OPTIMAL TRIANGULATION OF POLYGONS

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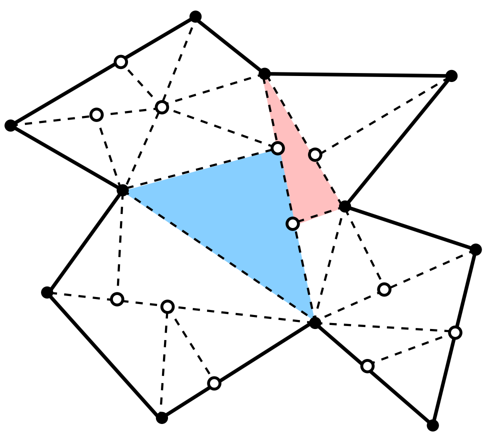
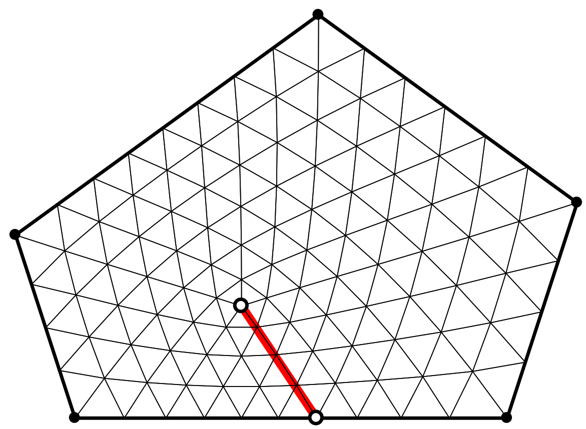
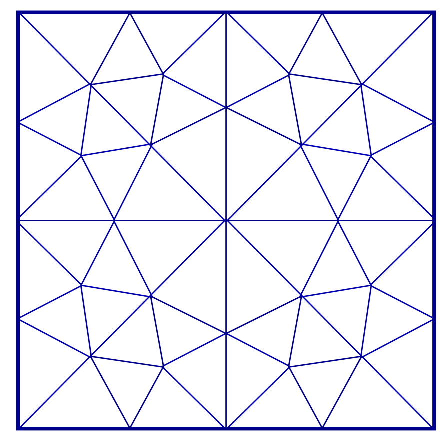
SBU Comp. Geom. Problem Group, Oct 4 2022

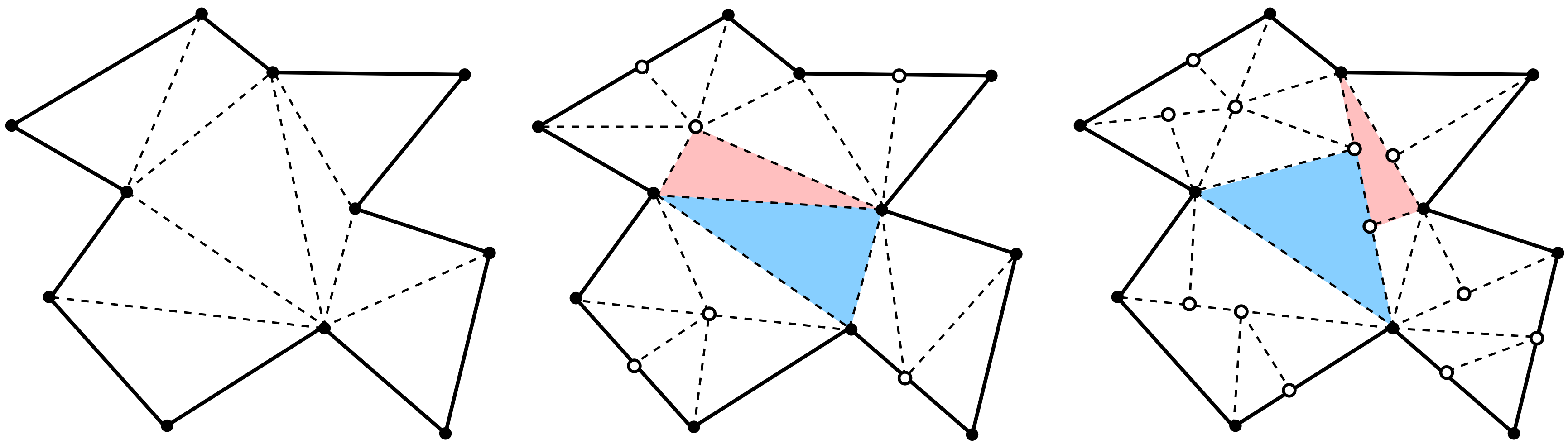






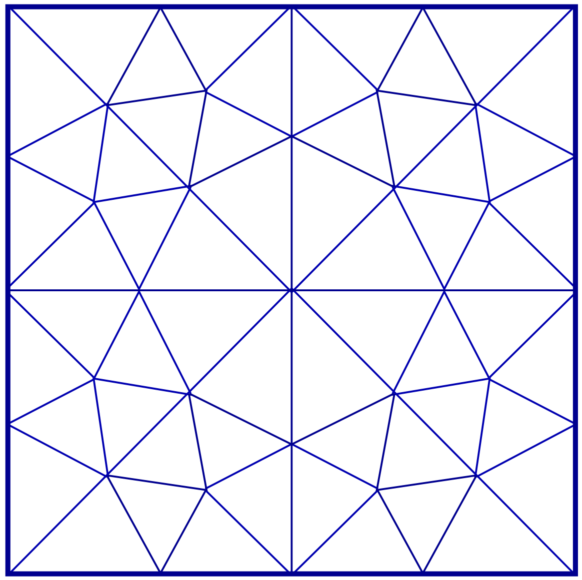
THE PLAN

- Definitions and history
- Necessary conditions
- The theorem and corollaries
- Sketch of proof
- Related results and open problems

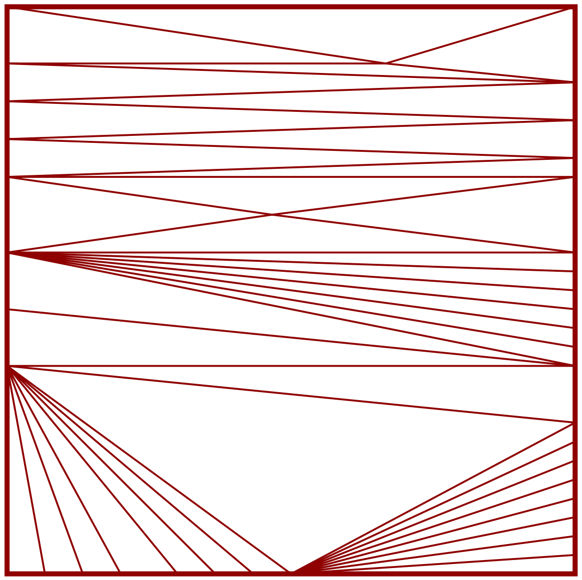


No Steiner Points — With Steiner Points — Dissection

Three types of triangulations



Good



Bad

Goal: make pieces as close to equilateral as possible.

Minimize the maximum angle (compute MinMax angle).

"Good" meshes improve performance of numerical methods.

Defn: acute triangle = all angles $< 90^\circ$.

Defn: nonobtuse triangle = all angles $\leq 90^\circ$.

Defn: ϕ-triangulation = all angles $\leq \phi$.

Defn: $\Phi(P) = \inf\{\phi : P \text{ has a } \phi\text{-triangulation}\}$.

Thm (Burago-Zalgaller, 1960): $\Phi(P) < 90^\circ$ all polygons.

"Every polygon has an acute triangulation."

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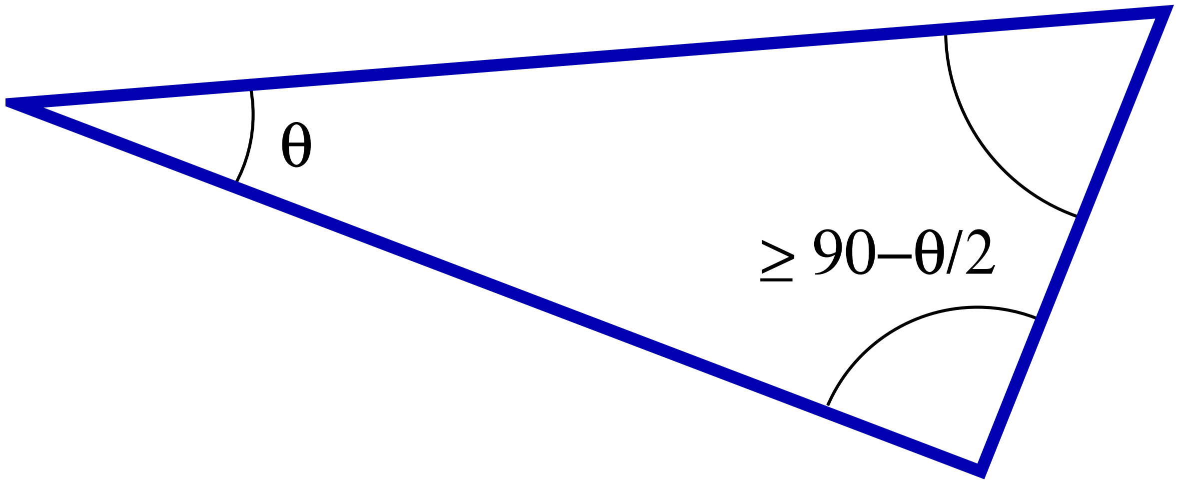
No bound $< 90^\circ$ works for **all** polygons.



Any triangle with an angle $\leq \theta$ also has an angle $\geq 90^\circ - \theta/2$.

Thm (Burago-Zalgaller, 1960): $\Phi(P) < 90^\circ$ all polygons.

"Every polygon has an acute triangulation."

Rediscovered by Baker-Grosse-Rafferty, 1988 (weaker version).

Much work on acute and non-obtuse triangulations by

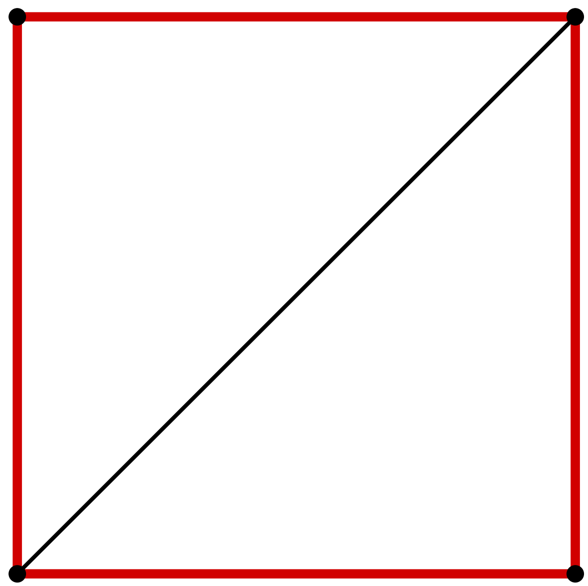
Barth,	Hirani,	Przytycki,	Tan,
Bern,	Itoh,	Ruppert,	Üngör,
Edelsbrunner,	Kopczyński,	Saalfeld,	VanderZee,
Eppstein,	Maehara,	Saraf,	Vavasis,
Erten,	S. Mitchell	Sheffer,	Yuan,
Gilbert,	Pak,	Shewchuk,	Zamfirescu,

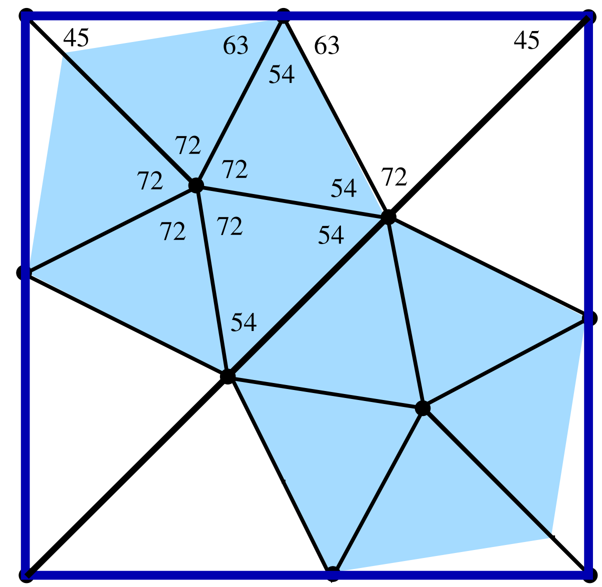
and many others (sorry if I omitted you).

Thm: every n-gon has an acute triangulation of size $O(n)$.

Burago-Zalgaller result first cited in CS literature around 2004.

Steiner points versus no Steiner points.





Consider triangulations of a square.

Without Steiner points, 90° is best angle bound.

Using Steiner points, a 72°-triangulation is possible.

We shall prove later this is best possible.

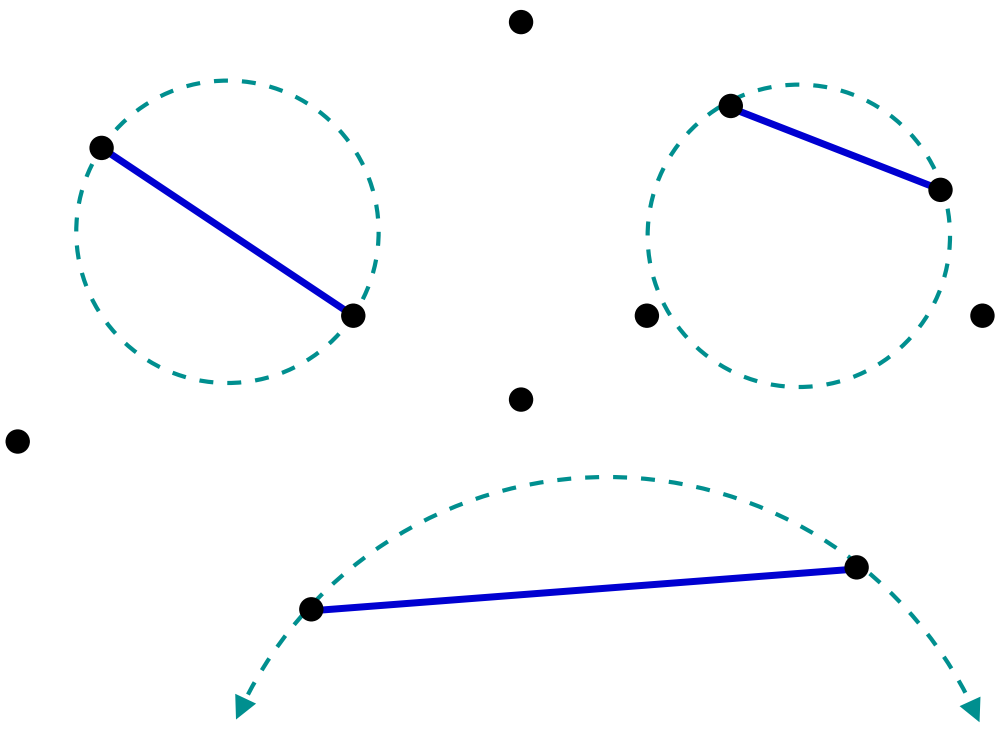
If no Steiner points, there are only finitely many triangulations of P.

MinMax and MaxMin triangulations clearly exist.

