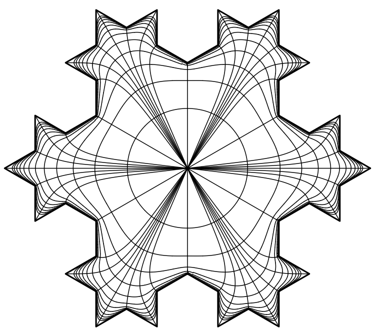
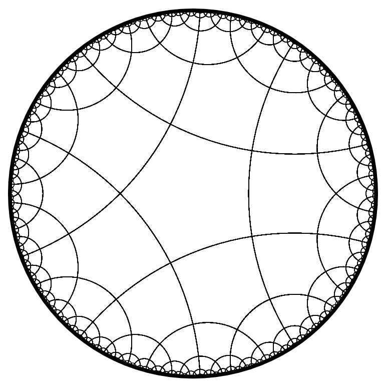
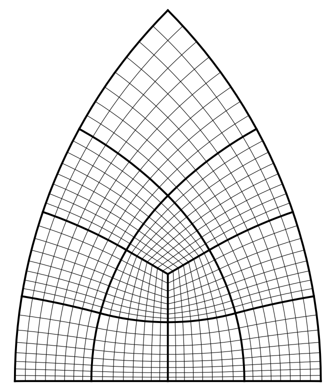
Conformal Maps, Optimal Meshes and Sullivan's Convex Hull Theorem

Christopher J. Bishop

Colloquium, March 3, 2011





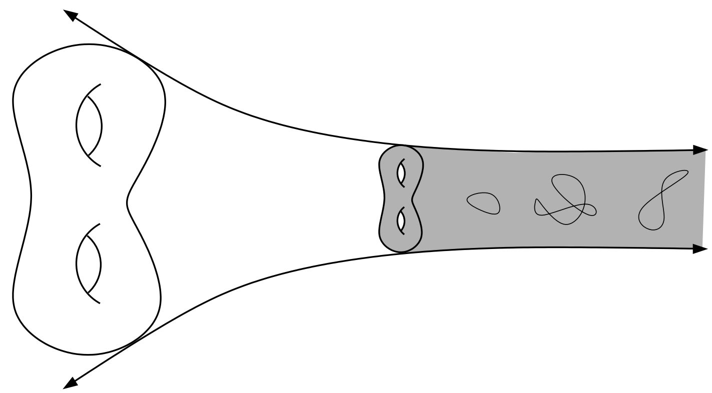


www.math.sunysb.edu/~bishop/lectures

Suppose M is a hyperbolic 3-manifold.

Convex hull of closed geodesics is the convex core, $C(M)$.

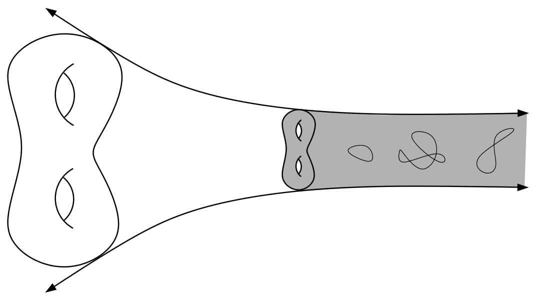
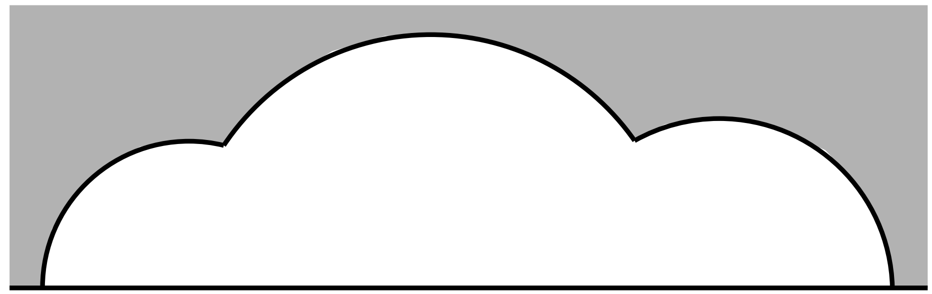
Boundary at infinity is a Riemann surface, ∂M.



Sullivan's convex hull theorem: $\partial C(M)$ and ∂M are bi-Lipschitz equivalent (hyperbolic metrics).

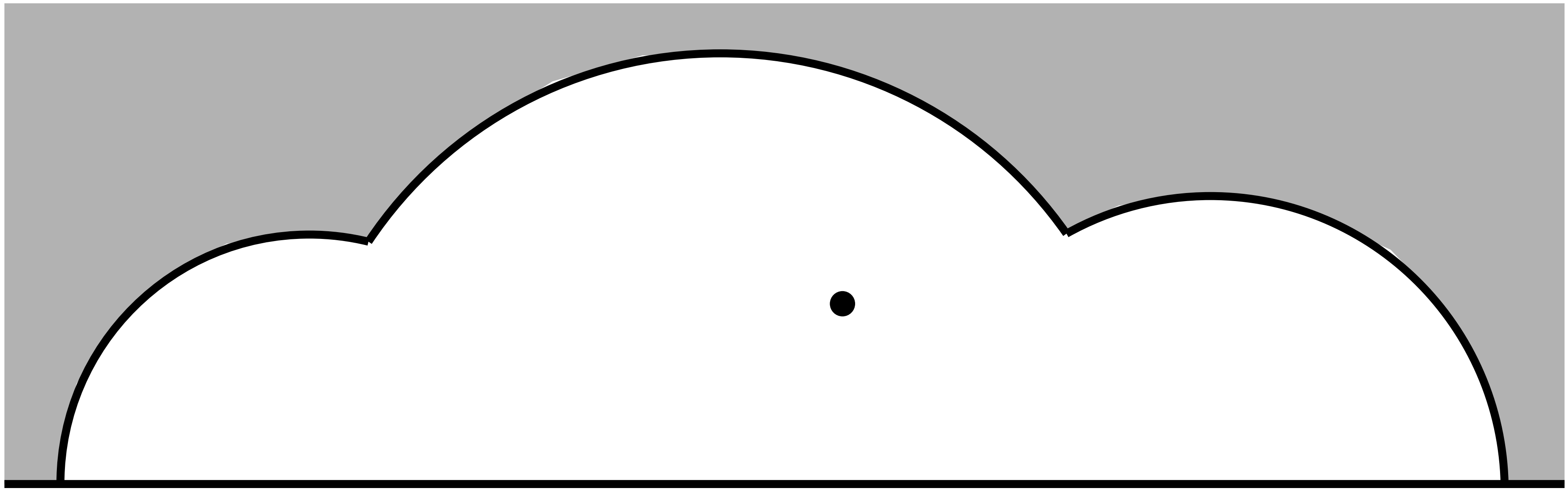
BiLipschitz: $A|x - y| \leq |f(x) - f(y)| \leq B|x - y|$.

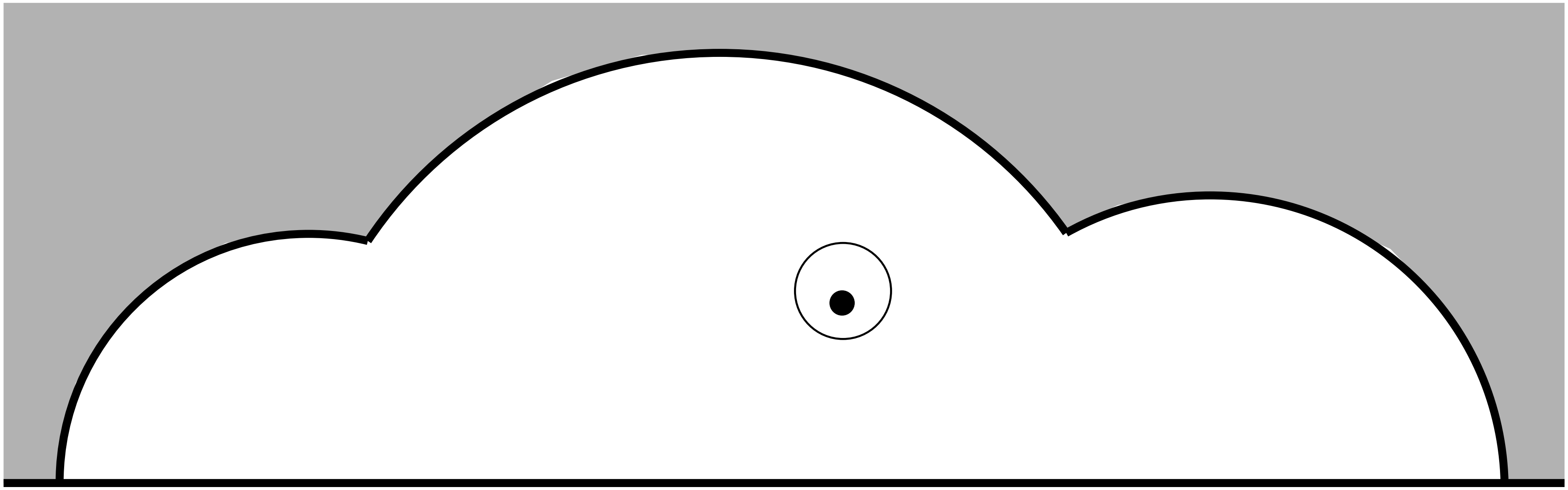
Pass to universal cover: upper half-space.

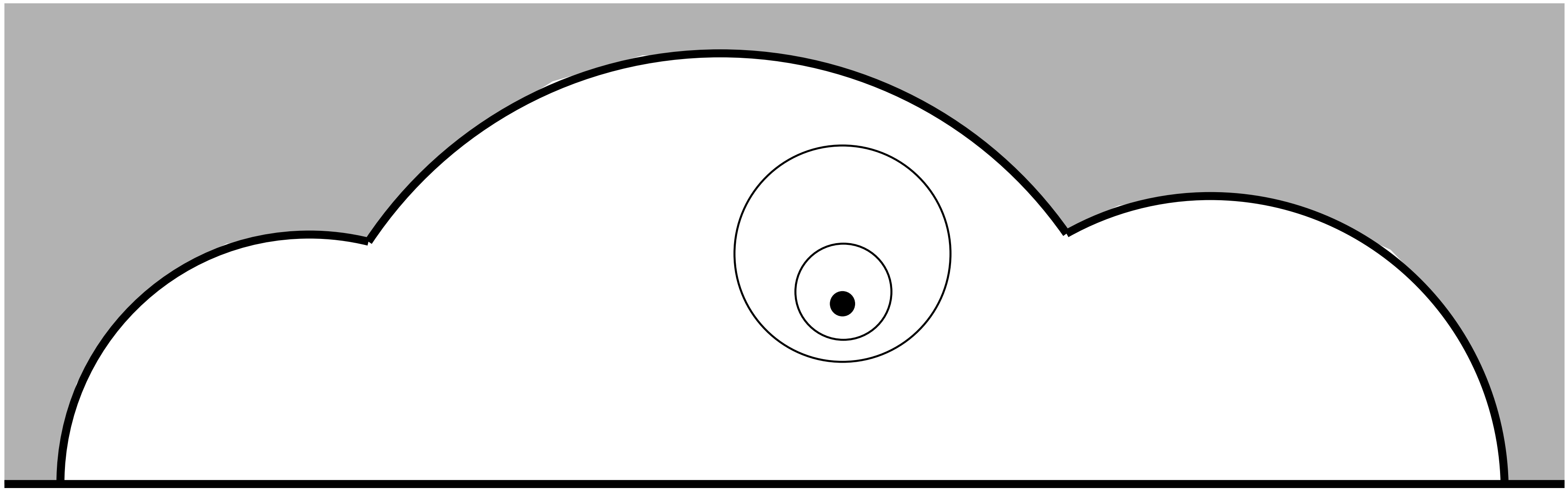



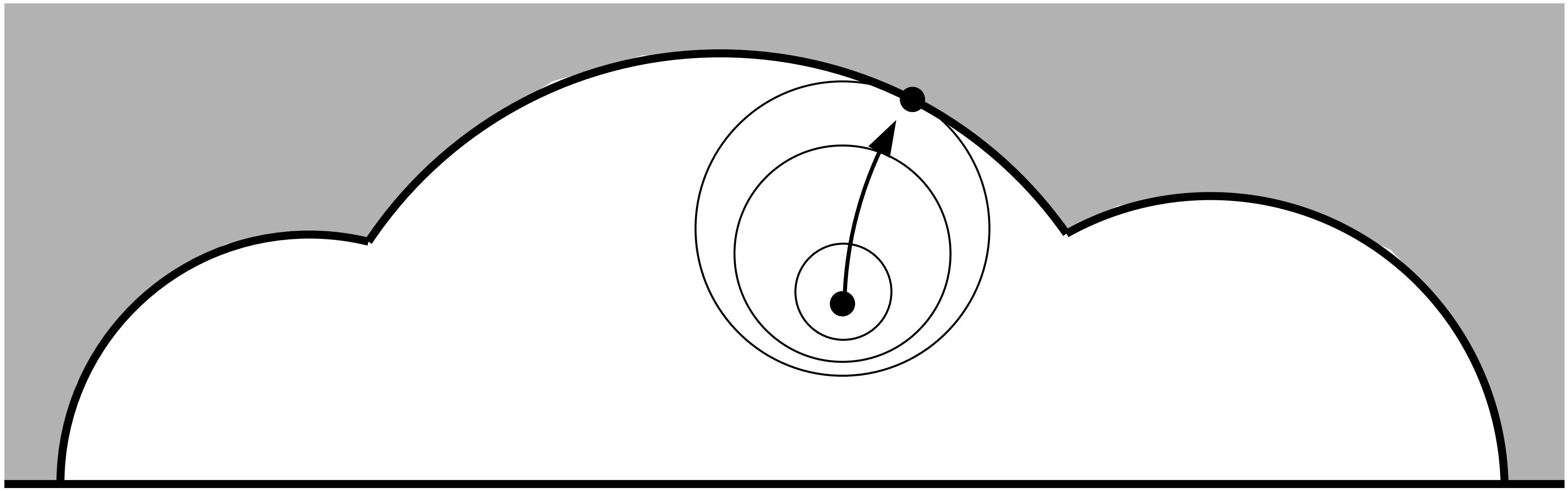
- ∂M becomes planar Ω.
- $C(M)$ (shaded) is convex hull of Ω^c.
- $M \setminus C(M)$ is union of hemispheres with base in Ω.
- Boundary is called **dome** of Ω.

