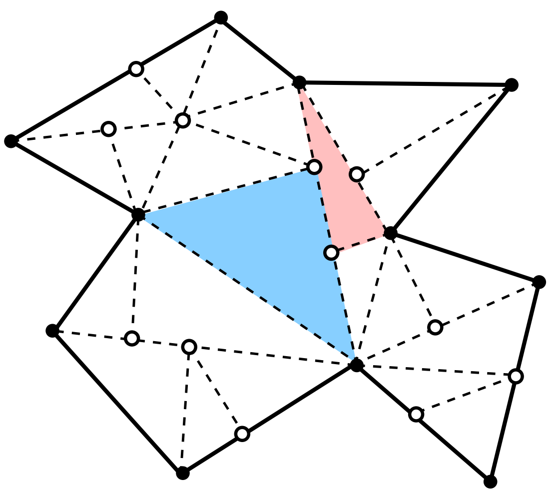
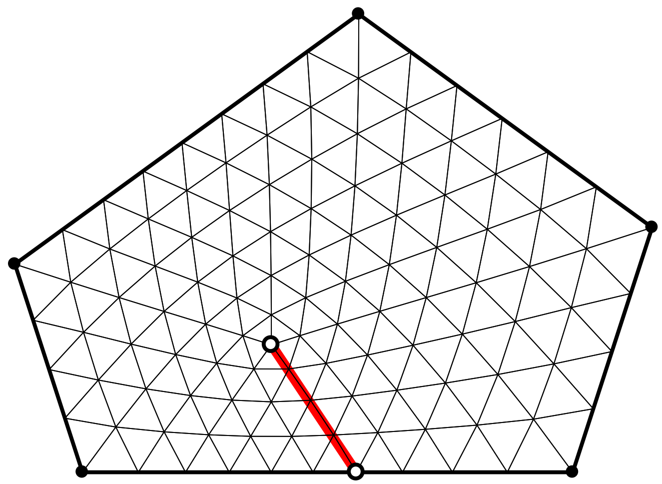
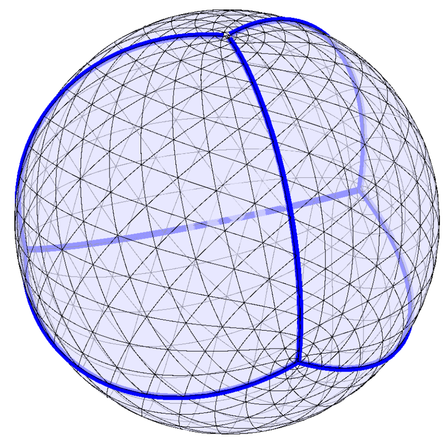
OPEN PROBLEMS ABOUT TRIANGLES

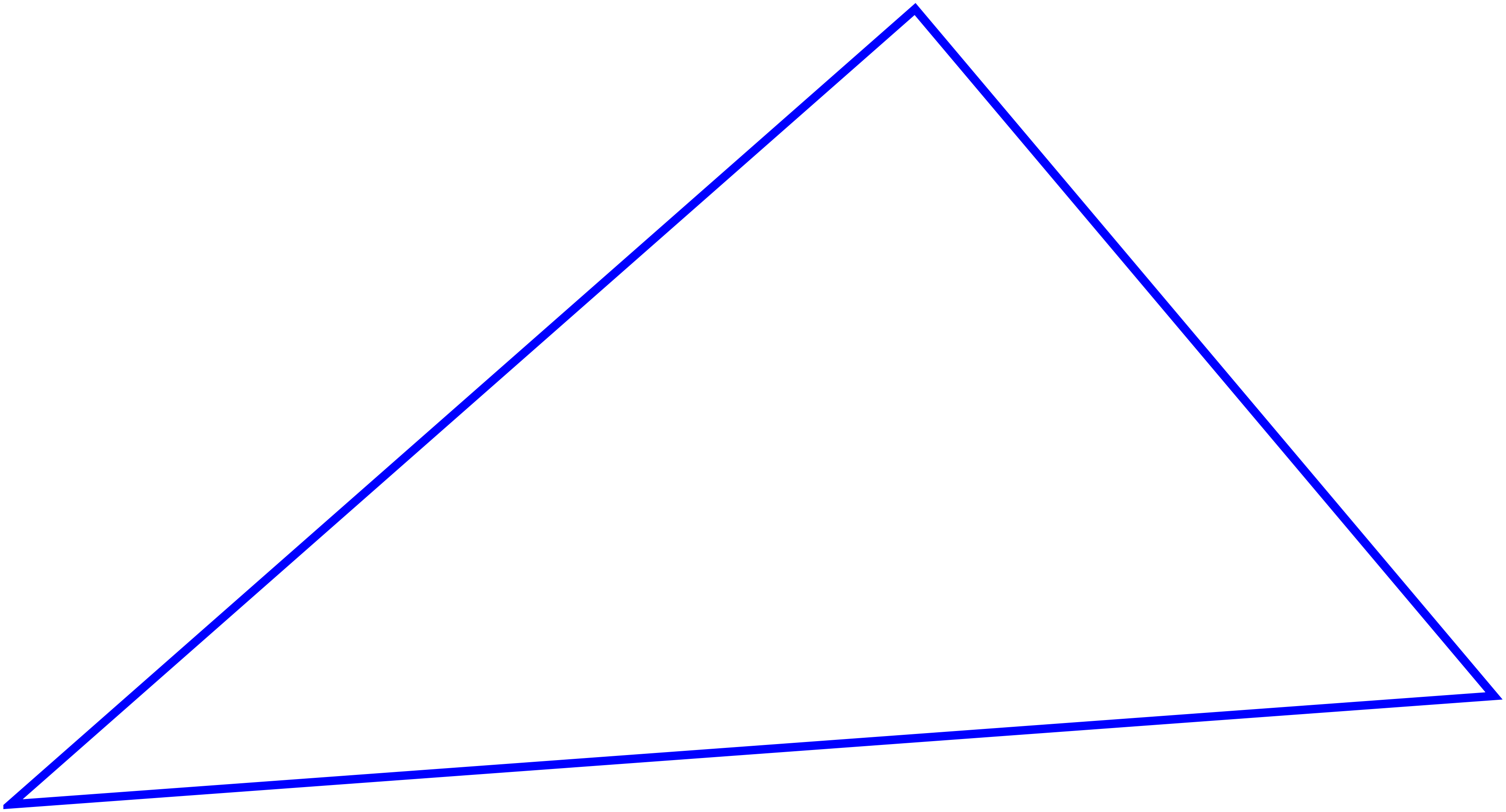
Christopher Bishop, Stony Brook University

1st and 2nd year seminar, 4-5pm, Math 5-127, Dec 4, 2023

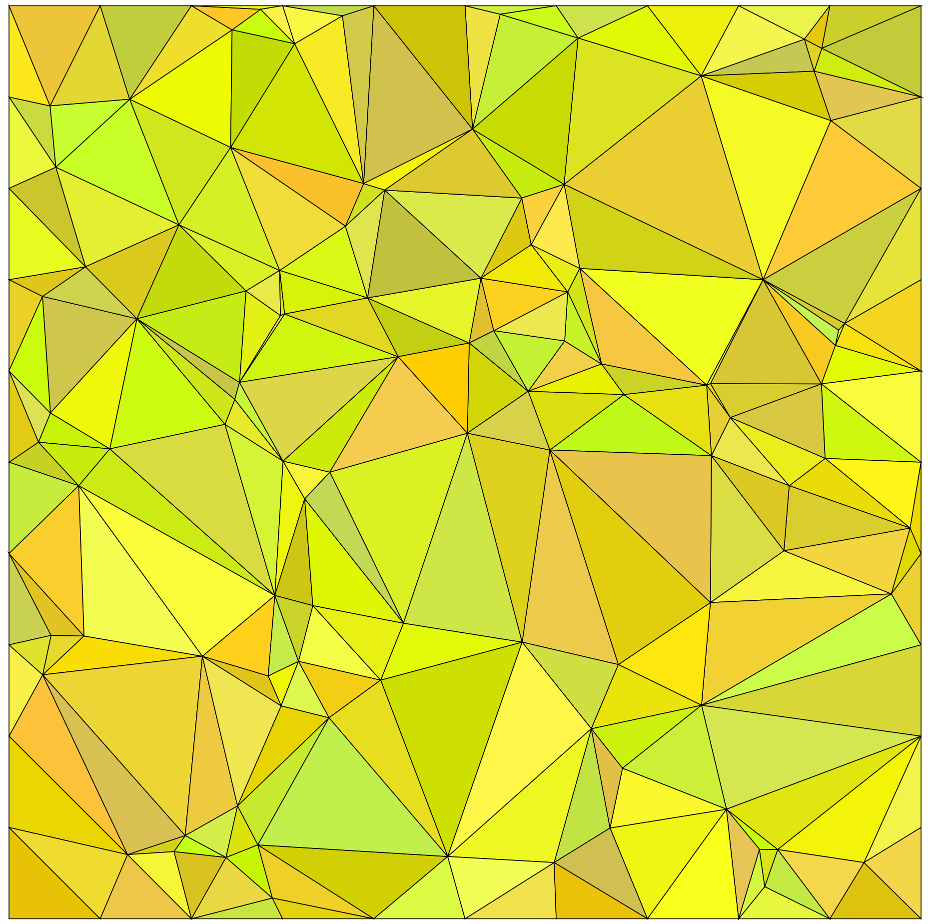








A triangle



A triangulation

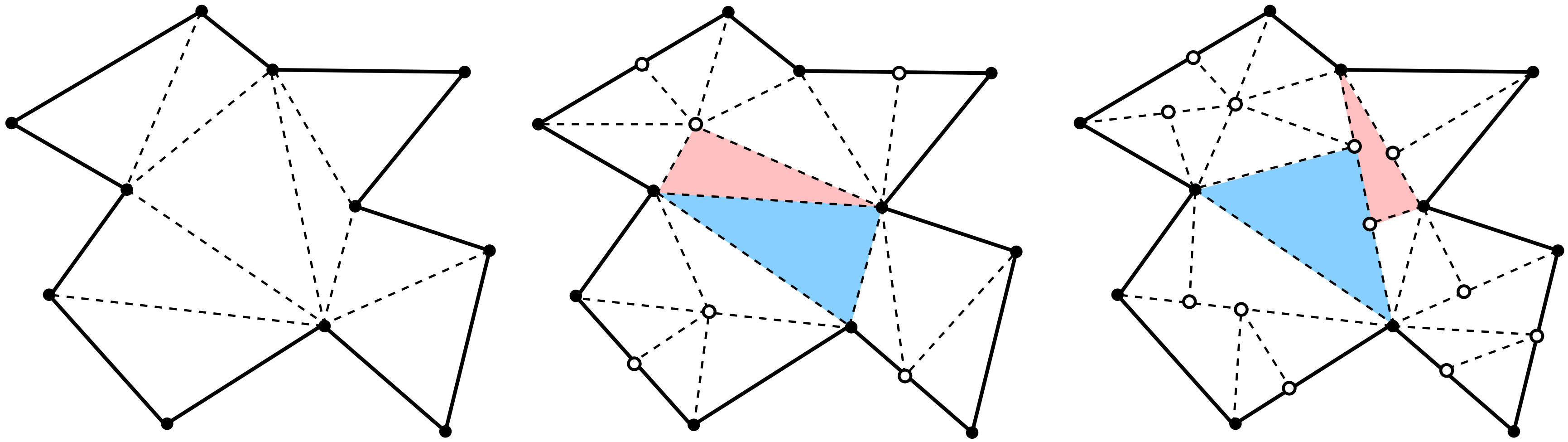
THE PLAN

- Triangulating polygons
- Triangulating planar graphs
- Triangulation flows
- Belyi's theorem and triangulations of Riemann surfaces
- Billards
- Mixing times

Triangulations of polygons

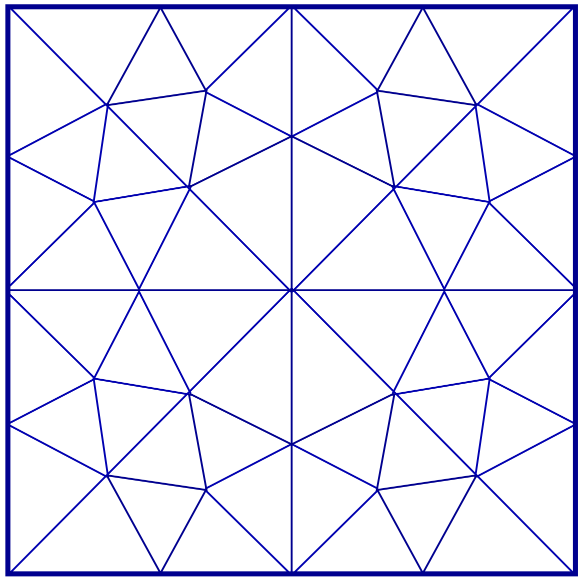
Triangulation of P = "Cut P into triangles."

Triangulation of P = a finite collection of triangles inside P with disjoint interiors and whose closures cover P.

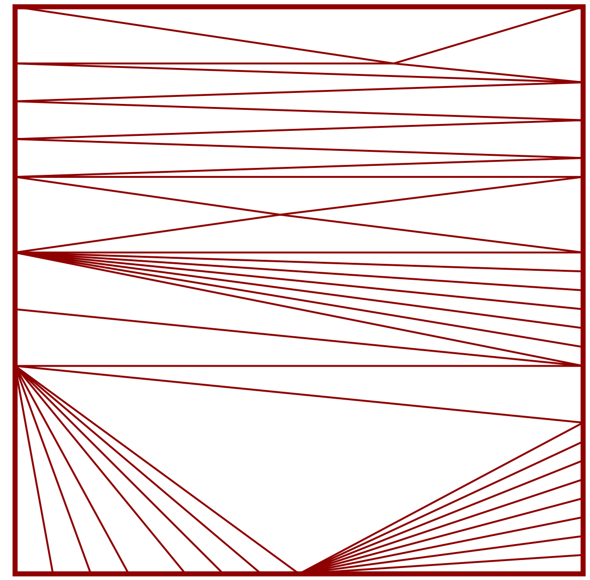


No Steiner Points　　With Steiner Points　　Dissection

Three types of triangulations



Good



Bad

Goal: make pieces as close to equilateral as possible.

Minimize the maximum angle (compute MinMax angle).

"Good" meshes improve performance of numerical methods.

Defn: acute triangle = all angles $< 90^\circ$.

Defn: nonobtuse triangle = all angles $\leq 90^\circ$.

Defn: ϕ-triangulation = all angles $\leq \phi$.

Defn: $\Phi(P) = \inf\{\phi : P \text{ has a } \phi\text{-triangulation}\}$.

Thm (Burago-Zalgaller, 1960): $\Phi(P) < 90^\circ$ all polygons.

"Every polygon has an acute triangulation."

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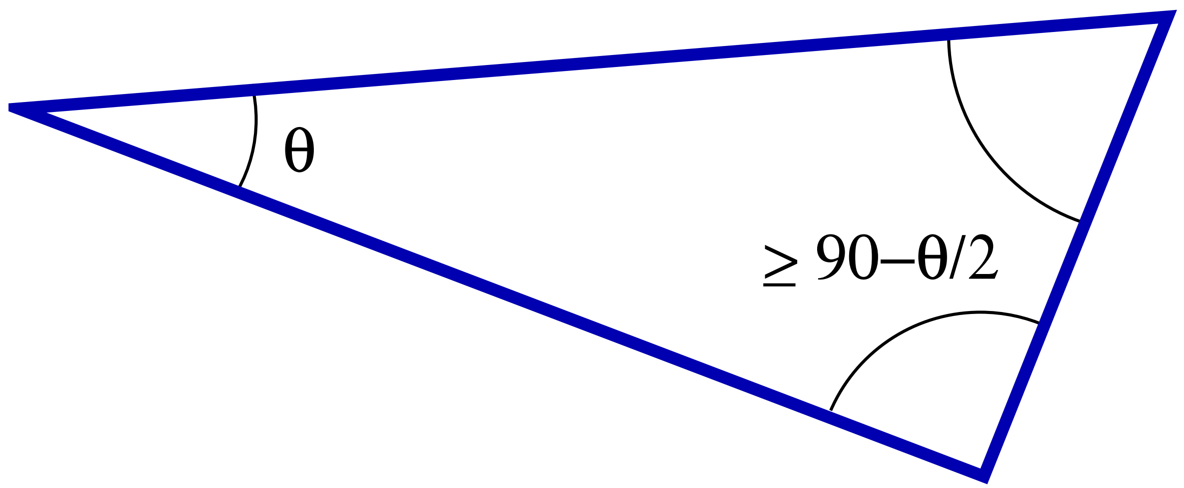
No bound $< 90^\circ$ works for **all** polygons.



Any triangle with an angle $\leq \theta$ also has an angle $\geq 90^\circ - \theta/2$.

Thm (Burago-Zalgaller, 1960): $\Phi(P) < 90^\circ$ all polygons.

"Every polygon has an acute triangulation."

Rediscovered by Baker-Grosse-Rafferty, 1988 (weaker version).

Much work on acute and non-obtuse triangulations by

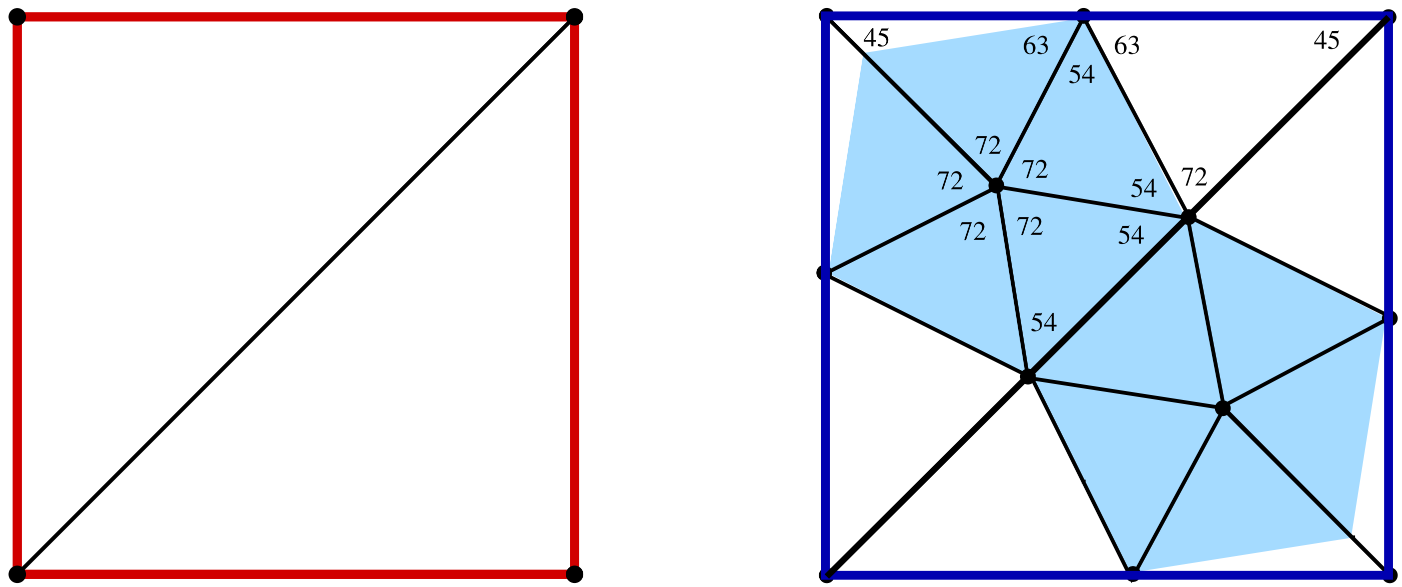
Barth,	Hirani,	Przytycki,	Tan,
Bern,	Itoh,	Ruppert,	Üngör,
Edelsbrunner,	Kopczyński,	Saalfeld,	VanderZee,
Eppstein,	Maehara,	Saraf,	Vavasis,
Erten,	S. Mitchell	Sheffer,	Yuan,
Gilbert,	Pak,	Shewchuk,	Zamfirescu,

and many others

Thm: every n-gon has an acute triangulation of size $O(n)$.

Burago-Zalgaller result first cited in CS literature around 2004.

Steiner points versus no Steiner points.



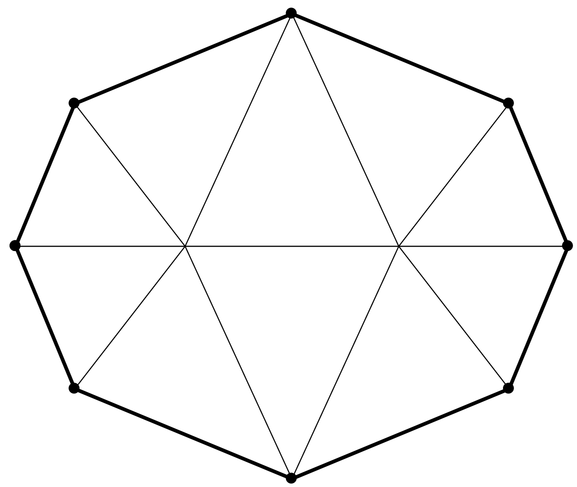
Consider triangulations of a square.

Without Steiner points, 90° is clearly best angle bound.

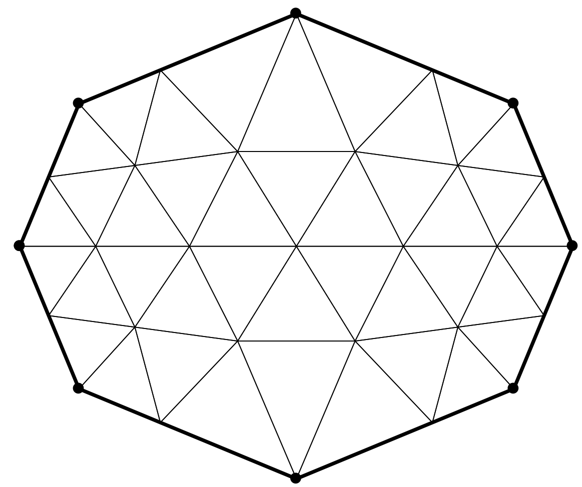
Using Steiner points, a 72°-triangulation is possible.

One can prove 72° is best possible.

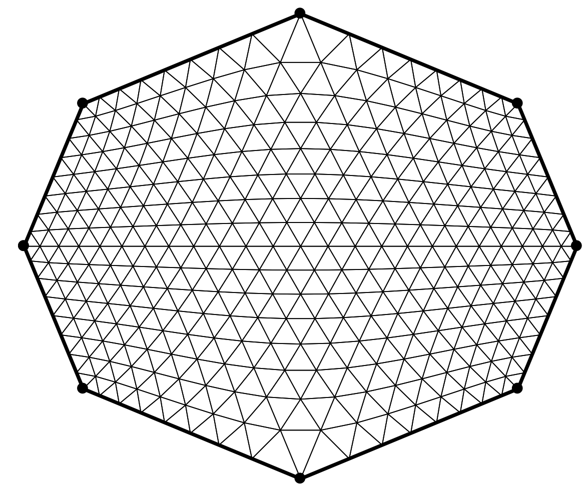
With Steiner points is the optimum attained?



$\theta_{\max} \approx 74.7482$



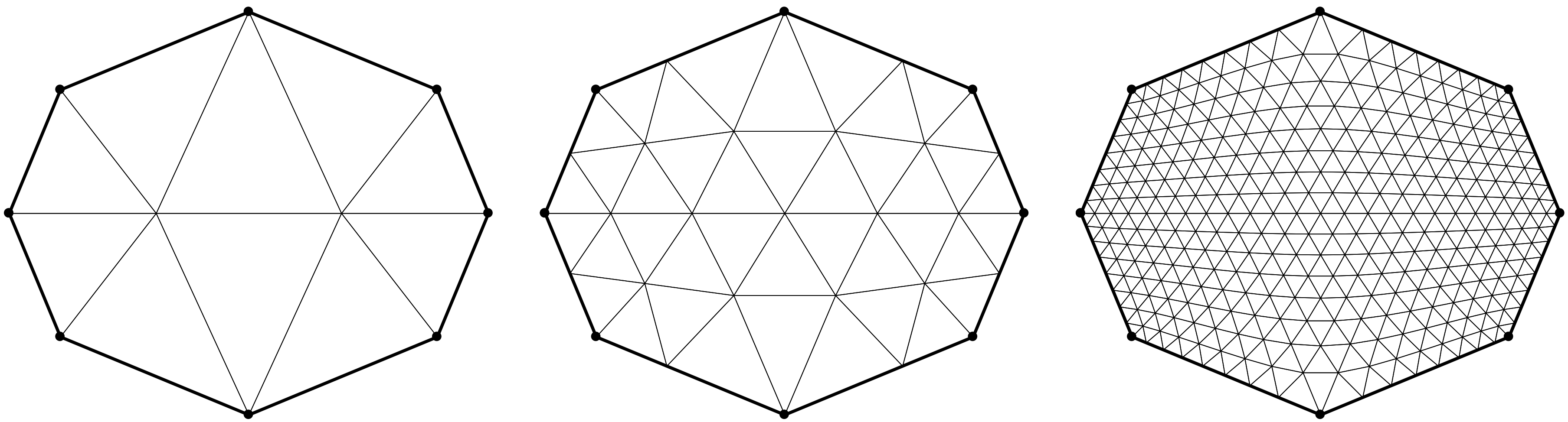
$\theta_{\max} \approx 70.3590$



$\theta_{\max} \approx 67.8690$

With Steiner points there are infinitely many possibilities.

Not obvious that optimal triangulation exists.



$\theta_{\max} \approx 74.7482$ $\theta_{\max} \approx 70.3590$ $\theta_{\max} \approx 67.8690$

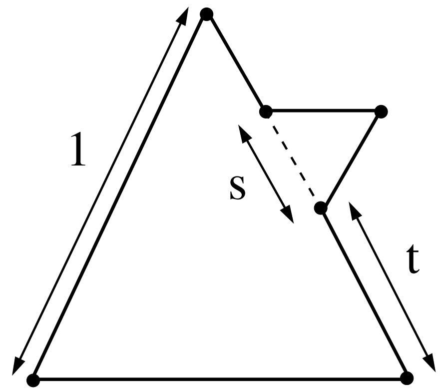
With Steiner points there are infinitely many possibilities.

Not obvious that optimal triangulation exists.

In case above, optimal angle is 67.5° and is attained.

But sometimes, the optimum bound is not achieved.

Defn: 60°-polygon = all angles are multiples of 60°.



Polygon has a dissection into two equilateral triangles.

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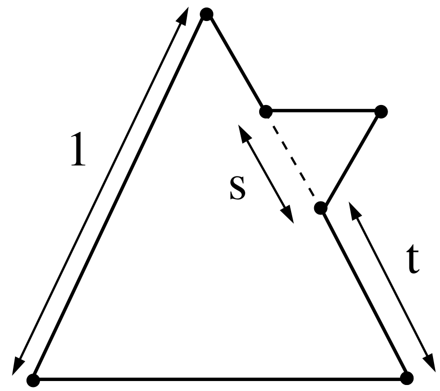
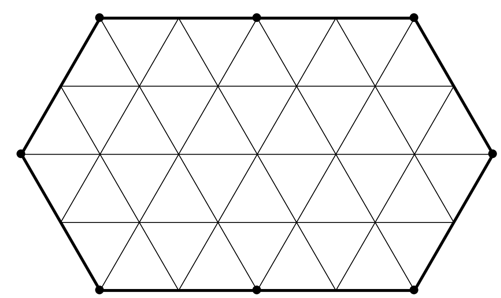


Polygon has a dissection into two equilateral triangles.

Claim: it need not have any equilateral triangulation.

Equilateral triangulation
$\Rightarrow$ all triangles the same size
$\Rightarrow$ edge lengths are integer multiples of triangle length
$\Rightarrow$ s/t is rational



Conclusion: 60°-dissection exists, but 60°-triangulation need not.

Thm: For 60°-polygons $\Phi(P) = 60^\circ$. Attained iff all side ratios rational.

Thm: For $60°$-polygons $\Phi(P) = 60°$. Attained iff all side ratios rational.

Thm: Except for $60°$-polygons, $\Phi(P)$ is attained.

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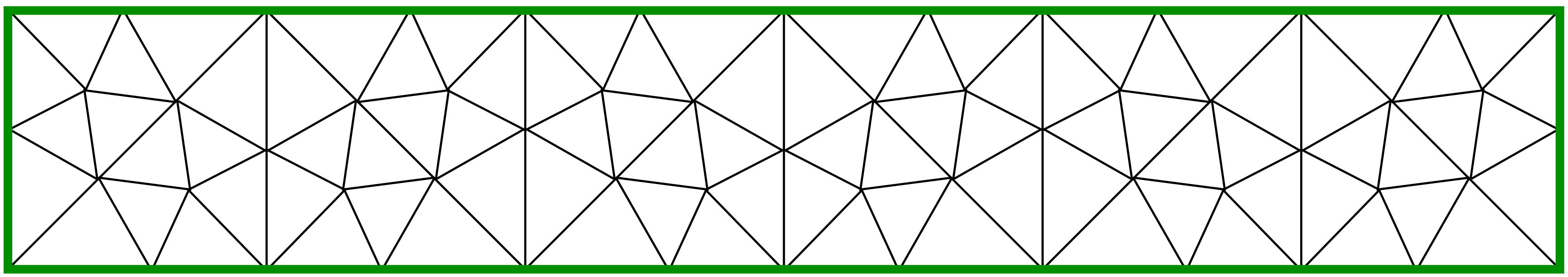

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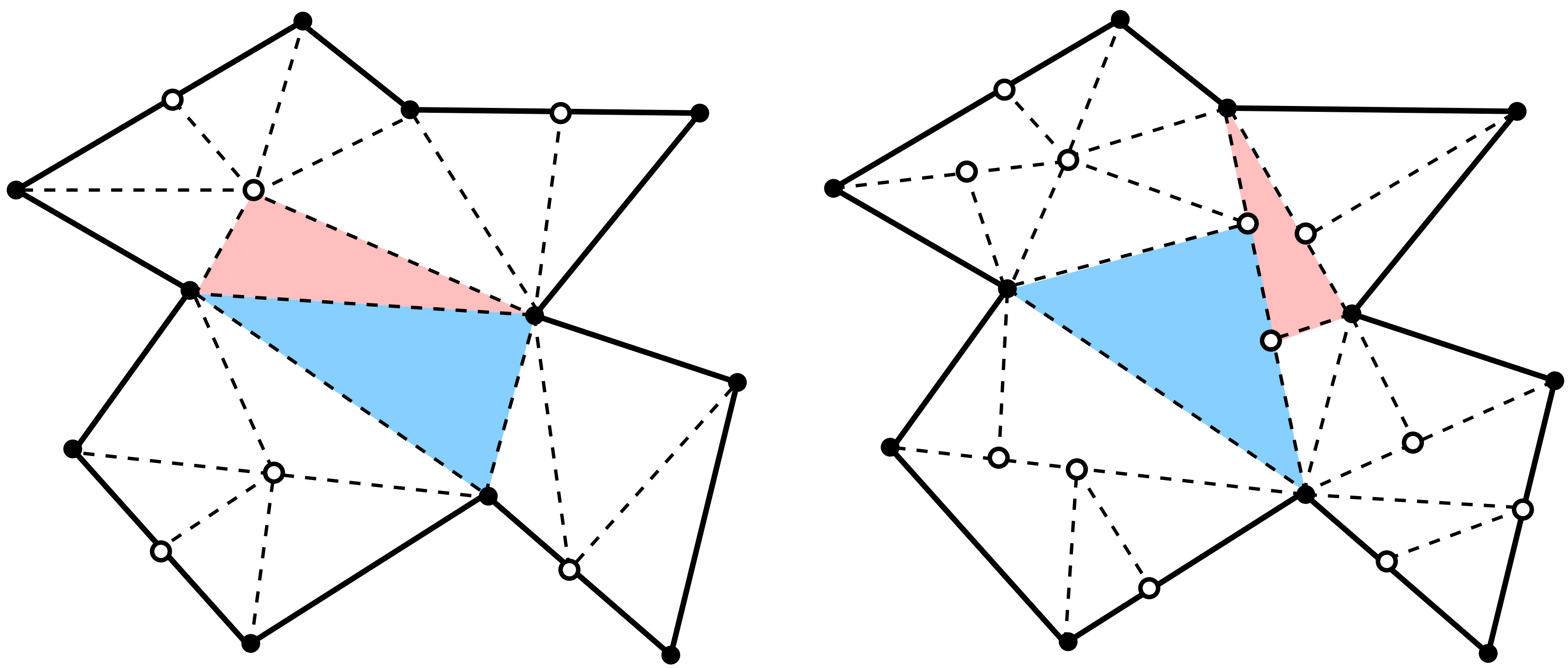
Thm: Triangulations and dissections give the same optimal bounds.