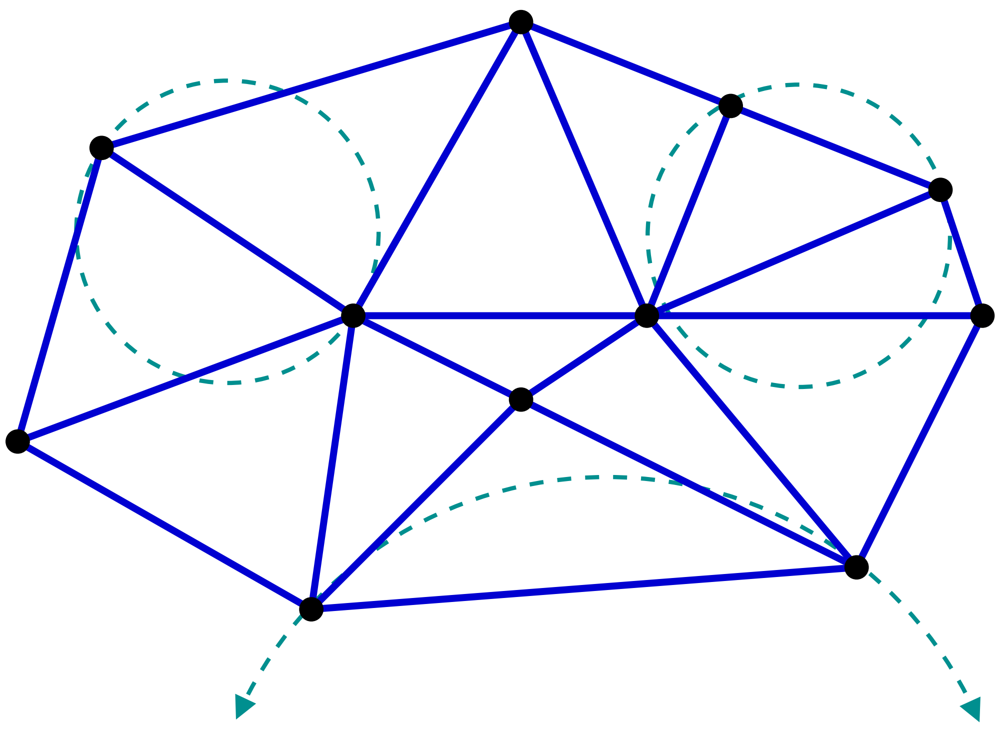
Delaunay triangulation solves MaxMin for point sets (C. Lawson, 1977).

Modification solves MaxMin for polygons. Takes $O(n \log n)$ time.

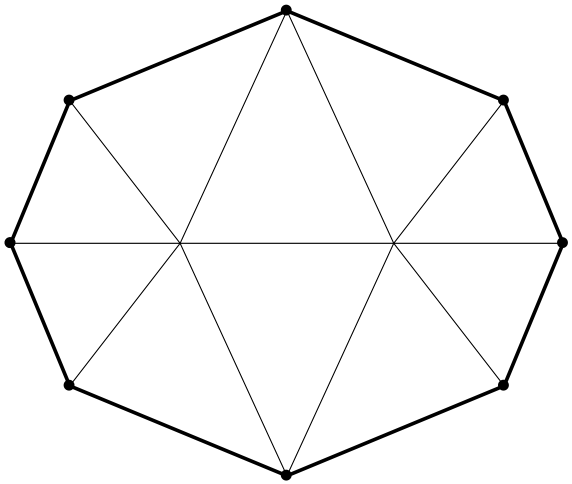
$O(n^2 \log n)$ algorithm for MinMax due to Edelsbrunner, Tan, Waupotitsch.



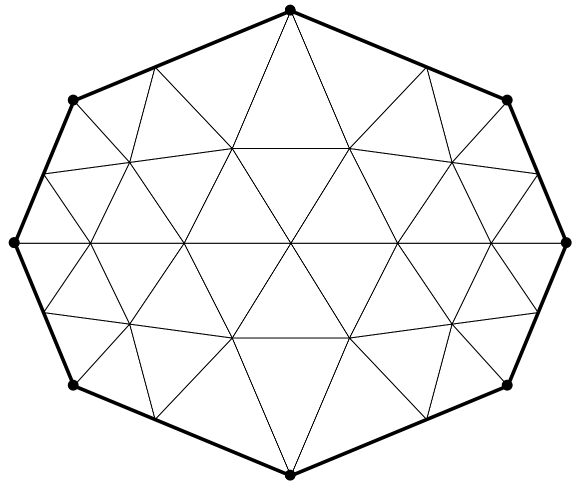
Given a point set V, a segment connecting two points in V is **Delaunay** if it is the chord of a disk missing V.



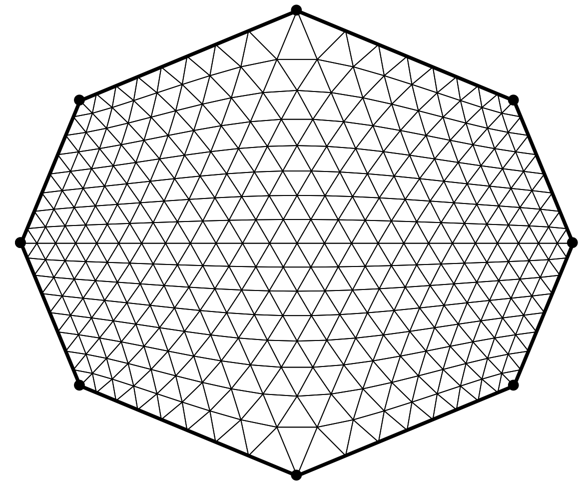
Delaunay edges of a point set form a triangulation of V.



$\theta_{\max} \approx 74.7482$



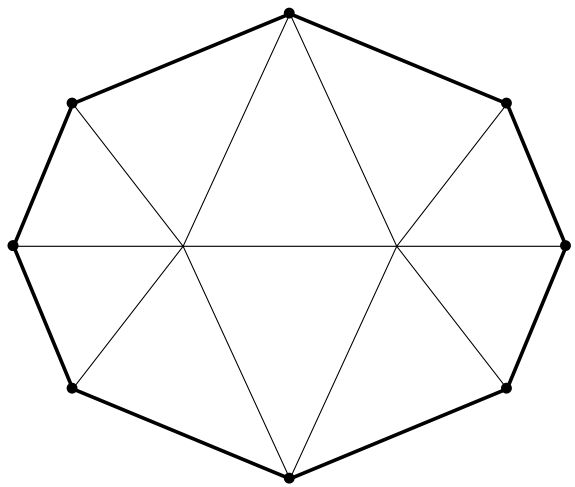
$\theta_{\max} \approx 70.3590$



$\theta_{\max} \approx 67.8690$

With Steiner points there are infinitely many possibilities.

Not obvious that optimal triangulation exists.



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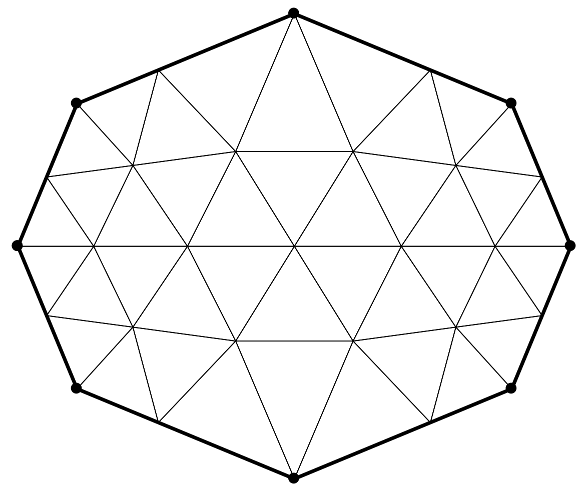

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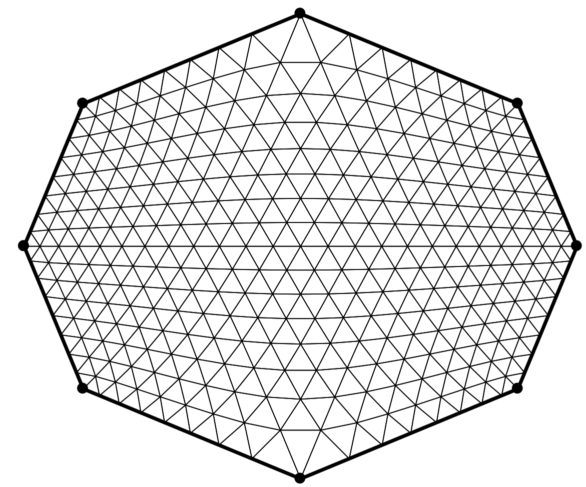


$\theta_{\max} \approx 67.8690$

With Steiner points there are infinitely many possibilities.

Not obvious that optimal triangulation exists.

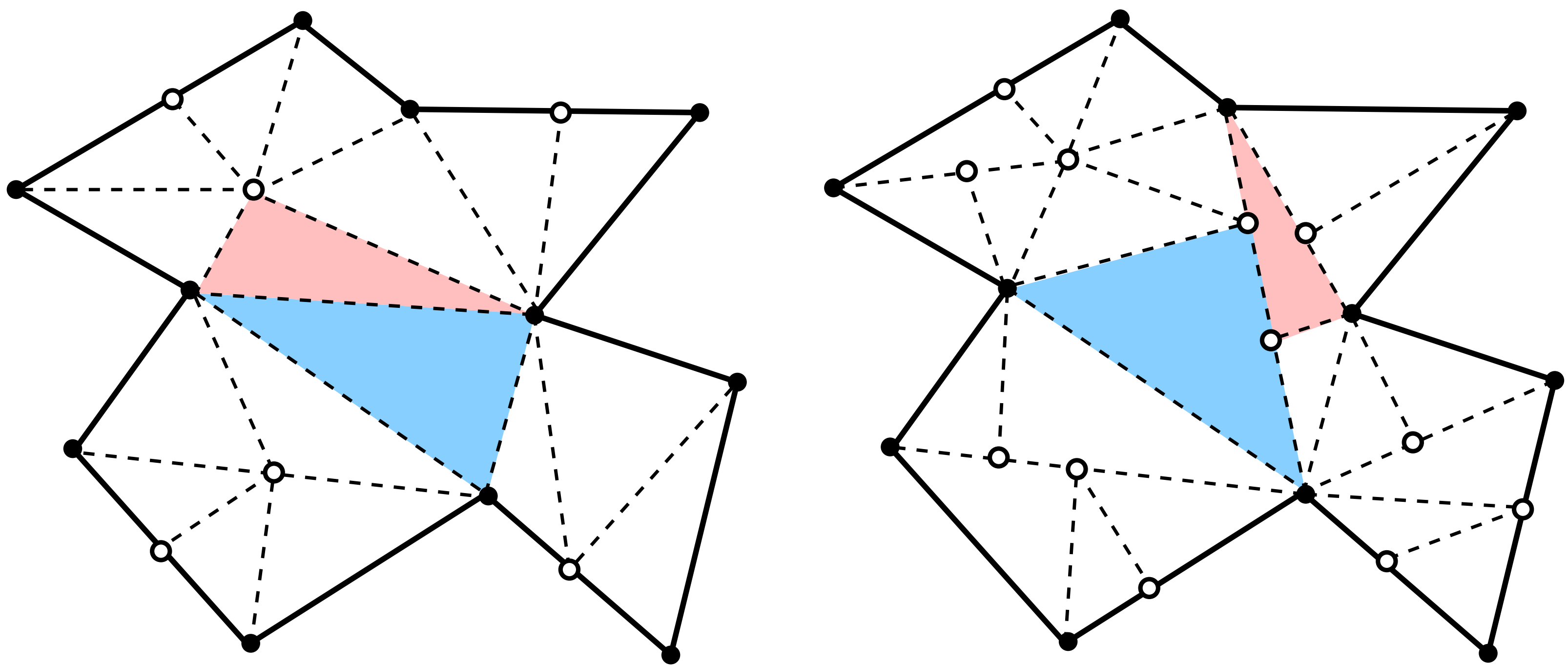
In case above, optimal angle is 67.5° and is attained.

But sometimes, the optimum bound is not achieved.

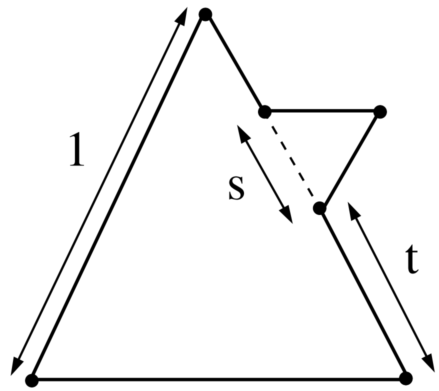
Dissections versus triangulations.

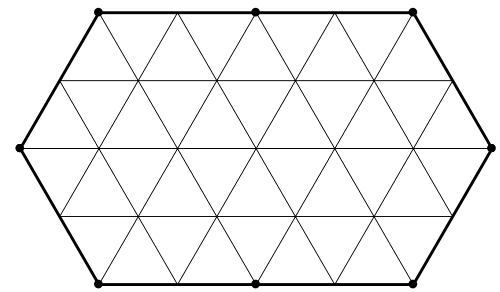
In a triangulation, triangles meet at vertices or full edges.

Dissections are more general, sometimes give better angles.



Defn: 60°-polygon = all angles are multiples of 60°.





Polygon on left has a dissection into two equilateral triangles.

Claim: it need not have any equilateral triangulation.

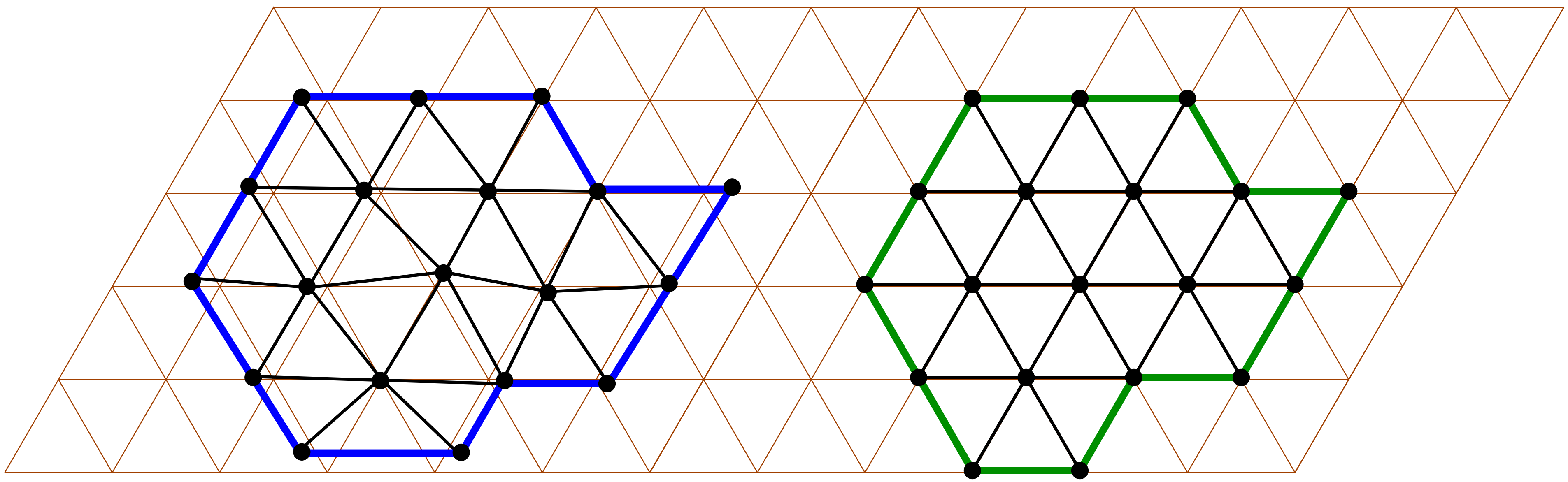
Equilateral triangulation
$\Rightarrow$ all triangles the same size
$\Rightarrow$ edge lengths are integer multiples of triangle length (right figure)
$\Rightarrow$ s, t are rational

Conclusion: a 60°-dissection exists, but a 60°-triangulation does not.

