An A_1 weight not comparable to any QC Jacobian

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copies of lecture slides available at
`www.math.sunysb.edu/~bishop/lectures`

Theorem: There is an A_1 weight w on $\mathbb{R}^2$ which is not comparable to any QC Jacobian.

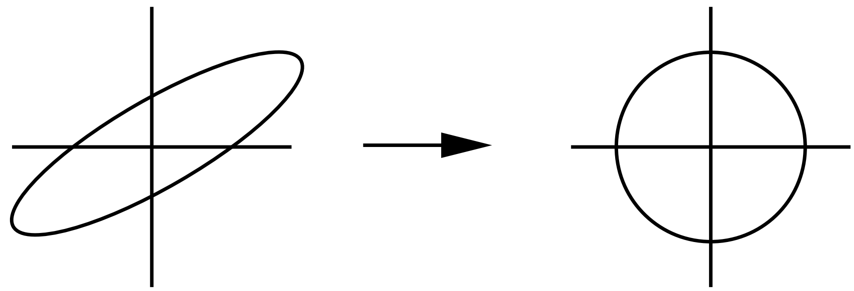
area$(\{x : w(x) > \lambda\}) \to 0$ as fast as we wish.

Theorem: There is a surface $S \subset \mathbb{R}^3$ that is Ahlfors 2-regular and locally linearly connected, but not bi-Lipschitz equivalent to $\mathbb{R}^2$.

Theorem: There is a set E with zero area and a $K > 1$ so that no function that tends to ∞ on E is comparable to the Jacobian of a K-QC map.

A mapping is K-quasiconformal if either:

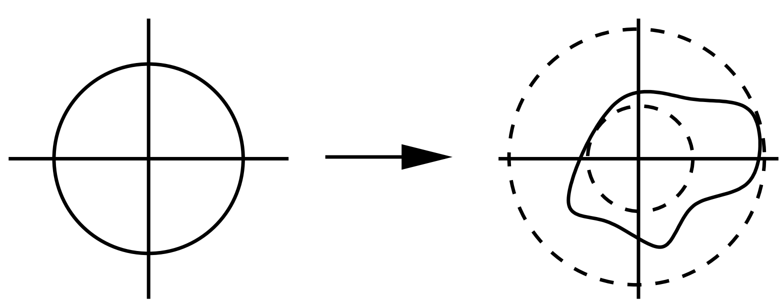
Analytic definition: $|f_{\bar{z}}| \leq \frac{K-1}{K+1}|f_z|$



$$f_z = \frac{1}{2}(f_x - i f_y),\ f_{\bar{z}} = \frac{1}{2}(f_x + i f_y).$$

Metric definition: For every $x \in \Omega$, $\epsilon > 0$ and small enough $r > 0$, there is $s > 0$ so that

$$D(f(x), s) \subset f(D(x, r)) \subset D(f(x), s(K+\epsilon)).$$



- The map is determined (up to Möbius maps) by

$$\mu_f = f_{\bar{z}}/f_z,$$

For μ with $\|\mu\|_\infty < 1$, there is a f with $\mu_f = \mu$.

- $\mu = 0$ iff f is conformal.

- K-QC maps form a compact family.

$w > 0$ is an A_1 weight if there is a $C < \infty$ so that

$$\frac{1}{\text{area}(B)} \int_B w \leq C \text{essinf}_{\text{B}} \text{w},$$

for every ball B in the plane.

A metric space is locally linearly connected (LLC) if there is a $C < \infty$ so that $B(x, r)$ can always be contracted to a point in $B(x, Cr)$.

X is Ahlfors 2-regular if Hausdorff 2-measure satisfies $r^2/C \leq \mathcal{H}_2(B(x, r)) \leq Cr^2$ for some $C < \infty$ which is independent of x and r.

Laakso gave surface which is LLC and Ahlfors 2-regular, but not BL equivalent to plane. His example is not embeddable in R^n or any uniformly convex Banach space.

