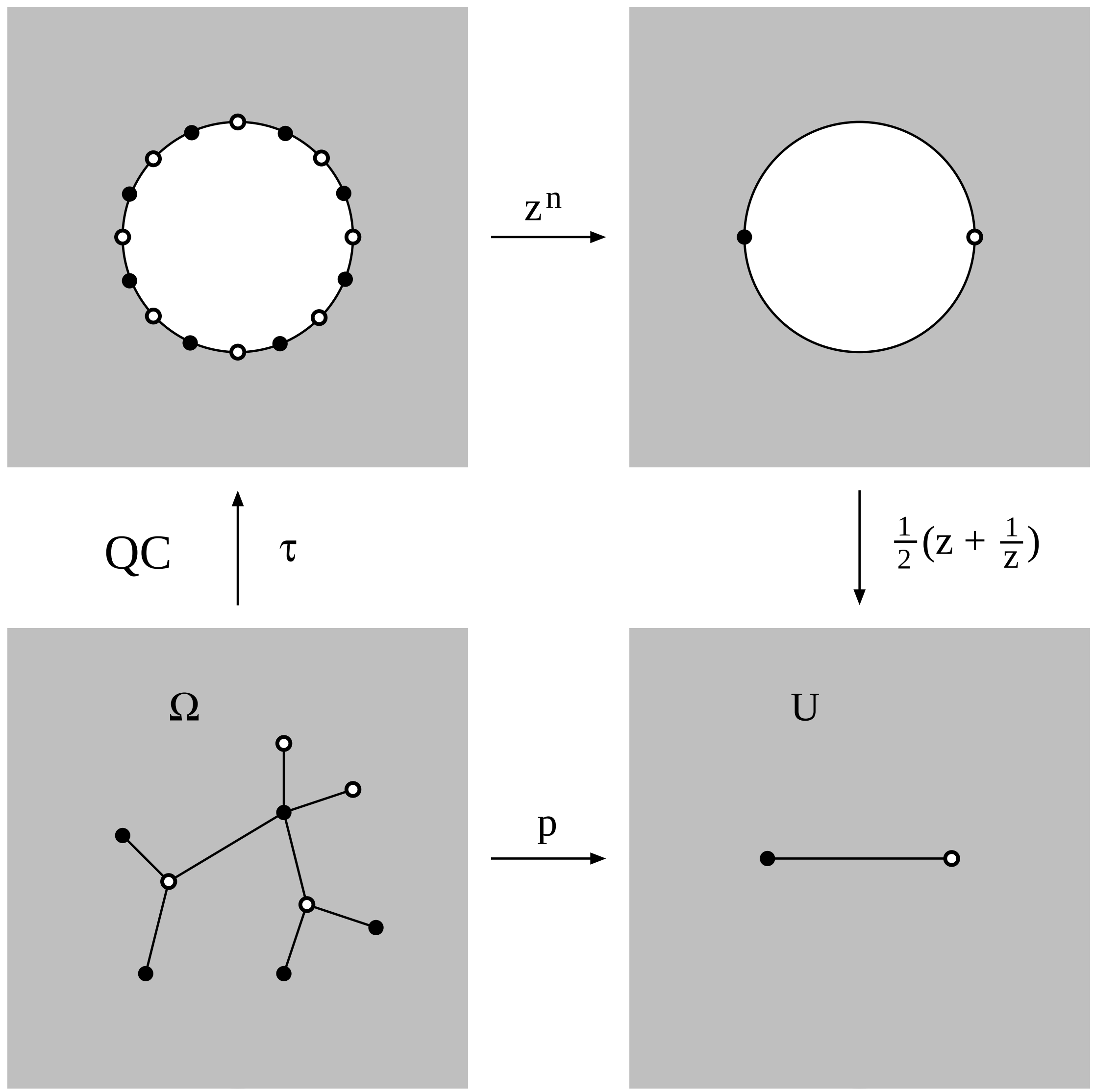
What if the tree is not balanced?

Replace conformal map by quasiconformal map.

Tree is QC-balanced if there is a QC map of tree complement to disk complement so that
(1) all edges map to equal length
(2) subsets of an edge map to two images of same length



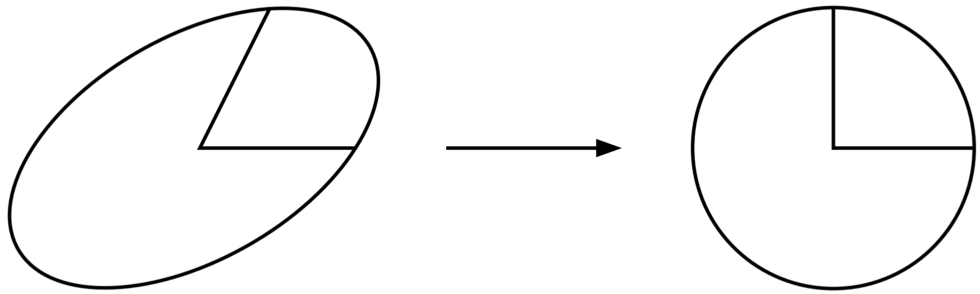
T is QC-balanced if this holds for a quasiconformal τ.



Every "nice" finite tree is QC-balanced.

quasiconformal = bounded angle distortion.

dilatation = $\mu_f = f_{\overline{z}}/f_z$ = measure of non-conformality

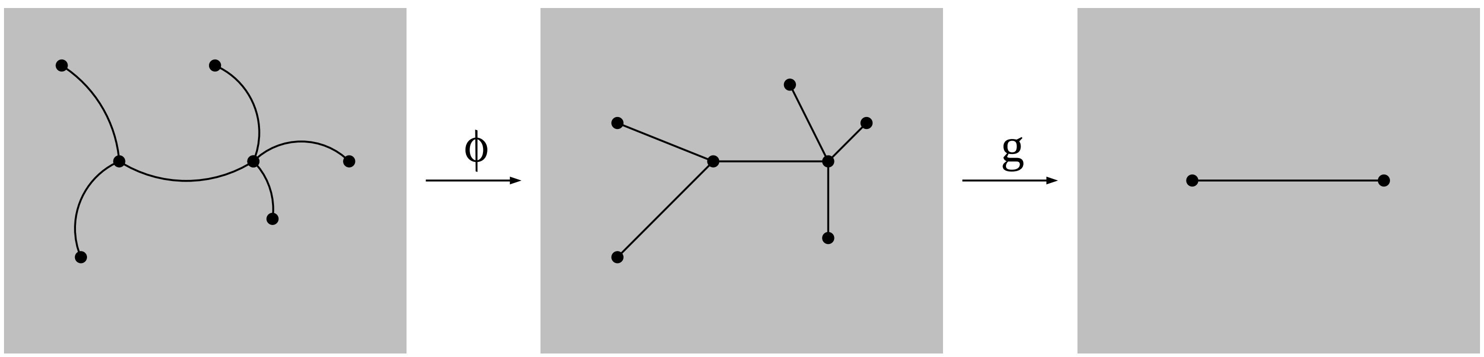


quasiregular $g = f \circ \phi$, where f is analytic, ϕ is QC.

T is QC-balanced iff $T = g^{-1}([-1,1])$ where g is quasi-Shabat (QR, finite-to-1, $\mathrm{CV}(g) = \pm 1$).

Thm: Any QC-balanced tree is the QC image of some conformally balanced tree.

Proof: Measurable Riemann Mapping Theorem



Cor: Every planar tree has a true form.

In Grothendieck's theory of *dessins d'enfants*, a finite graph of a topological surface determines a conformal structure and a Belyi function (a meromorphic function to sphere branched over 3 values).

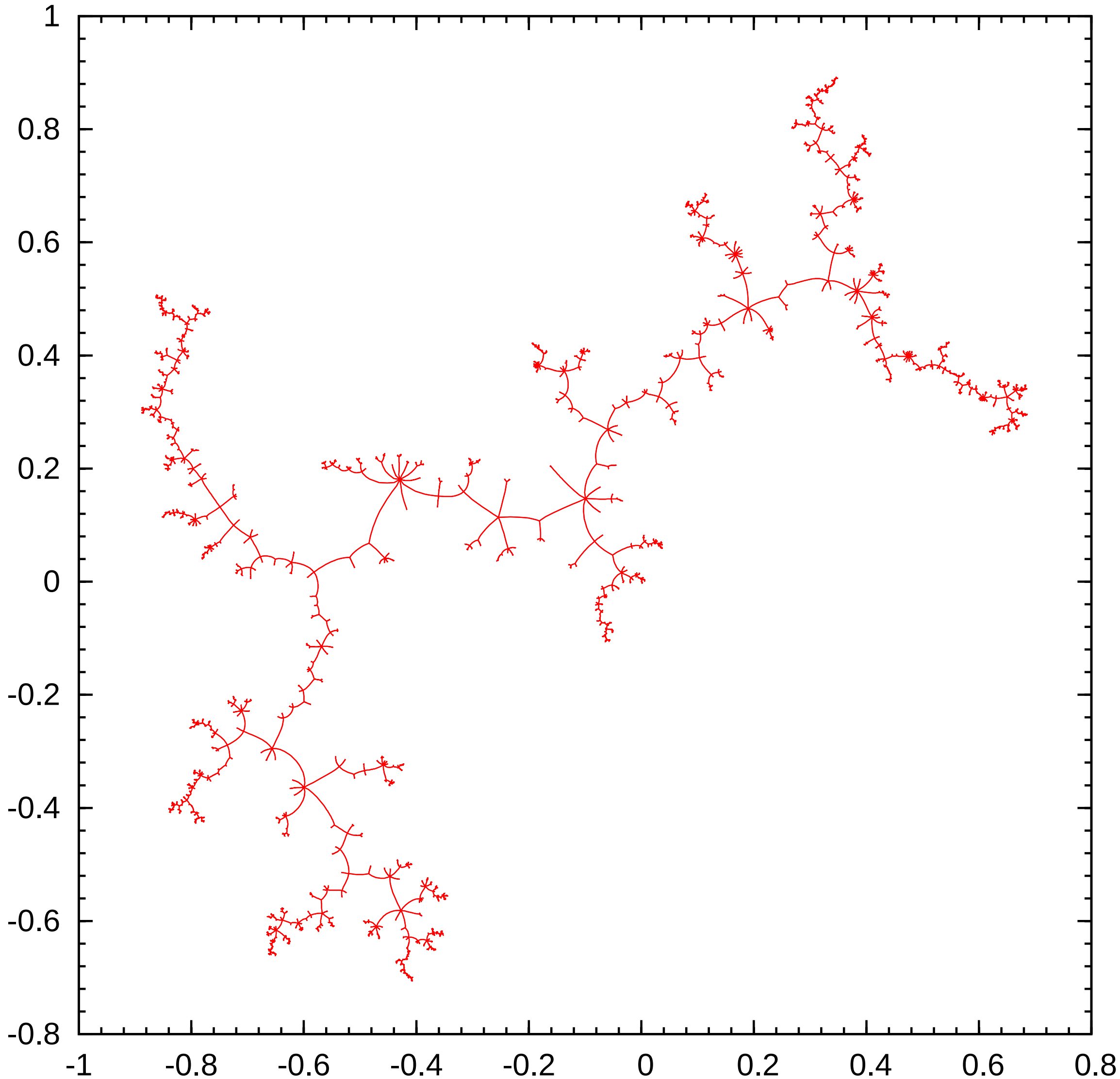
Shabat polynomial is special case of Belyi function. (branch points $= -1, 1, \infty$).

"conformally balanced tree" $=$ "true form of a tree".

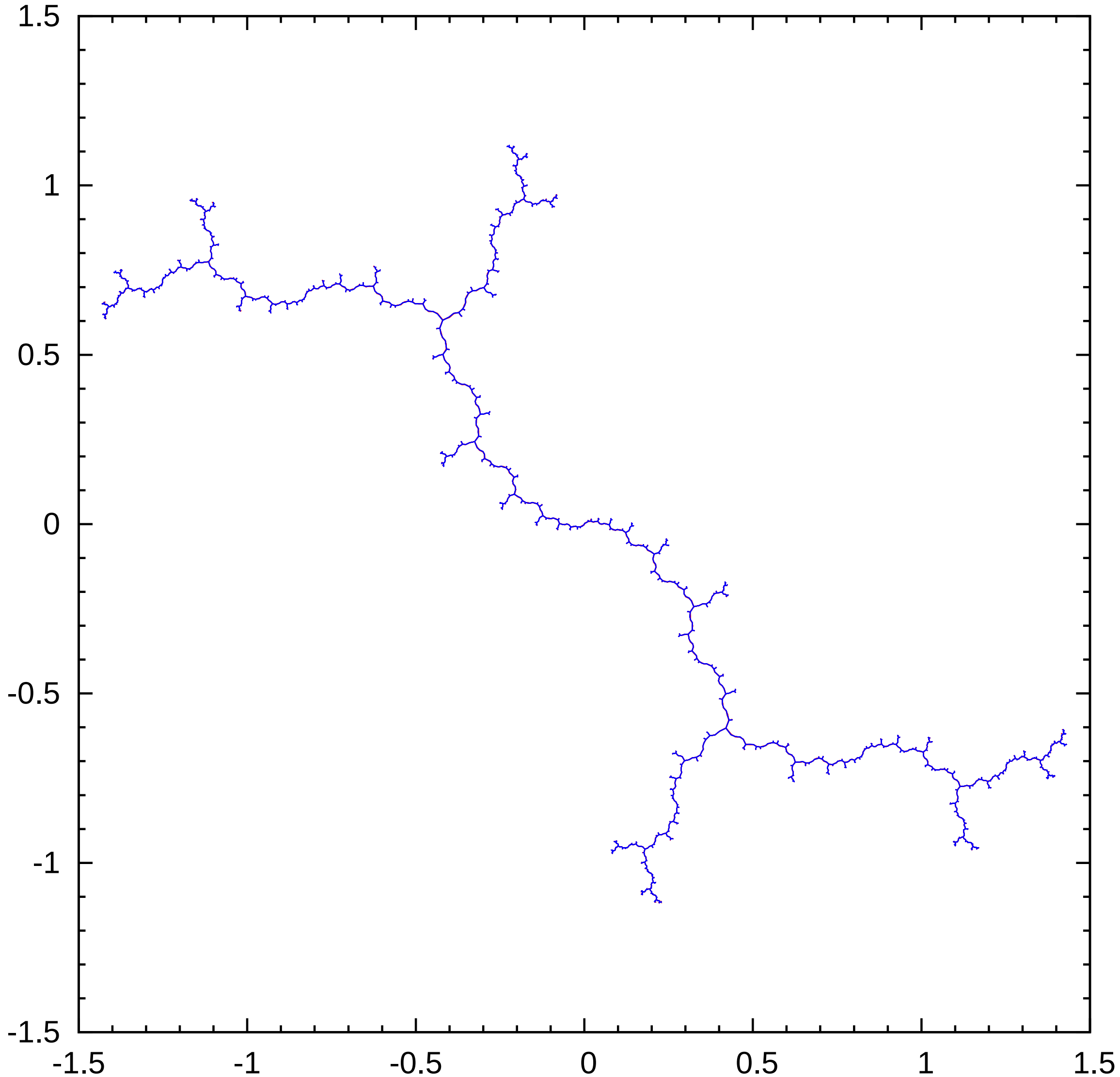
Is the polynomial computable from the tree?

Kochetkov, *Planar trees with nine edges: a catalogue*, 2007. “The complete study of trees with 10 edges is a difficult work, and probably no one will do it in the foreseeable future”.

Marshall and Rohde have approximated true trees with thousands of edges. They have catalogued all 95,640 true trees with 14 edges to 25 digits of accuracy. Can obtain 1000’s of digits.



Random true tree with 10,000 edges



Tree with dynamical combinatorics

Every planar tree has a true form.

In other words, all possible **combinatorics** occur.

What about all possible **shapes**?

Theorem: Every continuum is a limit of true trees.

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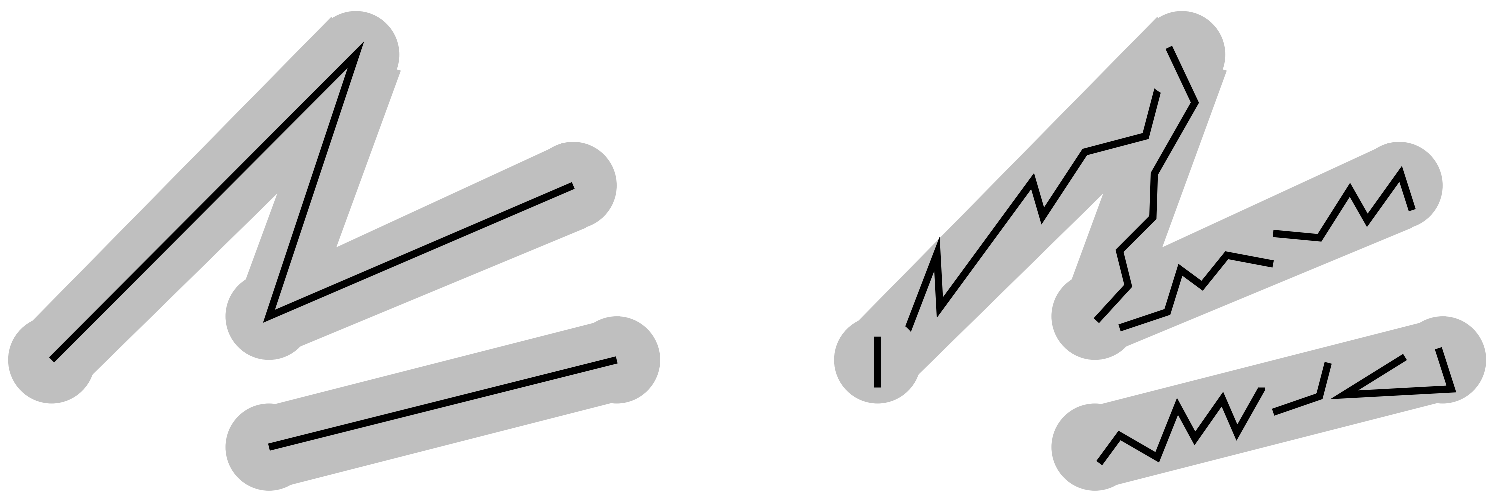
Limit in **Hausdorff metric**: if E is compact,

$$E_\epsilon = \{z : \mathrm{dist}(z, E) < \epsilon\}.$$

$$\mathrm{dist}(E, F) = \inf\{\epsilon : E \subset F_\epsilon, F \subset E_\epsilon\}.$$

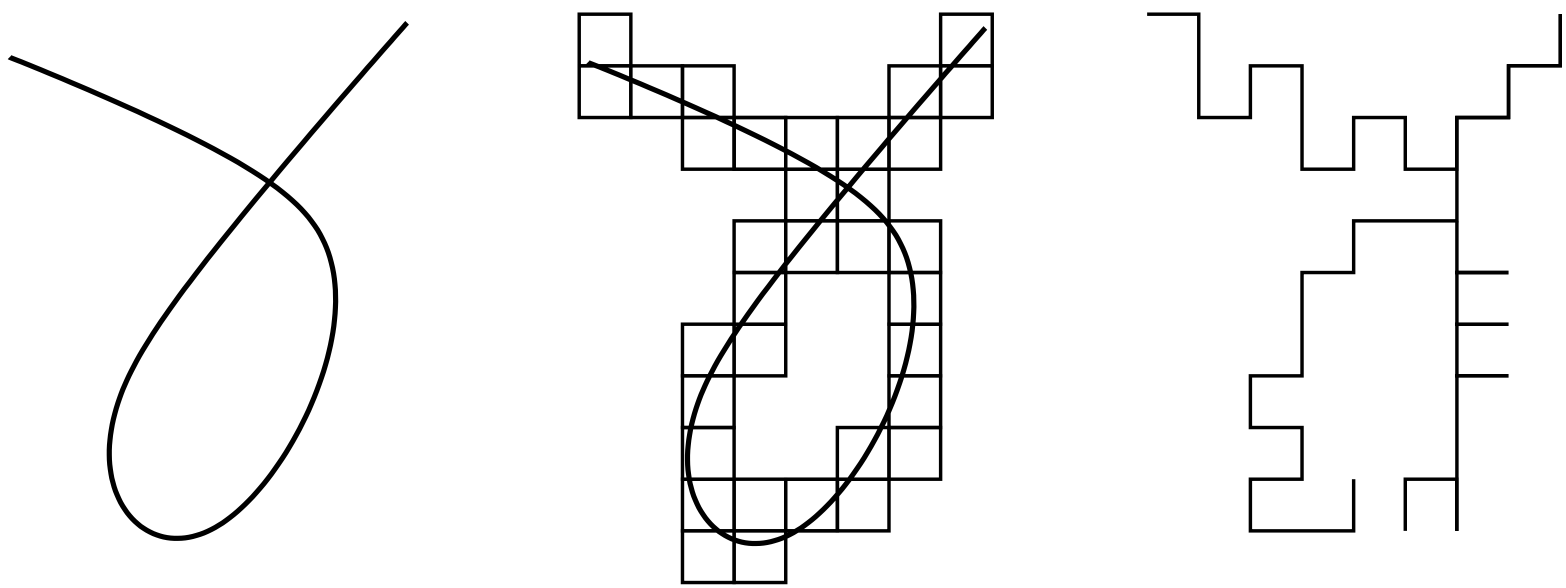


Theorem: Every continuum is a limit of true trees.

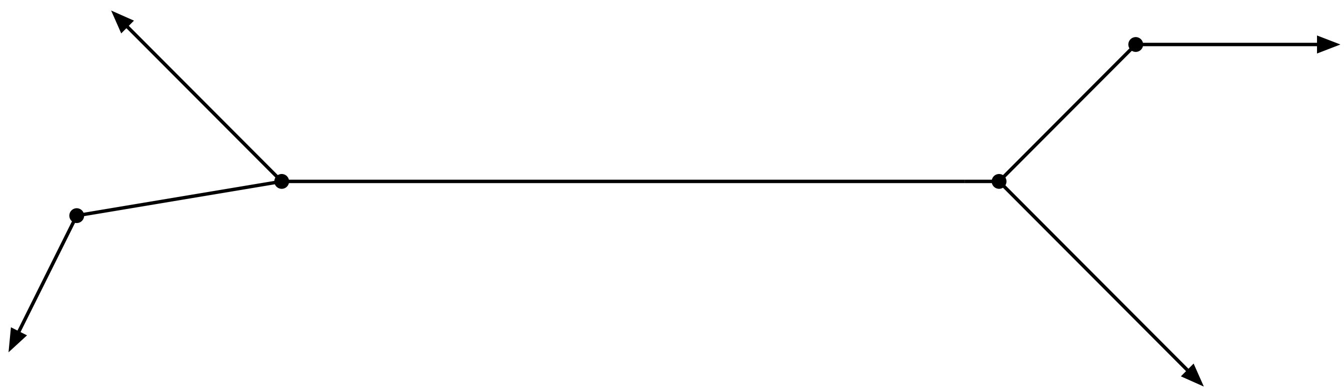
continuum = compact, connected set

Answers question of Alex Eremenko.

Enough to approximate certain finite trees by true trees.



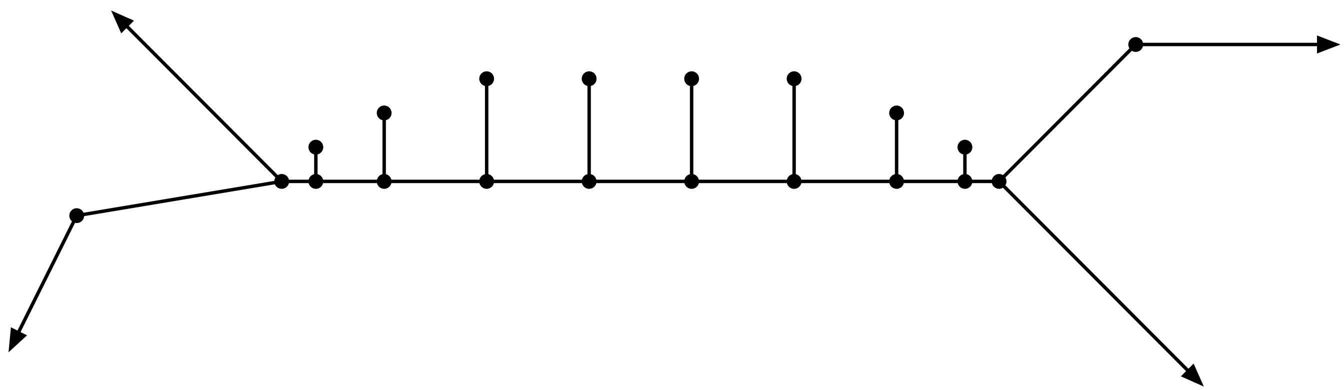
Idea: reduce harmonic measure ratio by adding edges.



Suppose upper side has larger harmonic measure.

(more likely to be hit by Brownian motion)

Idea: reduce harmonic measure ratio by adding edges.

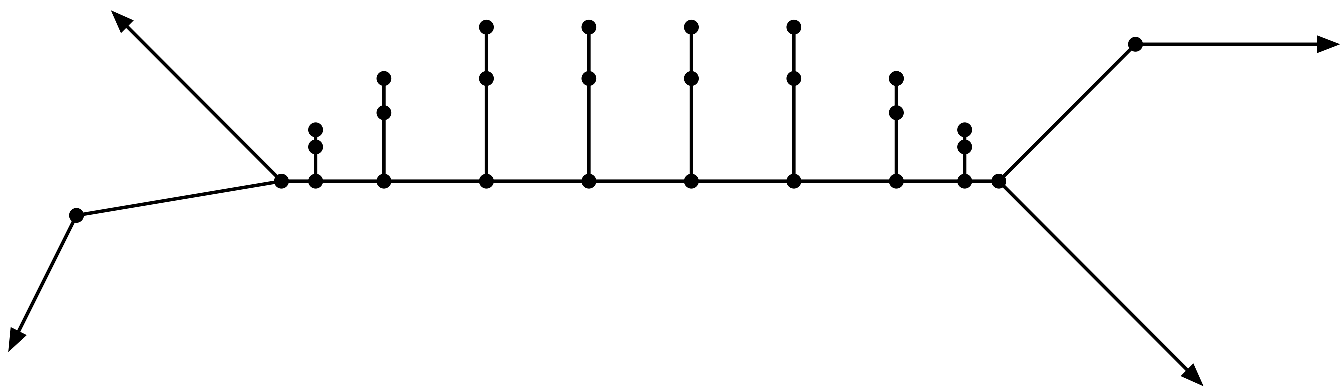


Harmonic measure of top side is reduced.

Roughly 3-to-1 reduction.

New edges are uniformly close to balanced.

Idea: reduce harmonic measure ratio by adding edges.

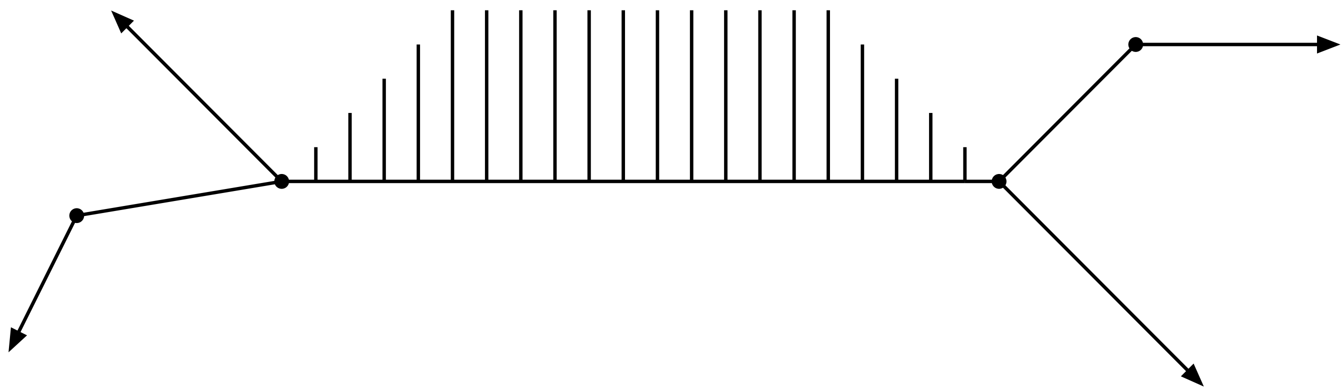


Higher spikes mean more reduction.

Need extra vertices to make edges have equal measure.