Lemma: For $60°$-polygons $\Phi(P) = 60°$.

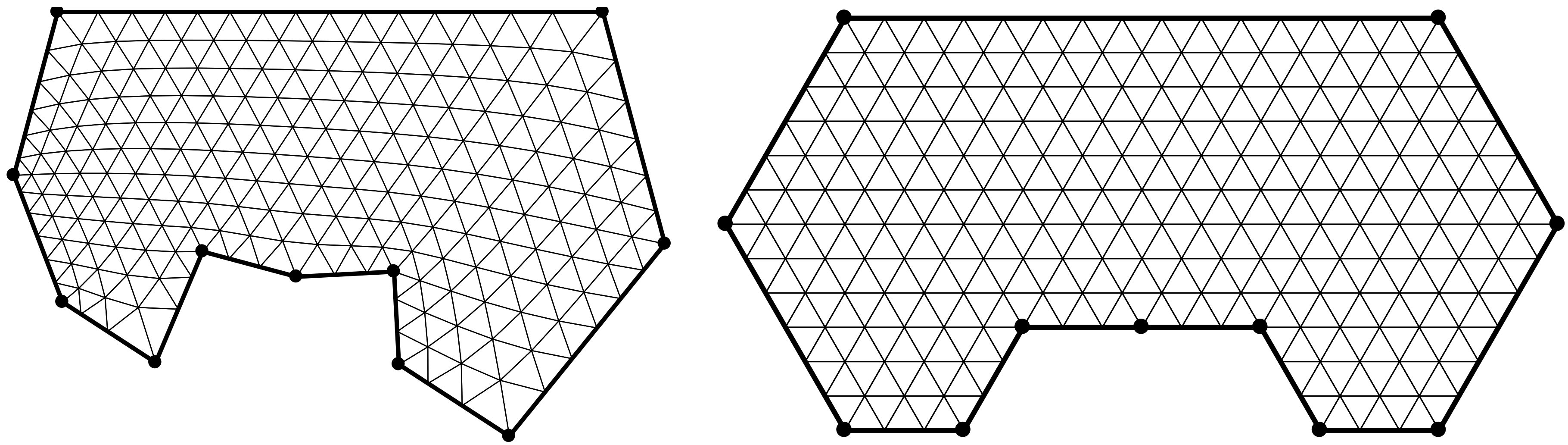
Sketch of Proof: Given P,

- Choose P' near P, with same angles and vertices on equilateral grid.
- Map $P' \to P$ using tiny angle distortion (quasiconformal map)
- Images of grid triangles all have angles $\leq 60° + \epsilon$.



Conclusion: here triangulations do as well as dissections, within ϵ.

Main idea: conformal images of 60°-polygons



We already know 60°-polygons have nearly equilateral triangulations.

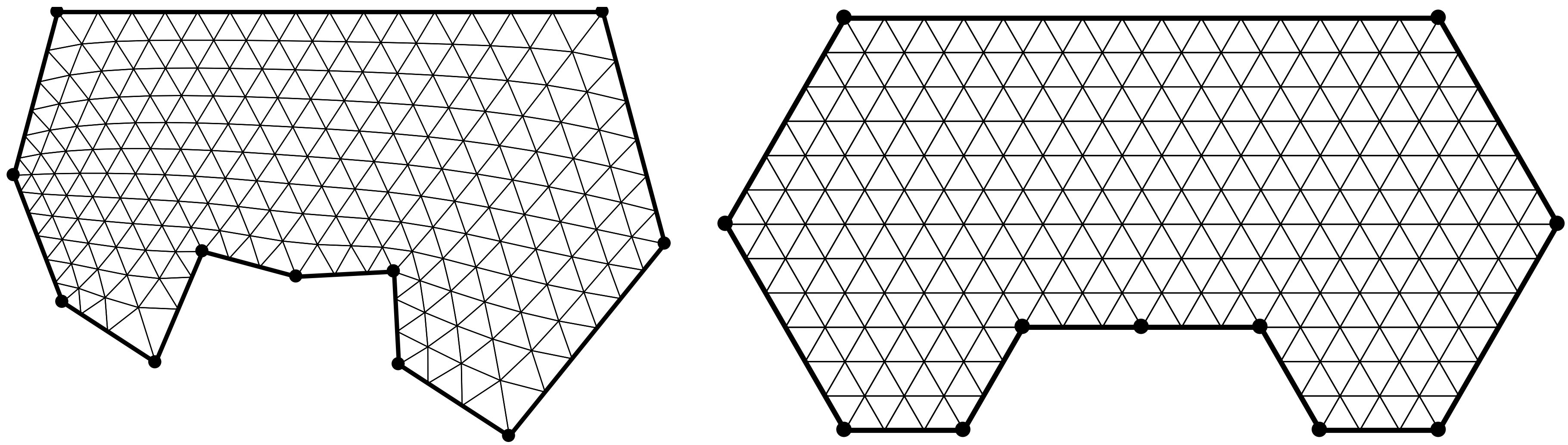
Given P, construct a 60°-polygon P' that "approximates" P.

Conformally map a nearly equilateral triangulation from P' to P.

Conformal = 1-1, holomorphic = preserves angles infinitesimally.

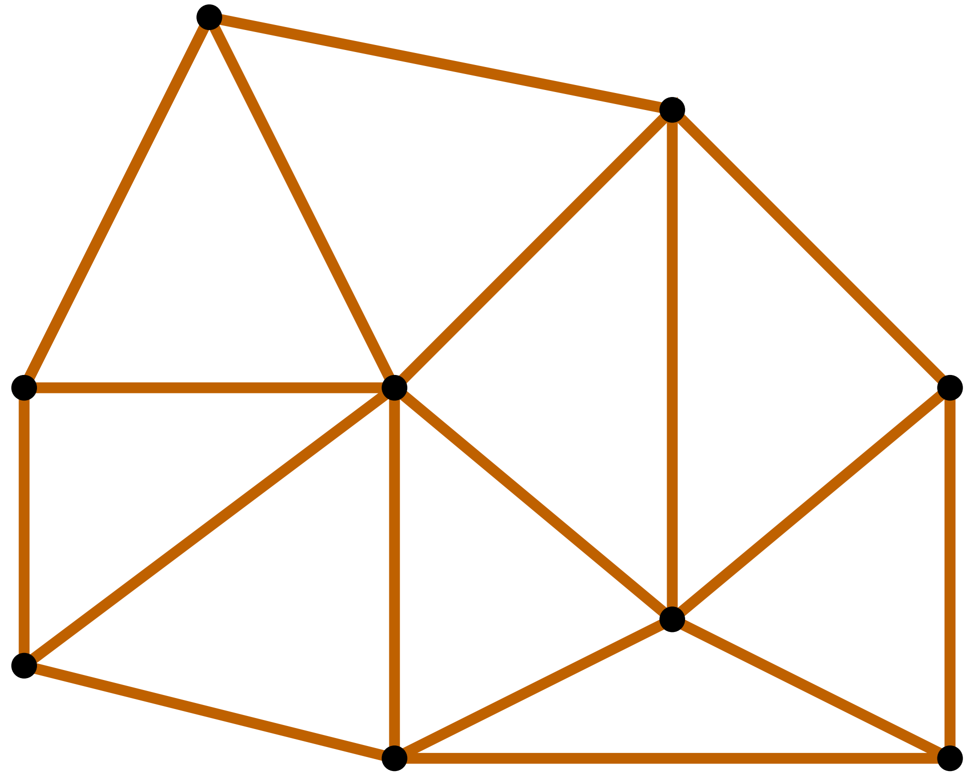
Map only vertices; then connect by segments. (Edge images are curved).

Main idea: conformal images of 60°-polygons



Problems to overcome (among others):

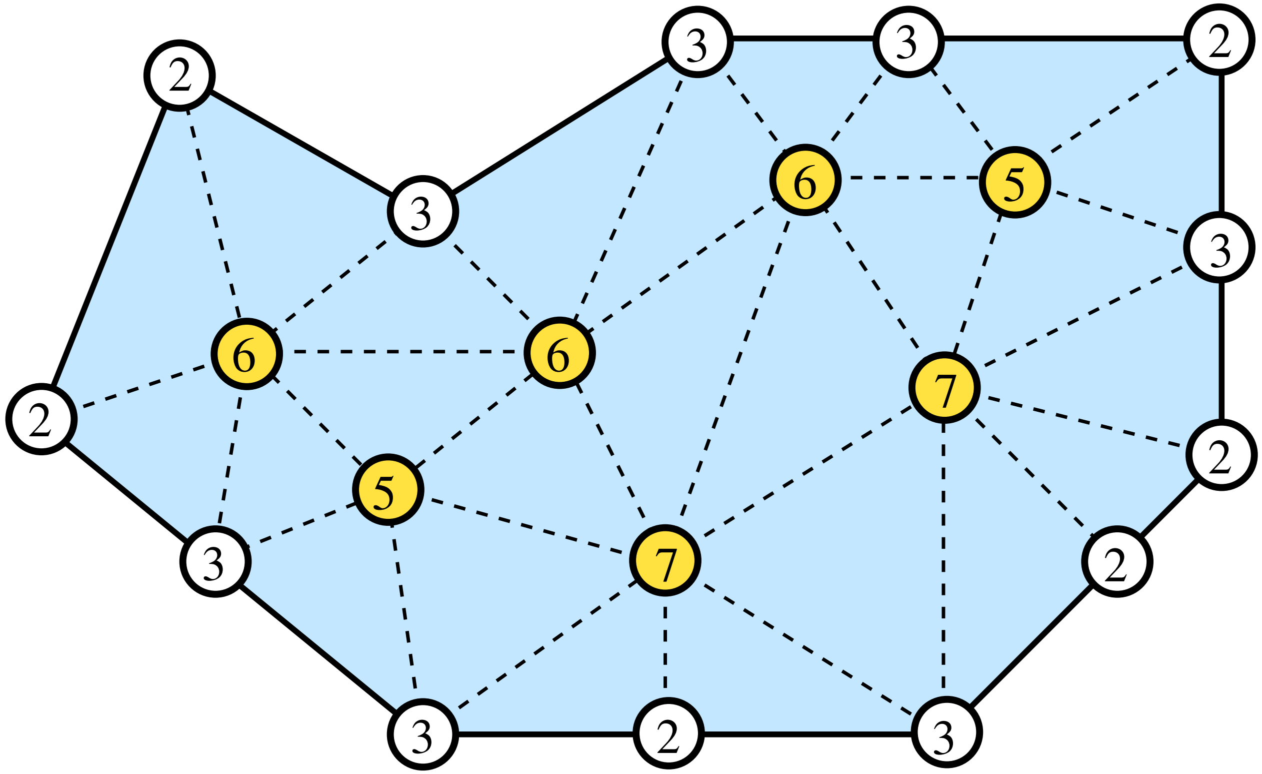
- must map vertices to vertices,
- bound angle distortion at positive scales,
- attain sharp bounds versus approximate them,
- Euler's formula sometimes forces vertices of degree 5 or 7.



Euler's formula: $F - E + V = 1$

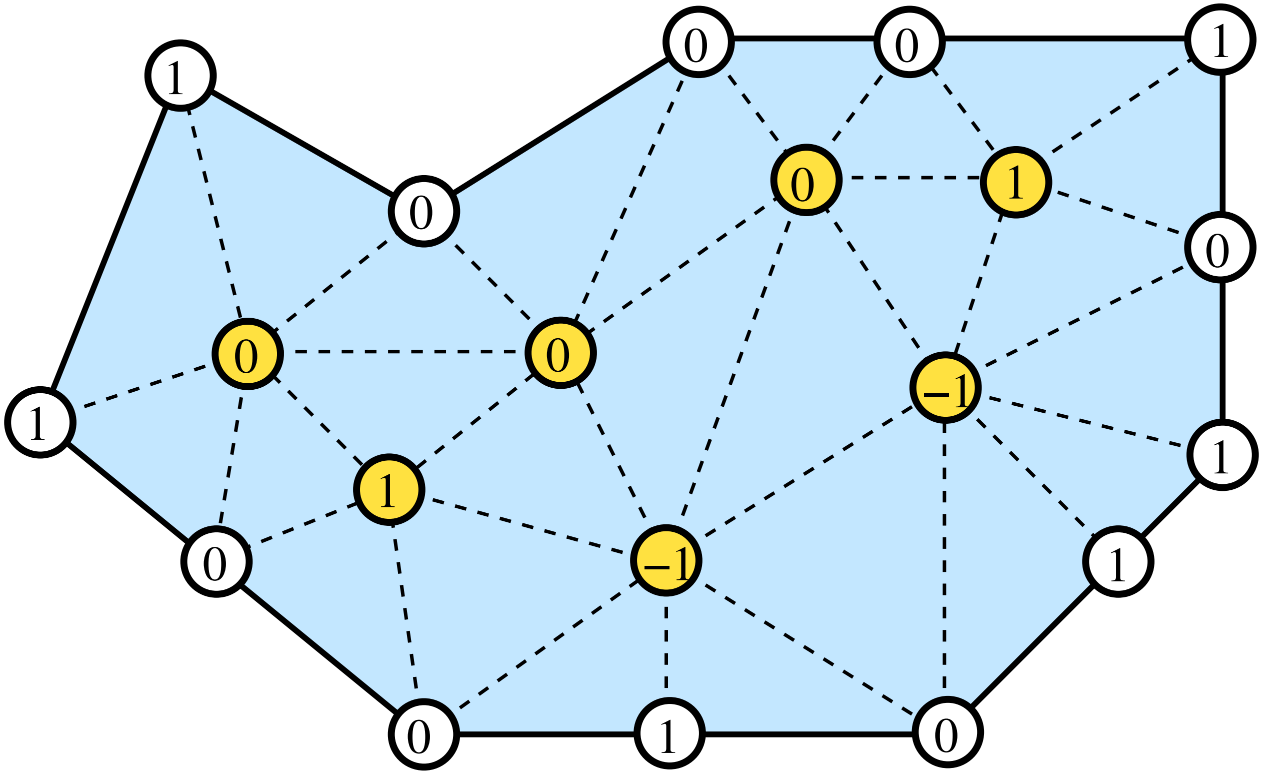
Faces - Edges + Vertices = 1

9 -17+9 = 1



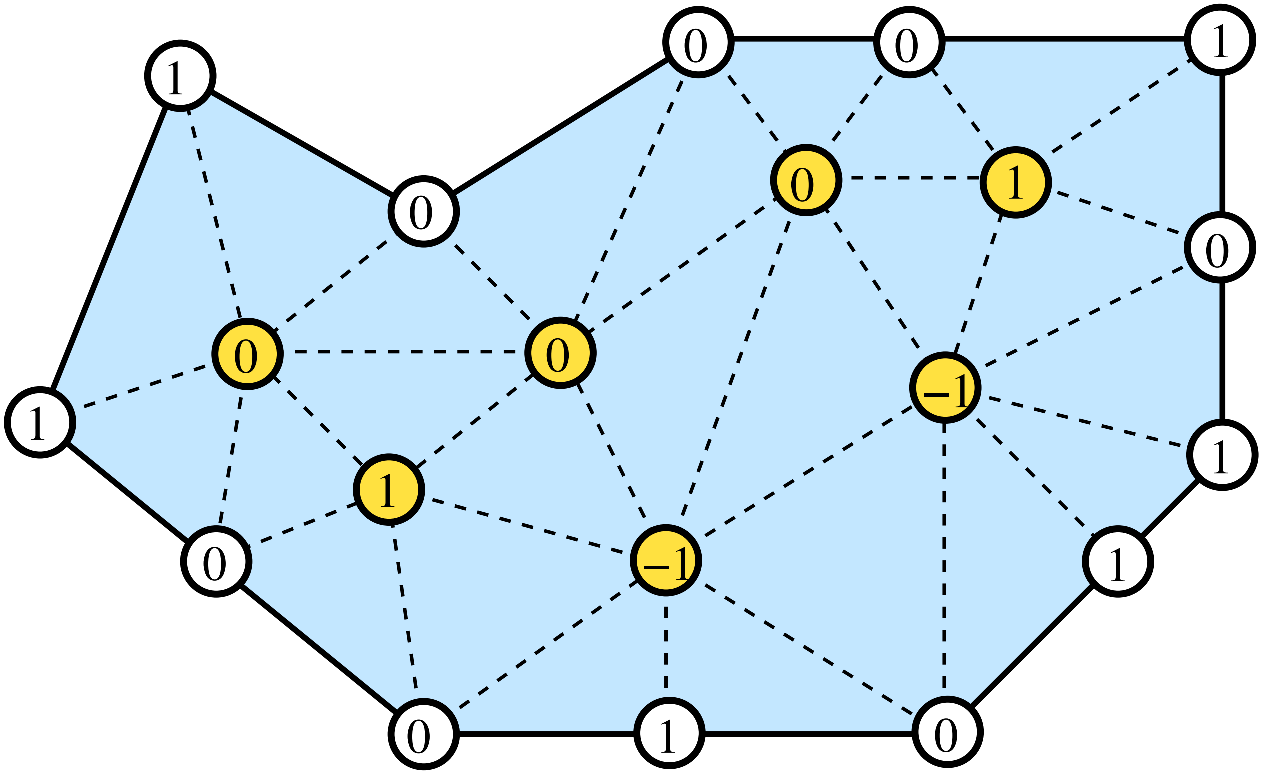
Let $L(v)$ = number of triangles with v as vertex.