Stephen Semmes proved that any A_1 weight is the Jacobian of an embedding of $\mathbb{R}^2$ into some R^n and the image is LLC and 2-regular.

Bonk and Kleiner showed that a complete metric space which is homeomorphic to $\mathbb{R}^2$, LLC and Ahlfors 2-regular is a quasisymmetric image of $\mathbb{R}^2$.

Bonk, Heinonen and Saksman observed that the Semmes embedding associated to w is biLipschitz equivalent to $\mathbb{R}^2$ iff the weight w is comparable to a quasiconformal Jacobian.

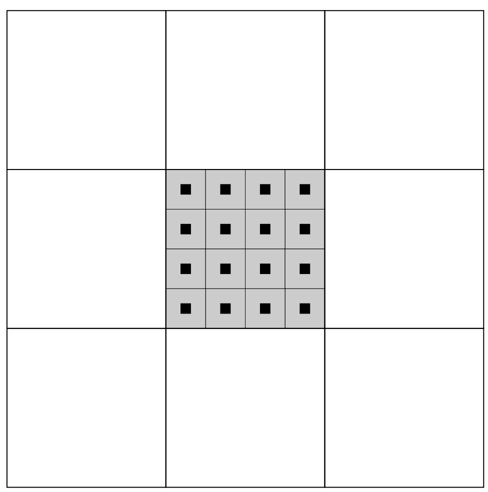
Idea of proof

Construct a set E and a weight $w \geq 1$ which blows up slowly near E. Show that if $J_f \sim w$, then $f(E)$ contains rectifiable curve γ.

Then f^{-1} has Jacobian 0 on γ, so $f^{-1}(E)$ is a point. This is contradiction, so no such f exists.

Definition of E: Fix odd integers $L \geq 3$, $N \geq 3$ and integer M.

- Given a square Q, divide it into L^2 subsquares.
- Divide center square into M^2 equal subsquares.
- Divide each of these into N^2 equal subsquares.
- Remove the M^2 center squares of size $1/LMN$.



1. $\text{area}(\Omega) = \text{area}(Q)(1 - (LN)^{-2})$.
2. Ω contains disjoint annuli around each omitted square with radii ratio $\sim N$.

Assume

$$\sum_n N_n^{-2} = \infty, \tag{1}$$

$$\sum_n N_n^{-3} < \infty, \quad \sum_n M_n^{-2} < \infty. \tag{2}$$

E.g., $L_n = 7$, $M_n = n$, $N_n = 3 + \frac{1}{8}\lfloor\sqrt{n}\rfloor$.

Define E_n inductively using these sequences.

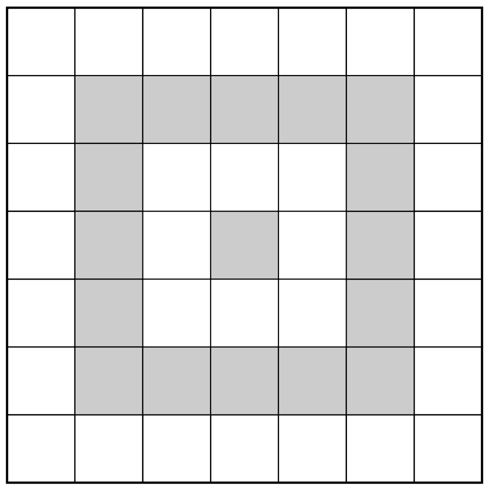
E **has zero area:** $L_n N_n \leq \sqrt{n}$ for n large, so

$$\text{area}(E_n) \leq \prod_{k=n_0}^{n} (1 - \frac{1}{k}) = O(n^{-1}).$$

If S is an nth generation square and $k > n$, then

$$\frac{\text{area}(E_k \cap S)}{\text{area}(S)} \leq C\frac{k}{n}.$$

Define ring squares, W, that separate the central square Q' from ∂Q.