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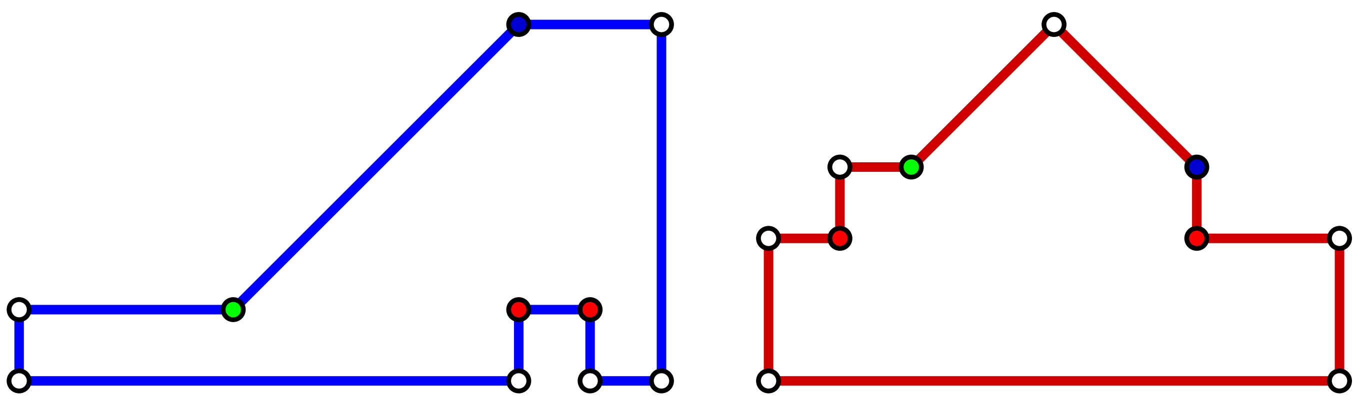
Thm: $\Phi(P)$ only depends on the set of angles. Not on their order around P or on the edge lengths.



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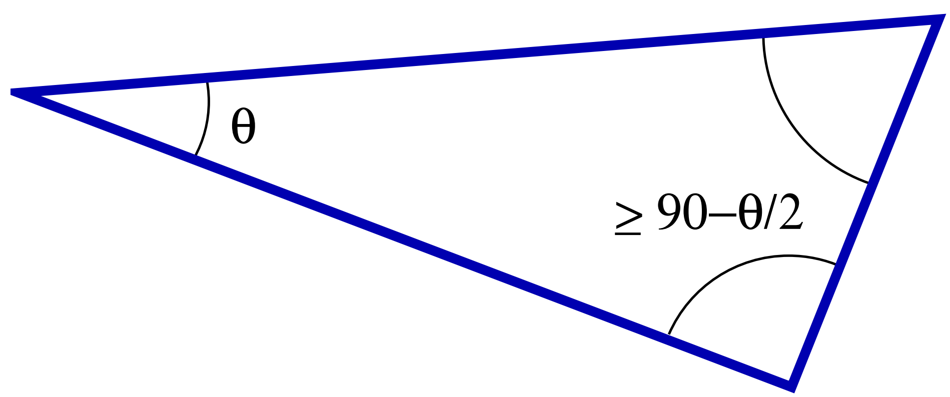
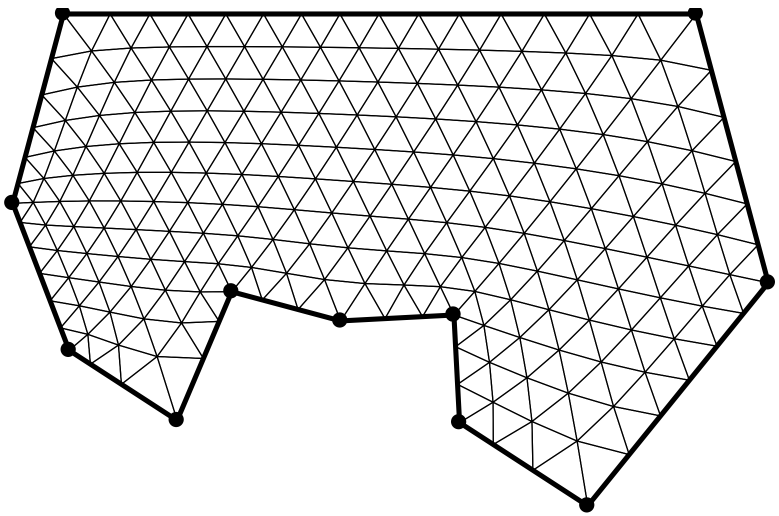
Thm: Triangulations and dissections give the same optimal bounds.

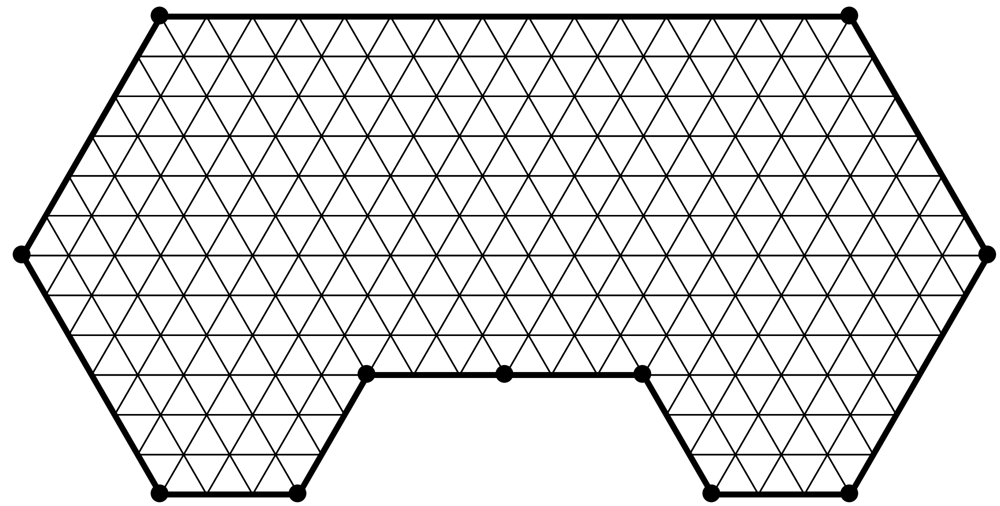
Thm: $\Phi(P)$ only depends on the set of angles. Not on their order around P or on the edge lengths.

Thm: $\Phi(P) \leq \max(72^\circ, 90^\circ - \frac{1}{2}\theta_{\min})$. $\theta_{\min}$ = min interior angle of P.



Main idea: conformal images of 60°-polygons





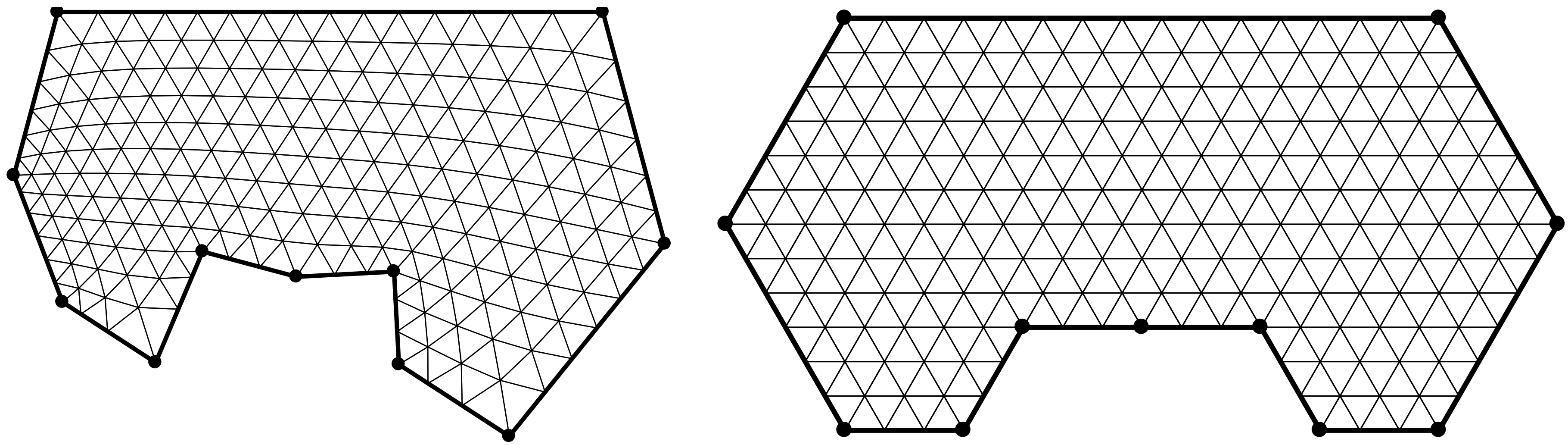
Given P, construct a 60°-polygon P' that "approximates" P.

Conformally map a nearly equilateral triangulation from P' to P.

Conformal = 1-1, holomorphic = preserves angles infinitesimally.

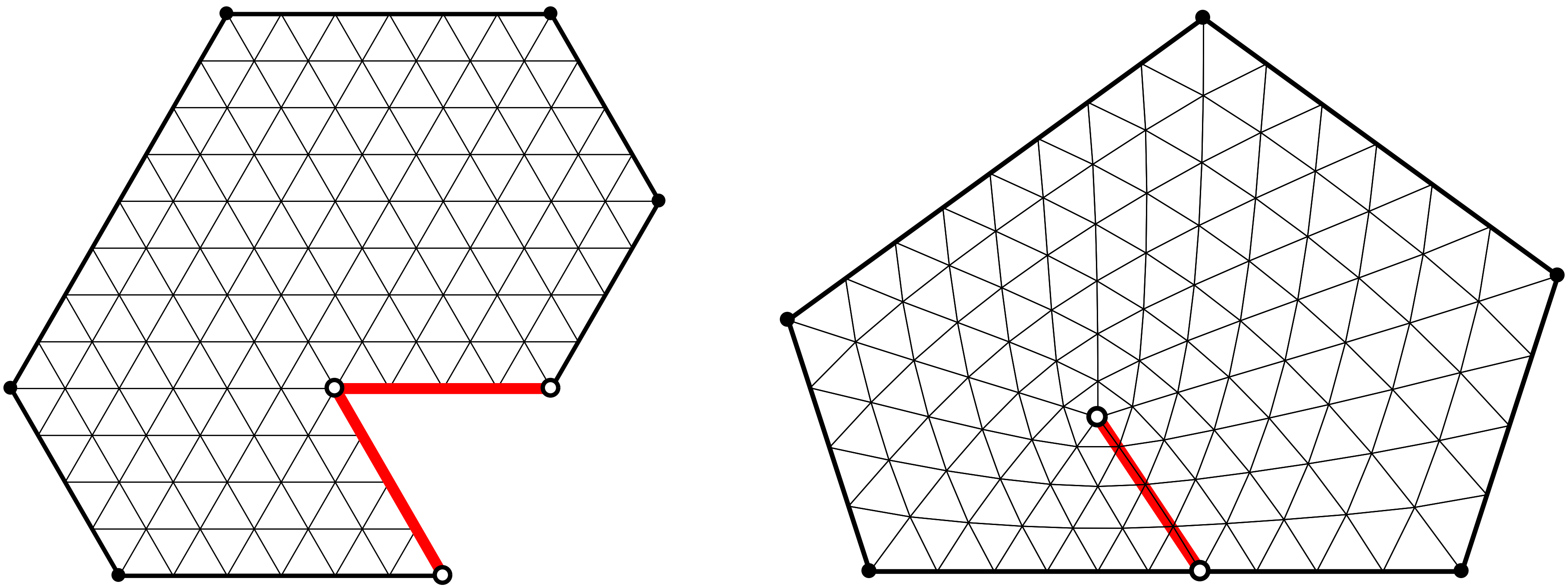
Map only vertices; then connect by segments. (Edge images are curved).

Main idea: conformal images of 60°-polygons



Problems to overcome (among others):

- must map vertices to vertices,
- bound angle distortion at positive scales,
- attain sharp bounds versus approximate them,
- **Euler's formula may force vertices of degree 5 or 7.**



Converting a boundary vertex to an interior vertex of degree 5.

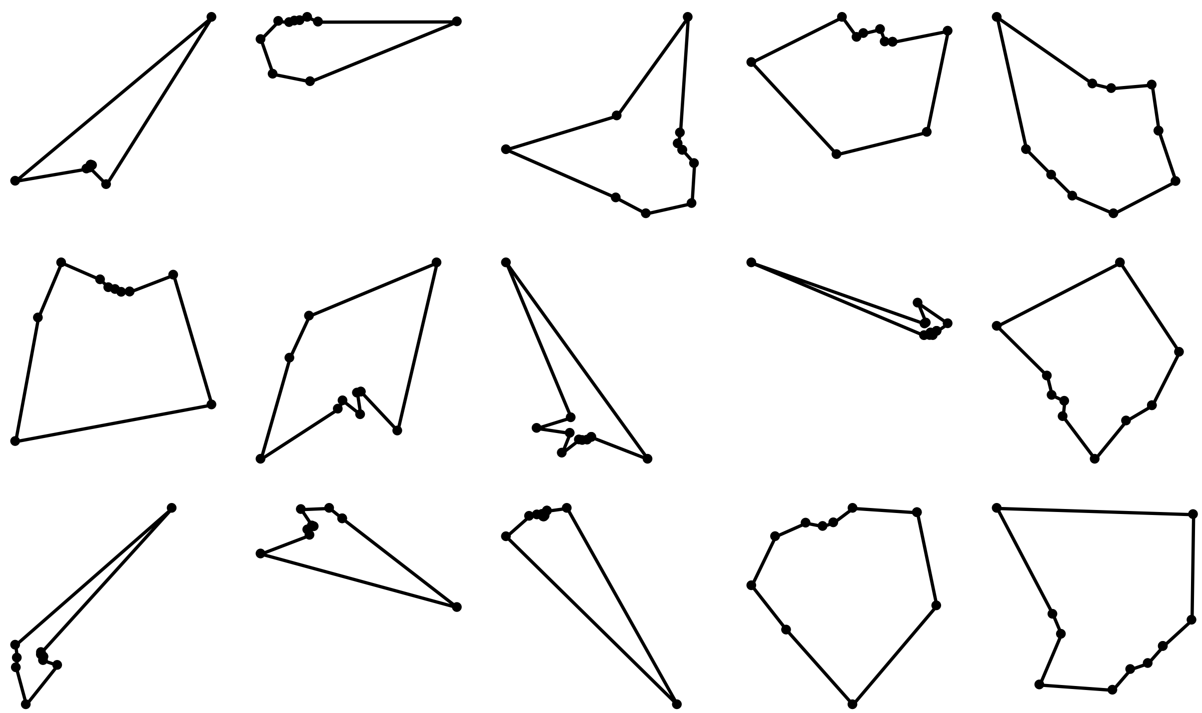
How do we know image is a triangulation (edges match up)?

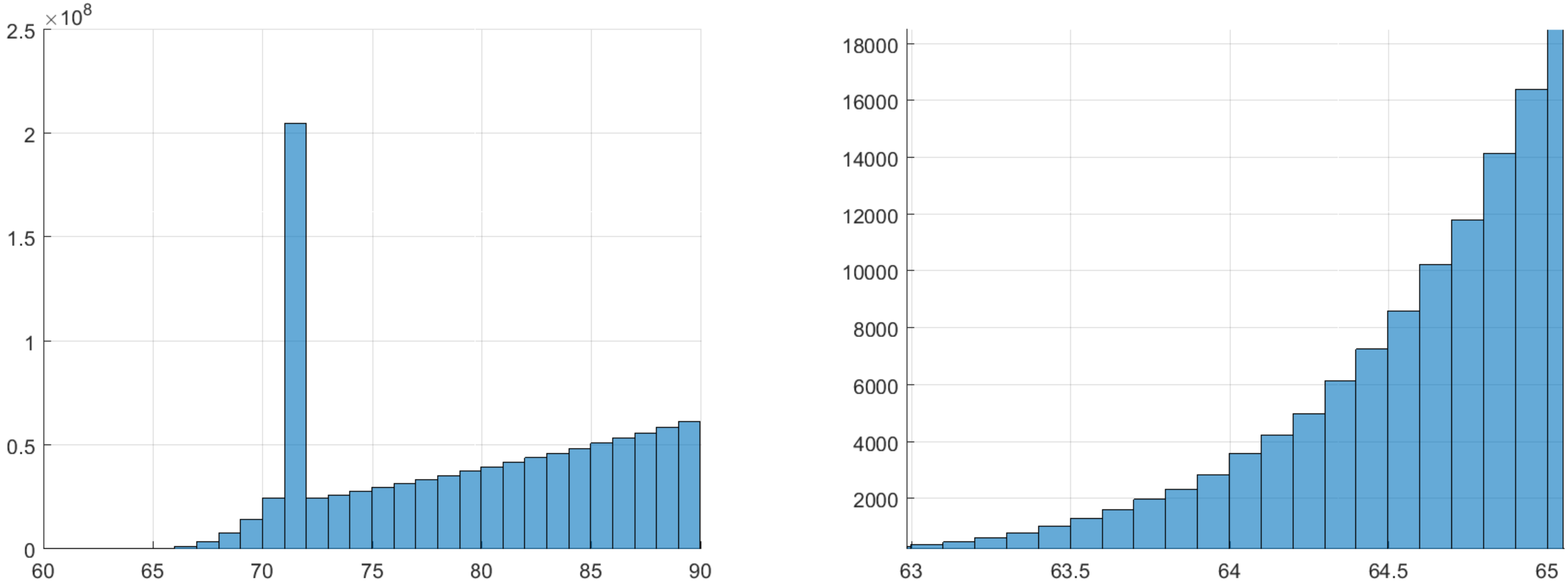
Thm: Put a topology on n-gons by thinking of them as a subset of $\mathbb{R}^{2n}$.

(a) The map $P \to \Phi(P)$ is continuous, so $\{P : \Phi(P) = \phi\}$ is closed.

(b) For n large, this set has interior iff $\phi = \frac{5}{7} \cdot 90°$ or $\phi = 72°$.

(c) Otherwise it has co-dimension ≥ 1.



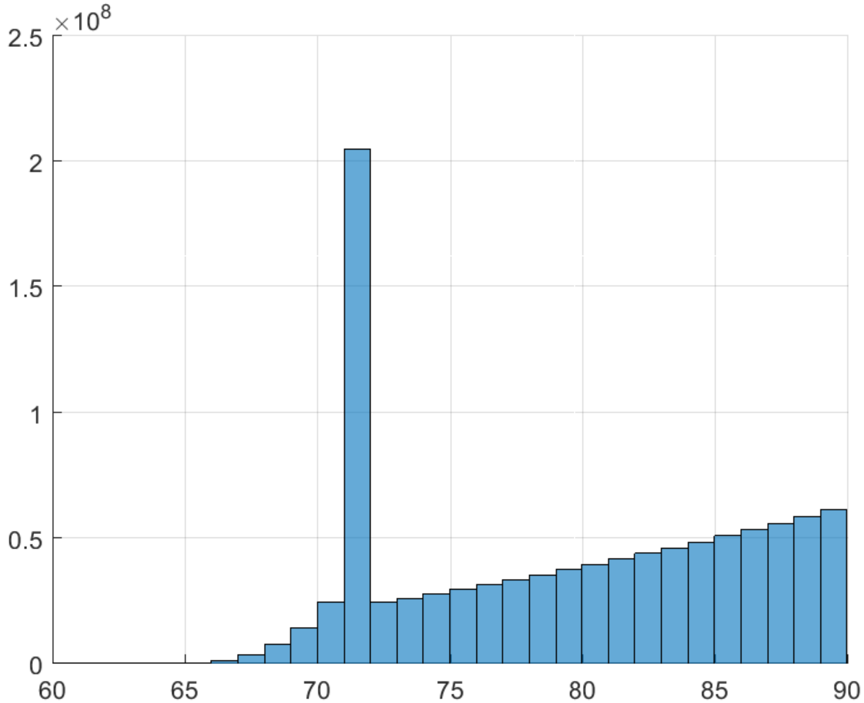


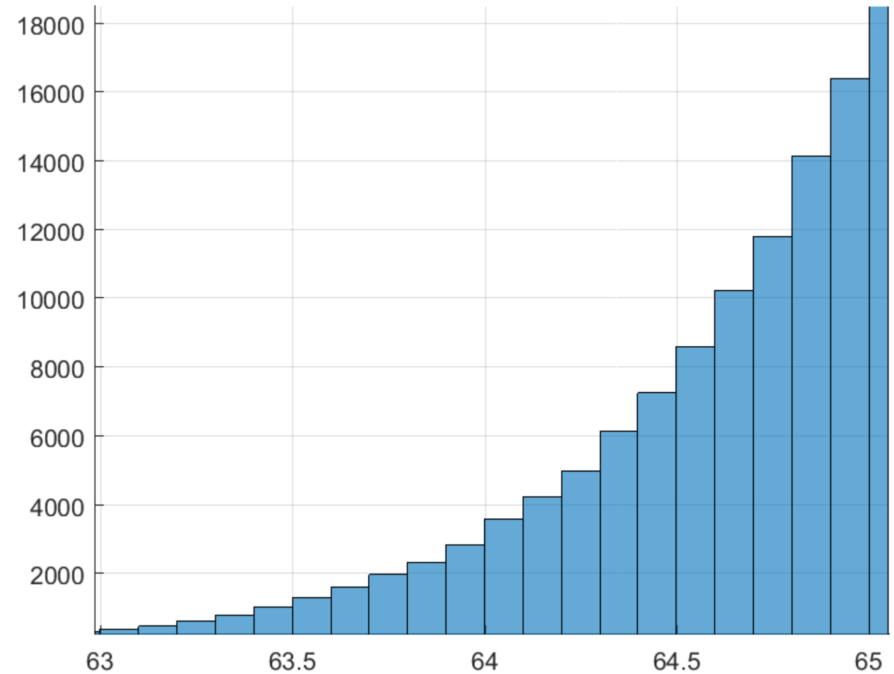
The distribution of optimal upper bounds over 10^9 random samples.

On the left is a histogram based on 1° bins. The spike a 72° is evident.

On the right is an enlargement near 64° using $.1^\circ$ bins.

No spike at $\frac{5}{7} \cdot 90^\circ \approx 64.26^\circ$ is visible. Is $N = 10$ too small?





In these experiments I just chose angles at random (with correct sum).

Didn't choose edge lengths or check for self-intersections.

What is better model for random polygons?

More open problems about triangulations:

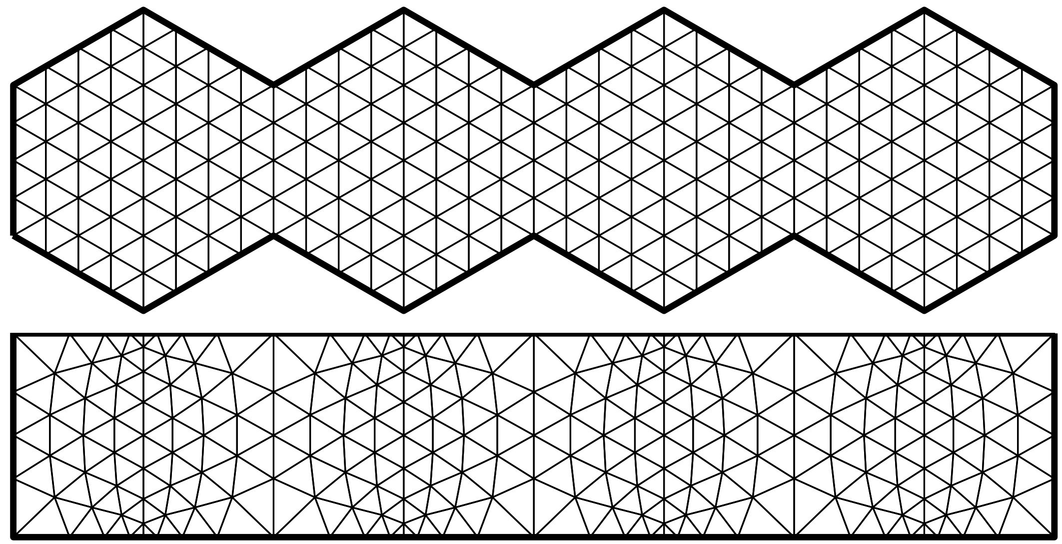
- How large are angle optimal triangulations?

- Minimal weight Steiner triangulations.

- Surfaces and solids.

- Triangulations of PSLGs (planar straight line graphs).

How many triangles does MaxMin solution need?



Proof of theorem gives exponentially many triangles for $1 \times R$ rectangle.

But good choice of $60°$-polygon P' above gives $O(R)$ triangles.

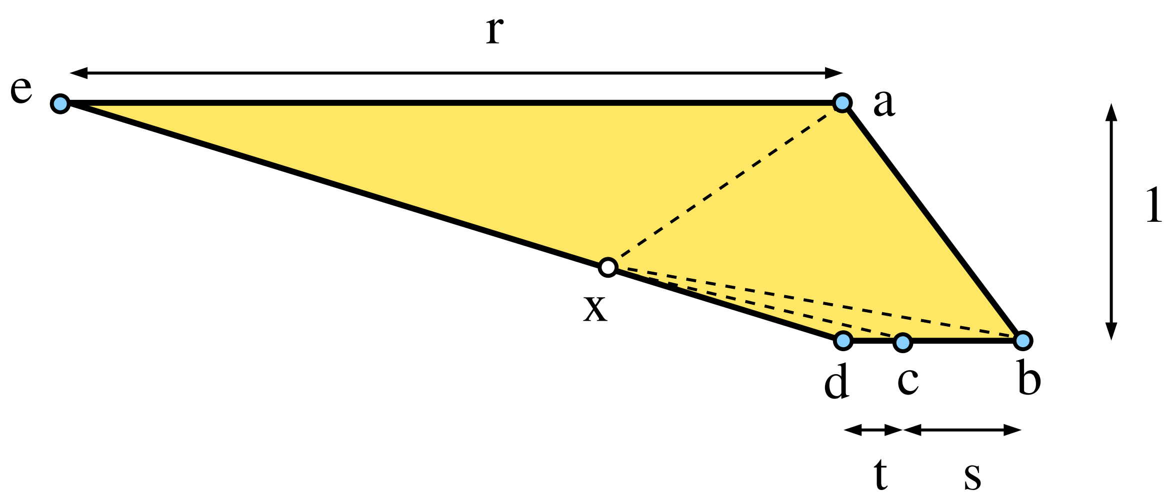
Estimate smallest number of triangles needed for general P?

Thick-thin decomposition of polygons may help.

Is exact minimum NP-hard to compute?

We saw a triangulation achieving MinMax angle usually exists.

A **minimal weight Steiner triangulation** (MWST) minimizes total edge length. It need not exist ($t \ll s \ll 1 \ll r$):



Question: Does a MWST exist for polygons in general position?

Without Steiner points, finding a MWT is NP-hard for point sets (Mulzer-Rote 2008) and $O(n^3)$ for polygons (Gilbert 1979, Klincsek 1980).

O(optimal) approximation of MWST is possible (Eppstein, 1994)

Burago-Zalgaller in 3 dimensions:

Does every polyhedron have an acute triangulation of polynomial size?

triangulation = tetrahedralization with dihedral angles $< 90°$.

Acute triangulation exists for unit cube $[0,1]^3$: 1370 tetrahedra.

No acute triangulation of cube in $\mathbb{R}^n$, $n \geq 4$.

Kopczynski-Pak-Przytycki 2009, VanderZee-Hirani-Zharnitsky-Guoy 2010.

2.5 dimensions:

Do polyhedral surfaces have polynomial sized acute triangulations?

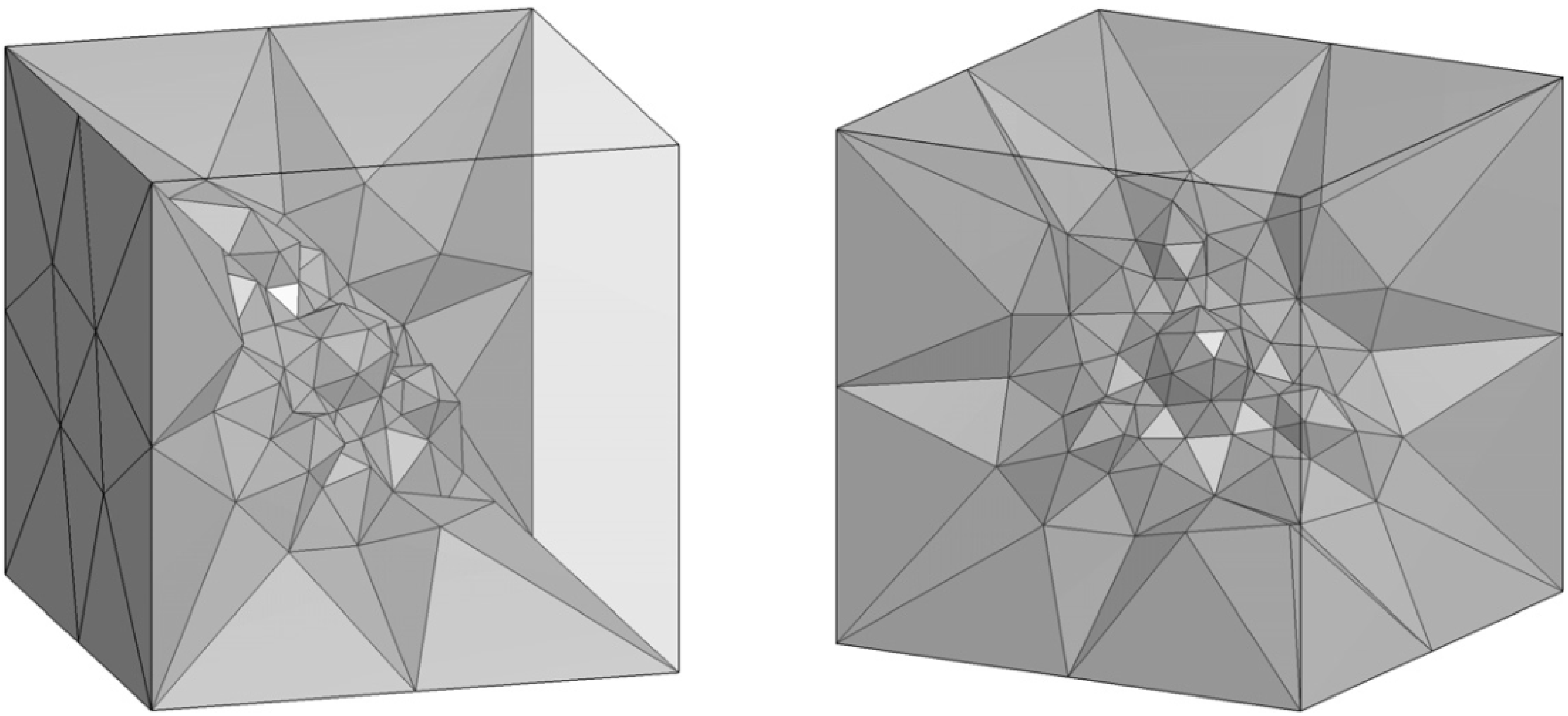
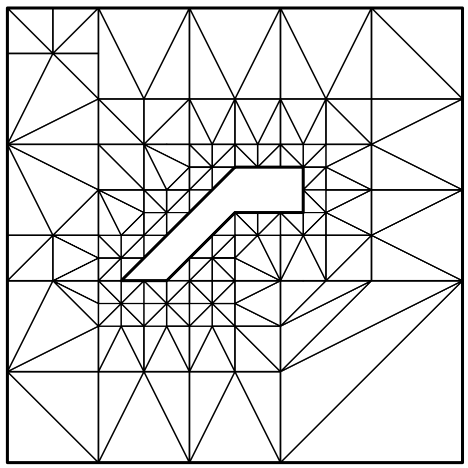
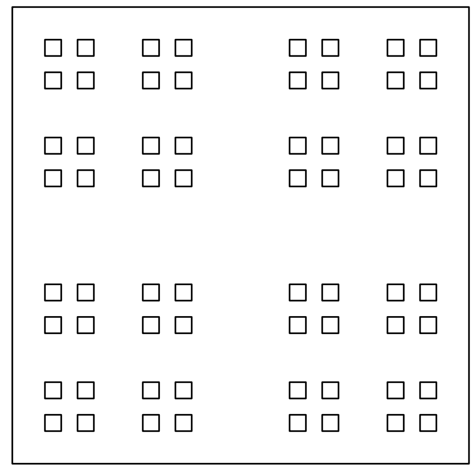


Fig. 1. Two views of a cutaway section of the first-known acute triangulation of the cube. The view at right is a 45° rotation about the z-axis from the view at left. On the left a 14-triangle triangulation of one of the square faces of the cube is visible. This 14-triangle triangulation of the square is used on each face of the cube.