In particular, this gives a labeling of P by positive integers.



Curvature of boundary vertex v: $\kappa(v) = 3 - L(v)$.

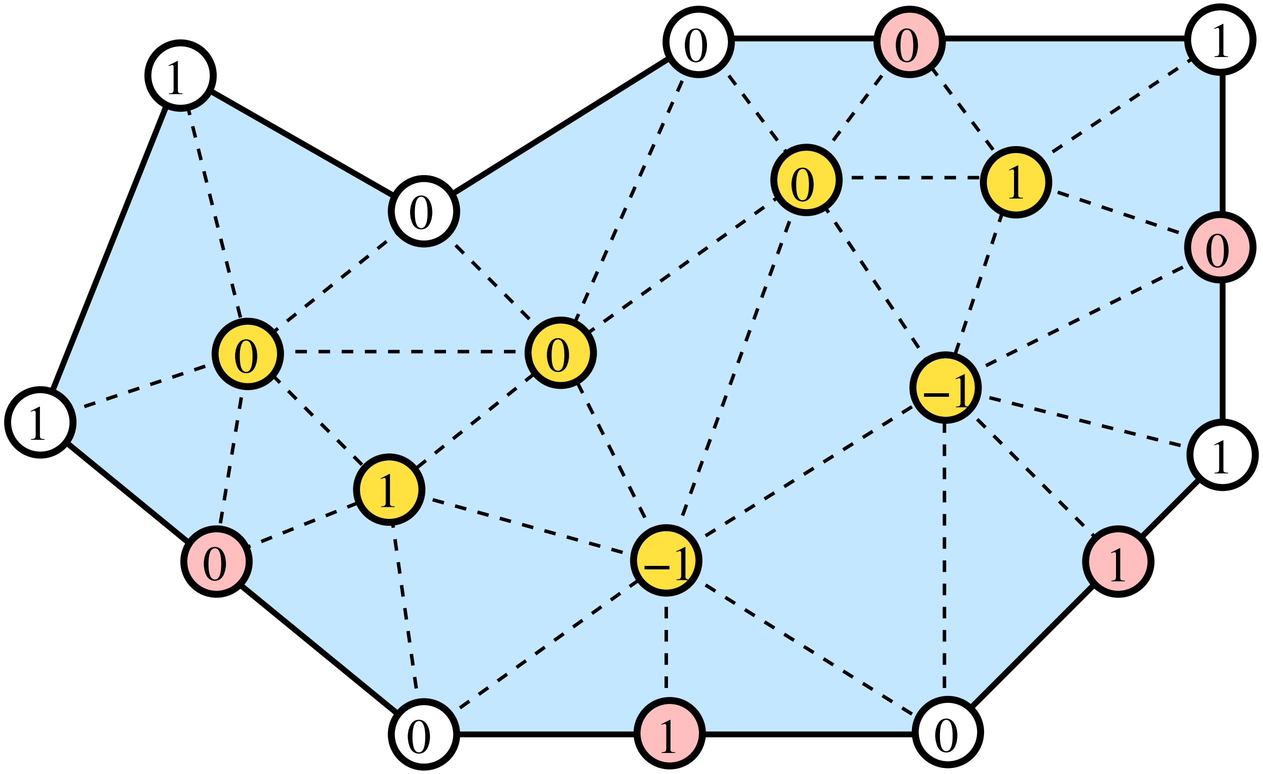
Curvature of interior vertex v: $\kappa(v) = 6 - L(v)$.



Euler's formula can be rewritten to look like Gauss-Bonnet:

$$\sum_{v \in \text{interior}} \kappa(v) = 6 - \sum_{v \in \text{boundary}} \kappa(v)$$

$$\kappa(\mathcal{T}) = 6 - \kappa(\partial \mathcal{T})$$

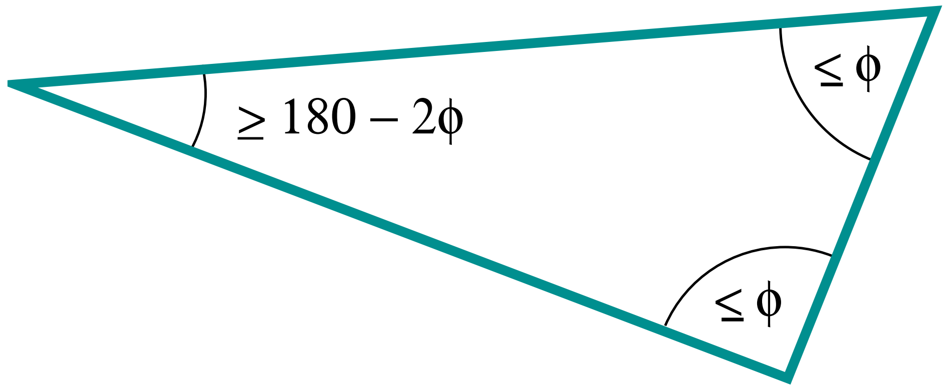


Define curvature of labeling L of vertices V of P (omit Steiner points):

$$\kappa(L) = 6 - \sum_{v \in P} \kappa(v).$$

Labelings of ϕ-triangulations have certain curvature restrictions.

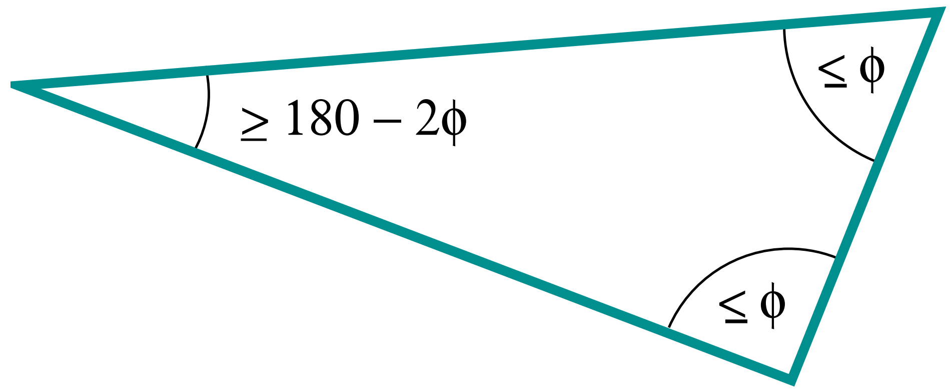
If a triangle has all angles $\leq \phi$, then all angles are $\geq 180^\circ - 2\phi$.



If a ϕ-triangulation has $L(v)$ triangles at vertex $v \in P$ of angle θ_v, then

$$L(v) \cdot (180^\circ - 2\phi) \;\leq\; \theta_v \;\leq\; L(v) \cdot \phi.$$

If a triangle has all angles $\leq \phi$, then all angles are $\geq 180^\circ - 2\phi$.



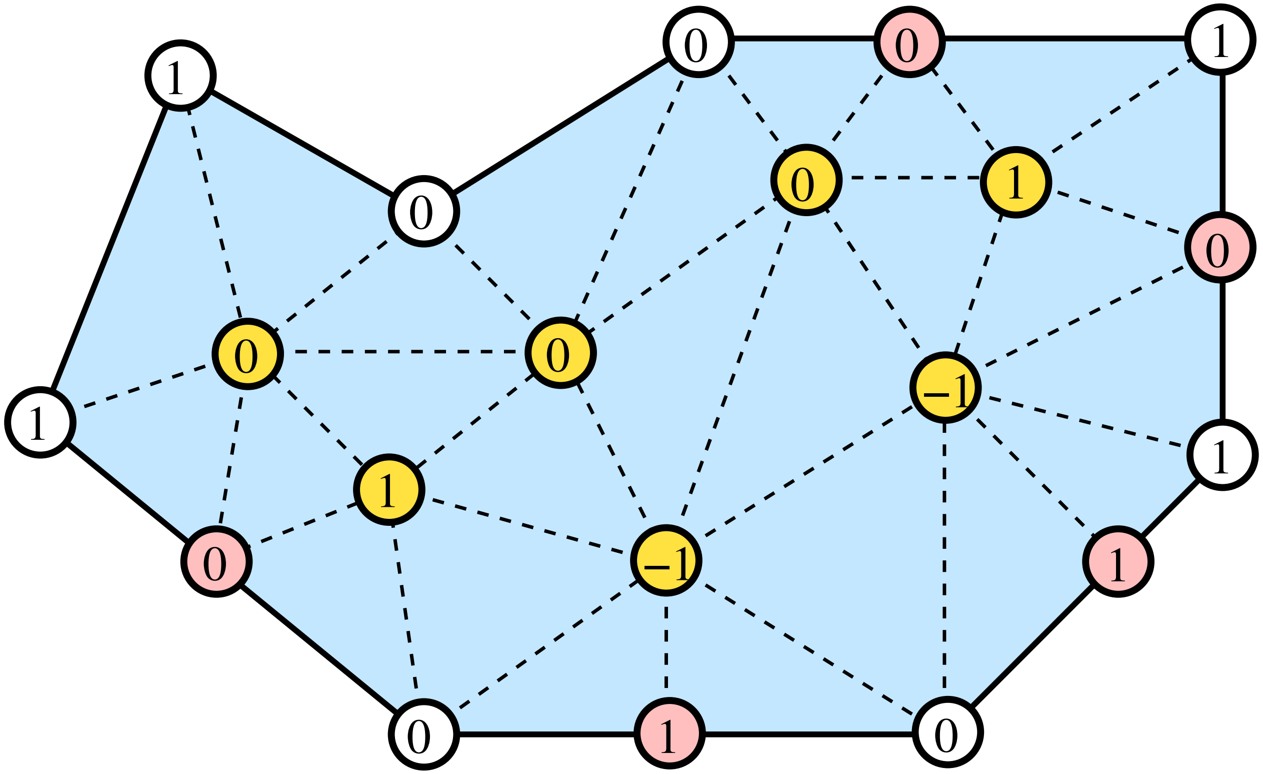
If a ϕ-triangulation has $L(v)$ triangles at vertex $v \in P$ of angle θ_v, then

$$L(v) \cdot (180^\circ - 2\phi) \ \leq \ \theta_v \ \leq \ L(v) \cdot \phi.$$

Defn: A labeling L of P is a **ϕ-labeling** if these inequalities hold, i.e.,

$$\frac{\theta_v}{\phi} \ \leq \ L(v) \ \leq \ \frac{\theta_v}{180^\circ - 2\phi}.$$

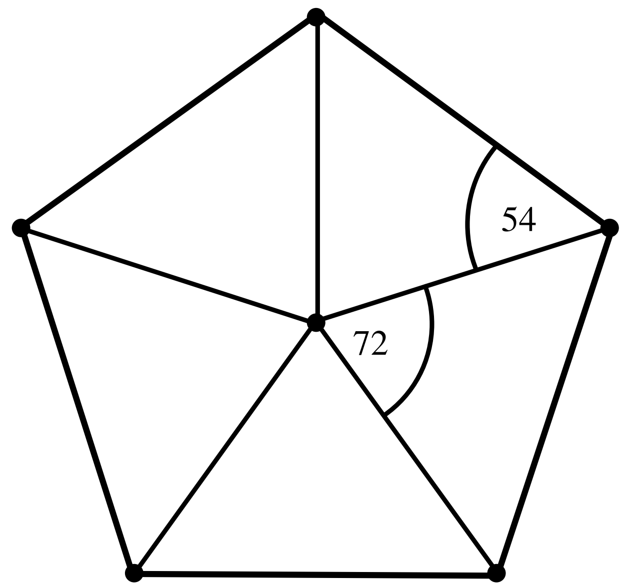
By definition, every ϕ-triangulation gives a ϕ-labeling.



For acute triangulations (angles $< 90°$) we must have

$$\kappa(L) \le \kappa(\mathcal{T})$$

since omitted boundary Steiner points have $L(v) \ge 3 \Rightarrow \kappa(v) \le 0$.

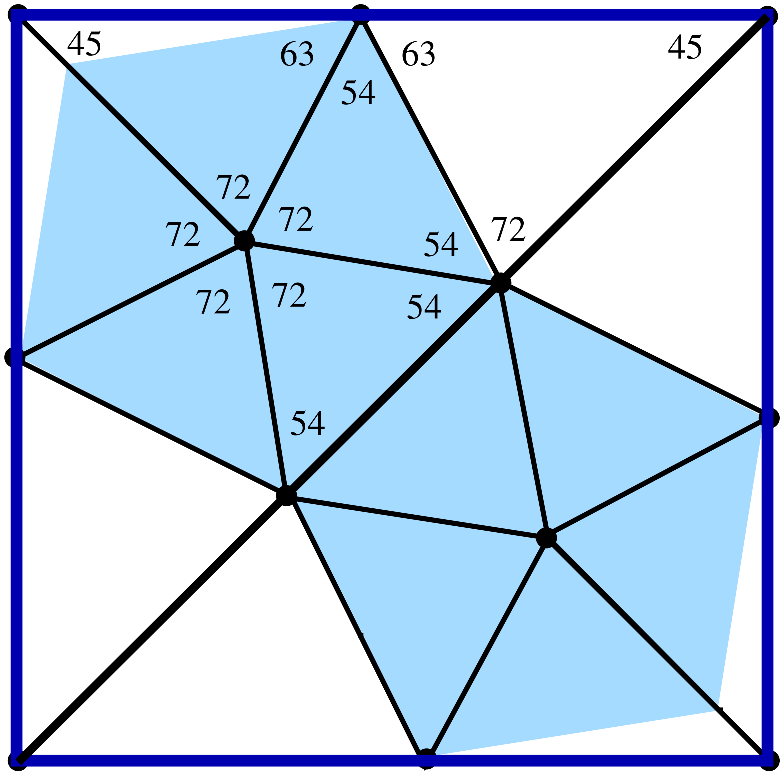


Suppose labeling L corresponds to a ϕ-triangulation. Then

- $\phi < 72^\circ \Rightarrow$ no degree ≤ 5 vertices $\Rightarrow \kappa(L) \leq \kappa(\mathcal{T}) \leq 0$.