By the compactness of K-QC maps,

$$\operatorname{dist}(f(W), \partial f(Q) \cup f(Q')) \geq C_K \operatorname{diam}(f(Q)),$$

The definition of the weight w:
Let F_n be s_n-neighborhood of E_n, s_n is side length of nth generation squares.

Let $A_0 = 1$ and $A_n = 1 + \sum_{k=0}^{n} a_k$ for $n > 0$.

Set $w(x) = 1$ for $x \notin F_1$ and set

$$w(x) = 1 + \sum_{k=1}^{\infty} a_k \chi_{F_k}(x)$$

otherwise. This is integrable as long as

$$\sum_{k=1}^{\infty} a_k \text{area}(F_k) = O(\sum_{k=1}^{\infty} \frac{a_k}{k}) < \infty.$$

Suppose S is nth generation square.

Suppose $a_k \le \frac{1}{k}$. Then $\sum_n^\infty \frac{a_k}{k} = O(\frac{1}{n})$.

$$\frac{1}{\text{area}(S)} \int_S w \le A_n + O(n \sum_{k=n}^{\infty} \frac{a_k}{k})$$
$$\le A_n + O(1) = O(A_n)$$

Since $A_n = \mathrm{essinf_{E_n} w}$, this is similar to the A_1 condition (but only for generation squares).

We can take $A_n \to \infty$ as slowly as we wish, so area$(\{x : w(x) > \lambda\}) \to 0$ as quickly as we wish, as claimed earlier.

w is an A_1 weight:

Suppose $B = B(x, r)$ and choose n so that $s_{n+1} \leq 4r < s_n$. Suppose $\inf_B w = A_k$. Then $B \subset F_k$, and $B \setminus F_{k+1} \neq \emptyset$.

If $k \leq n$, then $4r < s_n \leq s_k$ so $2B$ lies outside E_{k+1} and hence B lies outside F_{k+2}. Thus $w \leq A_{k+2}$ on B. Since $A_k \geq 1$ and $a_k \leq 1/2$ we have $A_{k+2} = A_k + a_{k+1} + a_{k+2} \leq 2A_k$.

If $k > n$, then $r \geq \frac{1}{4} s_{n+1} \geq \frac{1}{4} s_k$. Cover B by squares of size s_{k+1} inside $2B$. Average of w over B is at most twice average over all squares which is $2A_{k+1} \leq 4A_k$ by previous page.

Lemma: Suppose B_1 is a unit ball centered at the origin and $B_2 \subset B_1$ is the concentric ball with radius ϵ. Suppose f is a K-quasiconformal map of the plane which fixes 0 and ∞ and also suppose that the averages of the Jacobian, J_f, over the two balls are comparable with constant M. Then there is an $r > 0$ such that $f(B_1 \setminus B_2)$ contains a round Euclidean annulus with outer radius r and inner radius at most $\sqrt{M}\epsilon C_K^2 r$, where C_K is a constant that depends only on K.

The point is: if f has Jacobian comparable to w then $f(E_n)$ has annuli around its holes with approximately the same modulus that E_n does.

A "good path" for $f(E_n)$ is a polygonal path which is contained in $f(E_n)$ and all of whose vertices lie in sets of the form $f(\partial Q)$ where Q is a nth generation square.

Good Path Lemma: Suppose f is K-QC and $M^{-1} \le J_f/w \le M$. Then there are n_0, C_1, C_2 depending on K and M so that if $n \ge n_0$ and if γ is a good path for $f(E_{n-1})$, then there is a good path γ' for $f(E_n)$ such that