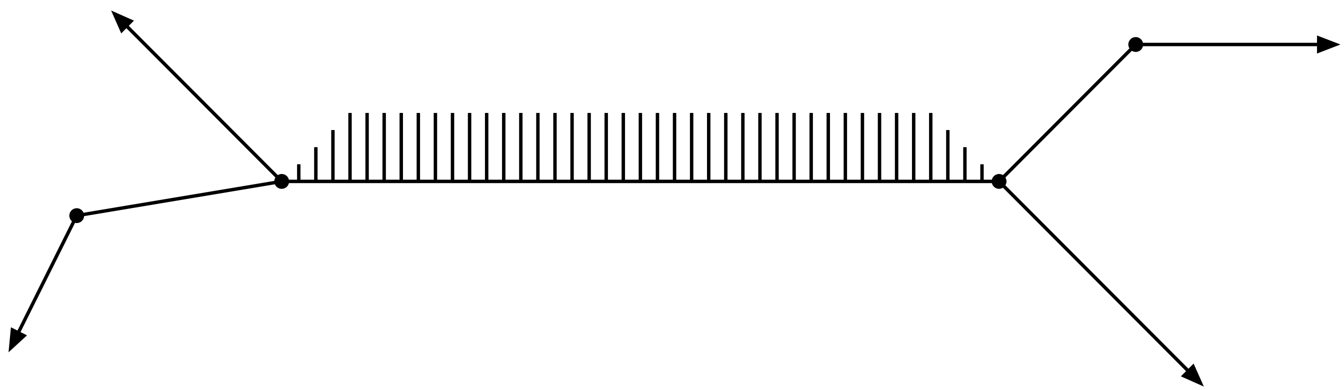
Roughly 5-to-1 reduction.

Idea: reduce harmonic measure ratio by adding edges.



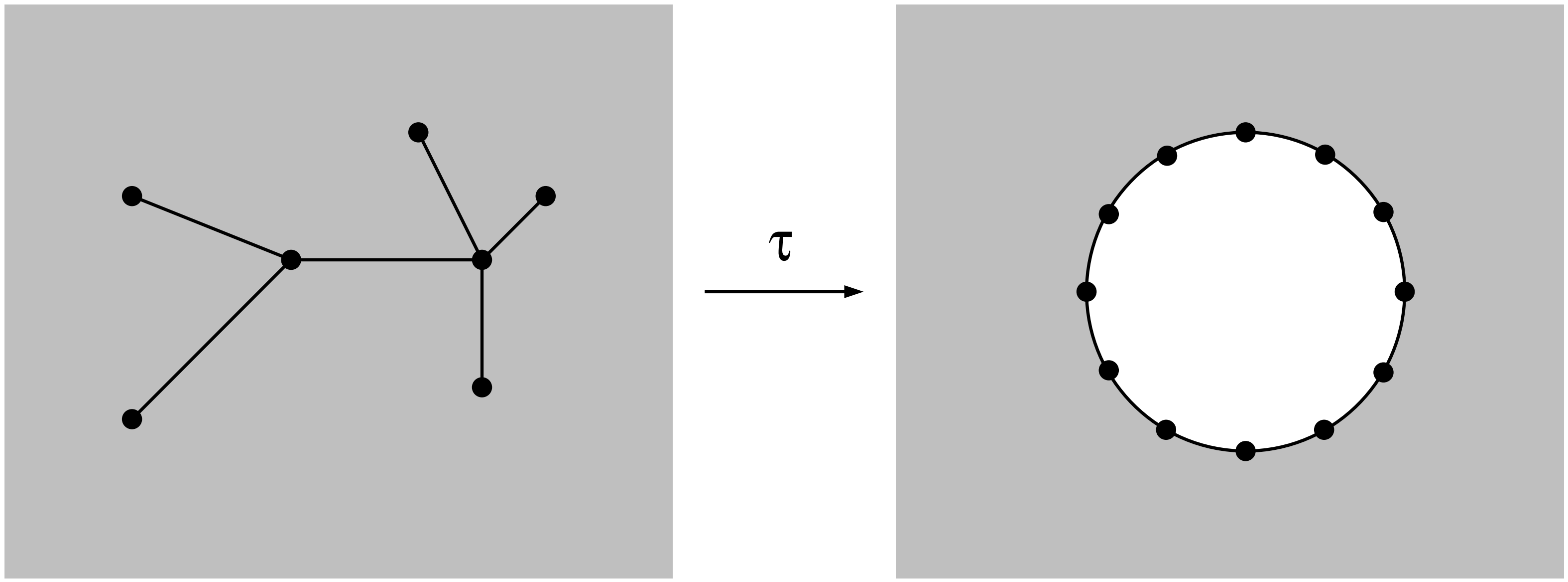
Eccentricity k gives reduction $\approx \exp(-\pi k)$.

Idea: reduce harmonic measure ratio by adding edges.



Any reduction can be achieved inside any neighborhood.

Adding edges to tree = adding "spikes" to circle.



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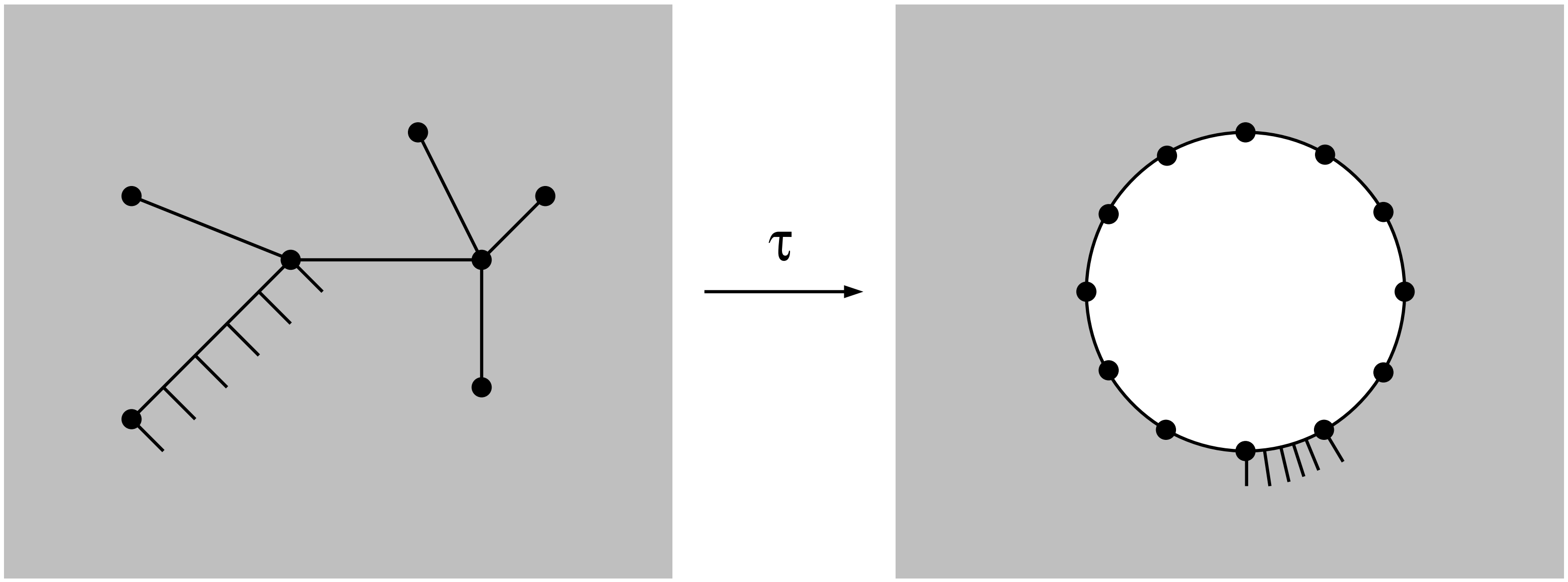
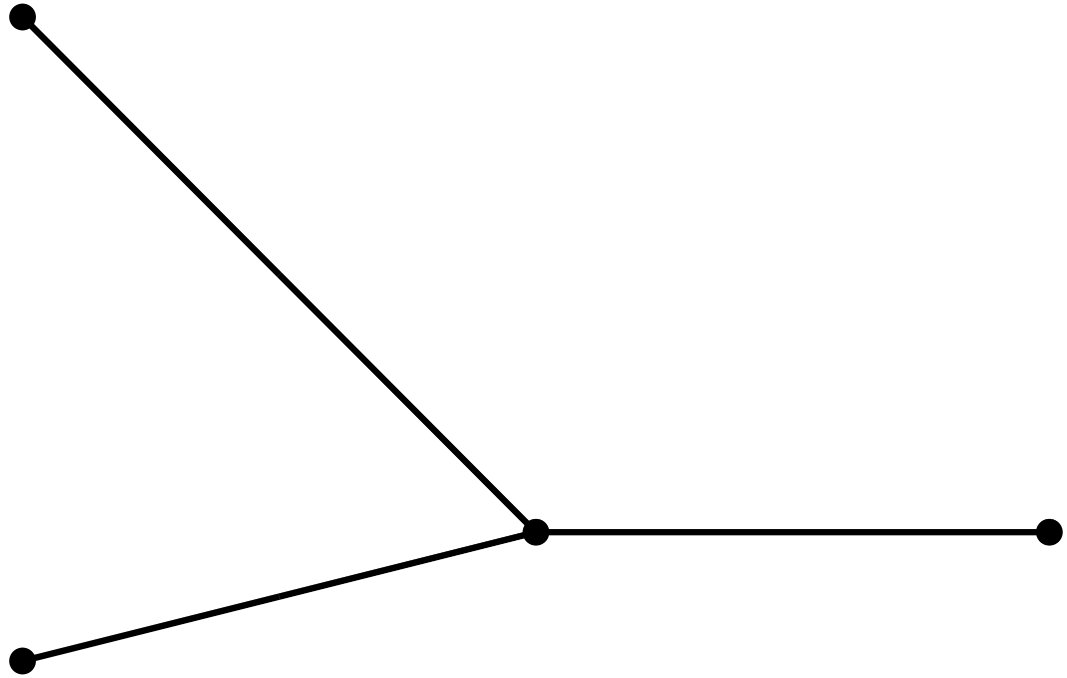
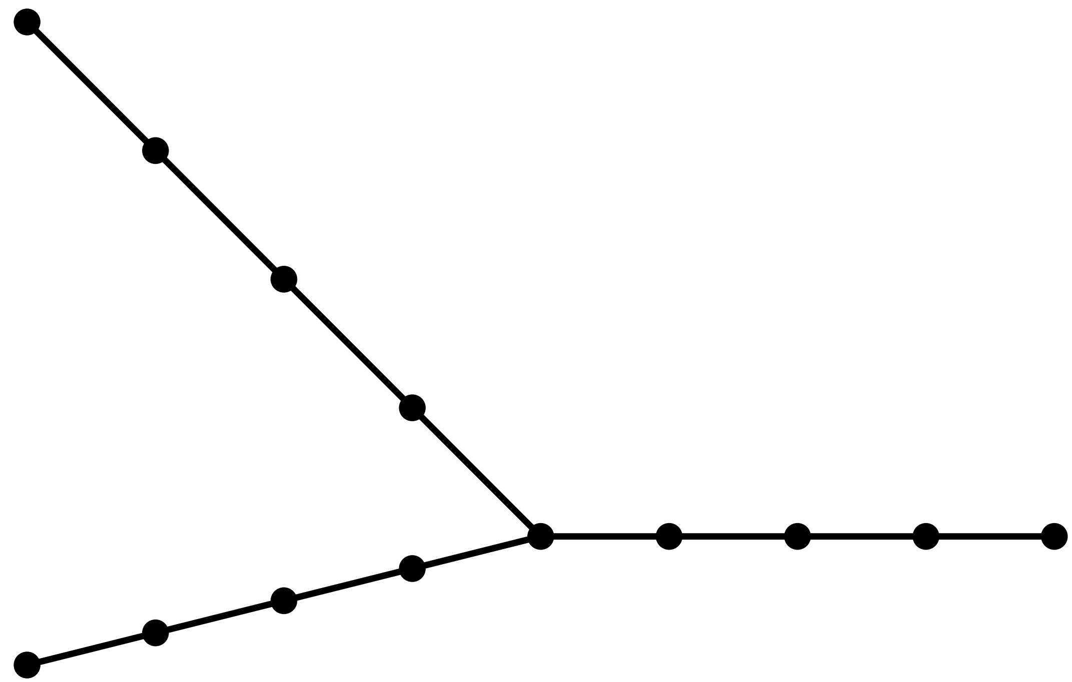


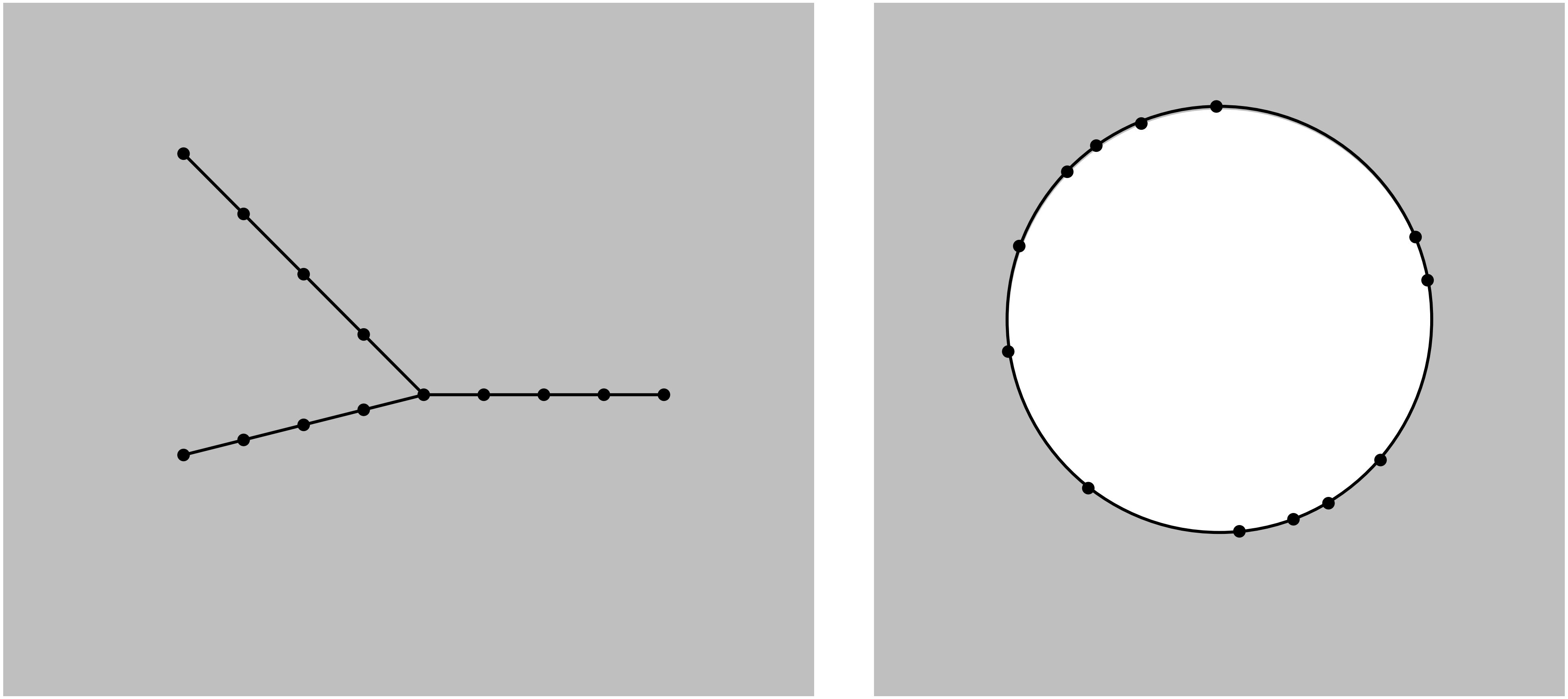
Spikes on circle are easier to deal with.



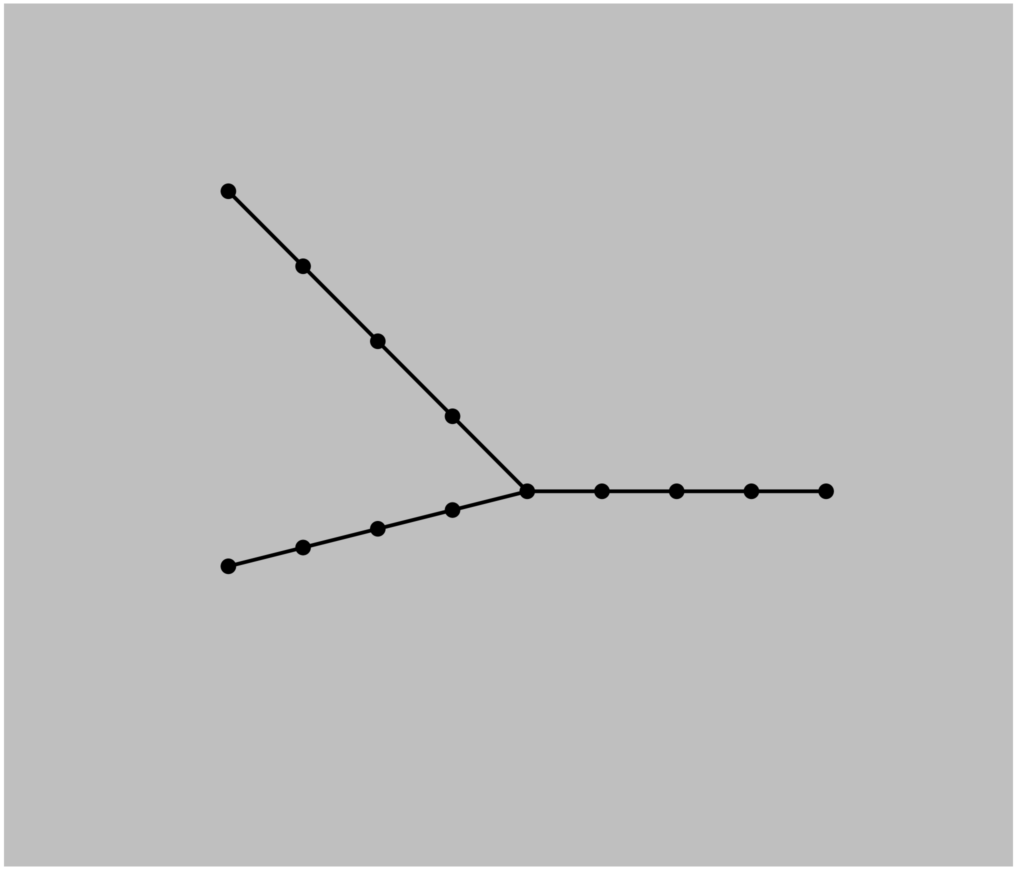
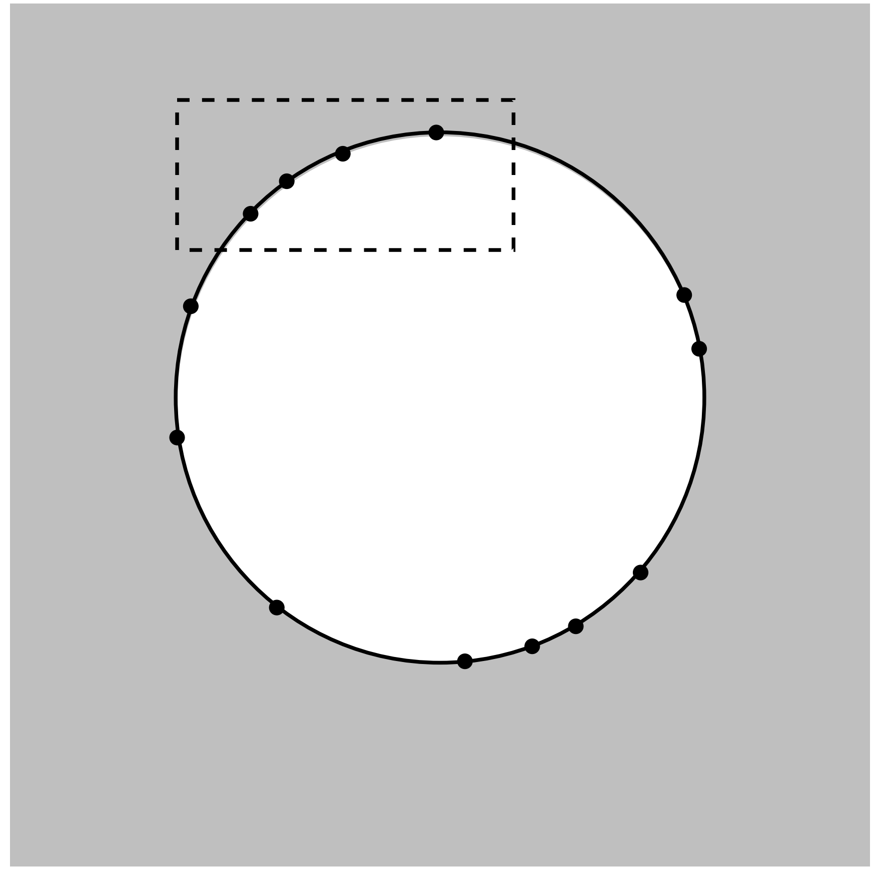
Start with a tree.



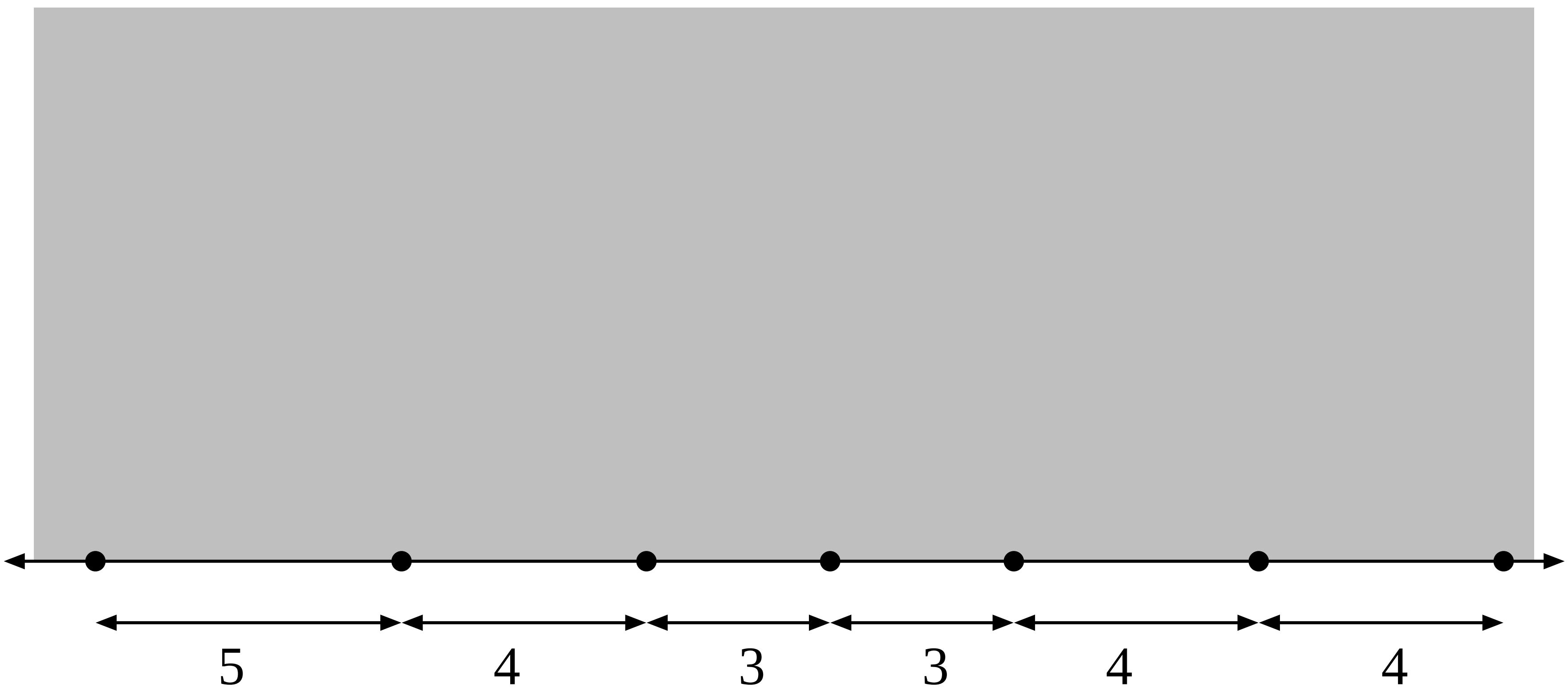
Subdivide edges (we will see why later).



Conformally map Ω to $\mathbb{D}^*$. Uneven distribution.

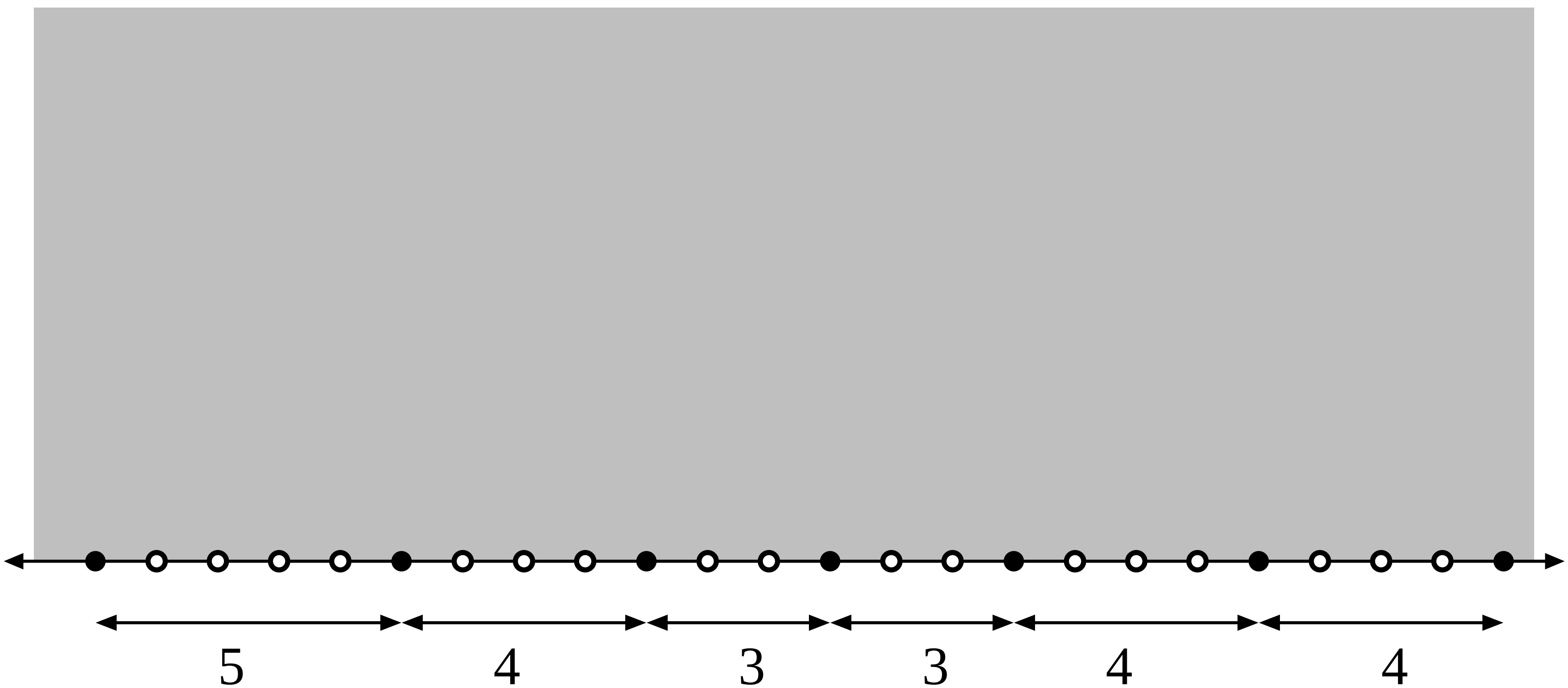
 

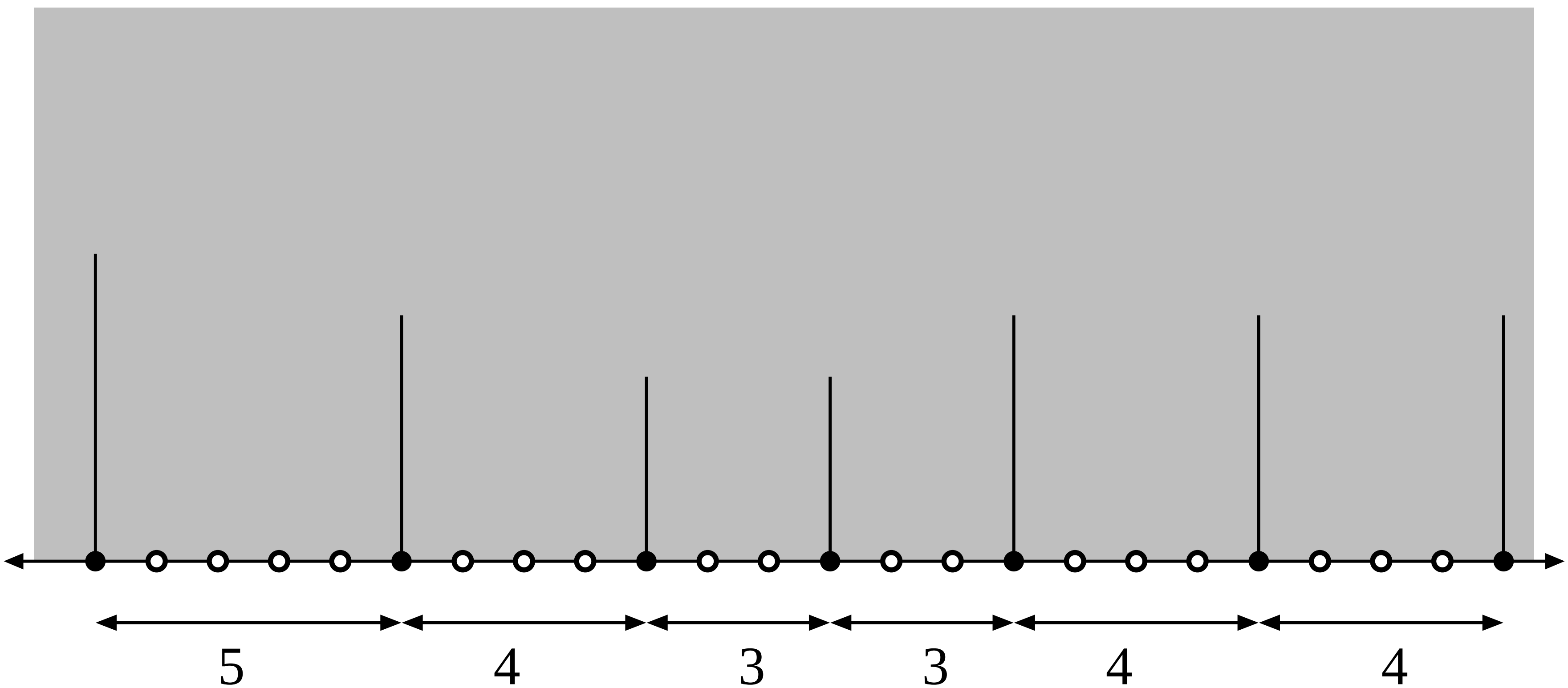
Let's blow-up a small segment.