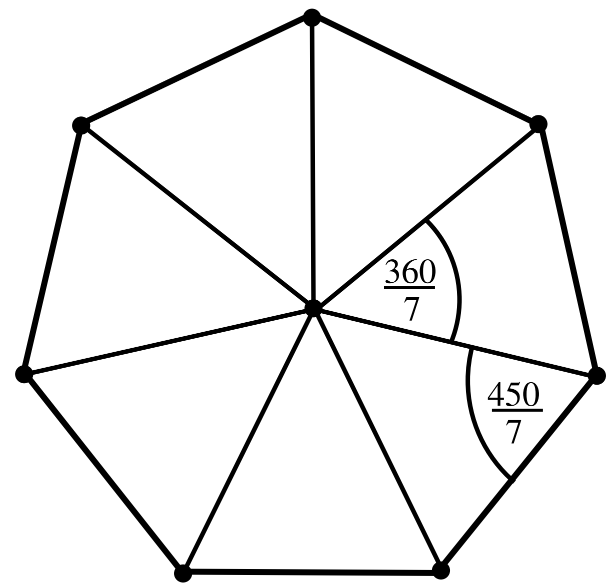
Example: for a square, $\Phi(P) = 72^\circ$.

Get $\Phi(P) \leq 72^\circ$ by explicit construction:



Converse:

- Suppose P has a ϕ-triangulation with $\phi < 72^\circ$.
- Euler $\Rightarrow$ there is a ϕ-labeling L of corners with $\kappa(L) \leq 0$.
- $\Rightarrow 6 - \sum \kappa(v) \leq 0$, so $\kappa(v) \geq 2$ for some corner.
- $\Rightarrow 3 - L(v) \geq 2$, so $L(v) = 1$.
- $\Rightarrow$ the triangulation has a 90° angle at corner $v \Rightarrow\Leftarrow$.



Suppose labeling L corresponds to a ϕ-triangulation. Then

- $\phi < (450/7)^\circ \Rightarrow$ no degree ≥ 7 interior vertices and only degree 3 Steiner boundary vertices $\Rightarrow \kappa(L) = \kappa(\mathcal{T}) = 0$.

To summarize: if a polygon P has a ϕ-triangulation, then the vertex set has ϕ-labeling L. Moreover,

$$\kappa(L) \leq 0 \text{ if } \phi < 72^\circ,$$

$$\kappa(L) = 0 \text{ if } \phi < (450/7)^\circ.$$

These necessary conditions observed by Gerver in 1984 (different notation).

Theorem: For $60^\circ < \phi < 90^\circ$, a polygon P has a ϕ-triangulation **iff**

1. $72^\circ \leq \phi < 90^\circ$ and P has a ϕ-labeling L of V_P,
2. $\frac{5}{7} \cdot 90^\circ \leq \phi < 72^\circ$, and P has a ϕ-labeling with $\kappa(L) \leq 0$,
3. $60^\circ < \phi < \frac{5}{7} \cdot 90^\circ$, and P has a ϕ-labeling with $\kappa(L) = 0$.

Cor: $\Phi(P) =$ infimum of ϕ for which P has a suitable ϕ-labeling.

Theorem: For $60^\circ < \phi < 90^\circ$, a polygon P has a ϕ-triangulation iff

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3. $60^\circ < \phi < \frac{5}{7} \cdot 90^\circ$, and P has a ϕ-labeling with $\kappa(L) = 0$.

Corollary: For $\phi > 60^\circ$, the following are equivalent:
(1) P has a $(\phi + \epsilon)$-triangulation for all $\epsilon > 0$.
(2) P has a ϕ-triangulation.

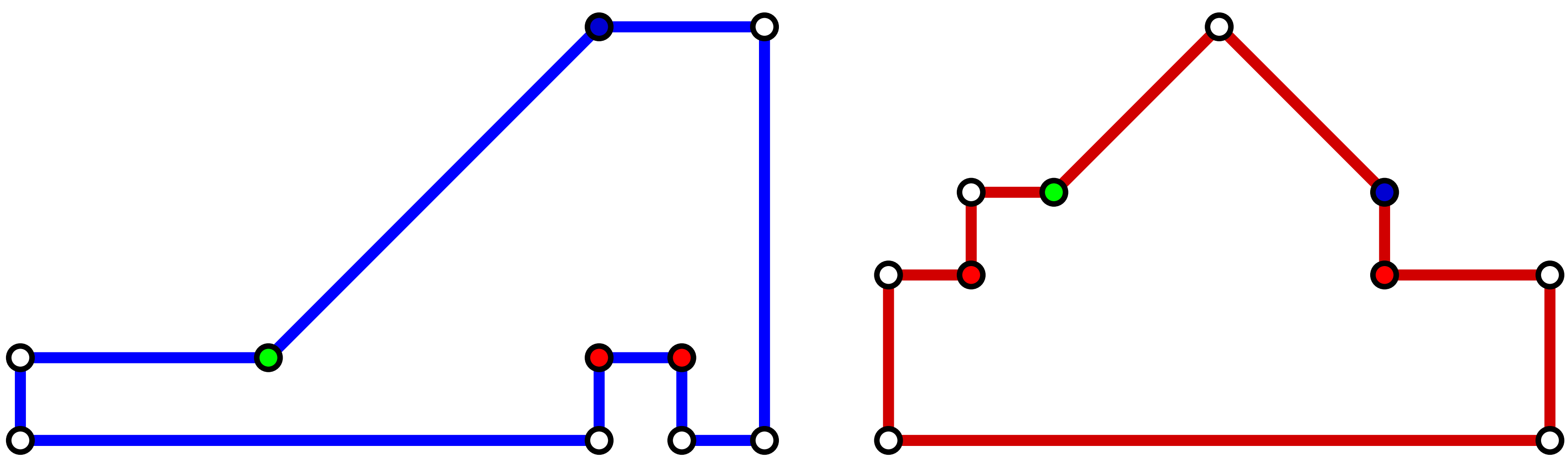
Equivalent: If P is not a 60°-polygon, then the angle bound $\Phi(P)$ is attained by some triangulation of P.

We proved earlier that $\Phi(P) = 60^\circ$ iff $P = 60^\circ$-polygon, and the bound is attained iff all side length ratios are rational.

Theorem: For $60^\circ < \phi < 90^\circ$, a polygon P has a ϕ-triangulation iff

1. $72^\circ \leq \phi < 90^\circ$ and P has a ϕ-labeling L of V_P,
2. $\frac{5}{7} \cdot 90^\circ \leq \phi < 72^\circ$, and P has a ϕ-labeling with $\kappa(L) \leq 0$,
3. $60^\circ < \phi < \frac{5}{7} \cdot 90^\circ$, and P has a ϕ-labeling with $\kappa(L) = 0$.

Cor: $\Phi(P)$ only depends on set of angles. Not order or edge lengths.



Theorem: For $60^\circ < \phi < 90^\circ$, a polygon P has a ϕ-triangulation iff

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3. $60^\circ < \phi < \frac{5}{7} \cdot 90^\circ$, and P has a ϕ-labeling with $\kappa(L) = 0$.

Gerver (1984) proved necessity when P only has ϕ-dissection.

Corollary: For $\phi > 60^\circ$, the following are equivalent:

(1) P has a ϕ-dissection.
(2) P has a ϕ-triangulation.

$\Rightarrow$ **Dissections and triangulations give same angle bound.**

Theorem: For $60^\circ < \phi < 90^\circ$, a polygon P has a ϕ-triangulation iff

1. $72^\circ \le \phi < 90^\circ$ and P has a ϕ-labeling L of V_P,
2. $\frac{5}{7} \cdot 90^\circ \le \phi < 72^\circ$, and P has a ϕ-labeling with $\kappa(L) \le 0$,
3. $60^\circ < \phi < \frac{5}{7} \cdot 90^\circ$, and P has a ϕ-labeling with $\kappa(L) = 0$.

Some geometric consequences of the proof:

Cor: If $\theta_{\min} \ge 36^\circ$, then $\Phi(P) \le 72^\circ$.

Cor: If $\theta_{\min} \le 36^\circ$, then $\Phi(P) = 90^\circ - \frac{1}{2}\theta_{\min}$.

Cor: $\Phi(P) = 72^\circ$ for any axis-parallel polygon.

Cor: If $\theta_{\min} \ge 144^\circ$ then $\Phi(P) = 72^\circ$. (This has interior!)

Theorem: For $60^\circ < \phi < 90^\circ$, a polygon P has a ϕ-triangulation iff

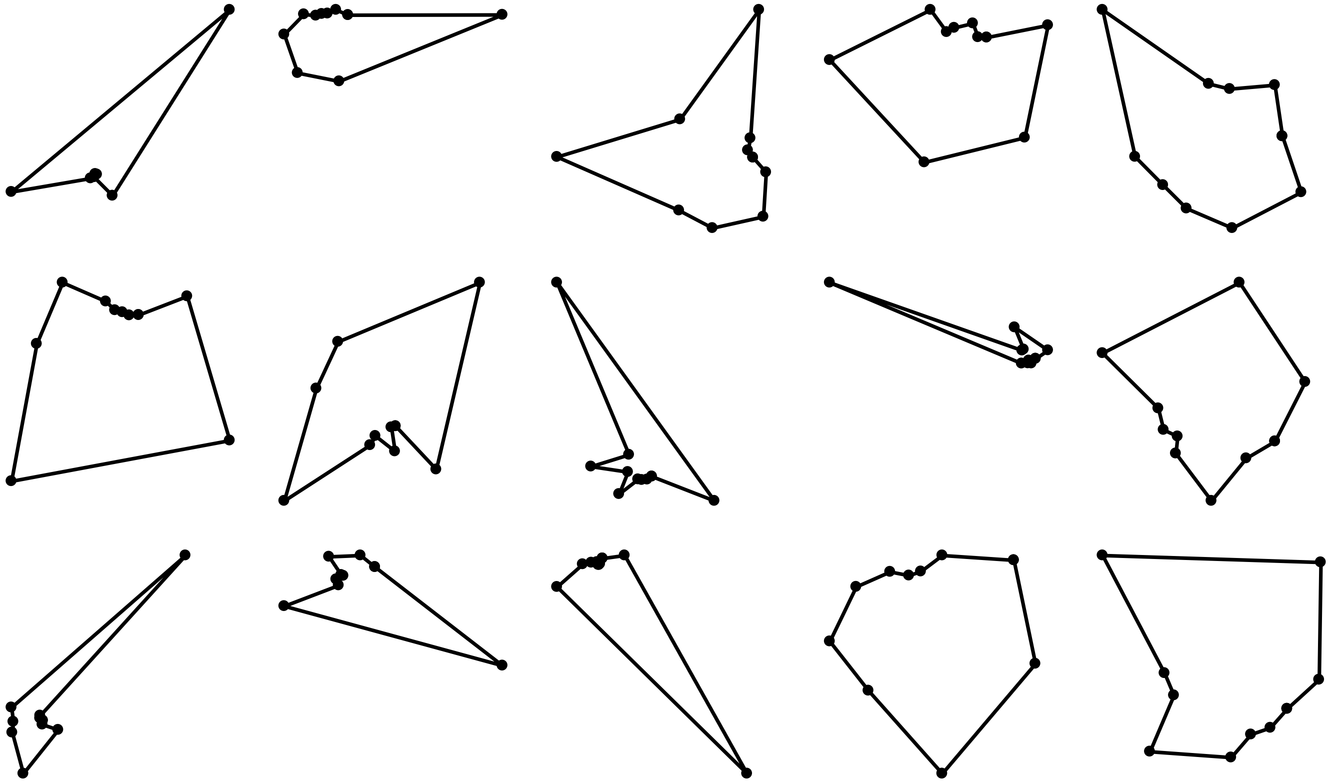
1. $72^\circ \leq \phi < 90^\circ$ and P has a ϕ-labeling L of V_P,
2. $\frac{5}{7} \cdot 90^\circ \leq \phi < 72^\circ$, and P has a ϕ-labeling with $\kappa(L) \leq 0$,
3. $60^\circ < \phi < \frac{5}{7} \cdot 90^\circ$, and P has a ϕ-labeling with $\kappa(L) = 0$.

Cor: Put a topology on n-gons by thinking of them as a subset of $\mathbb{R}^{2n}$.

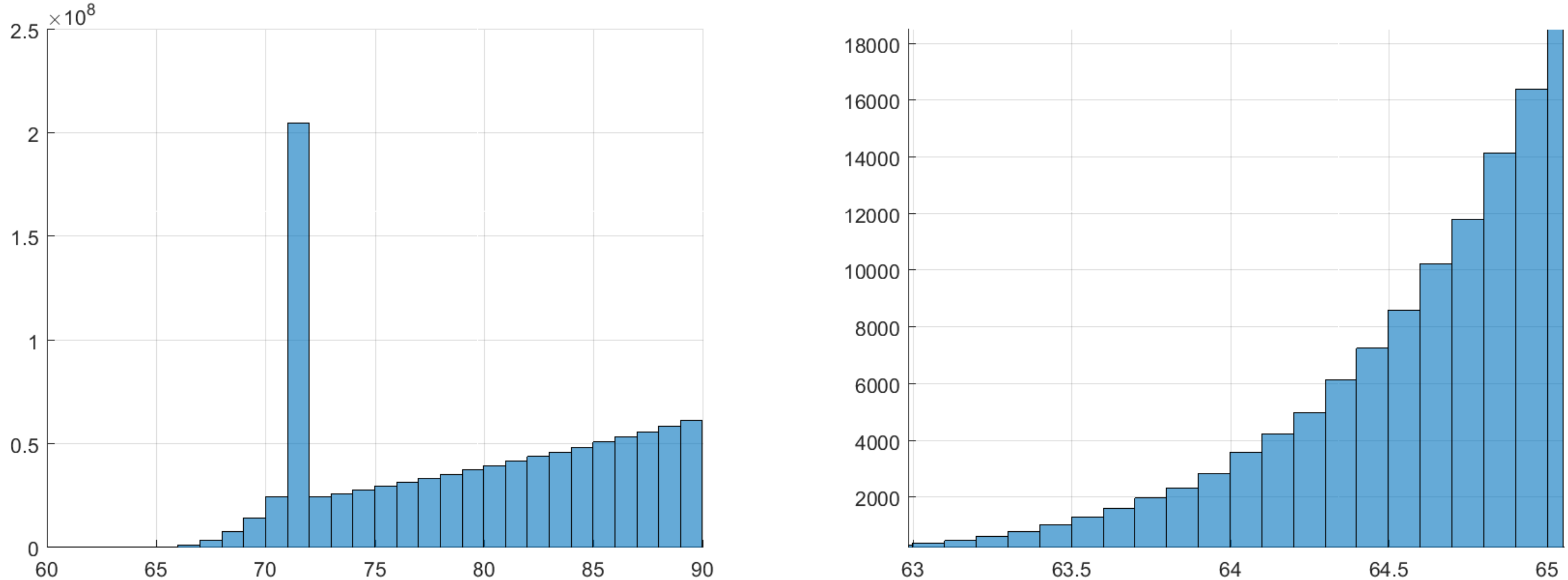
(a) The map $P \to \Phi(P)$ is continuous, so $\{P : \Phi(P) = \phi\}$ is closed.

(b) This set has non-empty interior iff $\phi = \frac{5}{7} \cdot 90^\circ$ or $\phi = 72^\circ$.

(c) Otherwise it has co-dimension ≥ 1.



Generating random 10-gons.