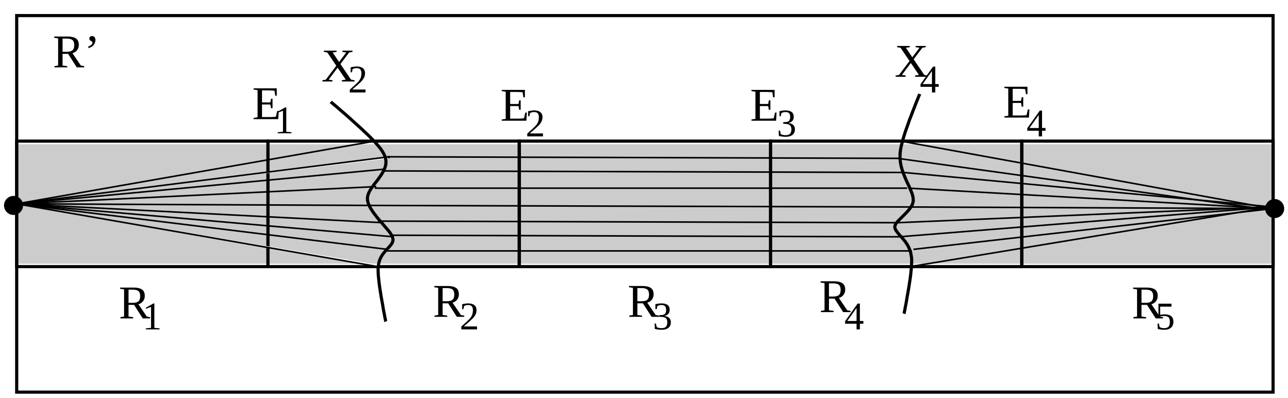
1. every vertex of γ is also a vertex of γ',
2. $\ell(\gamma') \le \ell(\gamma)(1 + C_1/M_n^2 + C_2/N_n^3)$.

We need $\sum_n N_n^{-2} = \infty$ to make E have zero area, and we need $\sum_n N_n^{-3} < \infty$ to produce a rectifiable arc in $f(E)$, so the gap between 2 and 3 is what allows our to example work.

Proof of the good path lemma

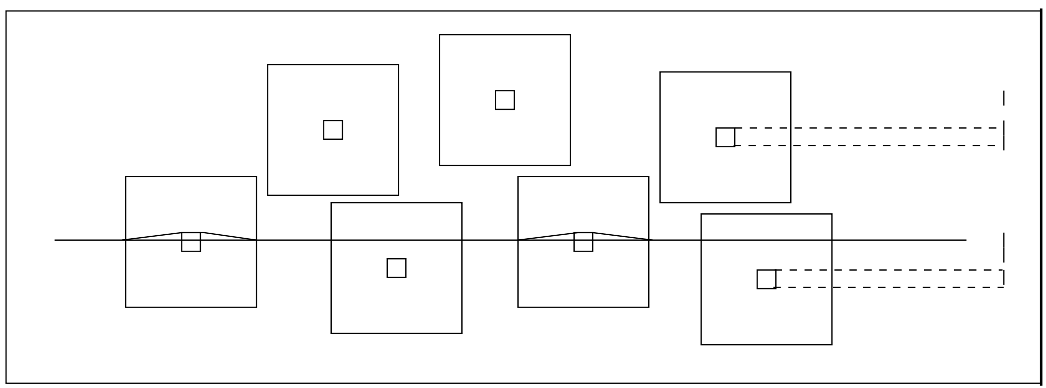
Assume γ is good for $f(E_{n-1})$ and modify to make good for $f(E_n)$. Assume it hits image of central square $f(Q')$. Inside $f(Q')$ are M^2 small holes $\{H_k\}$ each of size $\leq \delta$ inside disjoint squares $\{S_k\}$ of size $N\delta$.

Form family of "width" δ by varying γ parallel to self. Each has length at most $1 + M^{-2}$ times that of γ.



Let R be rectangle of width δ along γ and R' of width 3δ Thus

$$\sum_{S_k \cap R \neq \emptyset} \ell(S_k)^2 \leq N^{-2}\text{area}(R') \leq 3\delta/N^2.$$



Extra length $\leq \ell(H_k)/N$.