Planar Straight Line Graphs (PSLGs)

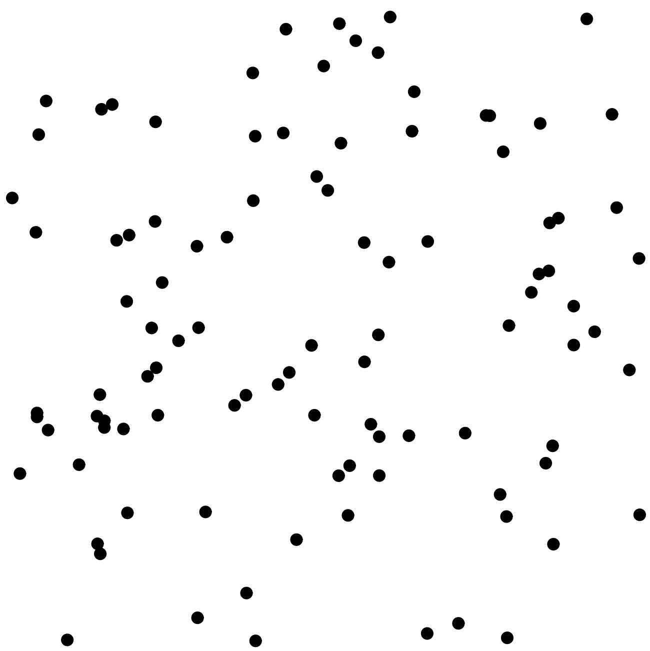
A **planar straight line graph** Γ (or **PSLG**) is finite union of points V and a collection of disjoint edges E with endpoints among these points.

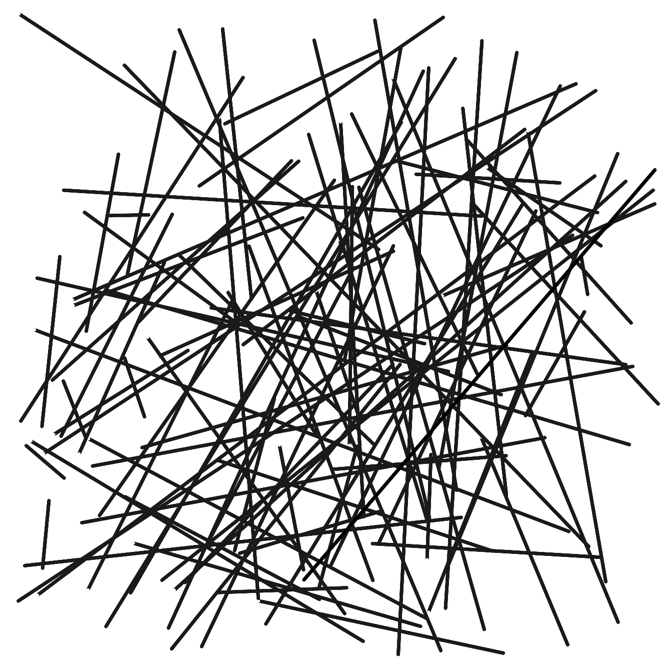


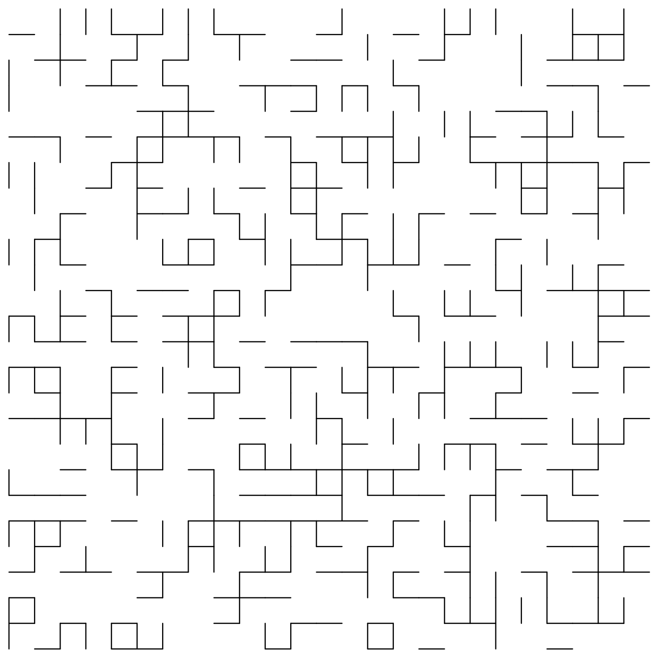


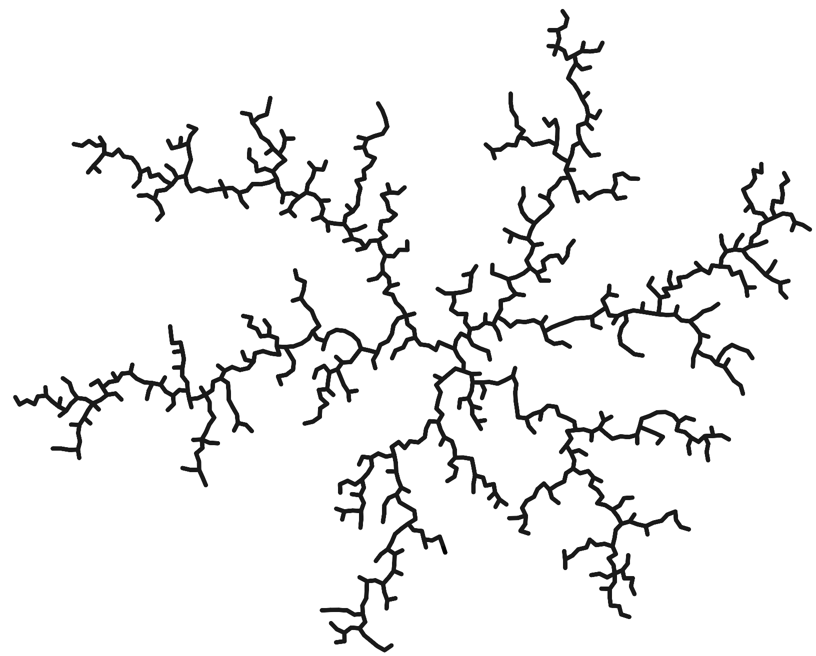
Generally let $n = |V|$ be the number of vertices.

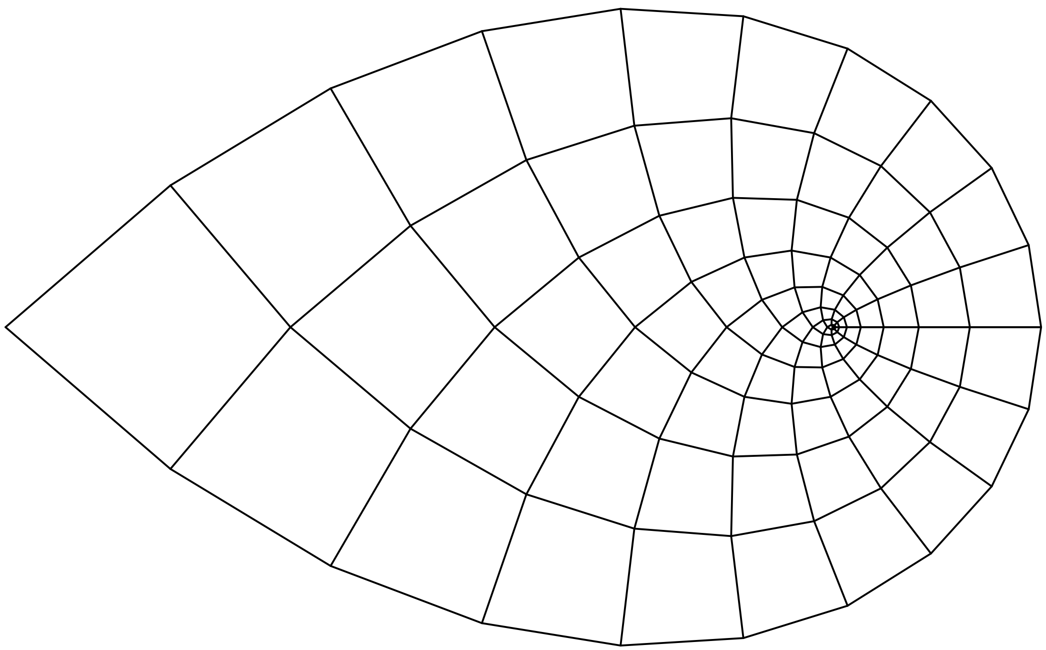
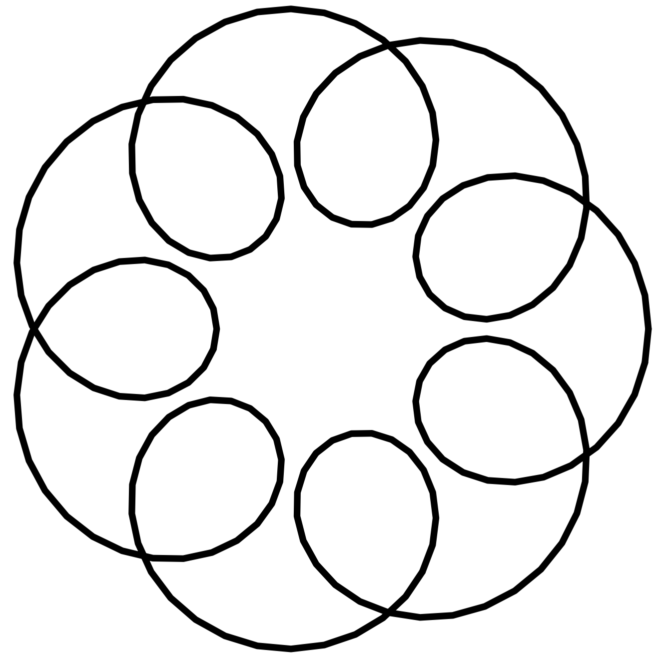
A simple polygon is a PSLG where edges form a closed cycle.



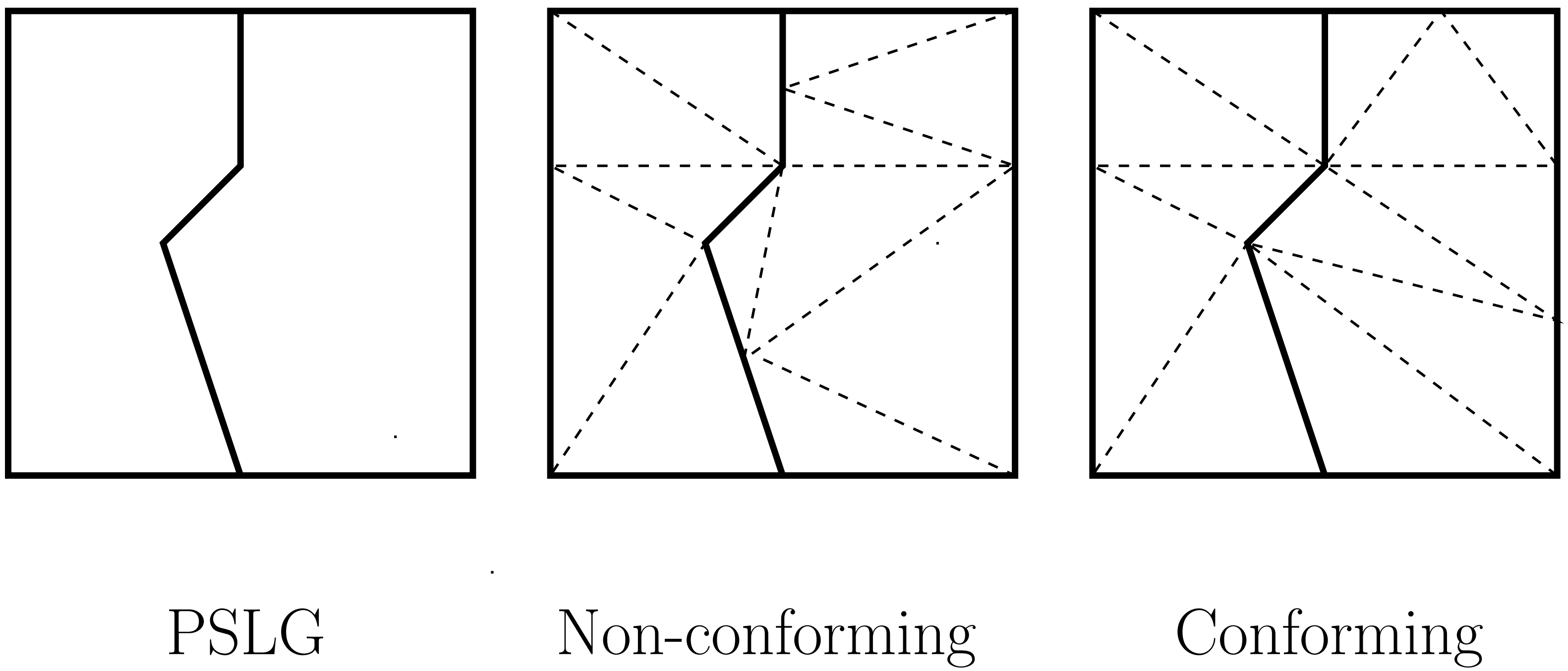




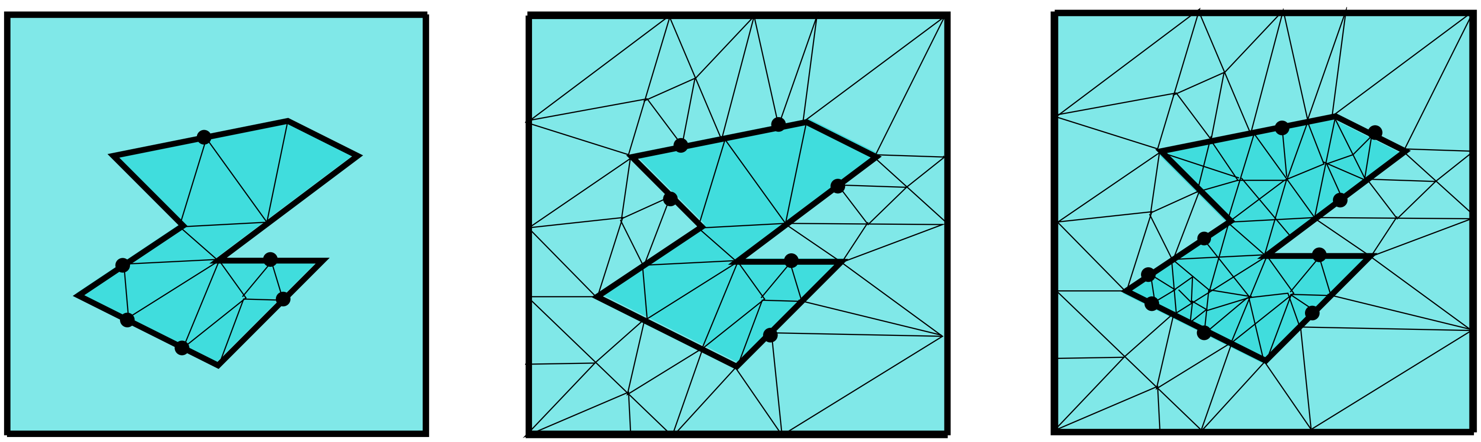


A **conforming triangulation** of a PSLG is a triangulation of each face, consistent across edges of the PSLG.



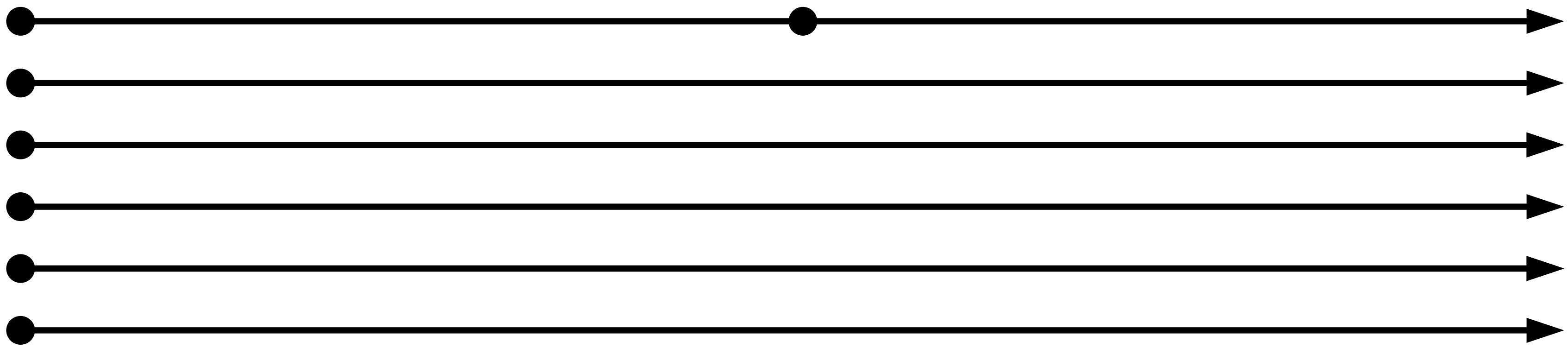
NOT = Non-Obtuse Triangulation = all angles $\leq 90°$.



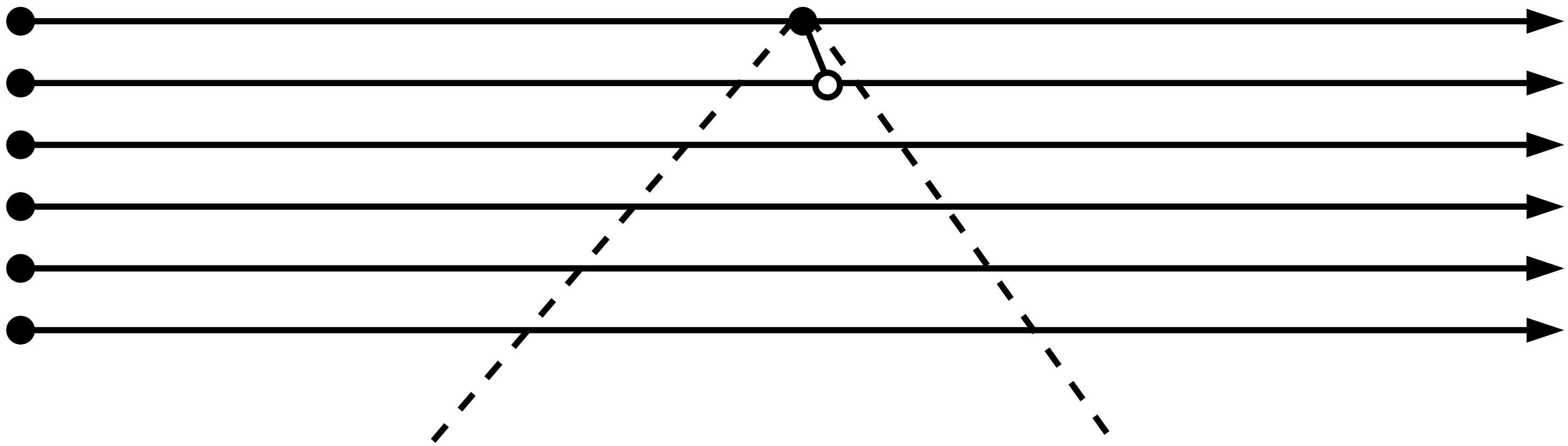
Triangulating a face may add extra boundary vertices.

Triangulating 2nd face may require re-triangulating 1st.

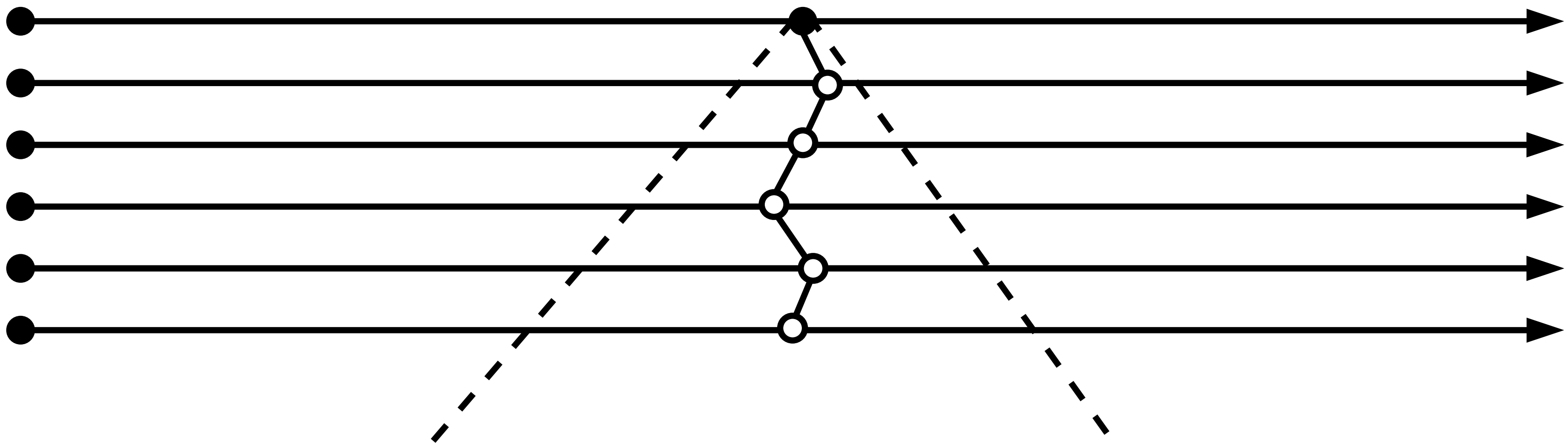
Does this process ever stop?



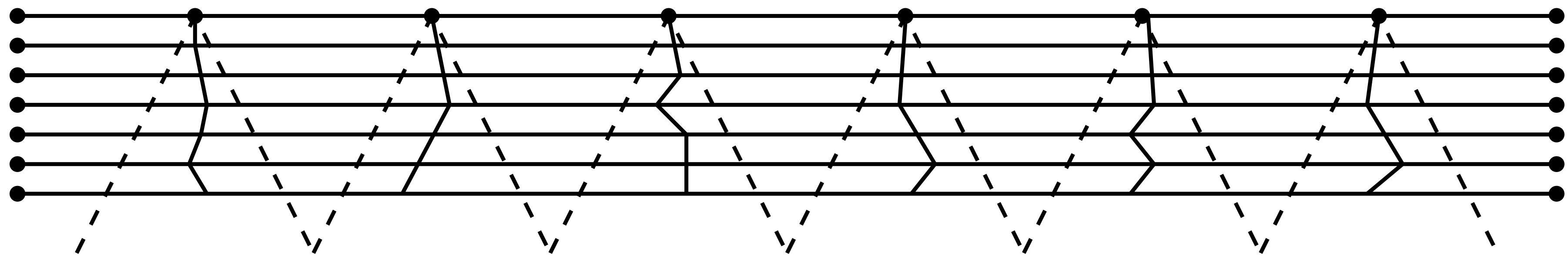
Consider a NOT for this PSLG (or any angle bound $< 180°$).



An edge must leave the vertex with the 90° wedge.



Iterating shows many new vertices, edges are needed.



A NOT for this PSLG needs $\gtrsim n^2$ triangles.

Burago-Zalgaller, 1960: Every PSLG has an NOT (no size bound).

S. Mitchell, 1993: Every PSLG has a 157.5°-triangulation, size $O(n^2)$.

Tan, 1996: Every PSLG has a 132°-triangulation, size $O(n^2)$.

Burago-Zalgaller, 1960: Every PSLG has an NOT (no size bound).

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NOT-Thm (B. 2018): Every PSLG has a NOT with $O(n^{2.5})$ elements.

Improves $O(n^3)$ for Delaunay triangulation by Edelsbrunner, Tan (1993).

First polynomial bound for NOTs of PSLGs.

Proof uses a “discrete closing lemma” for flows (described later).

Problems for PSLGs Γ:

- **NOT Conj:** Every PSLG has a NOT with $O(n^2)$ elements.
- Compute $\Phi(\Gamma)$ = MinMax angle for conforming triangulation of Γ.
- When is minimum MinMax angle attained?
- Give bounds on $\Phi(\Gamma)$ in terms of minimum angle $\theta_{\min}$ in Γ.

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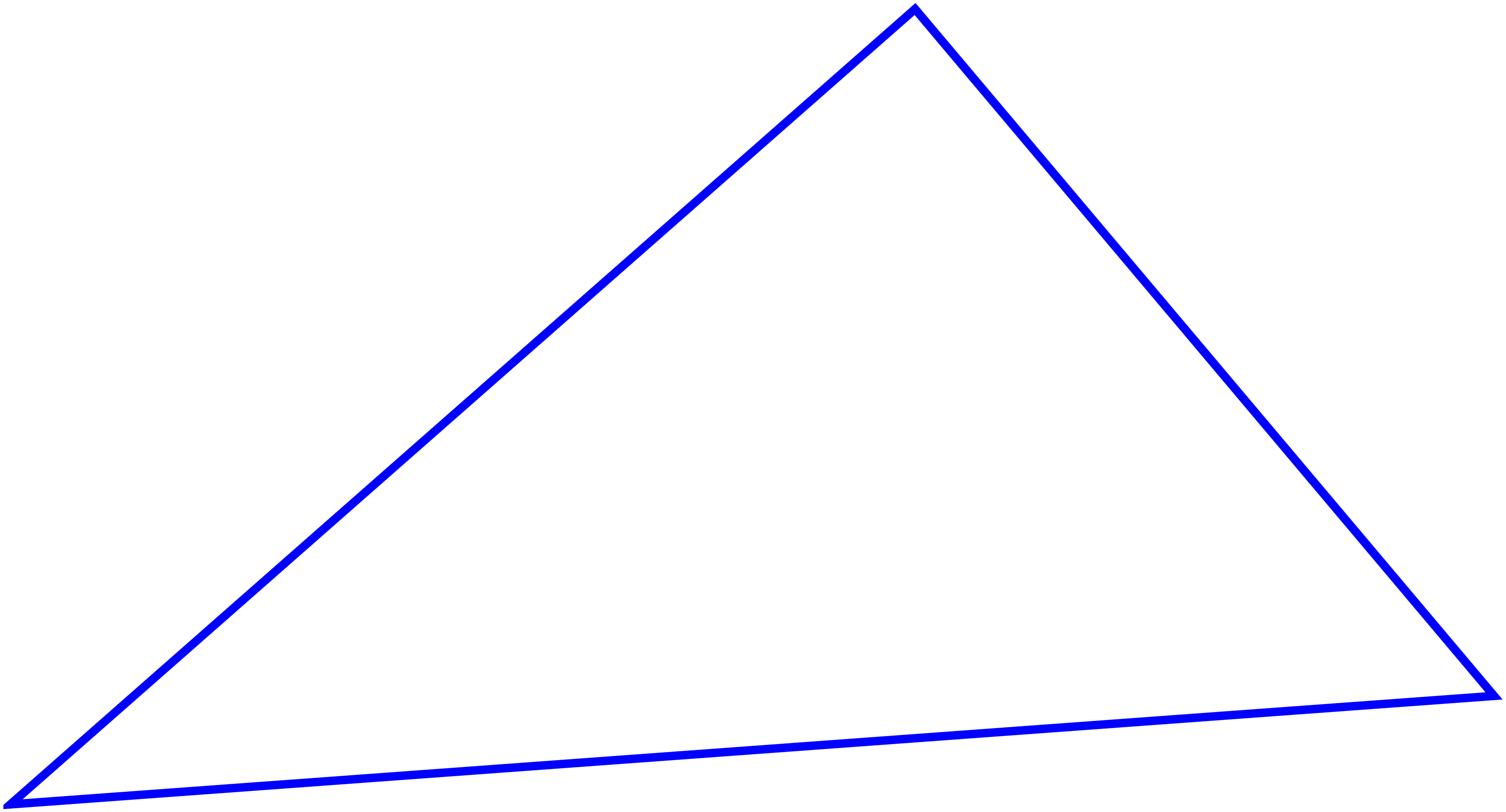
Best result so far: there is a $\theta_0 > 0$ so that

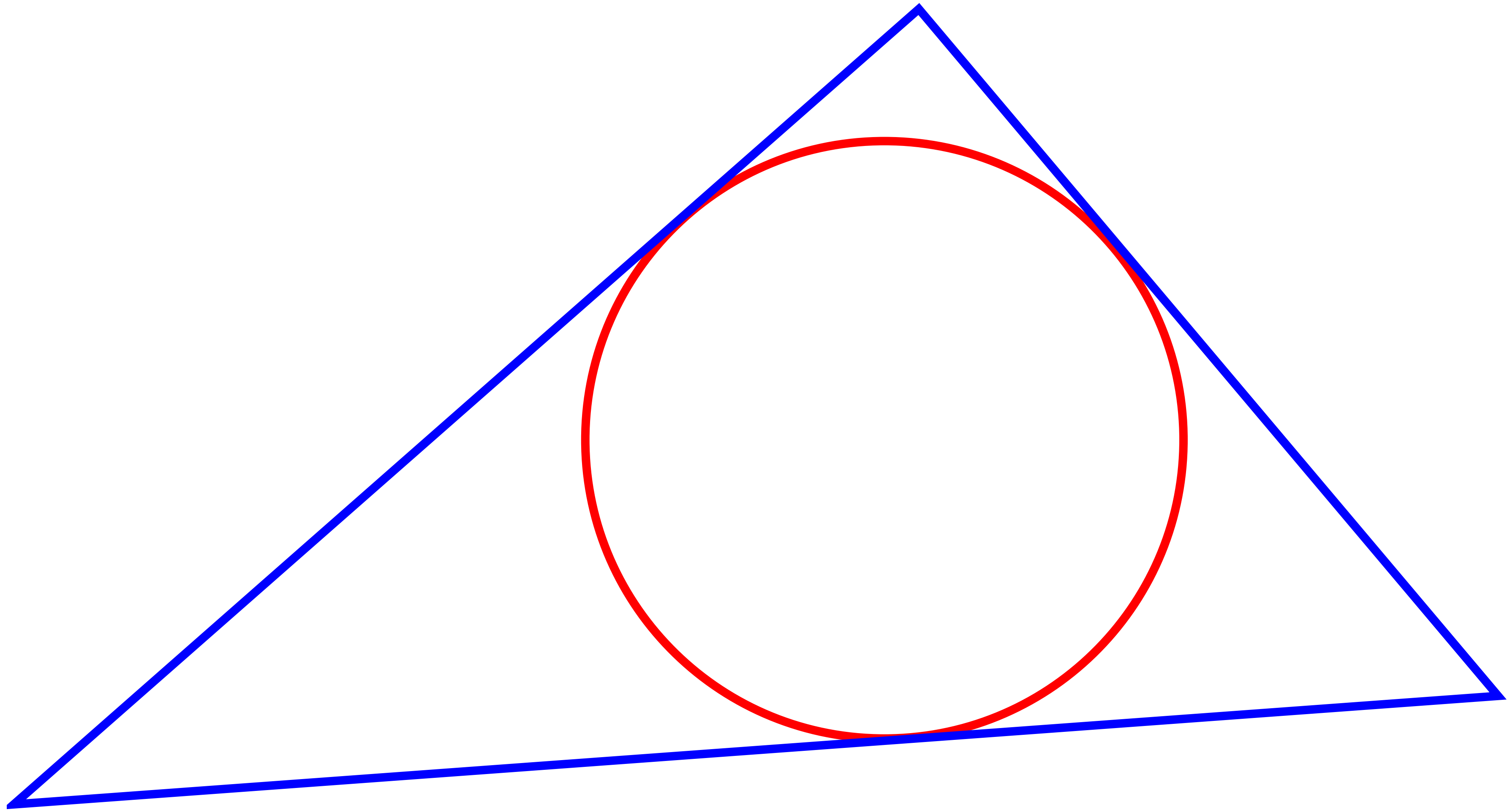
$$\Phi(\Gamma) \leq 90^\circ - \min(\theta_0, \theta_{\min})/2.$$

Uses compactness argument: θ_0 not explicit.

