DIMENSIONS OF JULIA SETS: FROM MANDELBROT TO MODELS

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Open University, Nov 19, 2024

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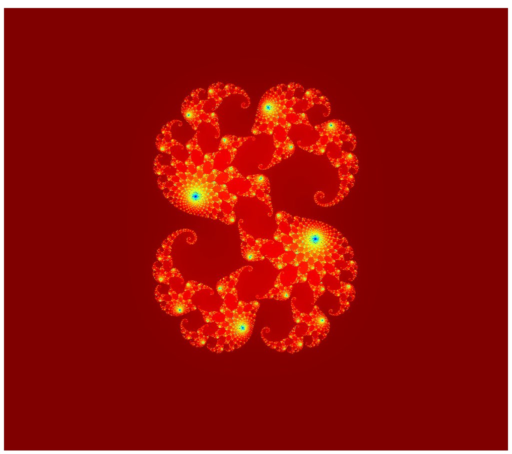
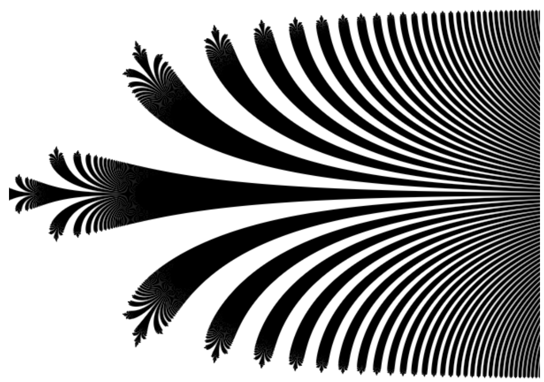
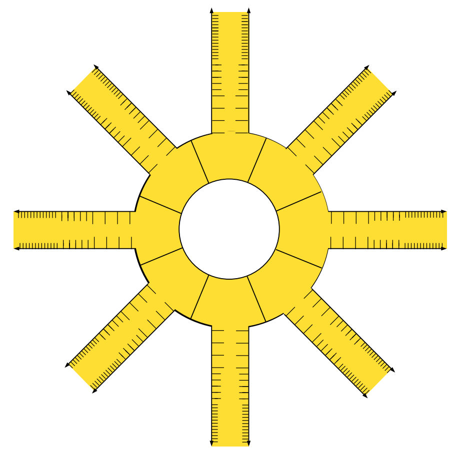






THE IDEA

Fractal dimensions measures how “large” a set is.

What are the possible dimensions of Julia sets?

For polynomials, all dimensions in $(0, 2]$ occur.

For non-polynomial entire functions, all values in $[1, 2]$ occur.

Hausdorff and packing dimension

Dimension = count number of small boxes needed to cover a set.

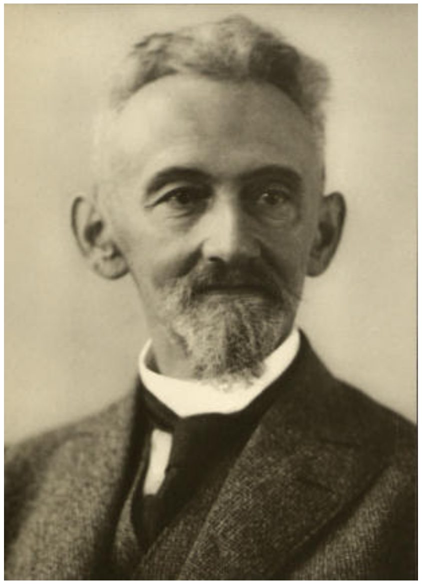
Packing dimension $\approx$ Minkowski dimension = covering by all ϵ-sized boxes

Hausdorff dimension = coverings using boxes $\leq \epsilon$ (more efficient)

Defn: α**-dimensional Hausdorff content**

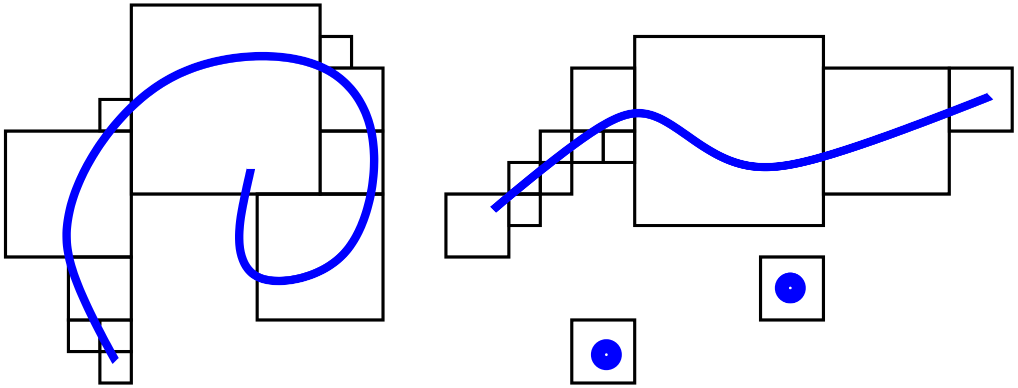
$$\mathcal{H}^{\alpha}_{\infty}(K) = \inf\{\sum_i |U_i|^{\alpha}\},$$

$\{U_i\}$ = cover of K, $|E|$ = diameter of a set E.
Decreasing function of α, $= 0$ for large α.



Felix Hausdorff

Defn: Hausdorff dimension $\dim(K) = \inf\{\alpha : \mathcal{H}^{\alpha}_{\infty}(K) = 0\}$.

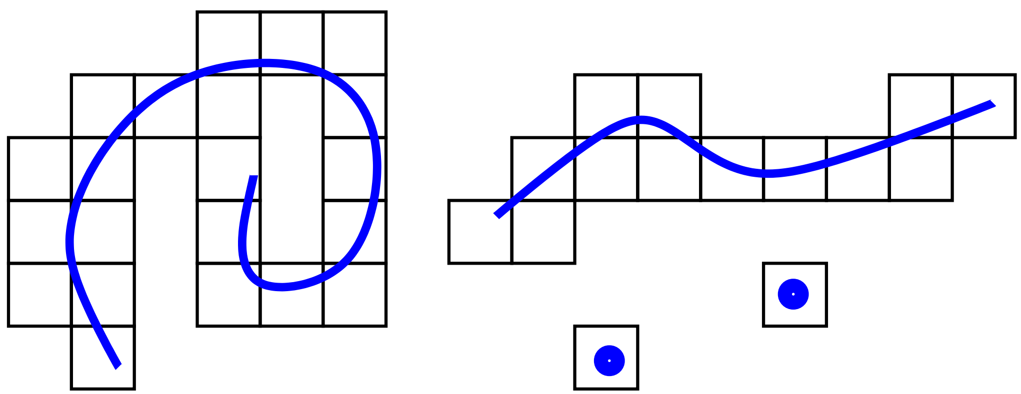


Defn: Minkowski dimension. If K is a bounded set, let $N(K,\epsilon)$ be number of ϵ-cubes needed to cover K.

We expect $N(K,\epsilon) \approx \epsilon^{-d}$, where $d = \dim$.

$$d = \log N(k,\epsilon)/\log(1/\epsilon).$$

Hermann Minkowski



Upper Minkowski dimension:

$$\overline{\mathrm{Mdim}}(K) = \limsup_{\epsilon \to 0} \frac{\log N(K, \epsilon)}{\log 1/\epsilon},$$

Two disadvantages:

- Not defined for unbounded sets
- Countable sets can have dimension > 0.

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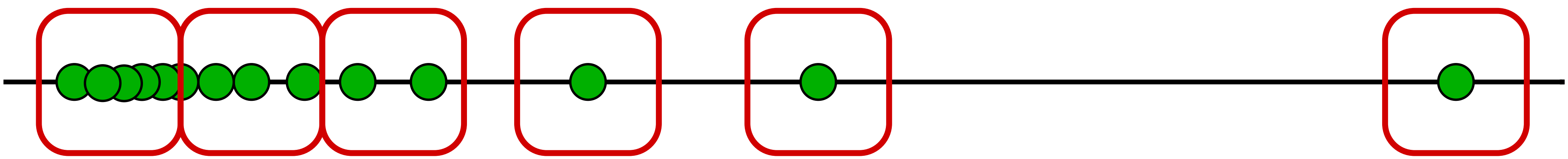

$\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$ needs $\sqrt{n}$ balls of size $1/n$ balls to cover.

This sequence has Minkowski dimension $= 1/2$.

Defn: packing dimension

$$\mathrm{Pdim}(A) = \inf \left\{ \sup_{j \geq 1} \overline{\mathrm{Mdim}}(A_j) : A \subset \bigcup_{j=1}^{\infty} A_j \right\},$$

where the infimum is over all countable covers of A.

Packing dimension $\leq d$ if it is a countable union of set of $\overline{\mathrm{Mdim}} \leq d$.

For today, we can think "Packing = Minkowski".

By definition, Hdim $\leq$ Pdim $\leq$ Mdim since

$$\mathrm{Hdim} \leq \underline{\mathrm{Mdim}} = \liminf_{\epsilon \to 0} \frac{\log N(K,\epsilon)}{\log 1/\epsilon} \leq \limsup_{\epsilon \to 0} \frac{\log N(K,\epsilon)}{\log 1/\epsilon} = \overline{\mathrm{Mdim}}$$

Equality holds for self-similar sets, but not in general.

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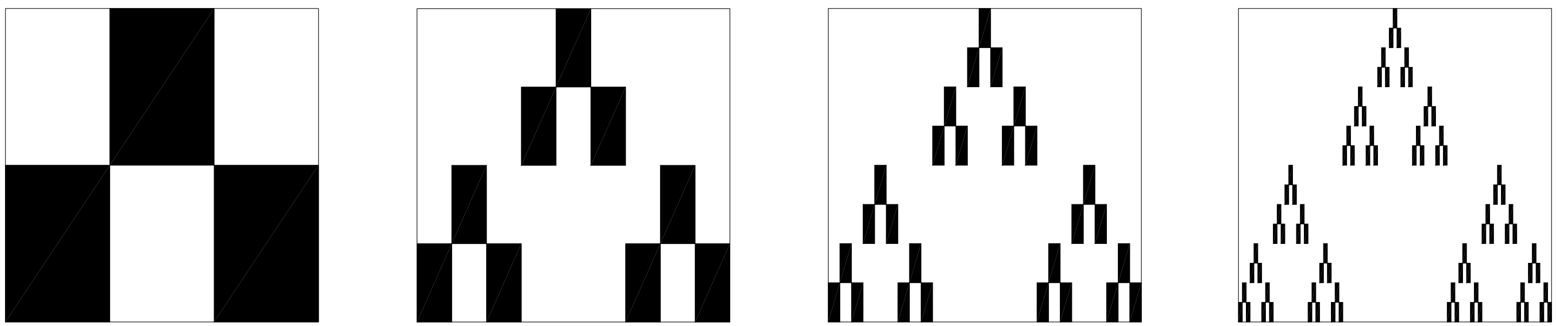
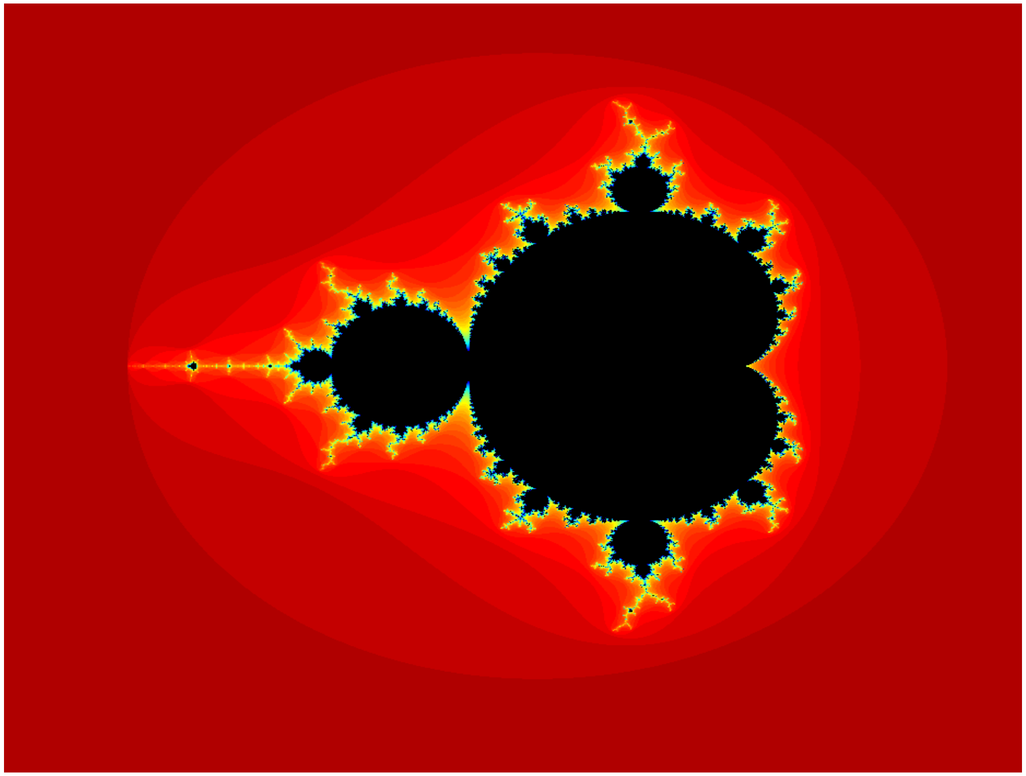


McMullen Set

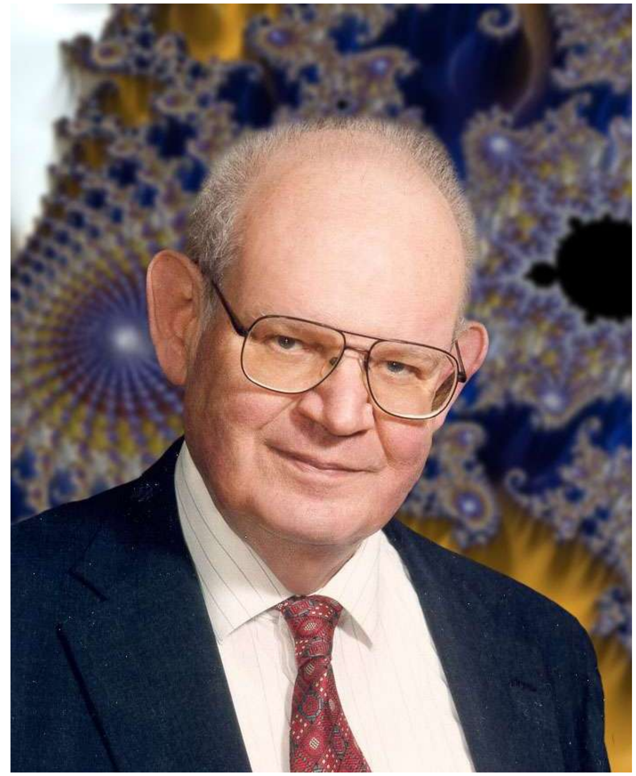
$$\mathrm{Hdim} = \log_2(2^{\log_3 2} + 1) \approx 1.34968$$

$$\mathrm{Pdim} = 1 + \log_3 \tfrac{3}{2} \approx 1.36907$$

Polynomial Julia sets



The Mandelbrot set



The Mandelbrot

The Brooks-Matelski set Robert Brooks

Who discovered the Mandelbrot set? Scientific American, 2009

Some terminology:

Entire $= f : \mathbb{C} \to \mathbb{C}$ is holomorphic on whole plane.

Polynomials $= a_0 + a_1 z + z^2 z^2 + \dots a_n z^n$.

Transcendental = not polynomial.

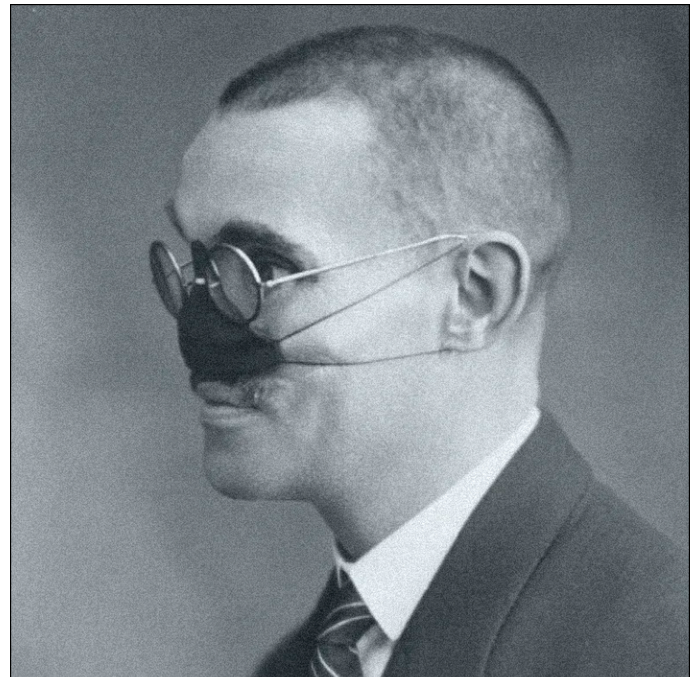
The iterates $\{f^n\}$ are **normal** on a disk D if every subsequence has a subsequence converging uniformly on D to a holomorphic limit (or $\equiv \infty$).

E.g., $\{z^2, z^4, z^8, \dots\}$ are normal on $\{|z| < 1\}$ and $\{|z| > 1\}$.

They are not normal on unit circle $\{|z| = 1\}$.

Fatou set = union of disks where iterates form a normal family

Julia set = complement of Fatou set = closure of repelling fixed points.