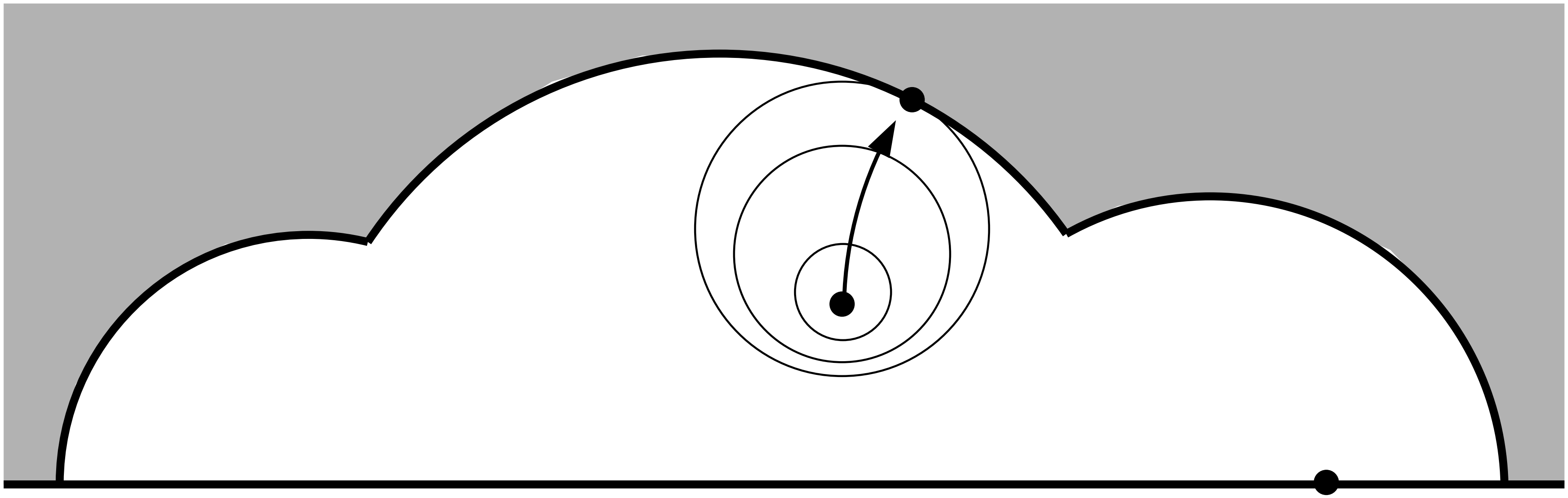
David Epstein and Al Marden: CHT is true for any simply connected Ω. (Ω and dome are biLipschitz equiv.)