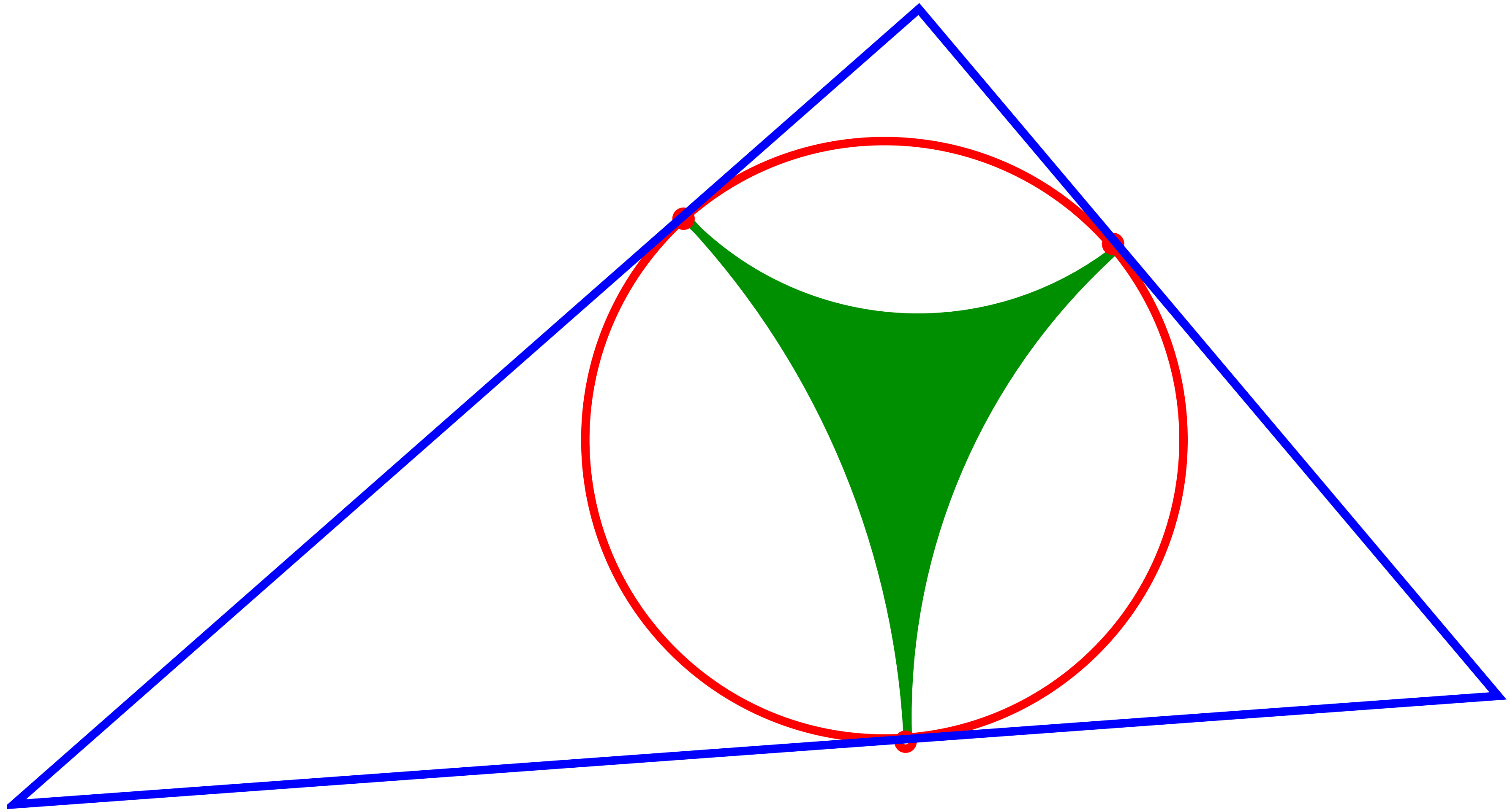
Flows associated to triangulations:



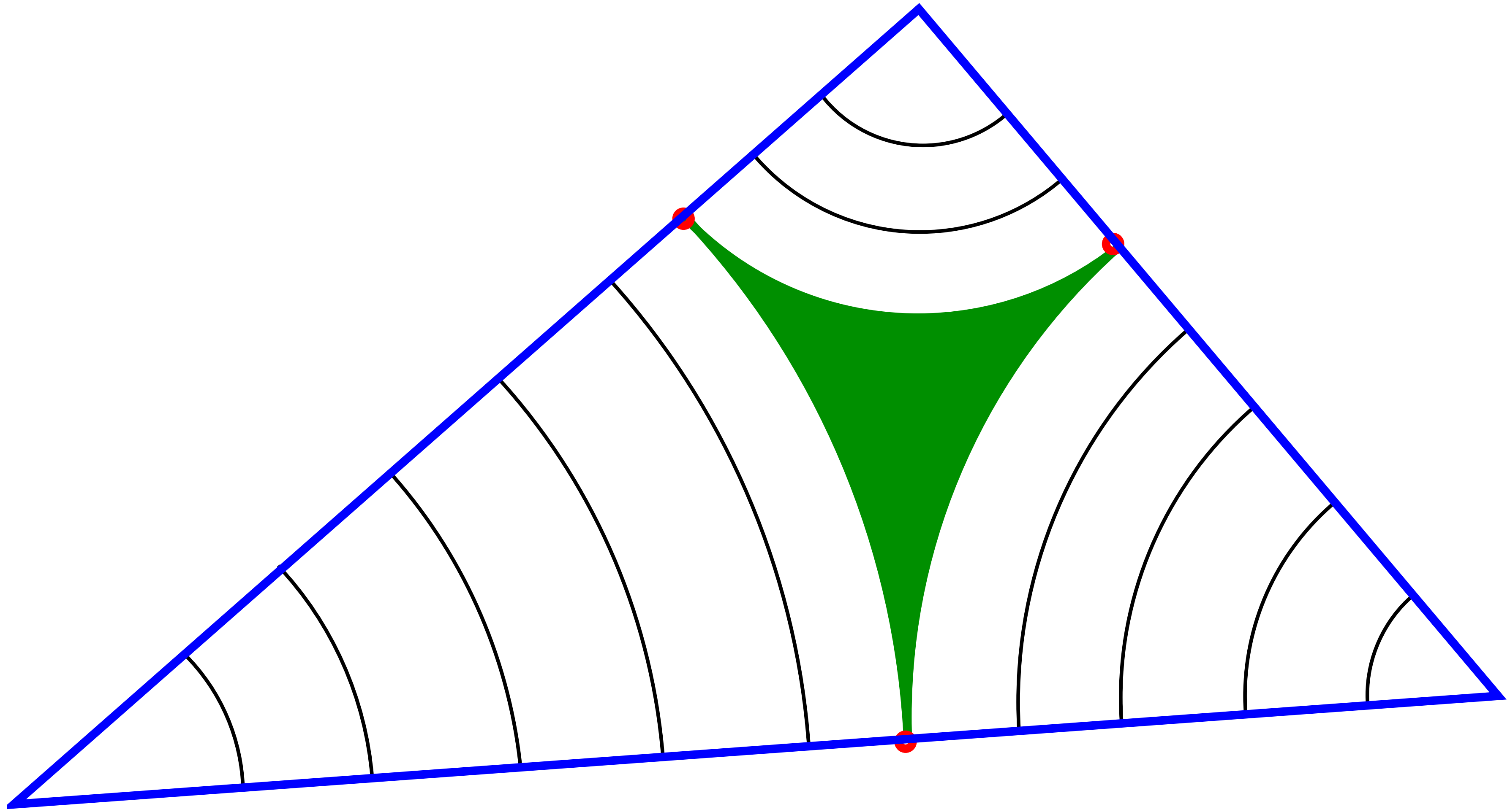
A triangle.



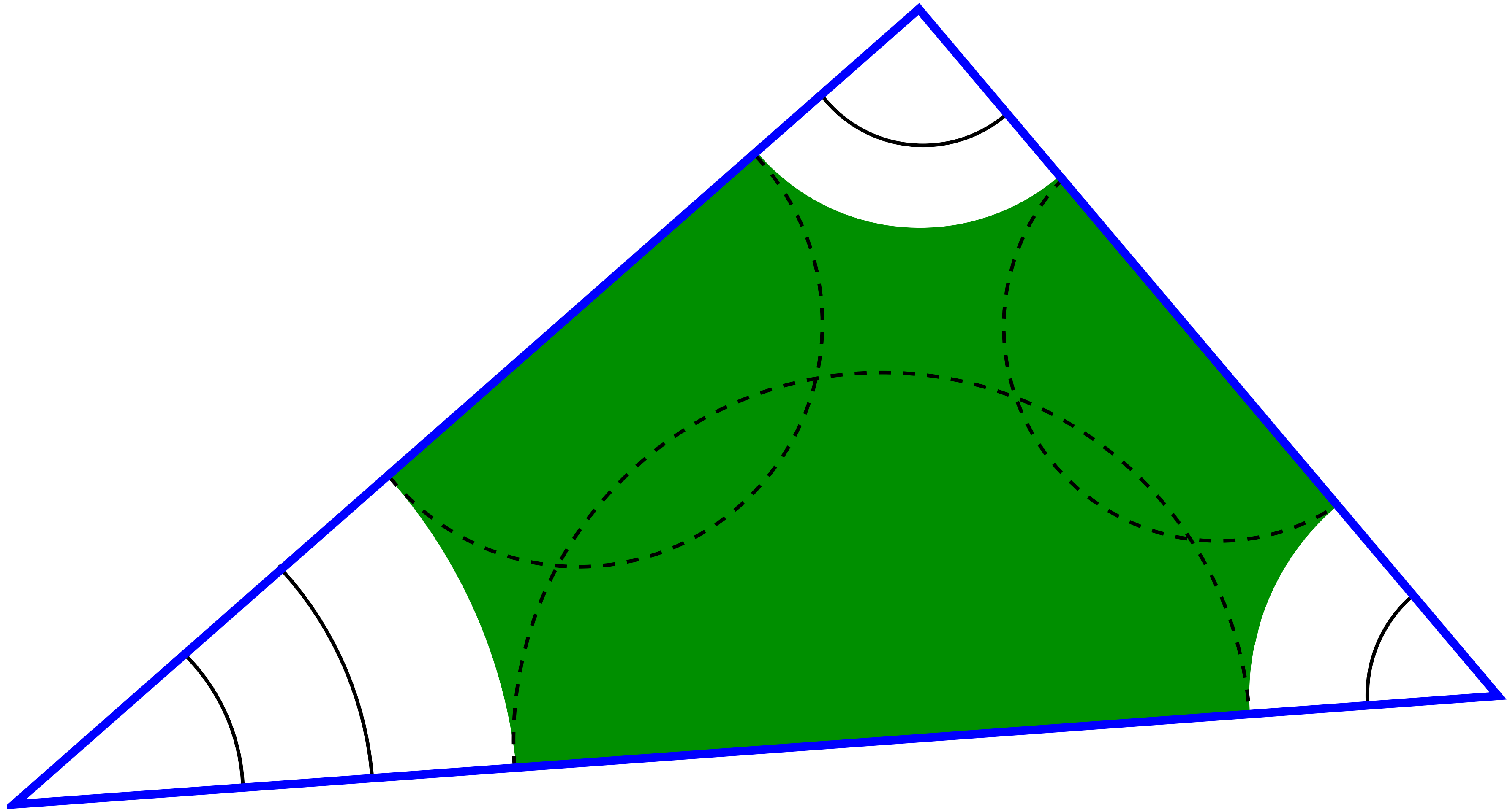
Its in-circle.



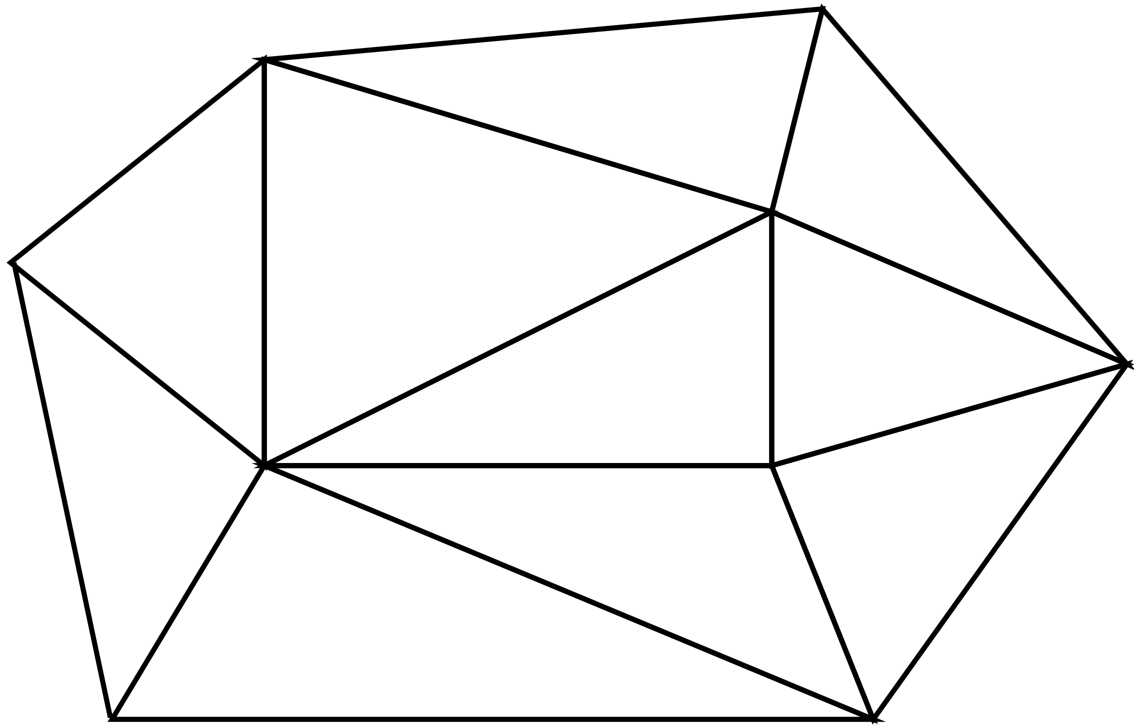
The central region and three sectors (thin version).



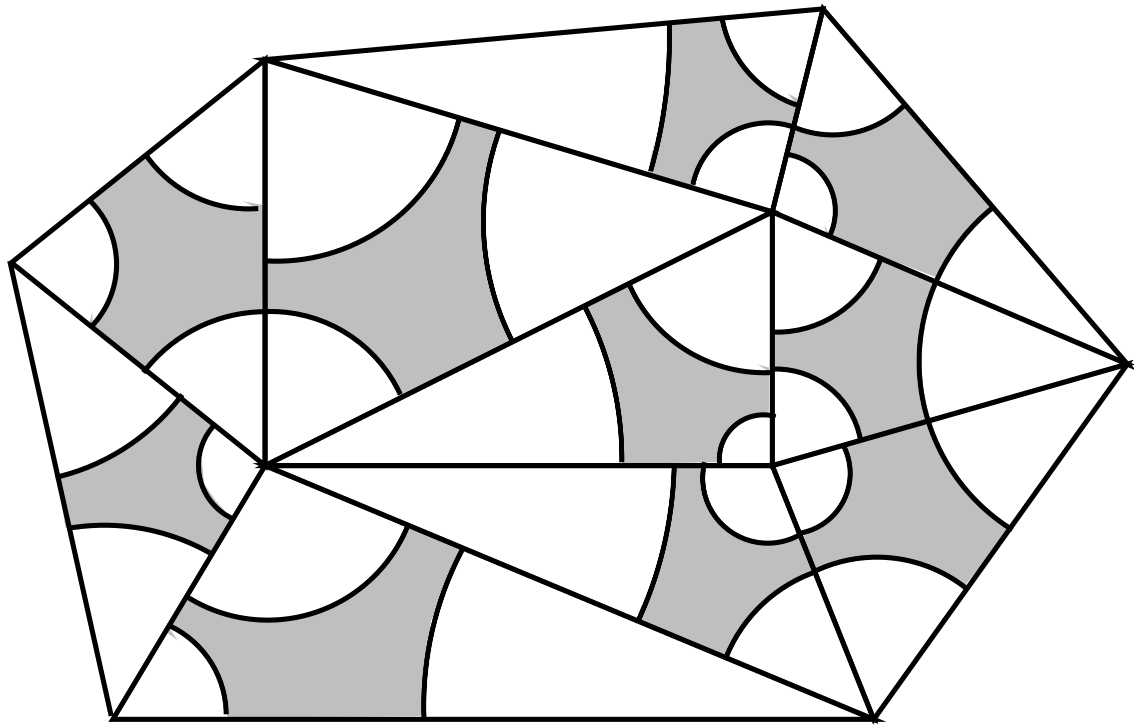
The three sectors are foliated by circular arcs.
Defines flow on a triangulation that stops at boundary or cusp point.



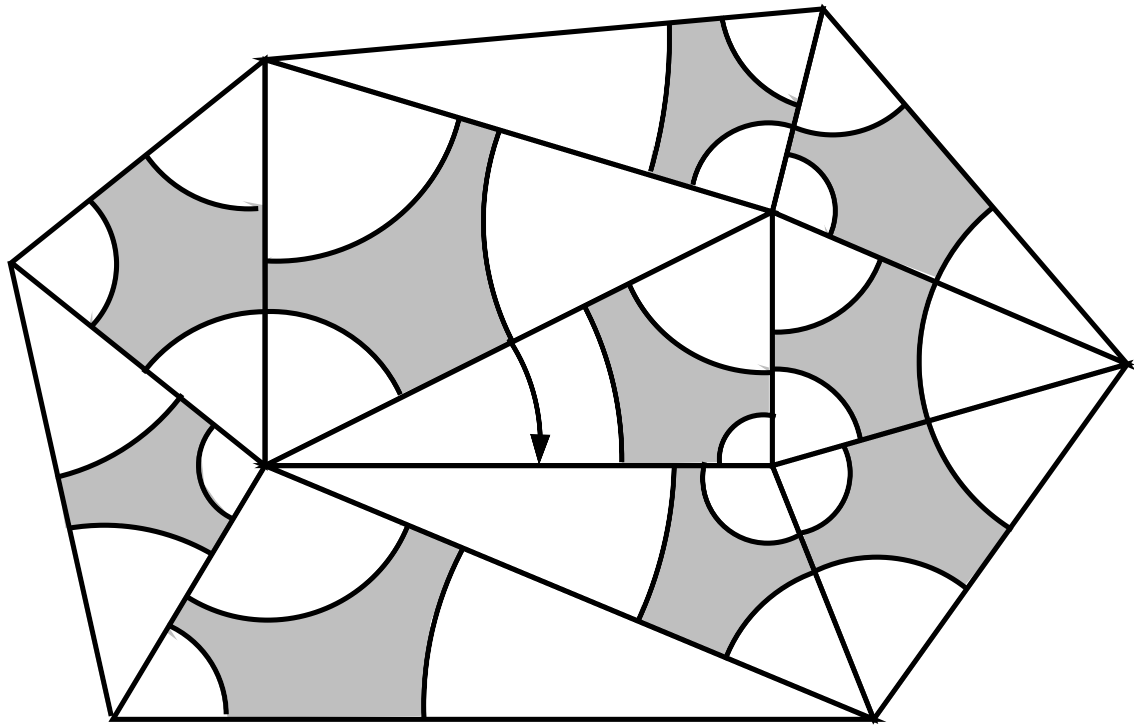
We can also will consider "thick" central regions.
Will need edges to be bases of half-disks contained inside triangle.



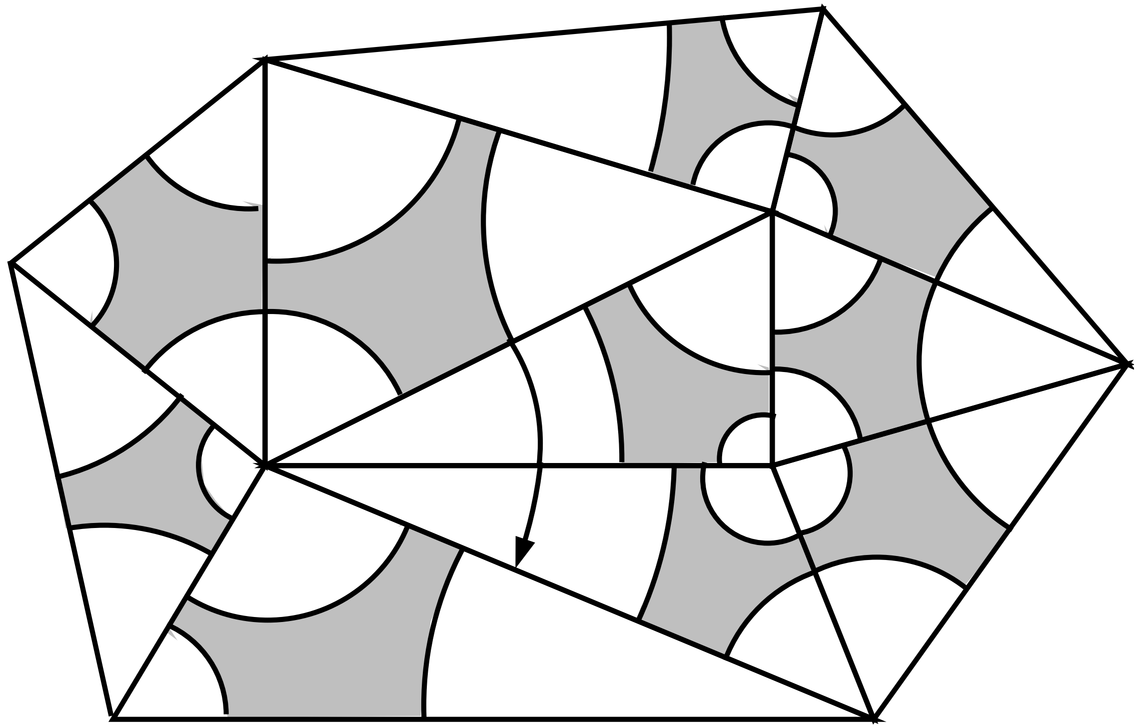
- Start with any triangulation.



- Start with any triangulation.
- Make central parts.



- Start with any triangulation.
- Make central parts.
- Propagate vertices until they leave thin parts.



- Start with any triangulation.
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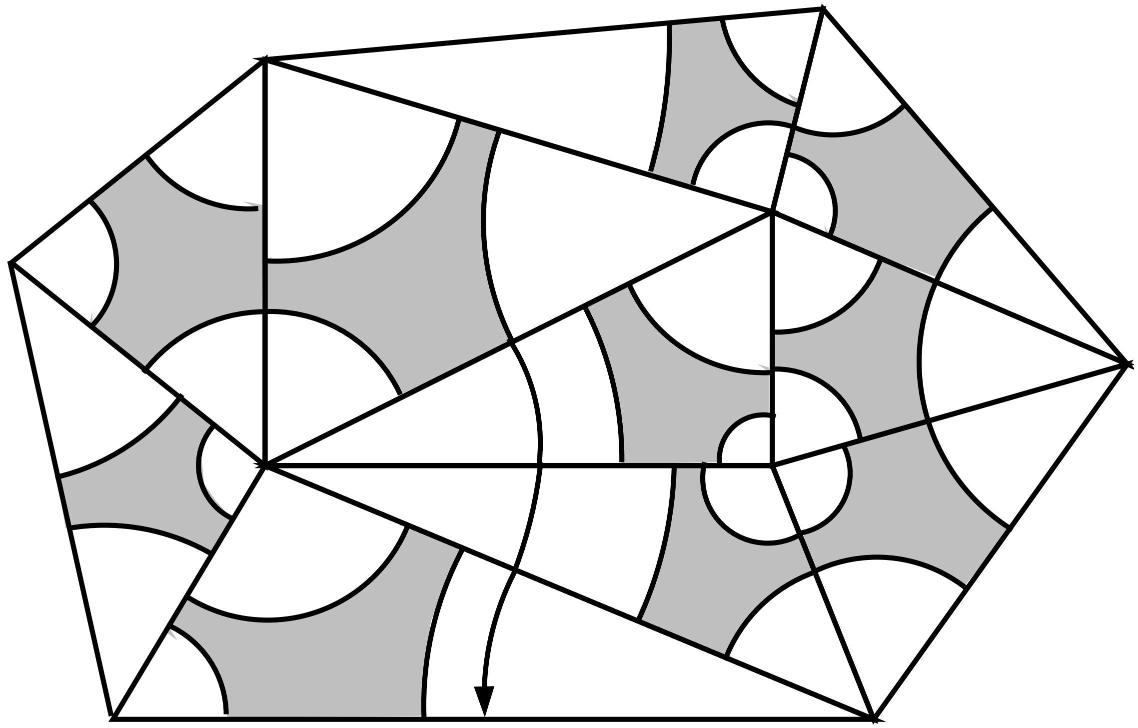

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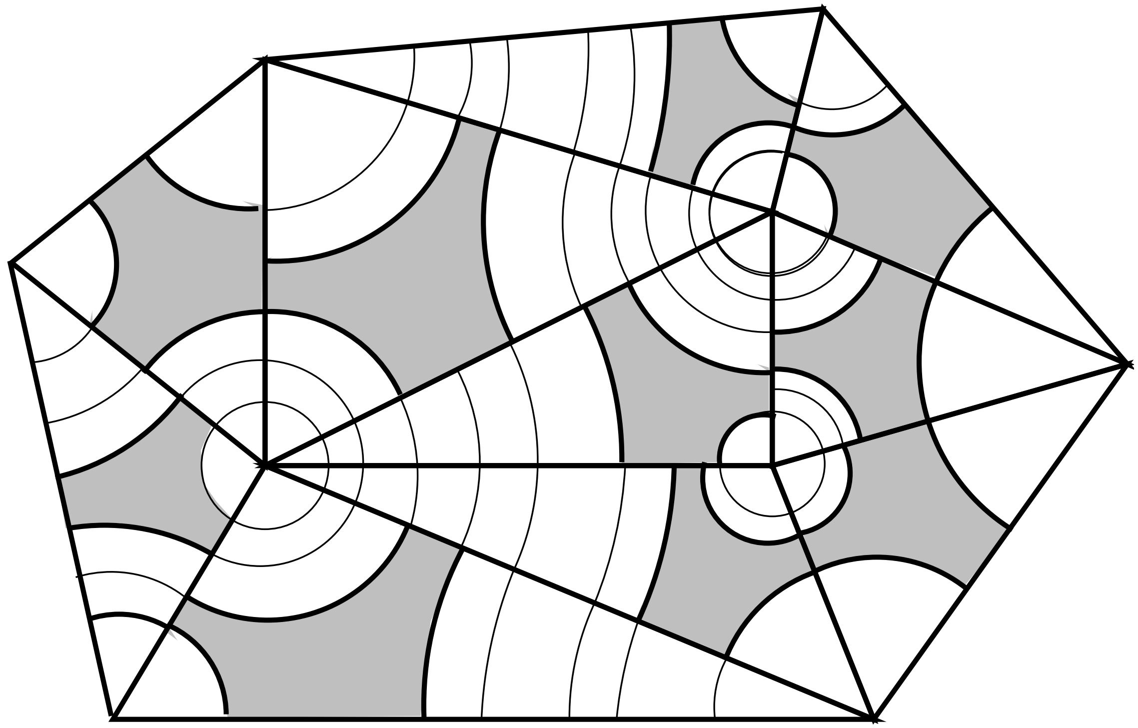

- Start with any triangulation.
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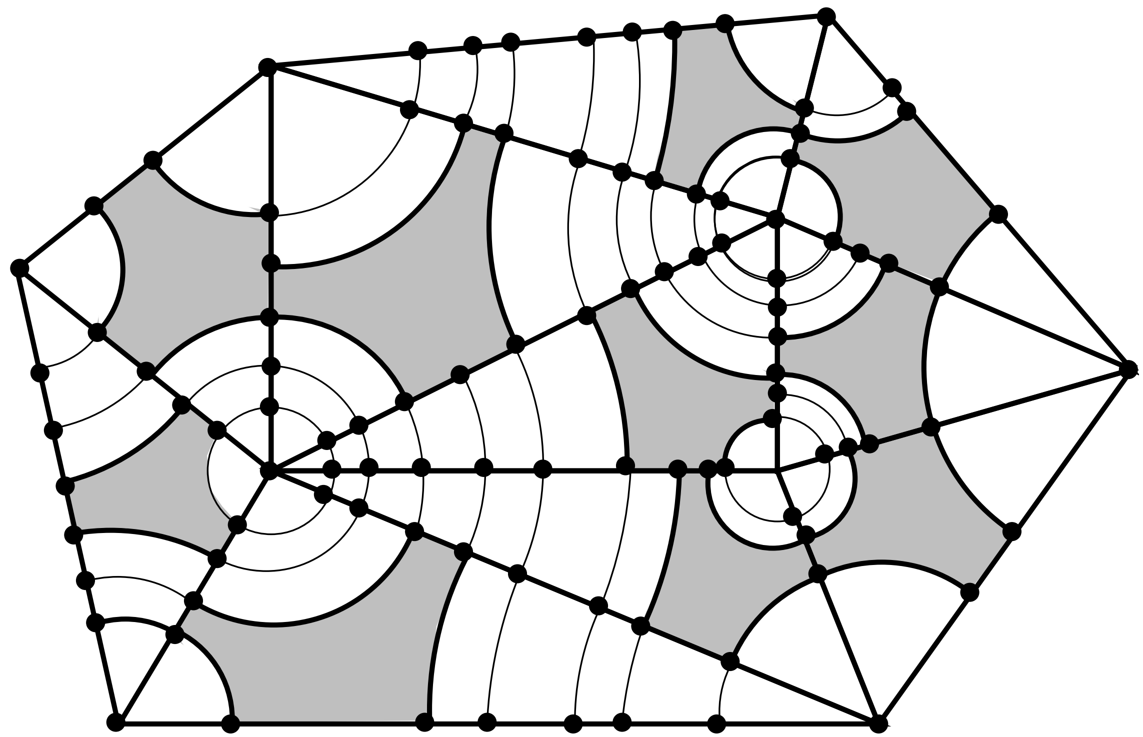
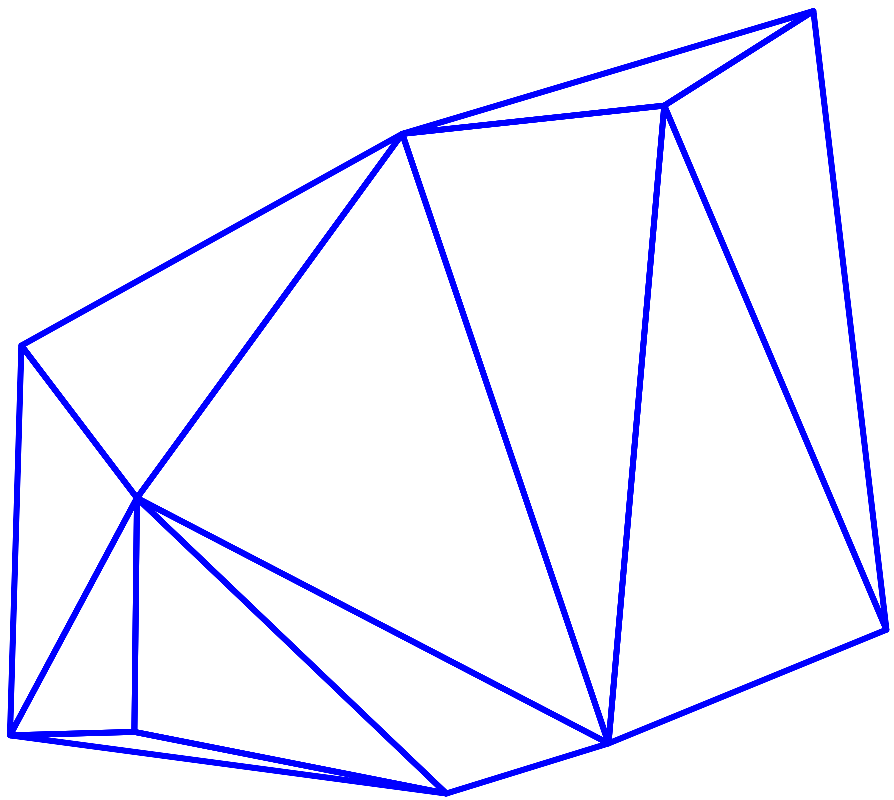
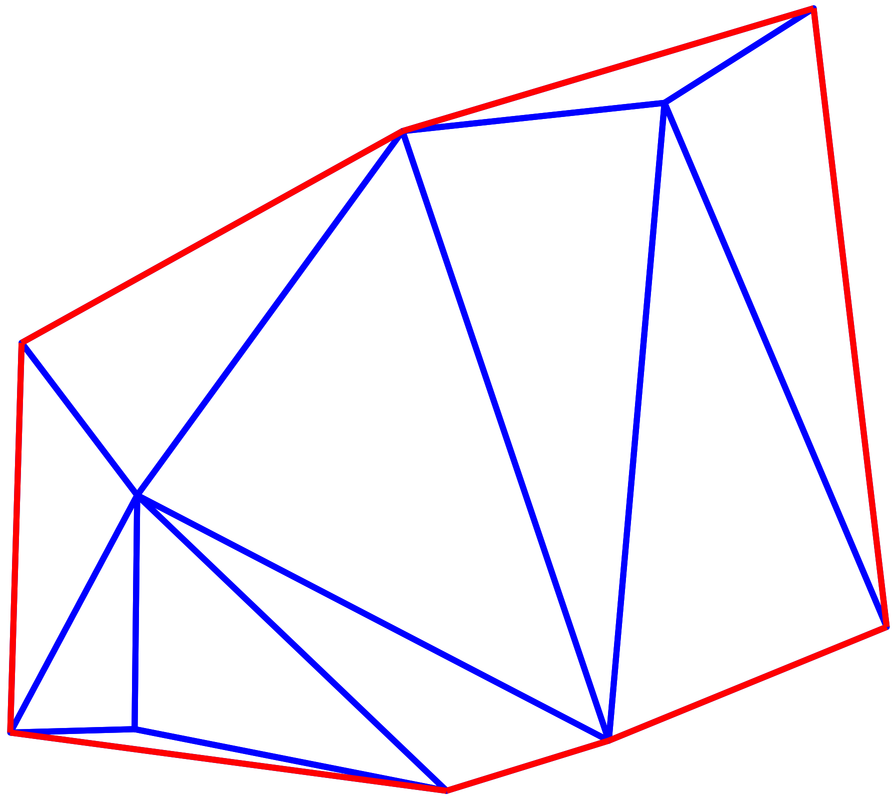