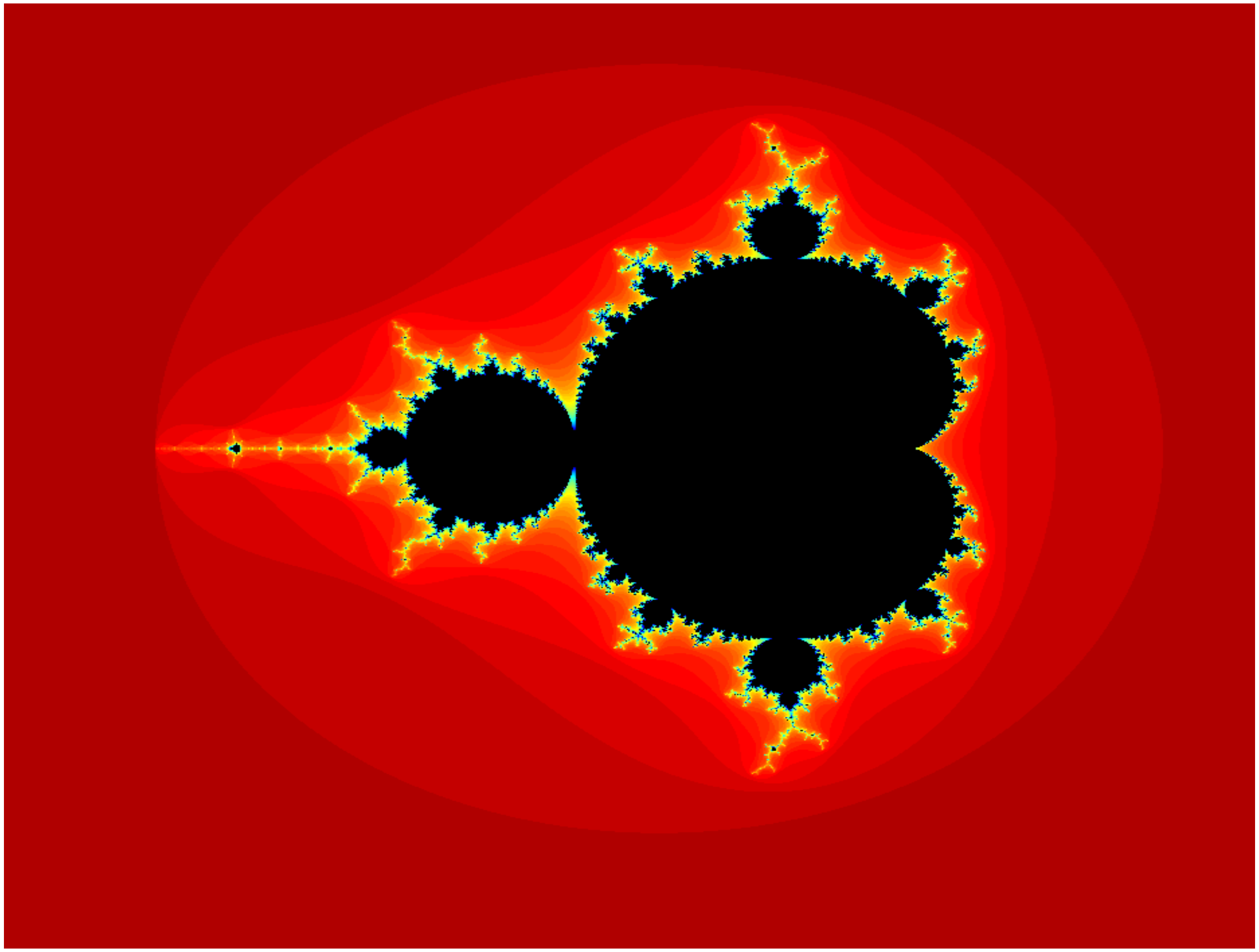
Pierre Fatou

Gaston Julia

Not to be confused with

LeFou

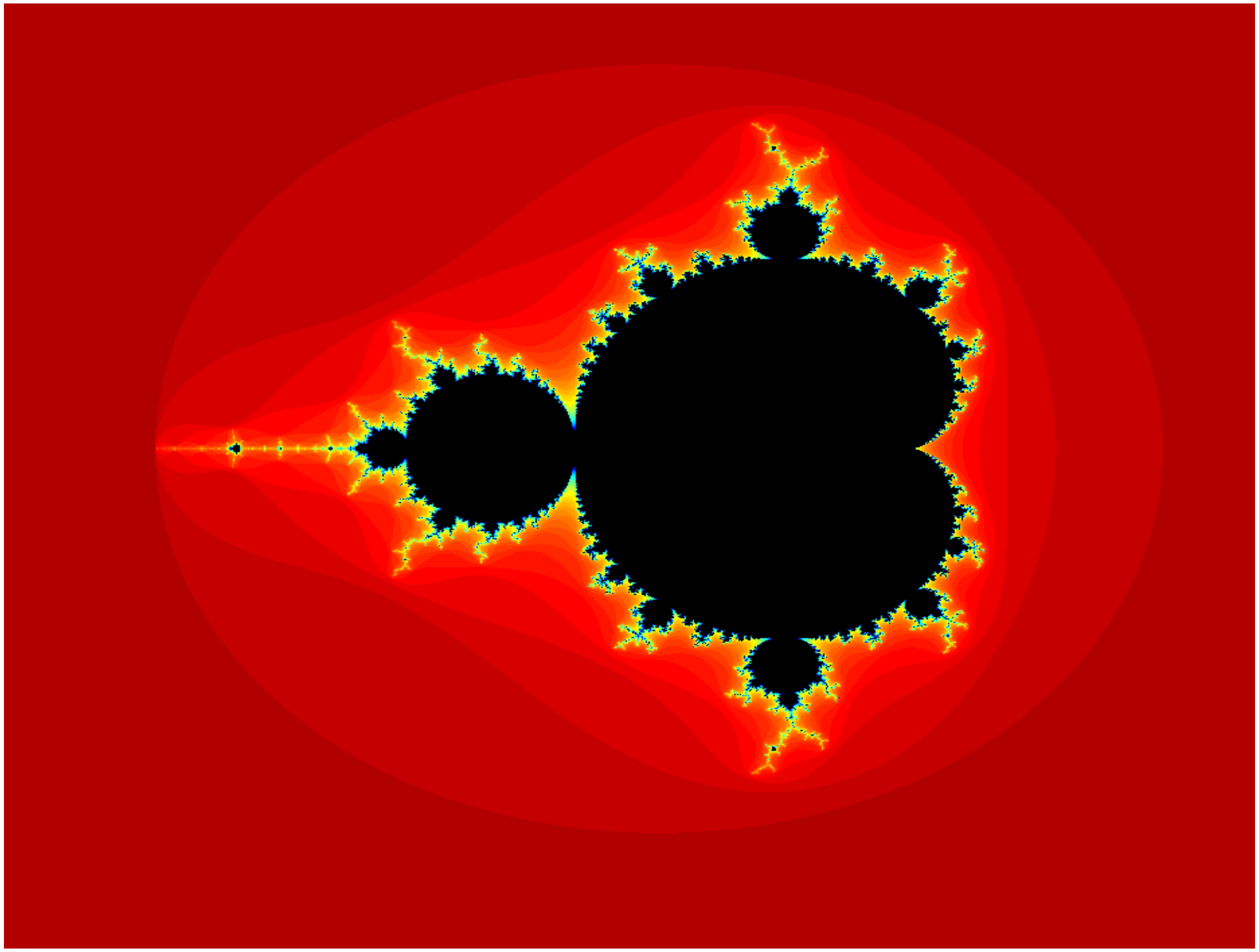
Gaston



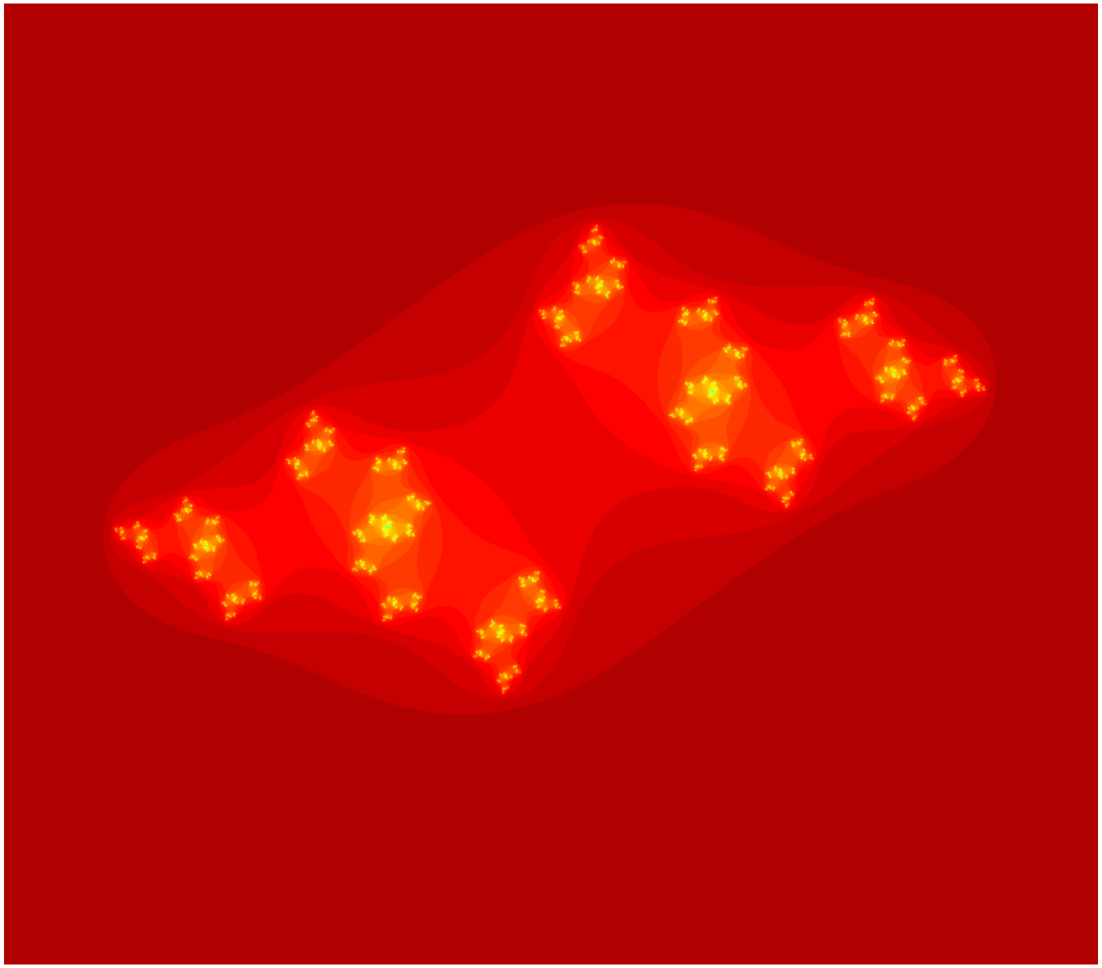
Mandelbrot set

= the set of c so that Julia set of $z^2 + c$ is connected.

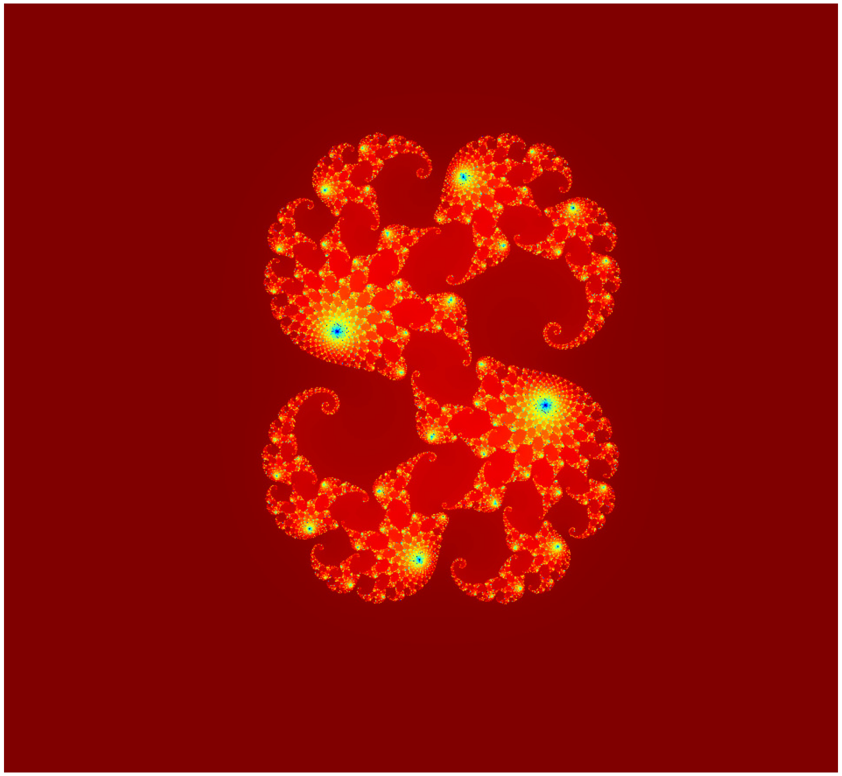
= the set of c so that critical orbit is bounded.



Outside Mandelbrot set, Packing dim = Hausdorff dim.
Both dimensions vary continuously.
$\mathrm{Dim}(\mathcal{J}_c) \to 0$ as $|c| \to \infty$.



Julia set with dimension near zero.



Shishikura (1994): $\mathrm{Dim}(\mathcal{J}_c) = 2$ for "most" $c \in \partial\mathcal{M}$.

$\mathrm{Dim}(\mathcal{J}_c) \to 2$ as $c \to \partial\mathcal{M}$ (most points).

Corollary: quadratic Julia sets take all dimensions in (0,2].

Xavier Buff and Arnaud Cheritat (2005) found quadratic examples with positive area.

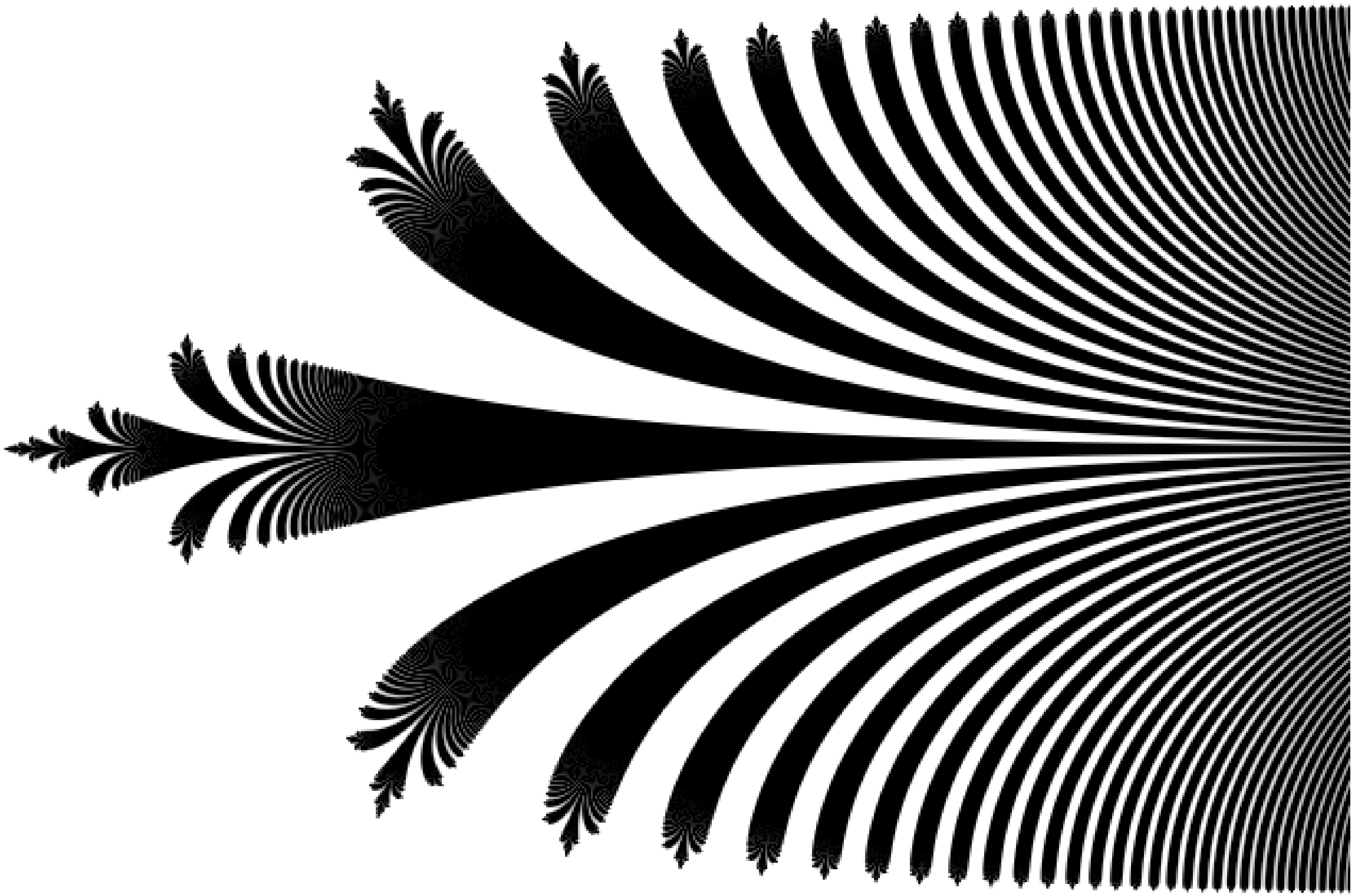
Hinkannen, '94: for polynomials, $\dim(\mathcal{J}) > 0$.

Thus $(0, 2]$ is the set of possible dimensions for polynomial Julia sets.

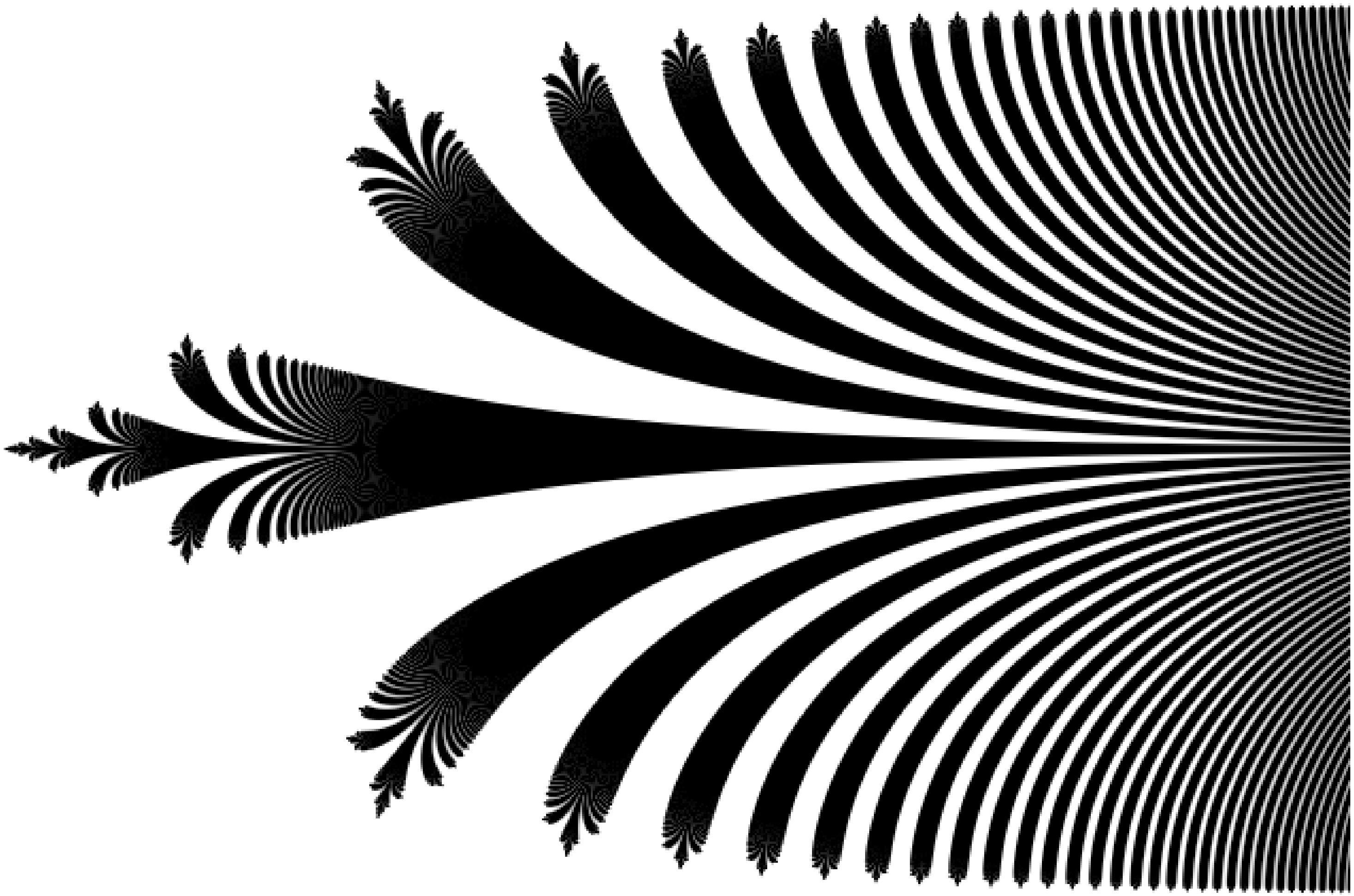
In examples discussed so far, Hausdorff = Packing.

Can they ever differ for polynomial Julia sets?