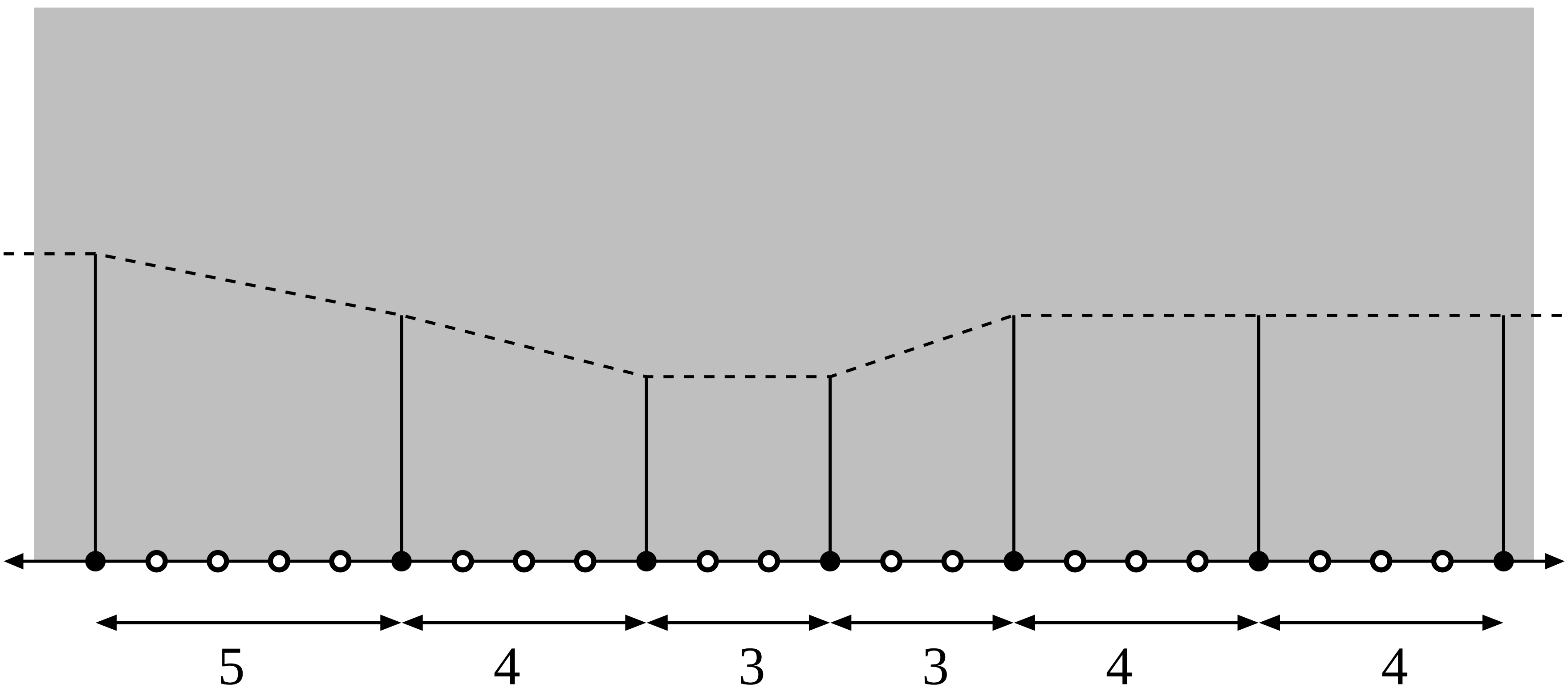
Assume gap sizes are integer multiples of some unit.

Also assume adjacent gaps have adjacent or equal sizes.

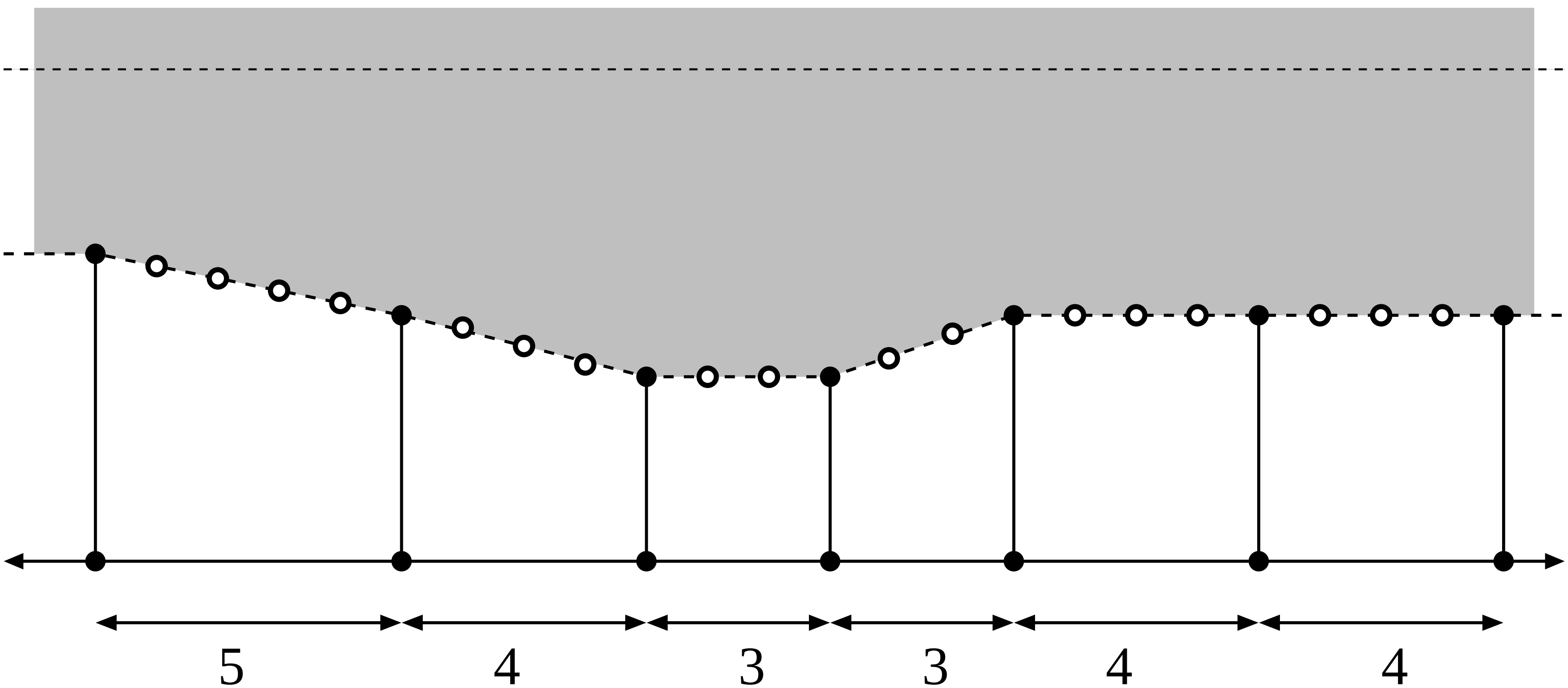

5
4
3
3
4
4



Add vertical spikes (height $\approx$ log(gap multiple)).

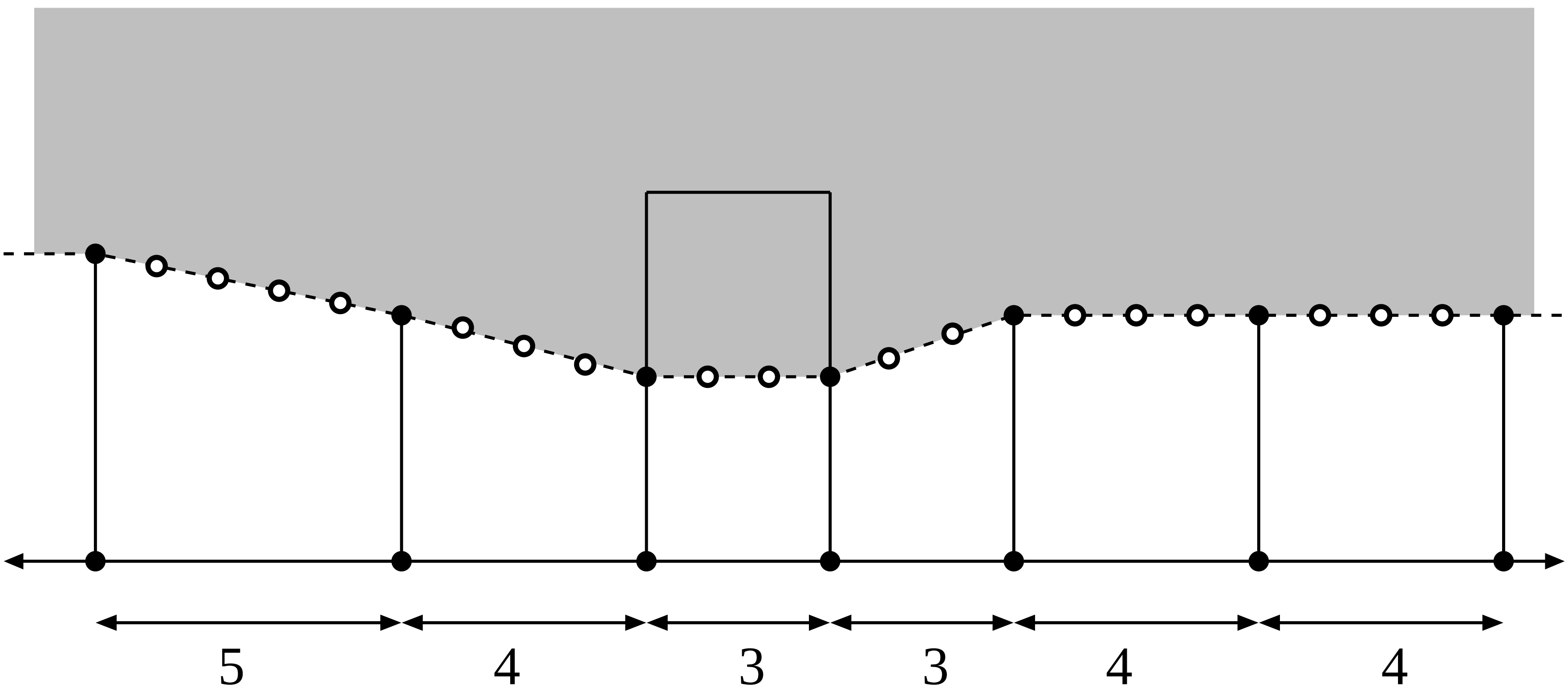


Connect spike tips (defines Lipschitz graph).

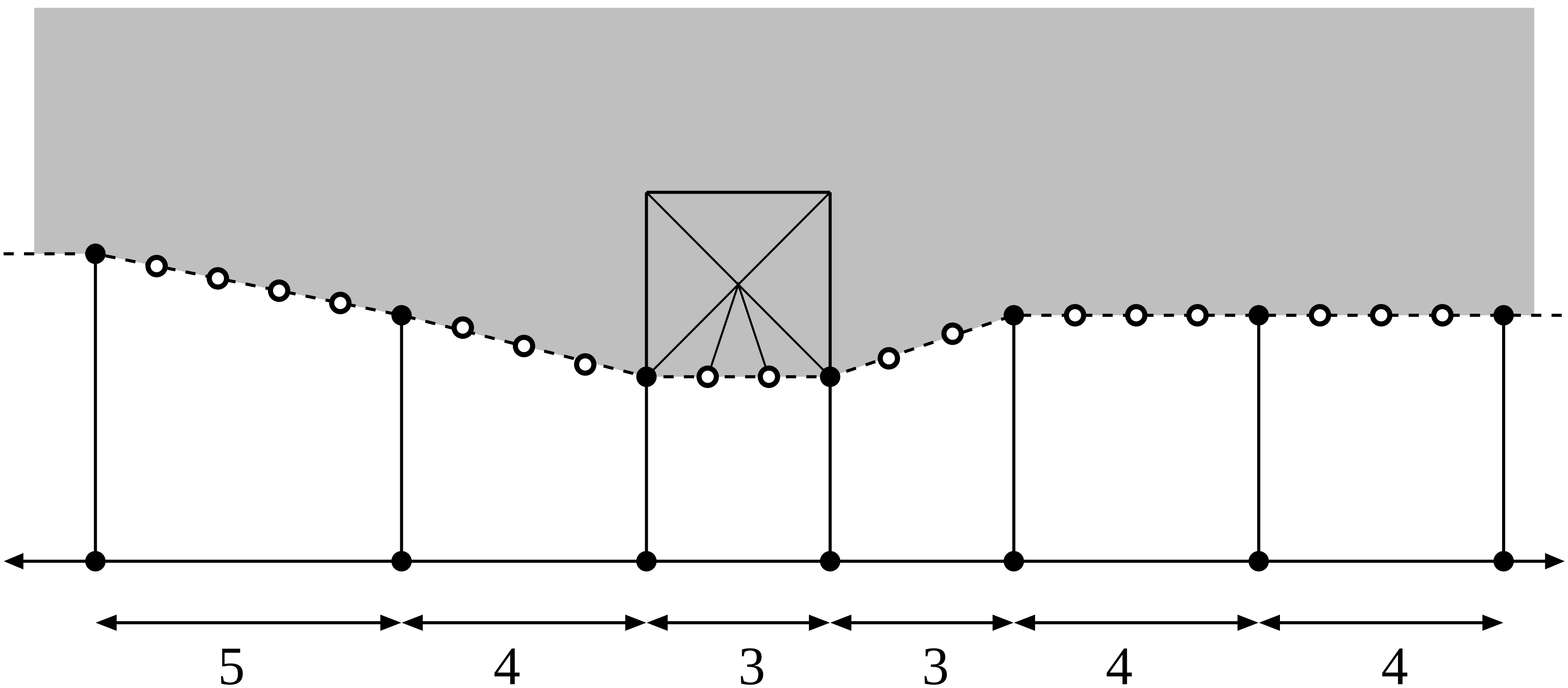


QC map $\mathbb{D}^*$ to shaded region.

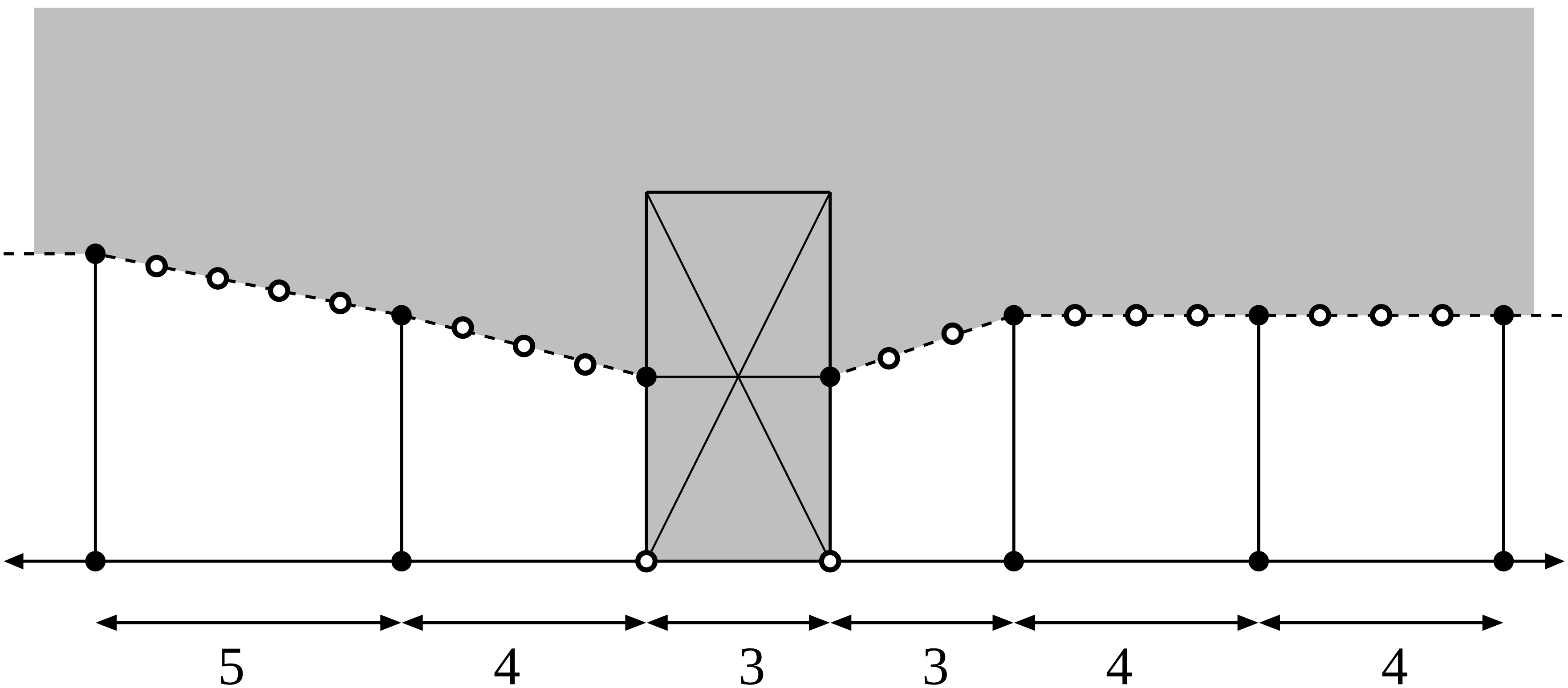
Identity (hence conformal) above dashed line.



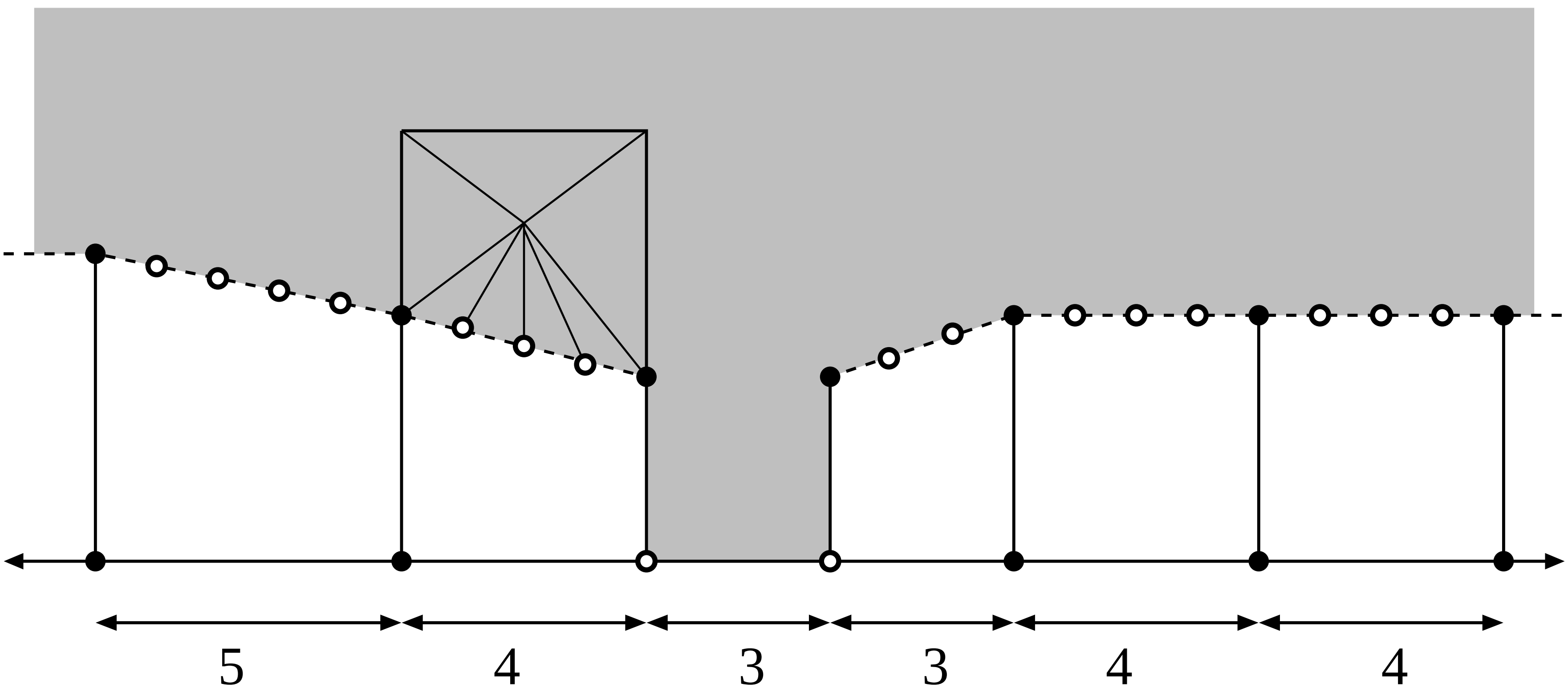
Choose box above region to be filled.



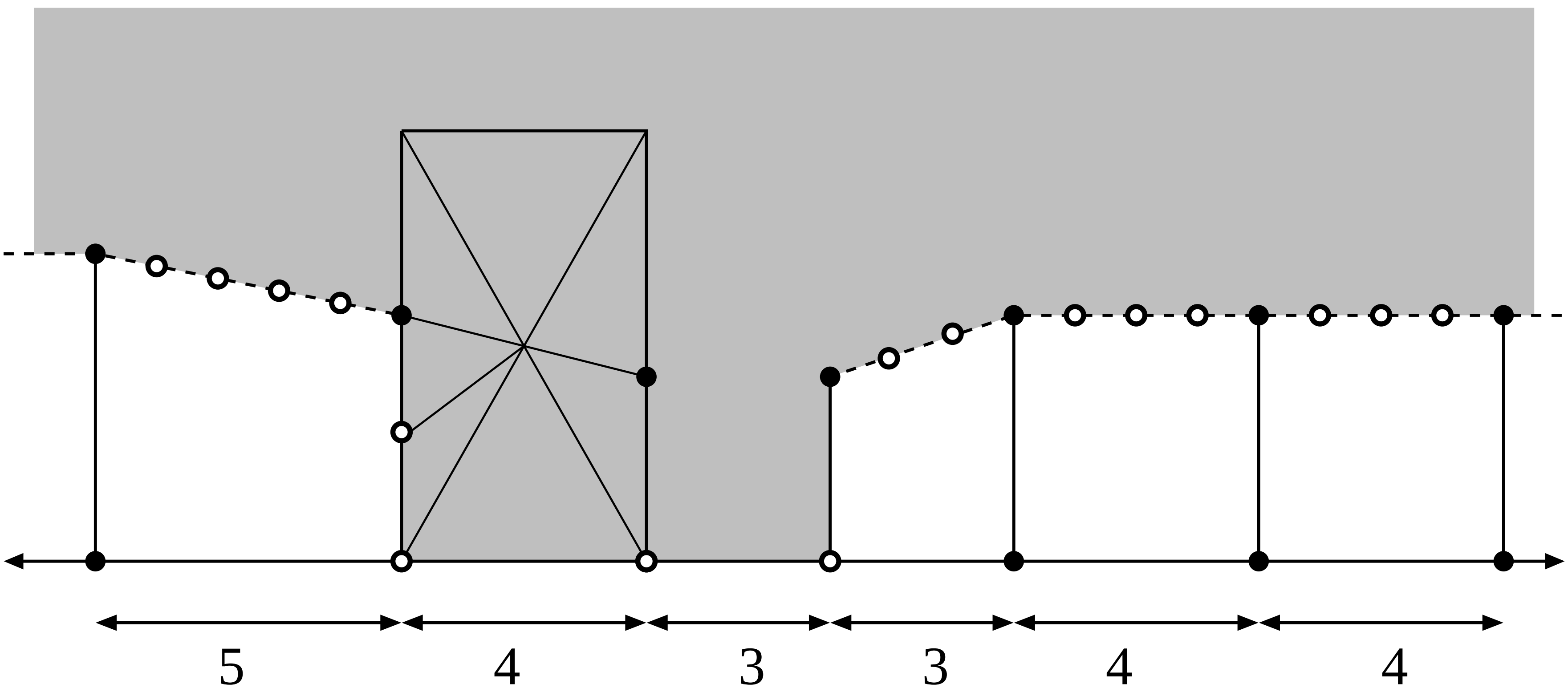
Triangulate box.