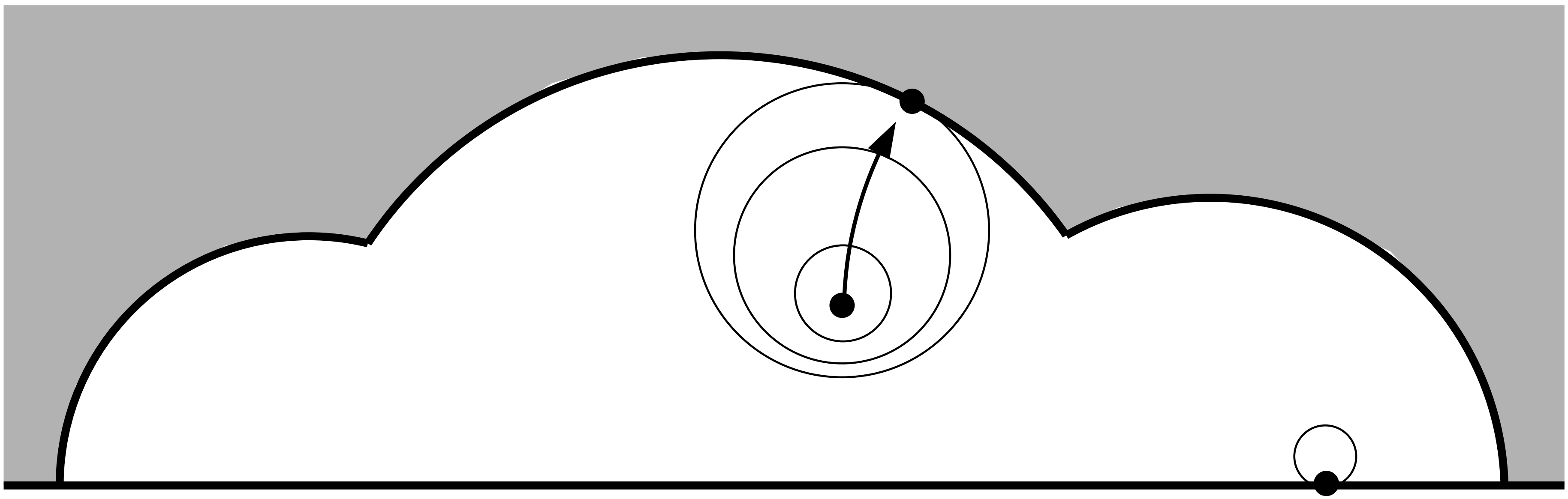


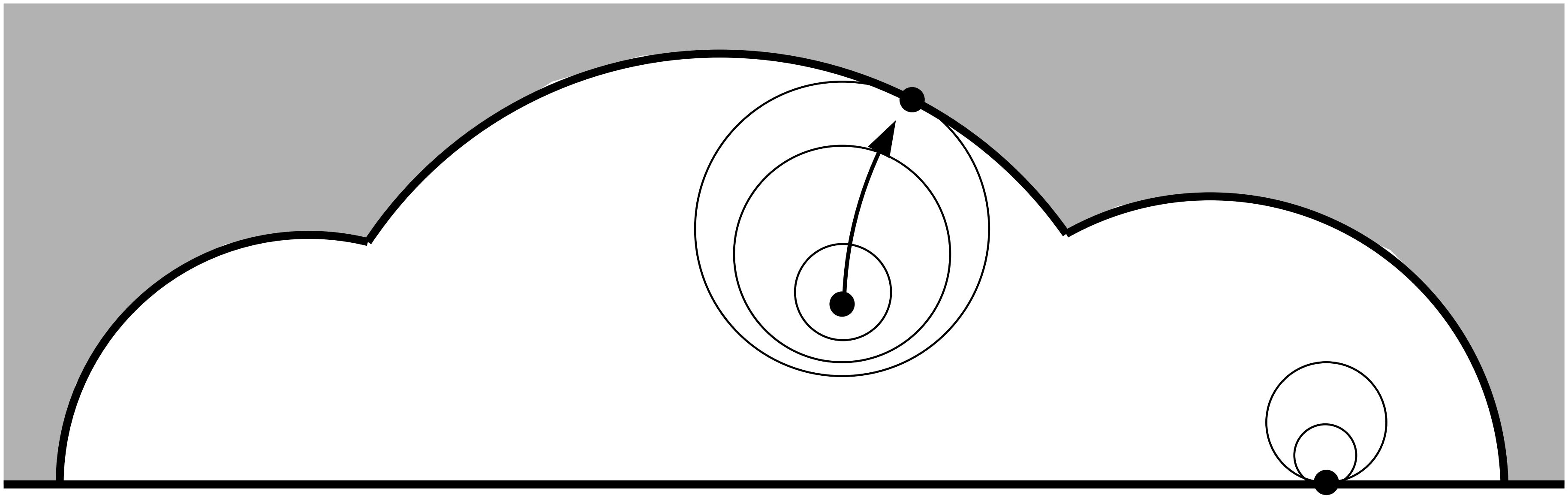


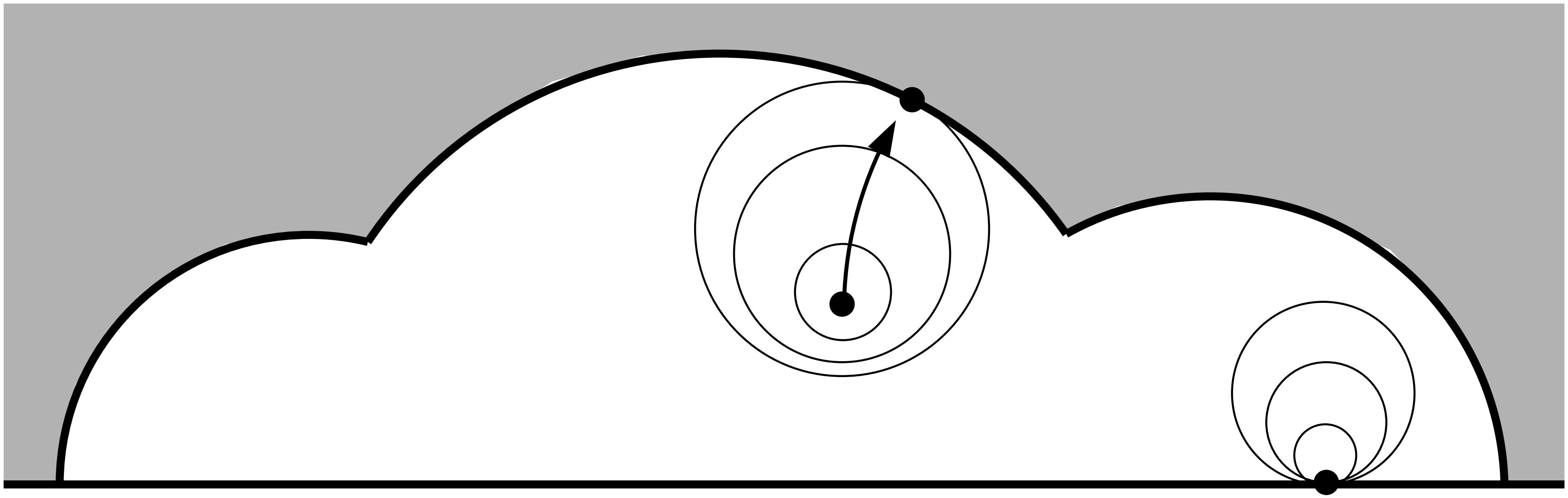


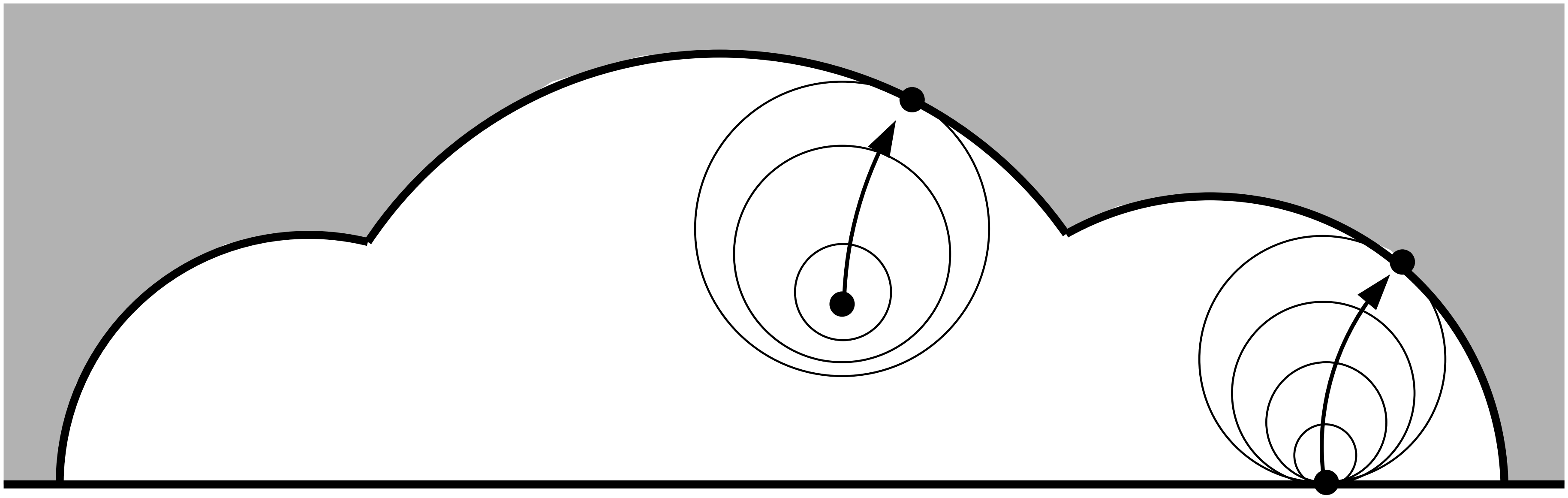


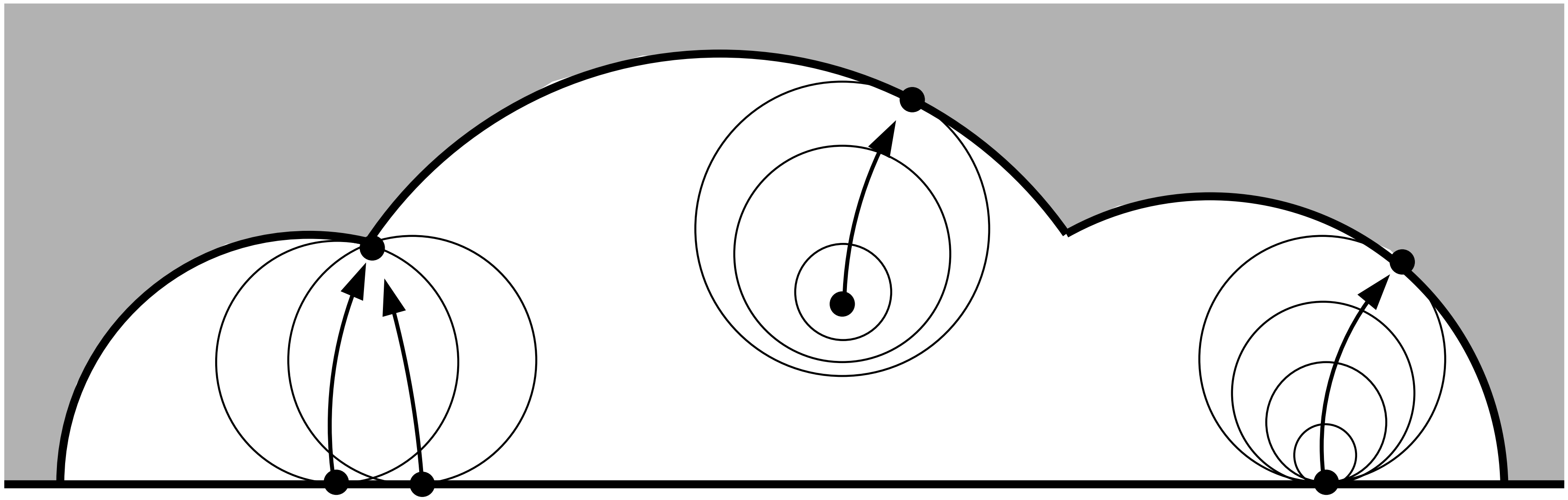


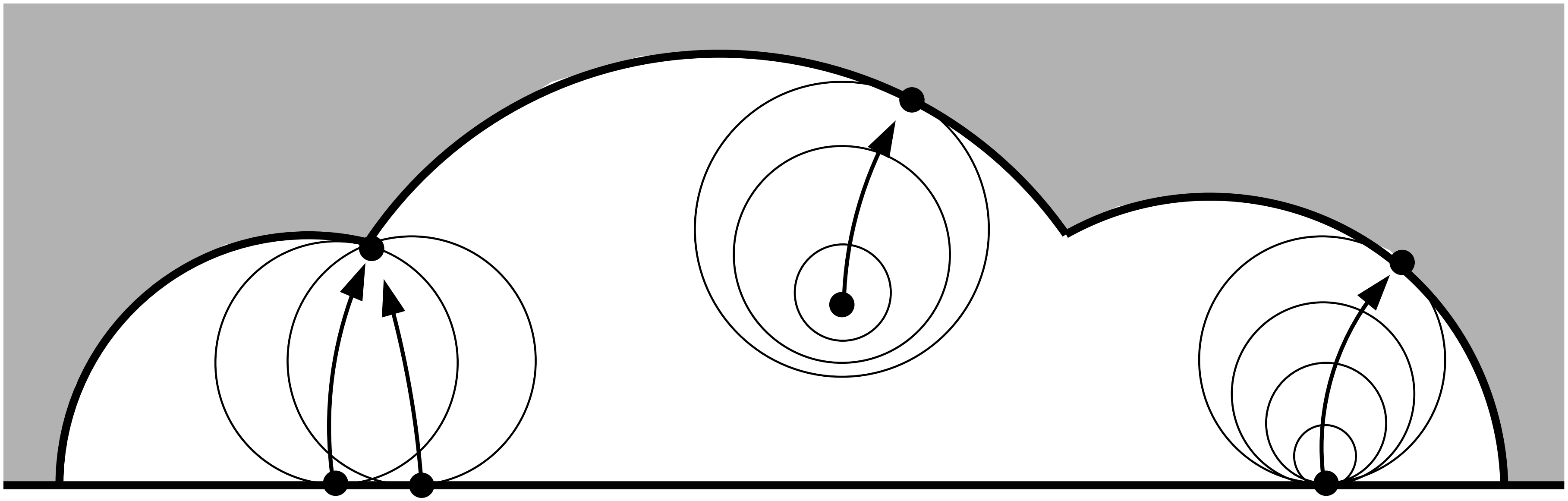








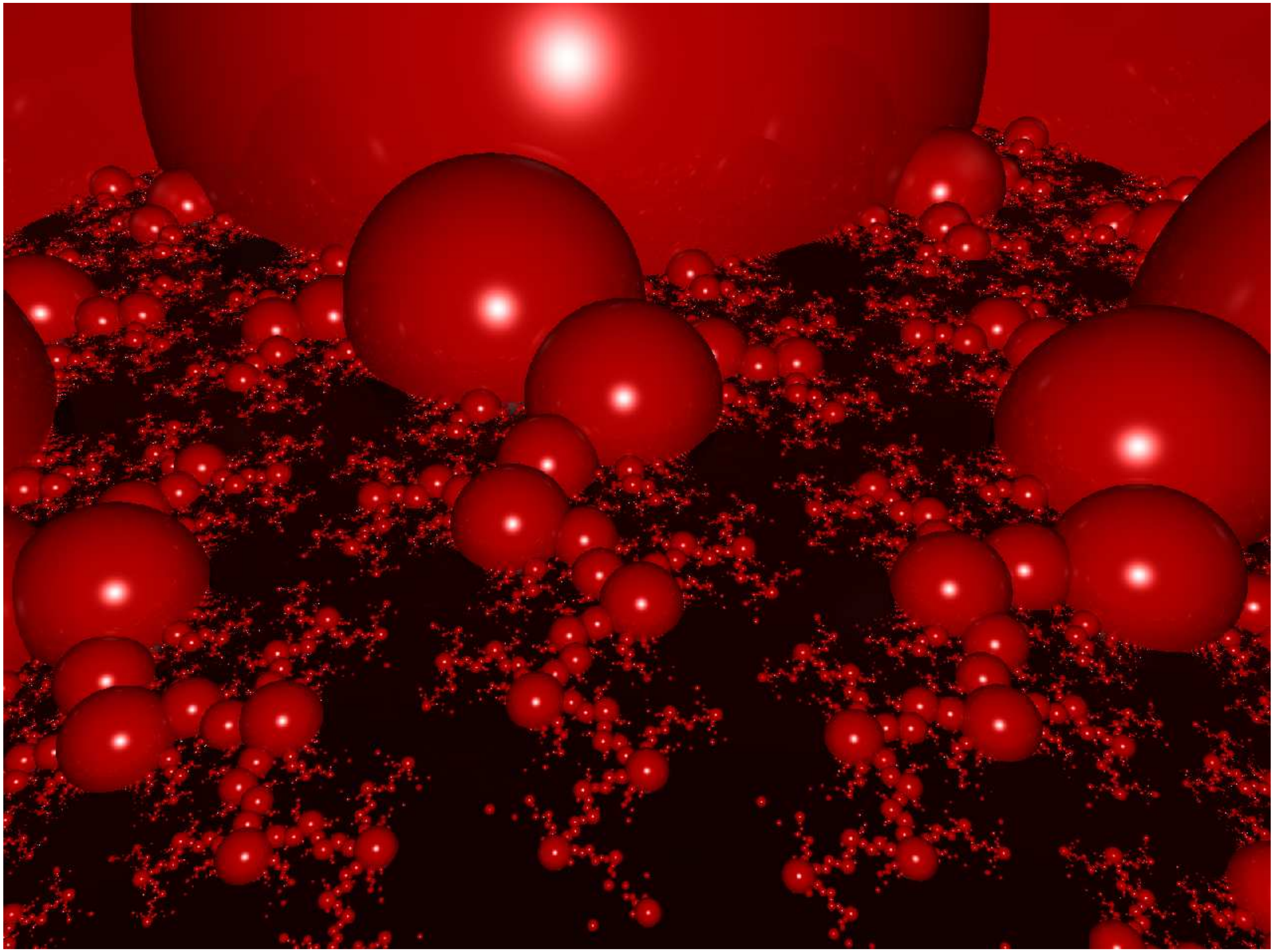




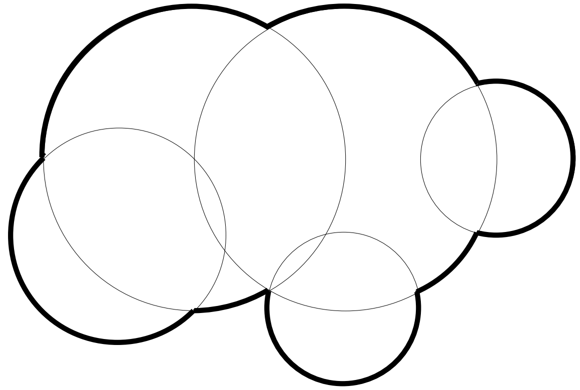
Nearest point retraction in hyperbolic space extends to map $R : \Omega \to S =$ Dome and is a quasi-isometry

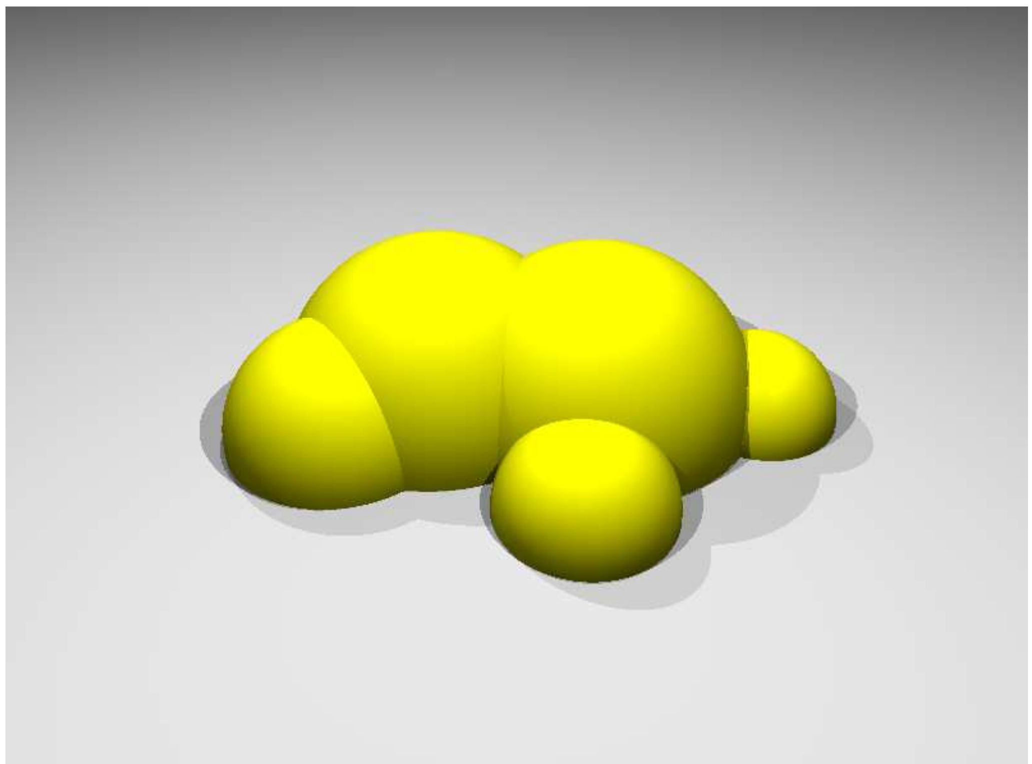
$$\frac{1}{A}\rho_\Omega(x,y) - B \le \rho_S(R(x),R(y)) \le A\rho_\Omega(x,y).$$

Standard techniques improve to biLipschitz or QC.

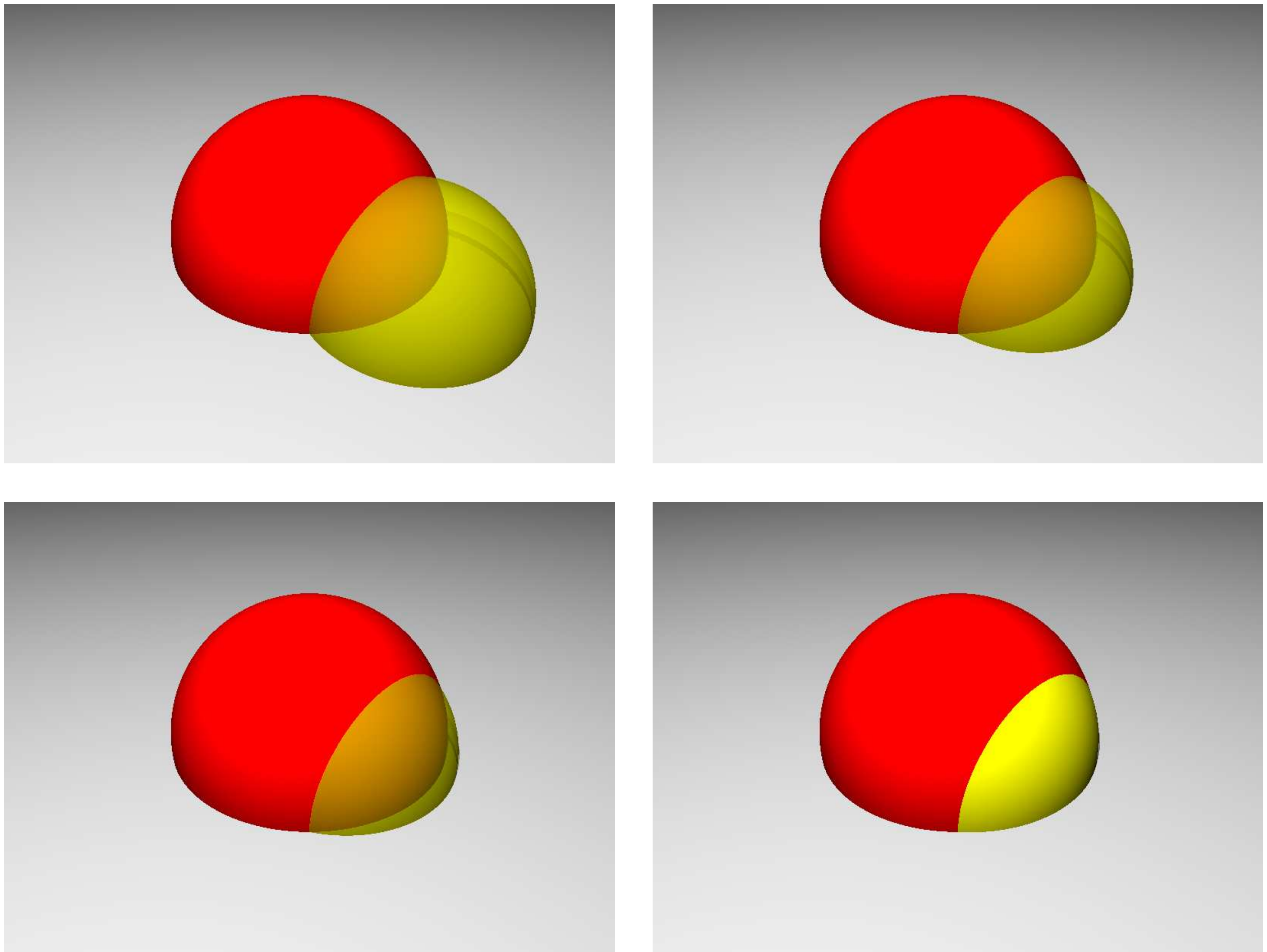


The dome of a geometrically infinite Kleinian group.