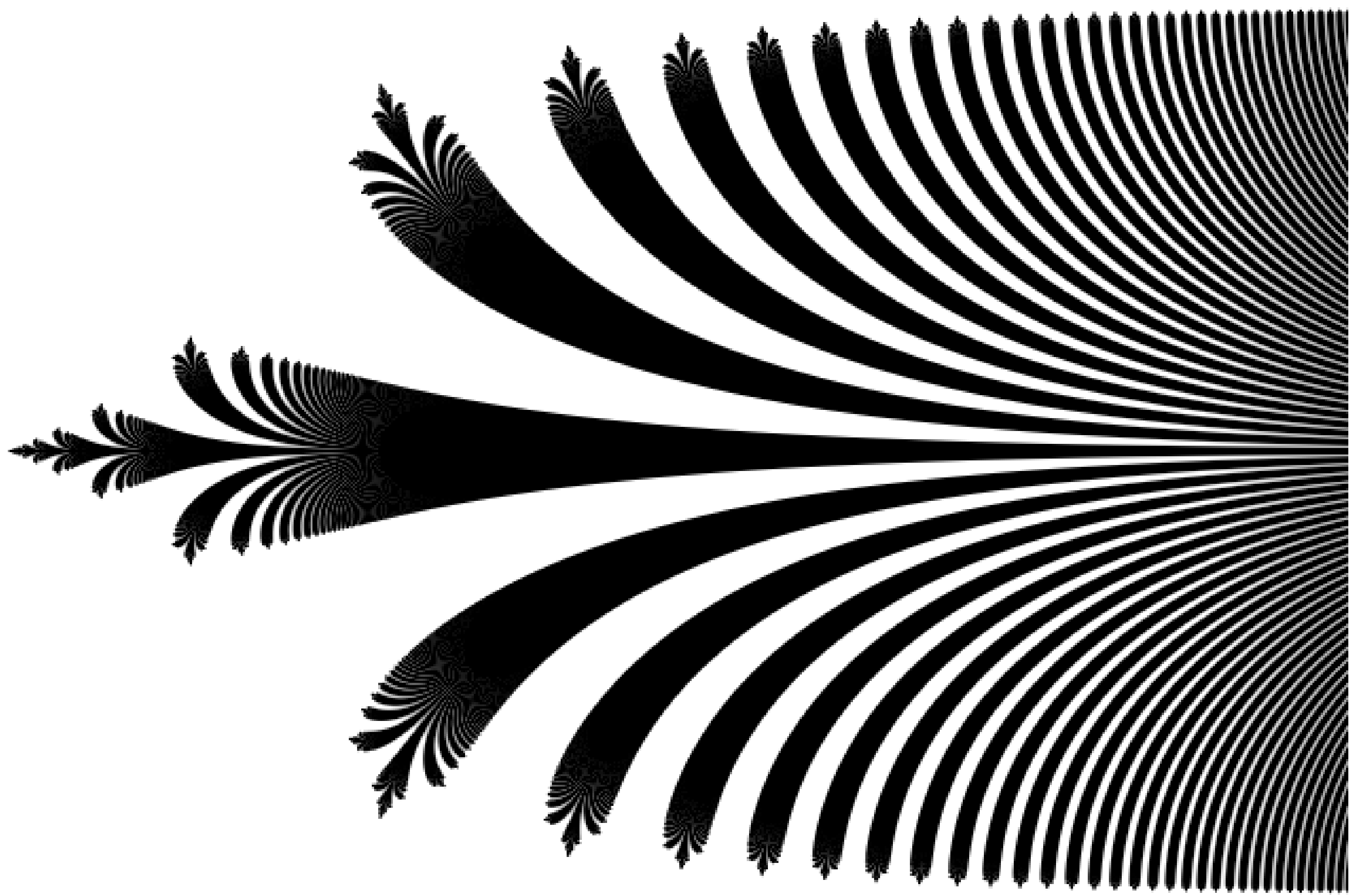
Transcendental Julia sets



Julia set of $(e^z - 1)/2$ by Arnaud Chéritat



Transcendental Julia sets are closed, non-empty, unbounded.



Julia set = union of paths to infinity. How typical is this?

Thm (Baker, '75): A transcendental Julia set can't be totally disconnected.

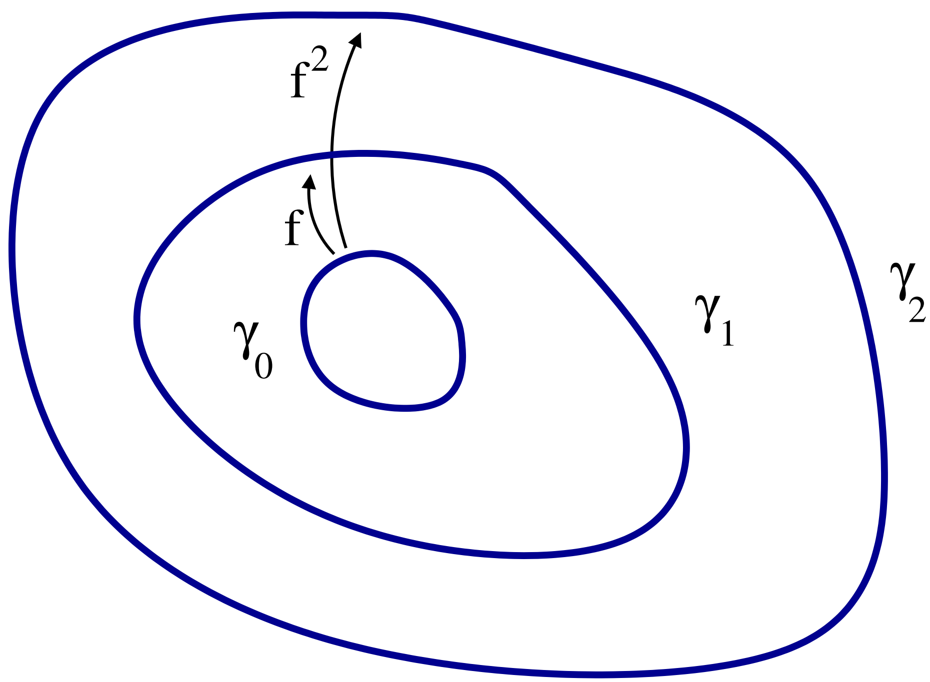
Cor: Every transcendental Julia set contains a non-trivial continuum.

Cor: Hdim ≥ 1 for all transcendental entire functions.

I.N. Baker

Sketch of Proof: Let $\mathcal{F}$ = Fatou set, $\mathcal{J}$ = Julia set.

1. Assume $\mathcal{F}$ is connected, unbounded, multiply connected.
2. Choose a loop $\gamma_0 \subset \mathcal{F}$ that surrounds a point of $\mathcal{J}$.
3. Fact: iterates $\gamma_n = f^n(\gamma_0)$ tend to infinity and "cover plane".



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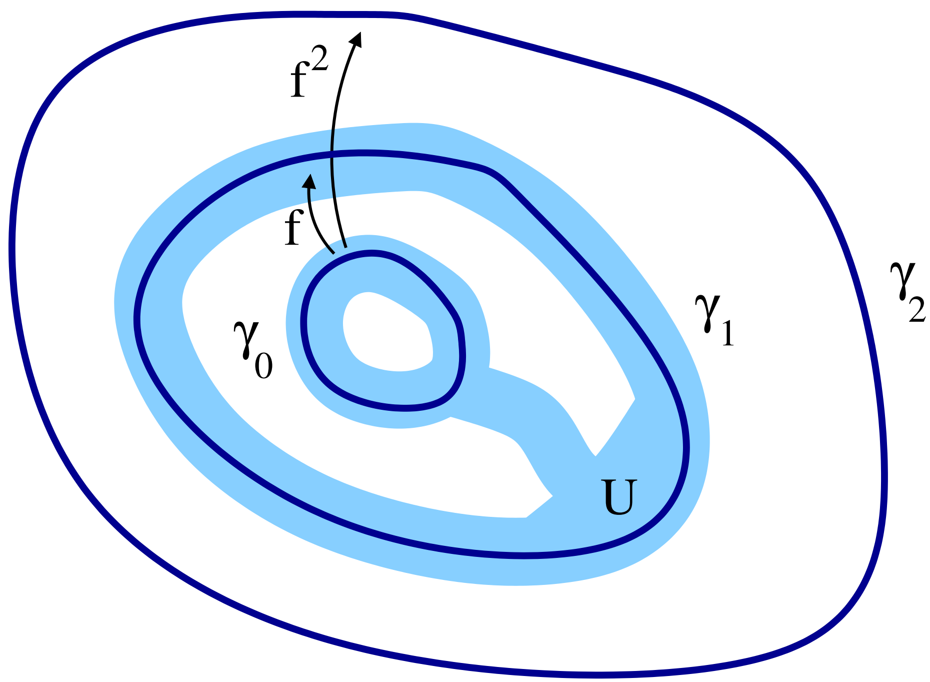
4. Choose connected open neighborhood U of $\gamma_0 \cup \gamma_1$.



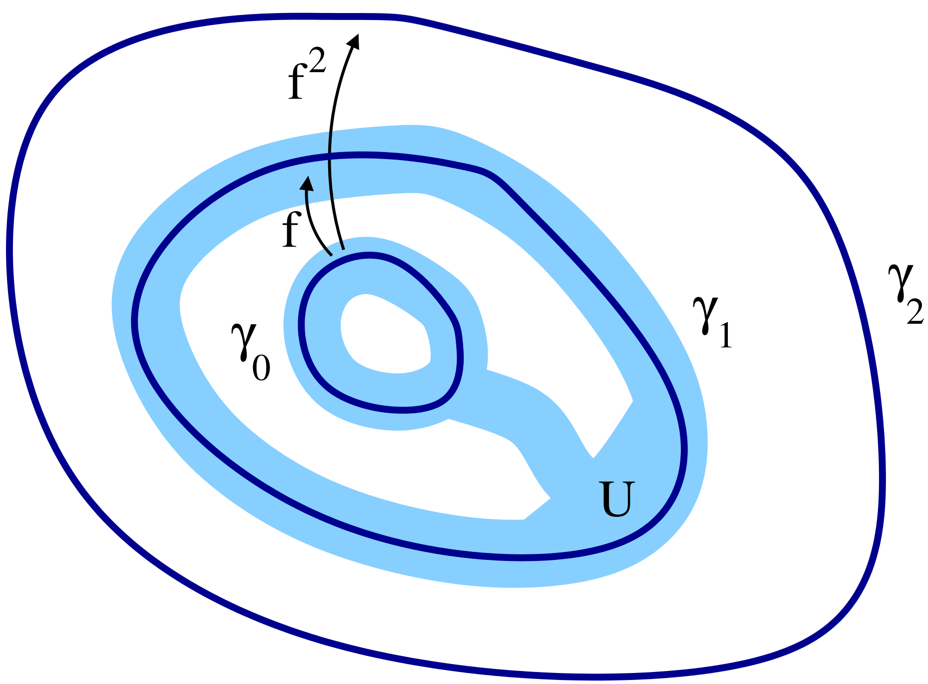
Sketch of Proof: Let $\mathcal{F}$ = Fatou set, $\mathcal{J}$ = Julia set.

5. $\log|f^n|$ is positive, harmonic on U. By Harnack's inequality

$$\sup_{\gamma_n} \log|f| = \sup_{\gamma_1} \log|f^n| \leq C \cdot \inf_{\gamma_0} \log|f^n| = C \cdot \inf_{\gamma_n} \log|z|$$

$$\Rightarrow \quad |f(z)| \leq |z|^C \text{ for all } z \in \gamma_n$$

6. This implies f is a polynomial.



Fact 3, part 1: Non-trivial loops escape

- Suppose curve γ in Fatou set surrounds a point of $\mathcal{J}$.
- If $\{f^n\}$ bounded on γ, also bounded on interior by max principle.
- Hence interior of γ in Fatou set, a contradiction.
- So a point in γ escapes. By normality all γ escapes.

Fact 3, part 2: Iterates of γ have non-zero index around 0

- Suppose not.
- Then minimum principle applies and interior of γ escapes.
- But γ surrounds $\mathcal{J}$ and hence surrounds a pre-periodic point.
- Contradiction.
- $\Rightarrow$ iterates of γ surround every compact set.

So for transcendental functions $\mathrm{Hdim}(\mathcal{J}) \in [1, 2]$.

Do all these values occur? Main cases are:

- $\mathrm{Hdim}(\mathcal{J}) = 2$
- $1 < \mathrm{Hdim}(\mathcal{J}) < 2$
- $\mathrm{Hdim}(\mathcal{J}) = 1$

Many transcendental functions have Hdim = Pdim = 2:

Misiurewicz, '81: $\mathrm{Hdim}(\mathcal{J}) = \mathbb{C}$ for $f(z) = e^z$.

McMullen, '87: $\mathrm{Hdim}(\mathcal{J}) = 2$, $\mathrm{area}(\mathcal{J}) = 0$ for $f(z) = (.3)e^z$.

McMullen, '87: $\mathrm{area}(\mathcal{J}) > 0$ for $f(z) = (.7)\sinh(z)$.

Michał Misiurewicz

Curt McMullen

Much harder to get $\mathrm{Hdim}(\mathcal{J}) < 2$:

Thm (Stallard, '97): All dimensions $1 < \mathrm{Hdim} < 2$ occur.

Her examples are in the Eremenko-Lyubich class (EL defined later).

Thm (Stallard, '96): For all $f \in \mathrm{EL}$ we have $\mathrm{Hdim}(\mathcal{J}) > 1$.

Gwyneth Stallard

Thm (Rippon-Stallard, '05): For $f \in \mathrm{EL}$ $\mathrm{Pdim}(\mathcal{J}) = 2$.

Cor: There are transcendental Julia sets with $1 < \mathrm{Hdim} < \mathrm{Pdim} = 2$.

Thm (Rippon-Stallard, '05): For $f \in$ EL Pdim$(\mathcal{J}) = 2$.

Cor: There are transcendental Julia sets with $1 <$ Hdim $<$ Pdim $= 2$.

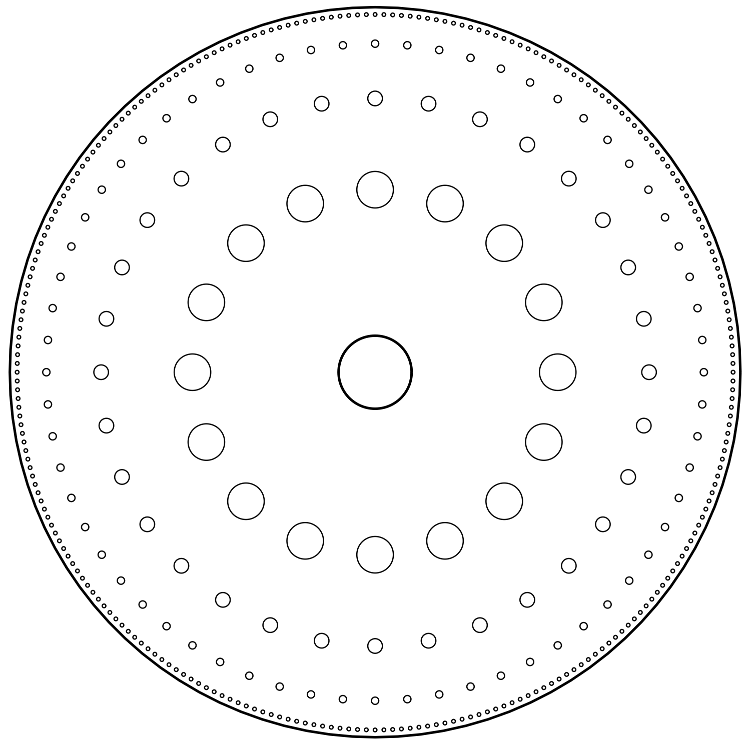
Three great British analysts:

Rippon, Stallard and Rippon-Stallard

Thm (Bishop, '18): Hdim = Pdim = 1 occurs.

Julia set has finite length on 2-sphere.

Julia set = union of C^1 loops and Cantor set with small dimension.

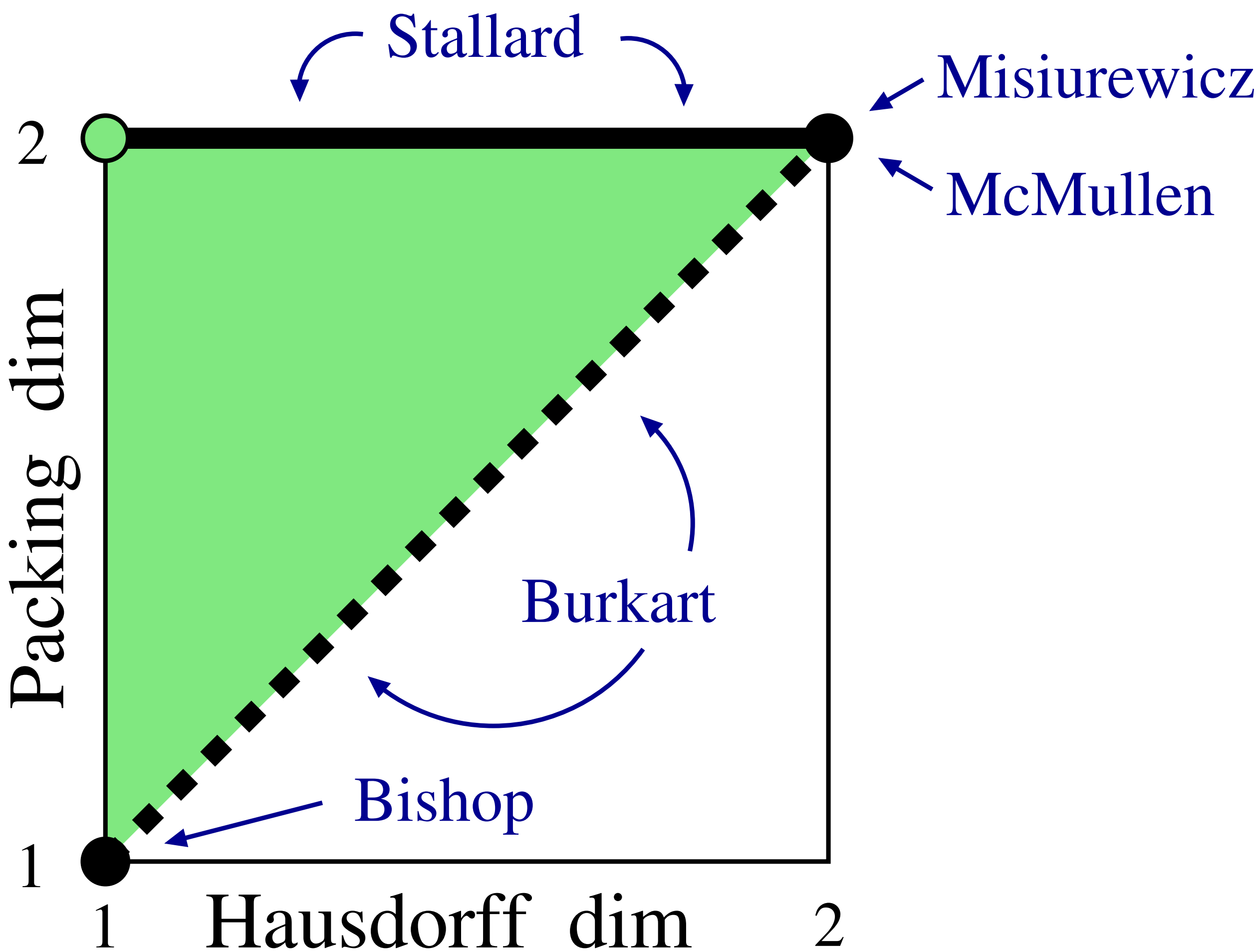


Sketch of typical Fatou component

In his 2021 Stony Brook PhD thesis, Jack Burkart proved:

Thm: For all $1 \leq s < t \leq 2$ we can have $s < \mathrm{Hdim} \leq \mathrm{Pdim} < t$.

Jack Burkart



Stallard
Misiurewicz
McMullen
2
1
Packing dim
Burkart
Bishop
1
Hausdorff dim
2

What is the Eremenko-Lyubich class?

More notation:

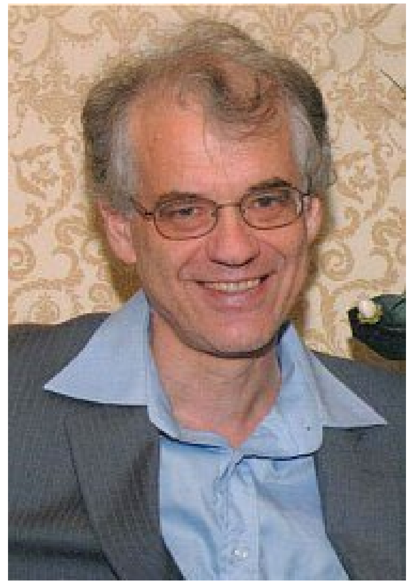
Critical value = image of a critical point.

Asymptotic value = limit of f along curve to infinity.