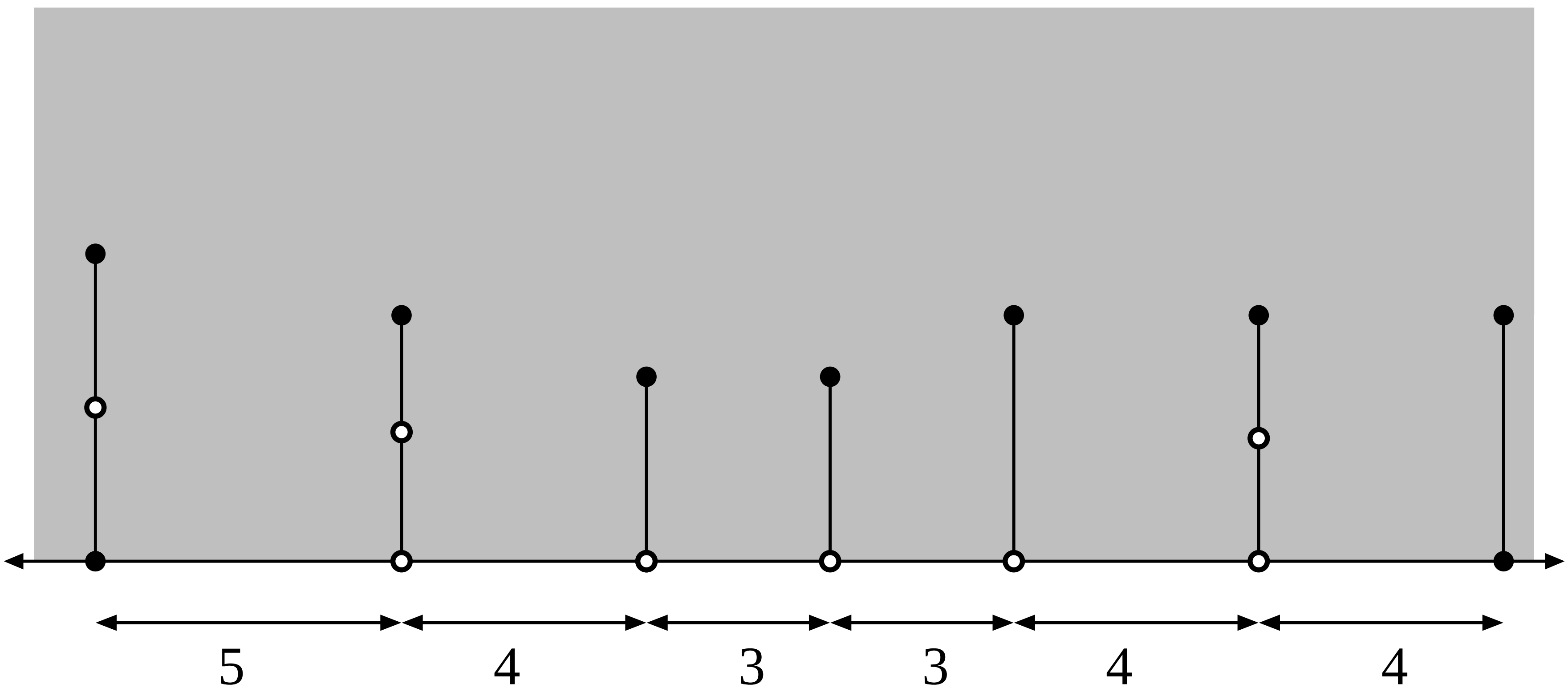
PL map is QC, linear on boundary, identity outside box.



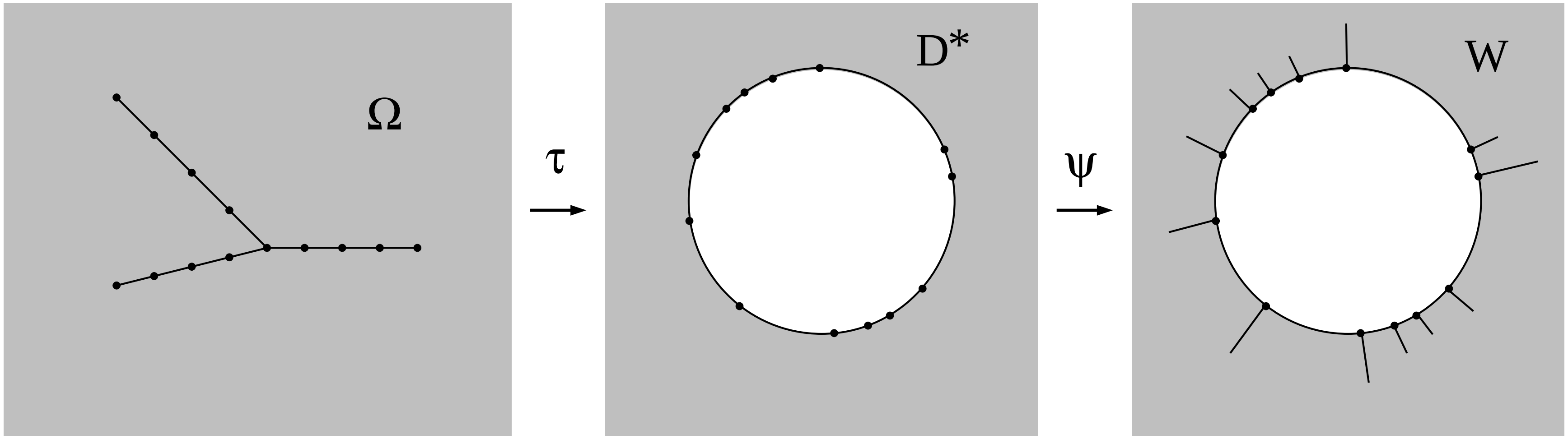
Similar procedure to next region. Triangulate . . .



…and PL map.



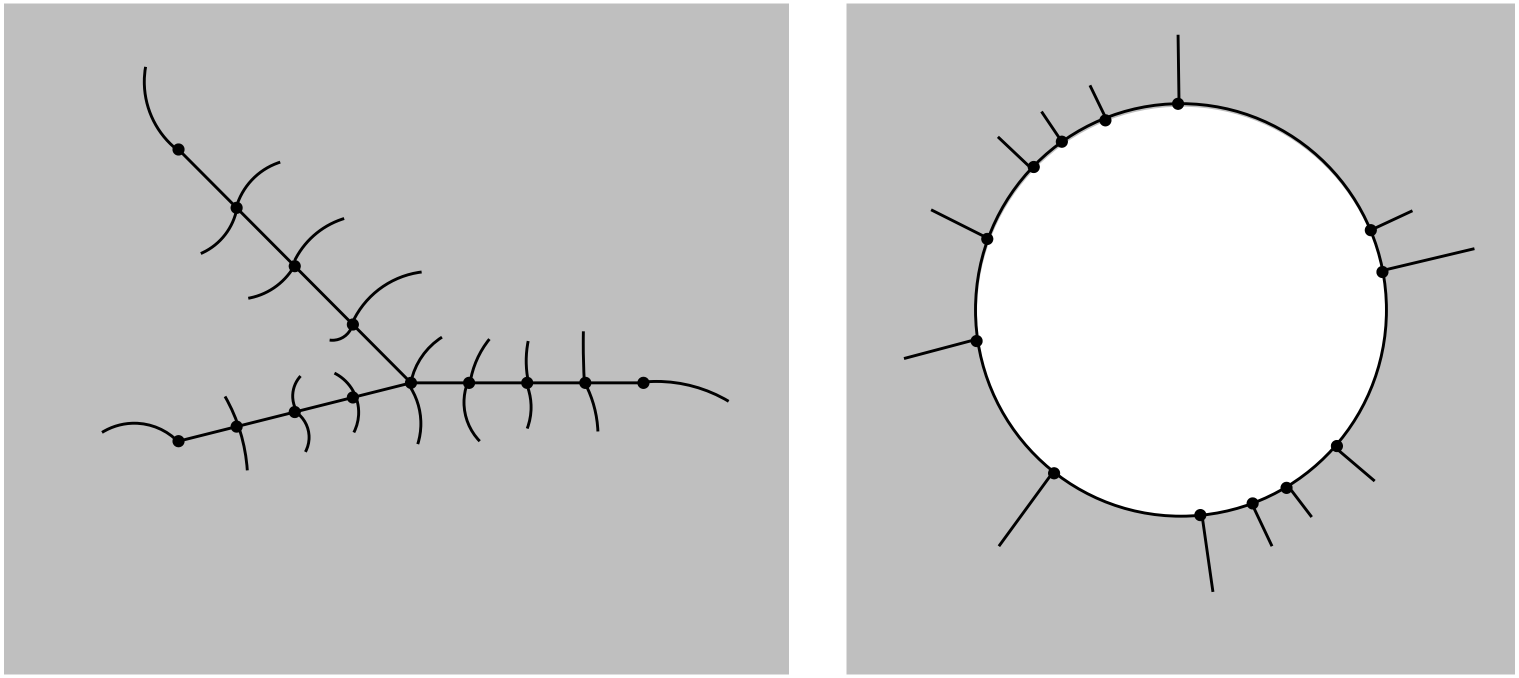
Fill in all regions. Can get uniform QC bounds.



τ is conformal map from $\Omega = \mathbb{C} \setminus T$ to $\mathbb{D}^*$

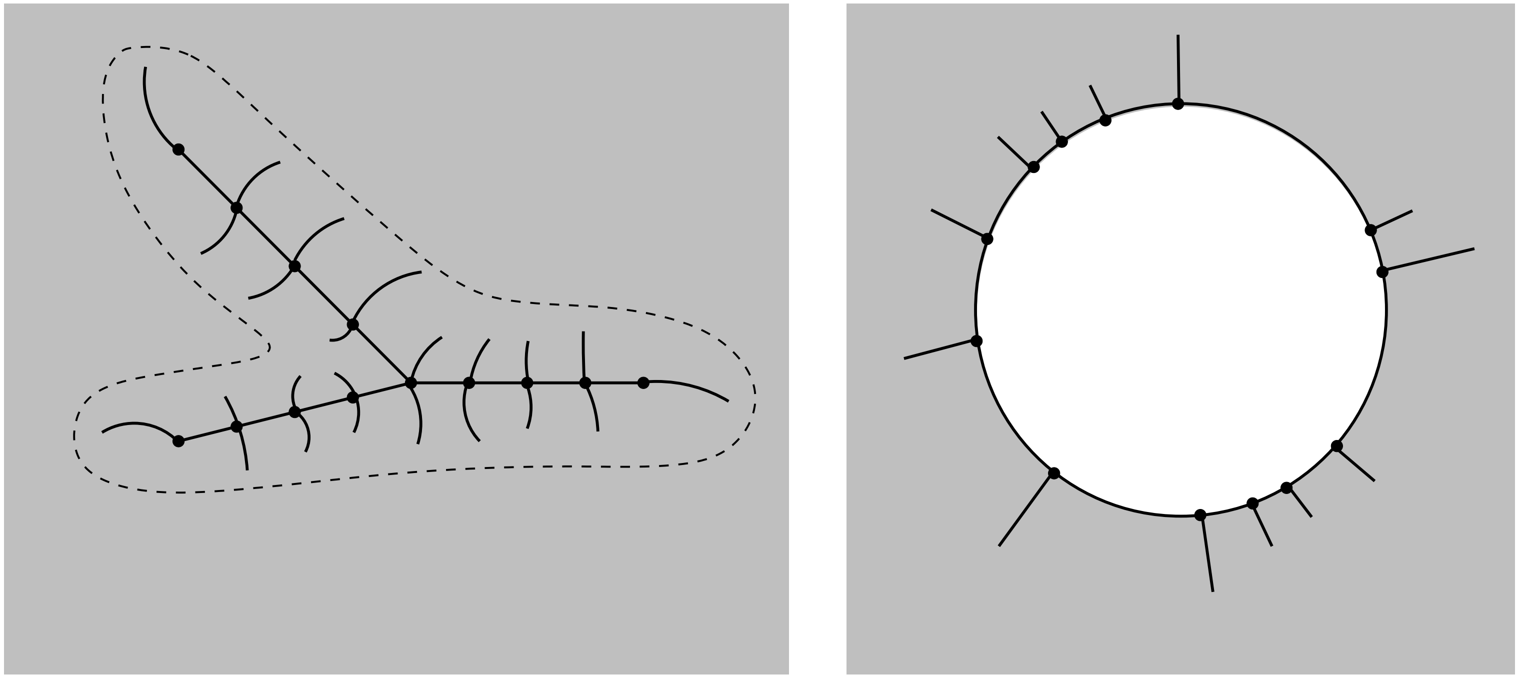
ψ is a QC map from $\mathbb{D}^*$ to $W = \mathbb{D}^* \setminus$ "spikes".

ψ is the "folding map"



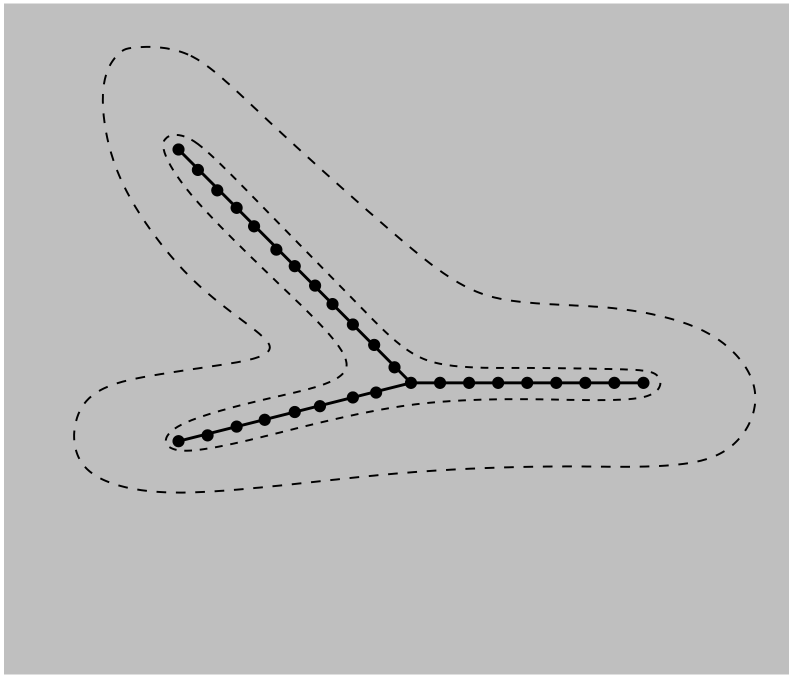
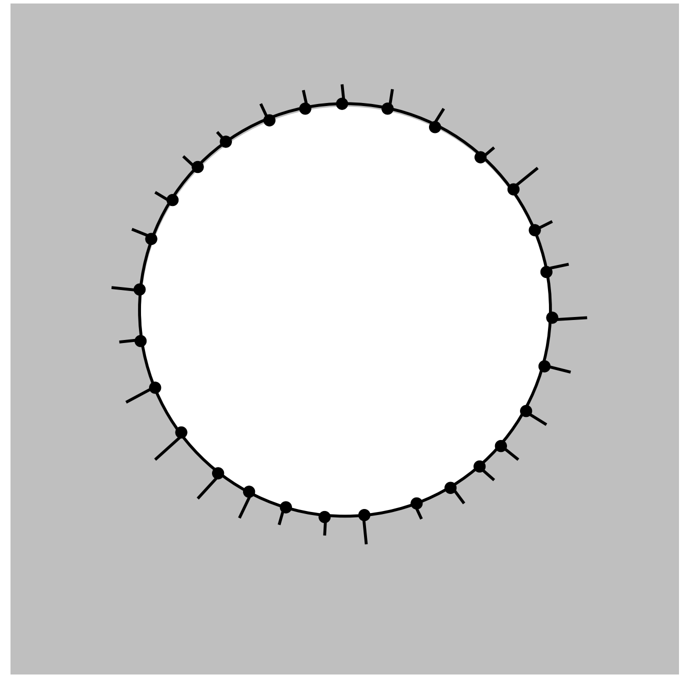
$T' = \tau^{-1}(\psi(\tau(T))) = \tau^{-1}(W)$ is a tree containing T.

By construction, T' is QC-balanced.



New tree lies in neighborhood of original tree.

QC dilatation supported in same neighborhood.

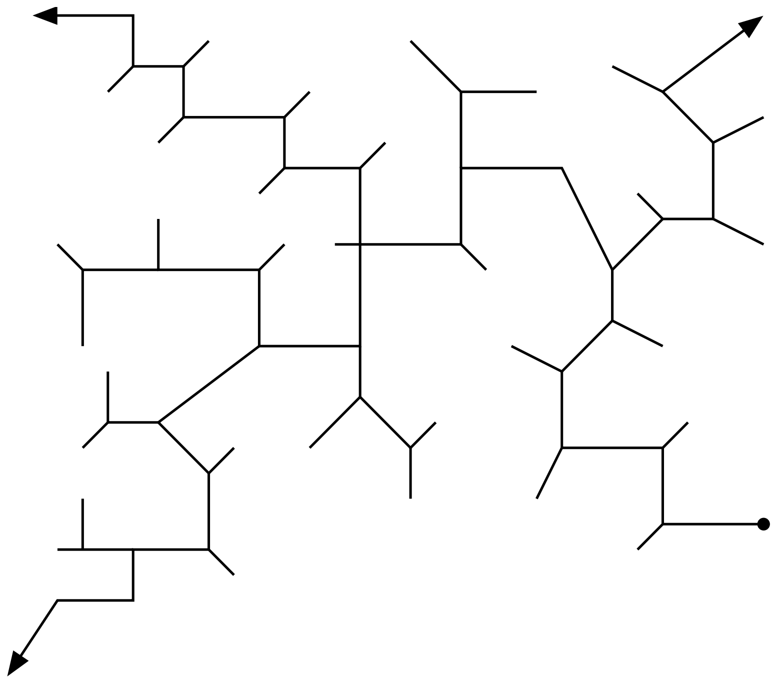
 

Further subdivision gives smaller neighborhood.

QC bound, small area $\Rightarrow$ MRMT correction $\approx$ identity.

Thus every tree is approximated by a true tree.

What about infinite trees?



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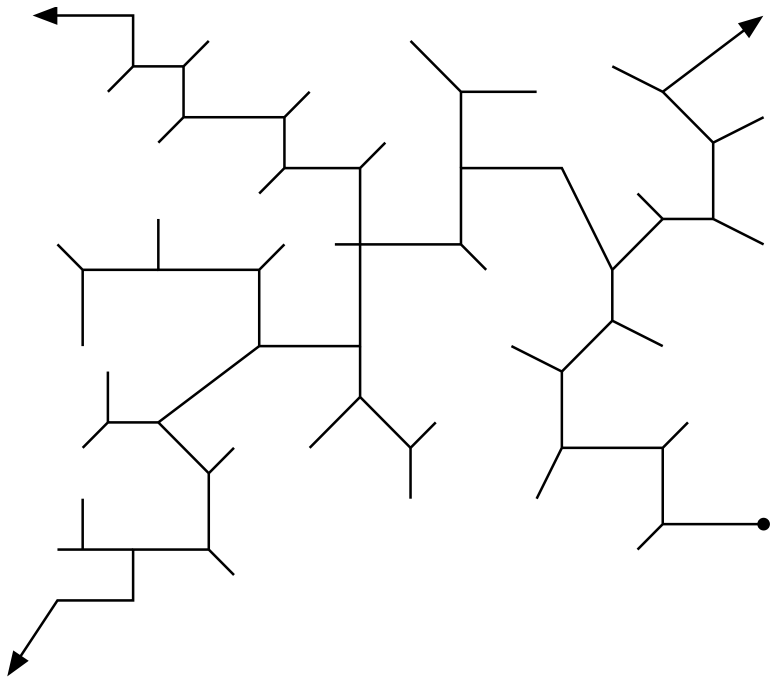
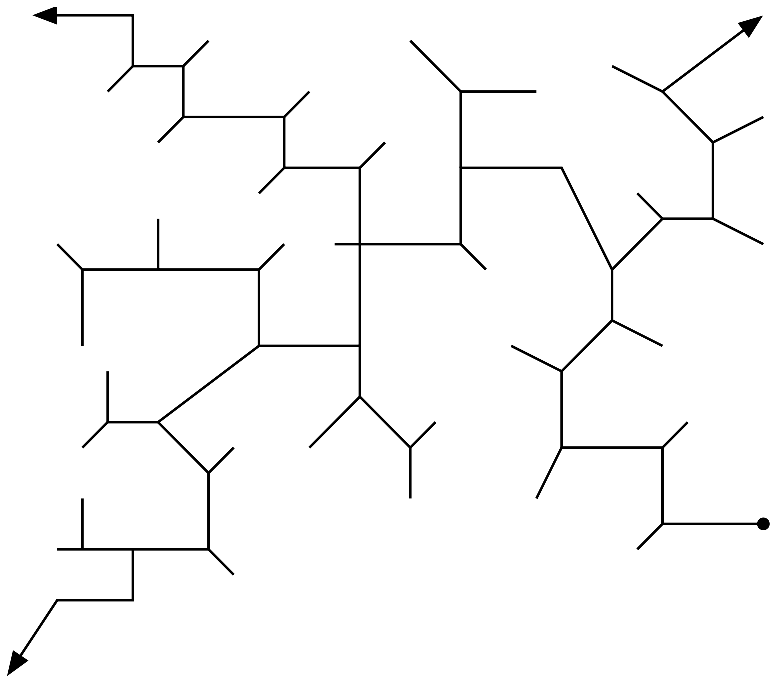


What does “balanced” mean now?

Harmonic measure from ∞ doesn’t make sense.

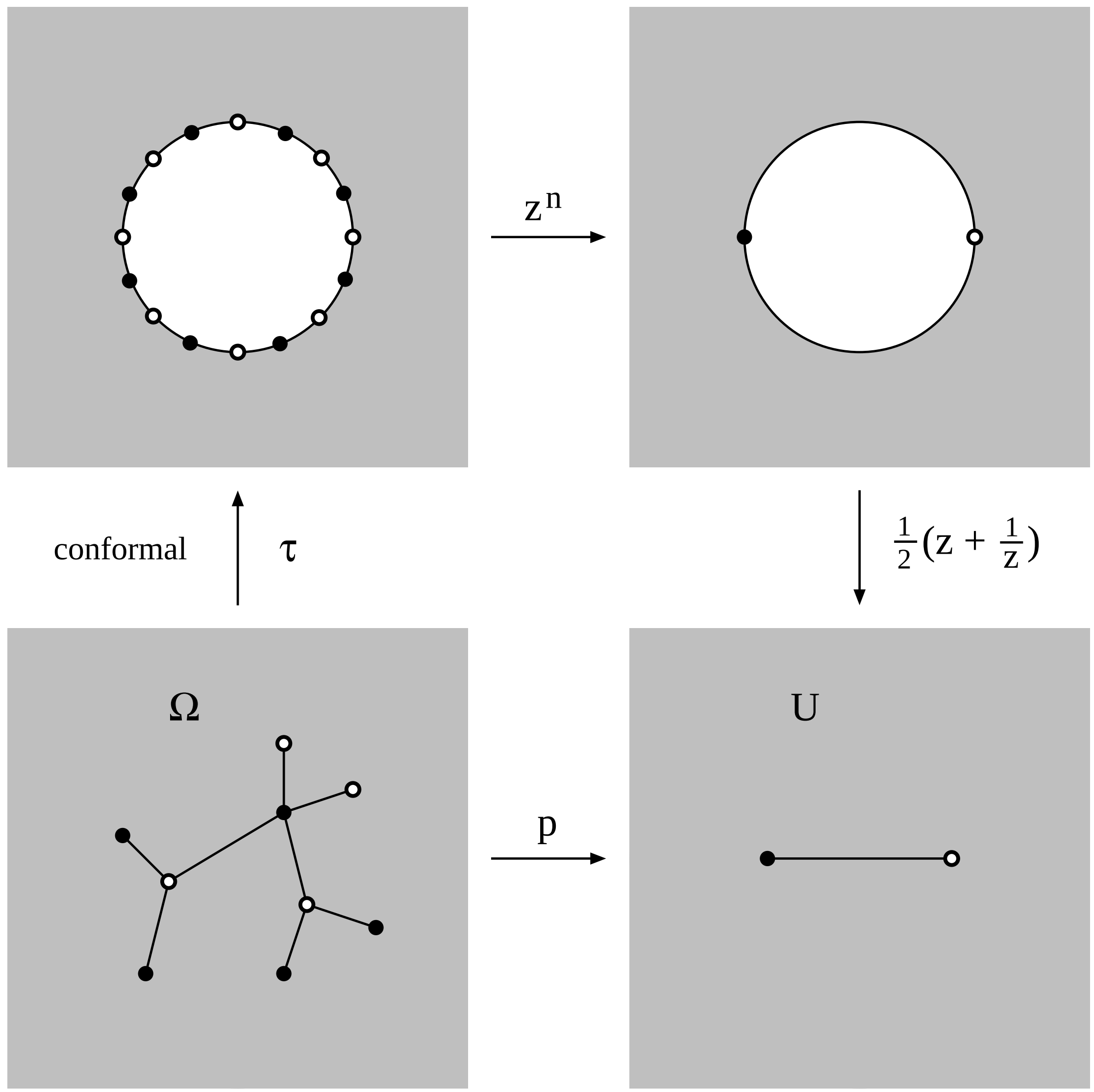
What about infinite trees?