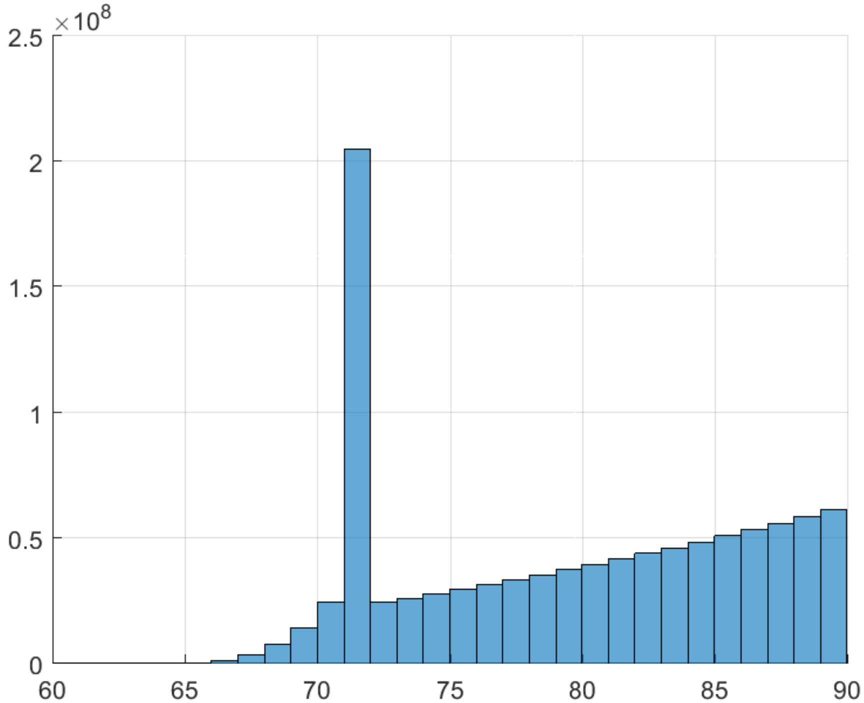


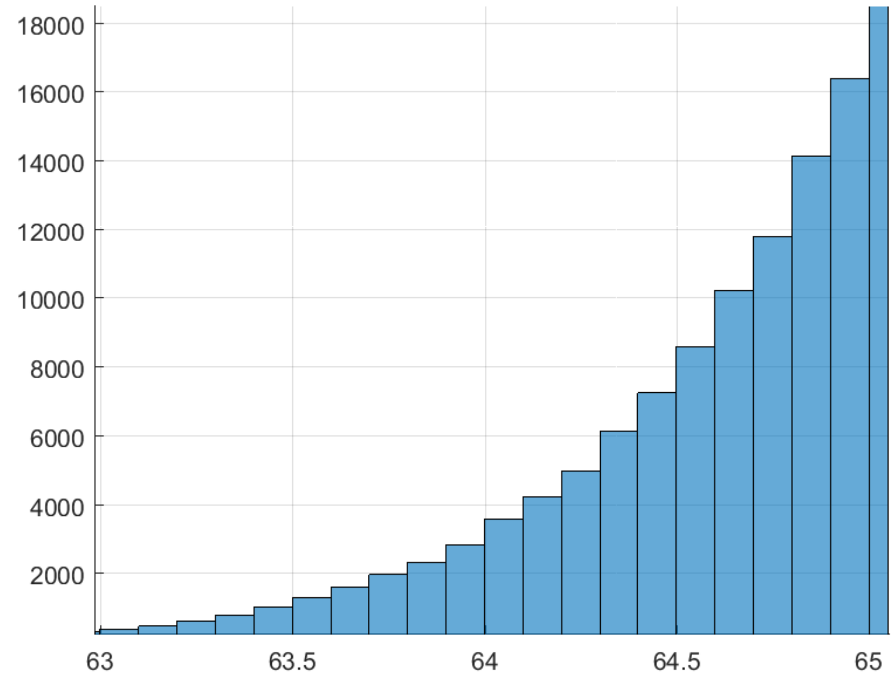
The distribution of optimal upper bounds over 10^9 random samples.

On the left is a histogram based on 1° bins. The spike a 72° is evident.

On the right is an enlargement near 64° using $.1^\circ$ bins.

No spike at $\frac{5}{7} \cdot 90^\circ \approx 64.26^\circ$ is visible.





In these experiments I just chose angles at random (with correct sum).

Didn't choose edge lengths or check for self-intersections.

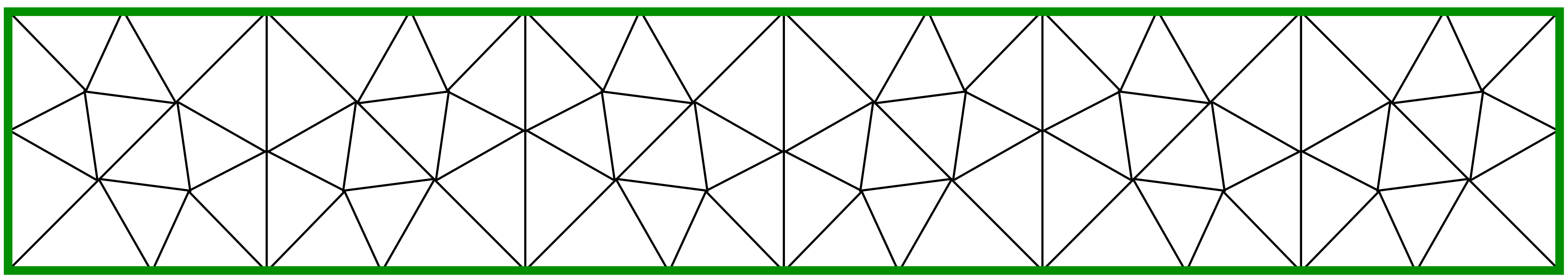
What is better model for random polygons?

Theorem: For $60^\circ < \phi < 90^\circ$, a polygon P has a ϕ-triangulation iff

1. $72^\circ \le \phi < 90^\circ$ and P has a ϕ-labeling L of V_P,
2. $\frac{5}{7} \cdot 90^\circ \le \phi < 72^\circ$, and P has a ϕ-labeling with $\kappa(L) \le 0$,
3. $60^\circ < \phi < \frac{5}{7} \cdot 90^\circ$, and P has a ϕ-labeling with $\kappa(L) = 0$.

Cor: For an N-gon $\Phi(P)$ can be computed in time $O(N)$.

However, $1 \times R$ rectangle needs $\gtrsim R$ triangles.



$\Rightarrow$ no bound for number of triangles in terms of N.

Sketch of $O(N)$ computation of $\Phi(P)$, $N =$ number of vertices:

Find $\theta_{\min}$ in time $O(N)$.

If $\theta_{\min} \leq 36^\circ$ then $\Phi(P) = 90^\circ - \theta_{\min}/2$. Done.

If $\theta_{\min} \geq 144^\circ$ then $\Phi(P) = 72^\circ$. Done.

If P is a 60° degree polygon then $\Phi(P) = 60^\circ$. Done.

Can now assume $36^\circ < \theta_{\min} < 144^\circ$ and $60^\circ < \Phi(P) \leq 72^\circ$.

After some work, computing $\Phi(P)$ reduces to finding two numbers:

$$\phi_\infty = \inf\{\phi : \exists\ \phi\text{-labeling }\},$$

$$\phi_0 = \inf\{\phi : \exists\ \phi\text{-labeling with } \kappa(L) = 0\}$$

Computing ϕ_∞ is easy: for v find the minimum ϕ so that either

$$\frac{\theta_v}{\phi} \quad \text{or} \quad \frac{\theta_v}{180 - 2\phi}$$

is an integer; this takes $O(1)$ work per vertex.

Then take maximum of these results ($O(N)$ work).

To compute ϕ_0 we rewrite it as

$$\phi_0 = \inf\{\phi : \min(f(\phi), 0) + \max(g(\phi), 0) = 0\}$$

where f, g are the monotone step functions:

$$f(\phi) = \sum_{v \in P} \inf\{k : 180 - 2\phi \leq \frac{\theta_v}{k} \leq \phi\} \quad \text{(decreasing)}$$

$$g(\phi) = \sum_{v \in P} \sup\{k : 180 - 2\phi \leq \frac{\theta_v}{k} \leq \phi\} \quad \text{(increasing)}$$

Note that $\phi_0 \in J = \{\ O(N) \text{ jump points of } f, g\ \}$.

The jump set J is known, but not sorted.

Need to evaluate these N-long sums at $O(N)$ values.

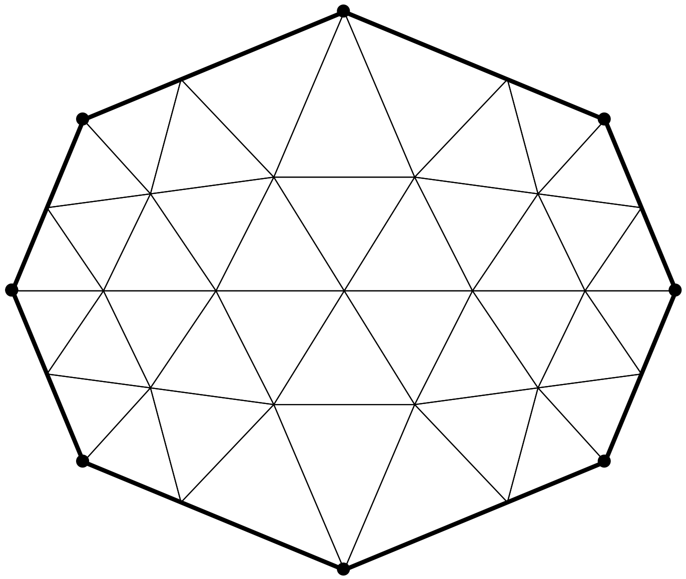
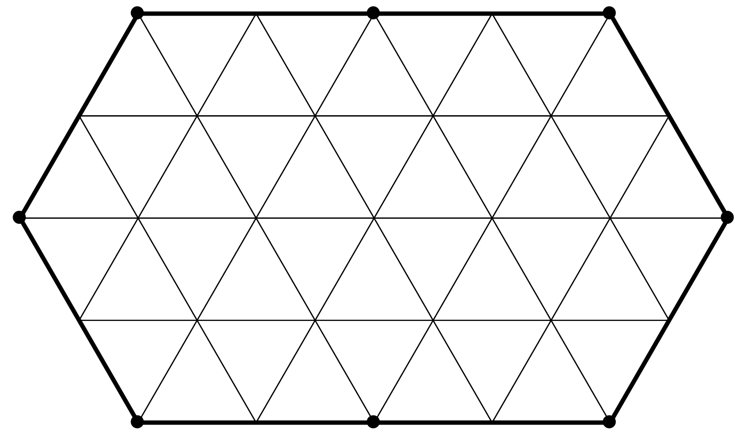
Does this require $O(N^2)$ work?

No, we can find $\phi_0 \in J$ in time $O(N)$ as follows:

- Find smallest, largest elements of J. Evaluate f, g.
- Find median of J by median-of-medians algorithm. Evaluate f, g.
- Decide if ϕ_0 is $\geq$ or $\leq$ median. Delete half of J.
- Repeat last two steps until ϕ_0 is found.
- Monotonicity implies new evaluations only use remaining points.

$\Rightarrow$ Work diminishes geometrically. Total is $O(N)$.

Idea behind proof of main theorem: conformal maps

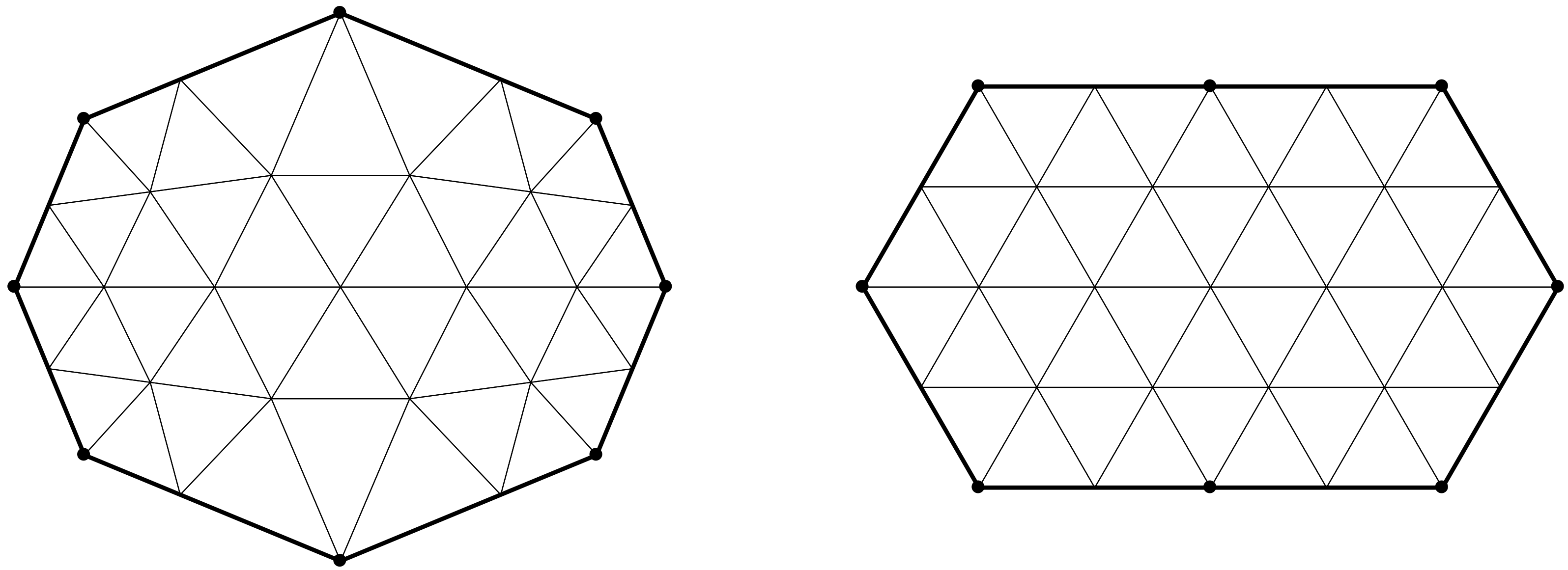
 

Given P with angles $\{\theta_k\}$, approximate by 60°-polygon P', angles $\{\psi_k\}$.