For example, 0 is an asymptotic value of e^z (along negative real axis).

$$\begin{aligned}\textbf{Singular set} \quad &= \quad \text{closure of critical values and finite asymptotic values} \\ &= \quad \text{smallest set so that } f \text{ is a covering map onto } \mathbb{C} \setminus S\end{aligned}$$

Alex Eremenko



Misha Lyubich

Andreas Speiser

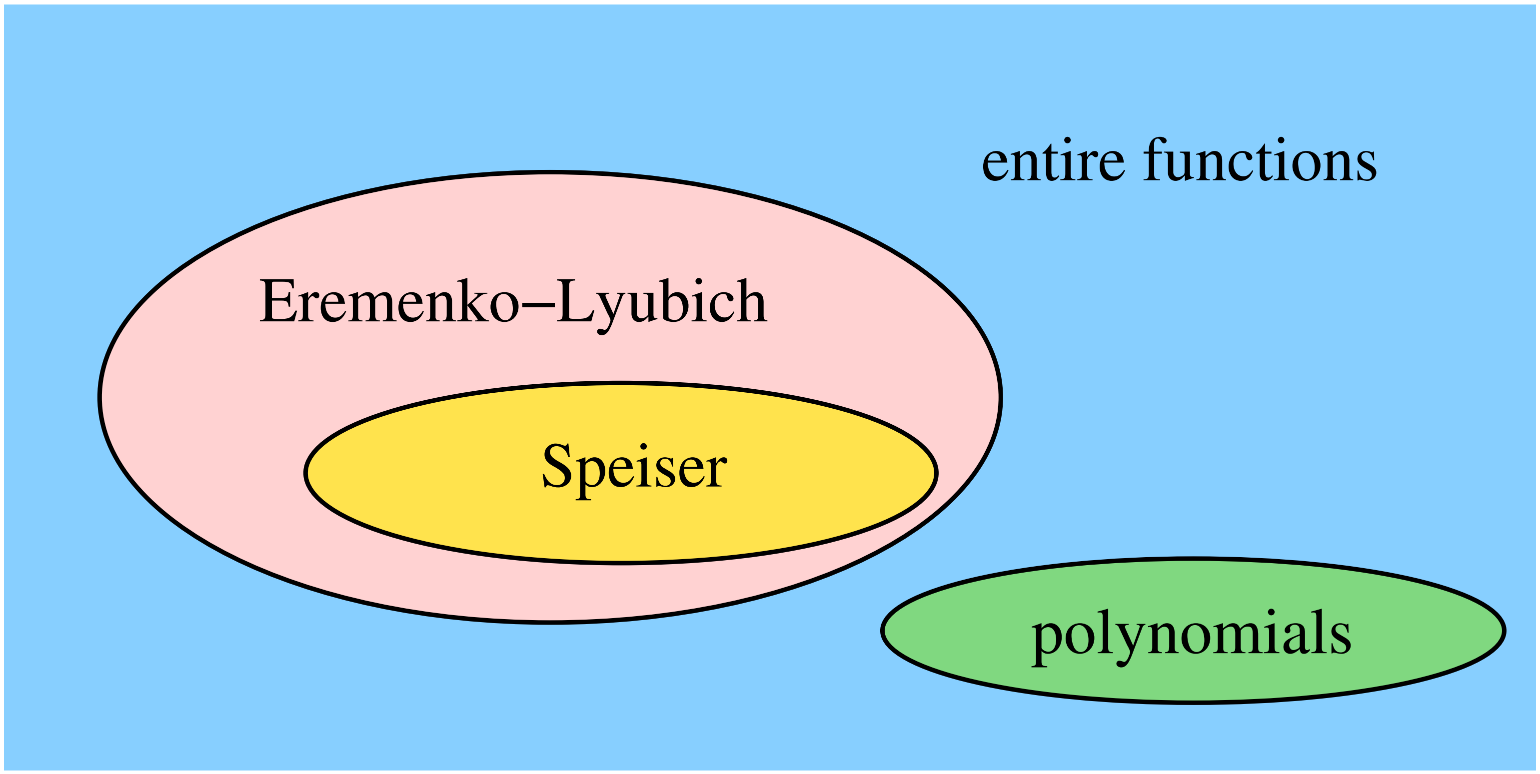
Eremenko-Lyubich class = entire with bounded singular set

Speiser class = entire with finite singular set

We exclude polynomials from both classes.

Stallard's examples of Hdim $\in (1,2)$ are in the Eremenko-Lyubich class.

Can we take them in the smaller Speiser class?



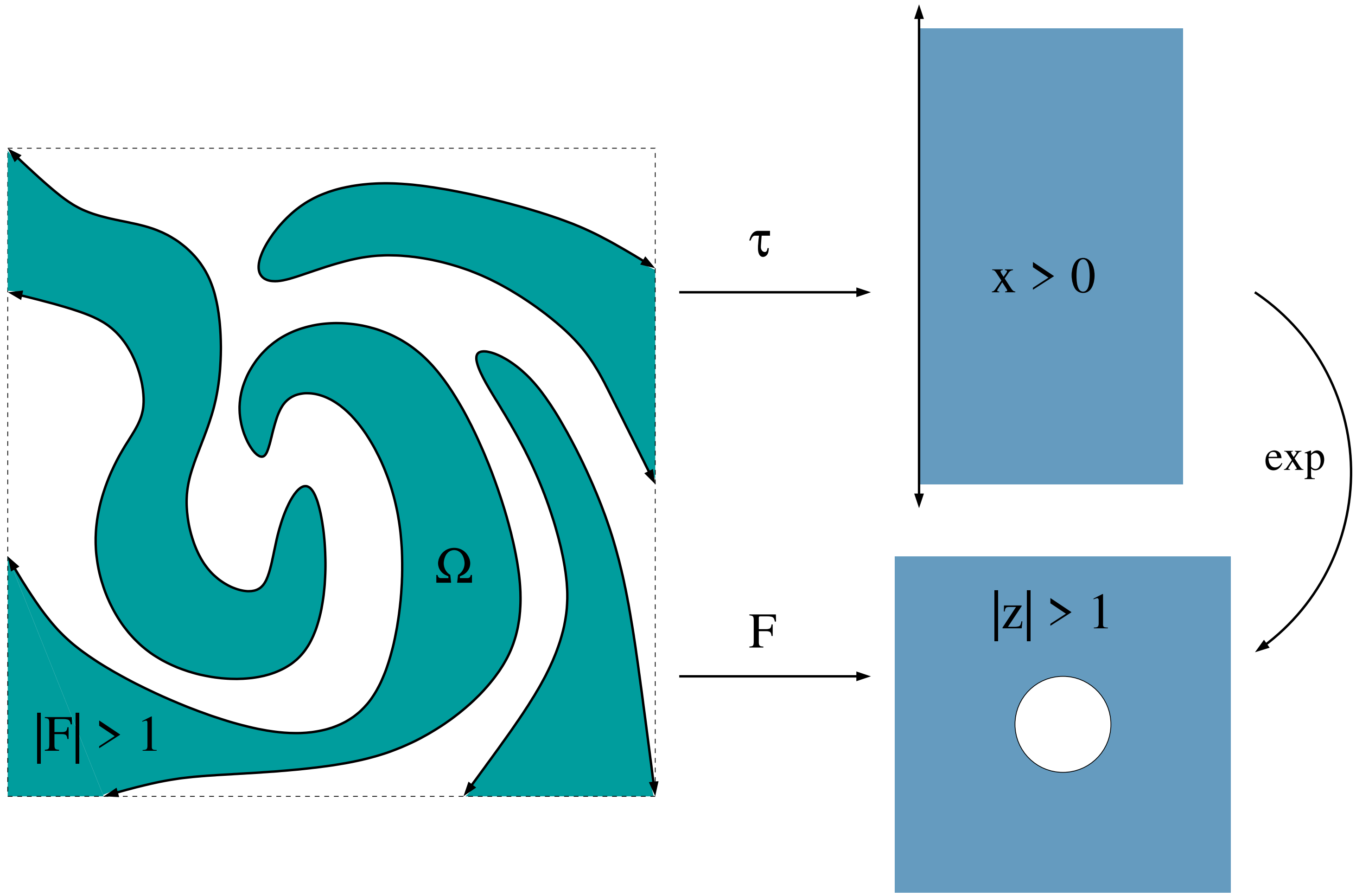
Simon Albrecht and I proved in 2020 that

Thm: There are Speiser functions $\{f_n\}$ with $\mathrm{Hdim}(\mathcal{J}_n) \searrow 1$.

But we don't know if all dimensions occur in Speiser class.

Simon Albrecht

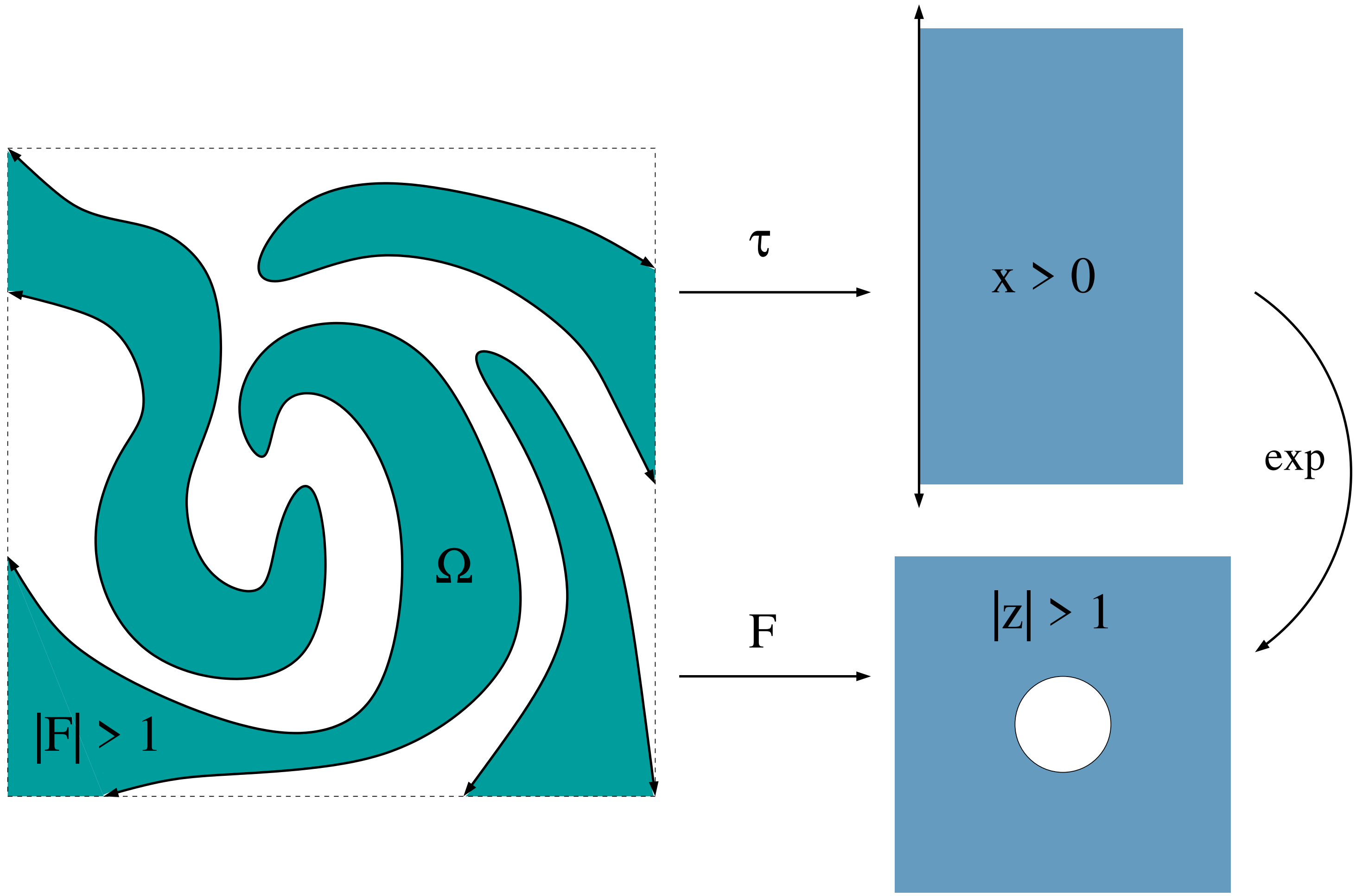
The structure of Eremenko-Lyubich functions



Suppose singular set is contained in $\mathbb{D} = \{|z| < 1\}$.

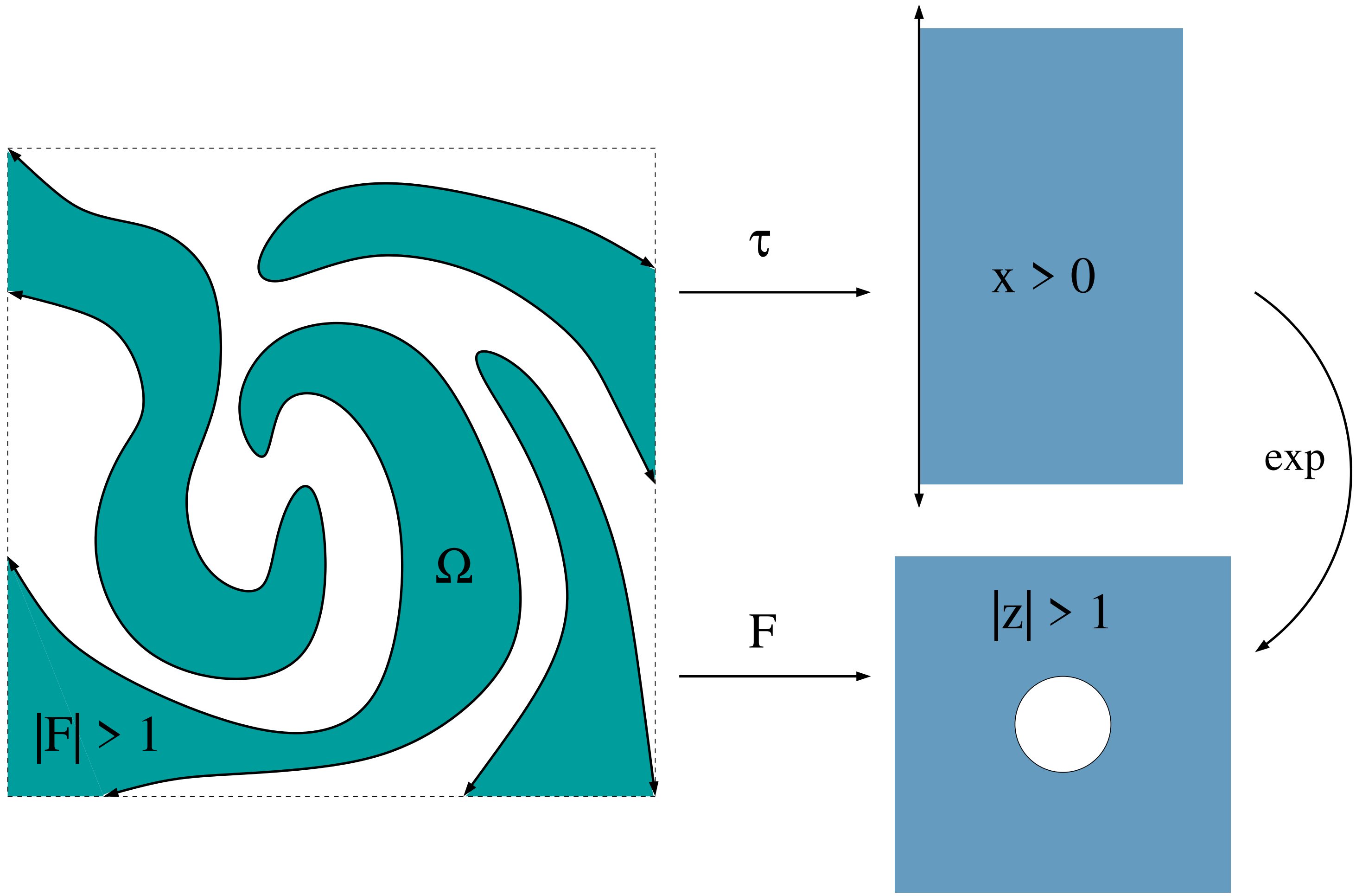
Each component of $F^{-1}(\mathbb{D}^*)$ is called a **tract**.

These are all simply connected and unbounded.



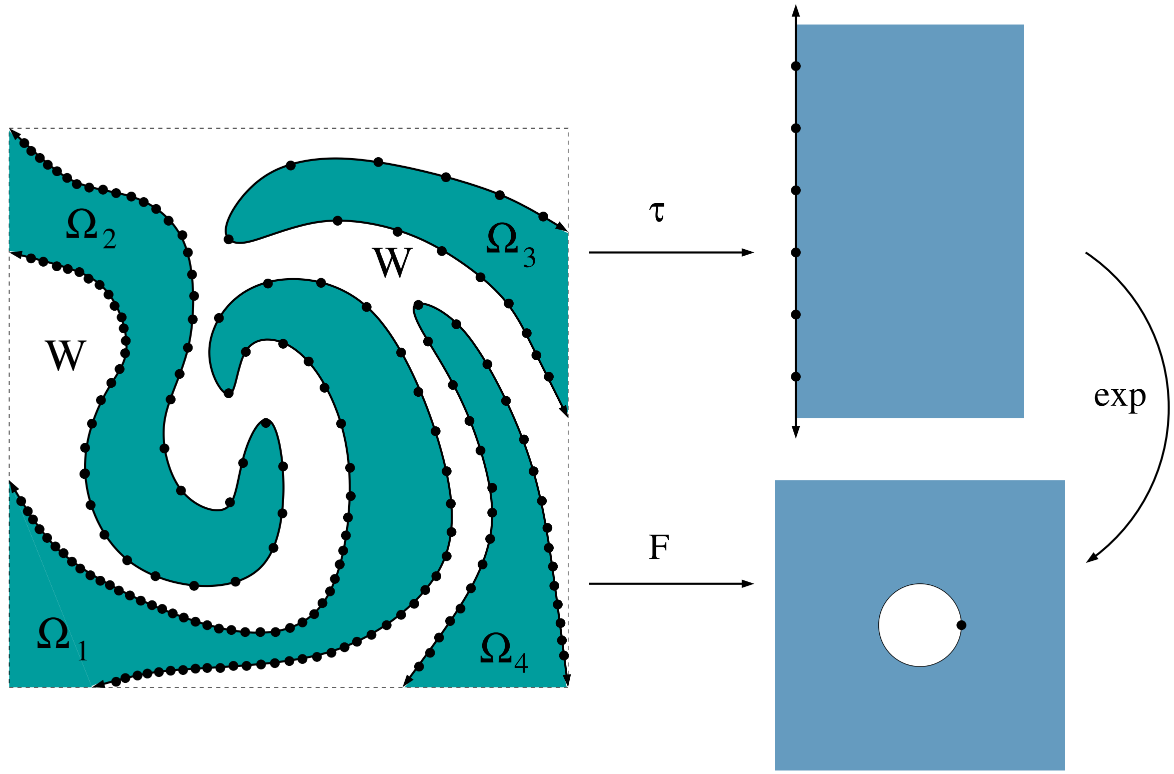
F is a covering map from each tract to $\mathbb{D}^*$.

F is a conformal map to right half-plane, followed by e^z.



Conversely, given any disjoint collection of tracts, and conformal maps from the tracts into the right hand-plane:

Models Thm: there is an EL-class function that approximates this map.



Idea of proof: ("Models for the Eremenko-Lyubich class")

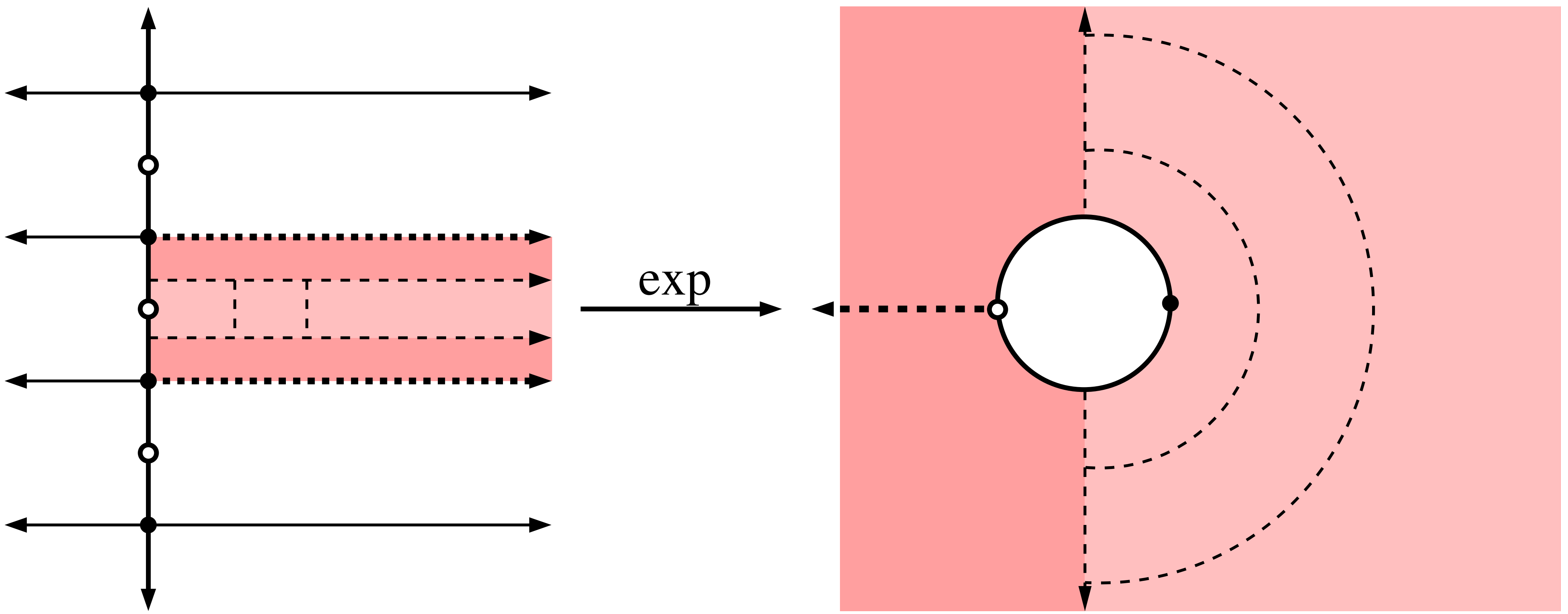
1. Replace tracts by smooth approximations.
2. Define $F = \exp(\tau) : \Omega \to \mathbb{D}^*$ using conformal maps τ.
3. Define holomorphic $W \to \mathbb{D}$ that "follows" F around $\mathbb{T}$.
4. Interpolate between maps on tract boundary. Get quasiregular map.
5. Solve Beltrami equation to get holomorphic map.