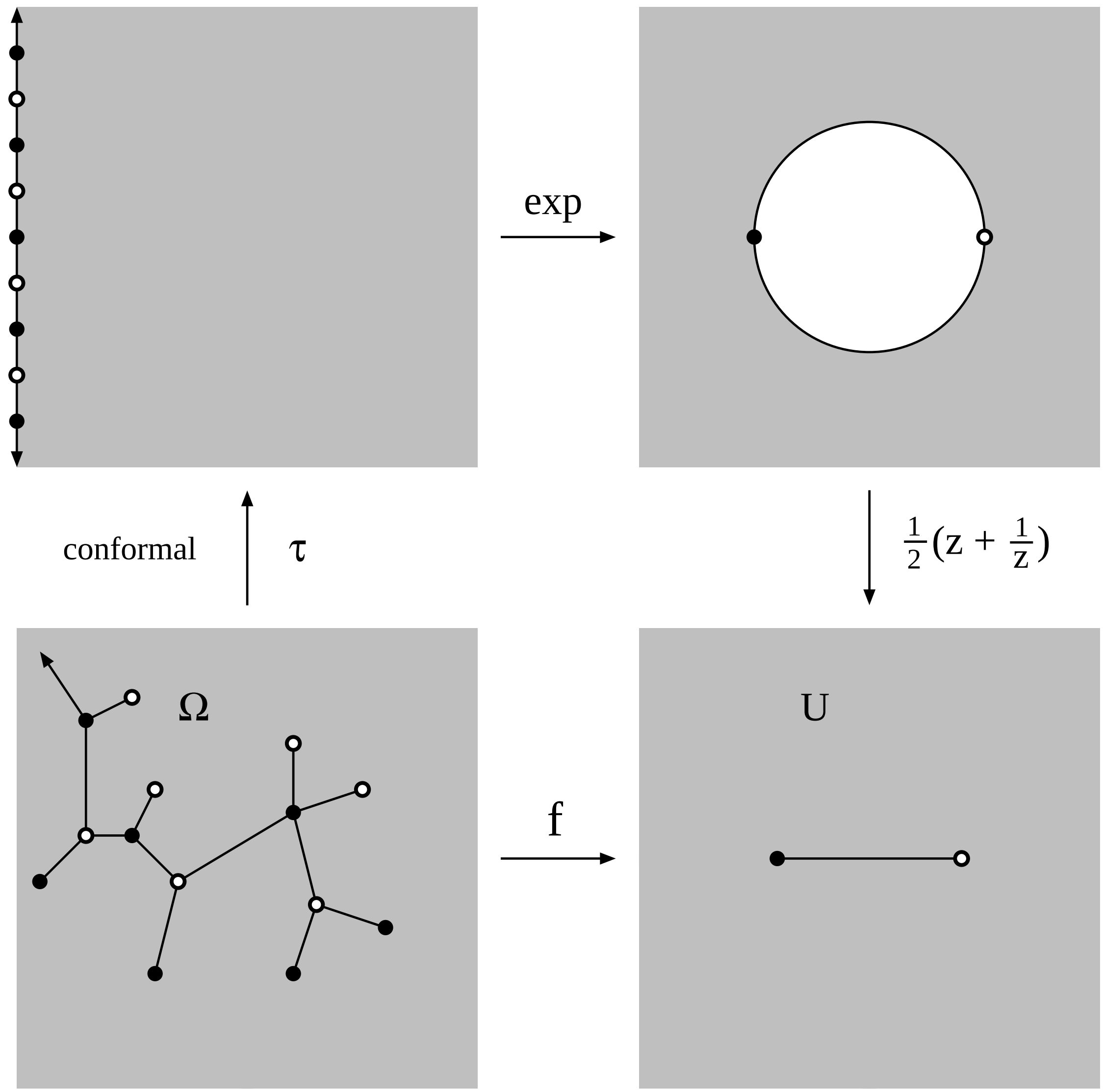
Main difference:

$\mathbb{C}\setminus$ finite tree = one annulus

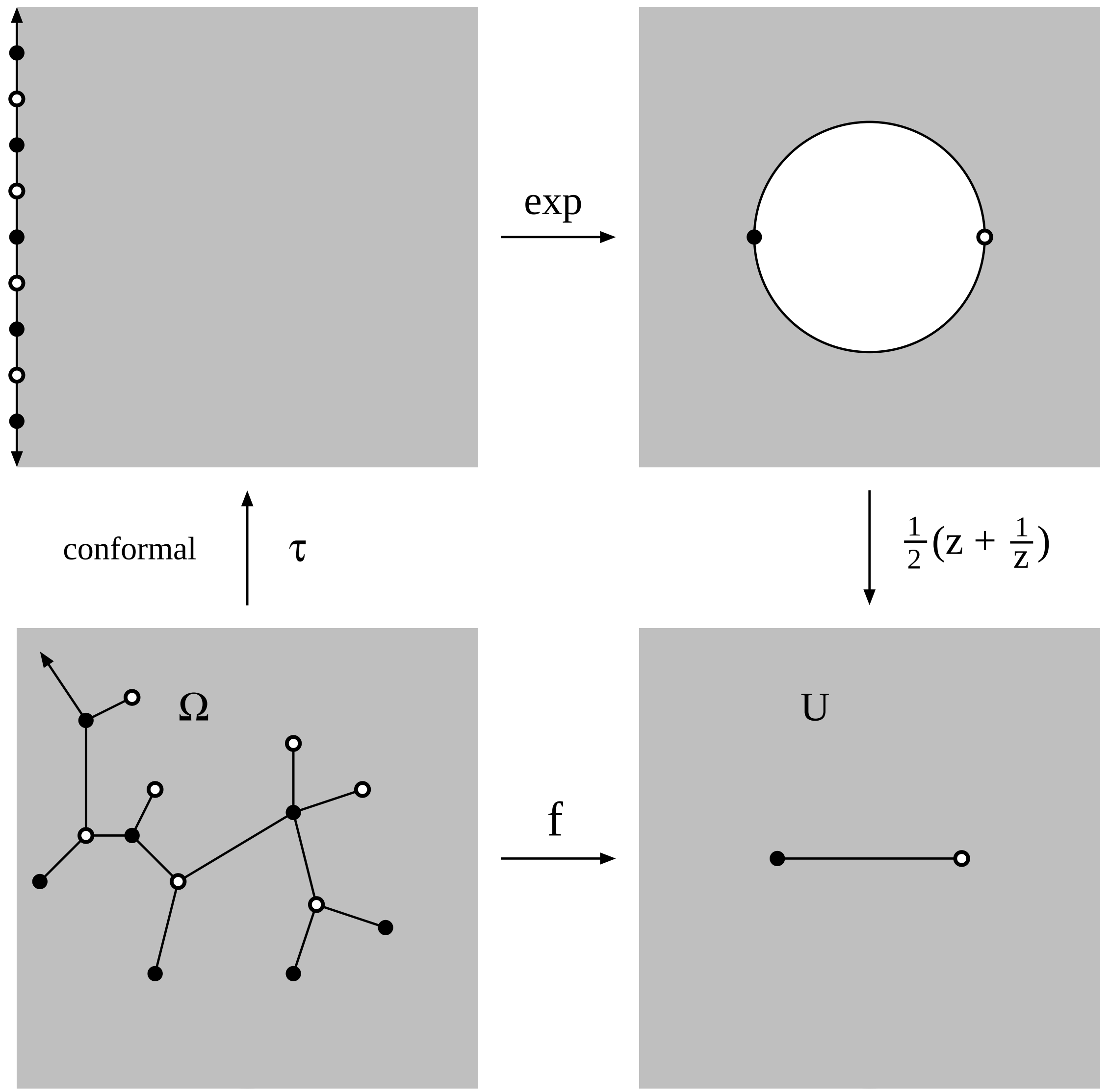
$\mathbb{C}\setminus$ infinite tree = ≥ 1 simply connected components



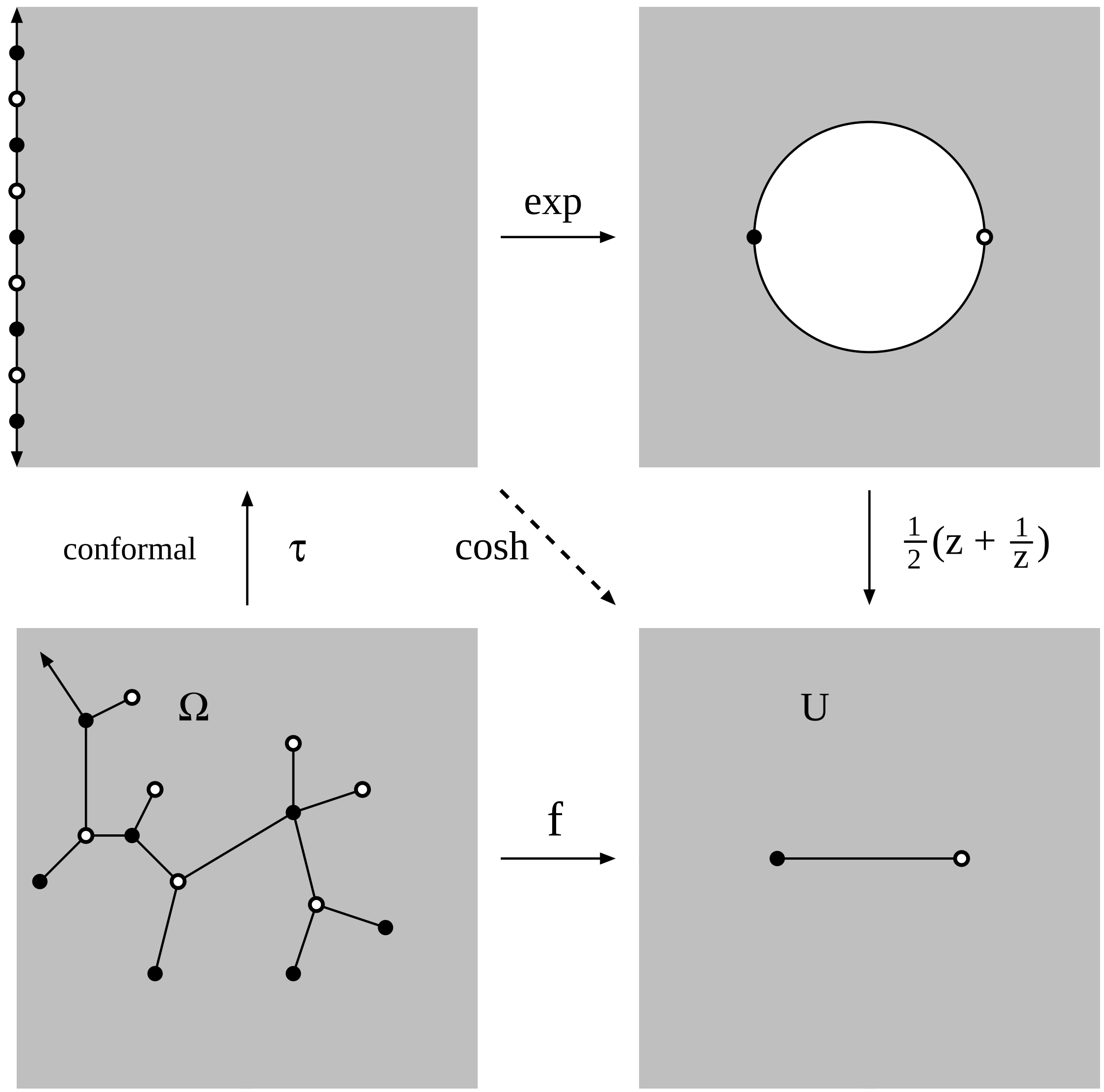
Recall finite case. Infinite case is very similar.



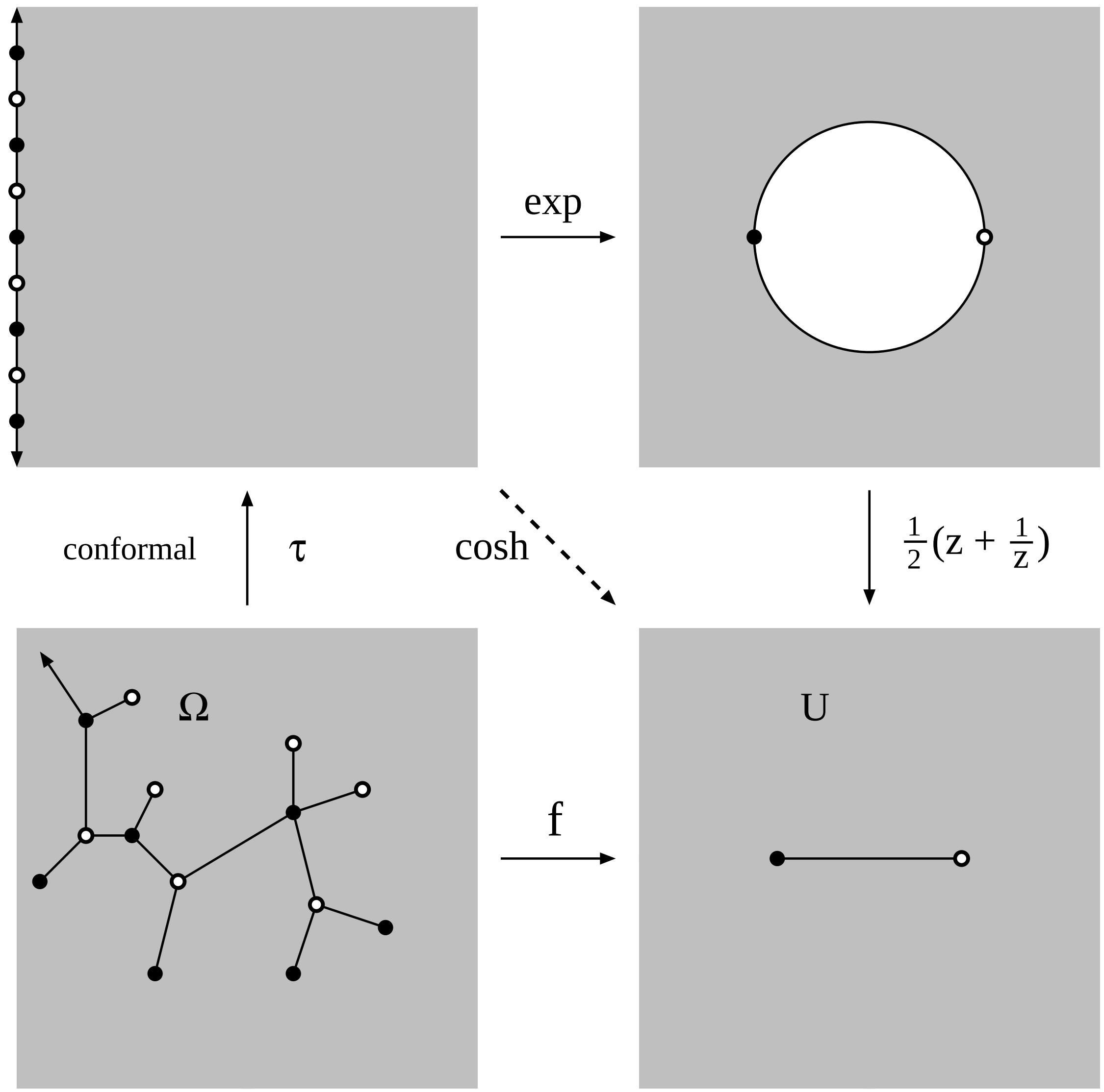
τ maps components of $\mathbb{C} \setminus T$ to right half-plane.



Pullback length to tree. Every side gets τ-length π.



Balanced tree $\Leftrightarrow f = \cosh \circ \tau$ is entire, $\mathrm{CV(f)} = \pm 1$.



Is every “nice” infinite tree QC-balanced?

Problem: Given an infinite tree T, build a quasi-regular g with $\mathrm{CV}(g) = \pm 1$ and $T \approx g^{-1}([-1, 1])$.

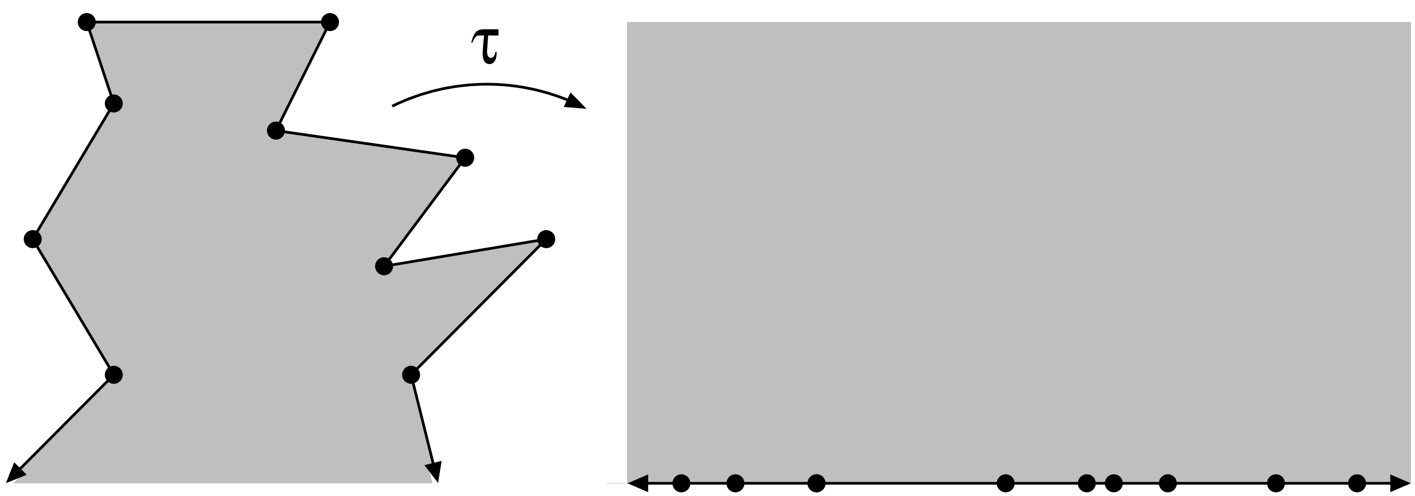
We make two assumptions about T.

Problem: Given an infinite tree T, build a quasi-regular g with $\mathrm{CV}(g) = \pm 1$ and $T \approx g^{-1}([-1,1])$.

We make two assumptions about T.

1. Adjacent sides have comparable τ-length (local)

2. τ-lengths have positive lower bound (global)



1. Adjacent sides have comparable τ-length.

This follows if T has **bounded geometry**:

- edges are uniformly C^2
- angles are bounded away from 0
- adjacent edges have comparable lengths
- non-adjacent edges satisfy $\mathrm{diam}(e) \leq C \mathrm{dist}(e, f)$.

This is usually **easy** to check.

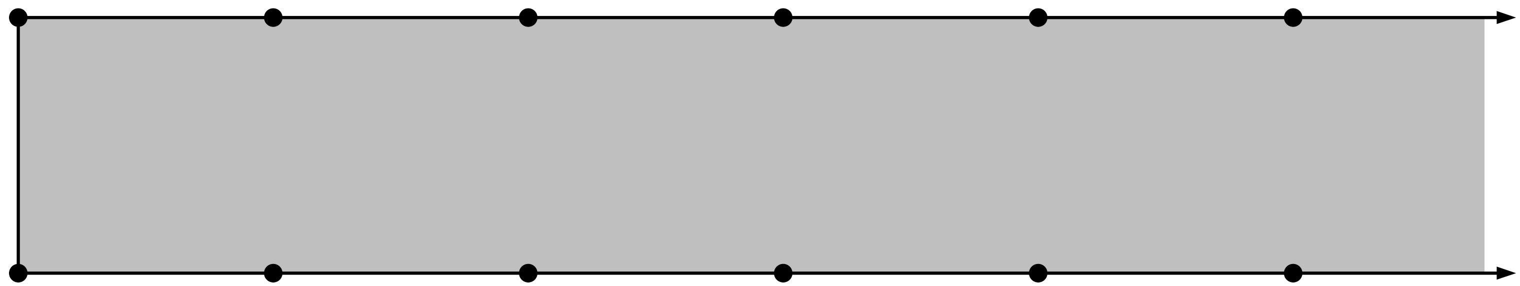
2. τ-lengths have positive lower bound.

Verifying this is usually straightforward.

e.g., read Garnett and Marshall *Harmonic Measure*.

Use standard estimates for hyperbolic metric, extremal length, harmonic measure ...

2. τ-lengths of sides have positive lower bound.



"inside" the τ-lengths grow exponentially. (good)

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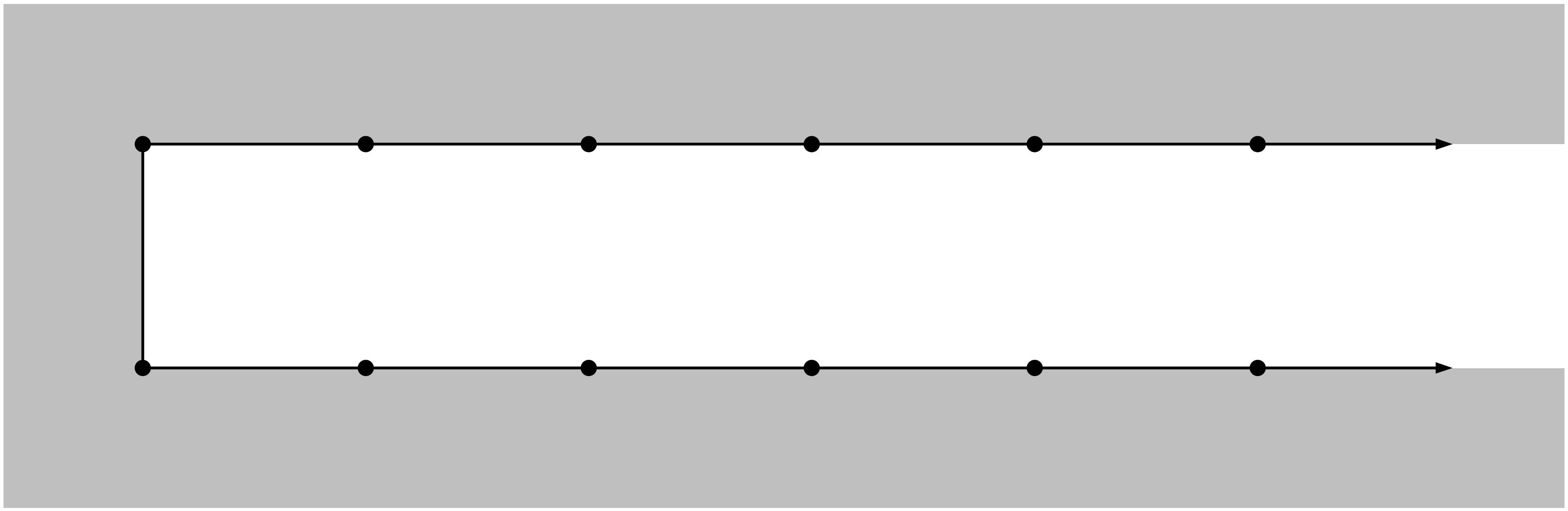
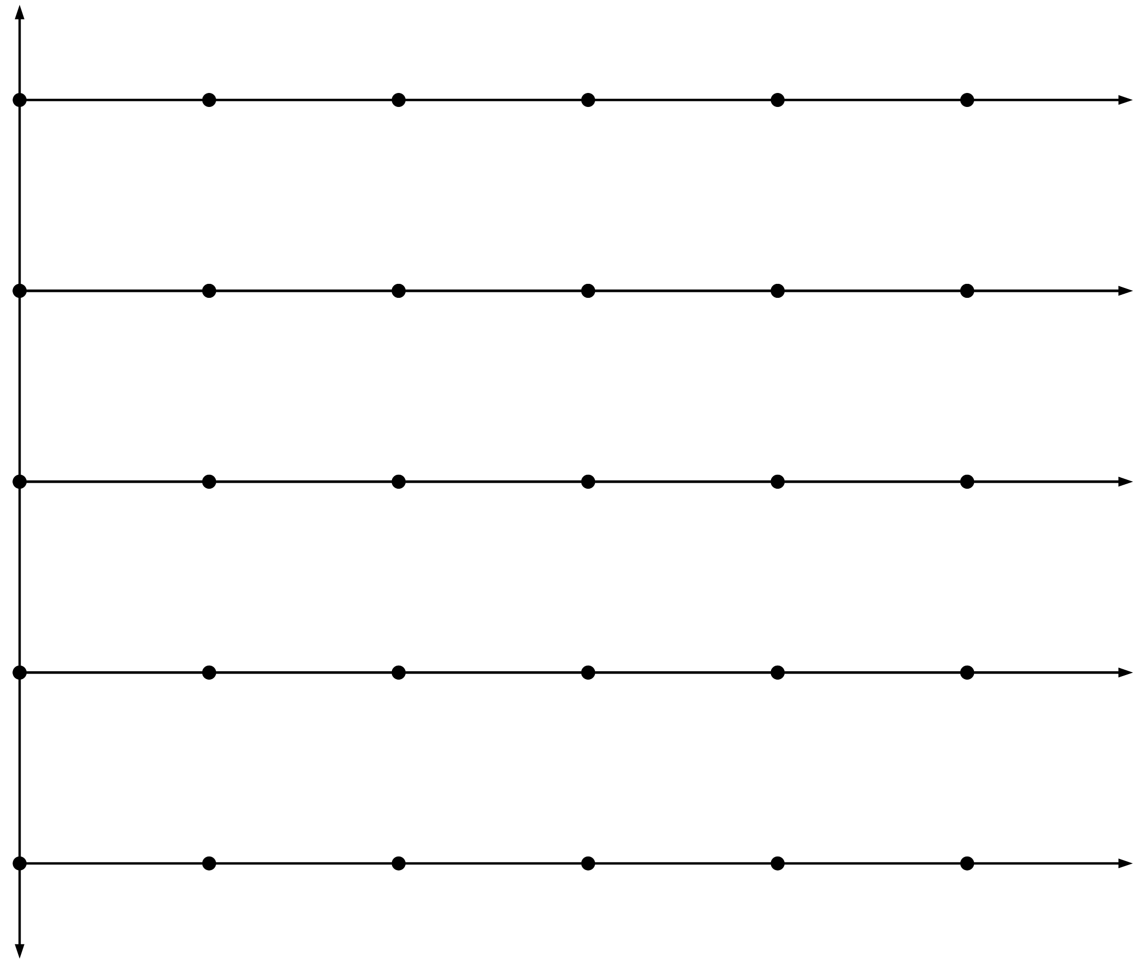


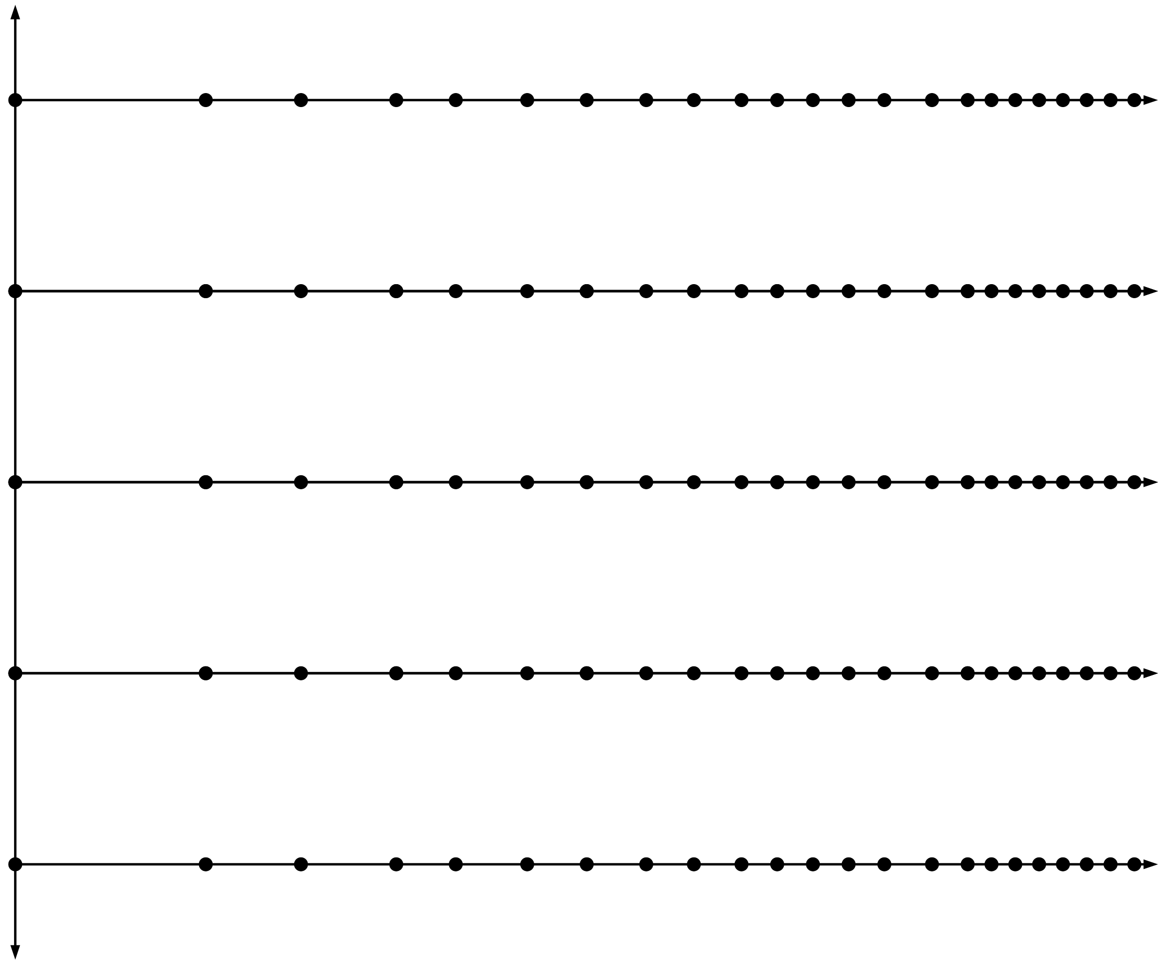
"inside" the τ-lengths grow exponentially. (good)

"outside" the τ-length decrease like $n^{-1/2}$. (bad)

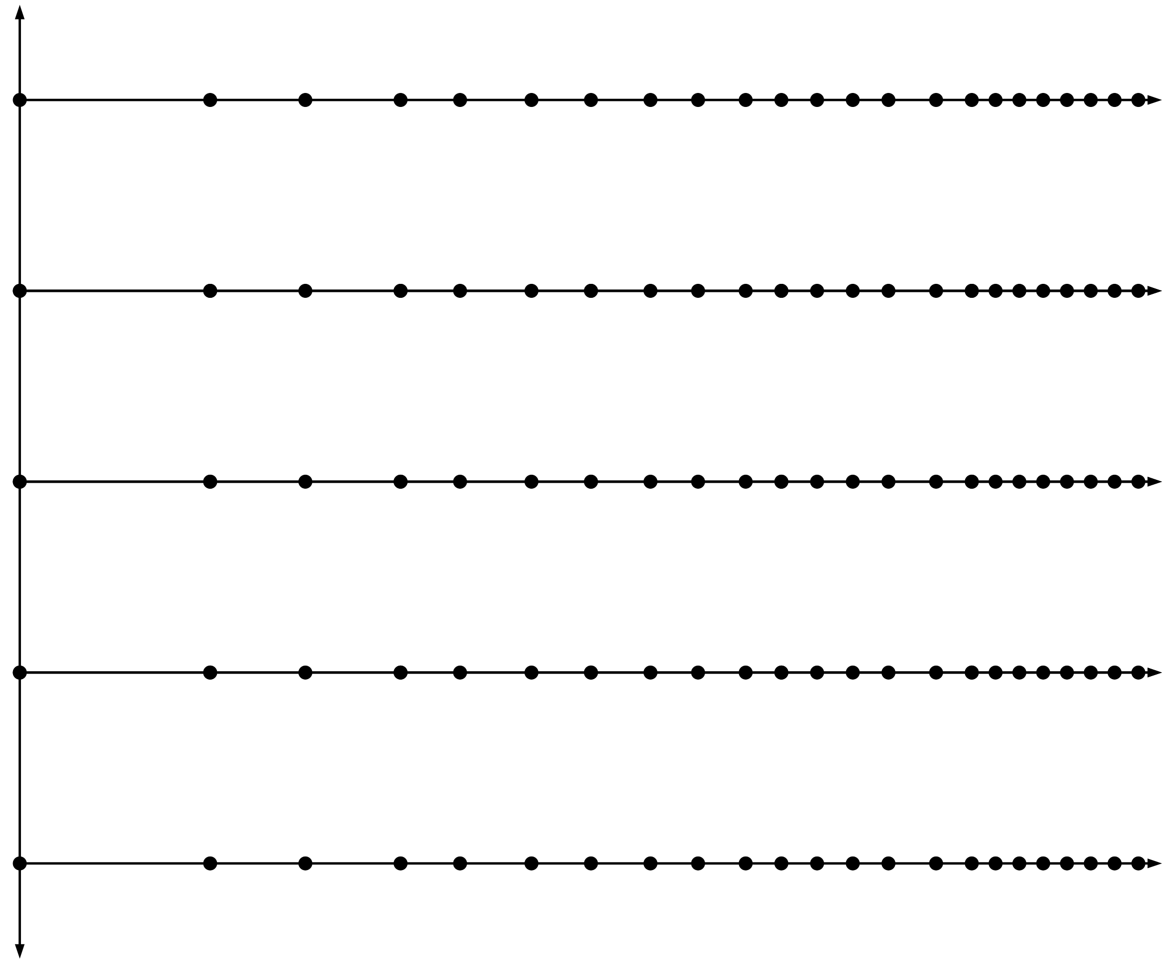
τ-lengths are **not** bounded below.