Rounding to nearest multiple of 60° often doesn't work: need

$$\sum \psi_k = (N-2) \cdot 180^\circ = \sum \theta_k.$$

Idea behind proof of main theorem: conformal maps

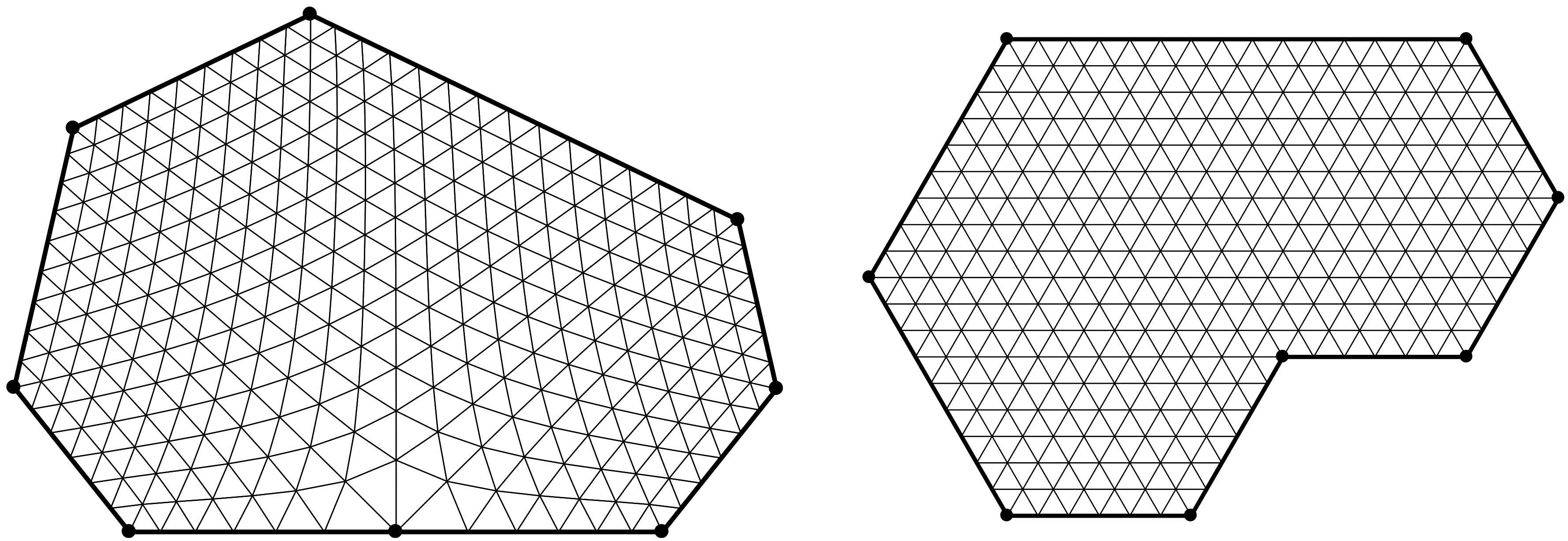


After angles $\{\psi_k\}$ are chosen, P' is built using Schwarz-Christoffel formula:

$$F(z) = A + C \int^z \prod_{k=1}^{n} (1 - \frac{w}{z_k})^{(\theta_k/\pi)-1} dw.$$

$F : \mathbb{D} \to P$ where $\{\theta_k\}$ = angles, $\{\mathbf{z}_k\}$ = vertex pre-images on unit circle.
P' has same z_k-parameters as P, new angles = $\{\psi_k\}$.

Idea behind main theorem: conformal maps



Map a (nearly) equilateral triangulation of P' to P.

Can prove worst angle distortion is at vertices $= \theta_k/\psi_k$.

Extra work needed to ensure angle bound attained, not just approximated.

But,...

This approach gives a triangulation with only degree 6 interior vertices.

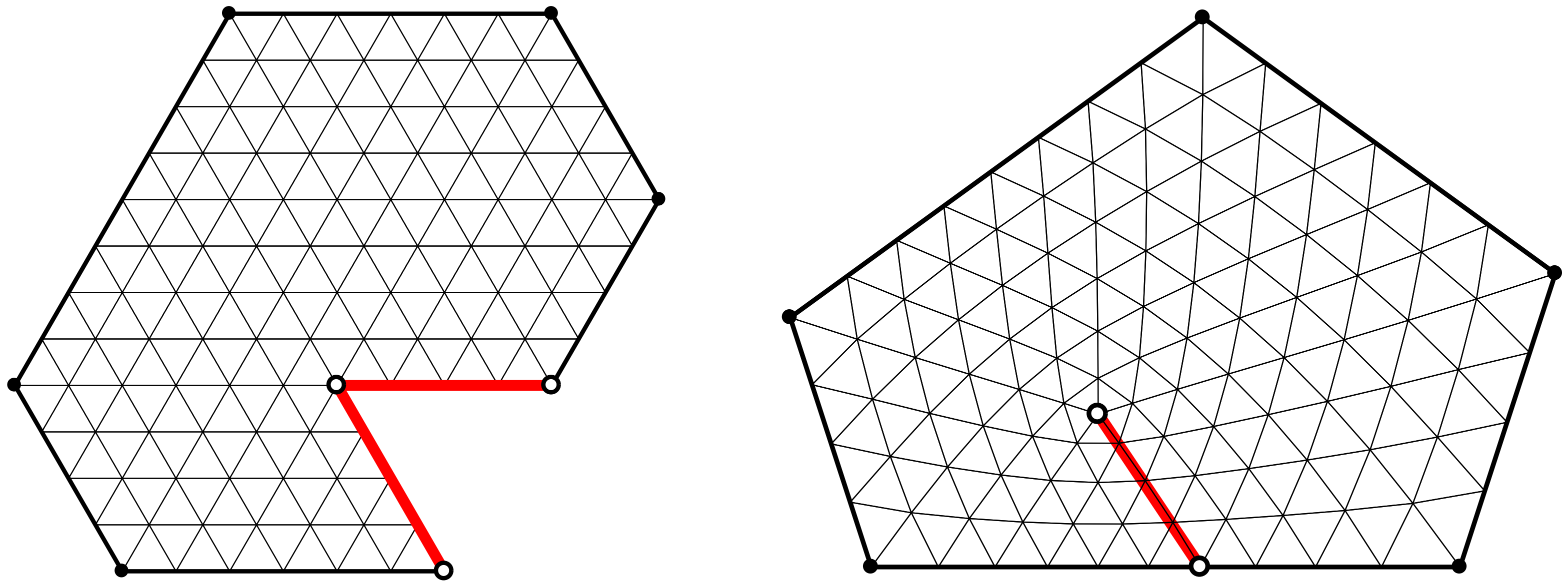
Only works when P has a zero curvature ϕ-labeling.

Let L minimize $|\kappa(L)|$ over ϕ-labelings of P.

If $\kappa(L) > 0$ there must be interior vertices of degree ≤ 5.

If $\kappa(L) < 0$ there must be interior vertices of degree ≥ 7.

Creating degree 5 vertices by folding:

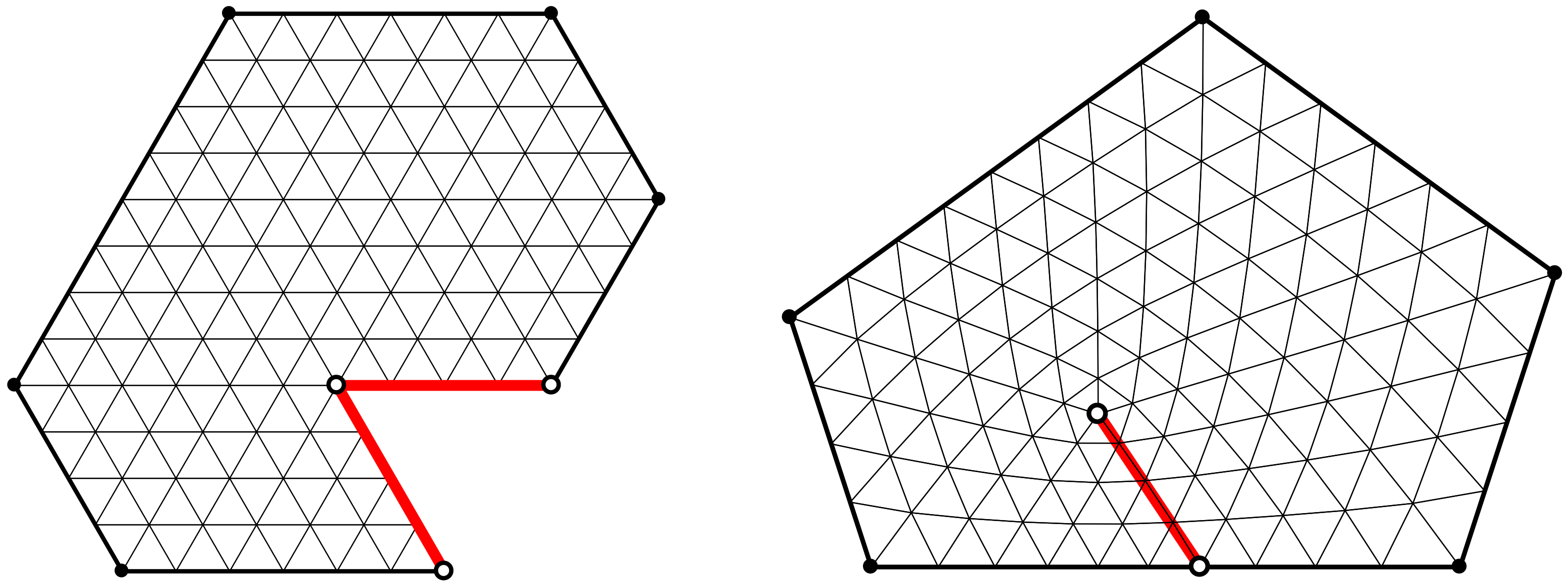


f maps P' to P with a slit removed; identifies boundary segments.

A degree 5 interior vertex is created.

But is this really a triangulation of P?

Creating degree 5 vertices by folding:

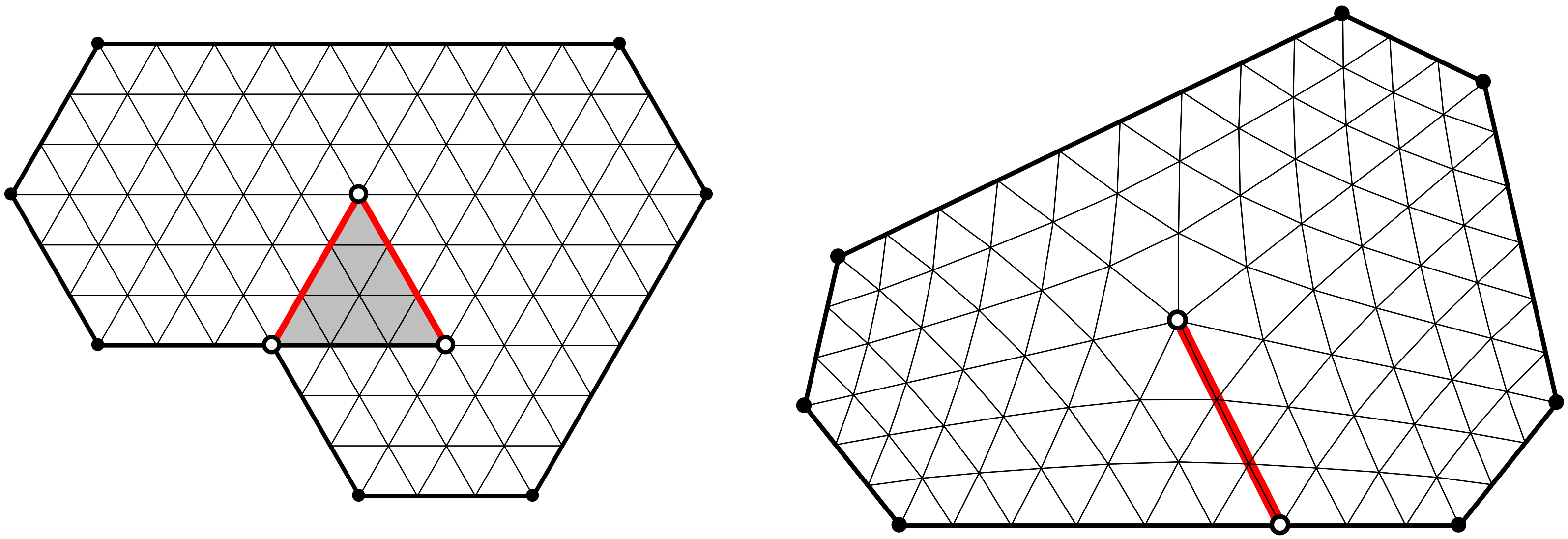


Technical difficulty: Image triangulations must match up across slit.

Matching occurs if $|f'(w)| = |f'(z)|$ whenever $f(w) = f(z)$.

Differential equation can be solved explicitly (= conformal welding).

Solution gives a curved slit (tangent changes by 3° above).



Creating a degree 7 vertex requires P' to be Riemann surface.

All non-zero curvature cases can be handled.

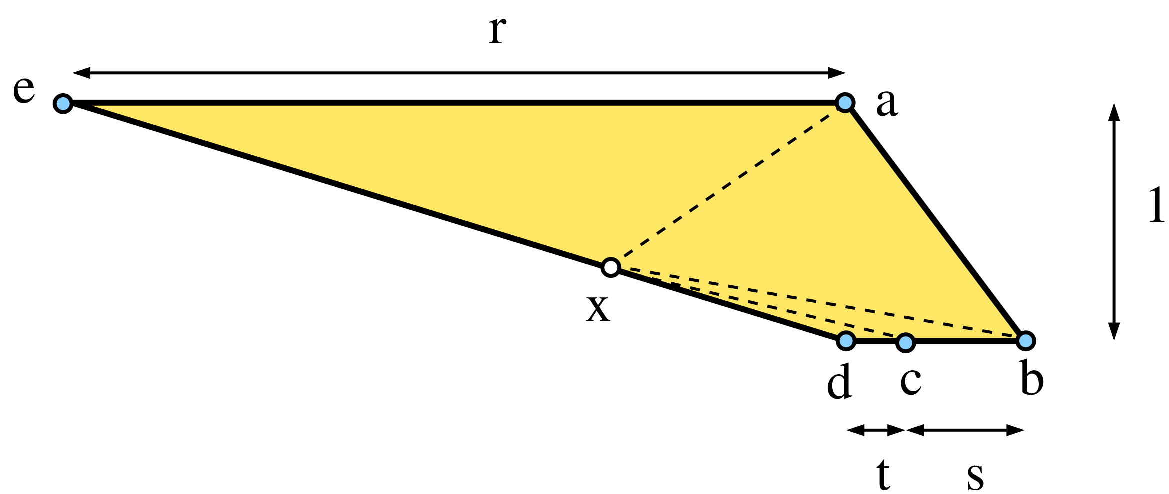
$\Rightarrow$ interior vertices are all degree six with $|\kappa(L)|$ exceptions

Open problems:

- Minimal weight Steiner triangulations.
- How large are optimal triangulations?
- Triangulations of PSLGs.
- Surfaces and solids.

We saw a triangulation achieving MinMax angle usually exists.

A **minimal weight Steiner triangulation** (MWST) minimizes total edge length. It need not exist ($t \ll s \ll 1 \ll r$):

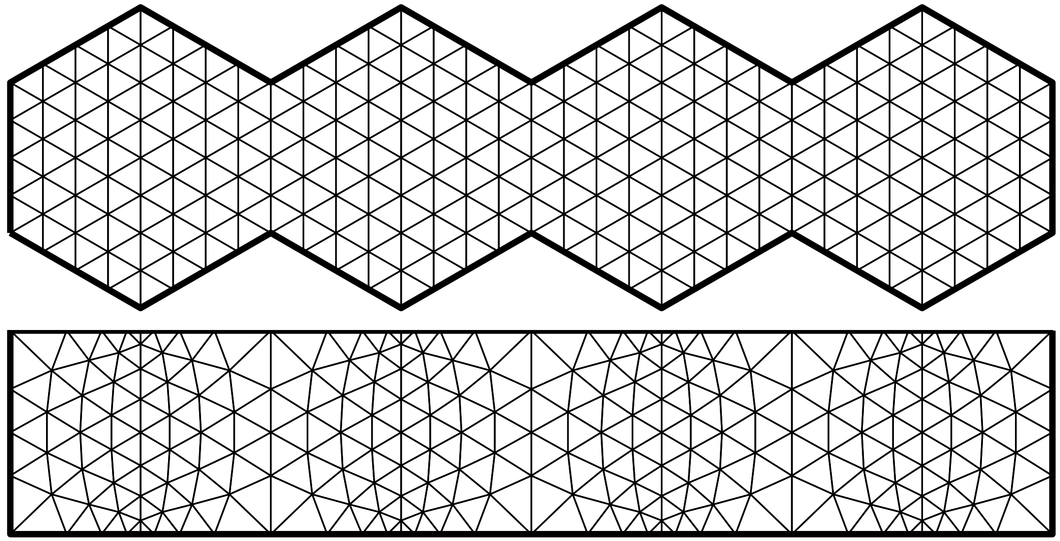


Question: Does a MWST exist for polygons in general position?