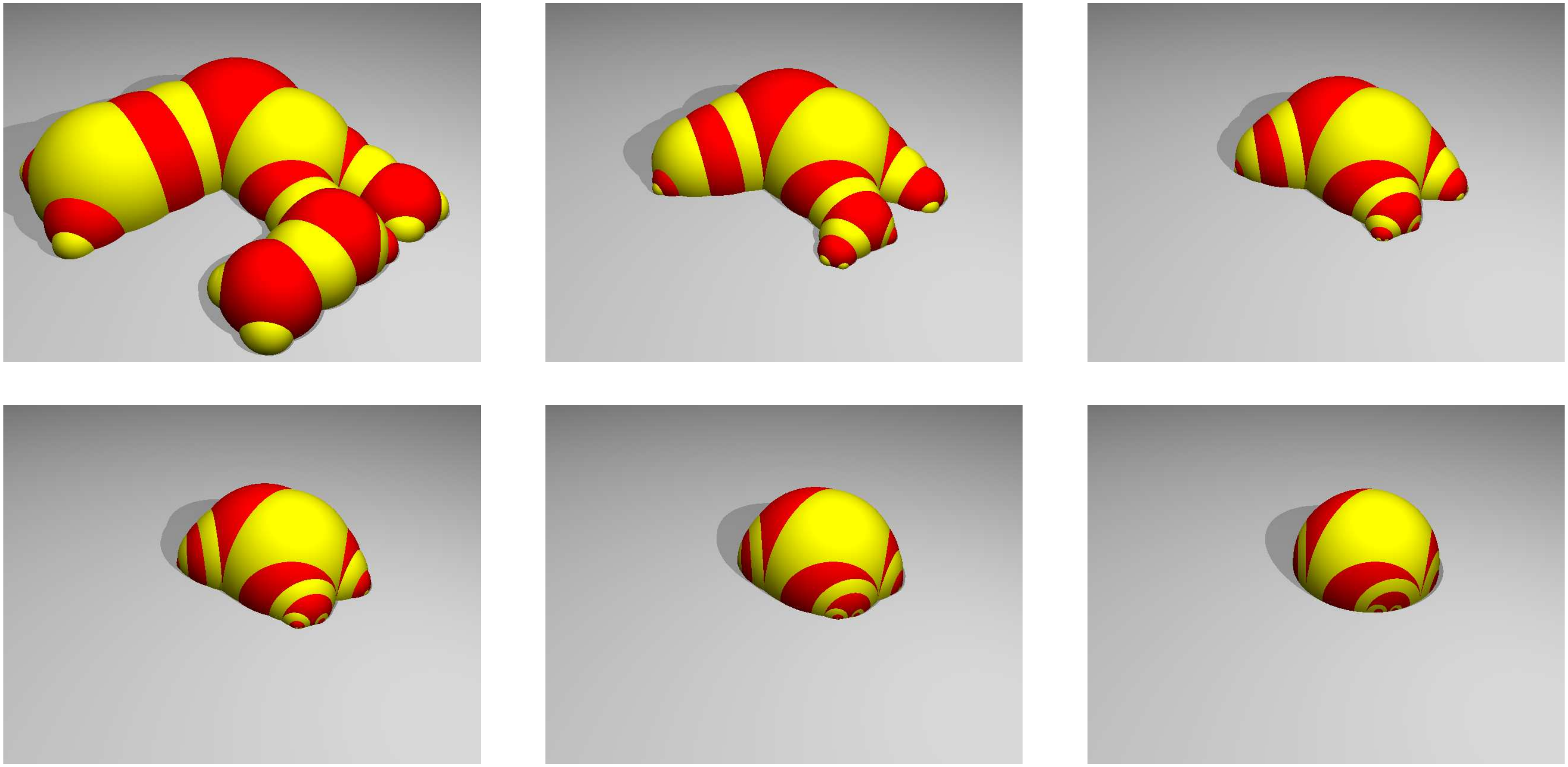




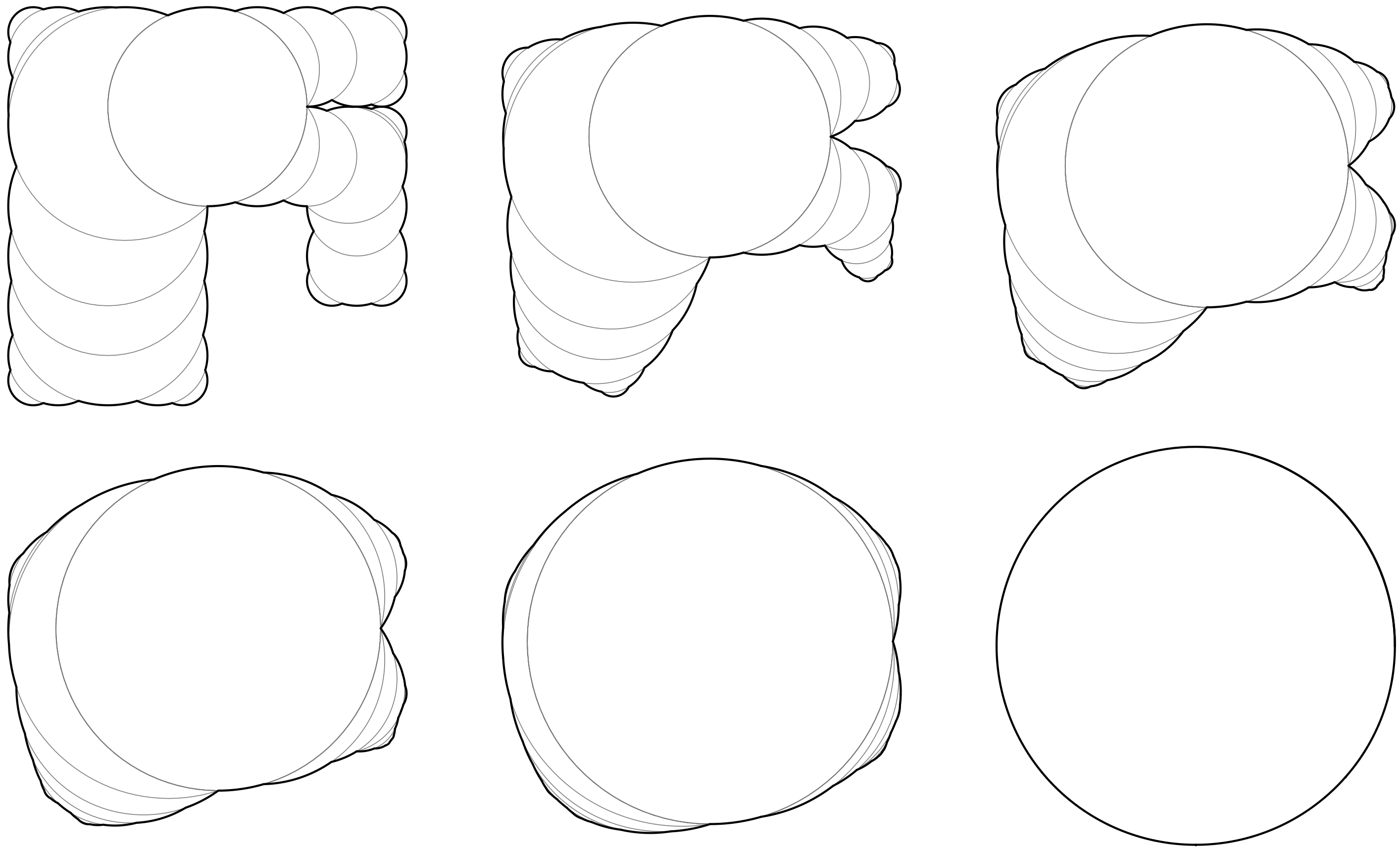
It is easy to map any dome conformally to a disk.



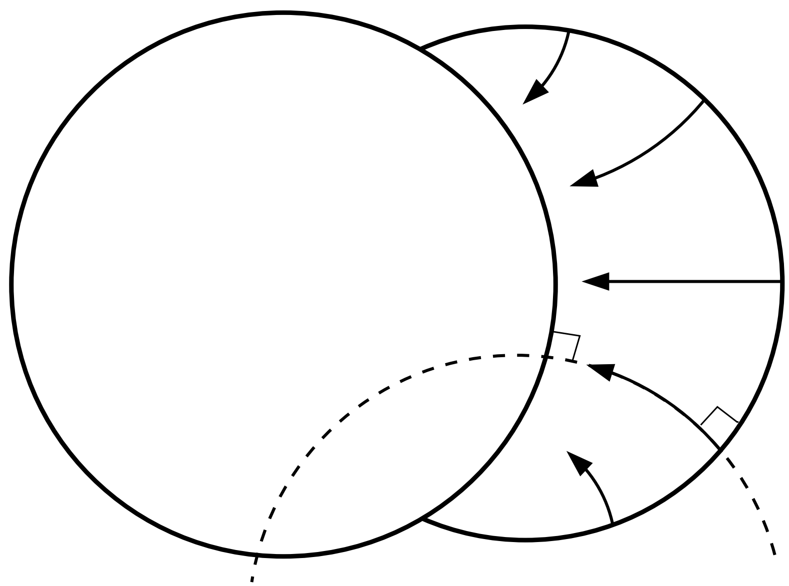
Iota map = isometry from a dome to hyperbolic disk.



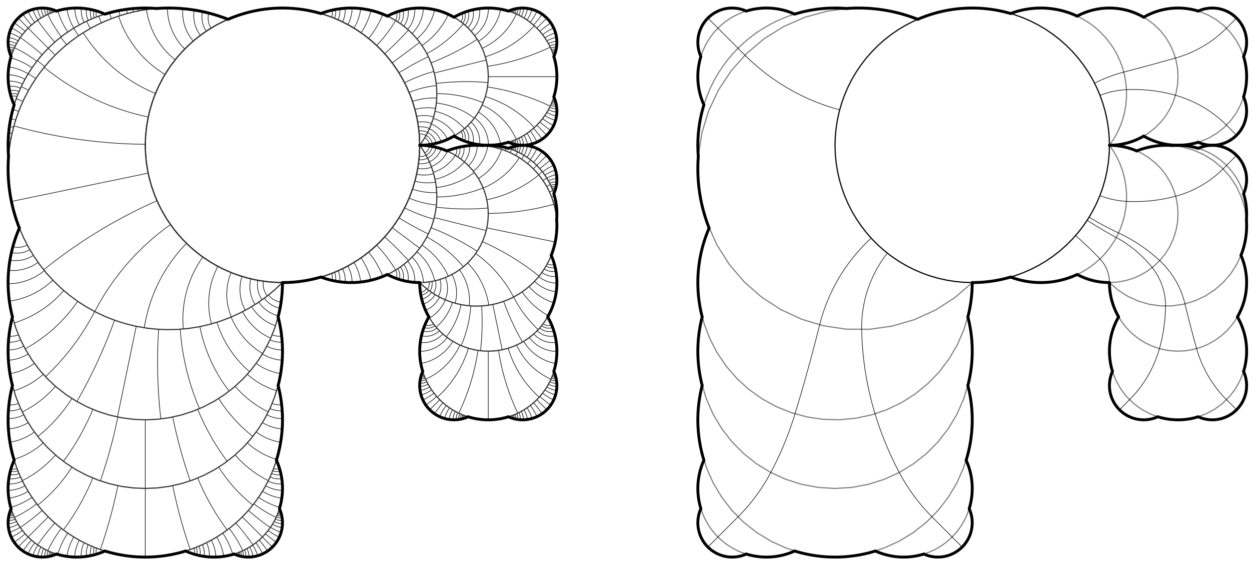
A dome is a hinged surface. We map it to a hemisphere by making all faces flush with each other. More interesting in hyperbolic space than Euclidean space because parallel postulate fails (more non-intersecting lines).



Flattening dome collapses crescent in base by collapsing orthogonal arcs.



Instead of collapsing all crescents at once, we may do one at a time, from leaves towards root.

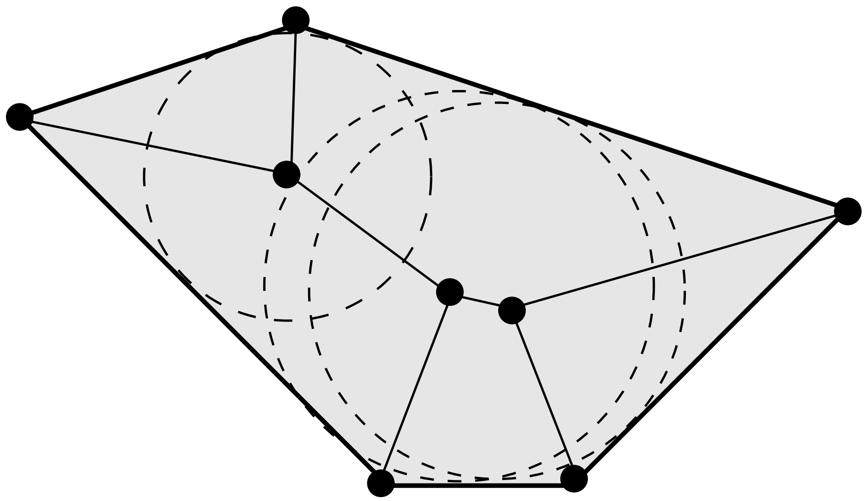


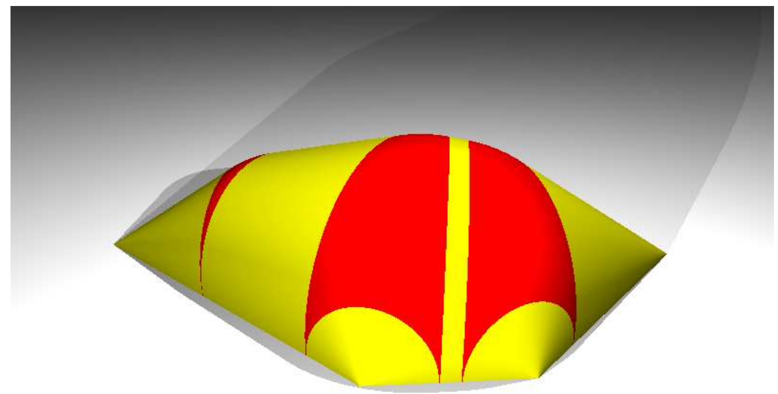
Defines a flow from boundary to disk along foliation of crescents by orthogonal arcs.

Taking limits, this flow exists for any domain.

Iota is closed related to medial axis in computer science.

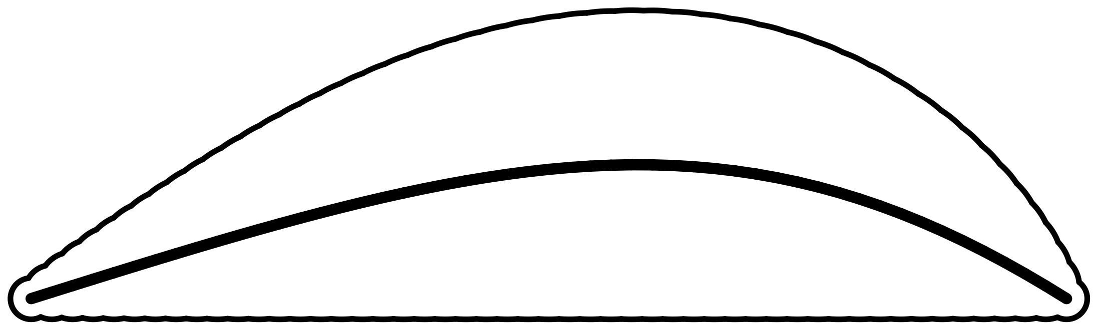
Medial axis is set of centers of subdisks of Ω that hit boundary in at least two points.