Integral of extra length over all modified paths is

$$\sum_j N^{-1}\ell(S_j)\ell(S_j) = \delta N^{-3}$$

At least one path has less than average length

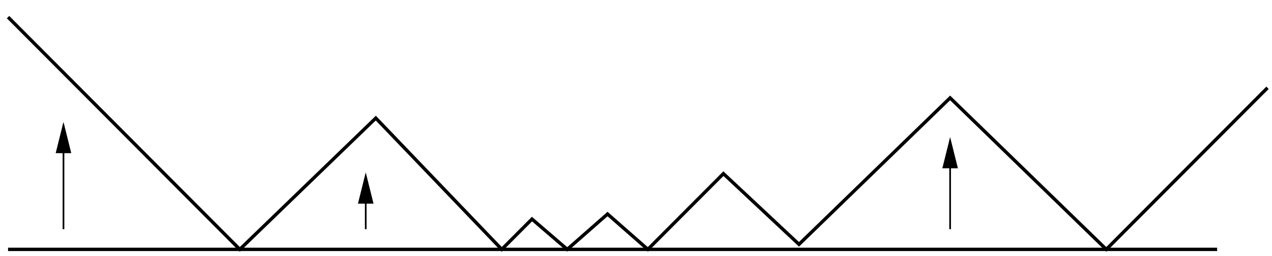
$$\leq \ell(\gamma)(1 + M^{-2})(1 + N^{-3})$$

If $\sum M_n^{-2} < \infty$ and $\sum N_n^{-3} < \infty$ then E has rectifiable curve.

Surface in R^3 not BL equivalent to plane

Our surface is image of plane under composition of two maps. The first is vertical projection to a Lipschitz graph,

$$P(x) = (x, \min(1, \mathrm{dist}(x, E))).$$



Second map is QC $\mathbb{R}^3 \to \mathbb{R}^3$ given by

Theorem: There is a non-increasing function f on $(0, \infty)$ which satisfies

$$\lim_{t \to 0^+} f(t) = \infty,$$

and a QC mapping $\Phi : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ such that each of the following conditions hold.

1. f can be taken to grow as slowly as we wish, e.g., given a sequence $s_n \searrow 0$ and any $\lambda > 1$ we may assume $f(s_{2n}) \le \lambda f(s_n)$.
2. There is a $C < \infty$ so that for all $z, w \in \mathbb{R}^2 \times \{0\}$,
$$\frac{1}{C} \le \frac{|\Phi(z) - \Phi(w)|}{|x - y| f(|z - w|)} \le C.$$
3. The map Φ is smooth on $\mathbb{R}^2 \times (0, \infty)$ and its Jacobian satisfies $J_\Phi(z, t) \sim f^3(t)$.

Question: Is there a compact set E of measure zero so that no A_1 weight w that blows up on E (i.e., so that $w(z) \to \infty$ as $z \to E$) can be comparable to a quasiconformal Jacobian?

Question: Is there a compact set E of measure zero so that no locally integrable w which blows up on E can be comparable to a quasiconformal Jacobian?

Question: Is there a compact set E of measure zero so that every quasiconformal image of E contains a rectifiable curve?

Question: Can the example in this paper have Hausdorff dimension < 2?

Problem: Characterize surfaces in R^n which are BL images of plane.

The w we construct is not Jacobian of planar map but is Jacobian of map onto surface in R^3. What property does plane have that surface does not? Traveling salesman theorem holds in plane. If it held on surface, our w could not be Jacobian of map onto surface.

Question: BL image of plane iff LLC + 2-regular + TST holds?

Question: TST holds iff curvature is a Carleson measure?

How do we make these precise?

Understand the figures and you understand the book.

John Garnett

"Ah!" replied Pooh. He'd found that pretending a thing was understood was sometimes very close to actually understanding it. Then it could easily be forgotten with no one the wiser...

Winnie-the-Pooh

A good lecture is the most ephemeral of triumphs.

T.W. Körner