To prove Stallard's theorem consider one tract $\approx$ half-strip:

Assume we have g in EL-class so that:

- $g(0) = 0$ and $|g(z)| < 1$ outside S = half-strip.

- $\{|z| < 1\}$ attracted to 0 (in Fatou set).



Definition of $\exp(z)$.

Inside S, $g(z) \approx \exp(\exp(z - K))$

This is conformal map of S to half-plane, followed by exp.

Fixed point $g(0) = 0$ attracts everything in complement of S

Indeed, $\mathcal{J}$ consists of points whose orbits stay in X forever:

$$\mathcal{J}(g) \subset \bigcap X_n,$$
$$X_n = \{z : |g^k(z)| \geq K, k = 1, \dots, n\}.$$

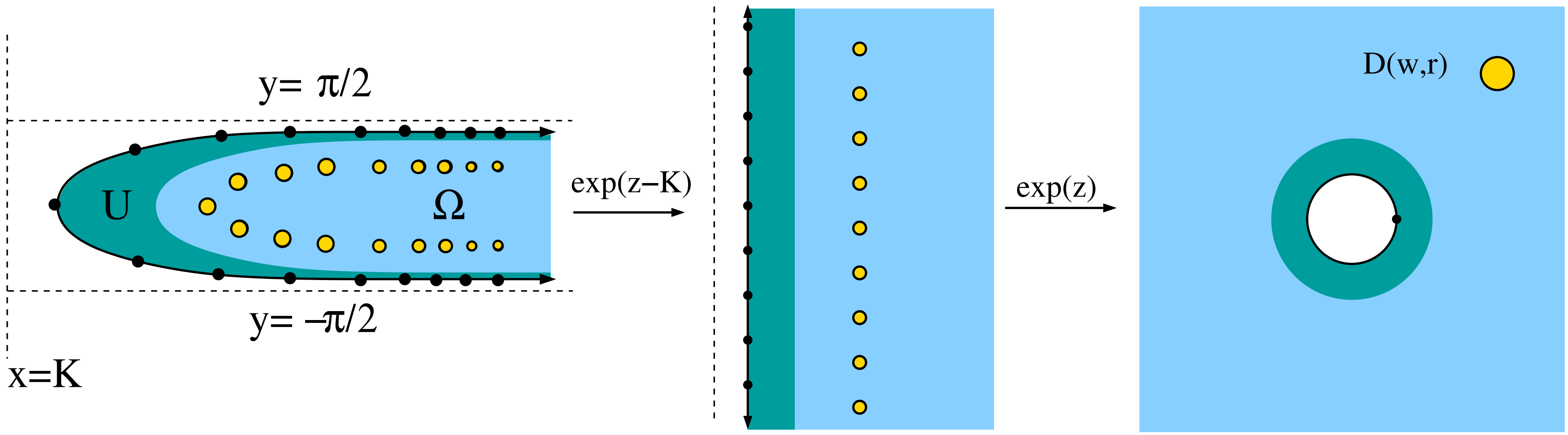
To prove $\dim(\mathcal{J}) \leq 1 + \delta$, choose K large enough so that

(1) $X_1 \subset \cup D_j$ where $\sum_j \mathrm{diam}(D_j)^{1+\delta} < \infty$,

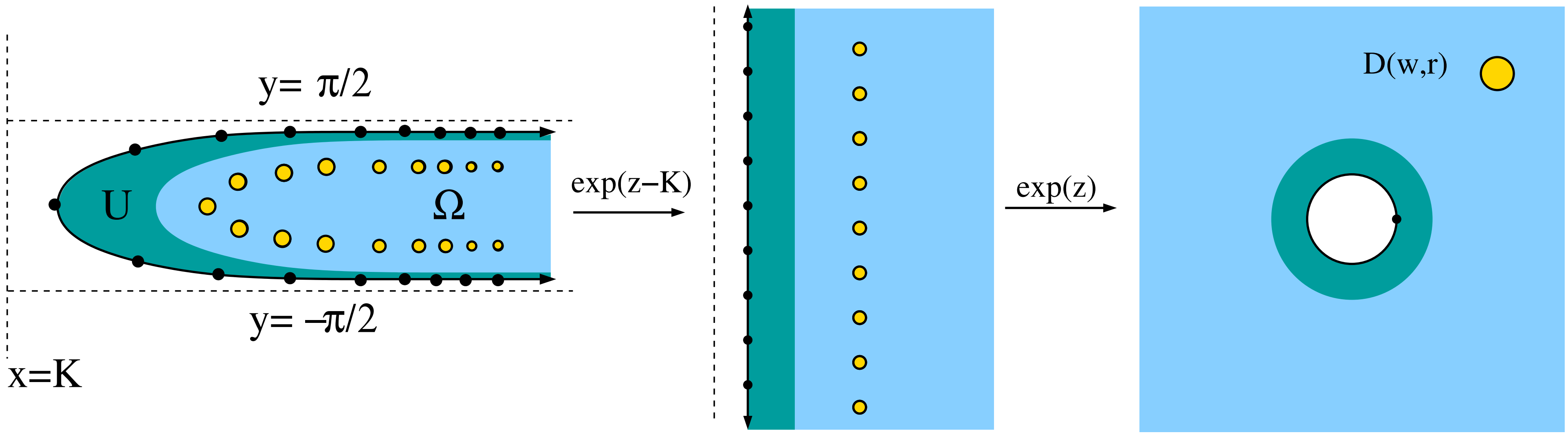
(2) each disk D has preimages W_k with

$$\sum_{W_k \in f^{-1}(D)} \mathrm{diam}(W_k)^{1+\delta} \leq \epsilon \cdot \mathrm{diam}(D)^{1+\delta}.$$

Iterating proves Julia set has zero $(1+\delta)$-content.



This is what preimages of one disk look like (restricted to strip).

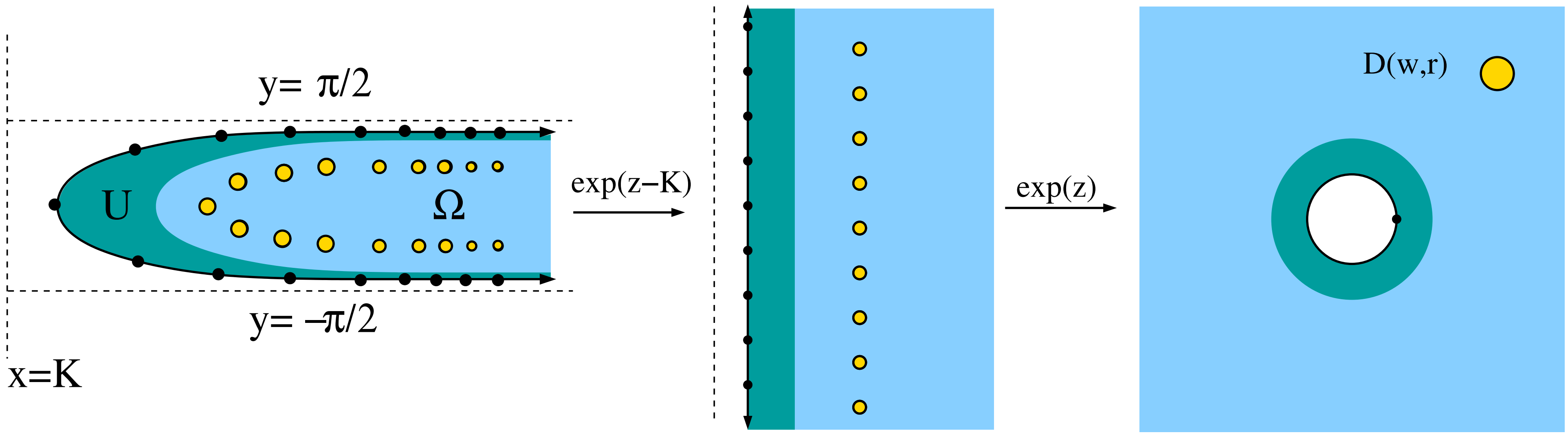


- First preimages: vertical stack on line $\{x = \log|w|\}$, diameters $O(r/|w|)$.

- Second preimages: disk at height $2\pi k$ has preimage of size

$$O\left(\frac{r}{|w|(\log|w| + 2\pi|k|)}\right).$$

These estimates only use $(\log z)' = 1/z$.

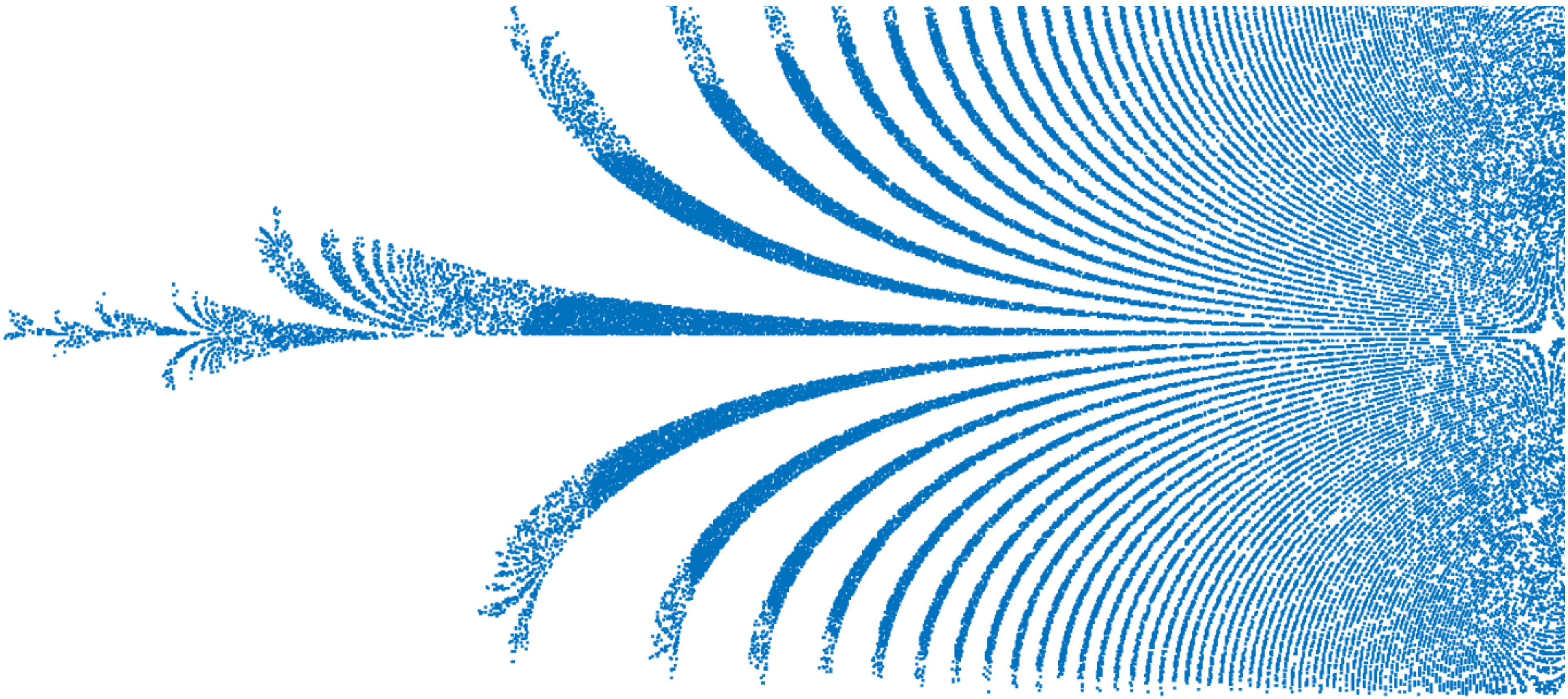


If K is large enough, preimages of $D(w,r)$ satisfy

$$\sum_{\text{preimages}} \operatorname{diam}^{1+\delta} \lesssim \left(\frac{r}{|w|}\right)^{1+\delta} \sum_k \frac{1}{(\log|w| + 2\pi|k|)^{1+\delta}}$$

$$\lesssim \left|\frac{r}{w}\right|^{1+\delta} \frac{1}{\delta \log^{1+\delta}|w|} \ll \left|\frac{r}{w}\right|^{1+\delta} \ll r^{1+\delta}$$

This proves $\dim(\mathcal{J}(g)) \leq 1+\delta$.



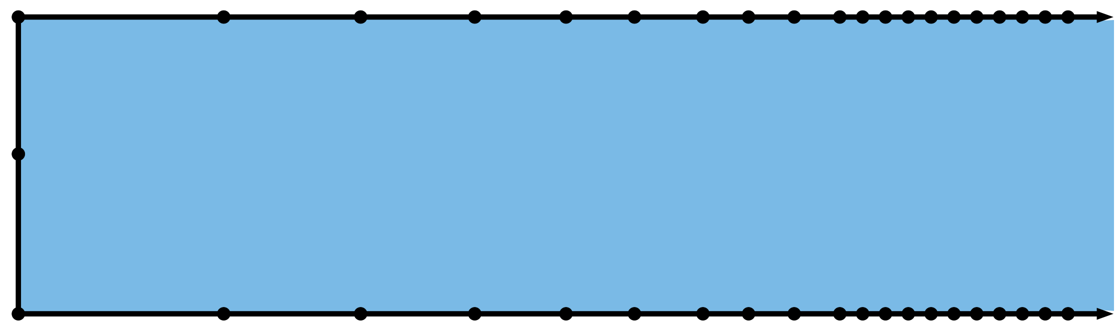
My attempt to draw the Julia set of the strip model.

Can you draw a better version?

Can we do the same thing in the Speiser class?

Can we do the same thing in the Speiser class?

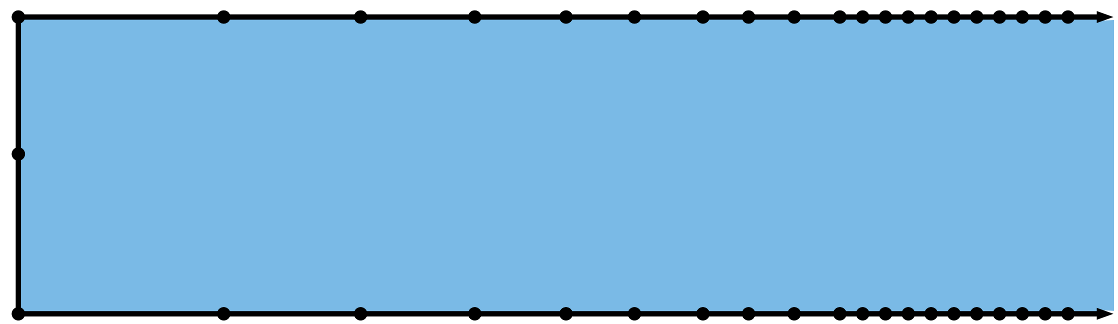
Not quite



To get $1 < d < 2$ in Speiser class, we want to repeat same argument.

Find g is Speiser class so that

- $g(z) \approx \exp(\exp(z - K))$ in half-strip
- $|g| < 1$ outside the half-strip



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Find g is Speiser class so that

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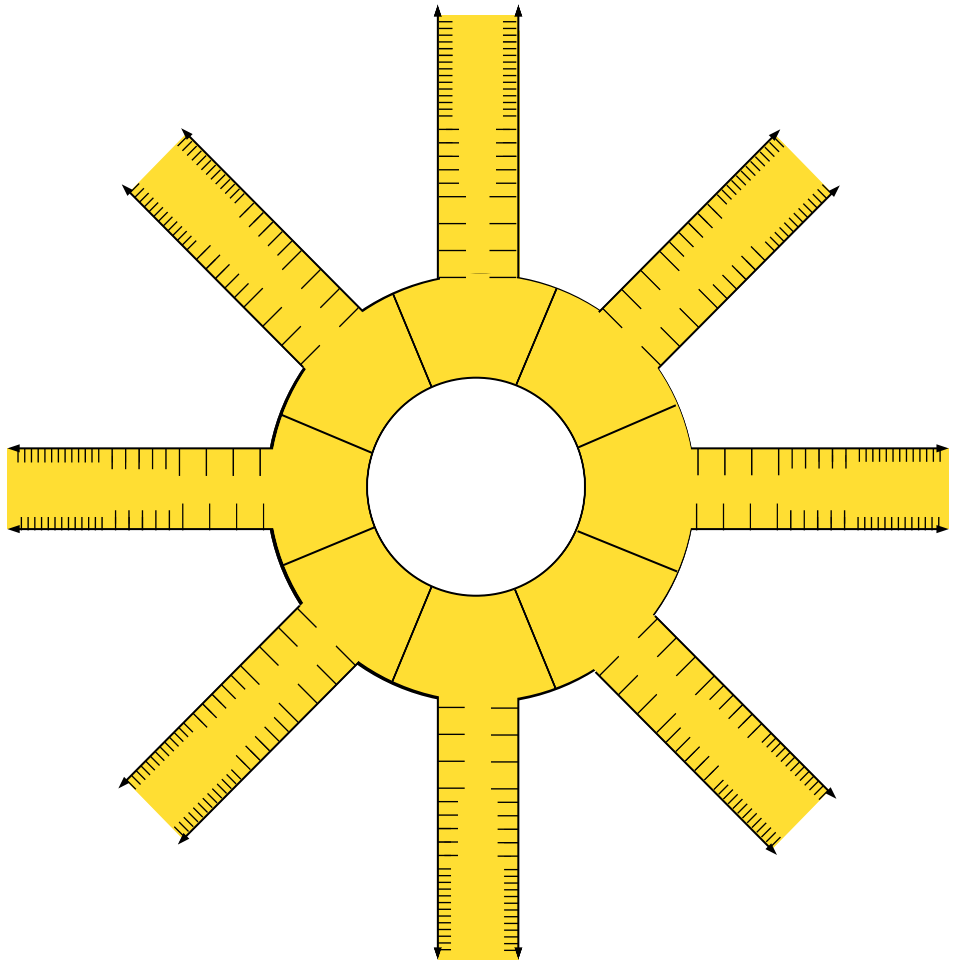
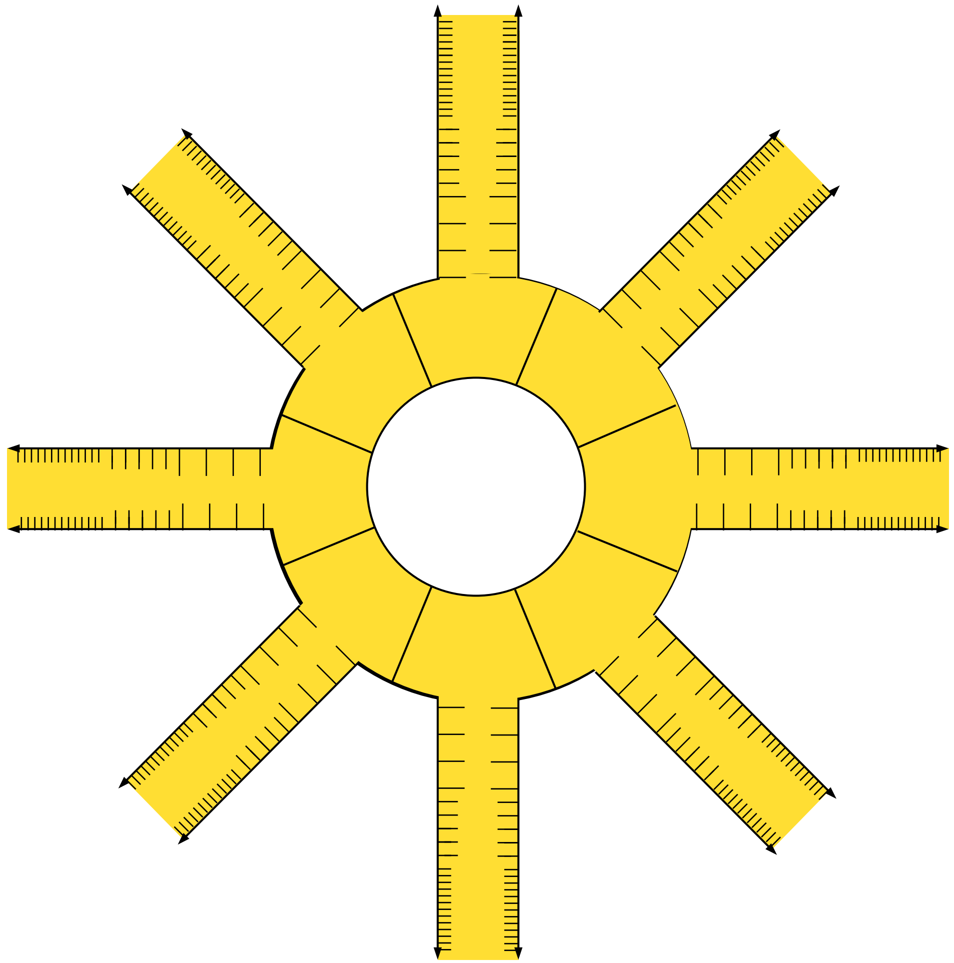
Unfortunately, no such g exists (B. 2017, "Models for the Speiser class").