Without Steiner points, finding a MWST is NP-hard for point sets (Mulzer-Rote 2008) and $O(n^3)$ for polygons (Gilbert 1979, Klincsek 1980).

O(optimal) approximation of MWST is possible (Eppstein, 1994)

How many triangles does MaxMin solution need?



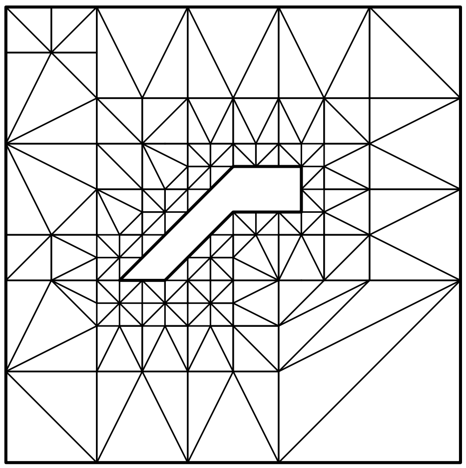
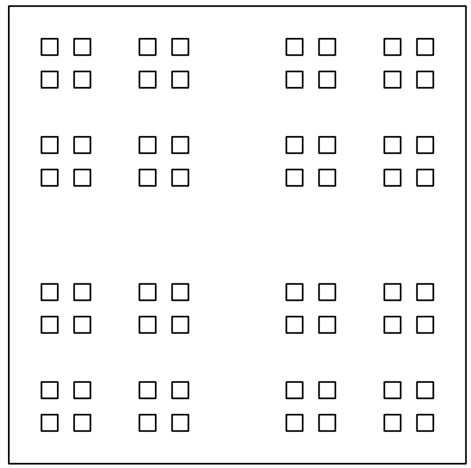
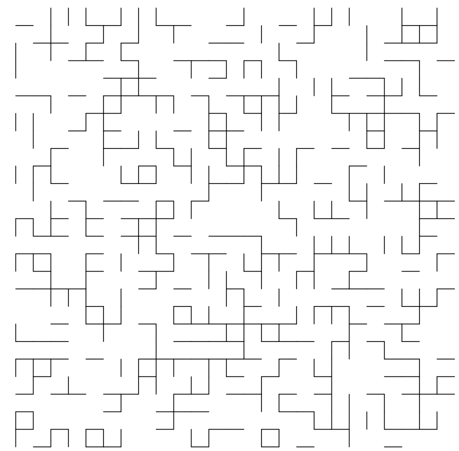
Proof of theorem gives exponentially many triangles for $1 \times R$ rectangle.

But good choice of 60°-polygon P' above gives $O(R)$ triangles.

Estimate smallest number of triangles needed for general P?

Is actual minimum needed NP-hard to compute?

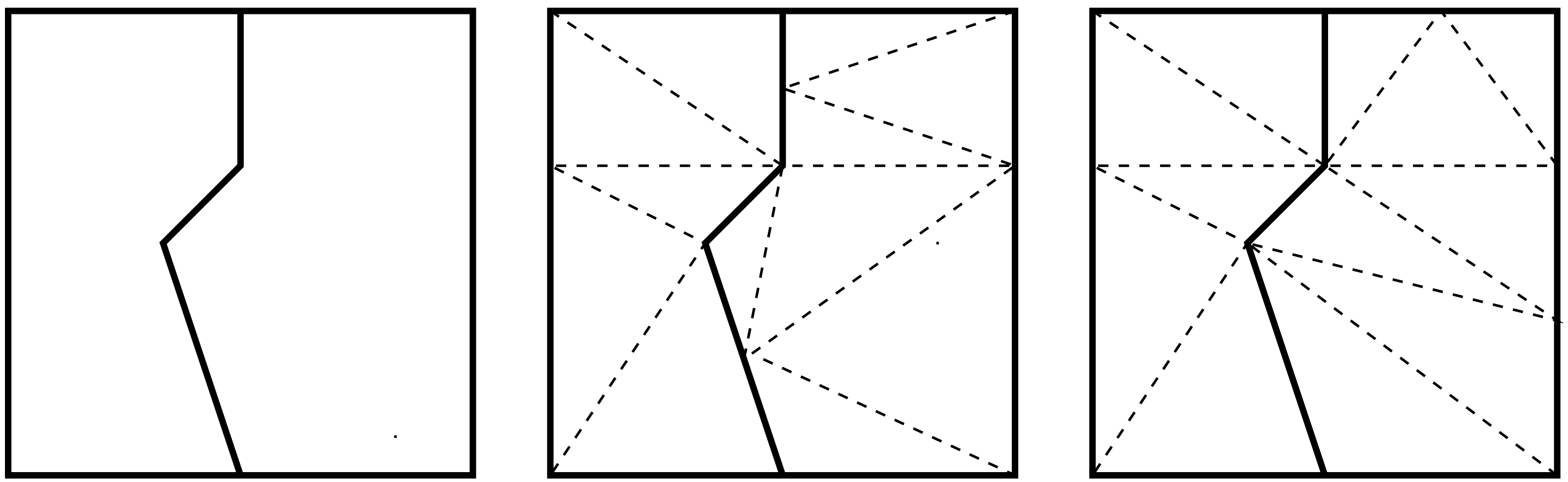
A **planar straight line graph** Γ (or **PSLG**) is finite union of points V and a collection of disjoint edges E with endpoints among these points.

Generally let $n = |V|$ be the number of vertices.

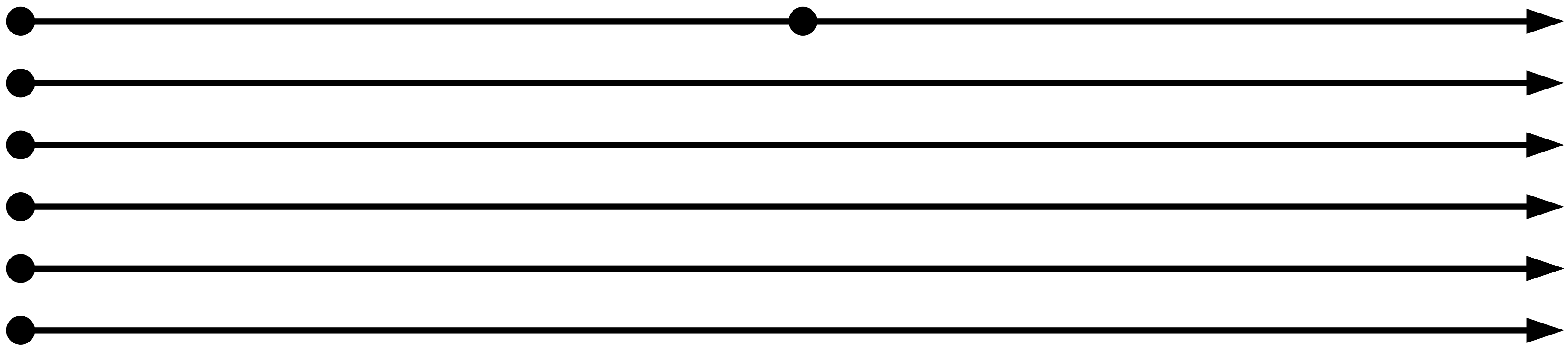
A simple polygon is a PSLG where edges form a closed cycle.

A **conforming triangulation** of a PSLG is a triangulation of each face, consistent across edges of the PSLG.

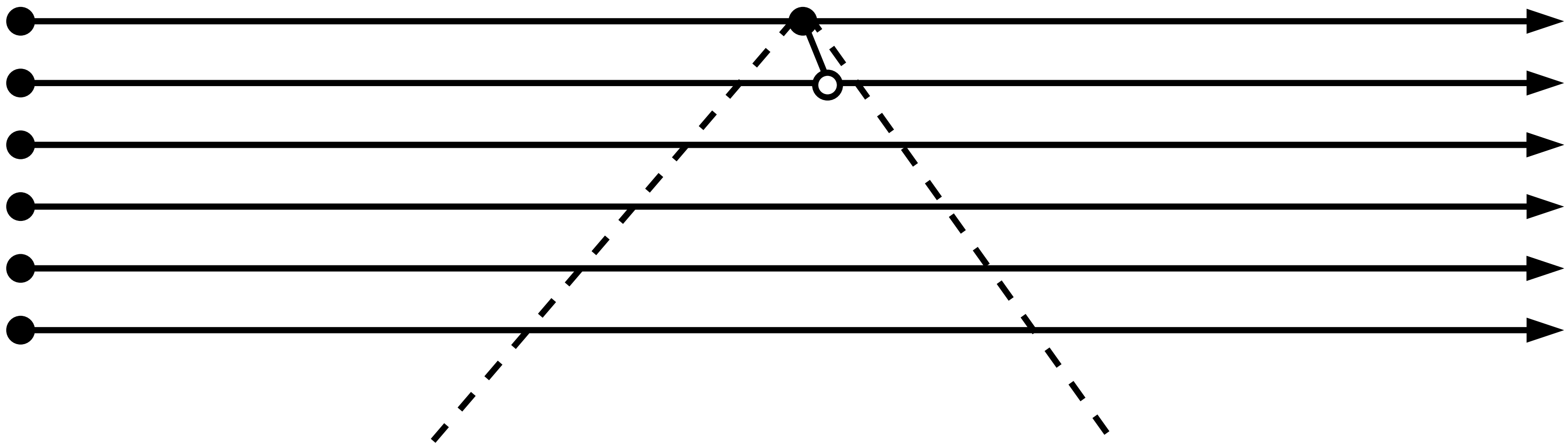


PSLG   Non-conforming   Conforming

NOT = Non-Obtuse Triangulation = all angles $\leq 90°$.



Consider a NOT for this PSLG (or any angle bound $< 180°$).



A edge must leave the vertex with the 90° wedge.

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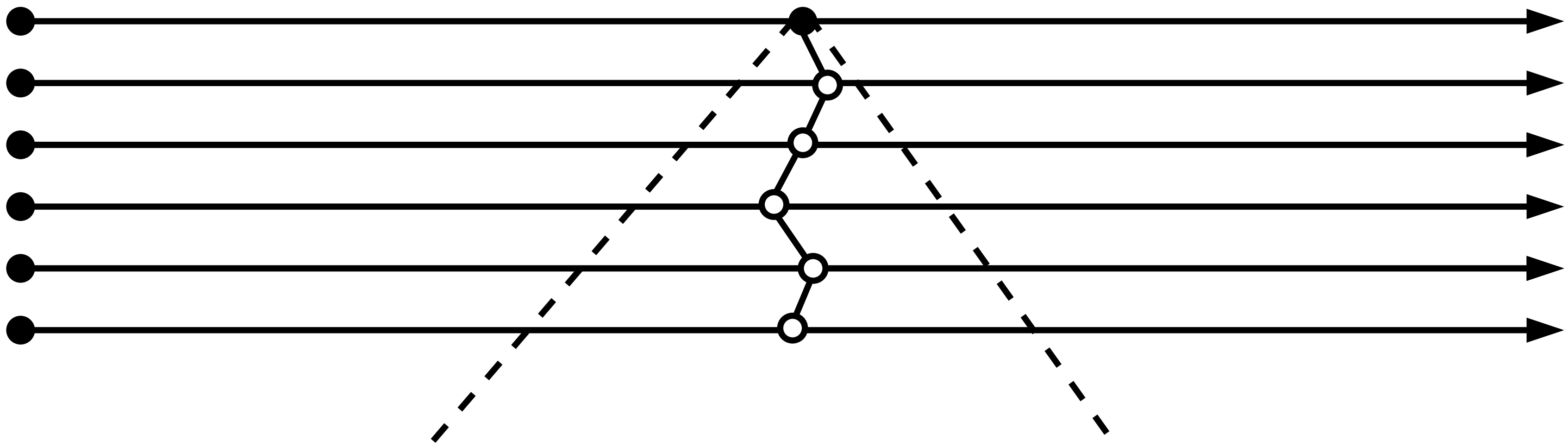

Iterating shows many new vertices, edges are needed.

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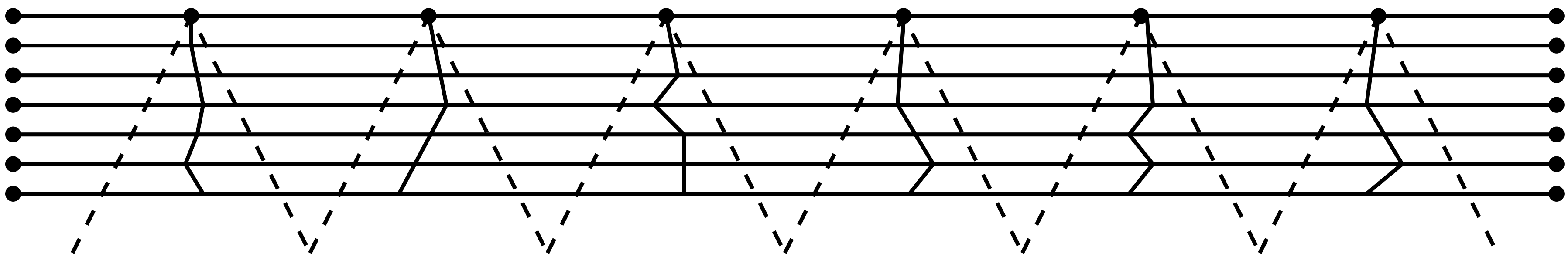

A NOT for this PSLG needs $\gtrsim n^2$ triangles.

Burago-Zalgaller, 1960: Every PSLG has an NOT (no size bound).

S. Mitchell, 1993: Every PSLG has a $157.5°$-triangulation, size $O(n^2)$.

Tan, 1996: Every PSLG has a $132°$-triangulation, size $O(n^2)$.

NOT-Thm (B. 2018): Every PSLG has a NOT with $O(n^{2.5})$ elements.

First polynomial bound for NOTs.

Improves $O(n^3)$ for Delaunay triangulation by Edelsbrunner, Tan (1993).

Problems for PSLGs Γ:

- **NOT Conj:** Every PSLG has a NOT with $O(n^2)$ elements.
- Compute $\Phi(\Gamma)$ = MinMax angle for conforming triangulation of Γ.
- When is minimum MinMax angle attained?
- Give bounds on $\Phi(\Gamma)$ in terms of minimum angle $\theta_{\min}$ in Γ.

Problems for PSLGs Γ:

- **NOT Conj:** Every PSLG has a NOT with $O(n^2)$ elements.
- Compute $\Phi(\Gamma)$ = MinMax angle for conforming triangulation of Γ.
- When is minimum MinMax angle attained?
- Give bounds on $\Phi(\Gamma)$ in terms of minimum angle $\theta_{\min}$ in Γ.

Best result so far: there is a $\theta_0 > 0$ so that

$$\Phi(\Gamma) \leq 90^\circ - \min(\theta_0, \theta_{\min})/2.$$

Uses compactness argument: θ_0 not explicit.

Main question in 3 dimensions:

Does every polyhedron have an acute triangulation of polynomial size?

triangulation = tetrahedralization with dihedral angles $< 90^\circ$.

Acute triangulation exists for unit cube $[0,1]^3$: 1370 tetrahedra.

No acute triangulation of cube in $\mathbb{R}^n$, $n \geq 4$.

See Kopczynski, Pak, Przytycki 2009 and VanderZee, Hirani, Zharnitsky, Guoy 2010.

Do polyhedral surfaces have polynomial sized acute triangulations?

Thanks for listening