Bounded geometry and τ-bounded.



Also bounded geometry and τ-bounded.



This tree is “better” than last one. Why?

If e is an edge of T and $r > 0$ let

$$e(r) = \{z : \mathrm{dist}(z, e) \le r \cdot \mathrm{diam}(e)\}$$

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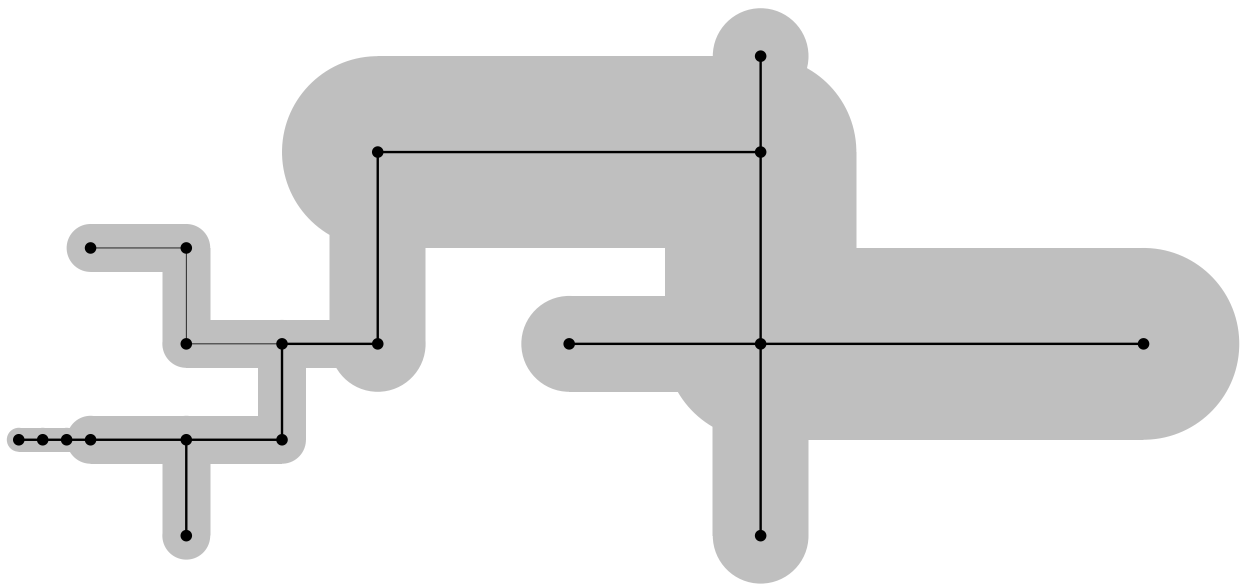
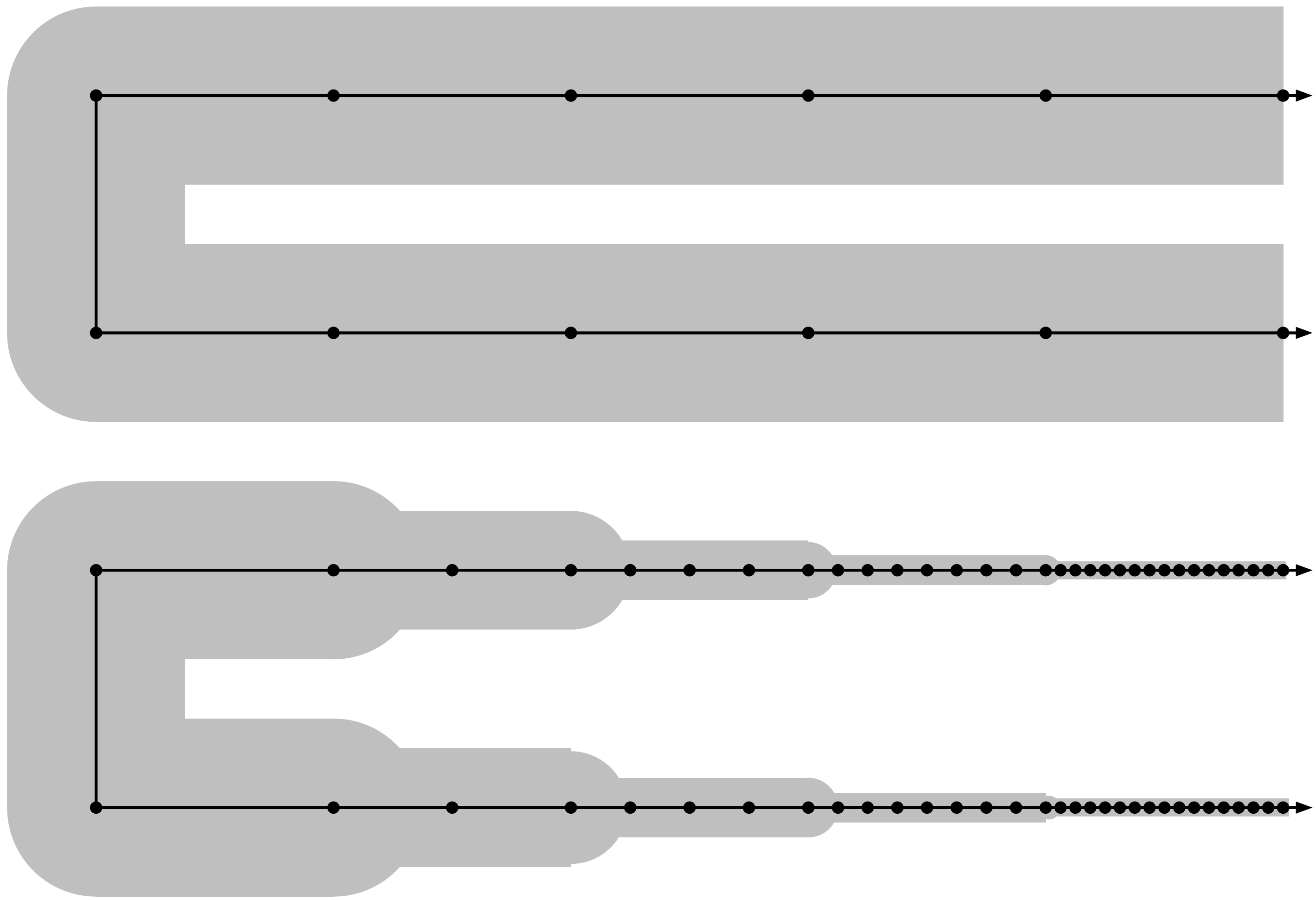
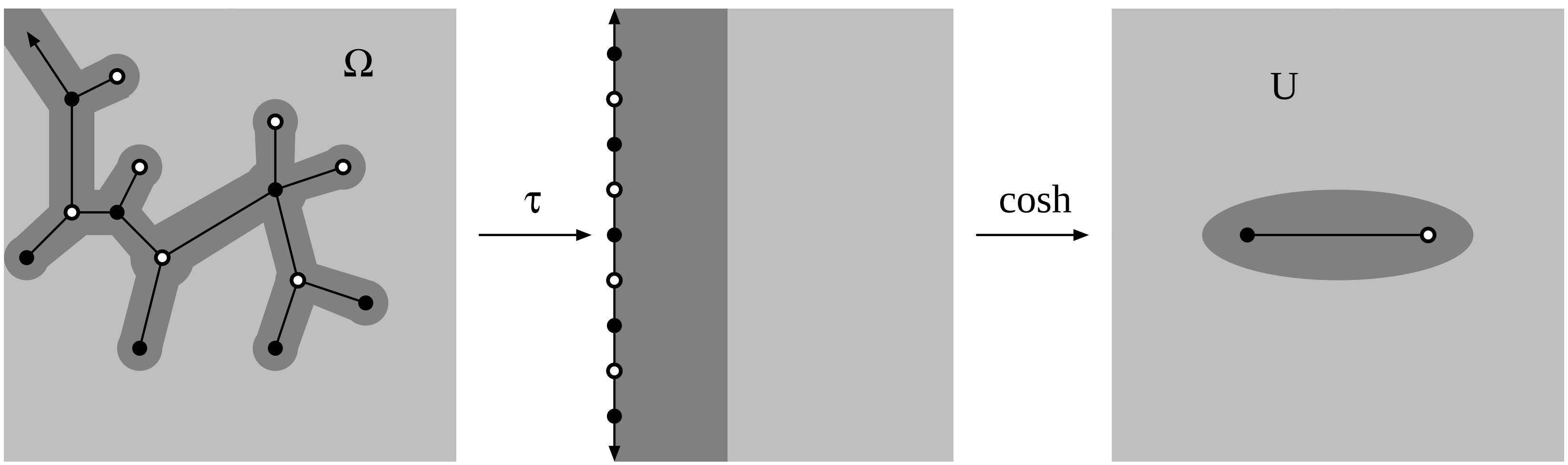
Define neighborhood of T: $T(r) = \cup\{e(r) : e \in T\}$.



We use QC maps with dilatations supported in $T(r)$.
If $T(r)$ is small, we get better control.



QC Folding Thm: Suppose T has bounded geometry and all τ-lengths $\geq \pi$. Then there is a quasiregular g s.t. $g = \cosh \circ \tau$ off $T(r)$ and $\mathrm{CV}(g) = \pm 1$.



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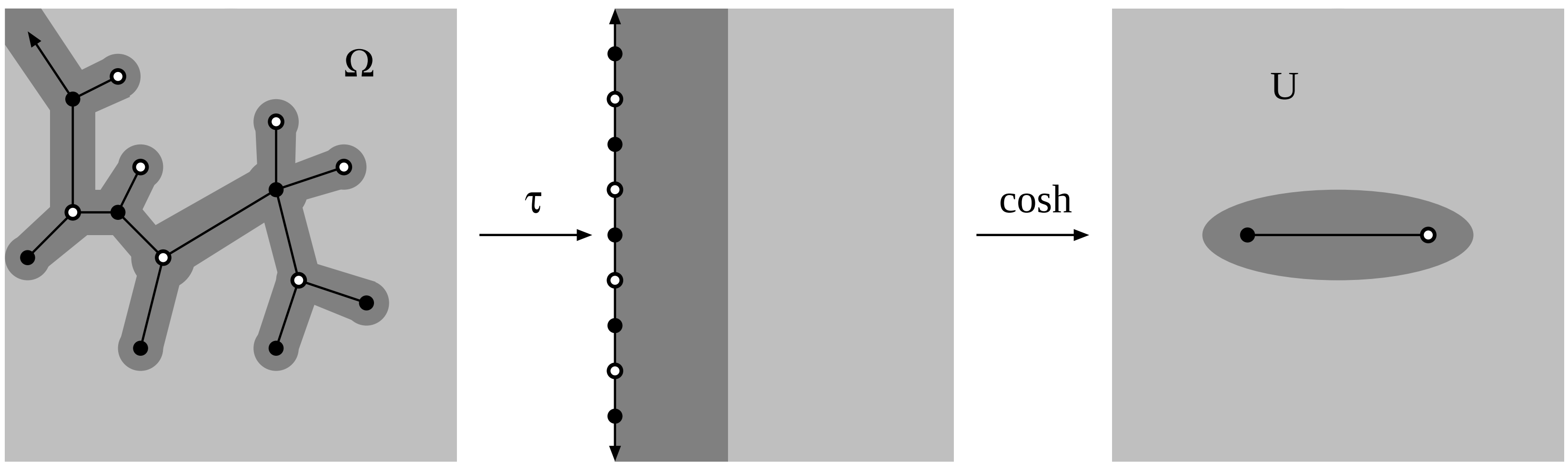


- $\exists$ QC-tree T' s.t. $T \subset T' \subset T(r)$.

QC Folding Thm: Suppose T has bounded geometry and all τ-lengths $\geq \pi$. Then there is a quasiregular g s.t. $g = \cosh \circ \tau$ off $T(r)$ and $\mathrm{CV}(g) = \pm 1$.

- $|\mu_\phi| \leq K$ and $\mathrm{supp}(\mu_\phi) \subset T(r)$.

- K and r depend only on bounded geometry constants.

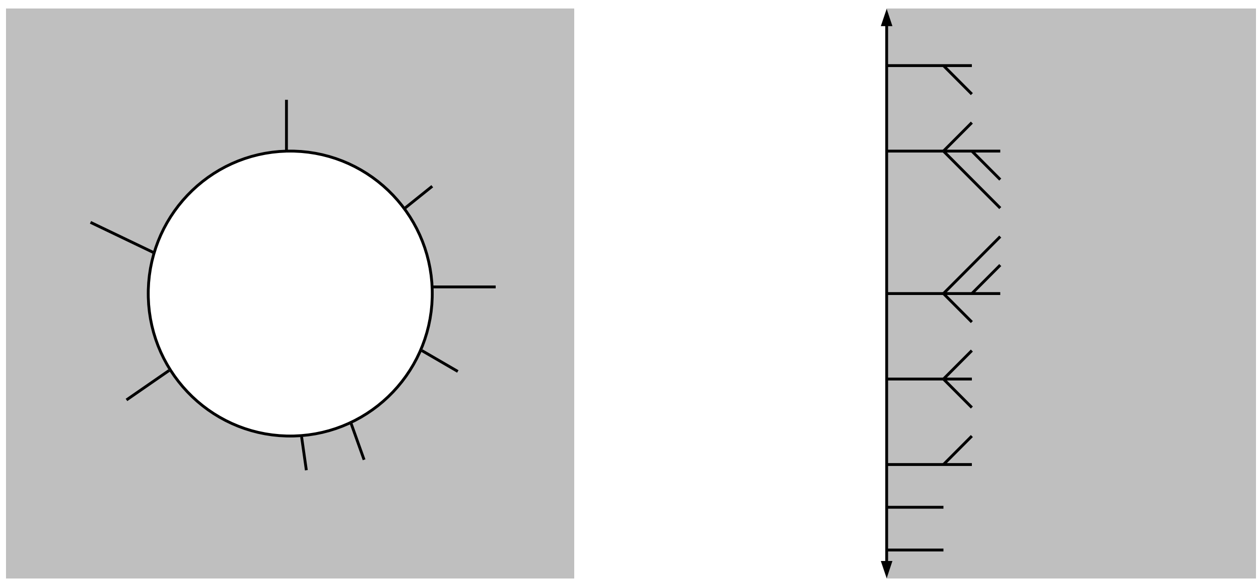
- Can shrink $T(r)$ by subdividing T and rescaling τ.

Cor: There is an entire function f with $\mathrm{CV}(f) = \pm 1$ so that $f^{-1}([-1, 1])$ approximates T.

Finite vs infinite case:

• The spikes in finite case could have large diameter, but we "shrunk" them by subdividing tree.

• In infinite case we add trees that may have unbounded complexity, but uniformly bounded diameter.



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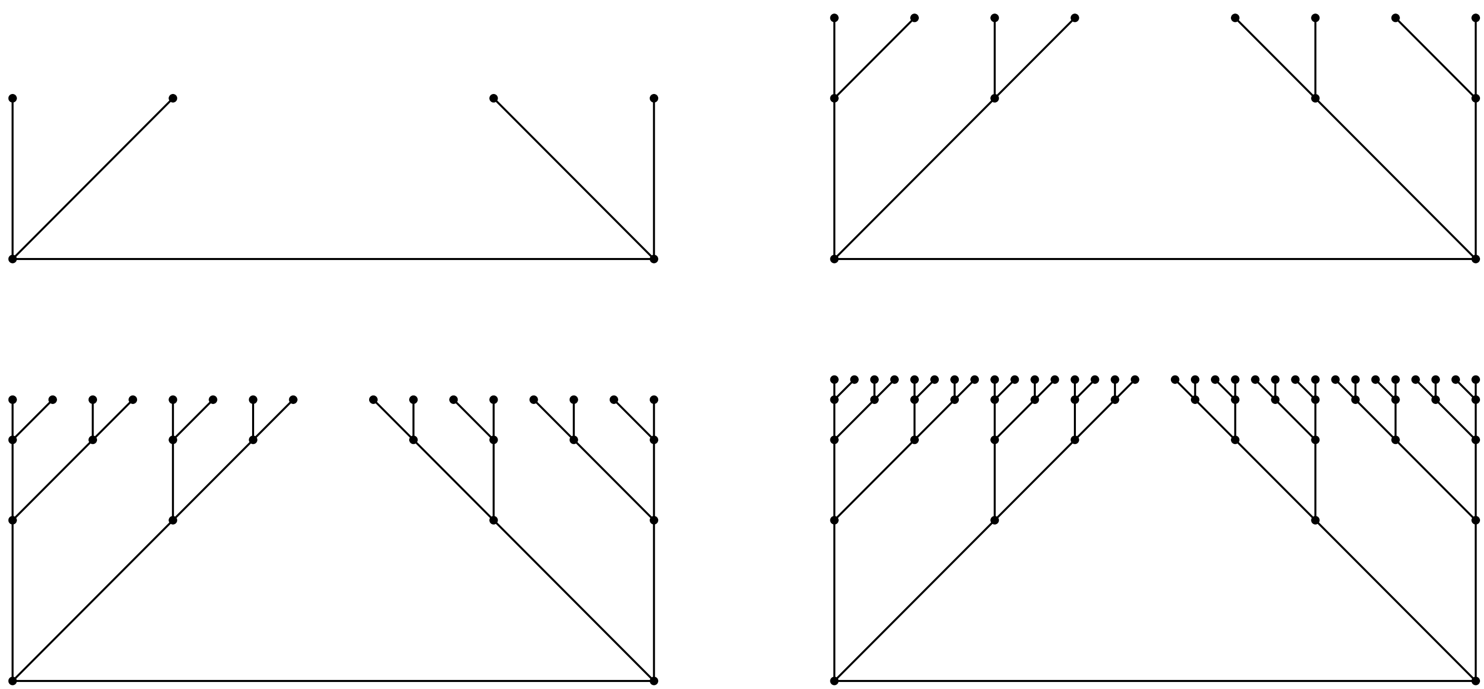
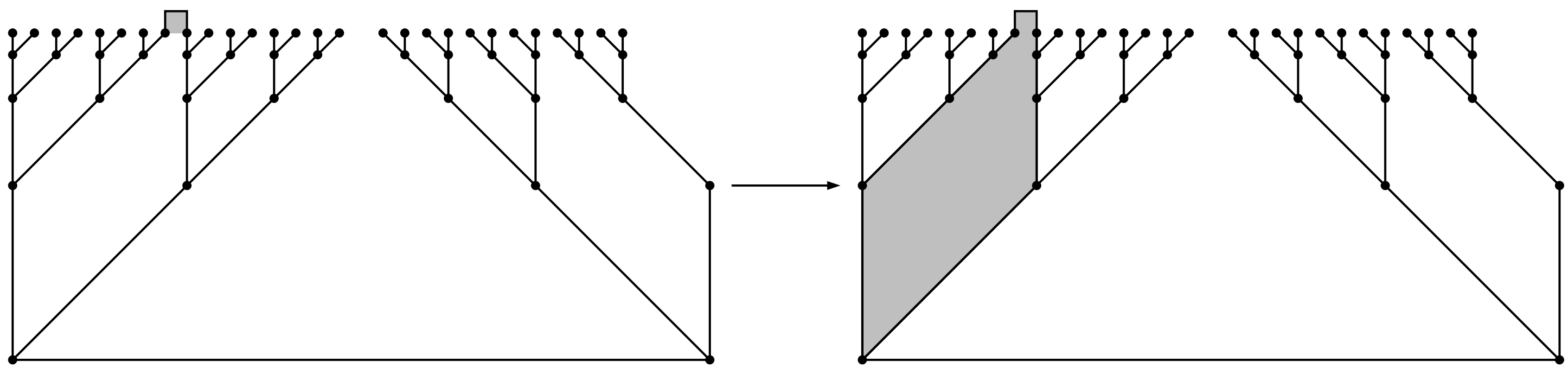
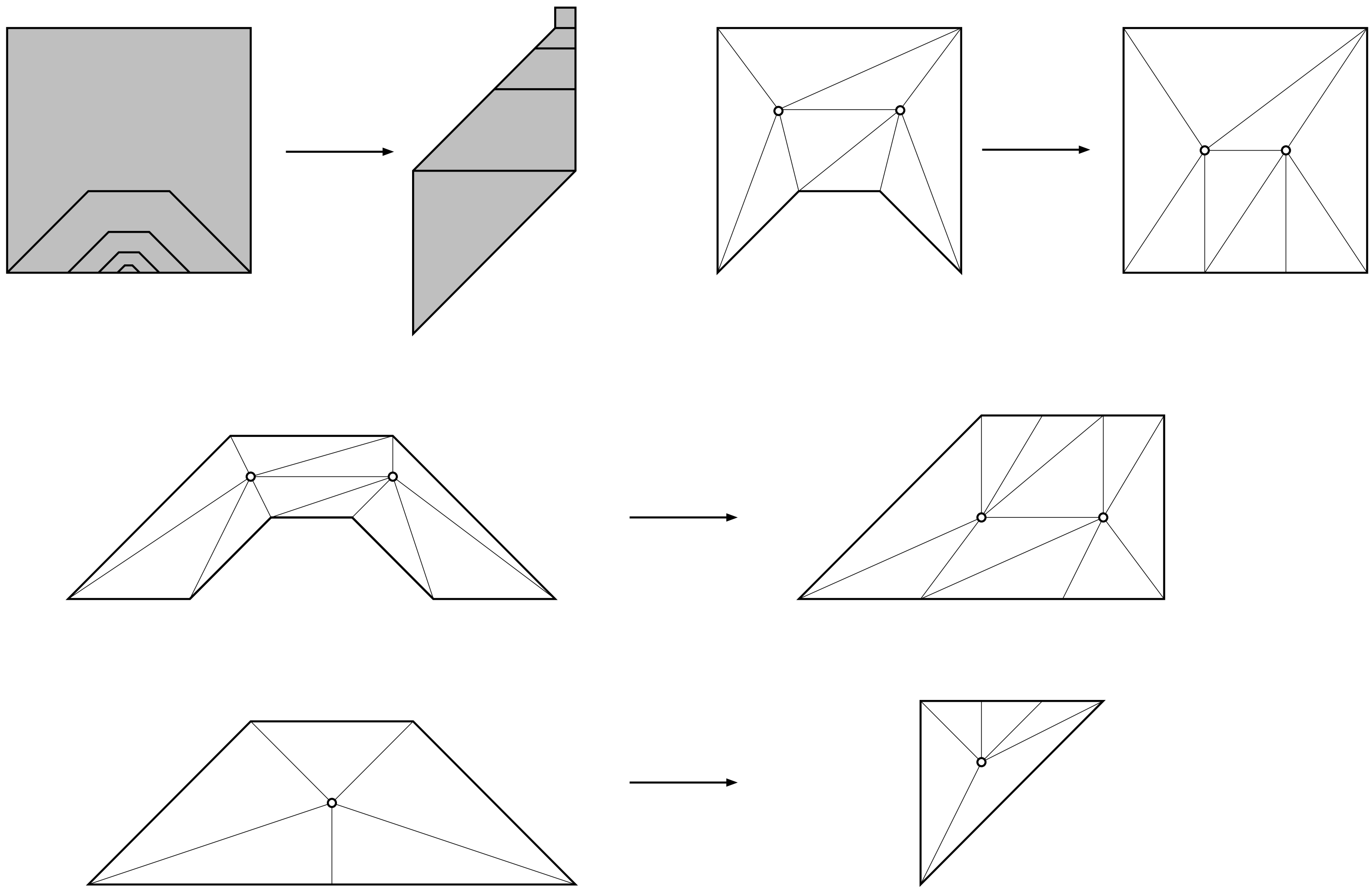


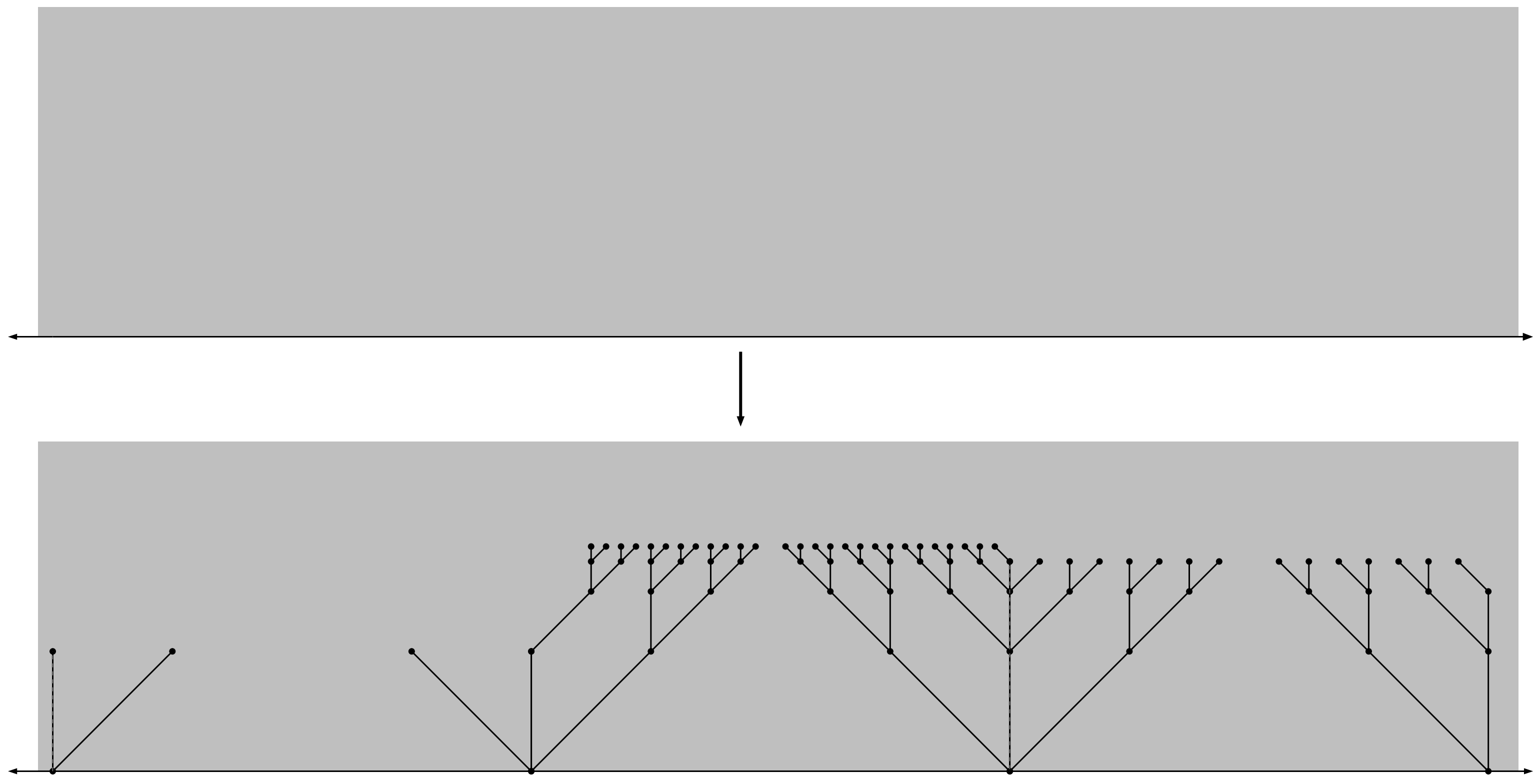
Rest of proof is pretty much the same.