Instead, we use several strip-like tracts arranged in a “star”.

As number of arms increases, dimension tends to 1.



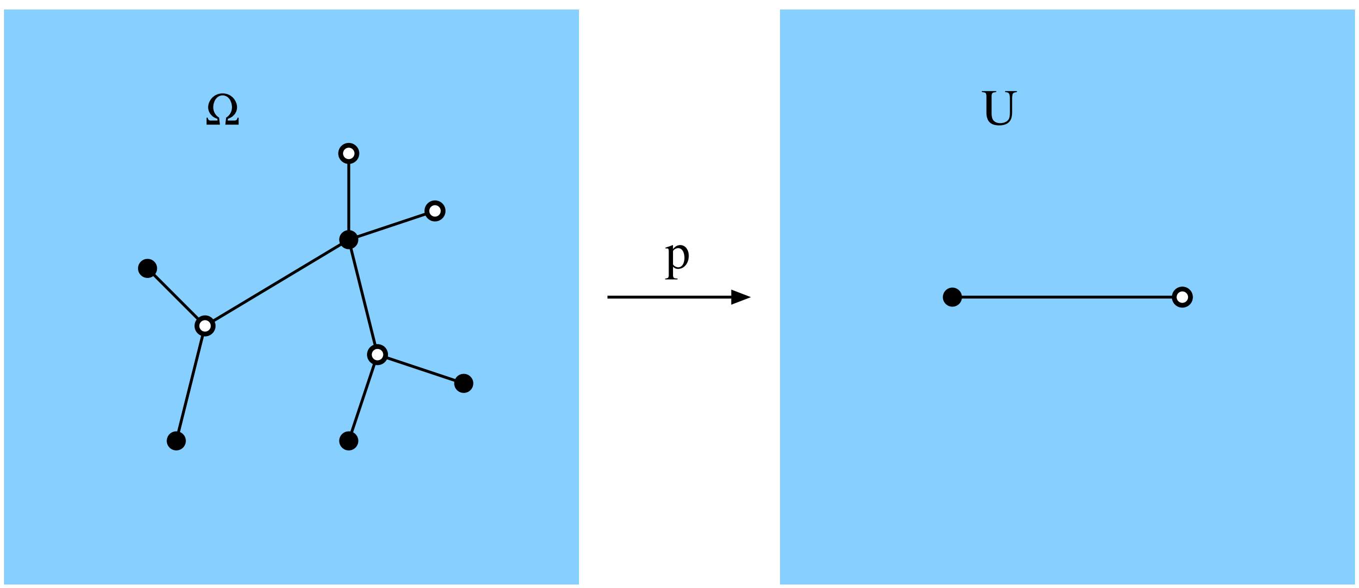
Why do several strips work when one strip fails?

This requires explaining the folding theorem.

Grothendieck's theory of *dessins d'enfants* associates a finite planar tree T to a polynomial P so that

1. p only has critical points at -1 and $+1$.
2. $T = p^{-1}([-1,1])$.

This gives a preferred way to draw any finite tree: its "true form".



$$\Omega = \mathbb{C} \setminus T \qquad\qquad U = \mathbb{C} \setminus [-1,1]$$

Quasiconformal Folding Theorem

Maps infinite planar trees to entire functions with critical values ± 1.

A finite tree has:

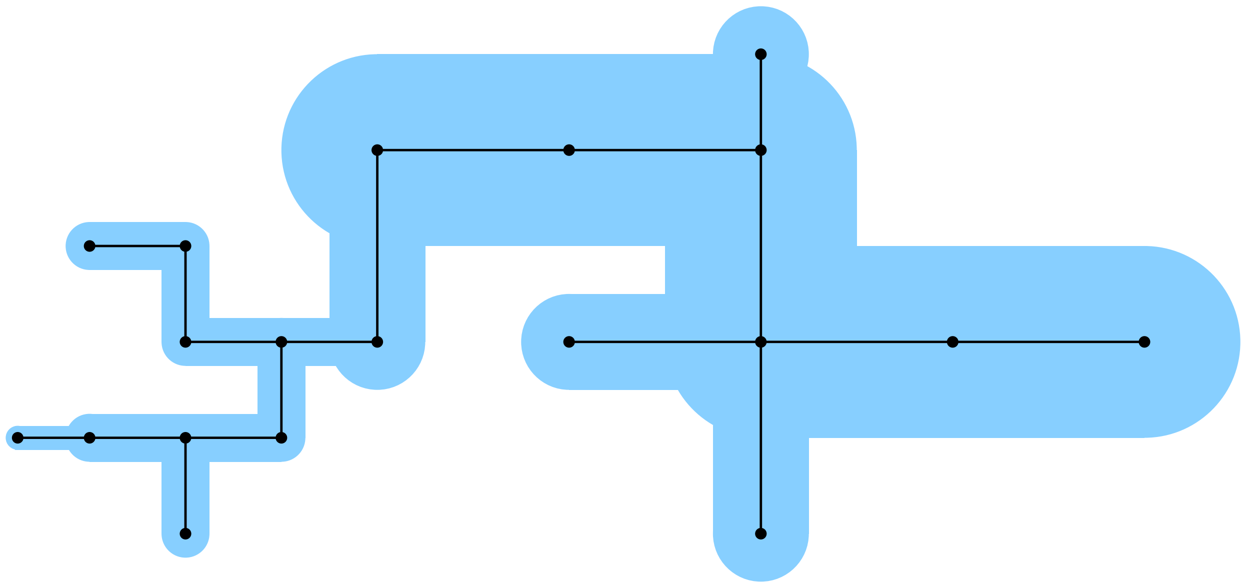
1. A bounded number of edges meeting at any vertex,
2. A shortest edge.

These can fail for infinite trees. Need corresponding assumptions.

If e is an edge of T and $r > 0$ let

$$e(r) = \{z : \mathrm{dist}(z, e) \leq r \cdot \mathrm{diam}(e)\}$$

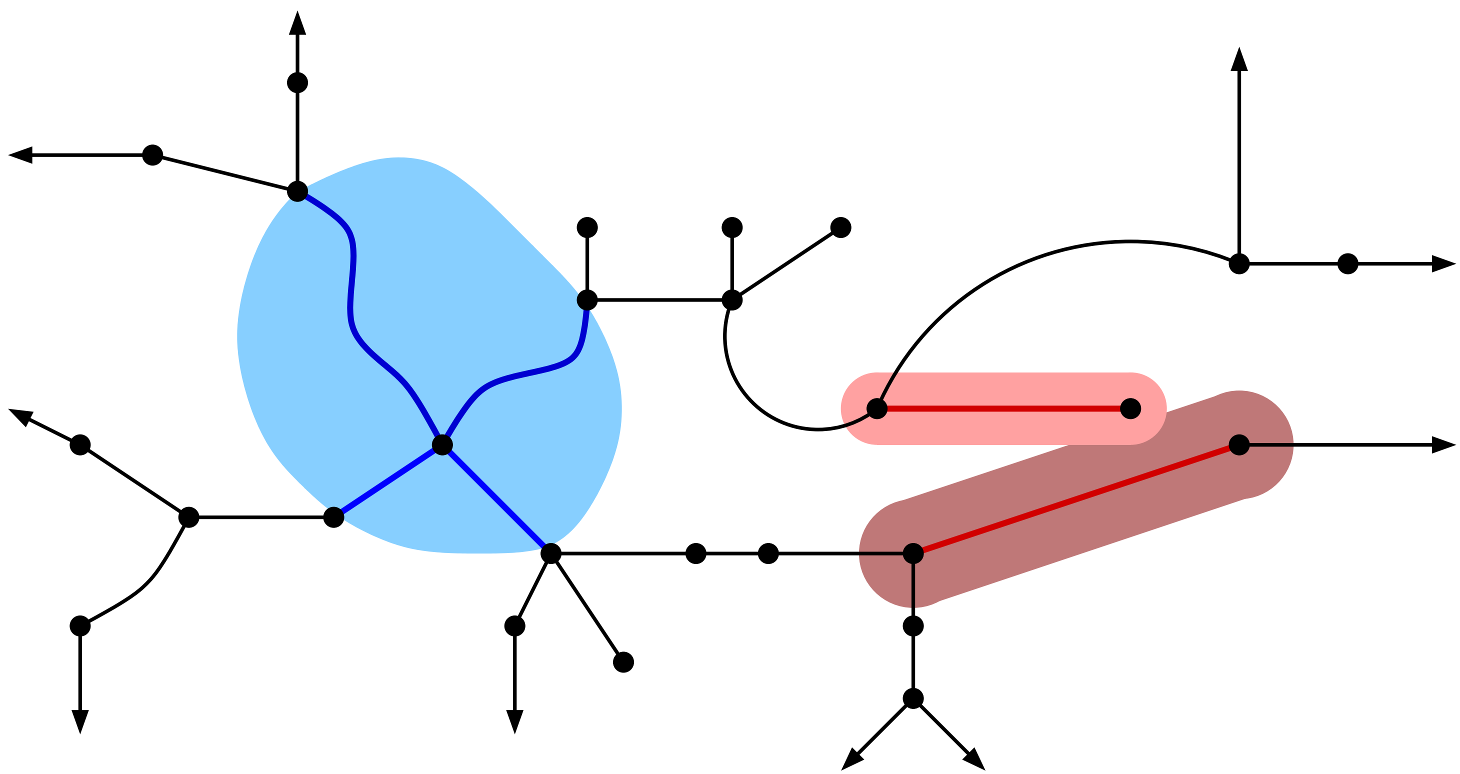
Define neighborhood of T: $T(r) = \cup\{e(r) : e \in T\}$.



(1) Bounded Geometry (local condition; easy to verify):

- edges are uniformly smooth.
- adjacent edges form bi-Lipschitz image of a star $= \{z^n \in [0, r]\}$
- non-adjacent edges are well separated,

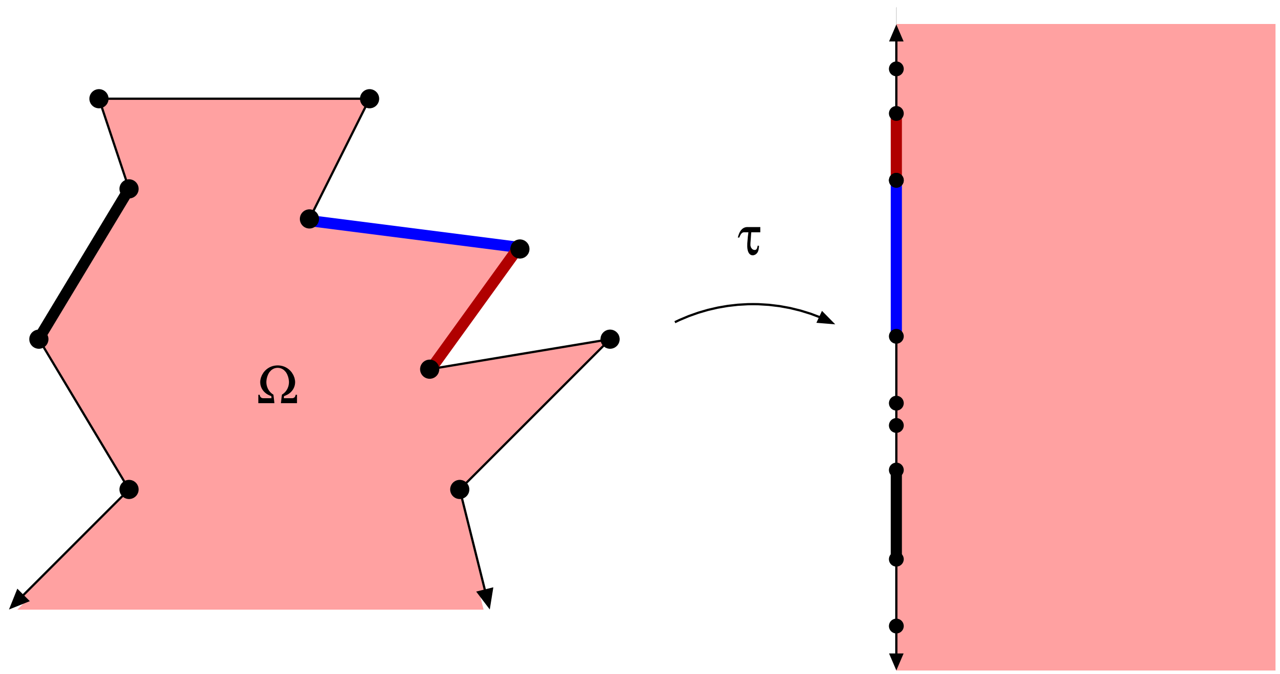
$$\mathrm{dist}(e, f) \geq \epsilon \cdot \min(\mathrm{diam}(e), \mathrm{diam}(f)).$$



(2) τ-Lower Bound (global condition; harder to check):

Complementary components of tree are simply connected.

Each can be conformally mapped to right half-plane. Call map τ.

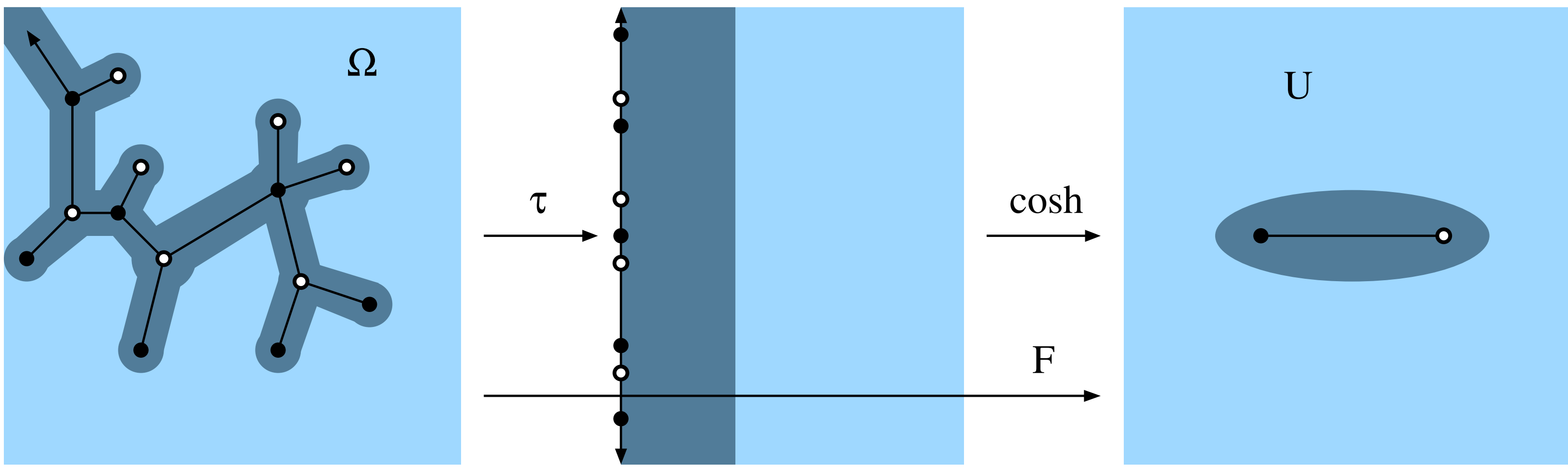


We assume all images have length $\geq \pi$.

Need positive lower bound; actual value usually not important.

Components are "thinner" than half-plane near ∞.

QC-Folding Theorem (B 2015): If T has bounded geometry and the τ-lower bound, then T can be approximated by a true tree in the following sense. Let $F = \cosh \circ \tau$. Then there is a K-quasiregular g and $r > 0$ such that $g = F$ off $T(r)$ (shaded) and $\mathrm{CV}(g) = \pm 1$.



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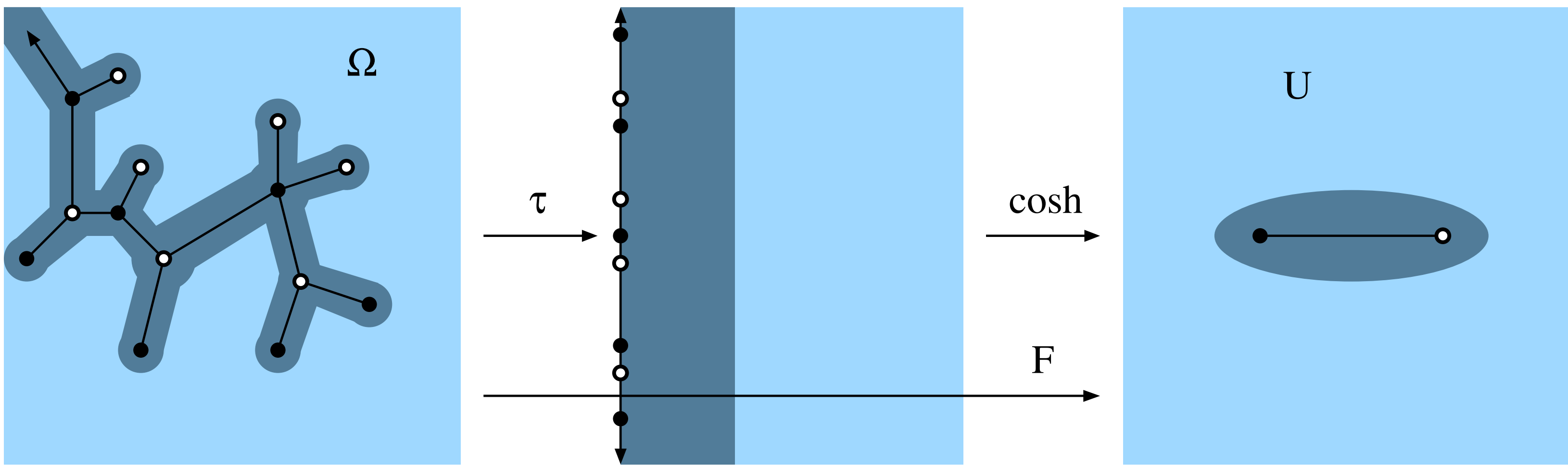
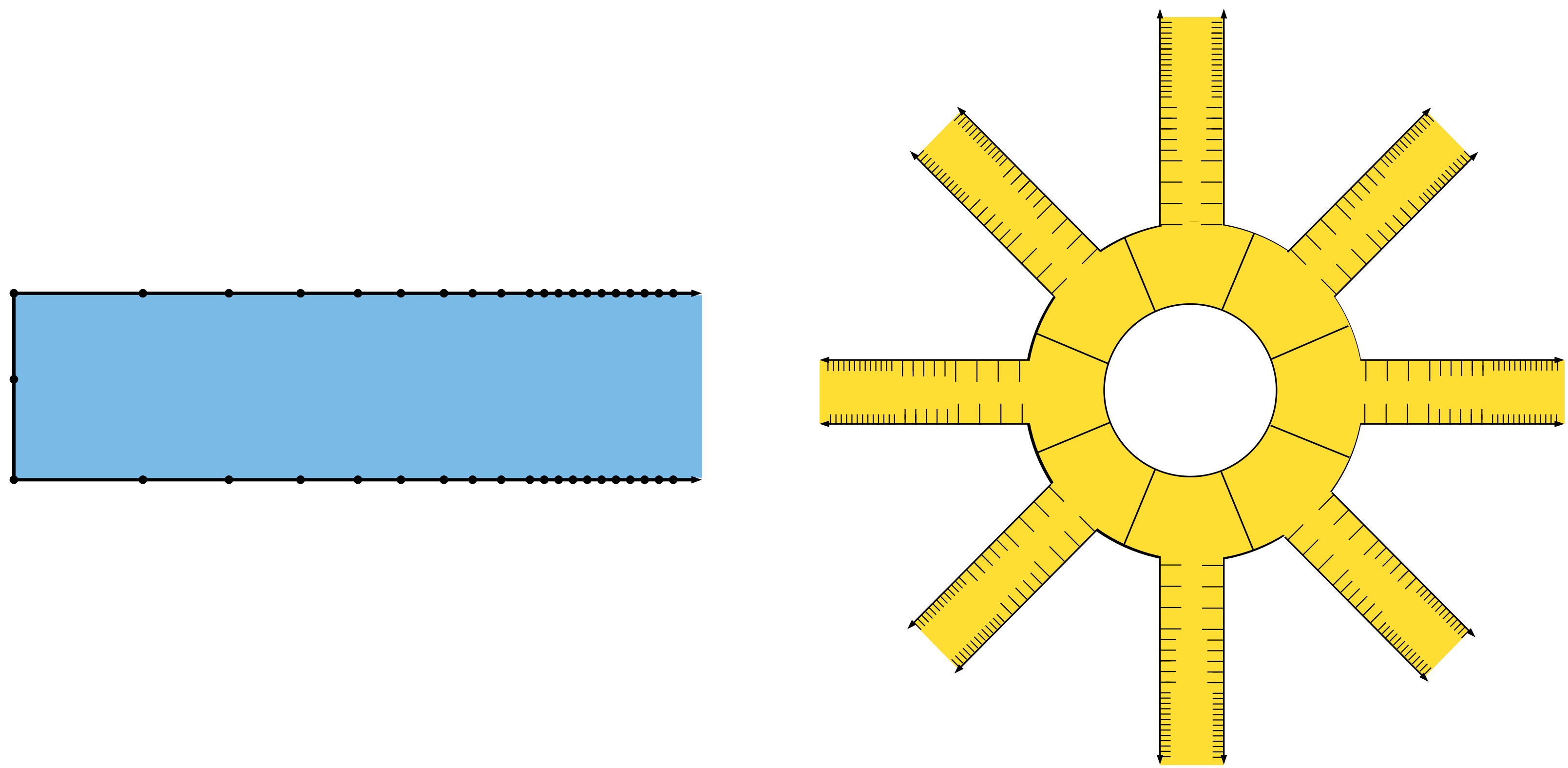


K and r only depend on the bounded geometry constants.