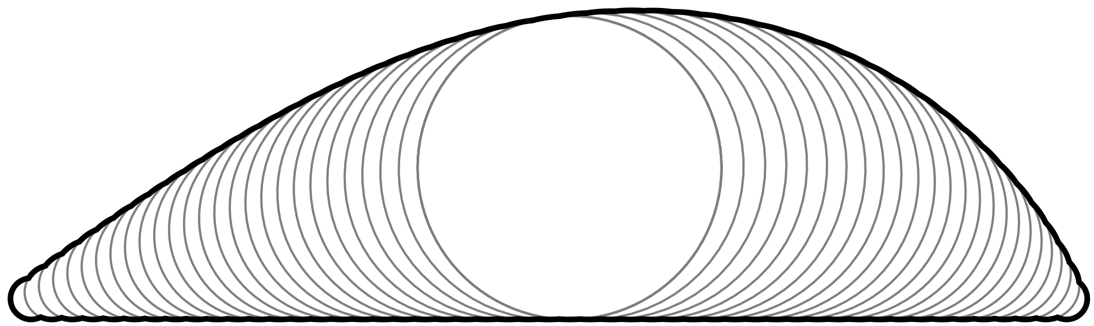


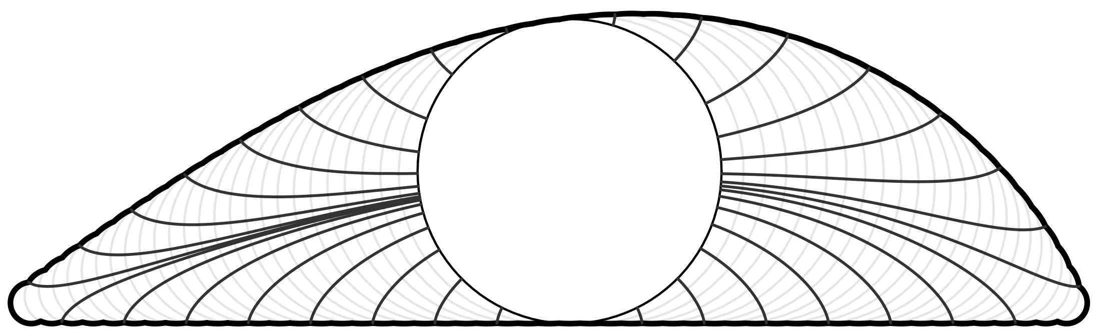


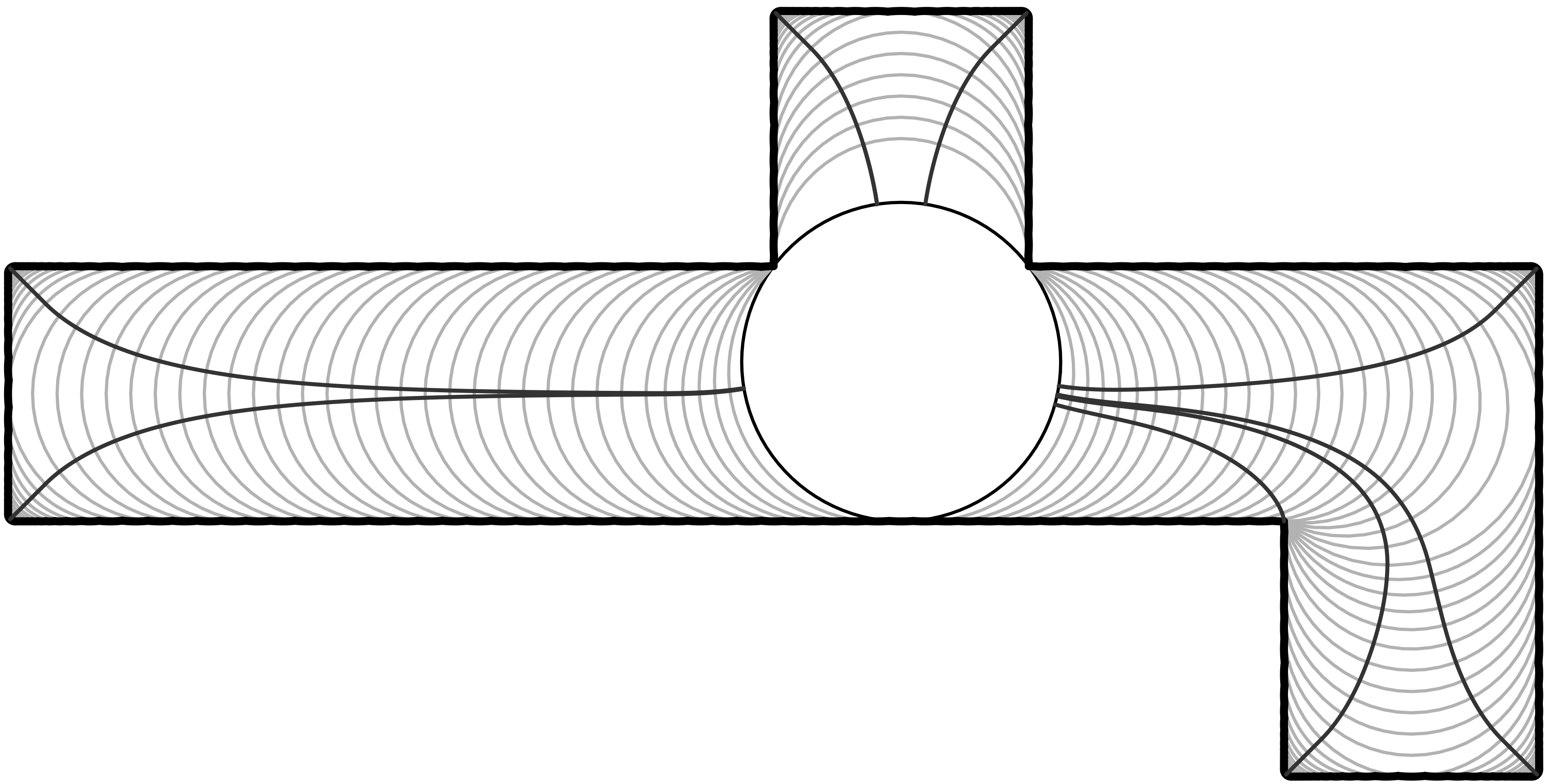
For n-gon it is computable in $O(n)$ time.

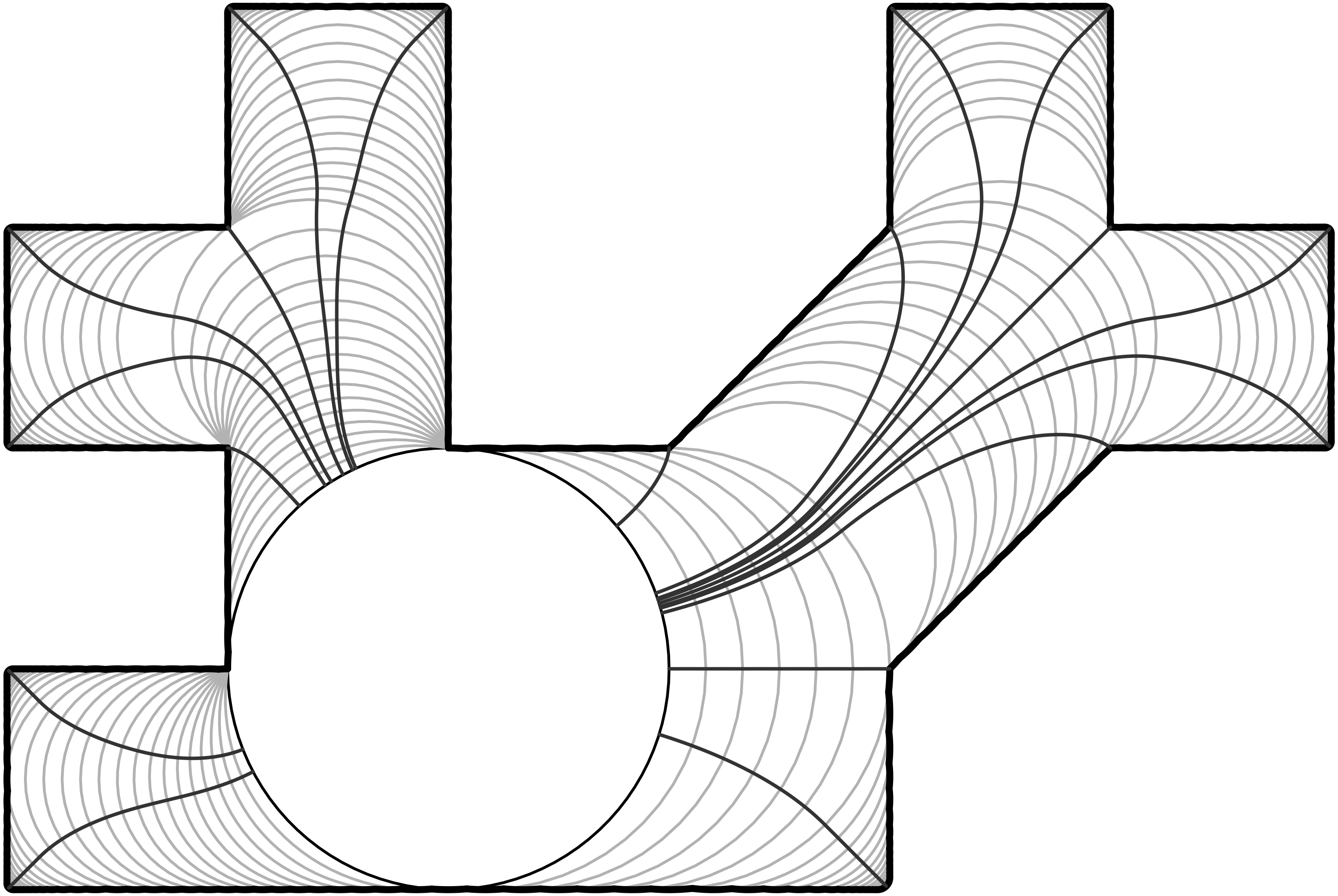
Iota is computable from Medial Axis in linear time.