We use PL maps to send squares into polygonal regions.



Detail of one filling map.



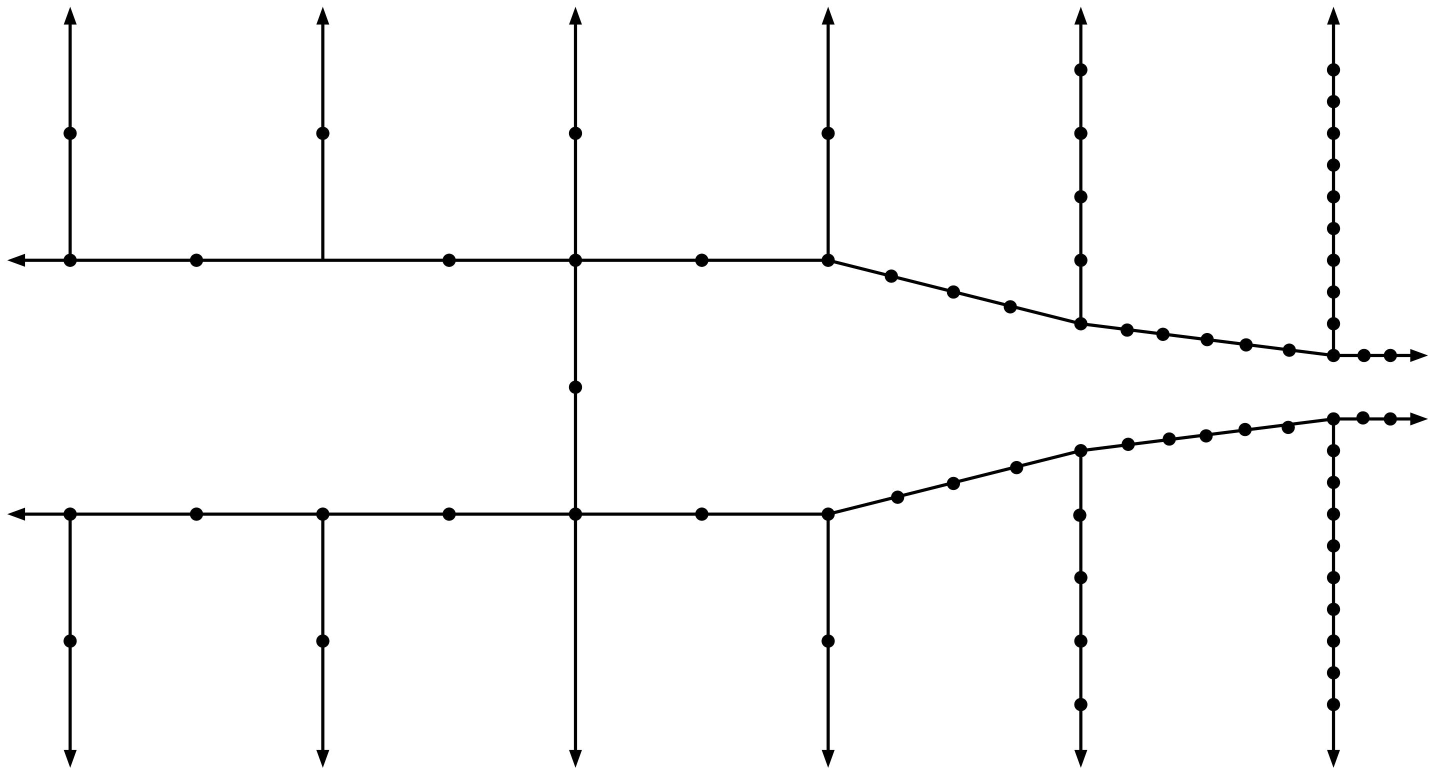


Folding map:

- is uniformly QC.
- is conformal except in strip along boundary.
- maps integer points on line to vertices of trees.
- is linear on boundary segments between vertices.

Rest of talk will describe applications of QC folding.

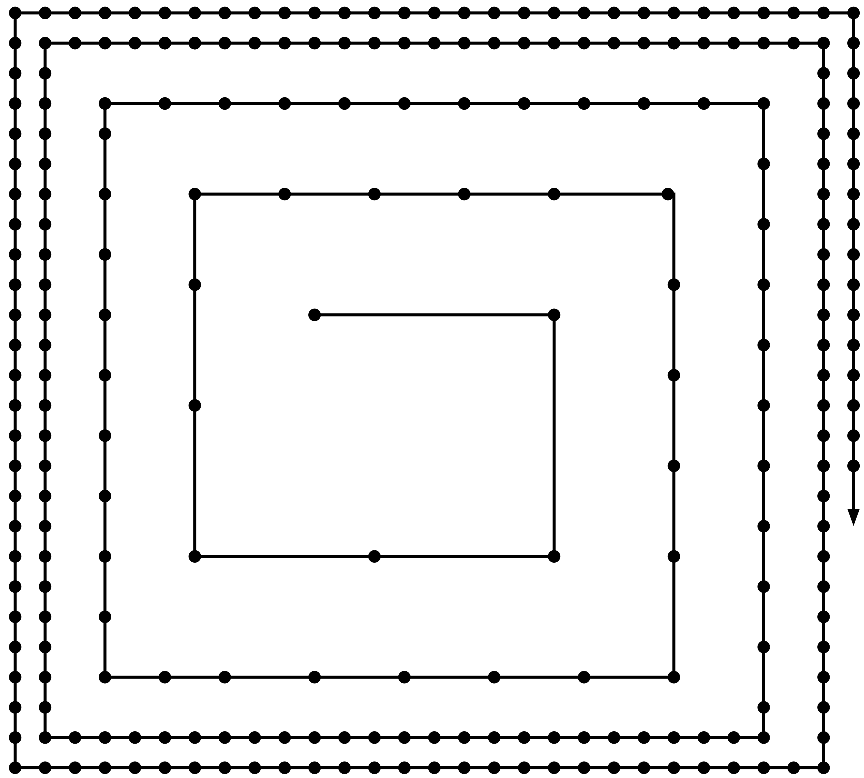
Rapid increase



$\exists f \in \mathcal{S}_2$ so $f(x) \nearrow \infty$ as fast as we wish.

First such example due to Sergei Merenkov.

Fast spirals



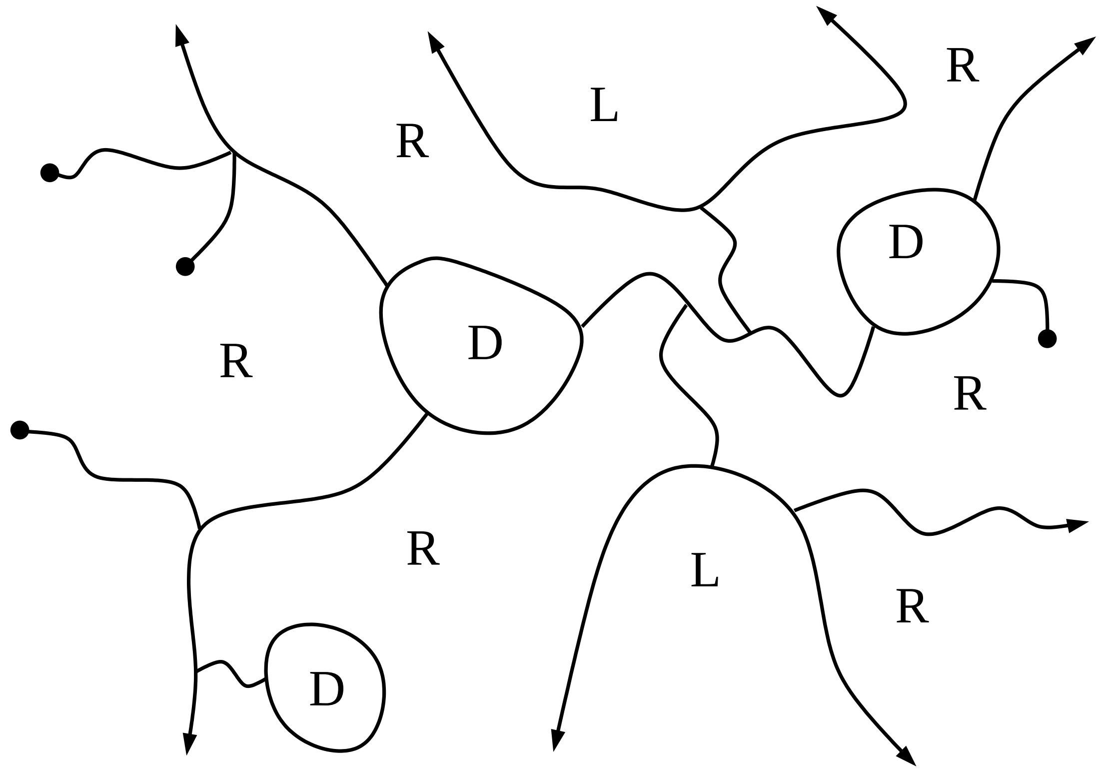
$\exists f \in \mathcal{S}_2$ so $\{|f| > 1\}$ spirals as fast as we wish.

So far, QC-folding always gives critical values ± 1.

All critical points have uniformly bounded degree.

A simple modification gives:

- high degree critical points
- critical values other than ± 1
- finite asymptotic values.

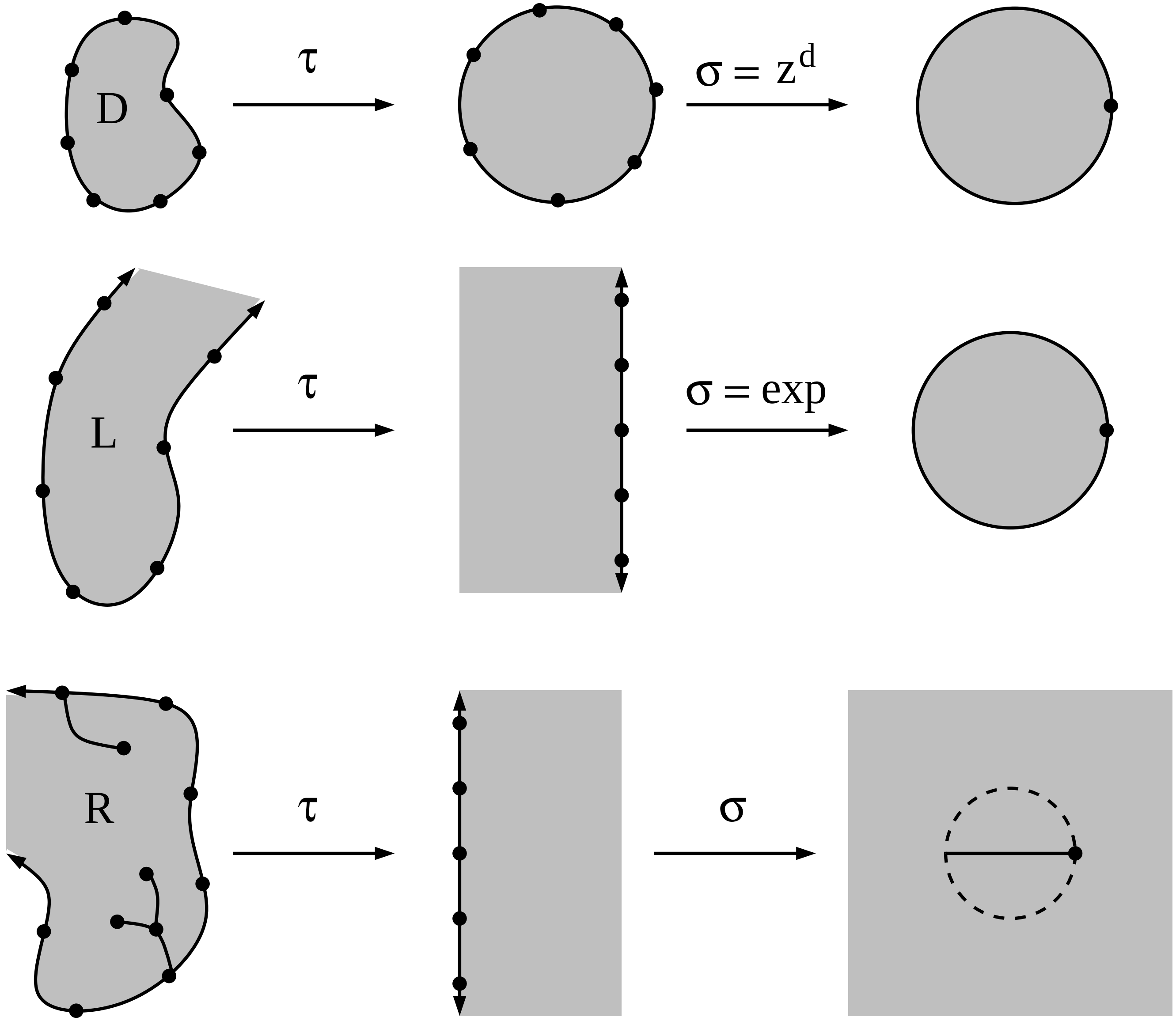


Replace tree by graph. Graph faces labeled D,L,R.

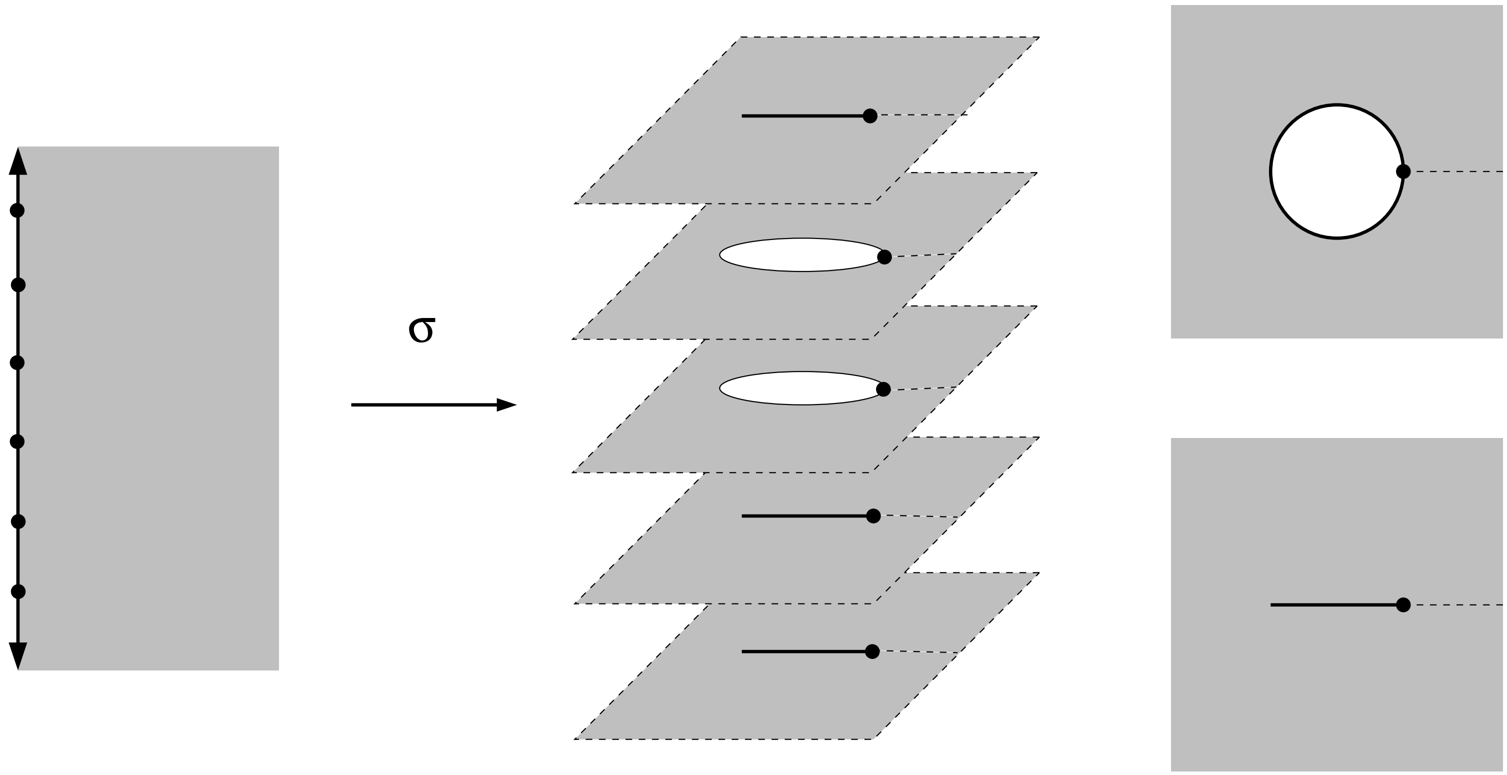
D = bounded Jordan domains (high degree critical points)

L = unbounded Jordan domains (asymptotic values)

D's and L's only touch R's.



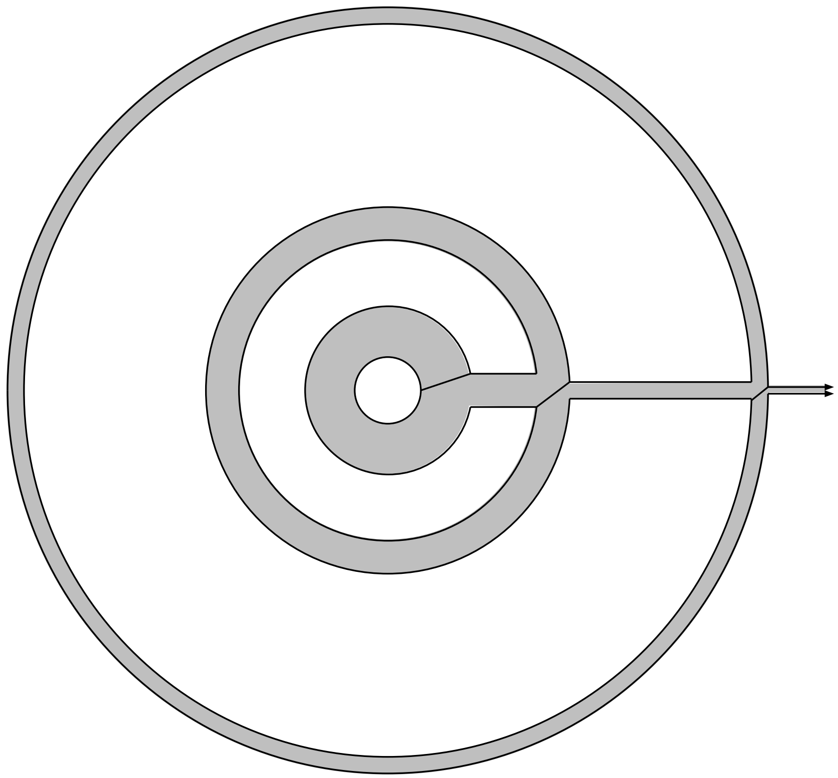
D
τ
σ = z^d
L
τ
σ = exp
R
τ
σ



On R components $\sigma = \cosh$ or $\sigma = \exp$ on boundary intervals depending on type of adjacent component.

Folding R-components gives QR map.

The area conjecture



One R-component, many D-components

$\exists f \in \mathcal{S}$ s.t. $\text{area}(\{z : |f(z)| > \epsilon\}) < \infty$ for all $\epsilon > 0$.

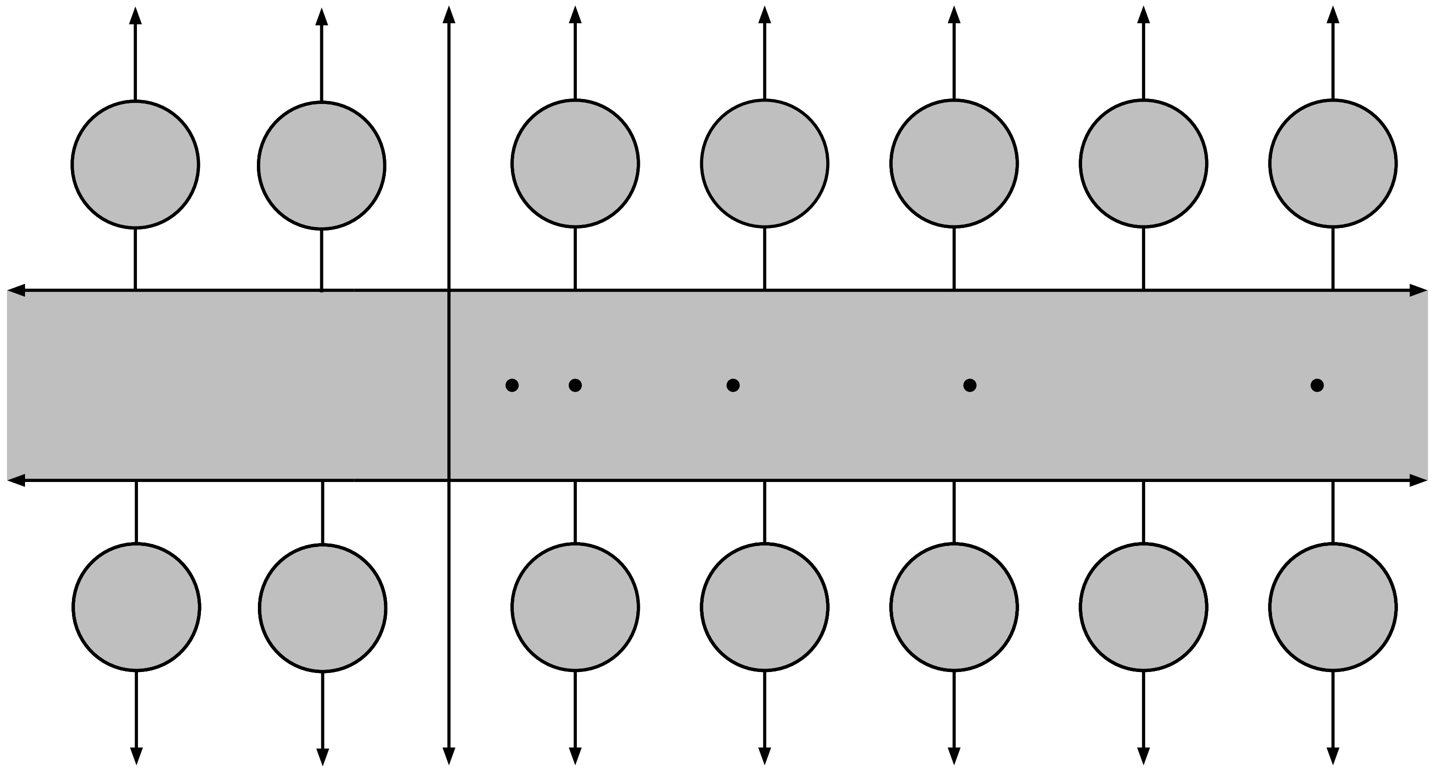
Bounded type wandering domain:

Thm: $\exists\ f \in \mathcal{B}$ that has a wandering domain.

wandering domain = non pre-periodic Fatou component

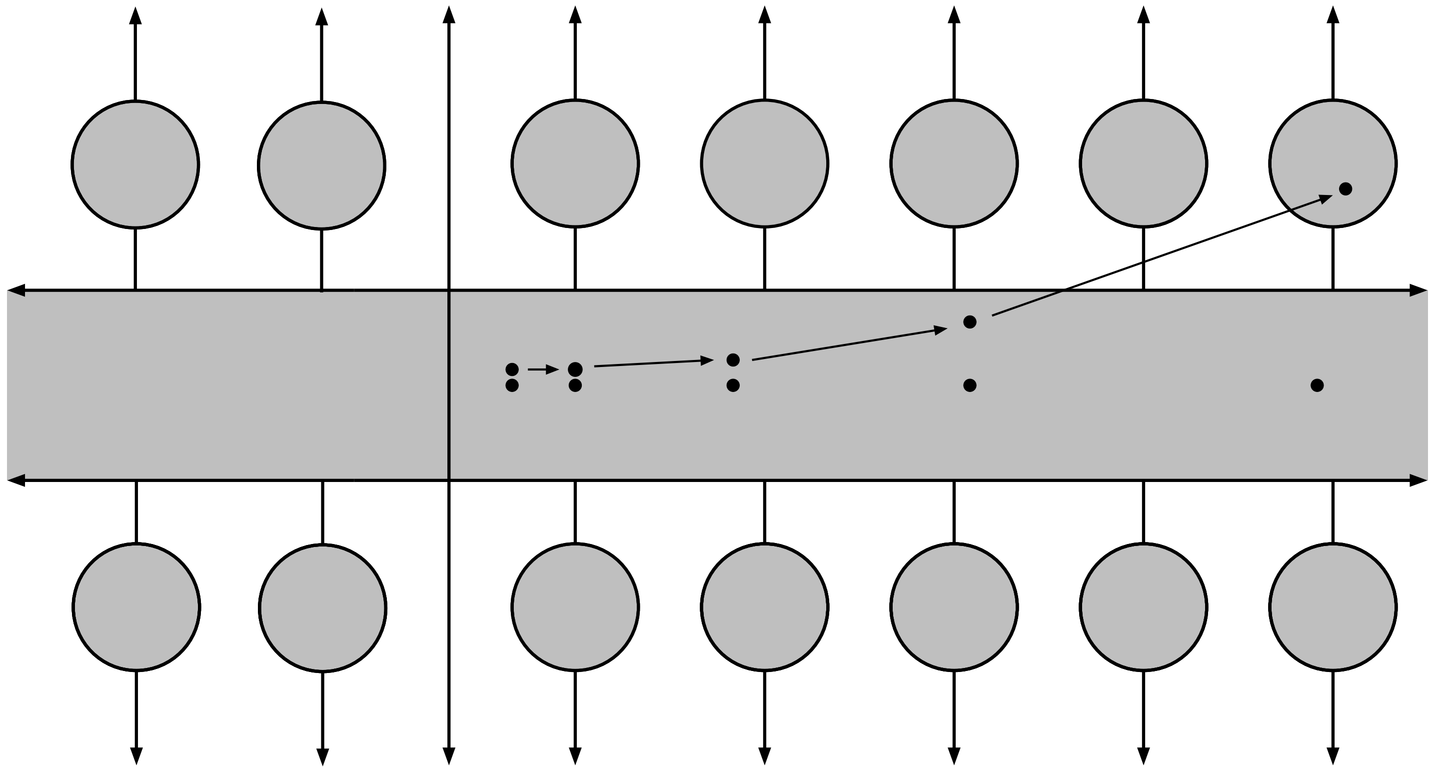
$\mathcal{B}$ = Eremenko-Lyubich class = bounded singular set

Sullivan's non-wandering theorem extended to $\mathcal{S}$ (finite singular set) by Eremenko-Lyubich, Goldberg-Keen.