$g = F$ on light blue.

F may be discontinuous across T. g is continuous everywhere.

g has critical points at vertices of T, plus extras depending on "τ-imbalance".



The map of strip complement to half-plane is $\approx z^{1/2}$

The map of "star" complement to half-plane is $\approx z^{N/2}$

Folding theorem applies when exponent is ≥ 1.

Open questions:

"Dim = 1" example has finite spherical Hausdorff 1-measure.

But it has infinite packing 1-measure.

$\Rightarrow$ Thus it is not subset of rectifiable curve on sphere.

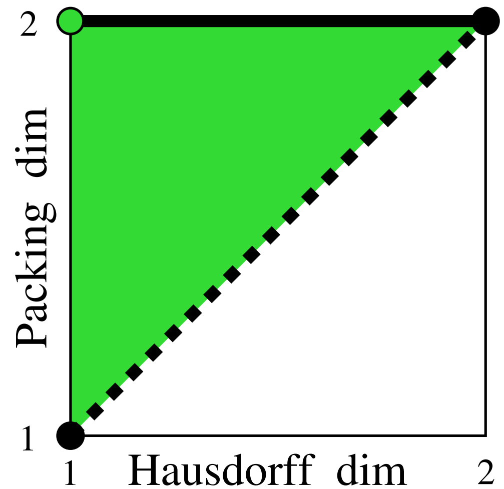
Can a transcendental Julia set lie on a rectifiable curve?

Open questions:

For Dim $= 1$ example, boundaries of Fatou components are C^1 curves.

Can they be better than C^1? C^2? C^∞?

Open questions: Black = known, Green = unknown



Which pairs (Hdim, Pdim) can occur for a transcendental entire function?

Can any pair $(s, s), 1 < s < 2$ occur?

Can we have Hdim = 1, Pdim = 2? Hdim = 1, 1 < Pdim < 2?

Does Hdim = Pdim hold for all polynomials?

Open questions:

Speiser class Julia sets take dimensions as close to 1 as desired.

Do they take all dimensions in $(1, 2]$?

Can we continuously deform examples so dimension sweeps out an interval?

Open questions:

Eremenko and Lyubich showed that for Speiser class f

$$M_f = \{g : g = \psi \circ f \circ \varphi \text{ for } \psi, \varphi \text{ QC } \}$$

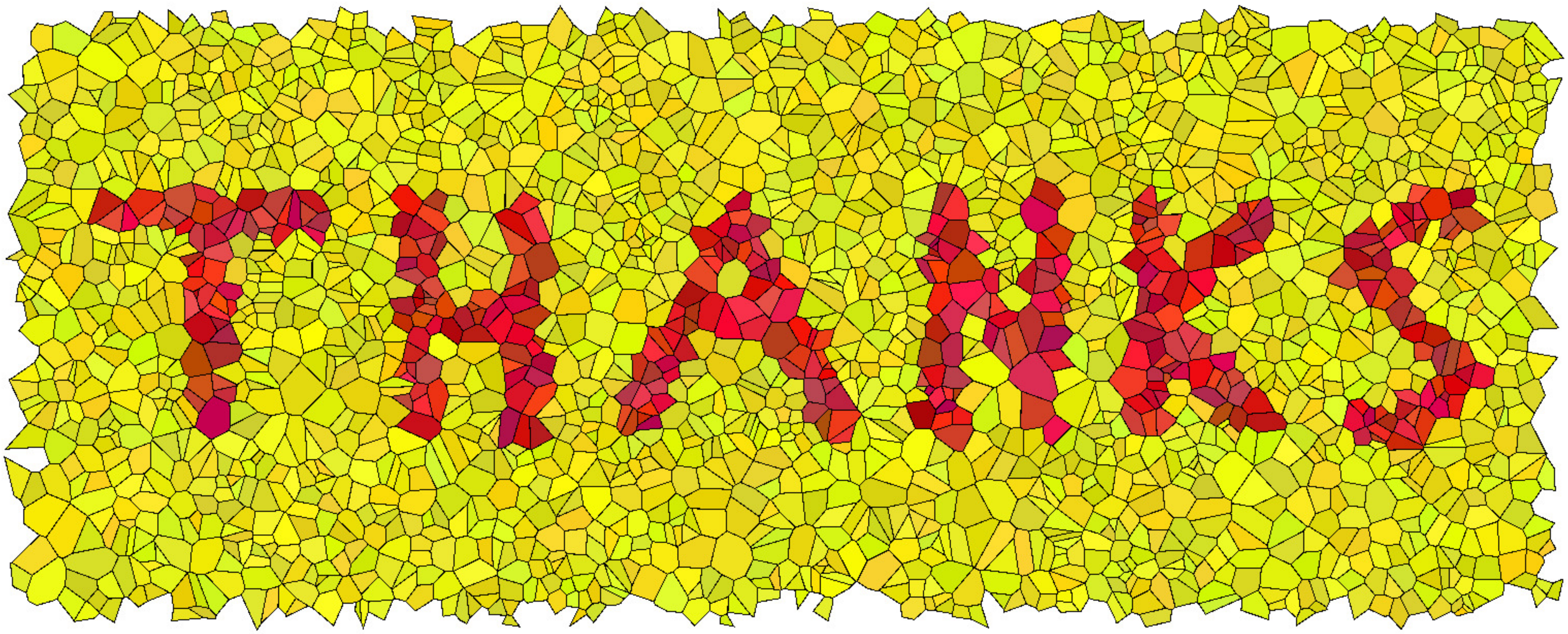
is a finite dimensional manifold.

$D(g) = \mathrm{Hdim}(\mathcal{J}(g))$ is a continuous function on M_f.

Often this is the constant 2 (e.g., finite order of growth).

Otherwise is it always non-constant?

Is the supremum over M_f always 2? (analog of Shishikura result)

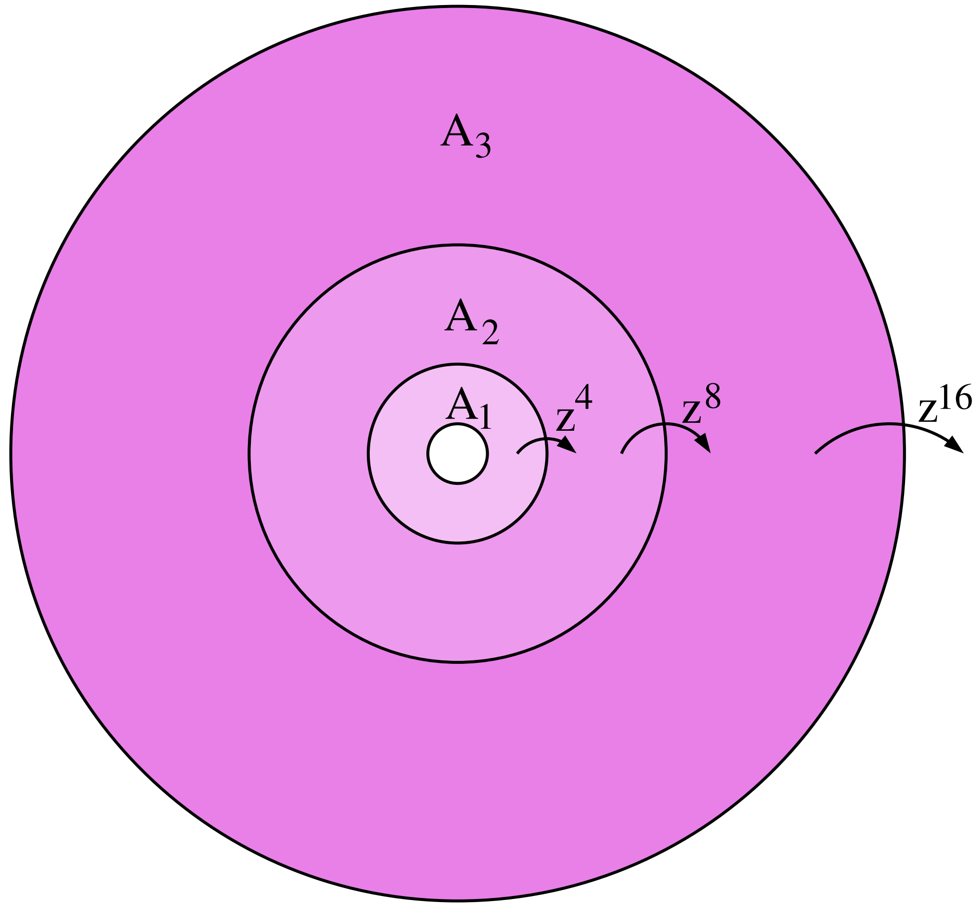

THANKS

Transcendental Julia sets with dimension 1

Theorem: $\mathrm{Hdim}(\mathcal{J}) = \mathrm{Pdim}(\mathcal{J}) = 1$ is possible. I gave example using infinite products. Somewhat technical.

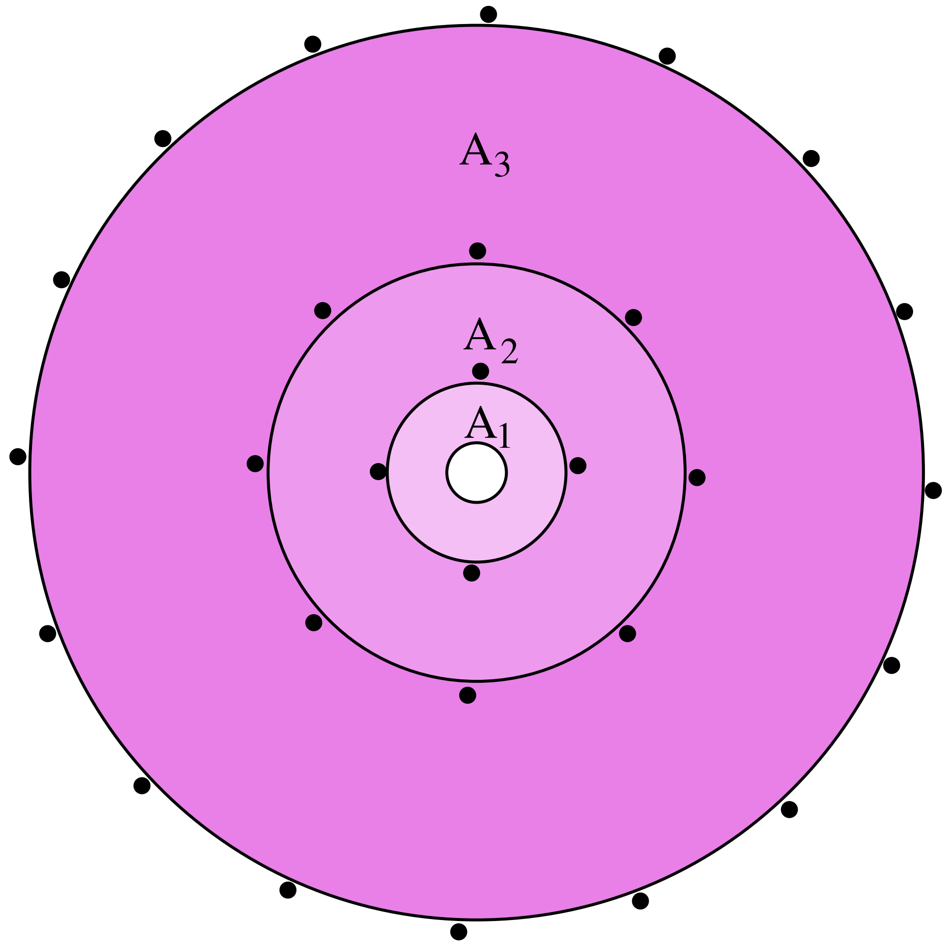
New, more geometric, proof by Burkart and Lazebnik ("folding-like").

Both proofs based on similar geometry.



Suppose we have annuli $\{r_k < |z| < r_{k+1}\}$.

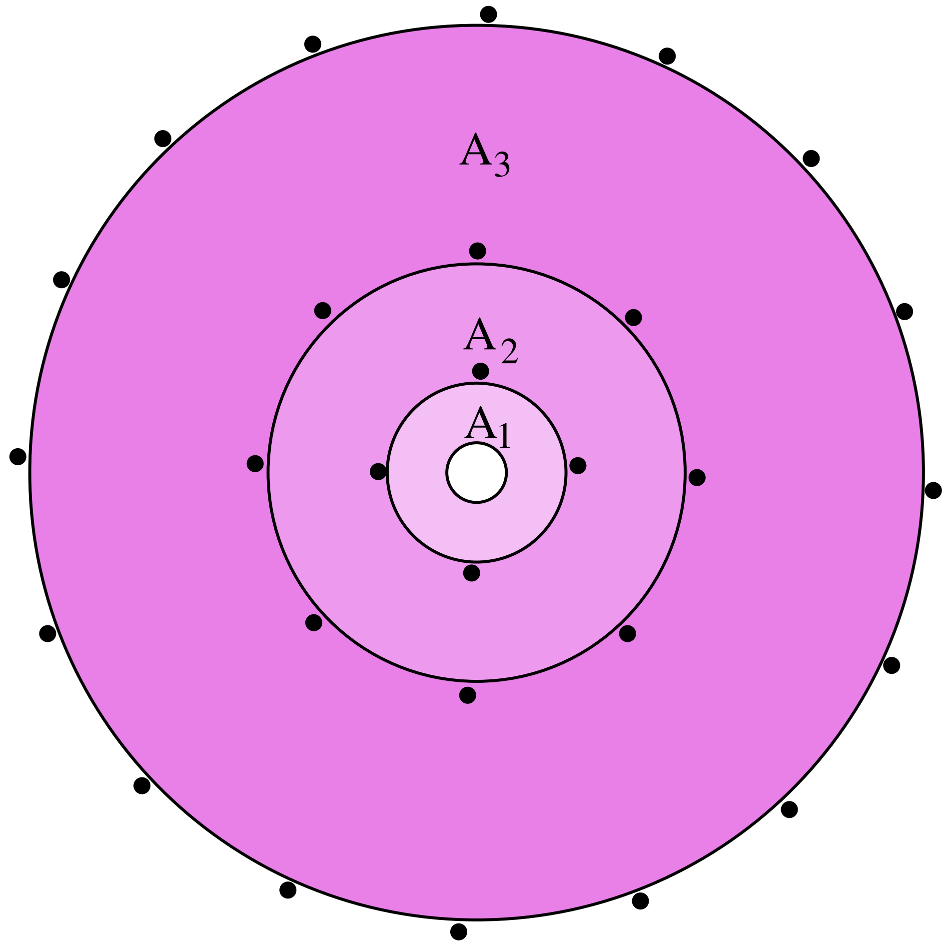
maps $z \to C_k \cdot z^{2^k}$ from A_k to A_{k+1}.



Approximate this by placing 2^k zeros evenly around kth circle.

Approximate polynomial p near origin so there is some Julia set near origin.

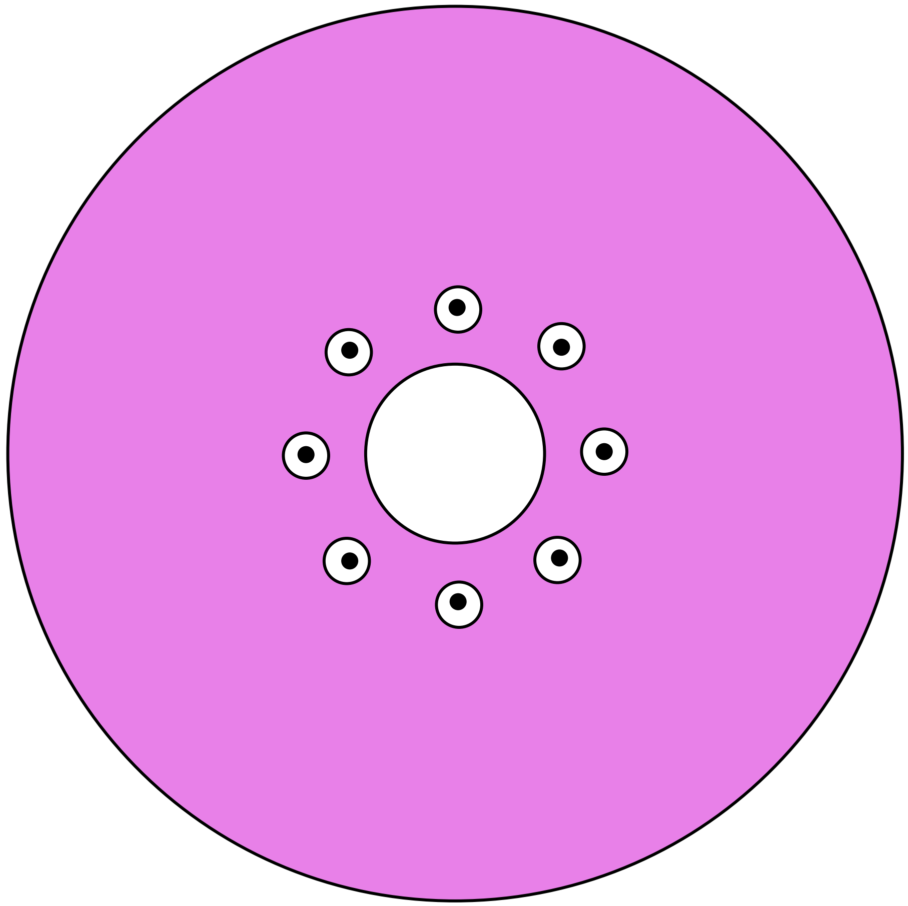
Annuli escape, each corresponds to a different Fatou component.



The kth annulus looks rotationally invariant.