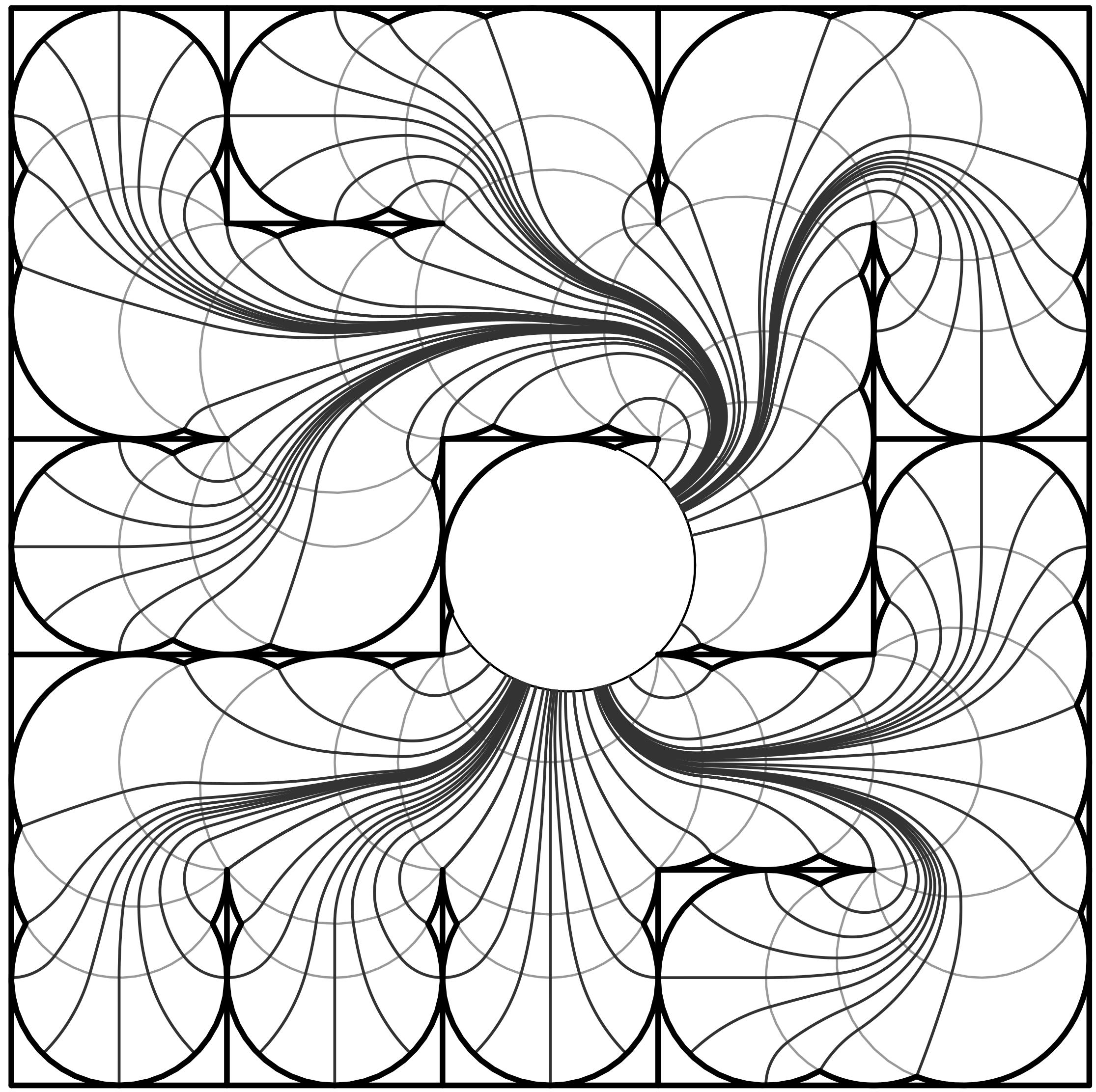


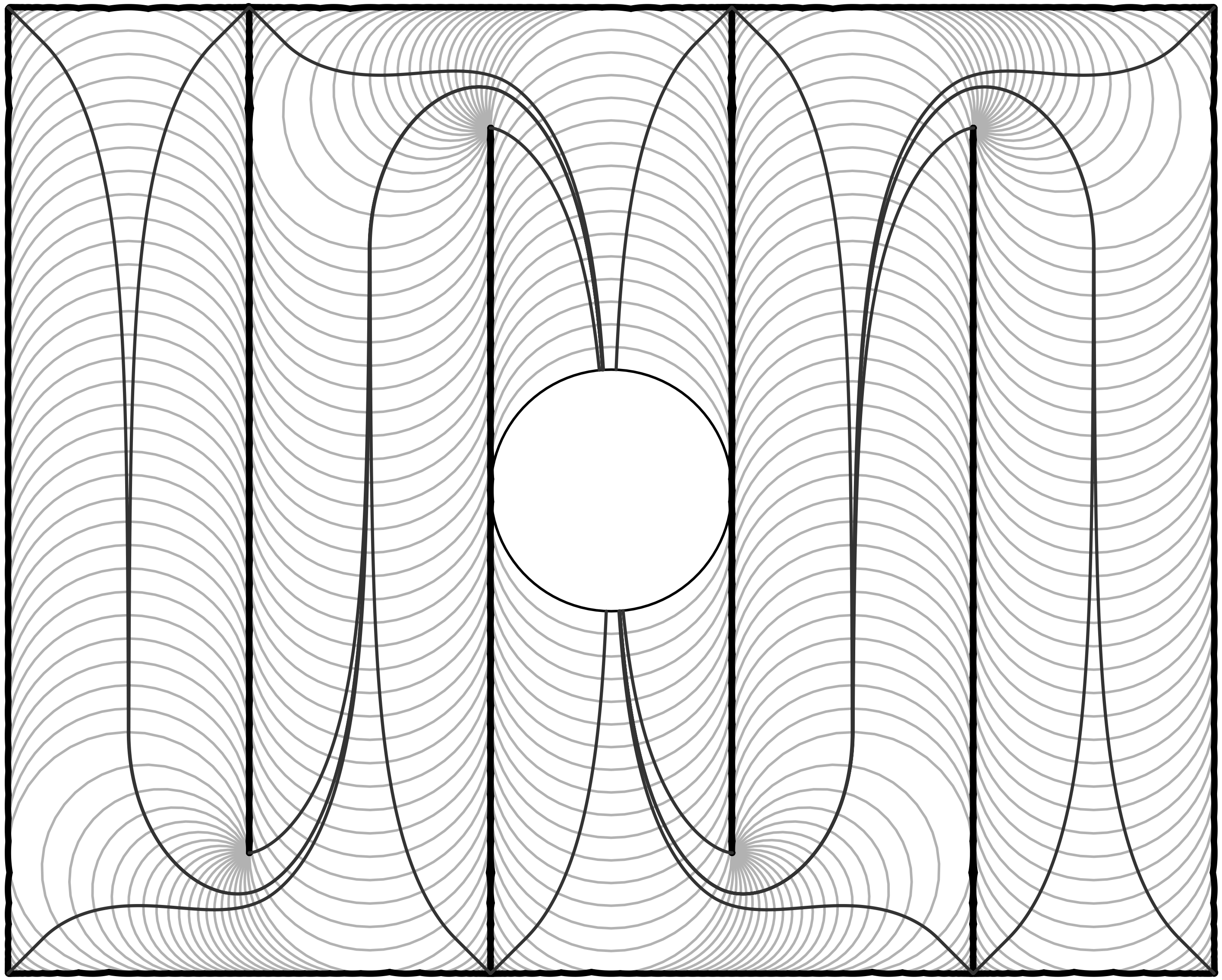


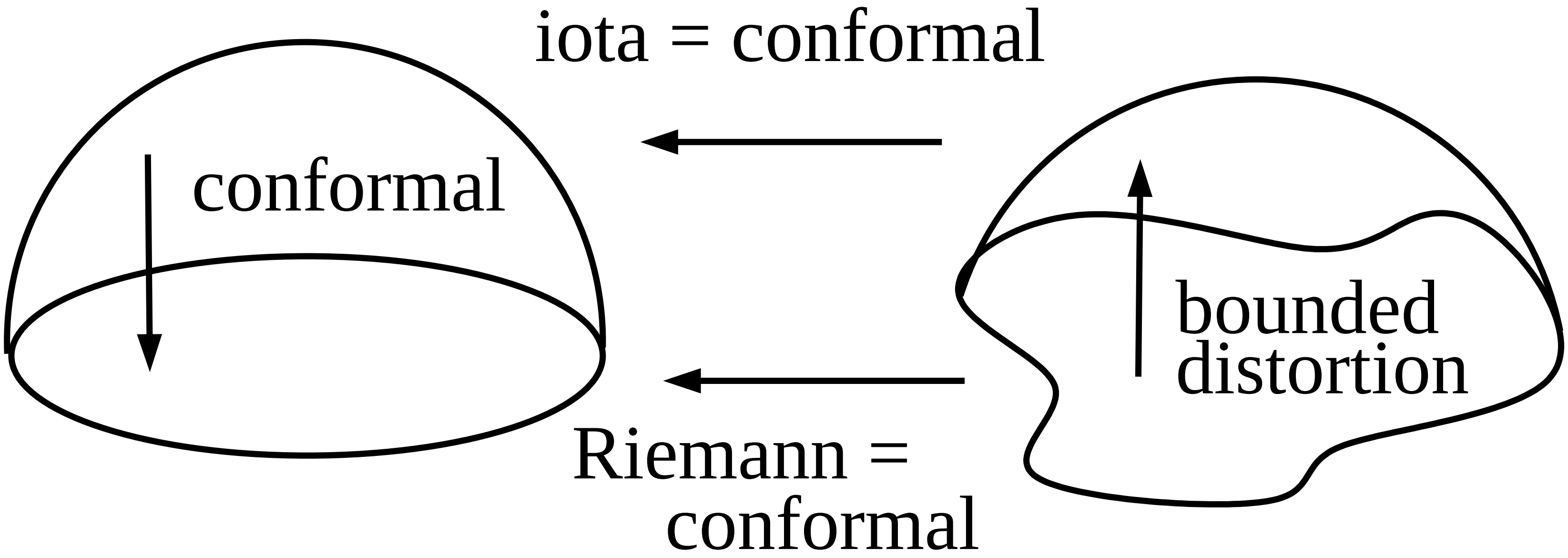












Iota ≈ Riemann Mapping by Sullivan's CHT.

Iota is computable in $O(n)$ time for n-gons.

Iota is locally Lipschitz (decreases boundary length).

Factorization Thm: A conformal map $f : \Omega \to \mathbb{D}$ can be written as $f = h \circ g$ where

- $g : \Omega \to \mathbb{D}$ is locally Lipschitz (Euclidean metrics)
- $h : \mathbb{D} \to \mathbb{D}$ is biLipschitz (hyperbolic metric)

Both maps are 8-quasiconformal.

Factorization Thm: A conformal map $f : \Omega \to \mathbb{D}$ can be written as $f = h \circ g$ where

- $g : \Omega \to \mathbb{D}$ is locally Lipschitz (Euclidean metrics)
- $h : \mathbb{D} \to \mathbb{D}$ is biLipschitz (hyperbolic metric)

Both maps are 8-quasiconformal.

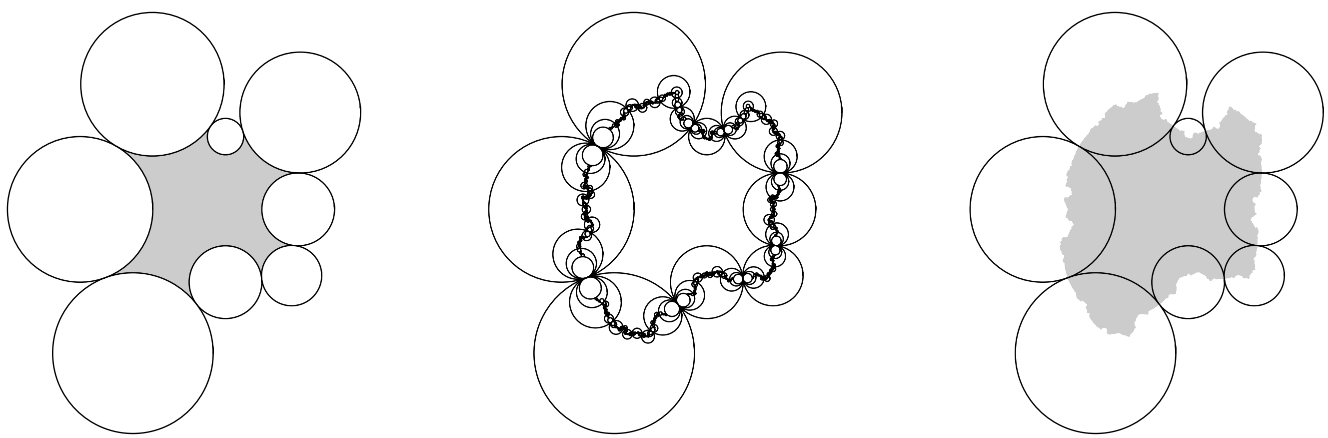
This has an application to conformal dynamics.

Factorization Thm: A conformal map $f : \Omega \to \mathbb{D}$ can be written as $f = h \circ g$ where

- $g : \Omega \to \mathbb{D}$ is locally Lipschitz (Euclidean metrics)
- $h : \mathbb{D} \to \mathbb{D}$ is biLipschitz (hyperbolic metric)

Both maps are 8-quasiconformal.

Bowen's Dichotomy: If Ω is simply connected and invariant under a discrete group G of Möbius transformations then $\partial\Omega$ is either a circle or $\dim(\partial\Omega) > 1$.



Factorization Thm: A conformal map $f : \Omega \to \mathbb{D}$ can be written as $f = h \circ g$ where

- $g : \Omega \to \mathbb{D}$ is locally Lipschitz (Euclidean metrics)
- $h : \mathbb{D} \to \mathbb{D}$ is biLipschitz (hyperbolic metric)

Both maps are 8-quasiconformal.

Bowen's Dichotomy: If Ω is simply connected and invariant under a discrete group G of Möbius transformations then $\partial\Omega$ is either a circle or $\dim(\partial\Omega) > 1$.

- Bowen: true if $R = \Omega/G$ is compact.
- Sullivan: true if R has finite area.
- B: true if R is recurrent for Brownian motion.
- Astala & Zinsmeister: false otherwise.

Factorization Thm: A conformal map $f : \Omega \to \mathbb{D}$ can be written as $f = h \circ g$ where

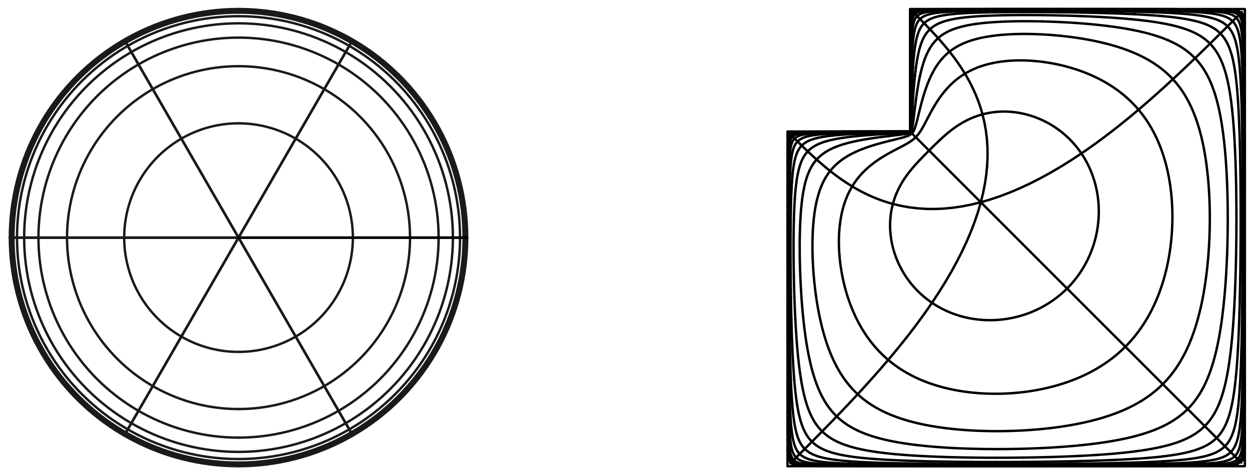
- $g : \Omega \to \mathbb{D}$ is locally Lipschitz (Euclidean metrics)
- $h : \mathbb{D} \to \mathbb{D}$ is biLipschitz (hyperbolic metric)

Both maps are 8-quasiconformal.

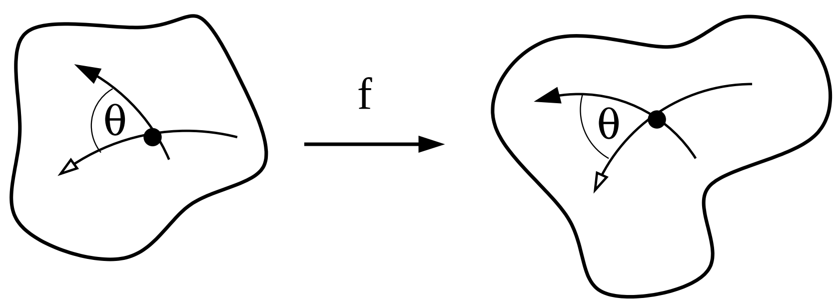
Bowen's Dichotomy: If Ω is simply connected and invariant under a discrete group G of Möbius transformations then $\partial\Omega$ is either a circle or $\dim(\partial\Omega) > 1$.

Idea of proof is to replace conformal map $\mathbb{D} \to \Omega$ (which both expands and contracts) by a QC map that only expands. Instead of discussing the technical details I will give some more recent applications the CHT.

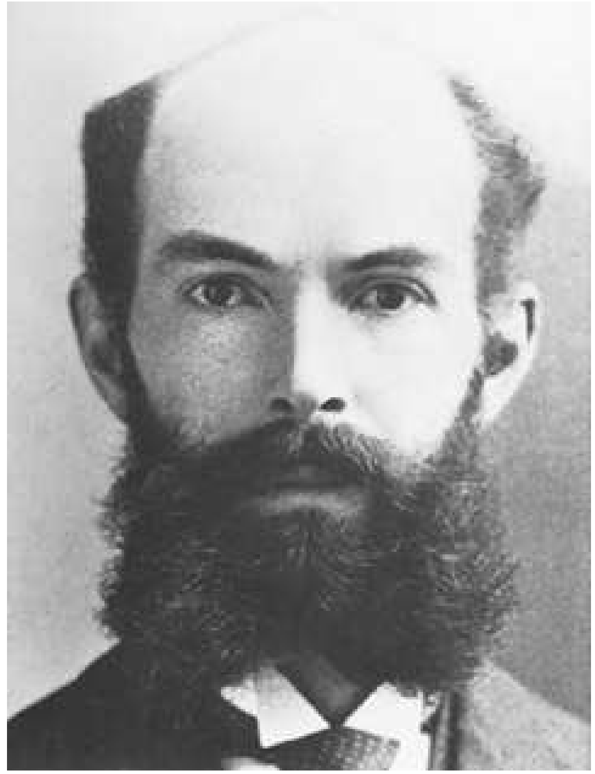
Riemann Mapping Theorem: If Ω is a simply connected, proper subdomain of the plane, then there is a conformal map $f : \mathbb{D} \to \Omega$.