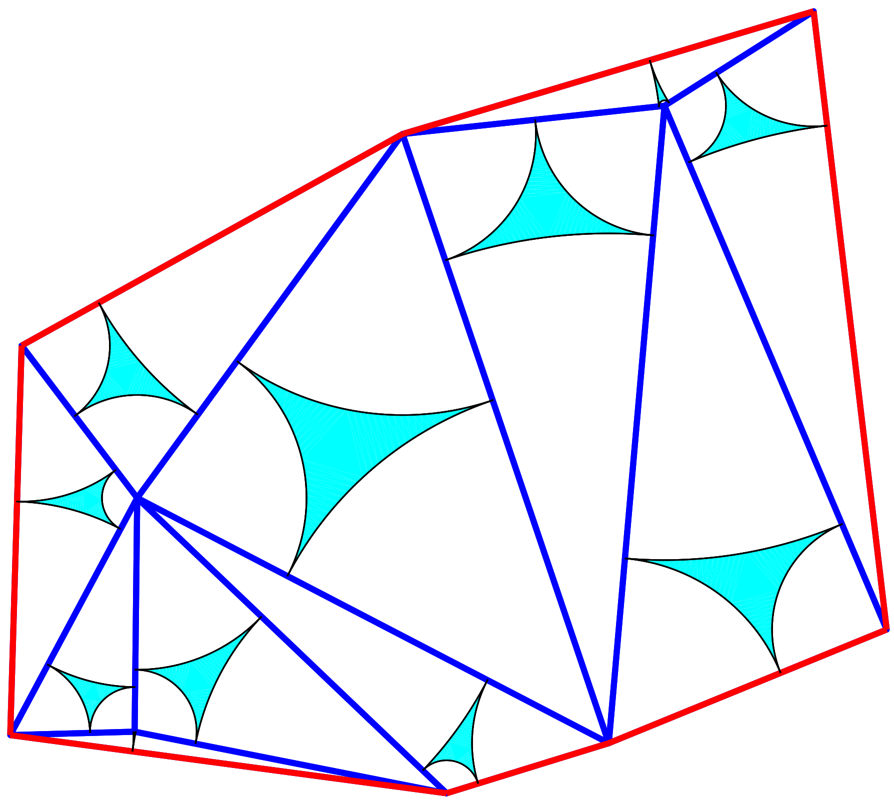
- Start with any triangulation.
- Make central parts.
- Propagate vertices until they leave thin parts.
- How many new points are created?



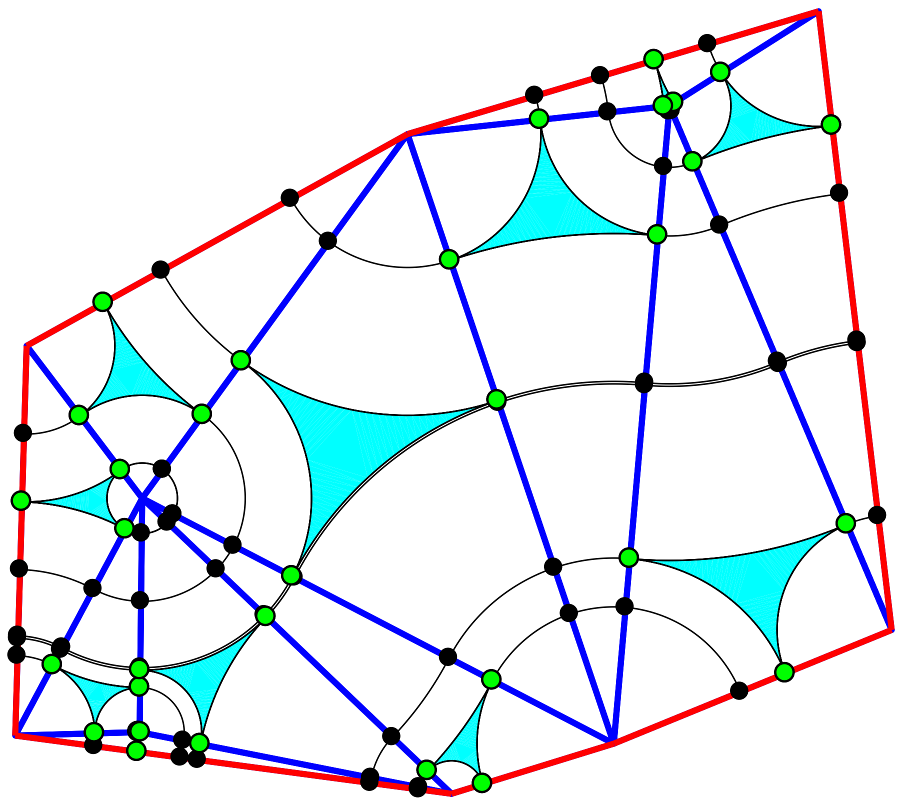
Delaunay triangulation of 10 random points,



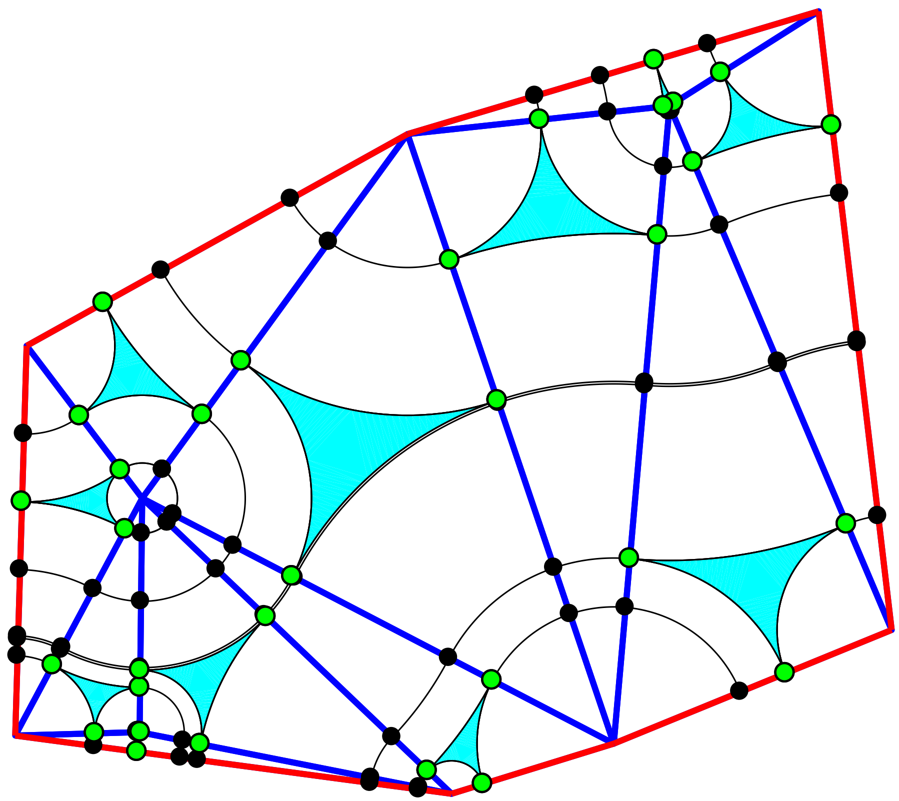
The boundary of the triangulation



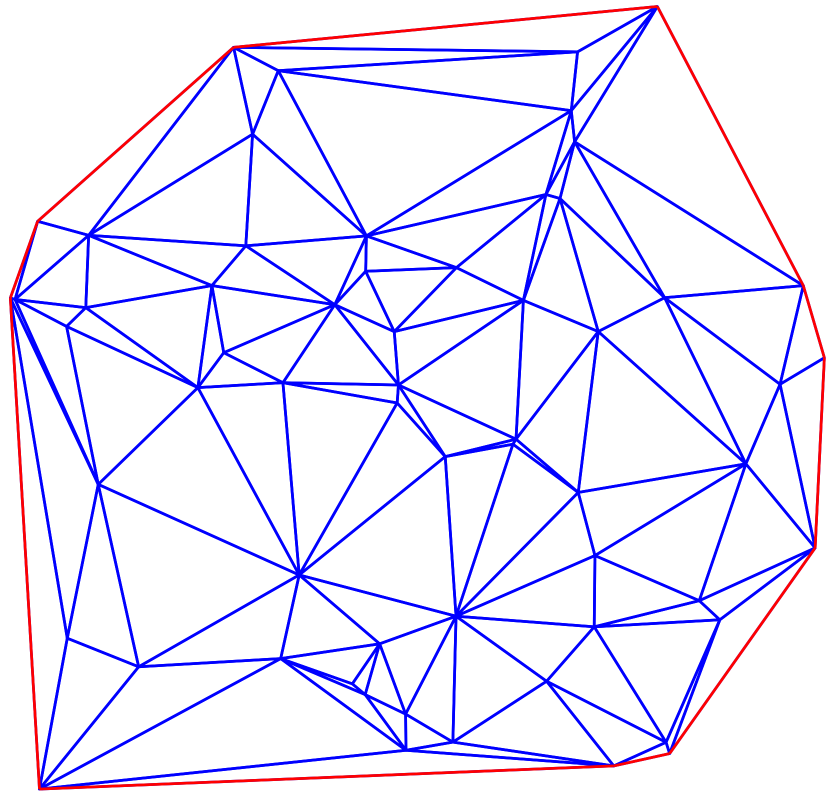
The central regions.



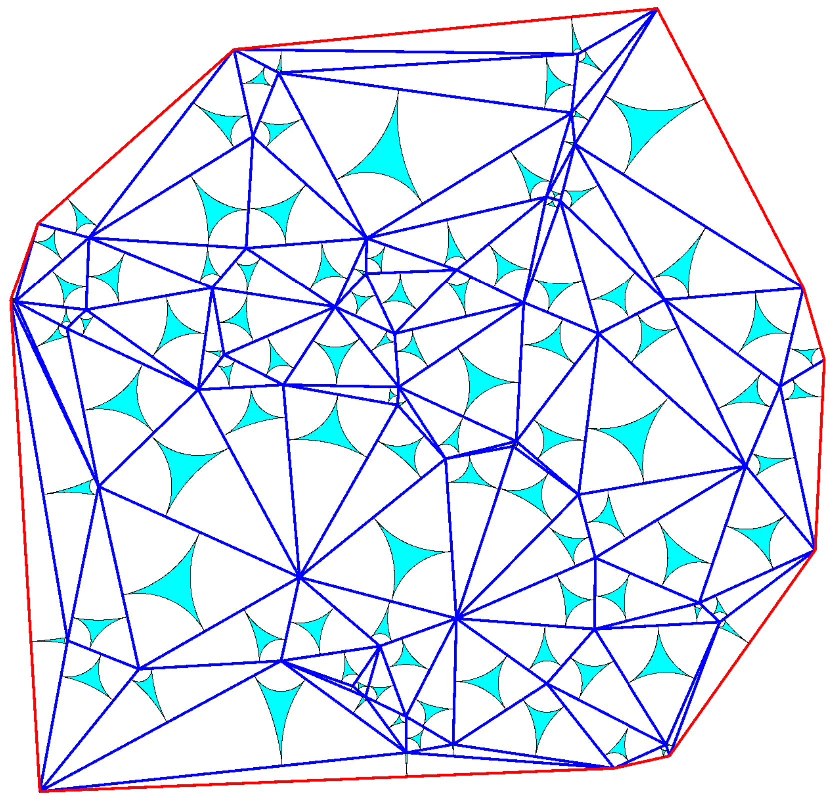
Propagation lines starting at all cusp points.



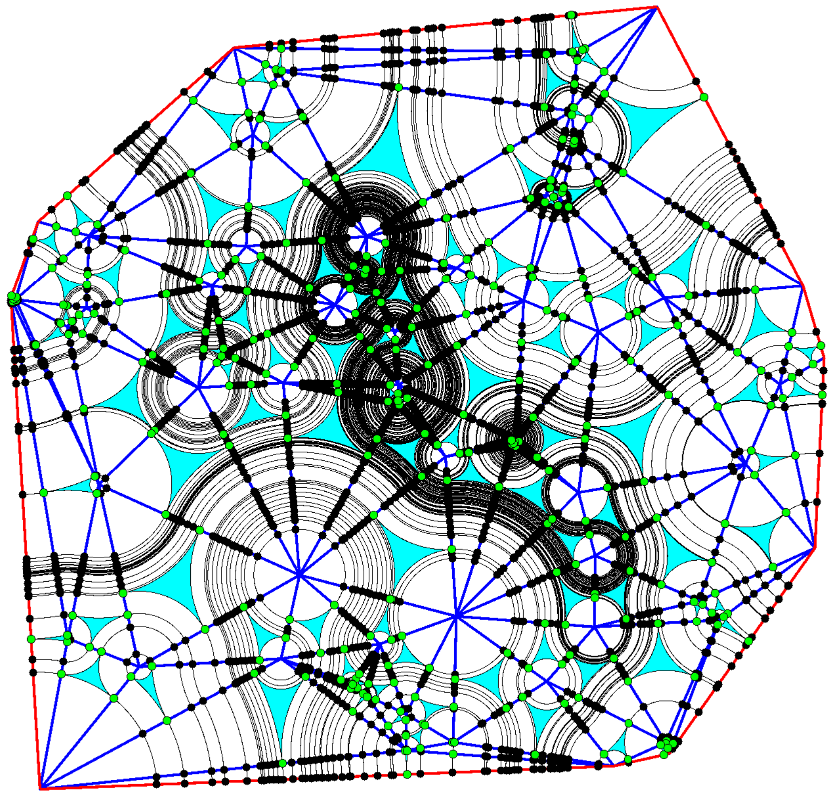
Propagation lines identify boundary points; induces tree.
Discontinuous, but piecewise length preserving.



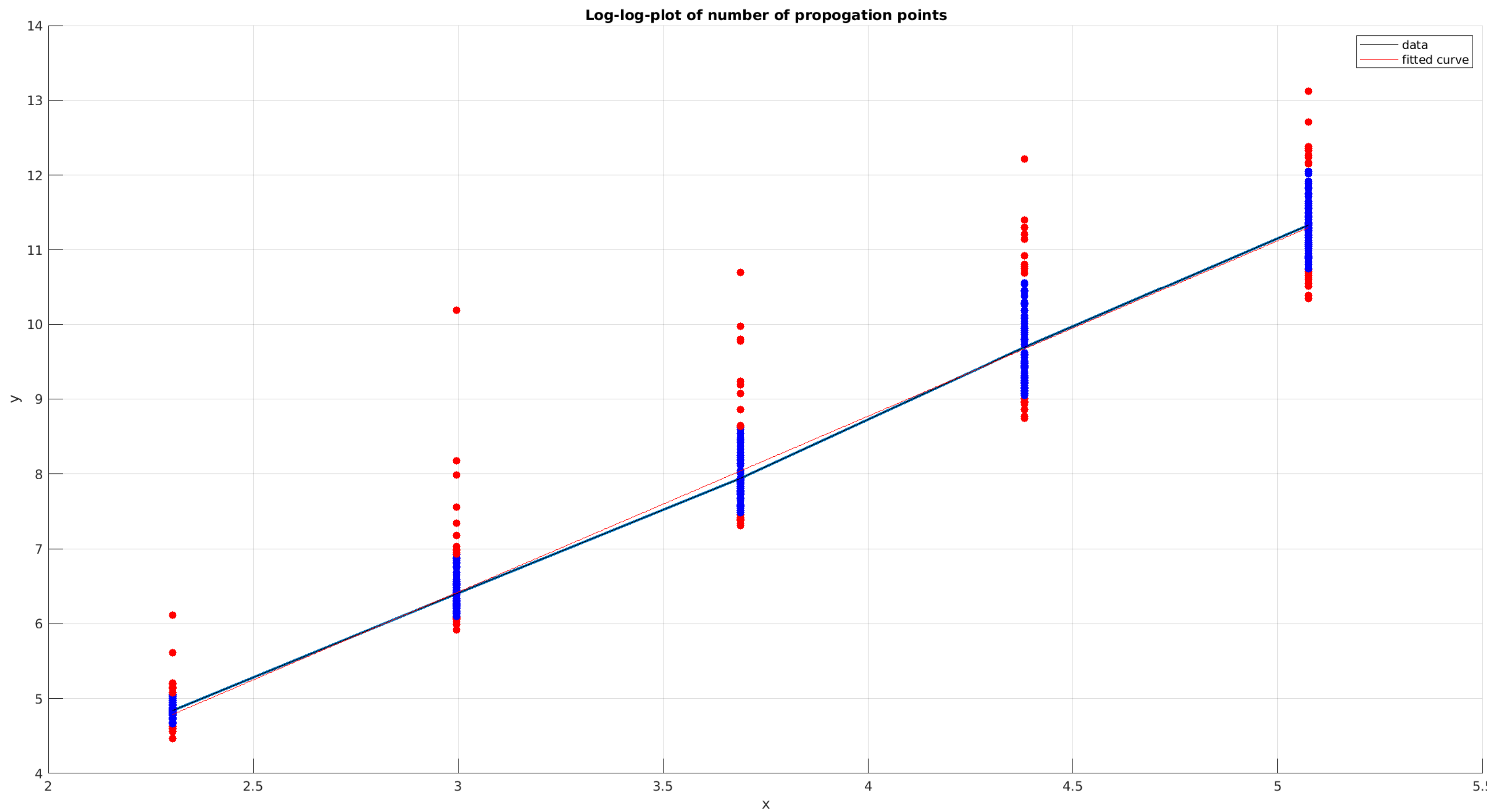
60 points



The central regions.



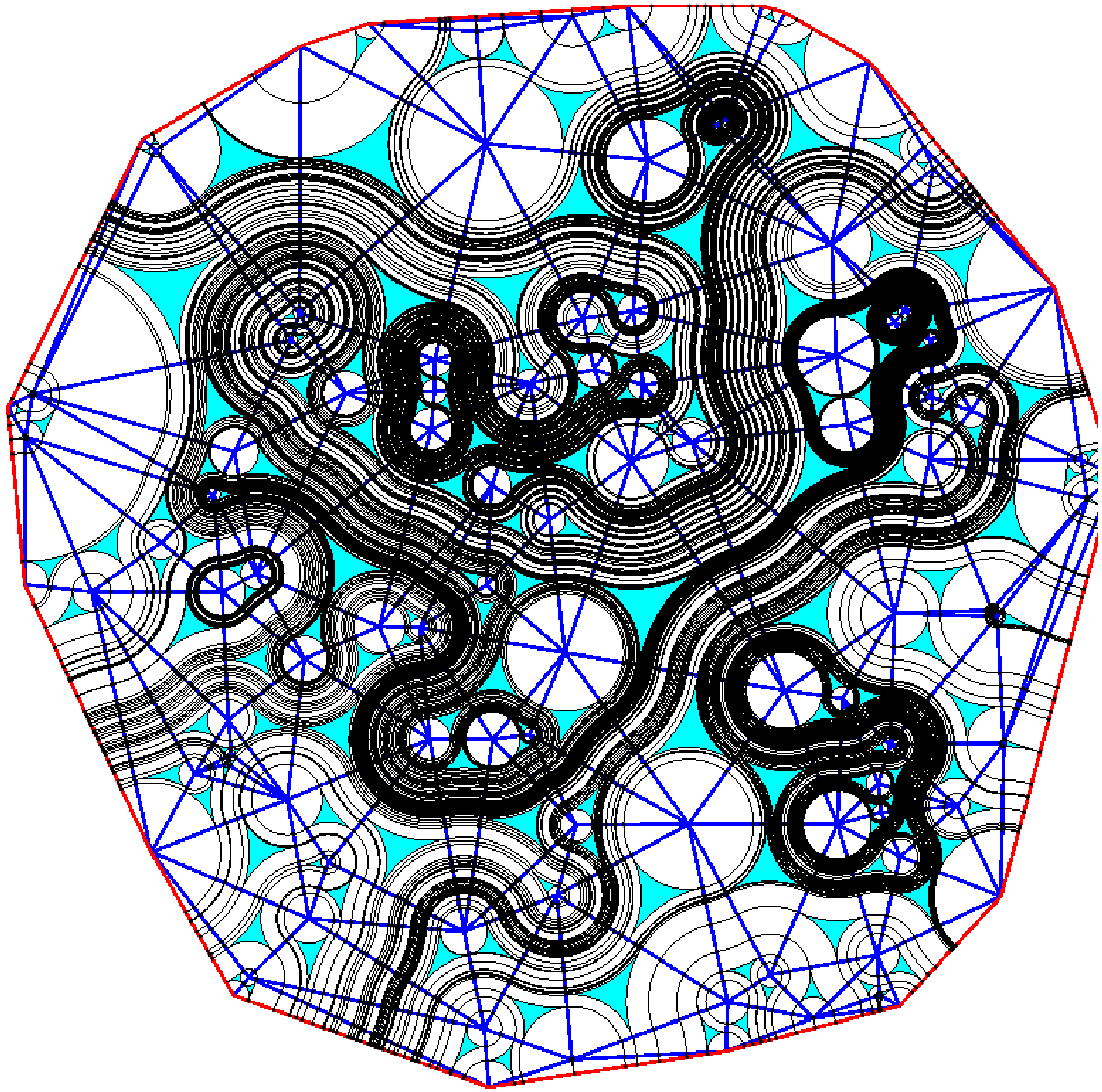
Propagation lines starting at all cusp points.

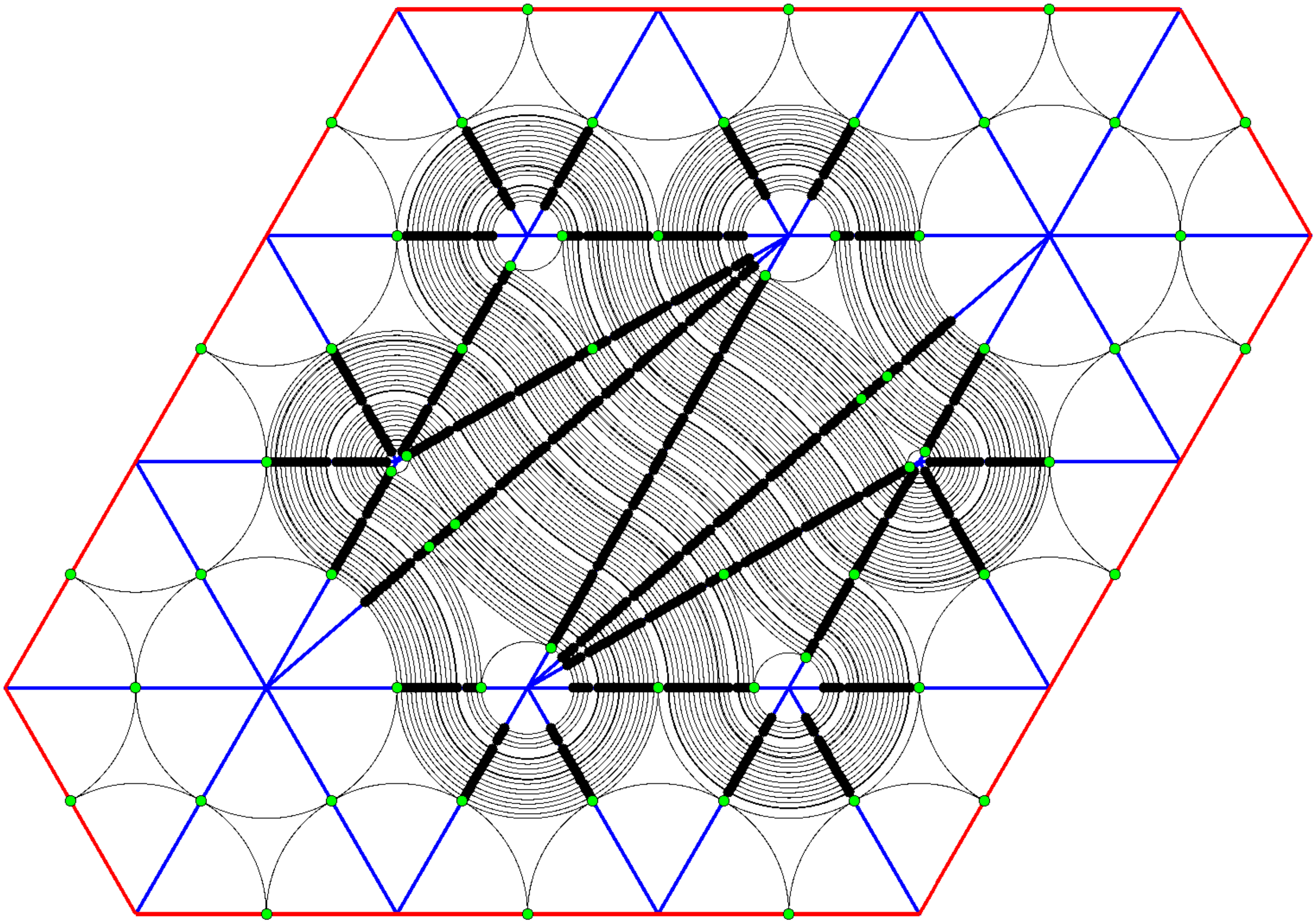


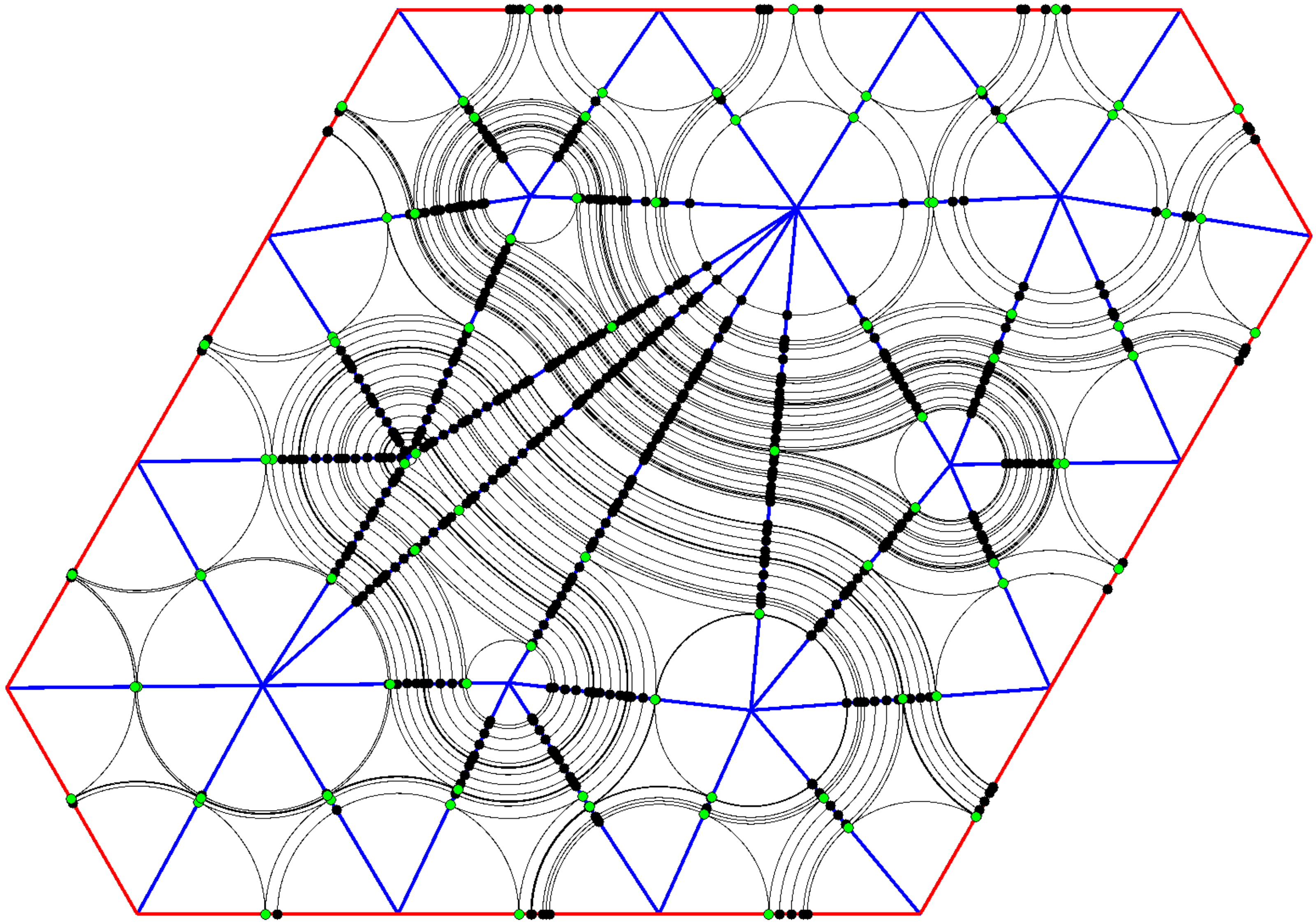
How many for triangulations of random point sets?

Log-log plot of number points created versus n.