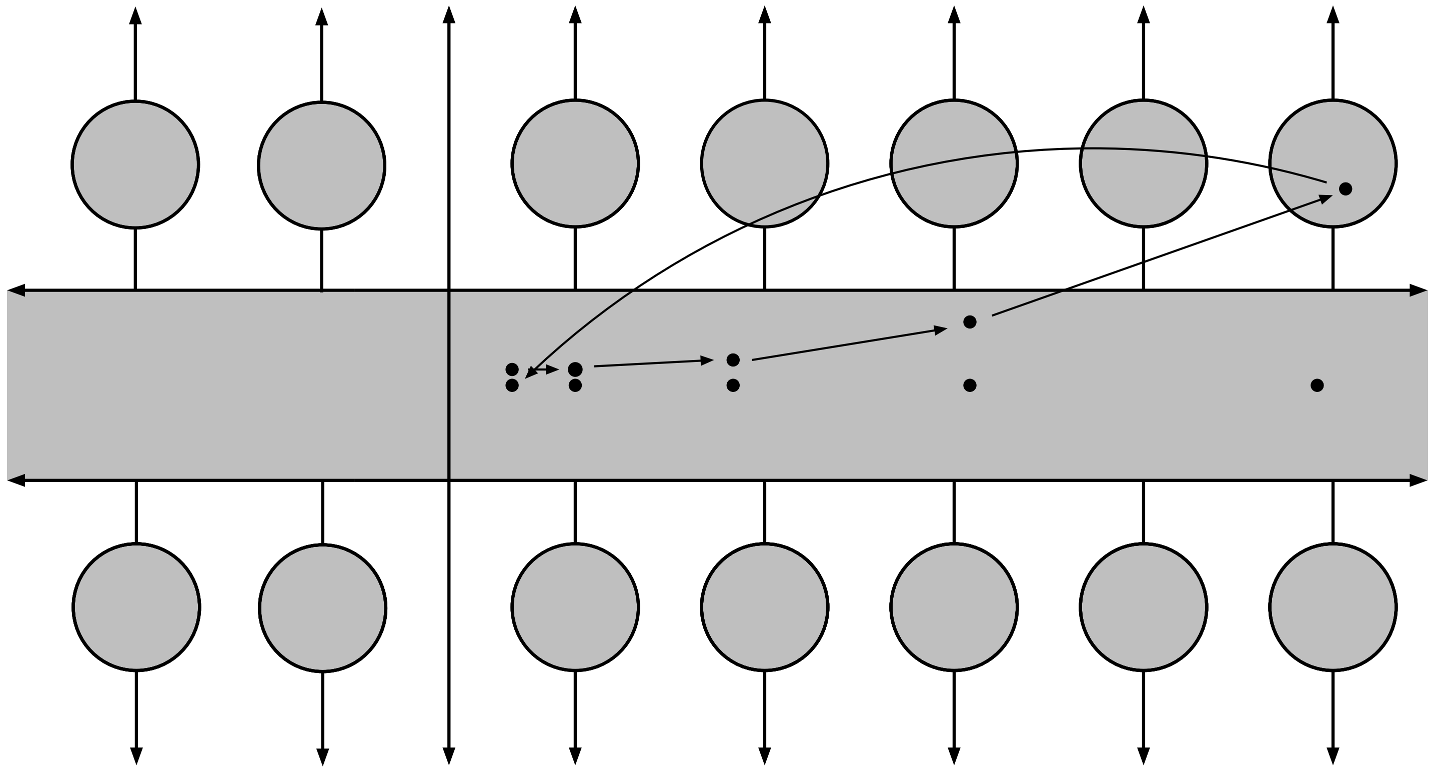
Uses only D and R components. Symmetric.

Dots are orbit of 1/2.



Orbit just above 1/2 diverges from real line.

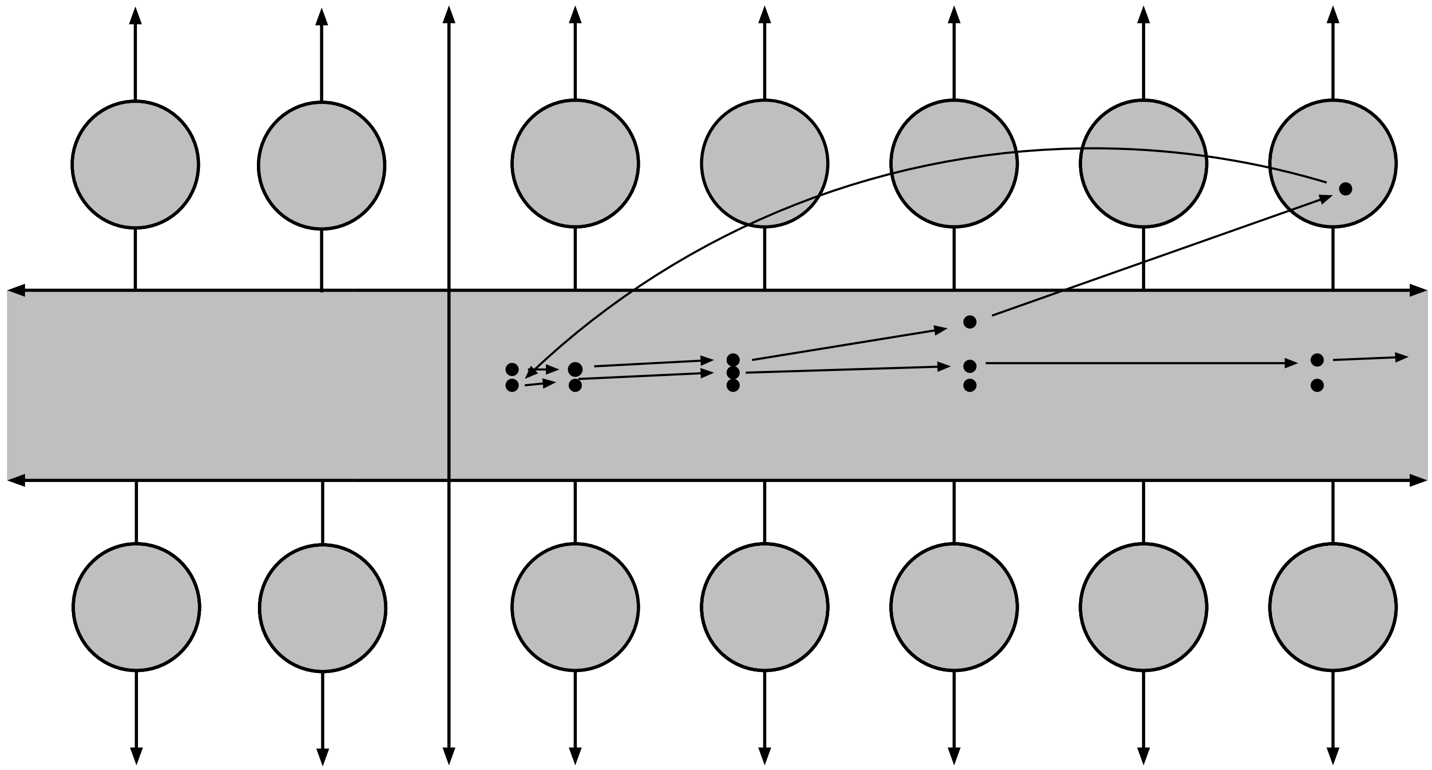
Can choose to land near center of a D-component.



D-component contains high degree critical point.

Critical value is just above 1/2.

But closer to 1/2 than previous starting point.



New orbit follows 1/2 longer before returning.

Orbit is unbounded, but not escaping.

Iterated disk has $\mathrm{diam}(D_n) \to 0$, so is in Fatou set.

Lemma: D_n's are in different Fatou components:

Proof: If not, there are $n < m$ so D_n, D_m are in same Fatou component. Then kth iterates D_{n+k}, D_{k+m} alway always land in same component.

Iterate until D_{k+m} returns near $1/2$; D_{k+n} is far away. Contradicts Schwarz lemma (hyperbolic distances decrease under iteration).

I will finish by mentioning some results that were inspired by QC folding, but don't use it in their proofs.

Adam Epstein's order conjecture

Order of growth:

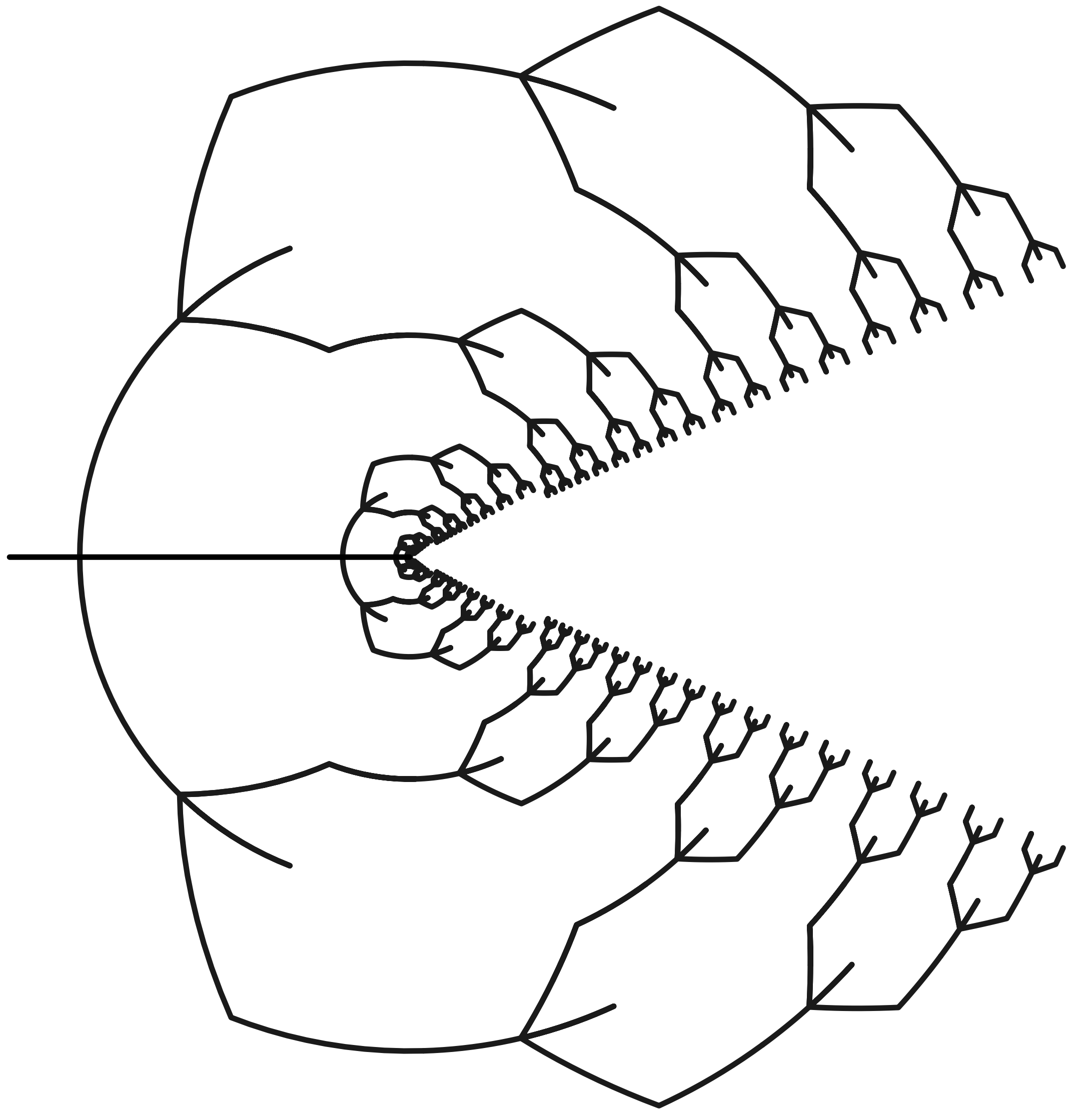
$$\rho(f) = \limsup_{|z|\to\infty} \frac{\log\log|f(z)|}{\log|z|}.$$

f, g QC-equivalent if $\exists$ QC ϕ, ψ s.t. $f \circ \phi = \psi \circ g$.

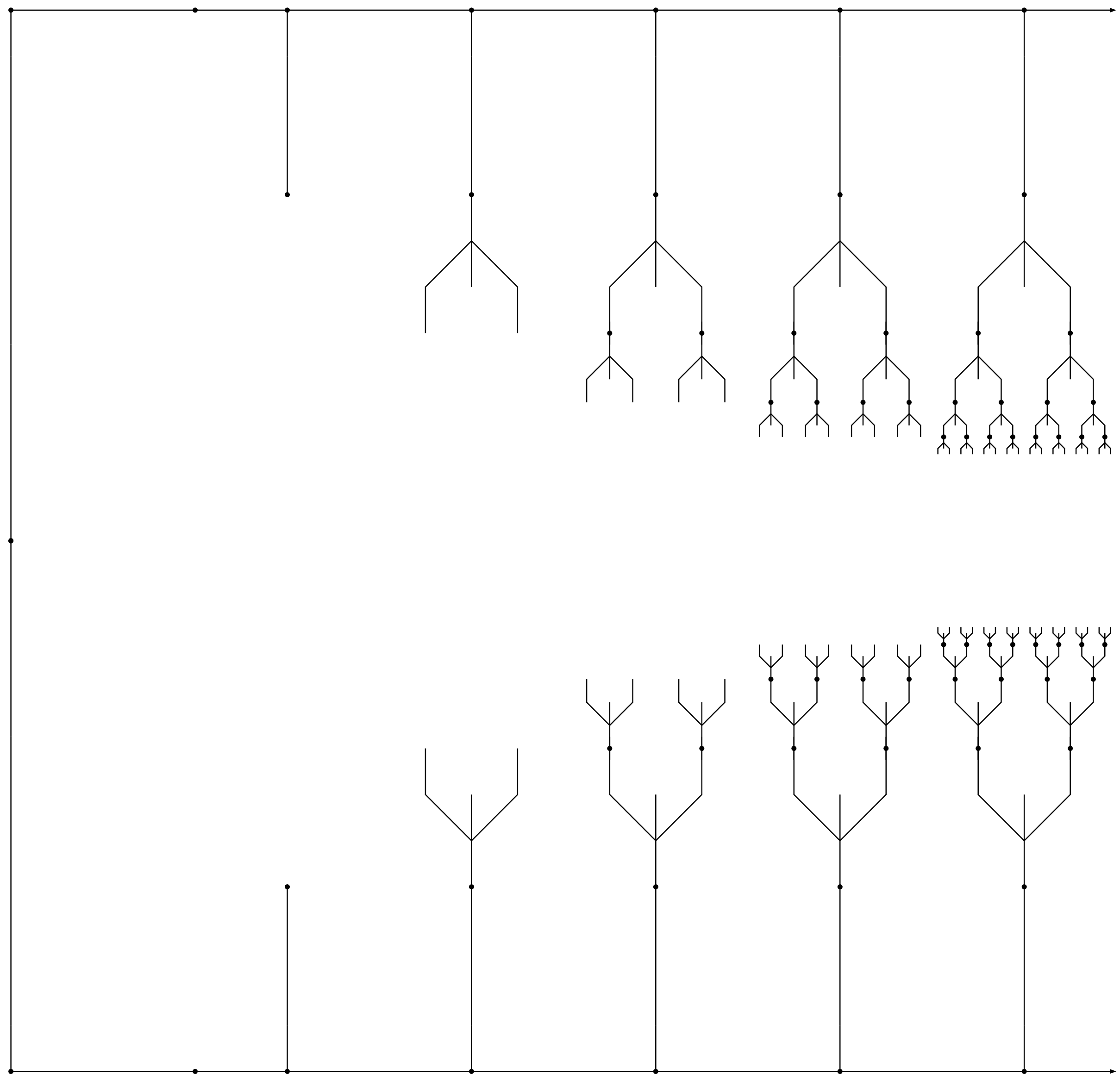
Question: $f, g \in \mathcal{S}$ QC-equivalent $\Rightarrow \rho(f) = \rho(g)$?

False in $\mathcal{B}$ (Epstein-Rempe)

No



Same domain in logarithmic coordinates



Baker (1975): if f is transcendental, then its Julia set contains a continuum, so dim ≥ 1.

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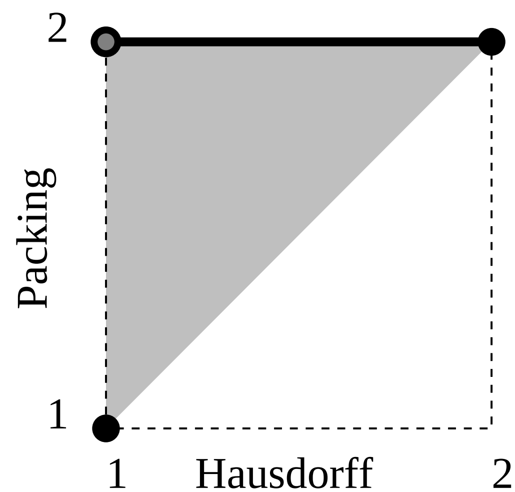
Dim=1 Theorem: There is a transcendental entire function f whose Julia set has dimension 1.

Also packing dimension $= 1$. First example P-dim < 2.

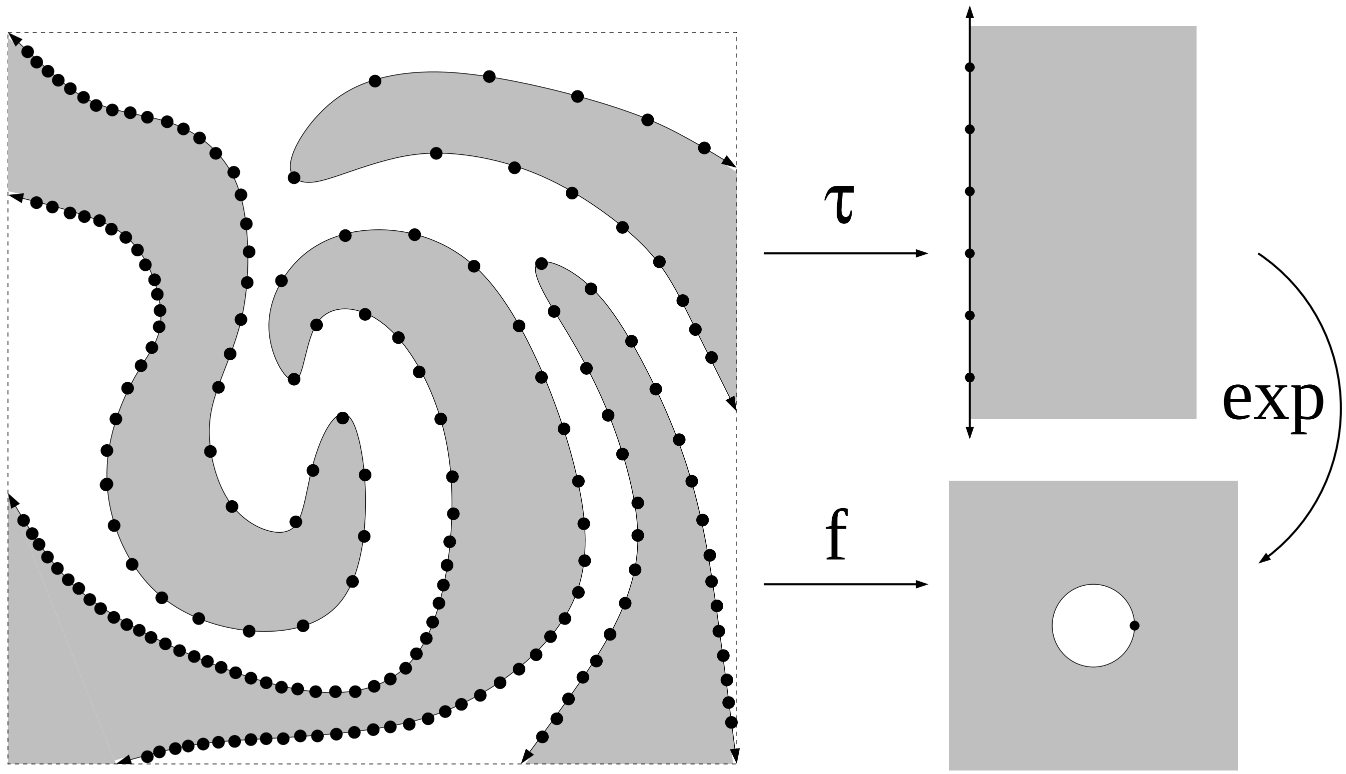
pairs of (Hdim,Pdim)

gray = possible

black = known



If $S(f) \subset \mathbb{D}$, then $\Omega = \{z : |f(z)| > 1\}$ is a disjoint union of unbounded analytic Jordan domains.



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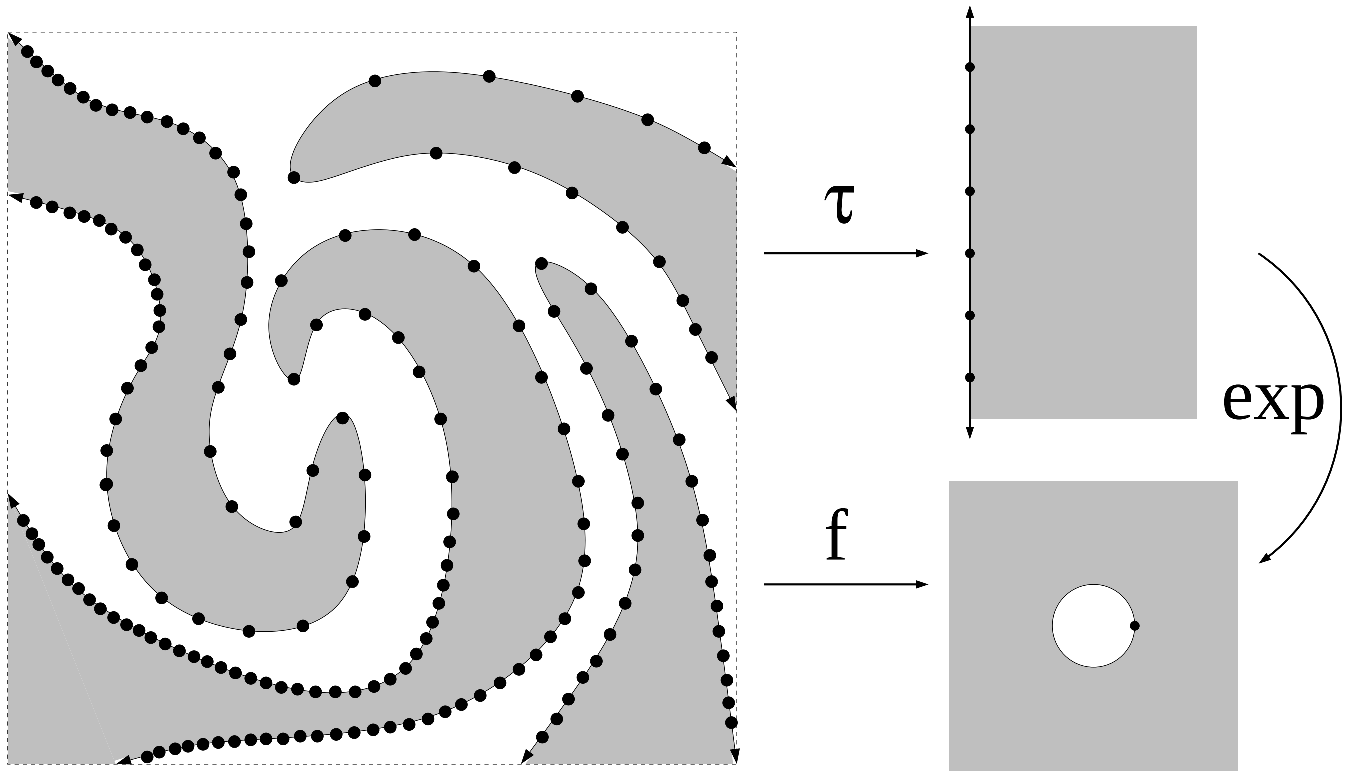

Any other restrictions?

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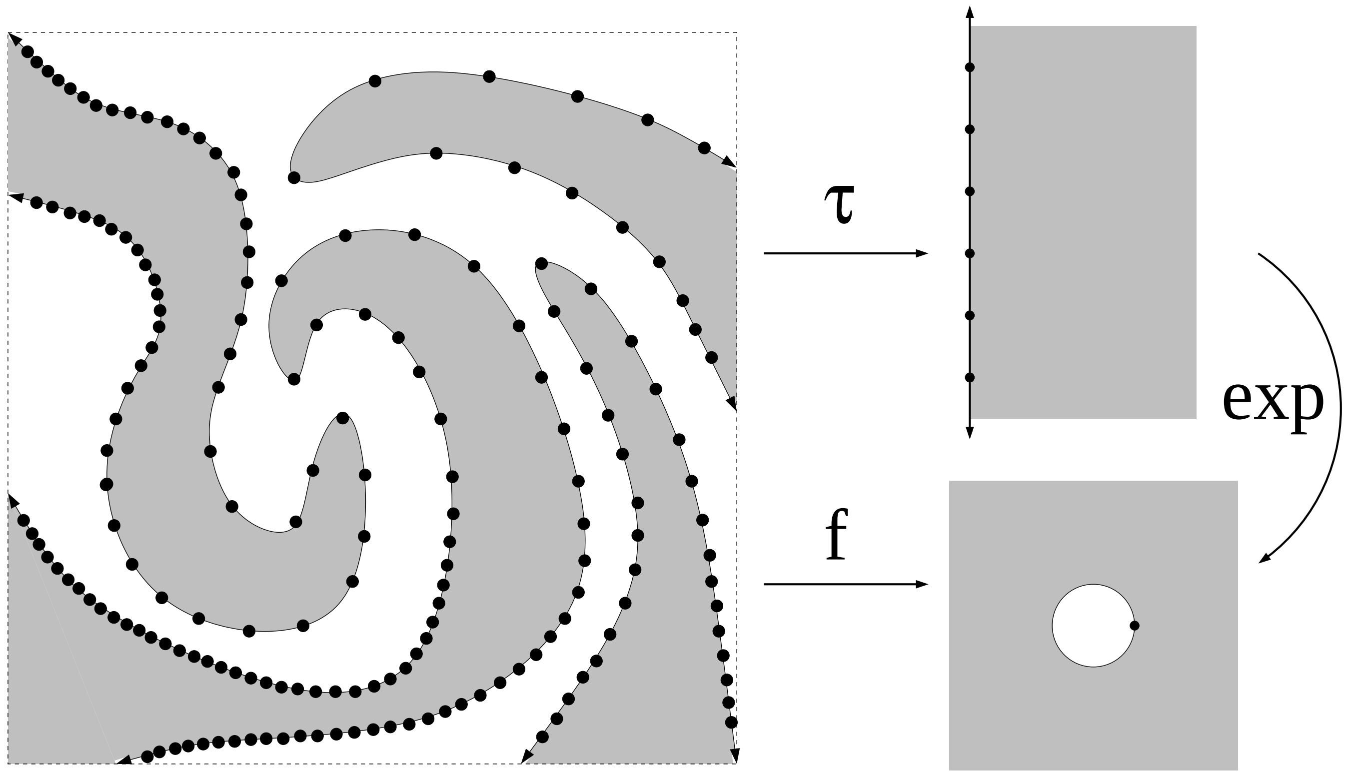
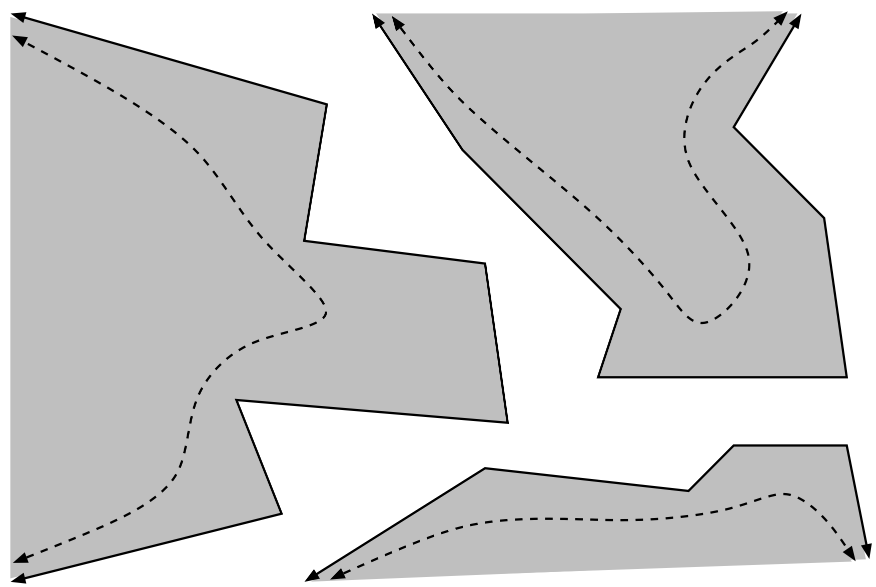


Any other restrictions? No.

Thm: Suppose Ω' is a disjoint union of unbounded Jordan regions and $\tau : \Omega' \to \{x > -\rho\}$ is conformal on each component. Then there is quasiregular g so that $g = \exp \circ \tau$ on $\Omega = \tau^{-1}(\{x > 0\})$ and $|g| < 1$ off Ω.



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Corollary: $\exists f \in \mathcal{B}$ and a QC ϕ so that $f \circ \phi = g$.

This is $\mathcal{B}$-version of the $\mathcal{S}^*$ folding theorem.

$\mathcal{S}$-version has geometrical assumptions; this does not.

Some $\mathcal{B}$-level-set approximates this. False for $\mathcal{S}$.

Idea of proof of $\mathcal{B}$-level-set theorem:

- $W = \text{int}(\mathbb{C} \setminus \Omega)$ is simply connected.
- Let $\Psi : W \to \mathbb{D}$ be Riemann map.
- Read Garnett's *Bounded Analytic Functions*
- Build Blaschke product so $B \circ \Psi \approx \exp \circ \tau$ on $\partial\Omega$.
- Define g by "glueing" $B \circ \Psi$ to $\exp \circ \tau$ across $\partial\Omega$.

Thanks for listening. Questions?