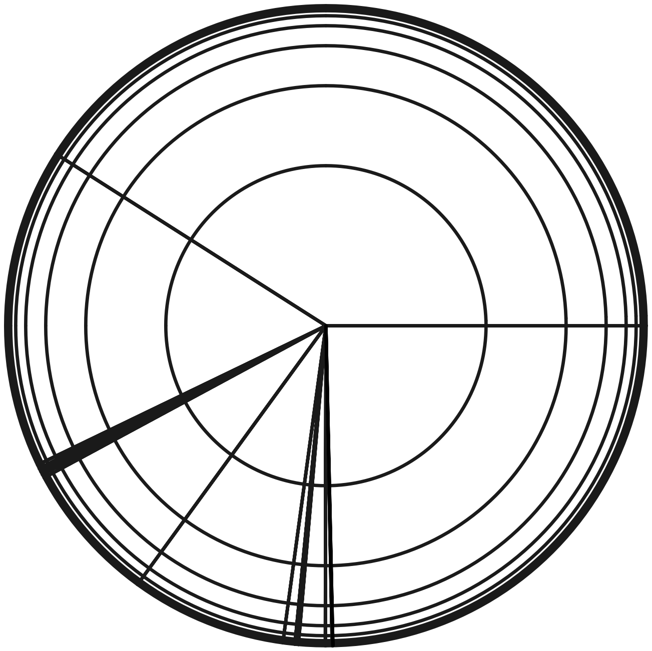
Conformal = angle preserving

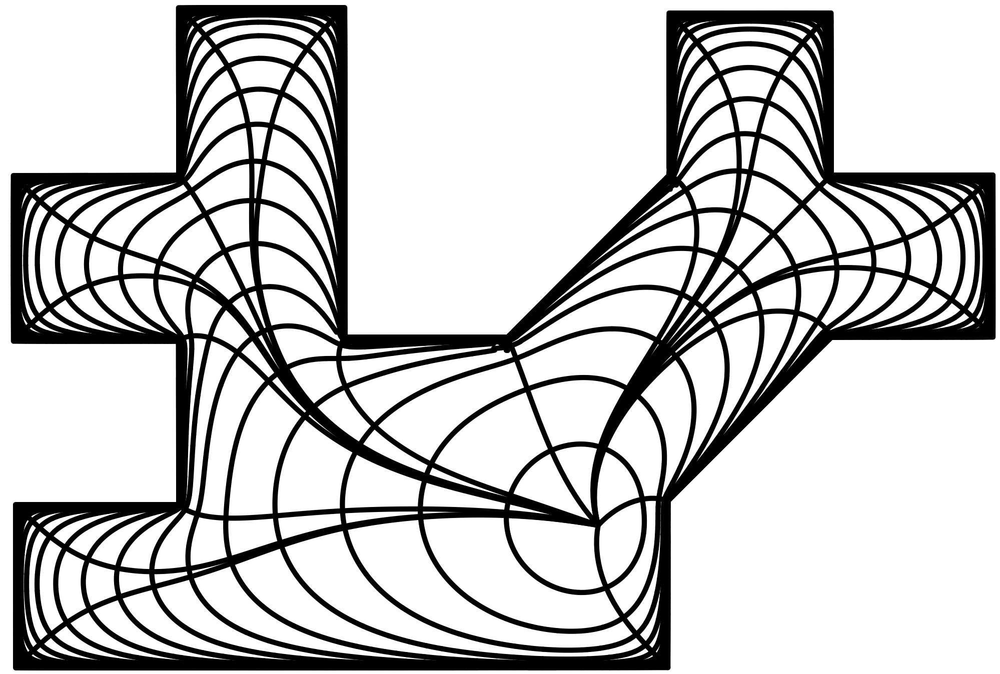


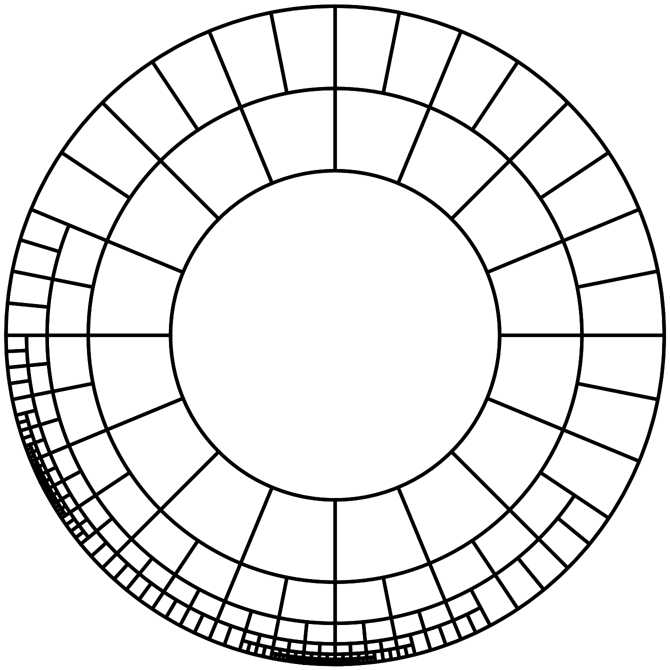
Our Founder
Stated RMT in 1851 thesis

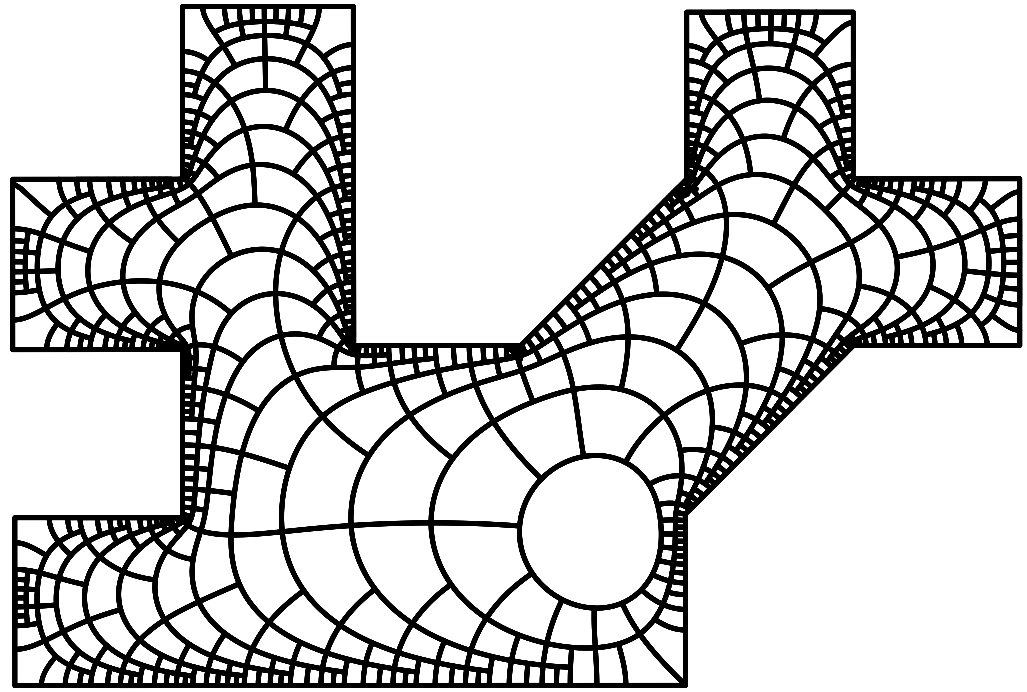


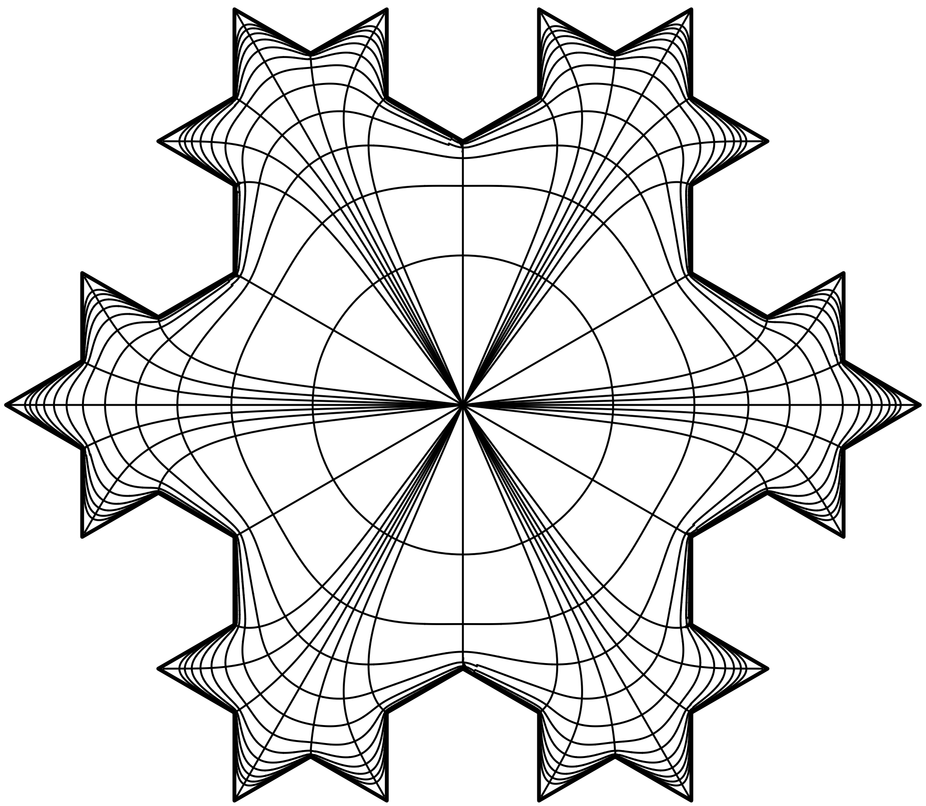
William Fogg Osgood
First proof of RMT, 1900

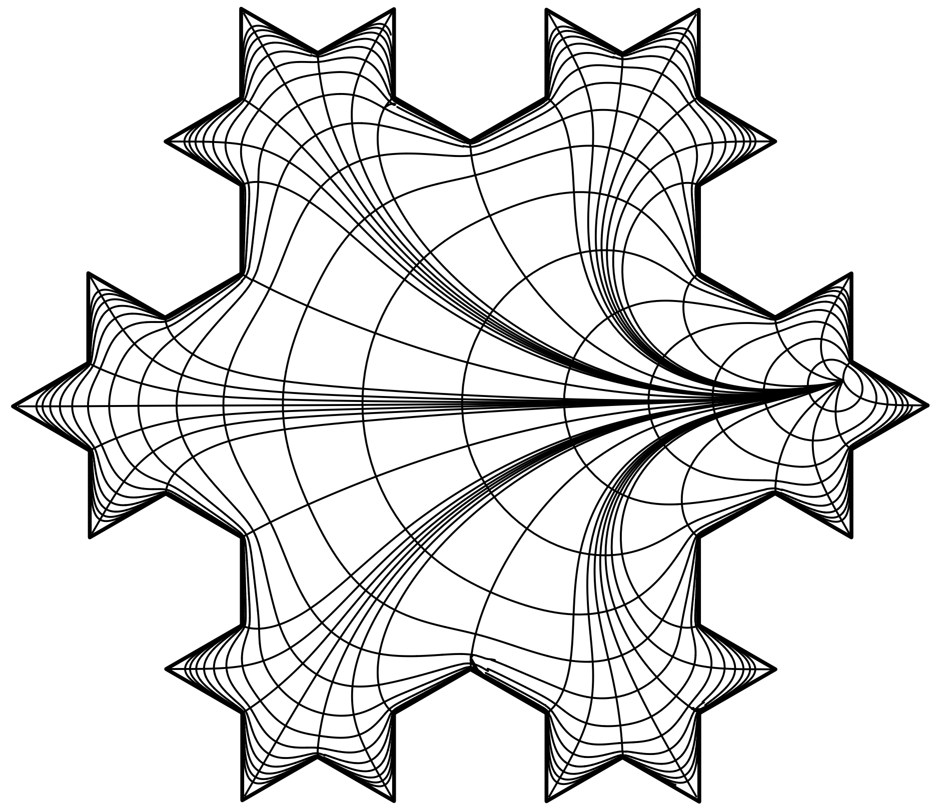


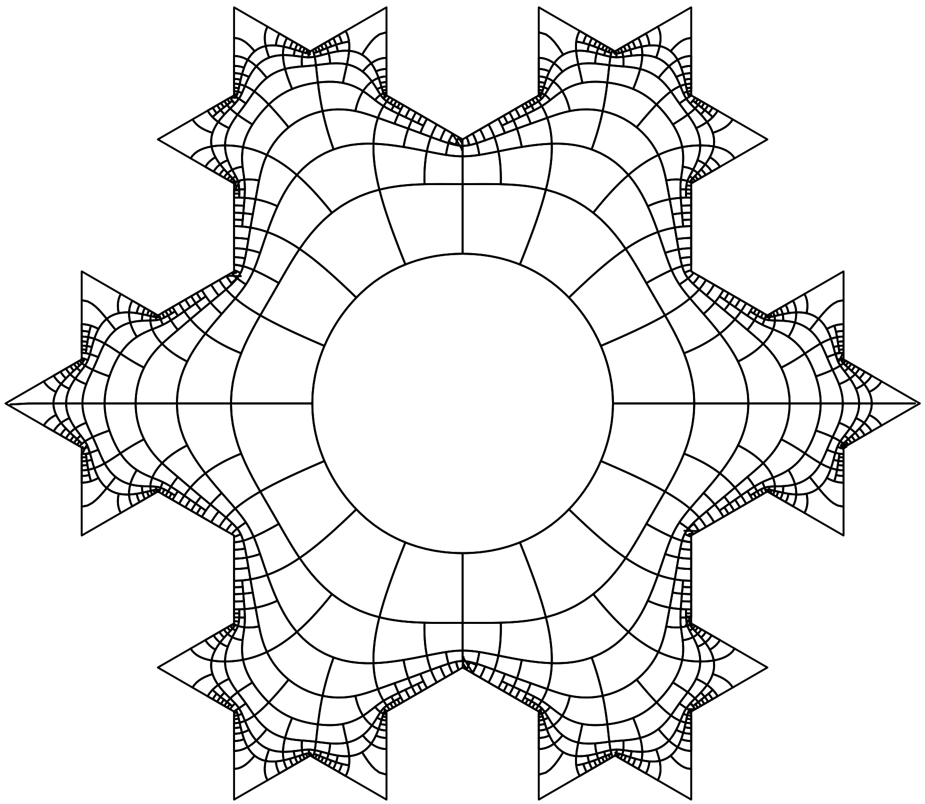
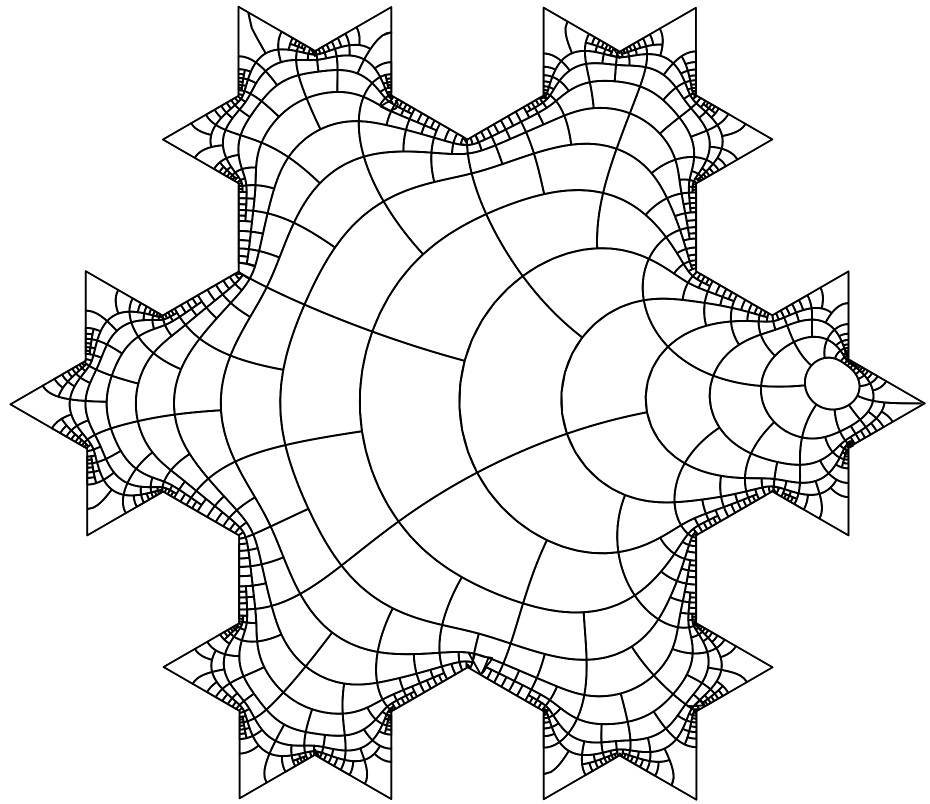