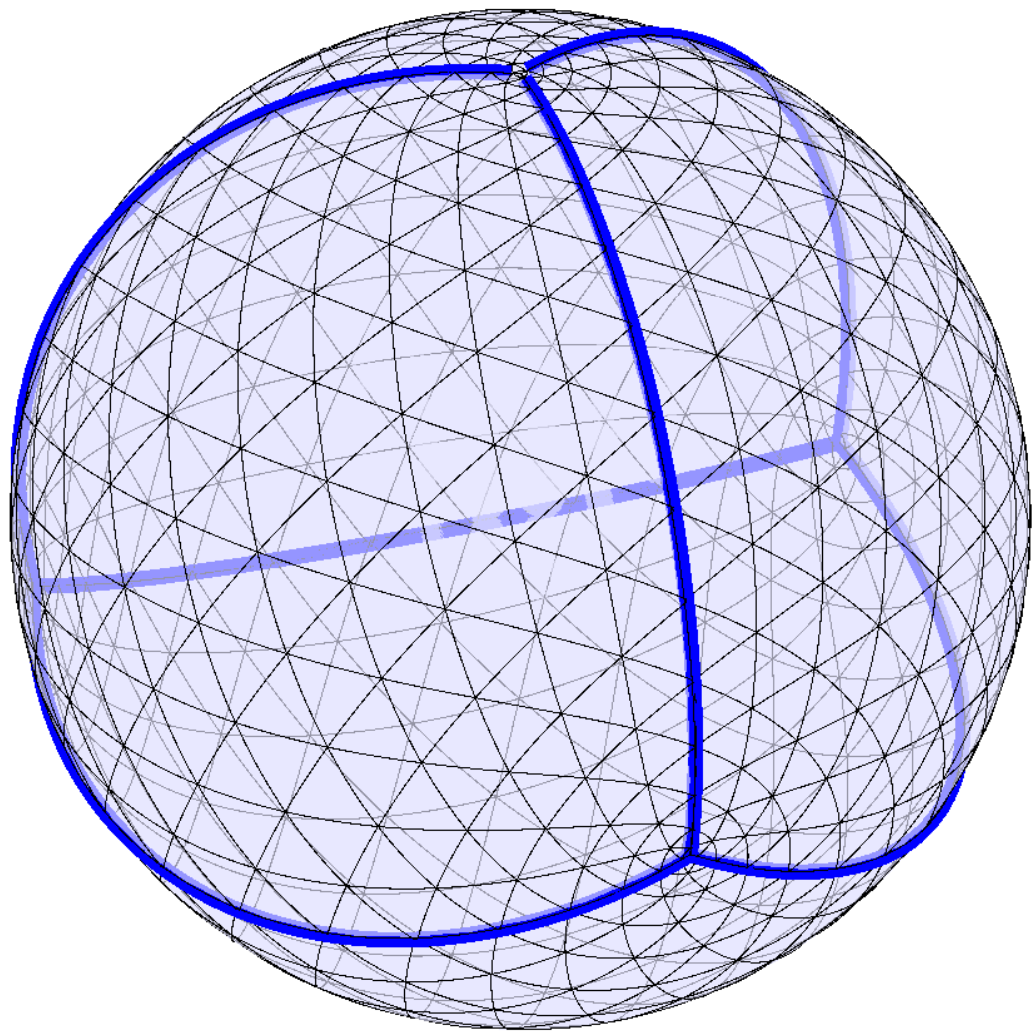
Slope ≈ 2.5







Equilateral triangulations of Riemann surfaces



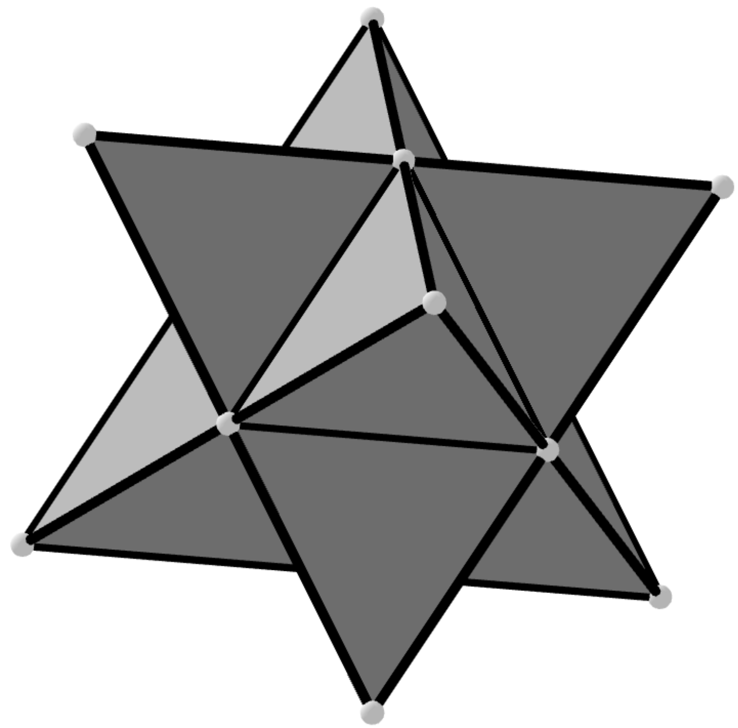
An equilateral triangulation of the 2-sphere

What do we mean by an equilateral triangulation of a Riemann surface?

There are several equivalent definitions.

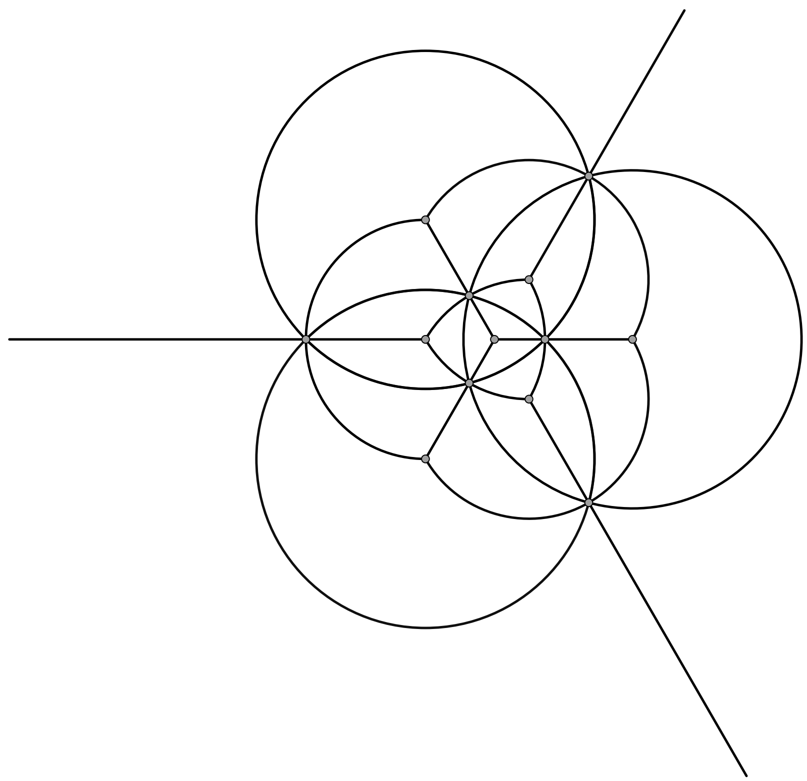
Defn 1: Build a compact surface by gluing a finite number of planar equilateral triangles along edges.

This gives a Riemann surface with a triangulation.



Defn 1: Build a compact surface by gluing a finite number of planar equilateral triangles along edges.

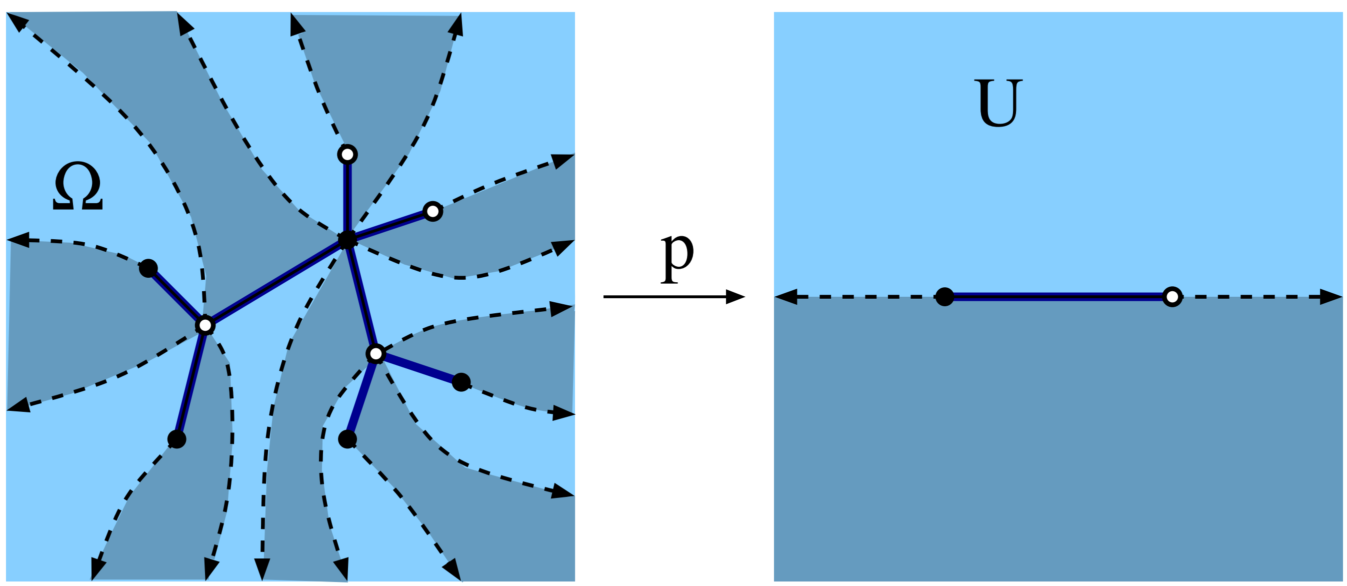
This gives a Riemann surface with a triangulation.



Defn 2:

Belyi function $=$ holomorphic map from Riemann surface with 3 critical values (usually $0, 1, \infty$).

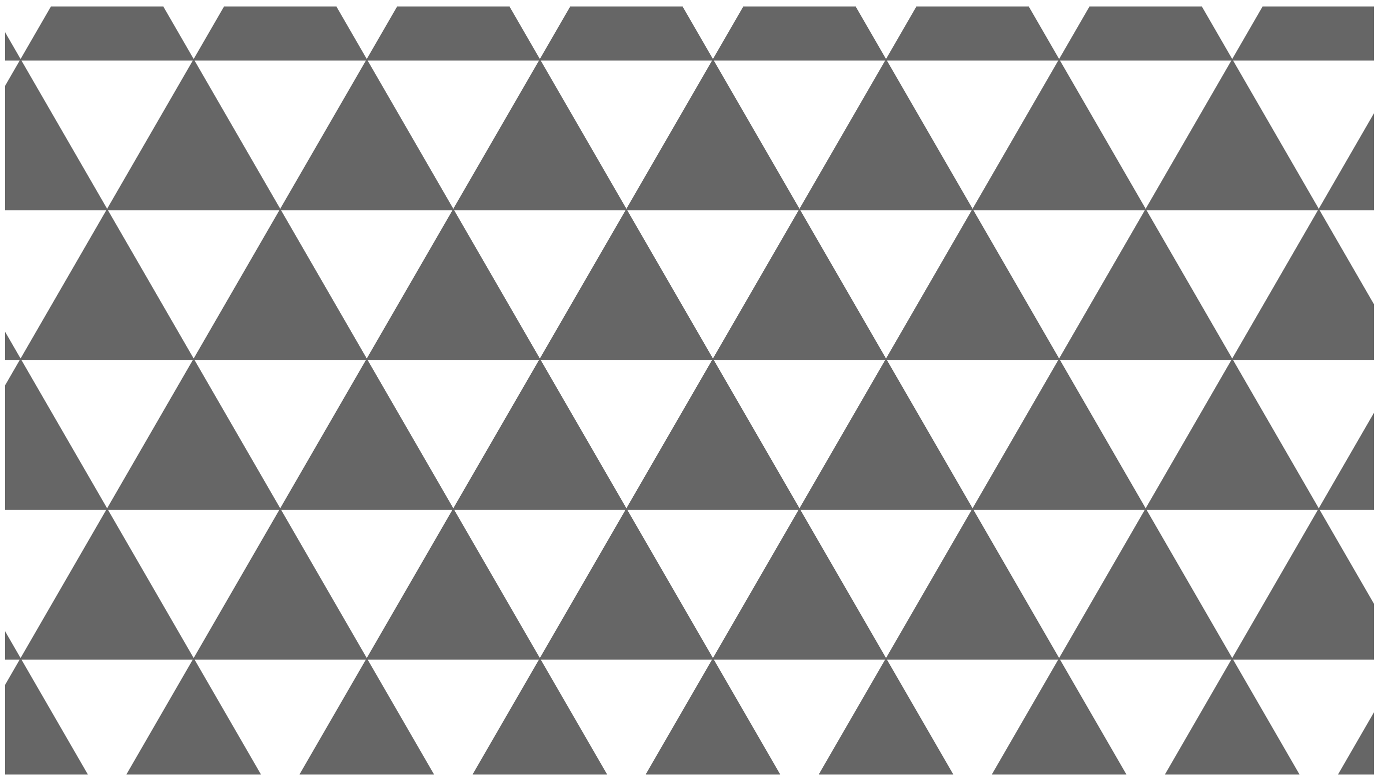
Theorem (Voevodsky-Shabat): A Riemann surface can constructed from equilateral triangles iff it has a Belyi function.



Triangles are inverse images of upper and lower half-planes.

Defn 3:

In a equilateral triangulation of plane, adjacent triangles are reflections.



Any single triangle generates whole triangulation via repeated reflections.

Defn 3:

In a equilateral triangulation of plane, adjacent triangles are reflections.

A triangulation of a Riemann surface is equilateral if any two triangles sharing an edge are exchanged by an anti-holomorphic map fixing that edge pointwise.

There are only finitely many distinct ways to glue n triangles together.

So only countably many compact surfaces have equilateral triangulations.

Belyi's theorem characterizes which ones do.

Theorem: The following are equivalent:

(1) R has an equilateral triangulation.

(2) R has a Belyi function (holomorphic, three critical values).

(3) R is algebraic.

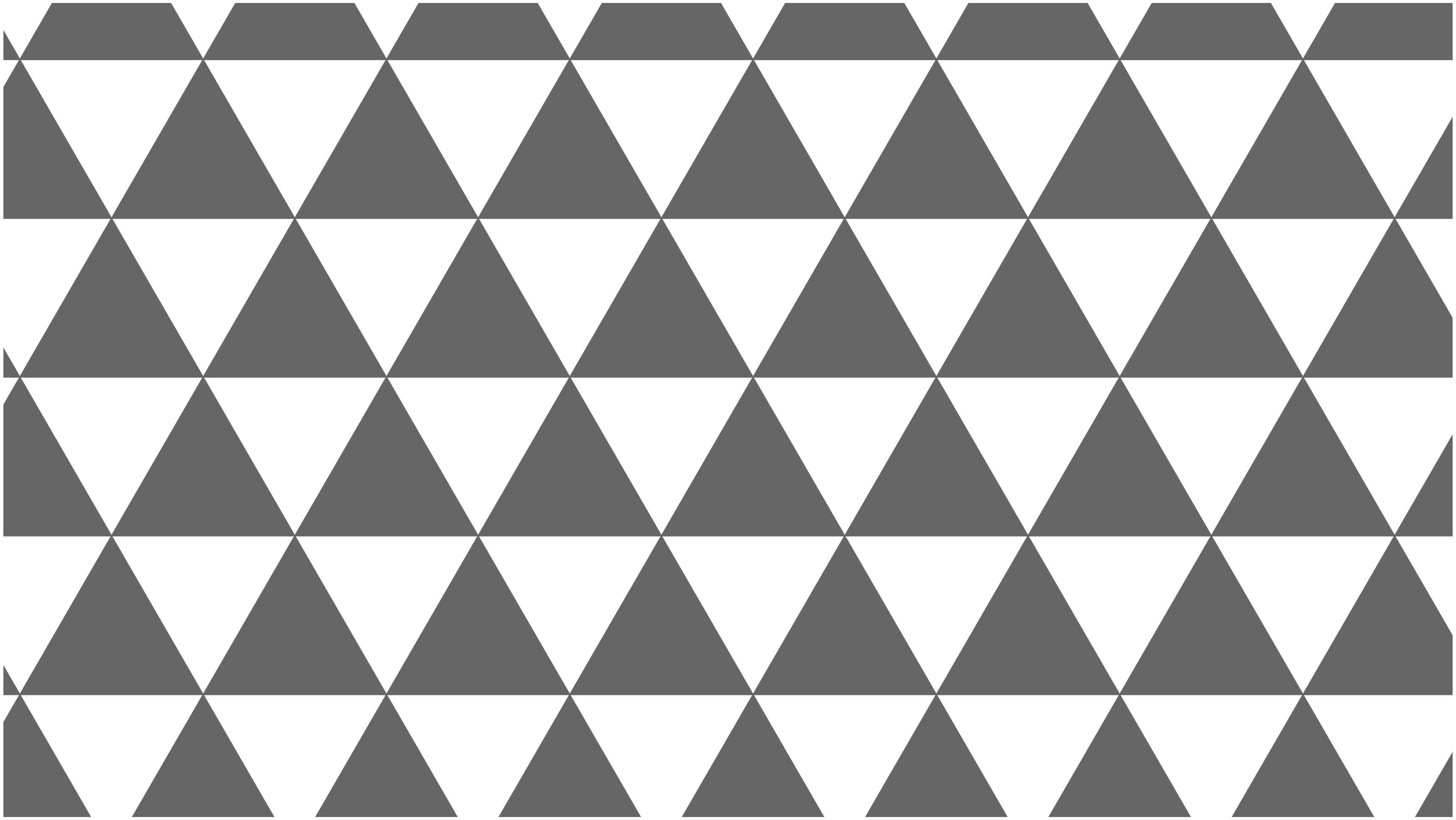
Every compact Riemann surface is zero set of a polynomial in several variables. Is algebraic if coefficients of P are in $\overline{\mathbb{Q}}$, algebraic closure of $\mathbb{Q}$.

Foundation of Grothendieck's theory of *dessins d'enfants*, connecting Galois theory, Riemann surfaces and combinatorics.

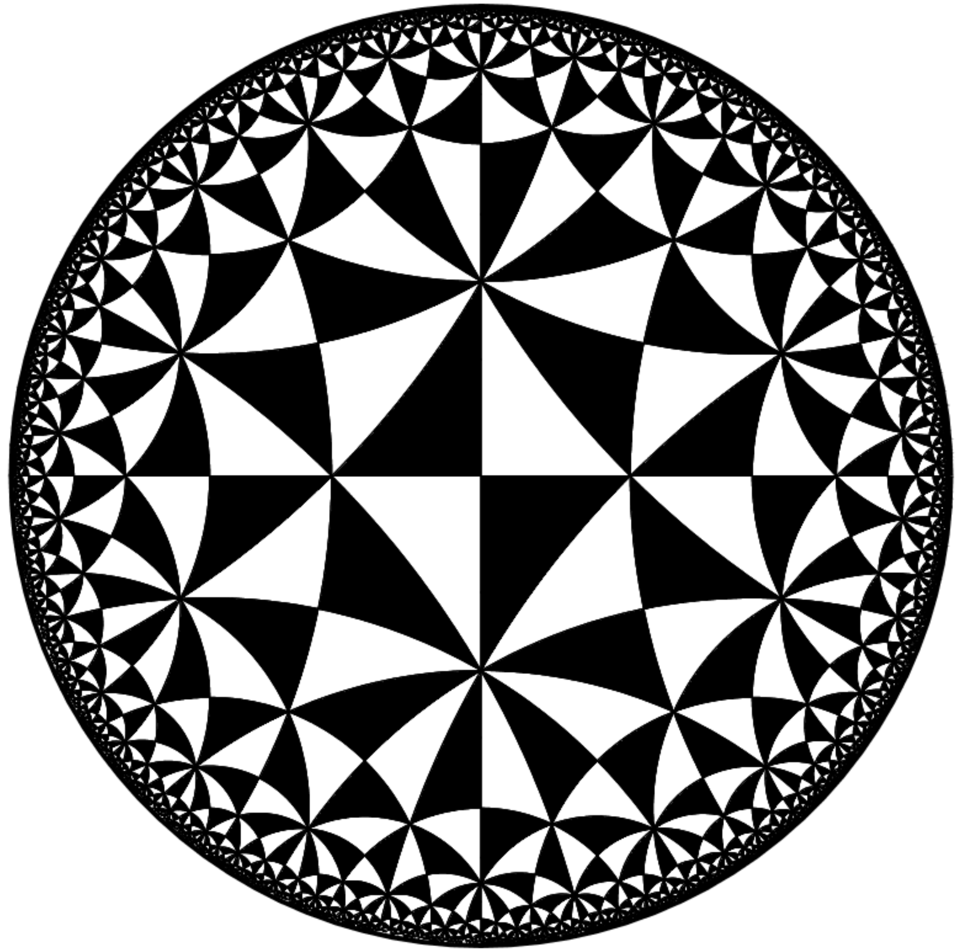
What about non-compact Riemann surfaces?

Now we can use countably many triangles.

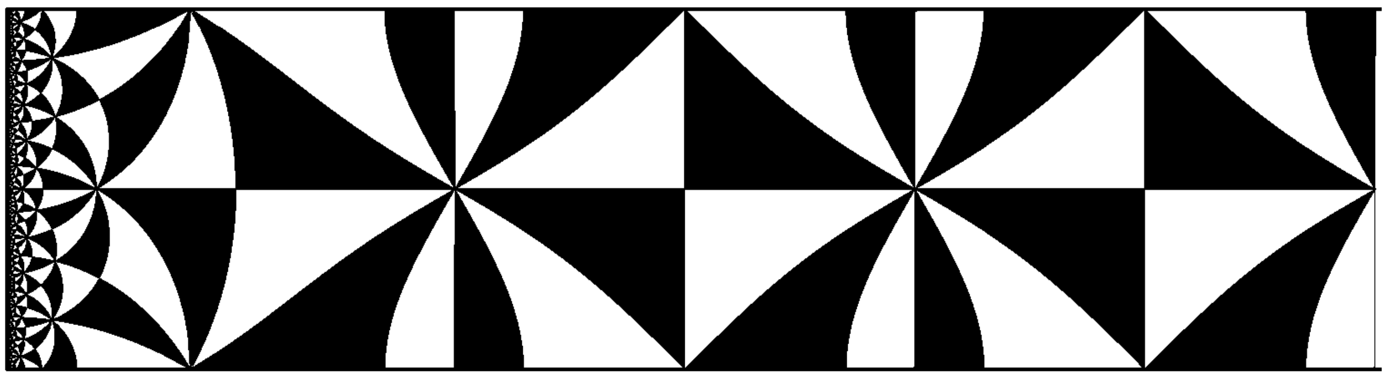
$\Rightarrow$ uncountably many way to glue them together.



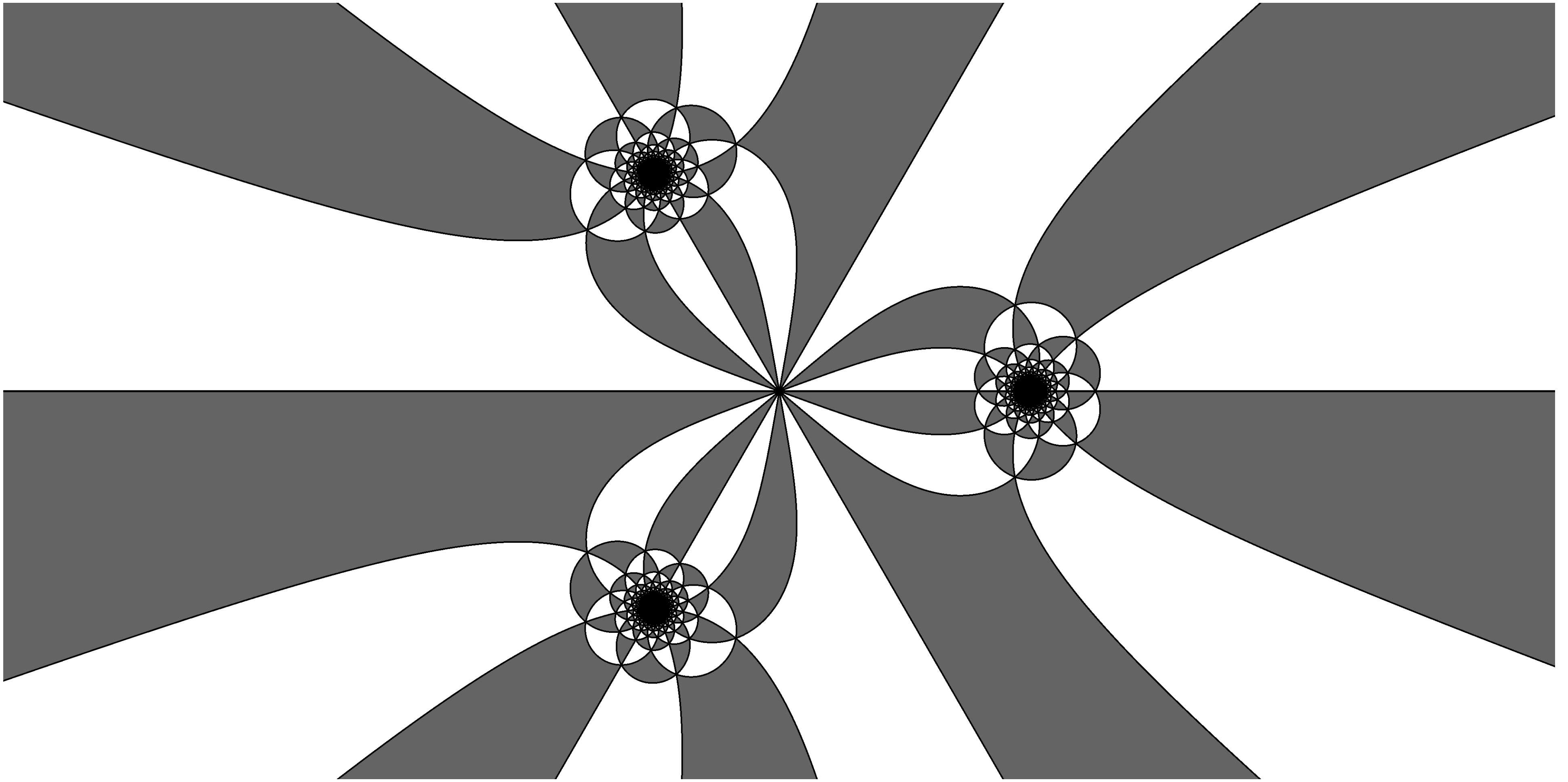
The plane



The disk



The punctured disk (identify top and bottom sides)



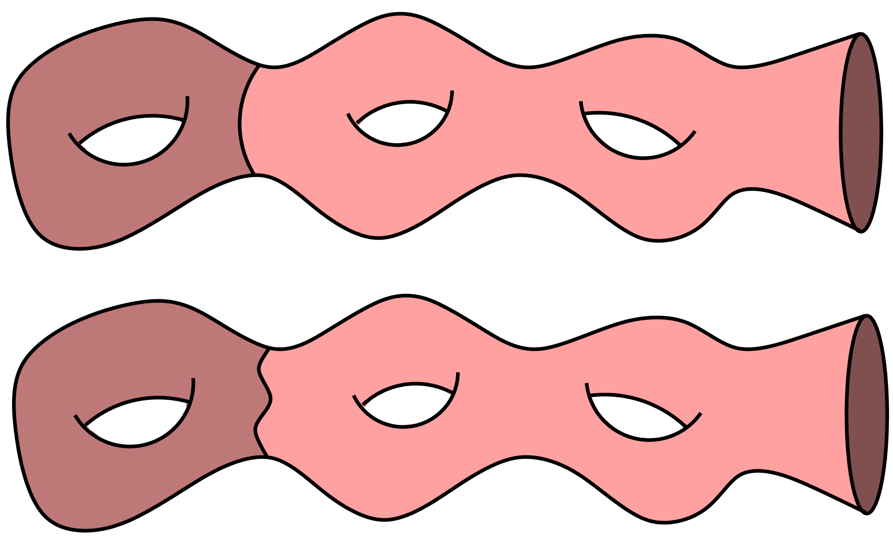
The trice punctured sphere

Thm (B-Rempe): Every non-compact surface has a Belyi function.

Thm (B-Rempe): Every non-compact surface has a Belyi function.

Idea of proof:

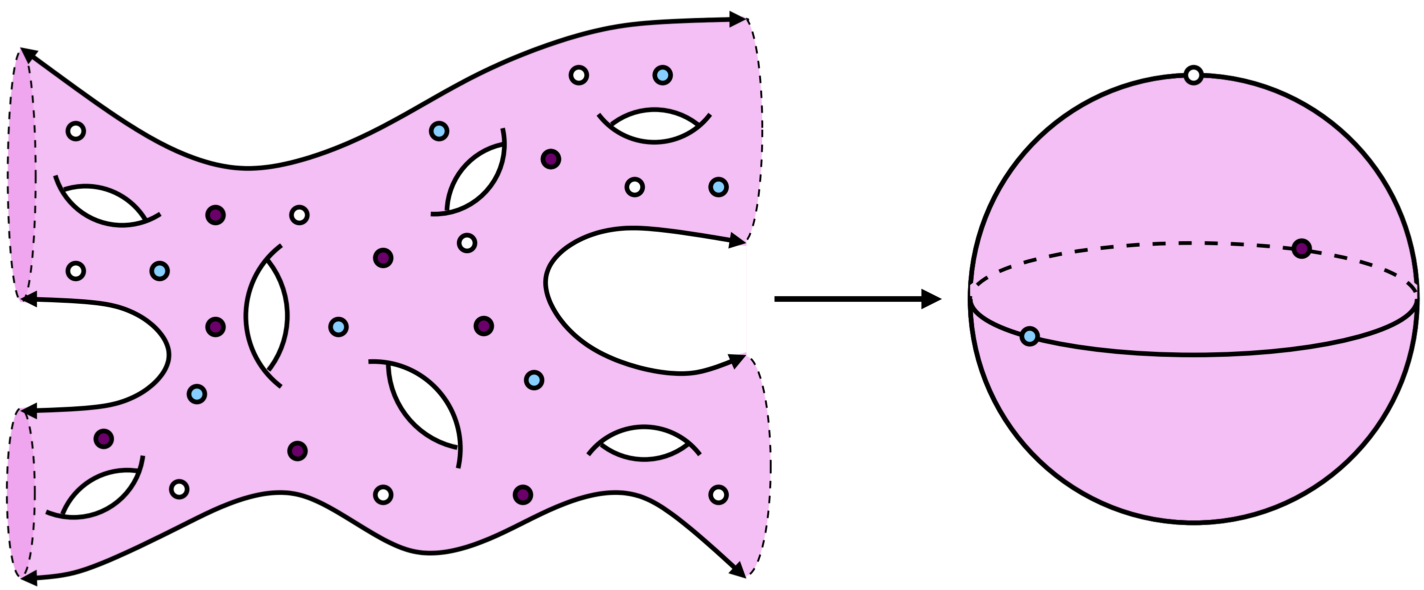
- Write R as union of compact, bordered pieces.
- Compact pieces can be approximated by triangulated surfaces.
- Conformal structure is changed, but as little as we wish.
- **Key fact:** small perturbation $\Rightarrow$ triangulated pieces re-embed in S.
- Take limit.



Thm (B-Rempe): Every non-compact surface has a Belyi function.

Corollary: Every Riemann surface is a branched cover of the sphere, branched over finitely many points.

- For compact surfaces, this is Riemann-Roch.
- Compact, genus g sometimes needs $3g$ branch points.
- 3 branch points suffice for all non-compact surfaces.



Questions:

Teichmüller space parameterizes different conformal structures on a compact topological surface. Belyi surfaces are dense in this space.

How "evenly distributed" are they? Question raised by Mirzakhani.

2022 MIT thesis of Sahana Vasudevan estimates number of such surfaces in large balls in Teichmüller space.

What are properties of "random" triangulated surface? Much work in statistical physics, "Liouville quantum gravity".

Given triangulation of compact surface is is minimal? When do two triangulations give same surface?

More questions:

Every non-compact surface has many equilateral triangulations.

Is there an "optimal" one? Using largest triangles?