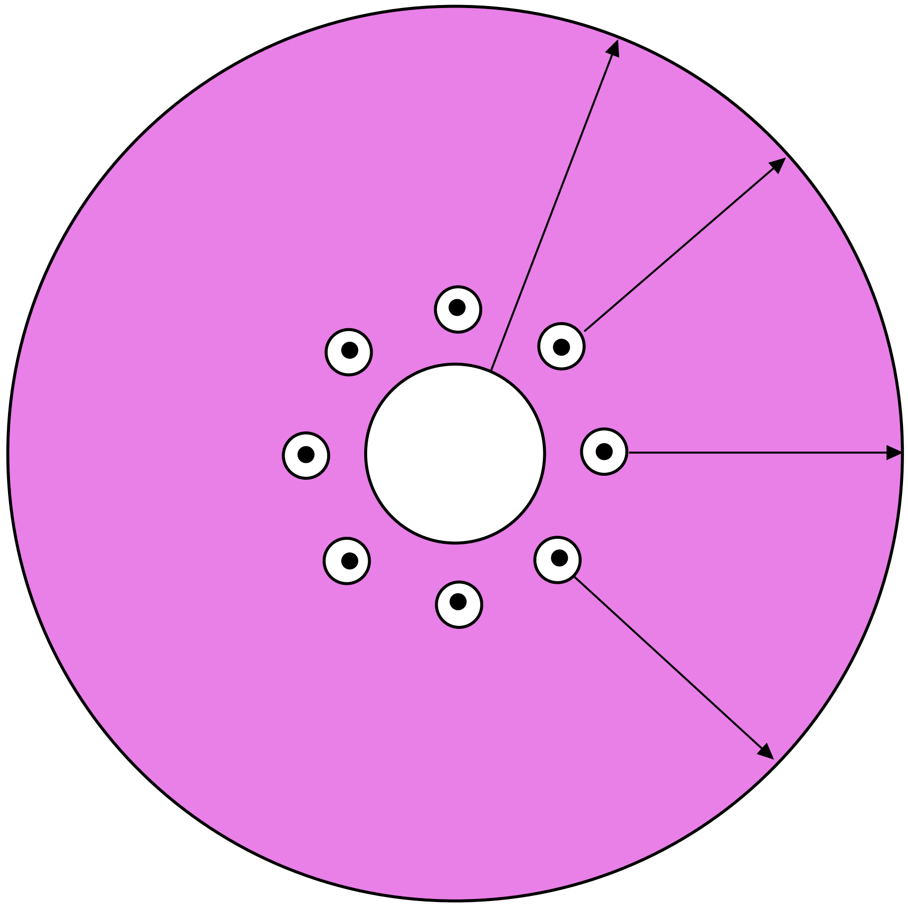
Other zeros are very, very close to 0 or ∞.

Its inner and outer boundaries should be nearly circular.

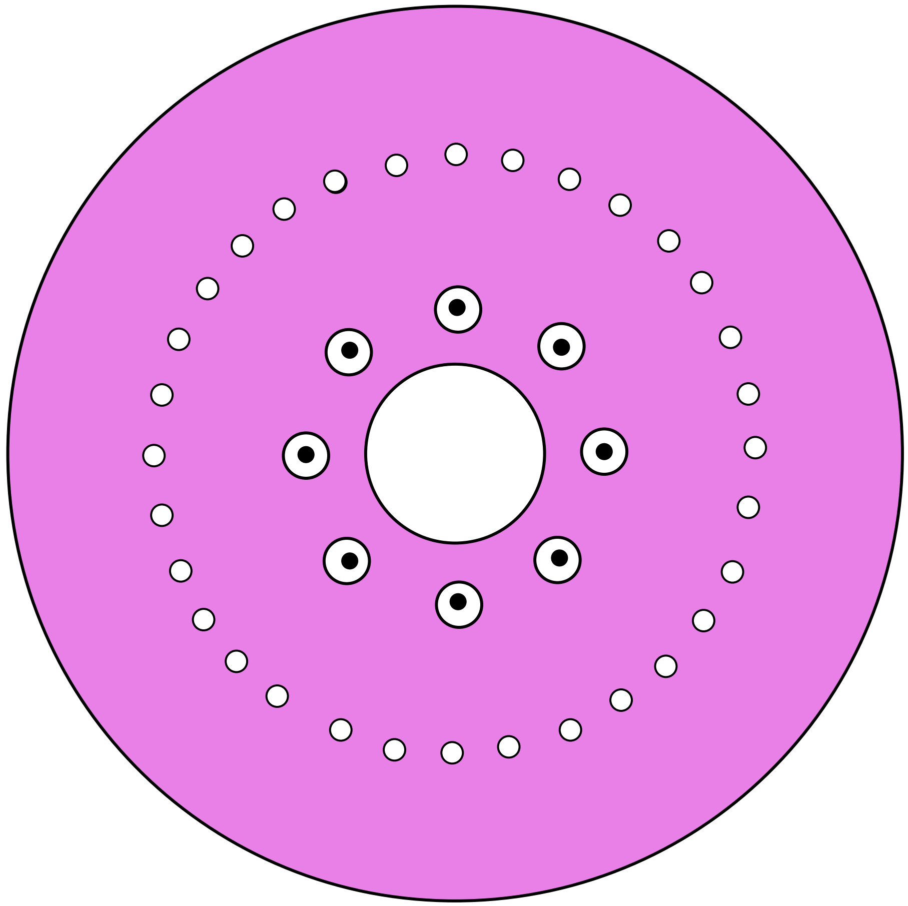


Fatou component has a boundary around each zero.

Must surround pre-image of component at zero.



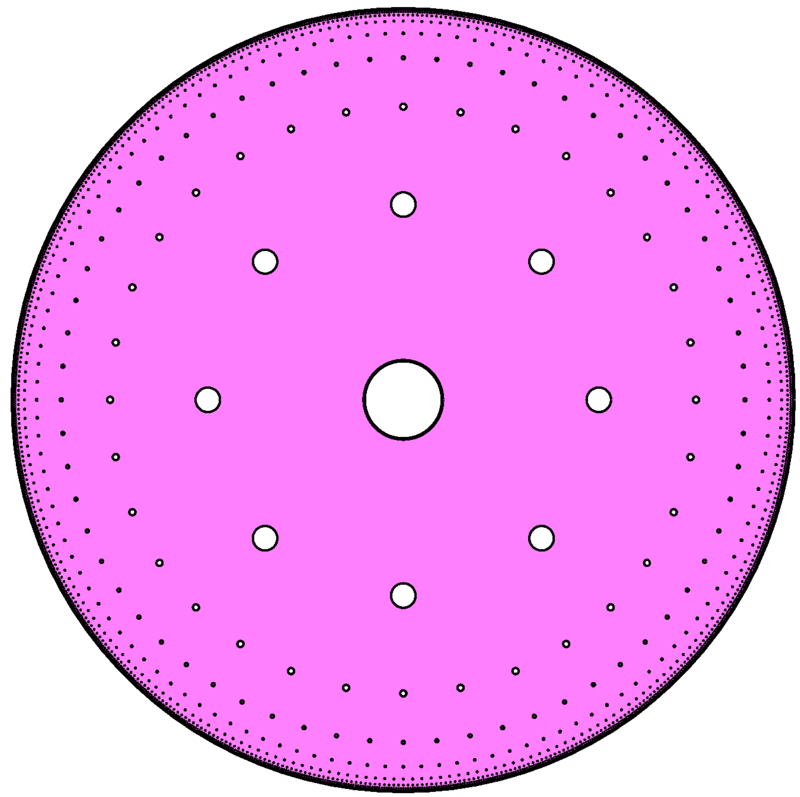
These boundaries and inner boundary map to outer boundary.



The kth annulus maps to $(k+1)st$ annulus.

The $(k+1)$st annulus also has ring of boundary components.

These have a preimage in the kth annulus; a second ring.



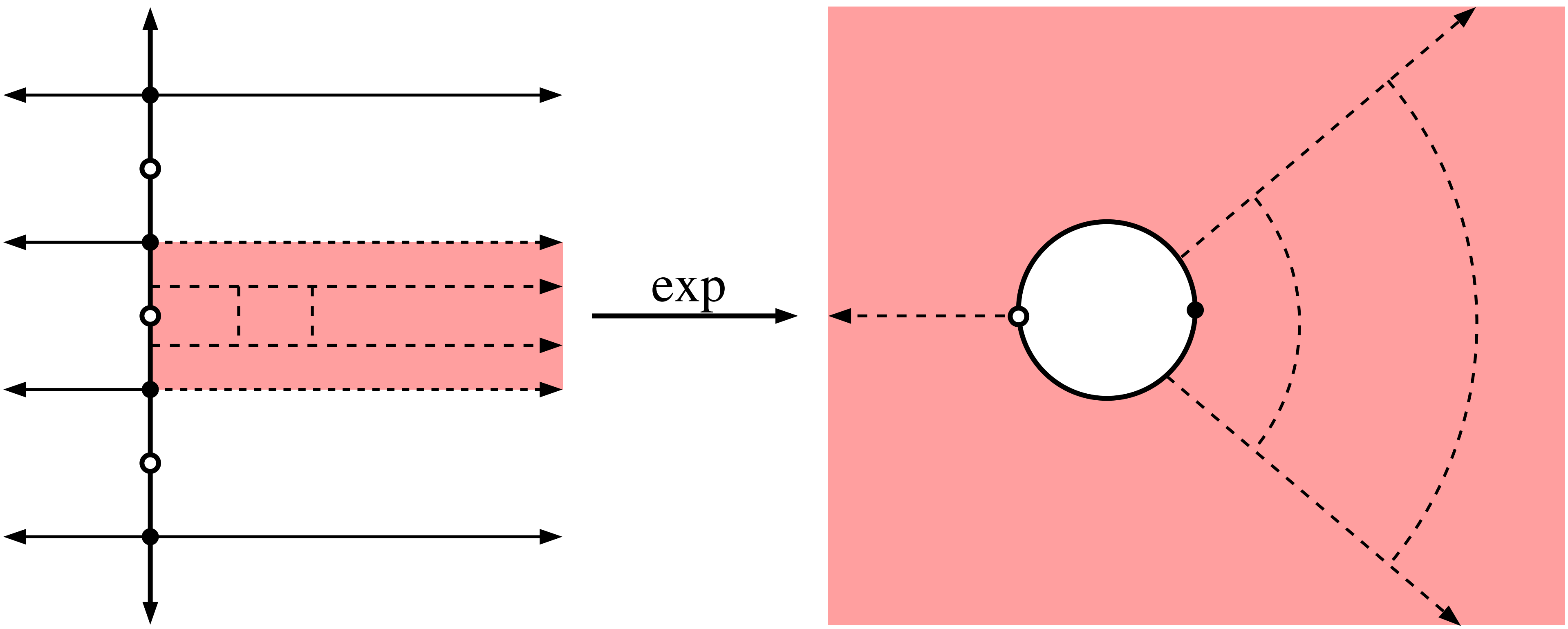
There is an infinite sequence of rings converging to outer boundary.

Estimates show component boundary is countable union of C^1 curves.

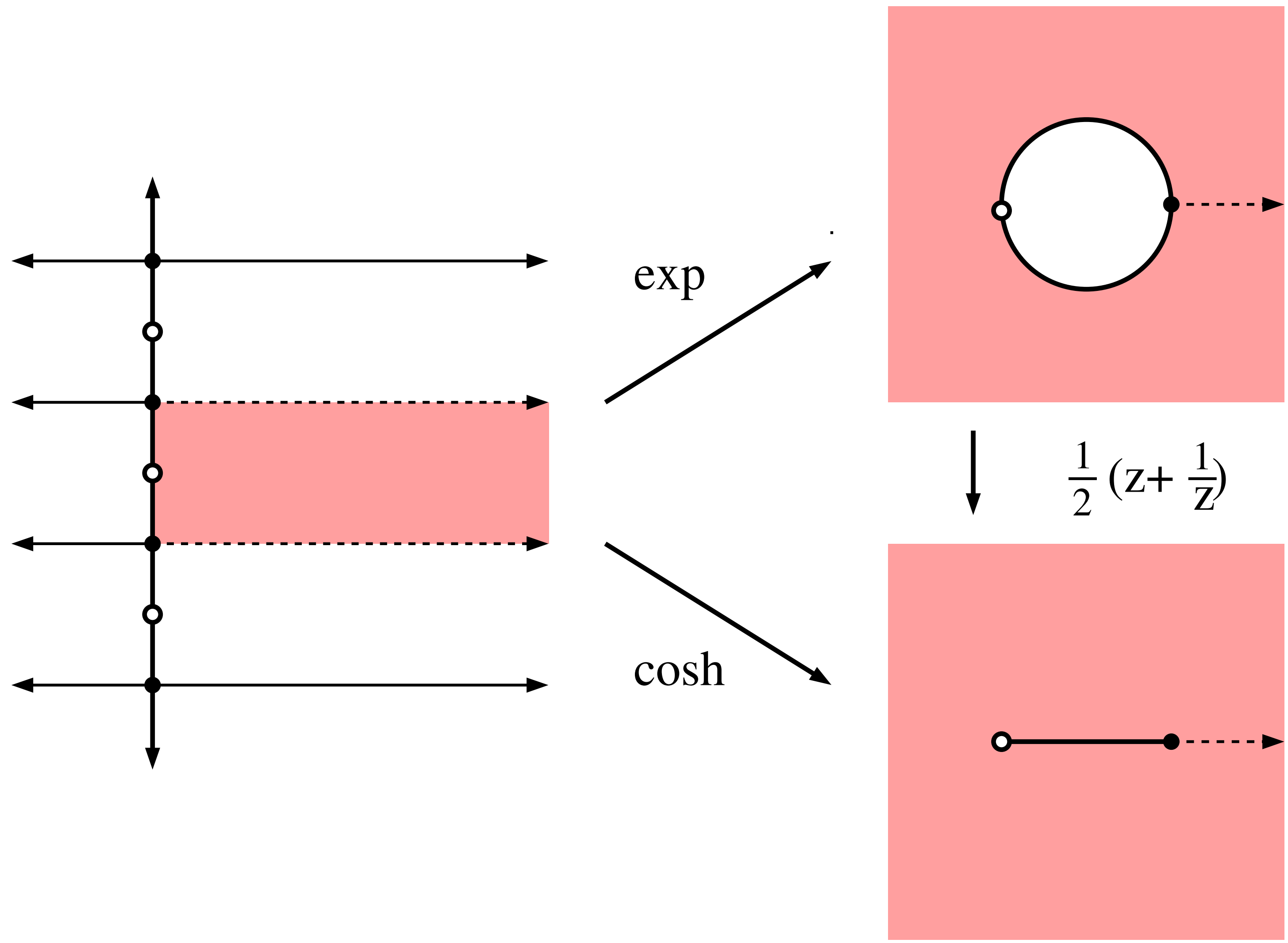
“Buried” points have small dimension $\Rightarrow \dim(\mathcal{J}) = 1$.

Defn: Escaping set $I(f) = \{z : f^n(z) \to \infty\}$.

Fact: In general, $\mathcal{J}(f) = \partial I(f)$. For f in EL-class, $\mathcal{J}(f) = \overline{I(f)}$.

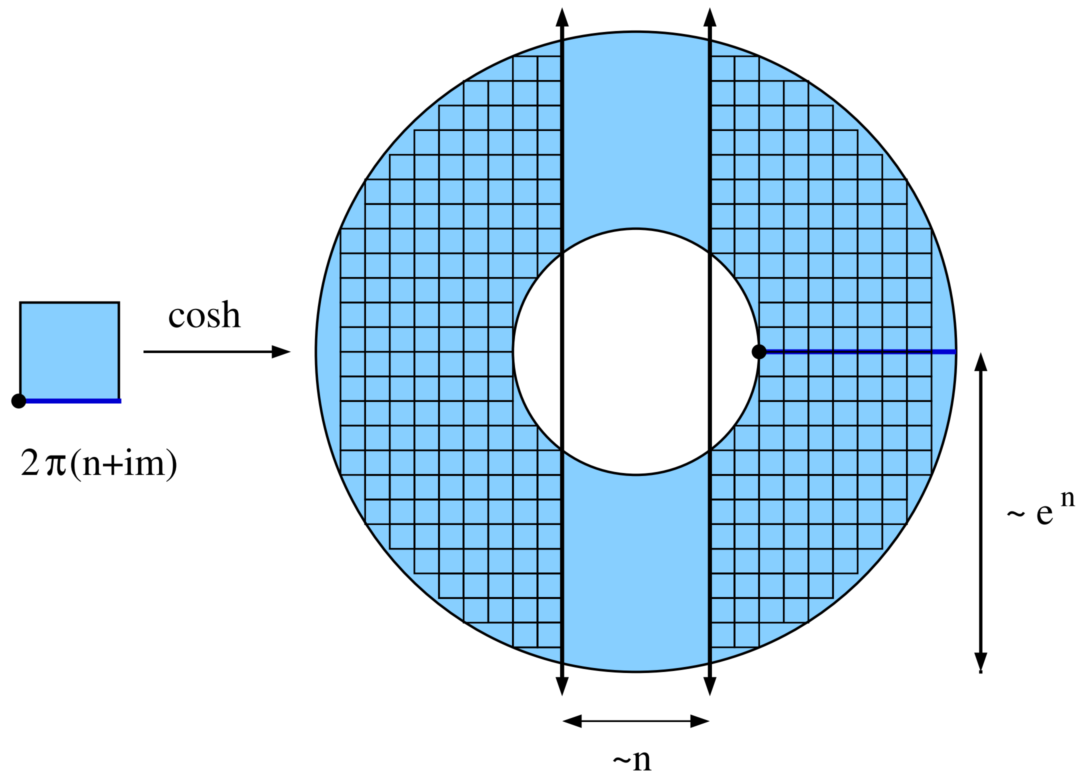


Definition of $\exp(z)$.



Definition of $\cosh(z)$. $\cosh(-x+iy) = \overline{\cosh(x+iy)}$

Proof that $\mathrm{area}(\mathcal{J}) > 0$ **for cosh:**

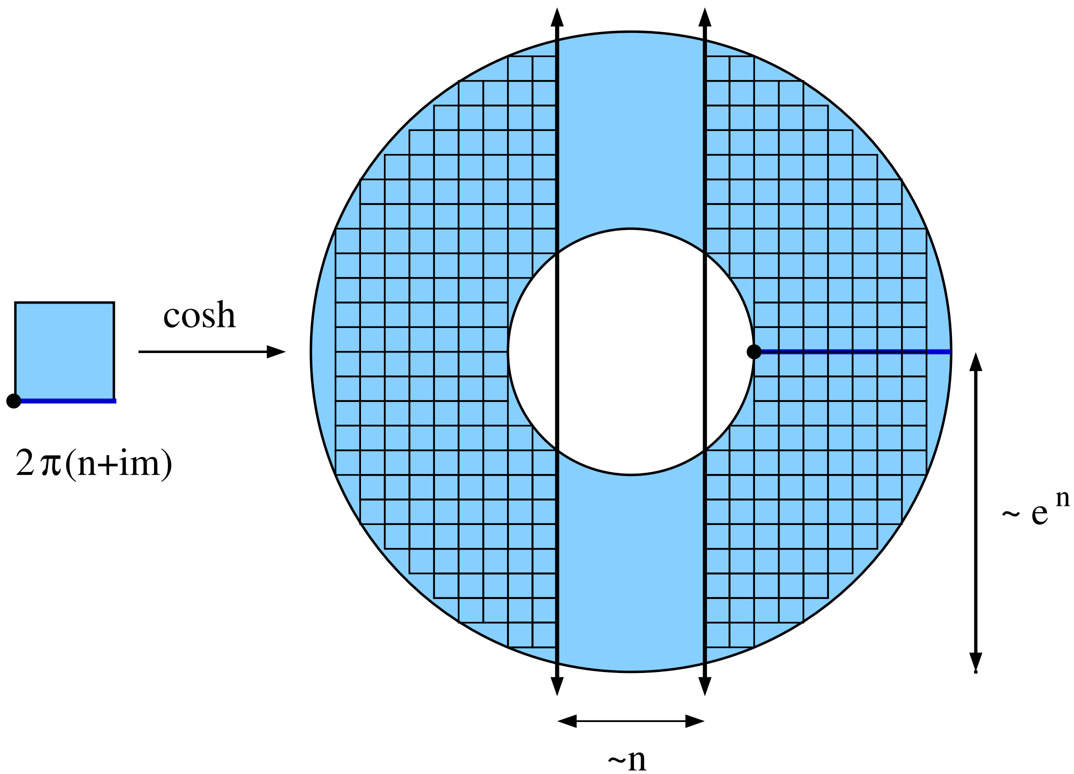


Let $S = 2\pi(n + im) + [0, 2\pi]^2$.

$\cosh(S)$ approximately covers annulus A_n of area $\simeq 2^{2|n|}$.

Annulus contains $\simeq e^{2|n|}$ disjoint translates of S.

Proof that $\mathrm{area}(\mathcal{J}) > 0$ **for cosh:**



Omit $\simeq |n| \cdot e^{|n|}$ squares near y-axis, $\simeq e^{|n|}$ near ∂A_n.

Remaining squares cover $1 - O(|n| \cdot e^{-|n|})$ area of annulus.

$\sum_{n>0} ne^{-n} < \infty \Rightarrow$ positive area escapes (so is in $\mathcal{J}$.)