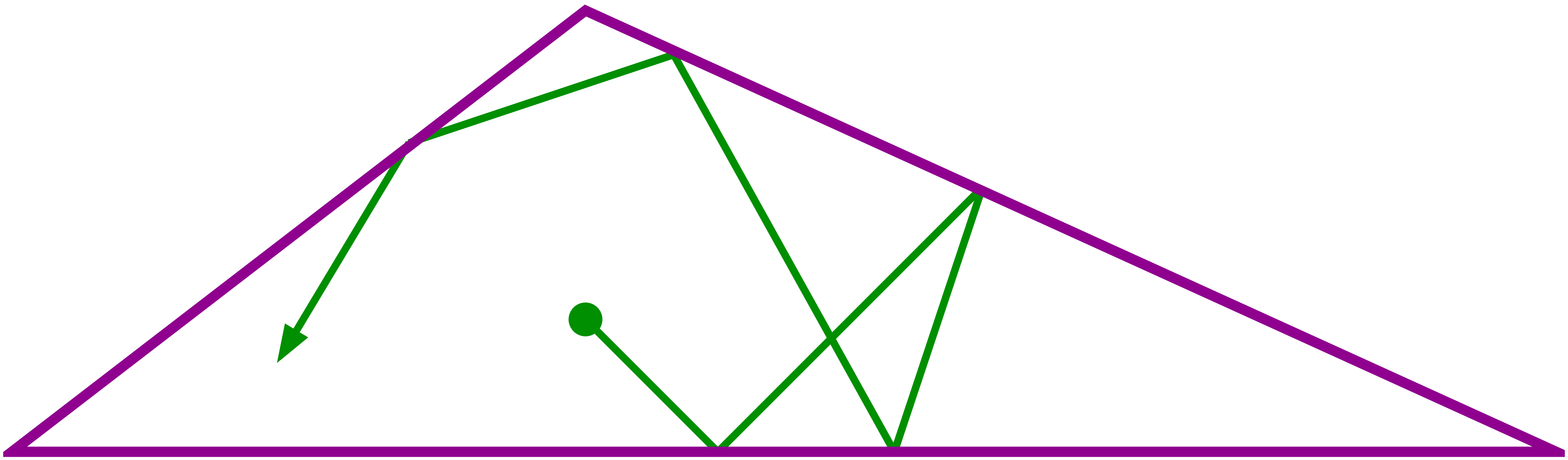
Injectivity radius at $z \in R$ is length of shortest non-trivial loop at z.

If injectivity radius of R is bounded away from zero, is there an equilateral triangulation with all diameters bounded away from zero?

How does one choose a "random" equilateral triangulation of an non-compact surface? Some work has been done on the plane.

Triangular billiards

Given initial point and direction form billiards path by reflecting off sides.



Known since 1775 that every acute triangle has a periodic orbit.

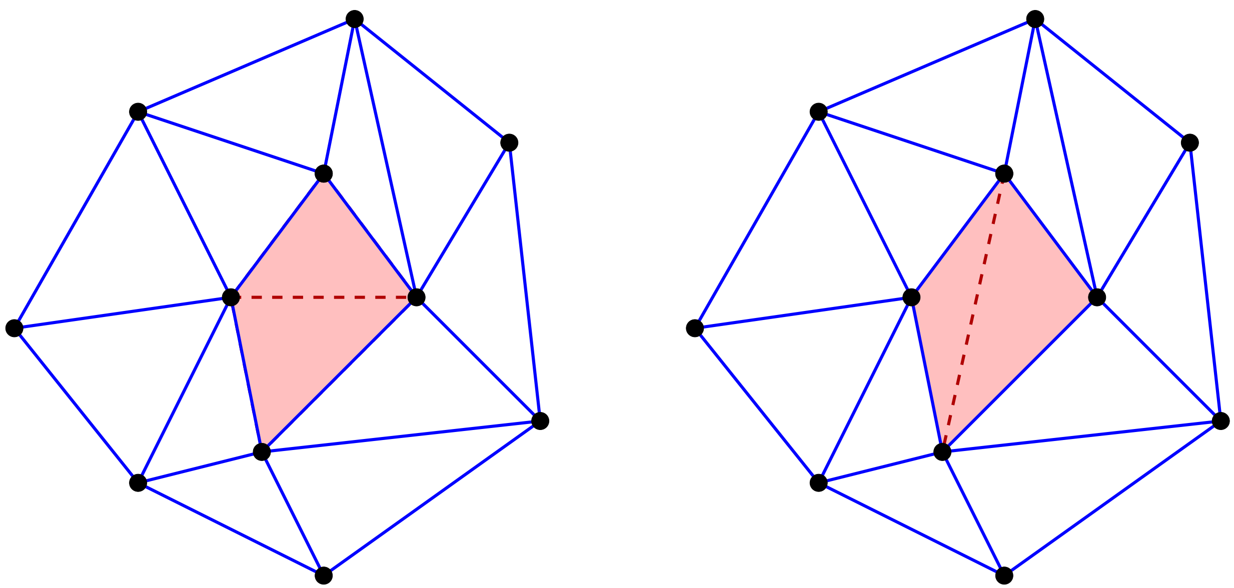
Rational triangles (angles in $\pi\mathbb{Q}$) have dense set of periodic orbits.

Unknown whether all obtuse triangles have a periodic orbit.

For any N, some obtuse triangles have minimal period $> N$.

Flip graph of a triangulation

Given a triangulation of a point set, if two adjacent triangles form a convex quadrilateral, we can flip the common edge to make a new triangulation.

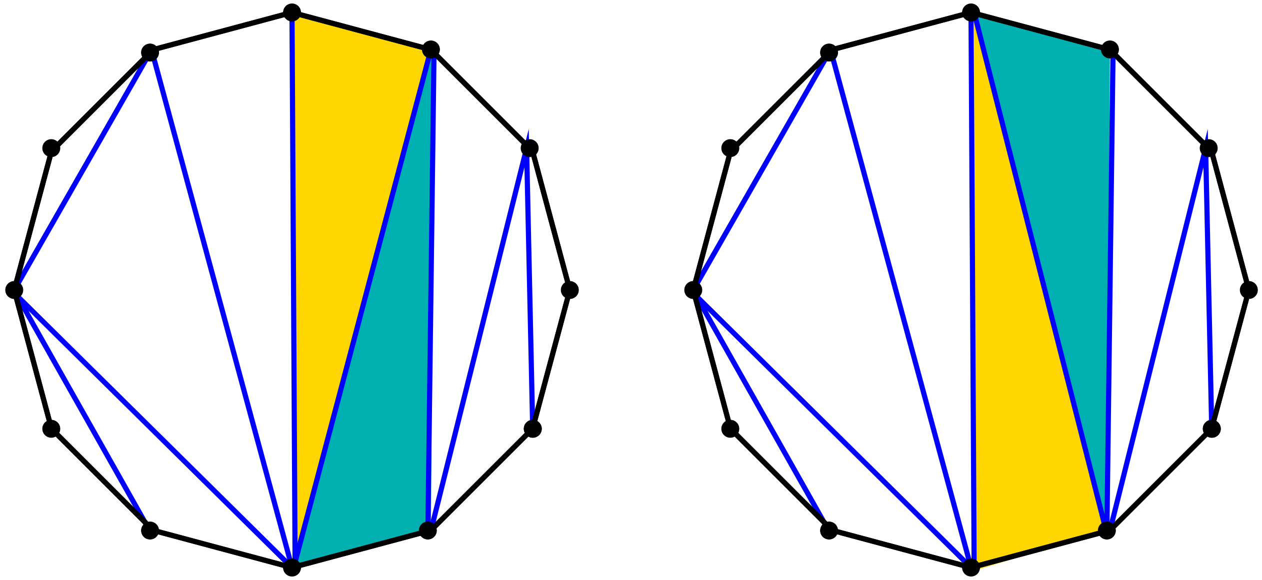


Triangulations form vertices of a graph; flipping gives edges.

Graph is connected. Any two triangulations can be joined by flips.

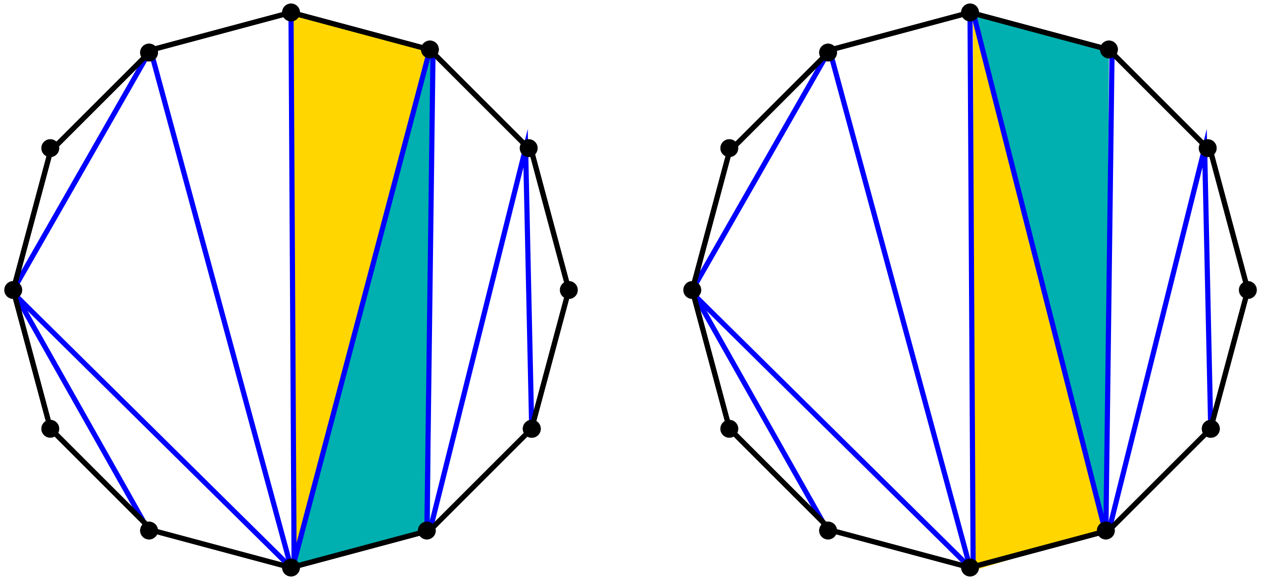
What is maximum possible diameter? Good bounds, but not proven sharp.

Mixing time: how many flips to create a "random" triangulation?



Triangulations of convex n-gons have many special properties.

- Triangles for a tree.
- Number of triangaulations is a Catalan number.
- Product structure: diagonal divides two smaller polygons.



Mixing time for convex polygons better understood than general case:

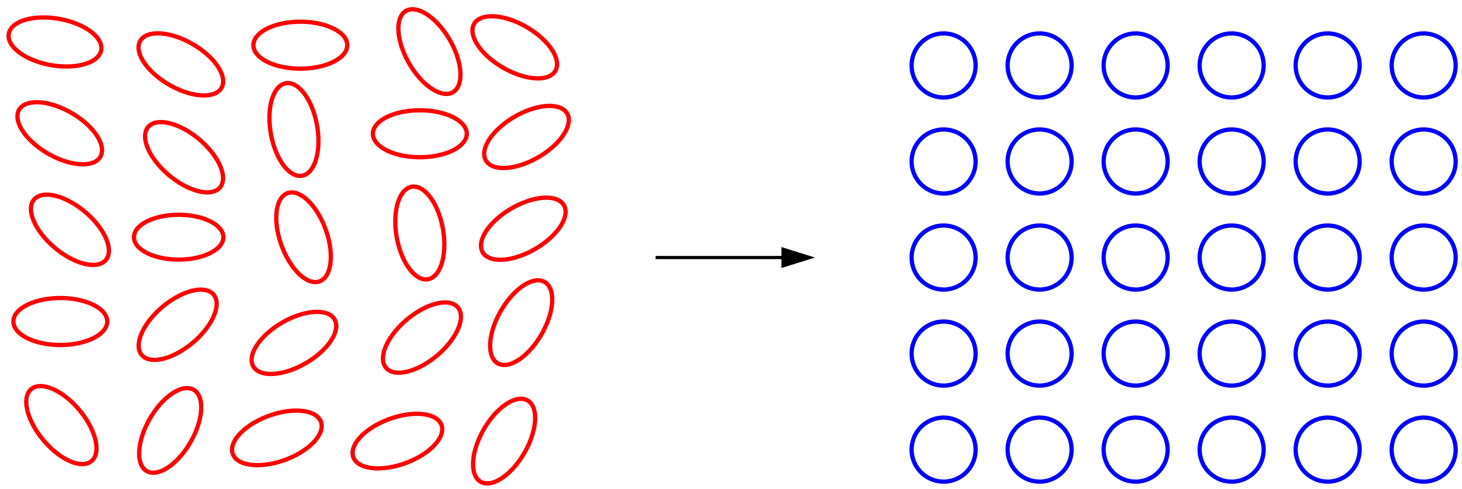
- $O(n^{3/2})$ is necessary. 1997 Molloy, Reed and Steiger
- $O(n^3 \log^3 n)$ suffices. 2023 Eppstein and Frishberg
- Proof depends on estimate of Cheegar constant of flip graph.
- Sharp estimate still open.


QUESTIONS?


ANSWERS?

Quasiconformal maps and Weil-Petersson curves

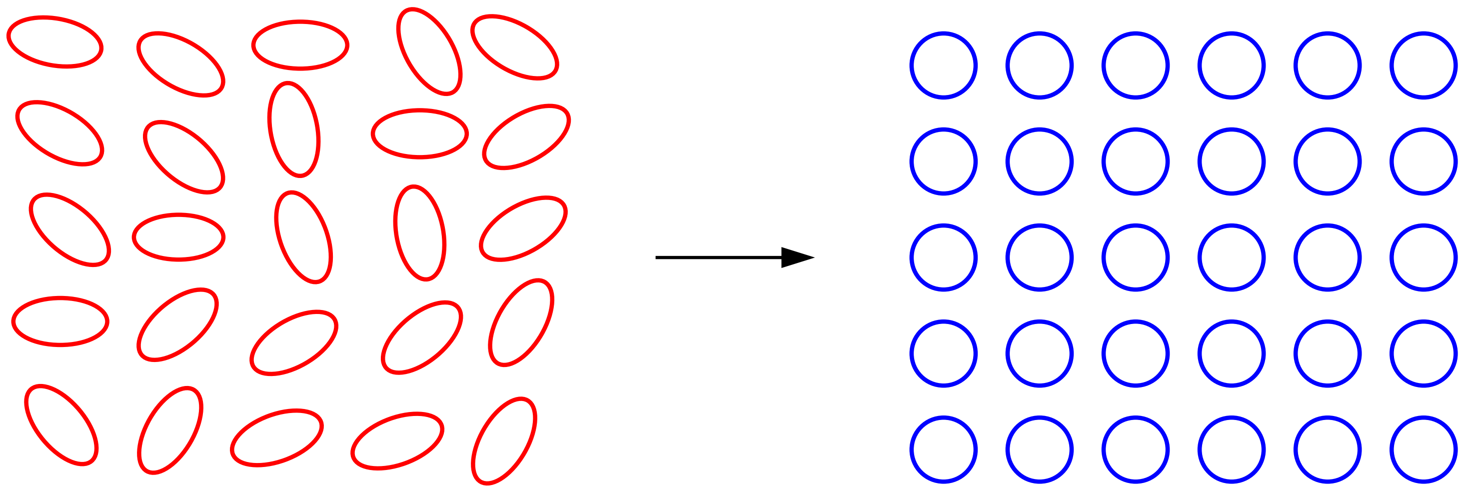
Diffeomorphisms send infinitesimal ellipses to circles.



Eccentricity = ratio of major to minor axis of ellipse.

K-quasiconformal = ellipses have eccentricity $\leq K$ almost everywhere

Diffeomorphisms send infinitesimal ellipses to circles.



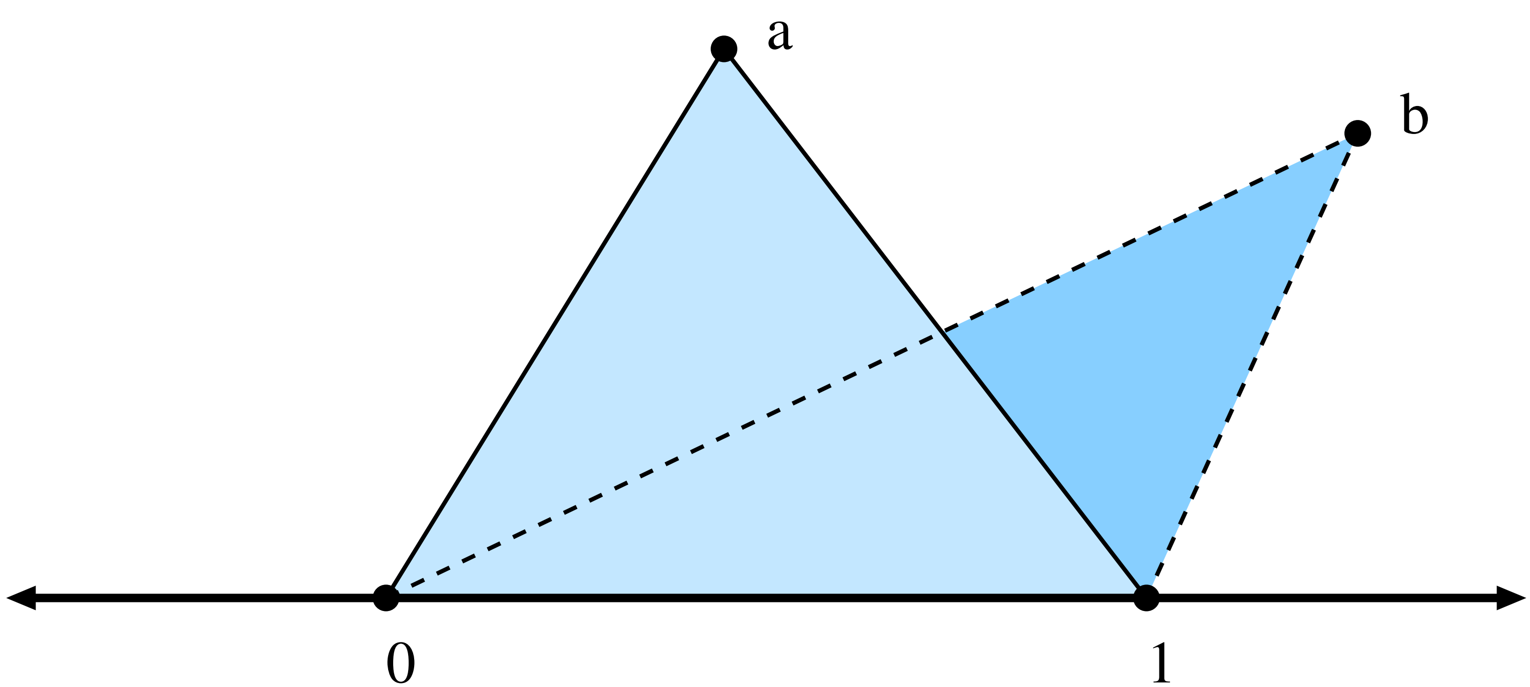
Eccentricity = ratio of major to minor axis of ellipse.

K-quasiconformal = ellipses have eccentricity $\leq K$ almost everywhere

Ellipses determined by dilatation $\mu = f_{\overline{z}}/f_z$ with $f_{\overline{z}}, f_z = \frac{1}{2}(f_x \pm i f_y)$.

$$|\mu| = \frac{K-1}{K+1} < 1, \qquad \arg(\mu) \text{ gives major axis.}$$

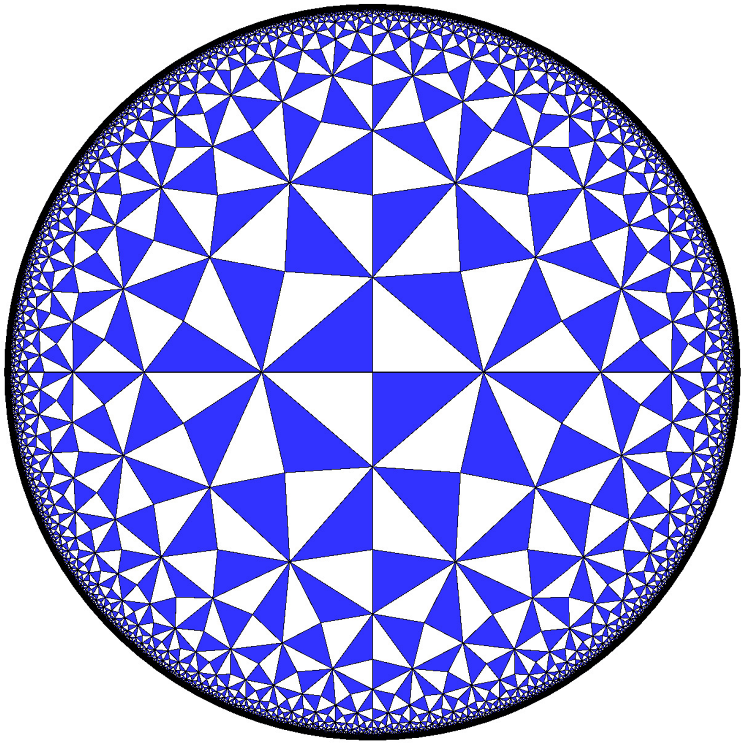
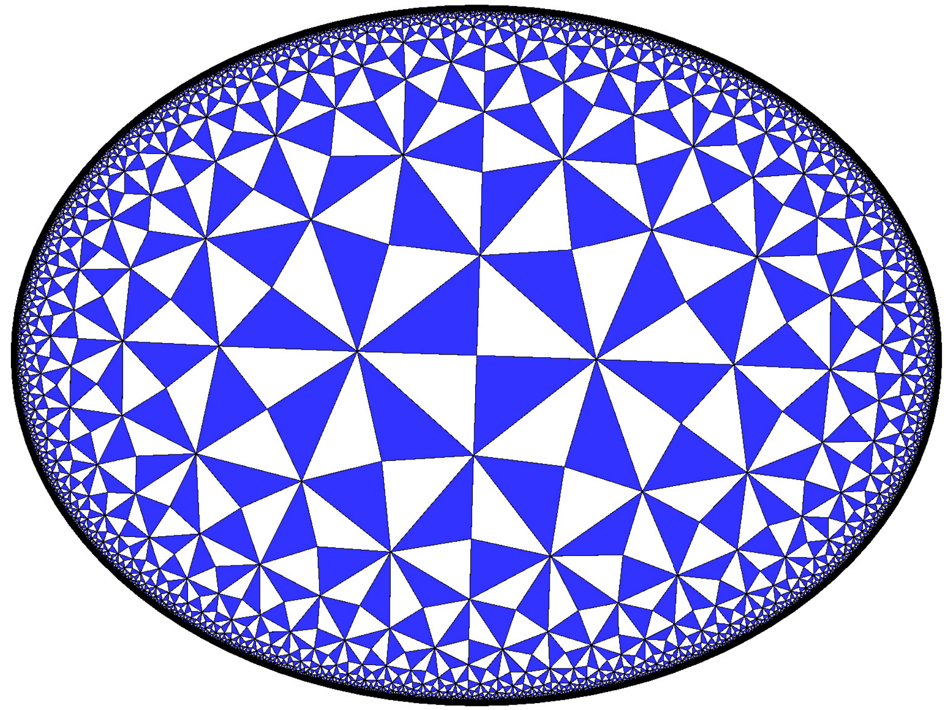
f is QC $\Leftrightarrow \|\mu\|_\infty < 1$. f is conformal = 1-1 holomorphic $\Leftrightarrow \mu \equiv 0$.



μ easy to compute for affine maps $f(z) = \alpha z + \beta \overline{z}$, $\qquad \mu = \beta/\alpha$.

Affine map between triangles $\{0, 1, a\}$ and $\{0, 1, b\}$ has constant dilatation

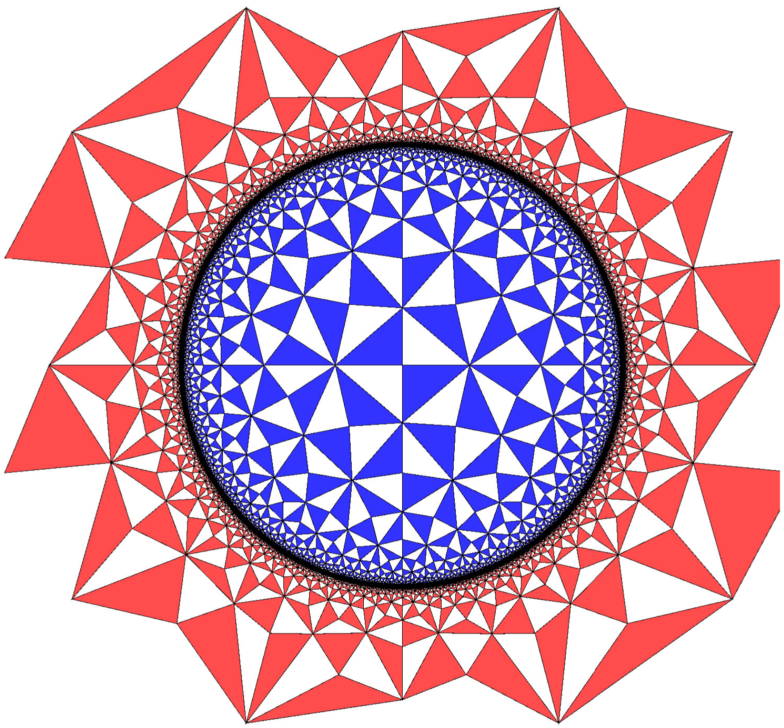
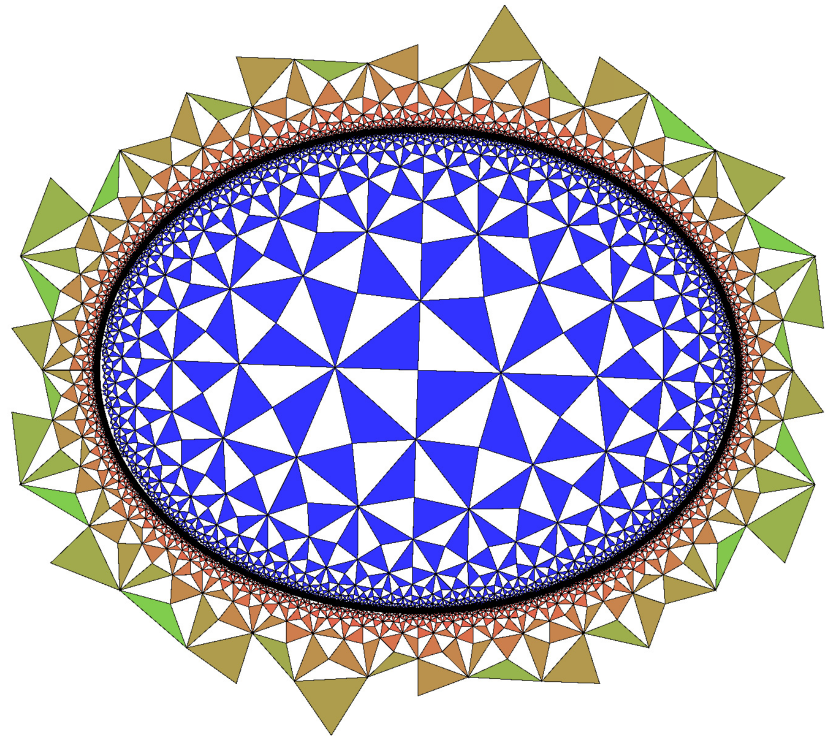
$$\mu = \frac{b - a}{b - \overline{a}}$$

Riemann Mapping Thm: any Jordan domain is conformal image of $\mathbb{D}$.

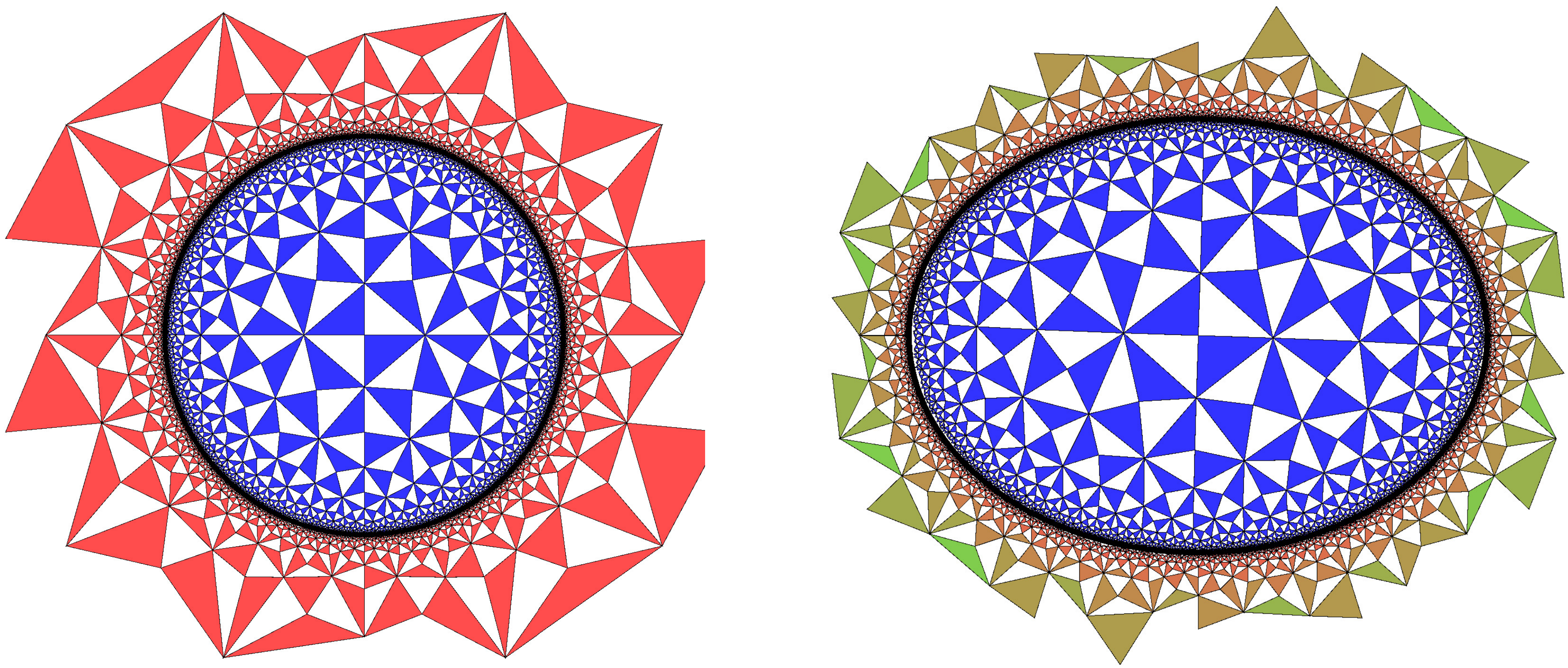
Liouville's Theorem $\Rightarrow$ any conformal map $\mathbb{C} \to \mathbb{C}$ is linear.

$\Rightarrow$ map above can't be extended to be conformal in whole plane.

Color distortion = angle distortion

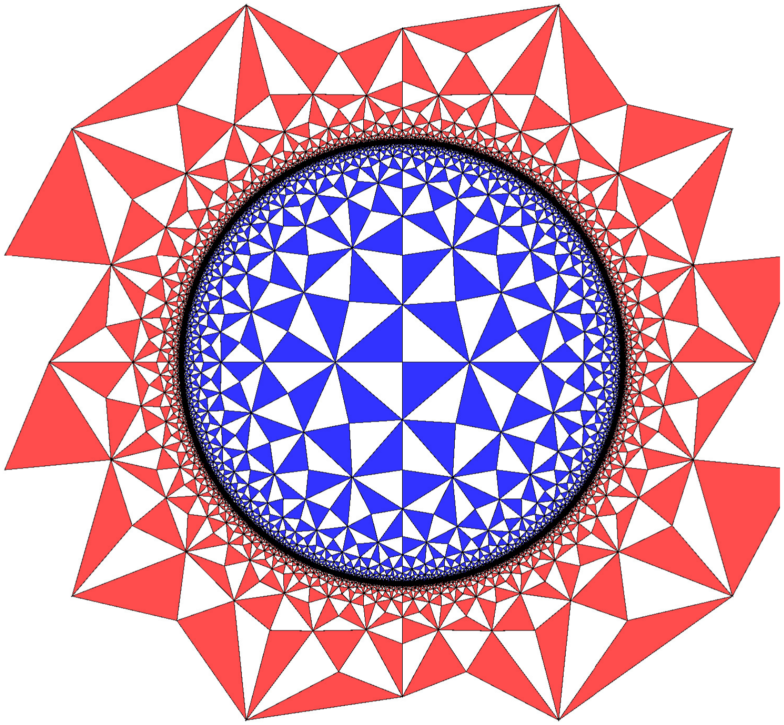
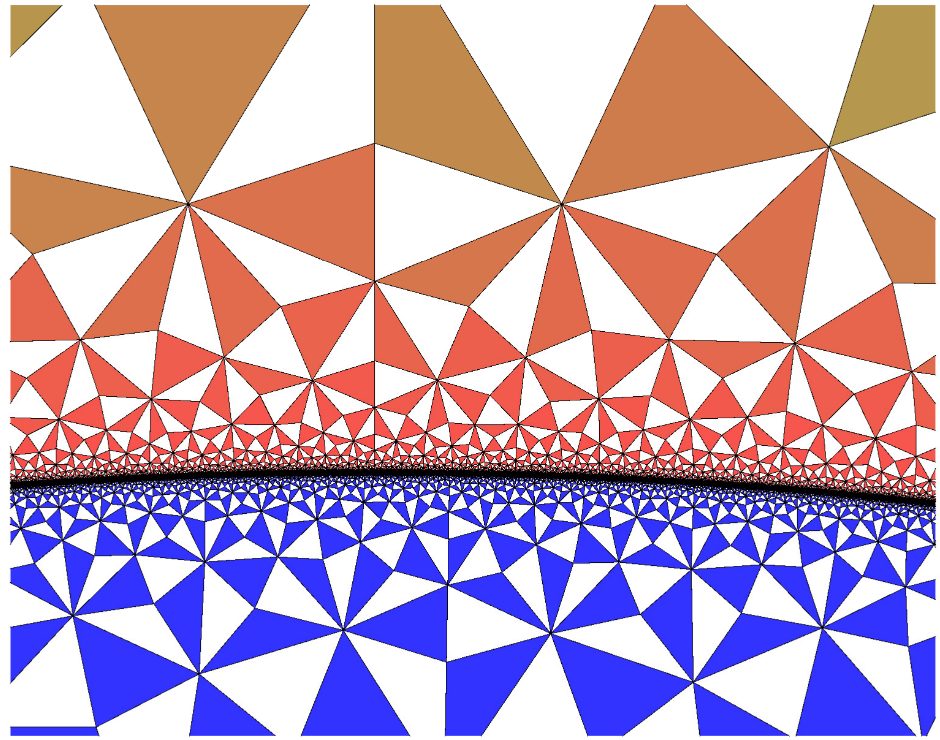
Measure distance of curve to circle using dilatations.



Color distortion = angle distortion

Teichmüller metric = maximum dilatation

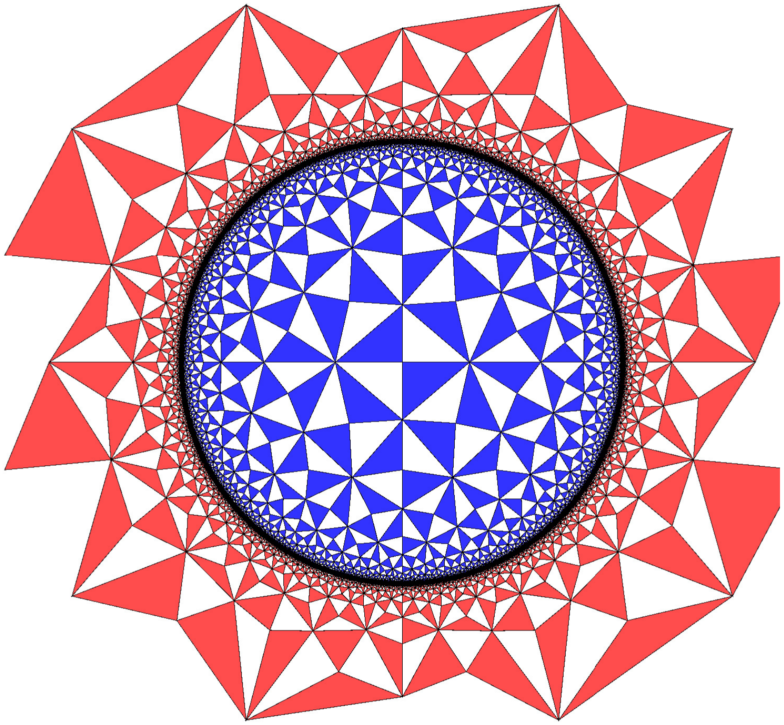
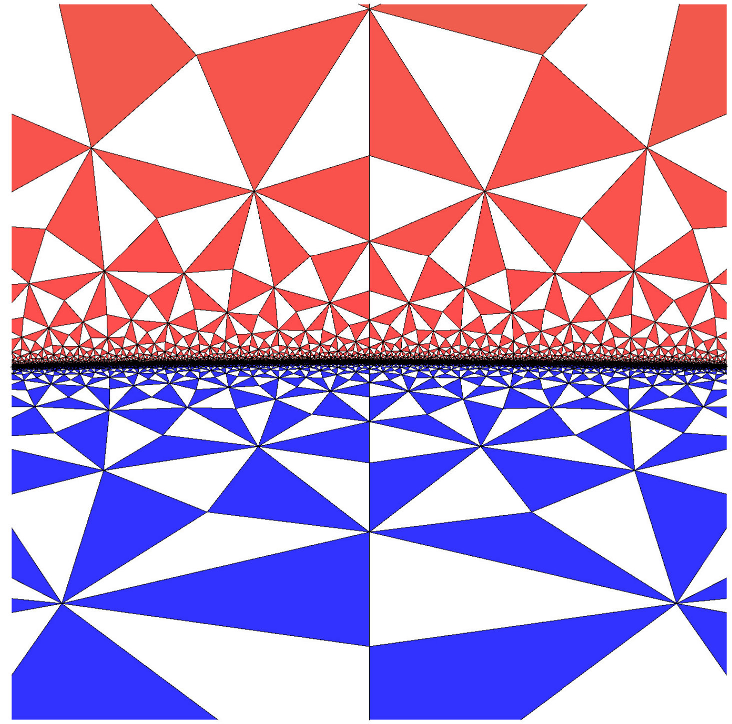
Weil-Petersson metric = $\sum(\text{dilatation})^2$

Distortion decreases near boundary (for smooth domains)

Teichmüller metric = maximum dilatation

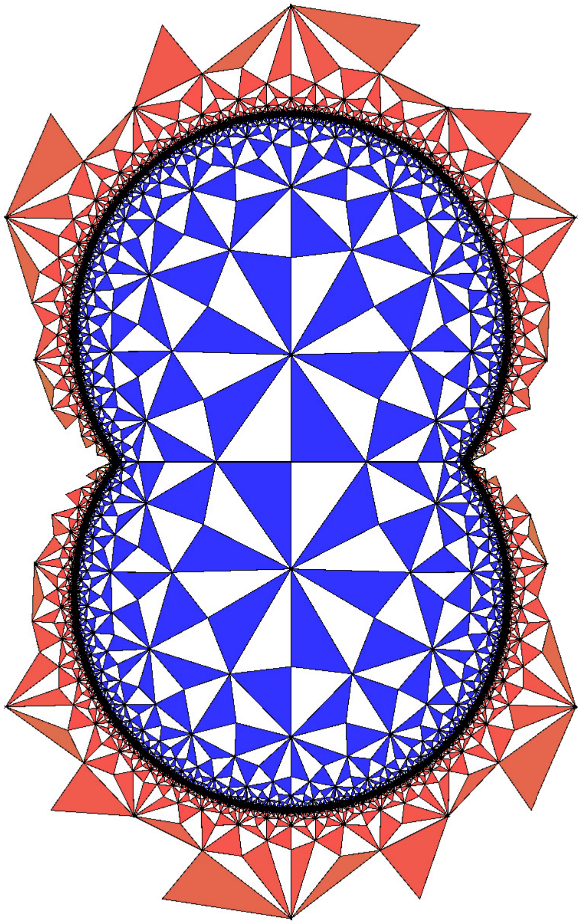
Weil-Petersson metric $= \sum(\text{dilatation})^2$

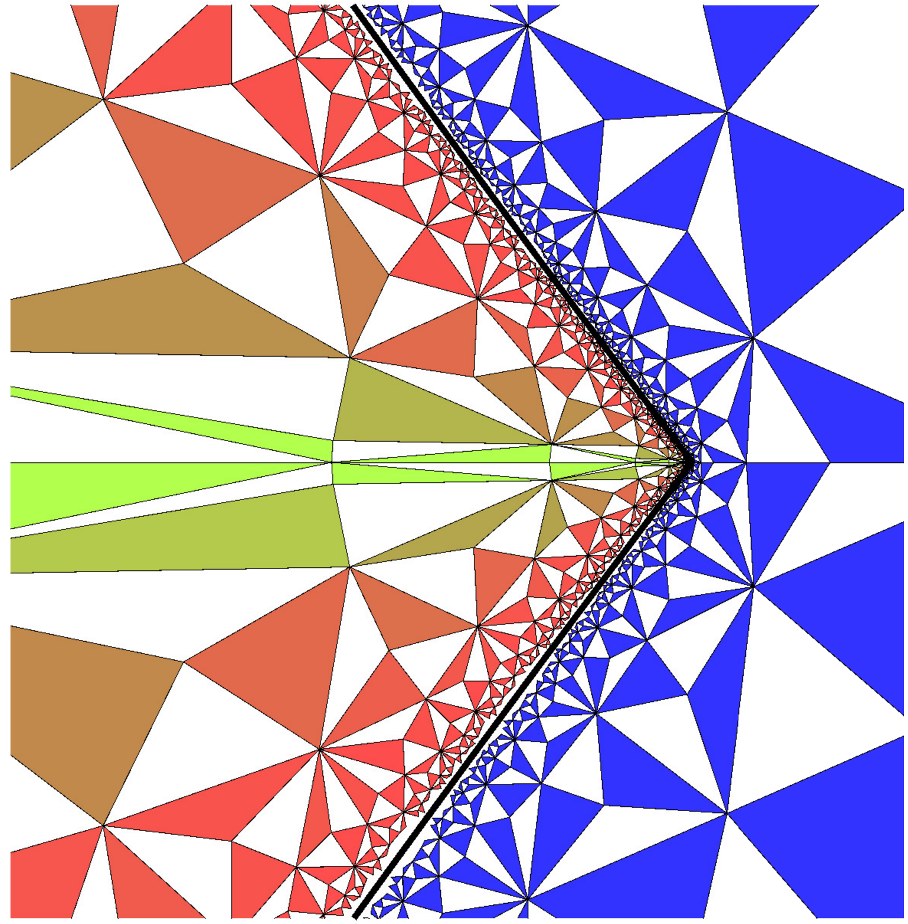
 

Quasicircles = maximum dilatation bounded

Weil-Petersson curves = $\sum(\text{dilatation})^2 < \infty$

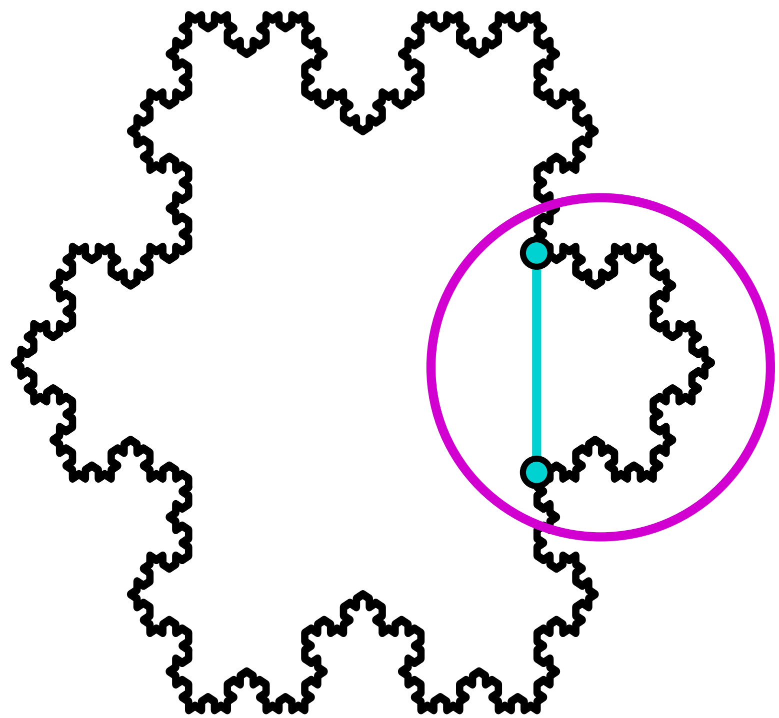
Quasicircles can be fractal, WP curves are finite length





Corners cause infinitely many triangles with large distortion

$$\text{WP sum} = \sum(\text{distortion})^2 = \infty$$



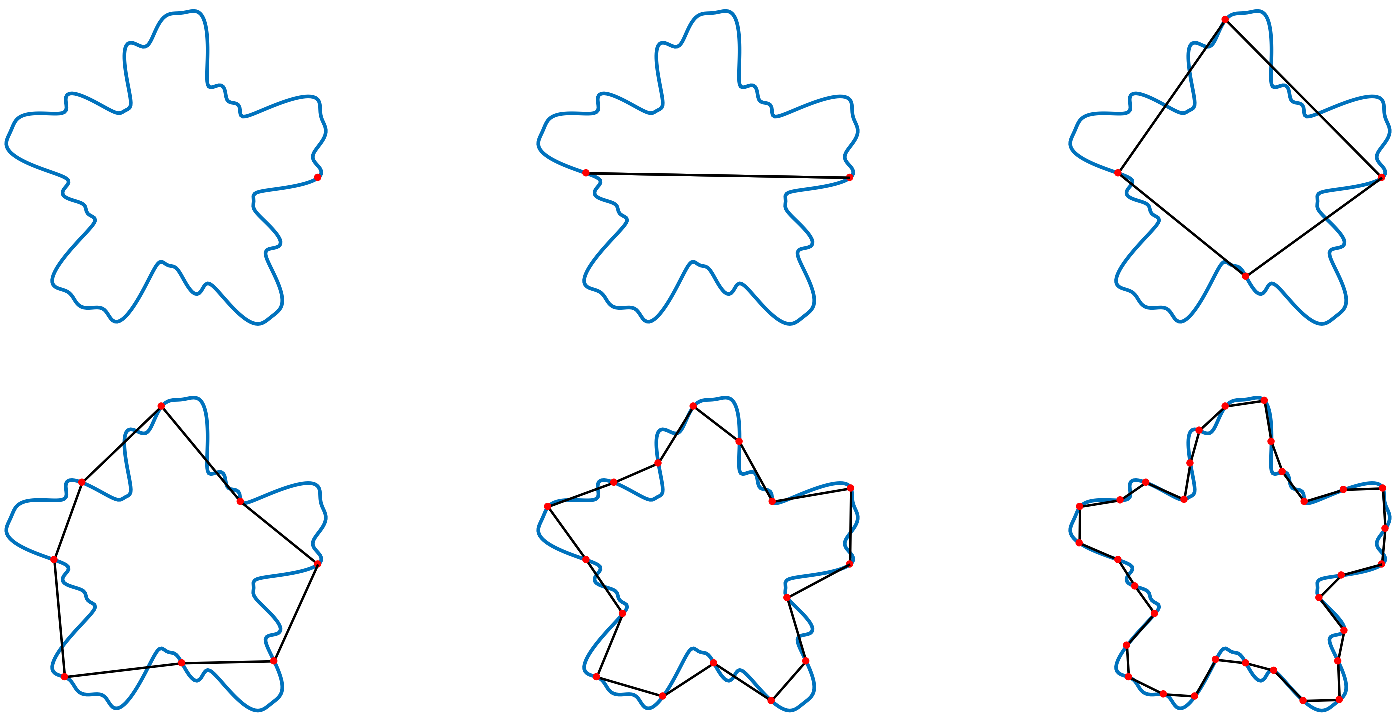
Γ is a quasicircle iff $\quad \mathrm{diam}(\gamma) = O(\mathrm{crd}(\gamma)) \quad$ for all $\gamma \subset \Gamma$.

$\mathrm{crd}(\gamma) = |z - w|$, z, w, endpoints of γ.

Geometric characterization of WP curves more recent (CB 2020).

Dyadic decomposition.

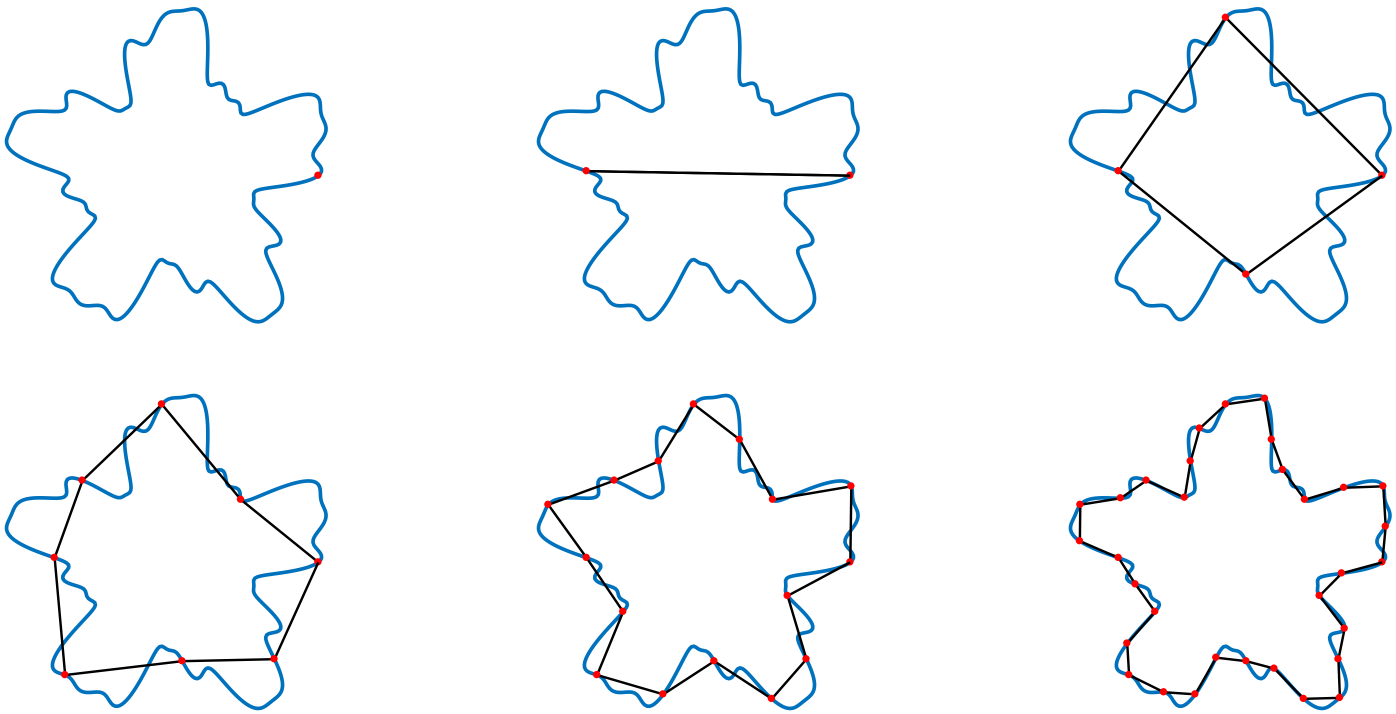
- Divide Γ into nested families of 2^n equal length arcs.
- Inscribe a polygon Γ_n at these points.
- Clearly $\ell(\Gamma_n) \nearrow \ell(\Gamma)$.



Theorem: Γ is Weil-Petersson if and only if

$$\sum_{n=1}^{\infty} 2^n \left[\ell(\Gamma) - \ell(\Gamma_n)\right] < \infty$$

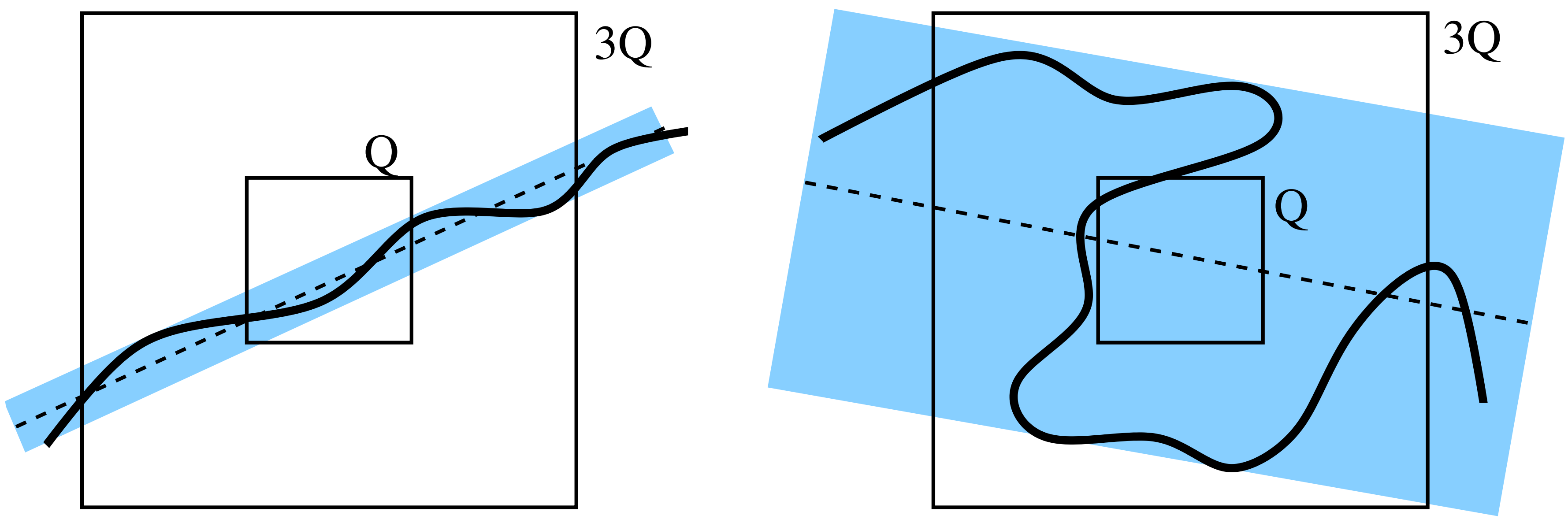
with a bound that is independent of the dyadic family.



Peter Jones's β-numbers:

$$\beta_\Gamma(Q) = \inf_L \sup\{\frac{\mathrm{dist}(z, L)}{\mathrm{diam}(Q)} : z \in 3Q \cap \Gamma\},$$

where the infimum is over all lines L that hit $3Q$.

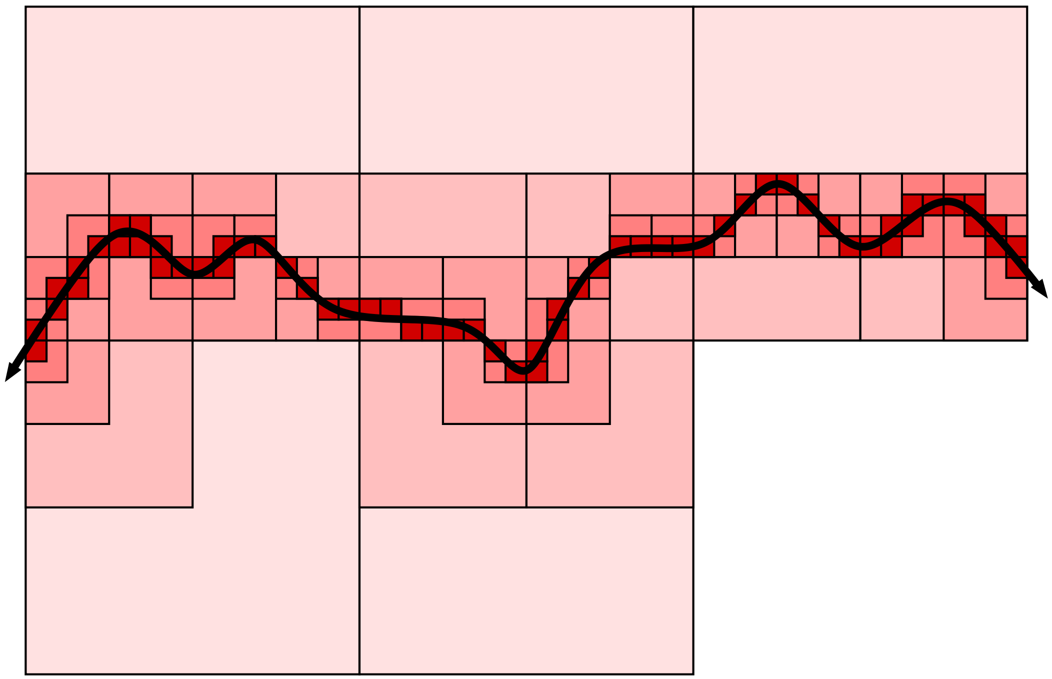


$\beta \approx 0$ $\beta \approx 1$

Jones invented the β-numbers for his traveling salesman theorem:

$$\ell(\Gamma) \simeq \mathrm{diam}(\Gamma) + \sum_Q \beta_\Gamma(Q)^2 \mathrm{diam}(Q),$$

where the sum is over all dyadic cubes Q in $\mathbb{R}^n$ hitting Γ.



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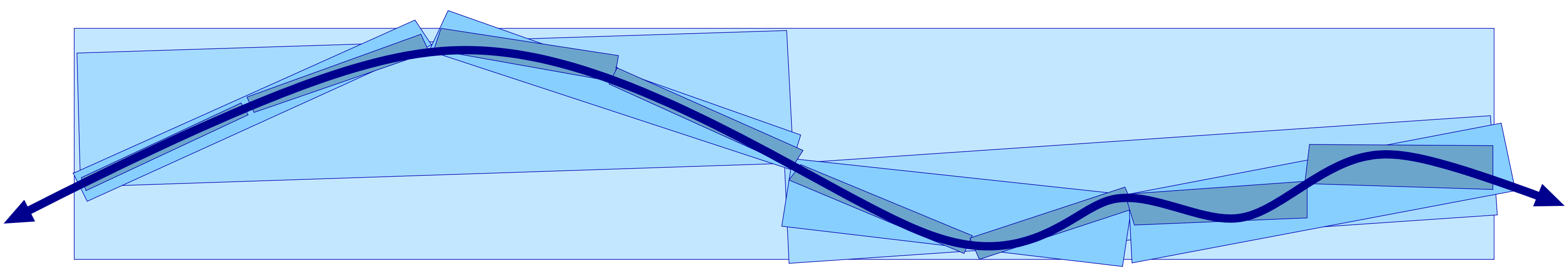
Theorem: Γ is Weil-Petersson iff $\sum_Q \beta_\Gamma(Q)^2 < \infty$.



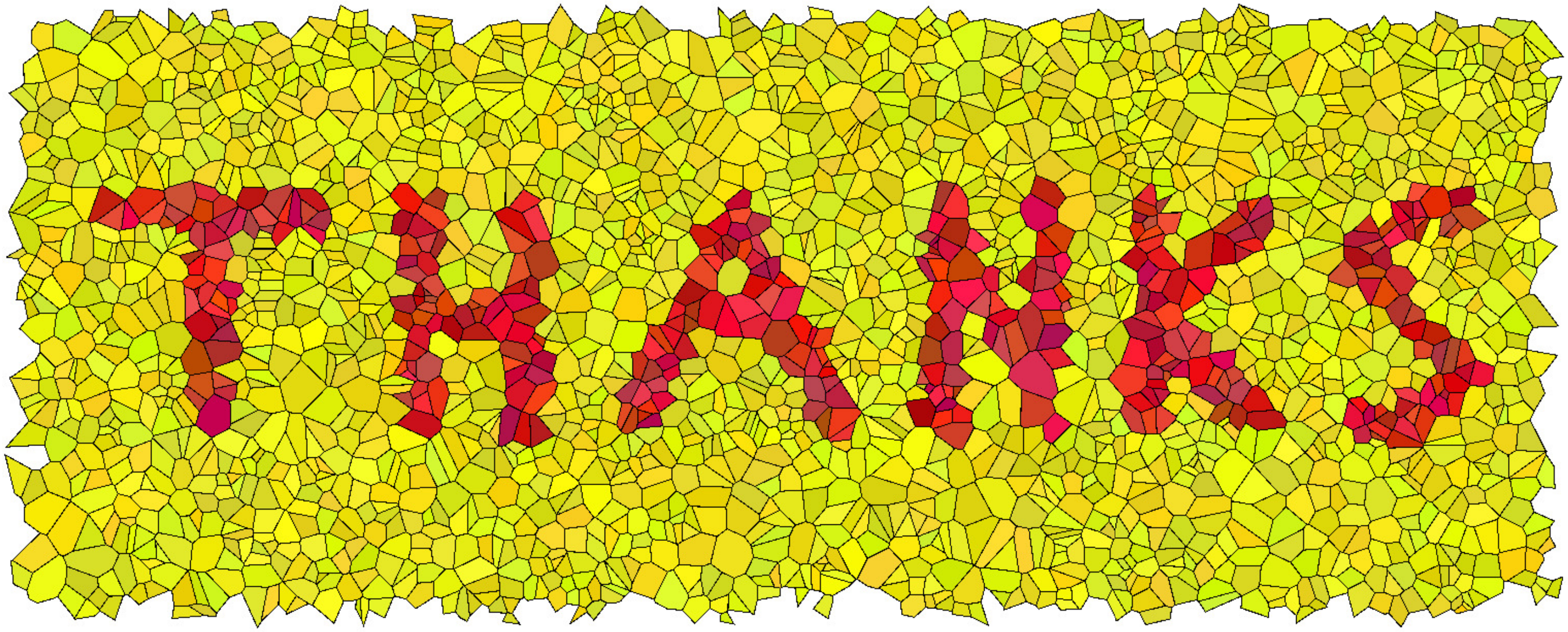
WP = "curvature in L^2, summed over all positions and scales".
= "rectifiable in scale invariant way".

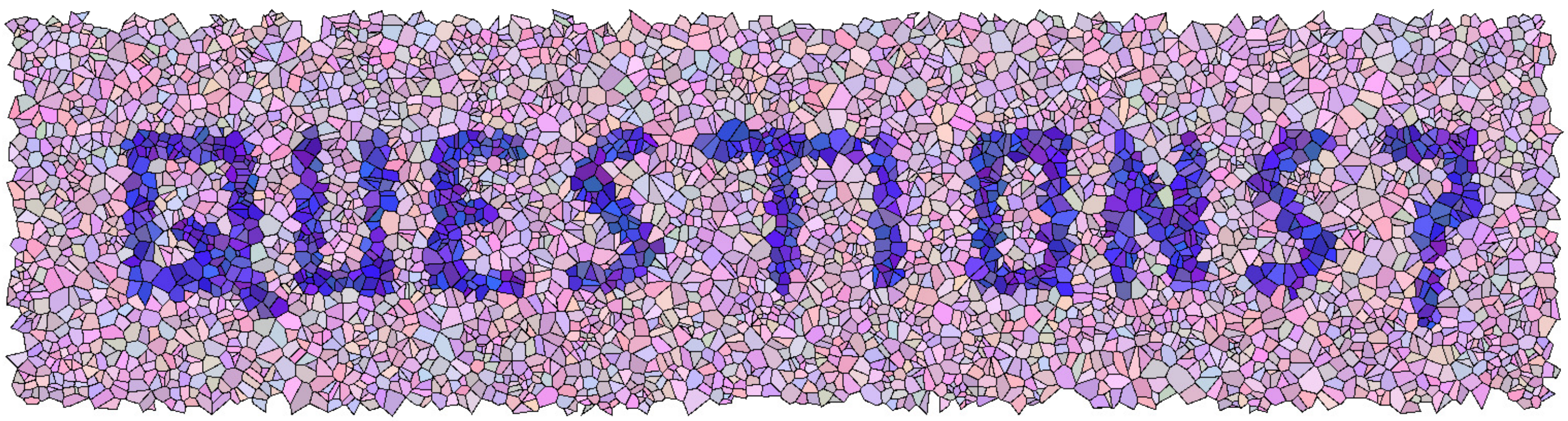
Many open questions about alternative characterizations of WP curves.

Connections to geometric measure theory, probability, knot theory, Sobolev spaces, string theory, hyperbolic geometry, minimal surfaces,...

What if $\sum(\text{distortion})^p < \infty, p \neq 2$?

What is analog for surfaces instead of curves?


THANKS


QUESTIONS?