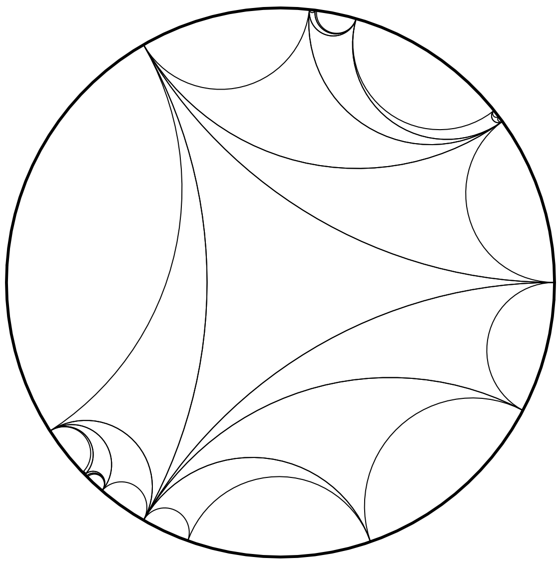


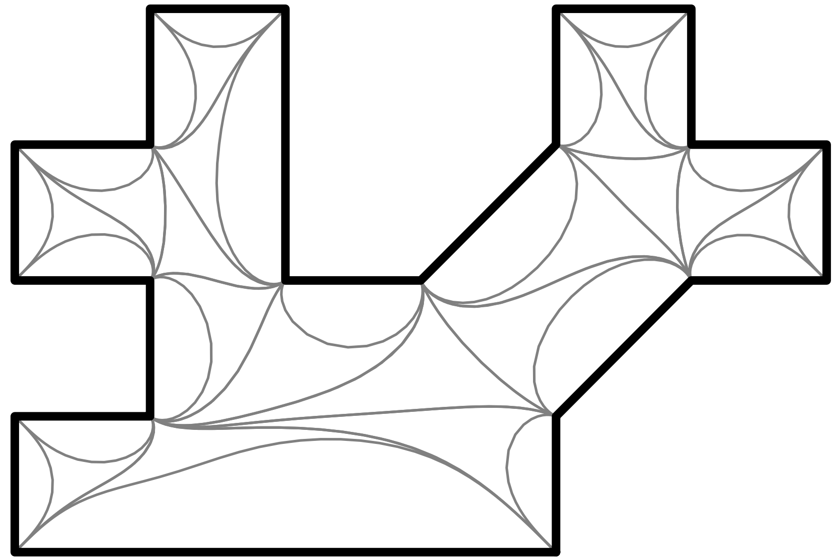


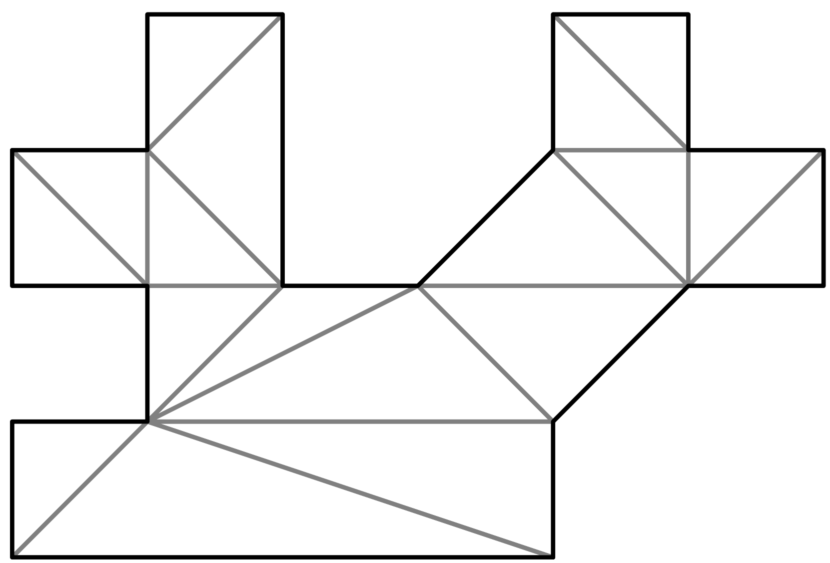










How much time is needed to compute a conformal map?

How much time is needed to compute a conformal map?

Theorem (DCG Sept 2010): We can compute a ϵ-conformal map onto an n-gon in $O(n \log \frac{1}{\epsilon} \log \log \frac{1}{\epsilon})$.

How much time is needed to compute a conformal map?

Theorem (DCG Sept 2010): We can compute a ϵ-conformal map onto an n-gon in $O(n \log \frac{1}{\epsilon} \log\log \frac{1}{\epsilon})$.

ϵ-conformal = ϵ-distortion of angles = $(1+\epsilon)$-QC.

Goal: Find good representation of map in time $O(n)$.

How much time is needed to compute a conformal map?

Theorem (DCG Sept 2010): We can compute a ϵ-conformal map onto an n-gon in $O(n \log \frac{1}{\epsilon} \log \log \frac{1}{\epsilon})$.

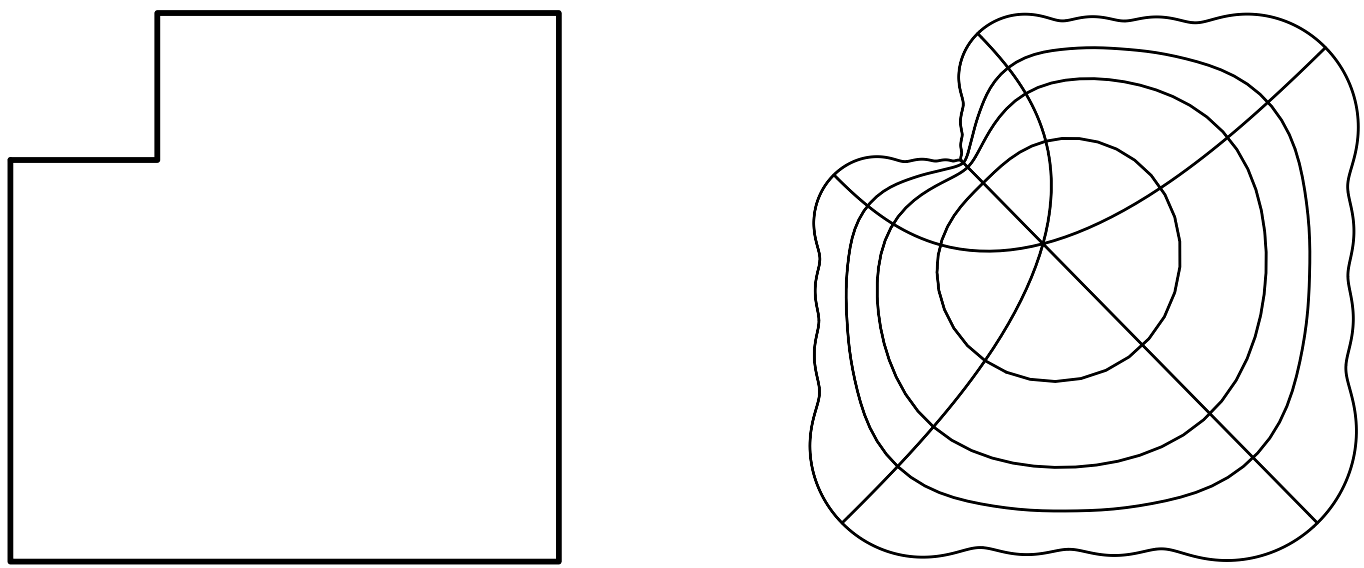
ϵ-conformal = ϵ-distortion of angles = $(1+\epsilon)$-QC.

Goal: Find good representation of map in time $O(n)$.

Proof of the fast mapping theorem:

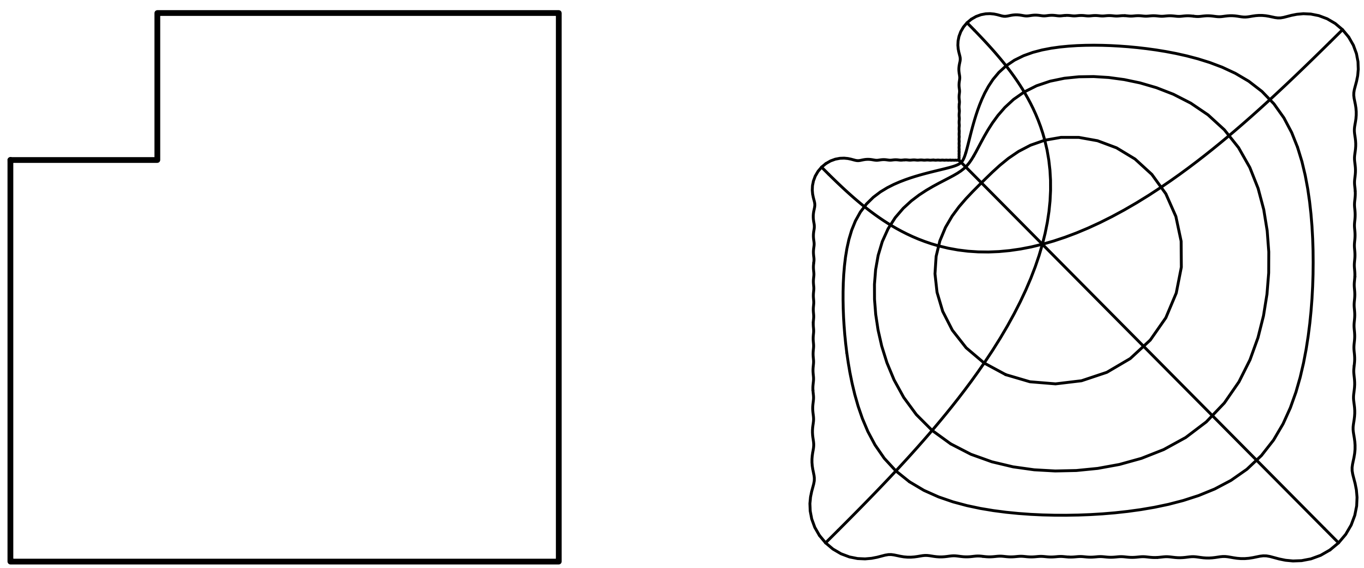
- Local representation of maps
- Newton's method for Beltrami's equation
- Use Iota for initial guess

Conformal maps have power series, but corners of polygon create singularities on circle. Convergence is slow.



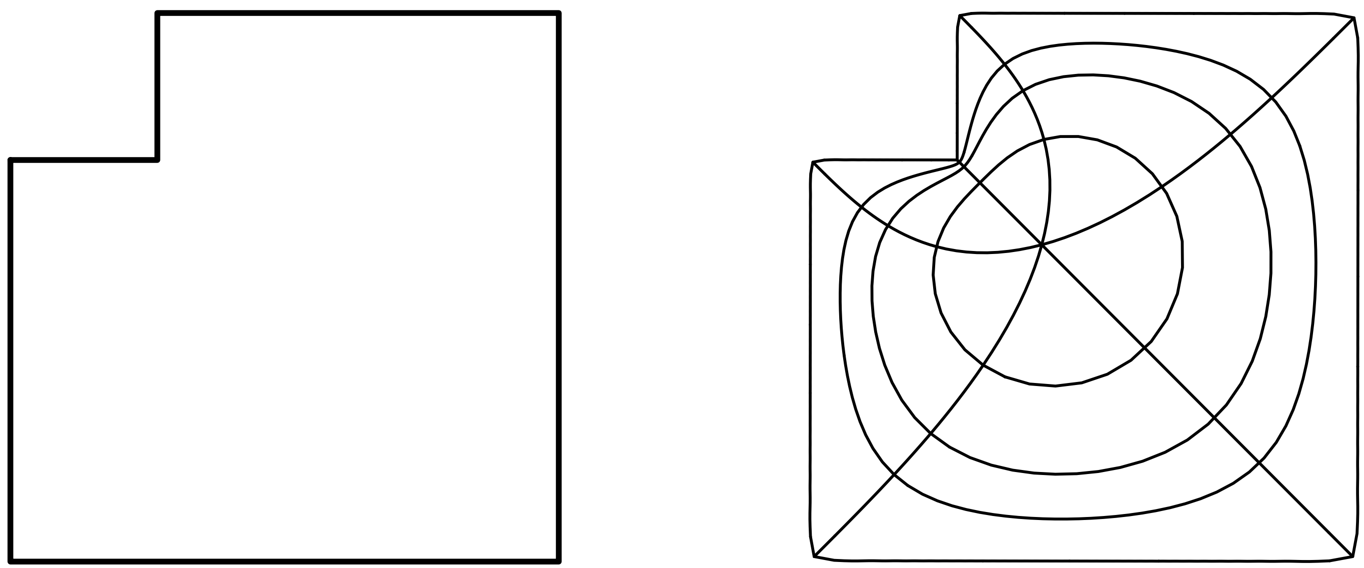
20 terms

Conformal maps have power series, but corners of polygon create singularities on circle. Convergence is slow.



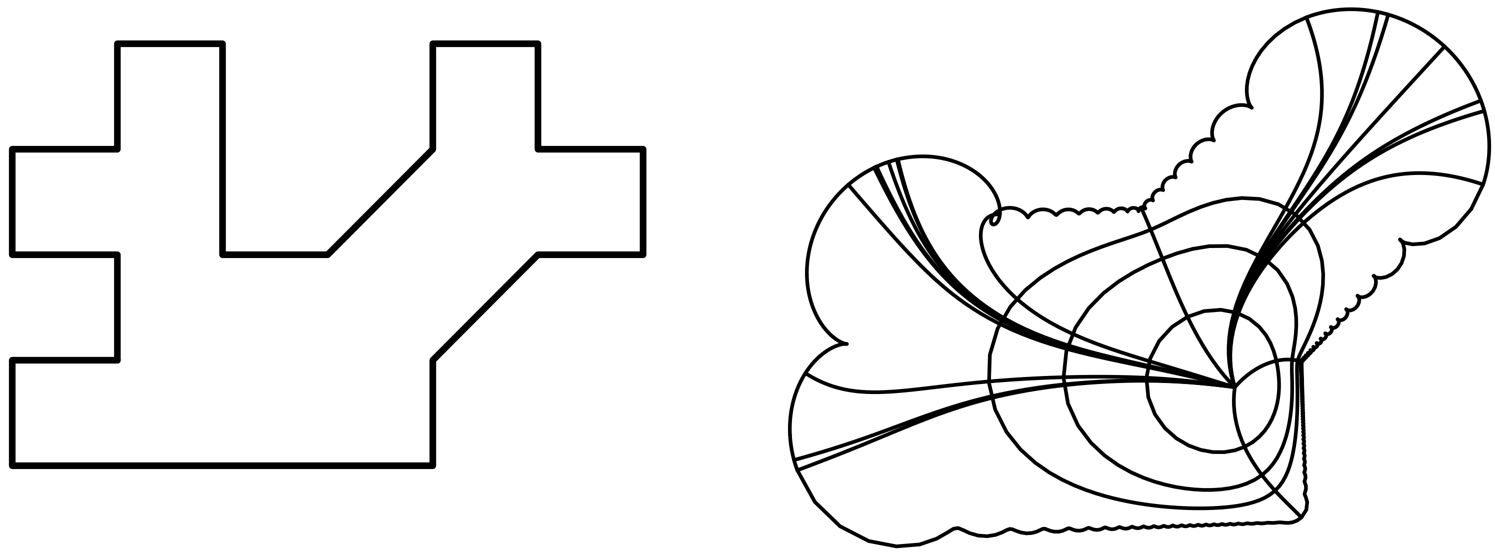
100 terms

Conformal maps have power series, but corners of polygon create singularities on circle. Convergence is slow.



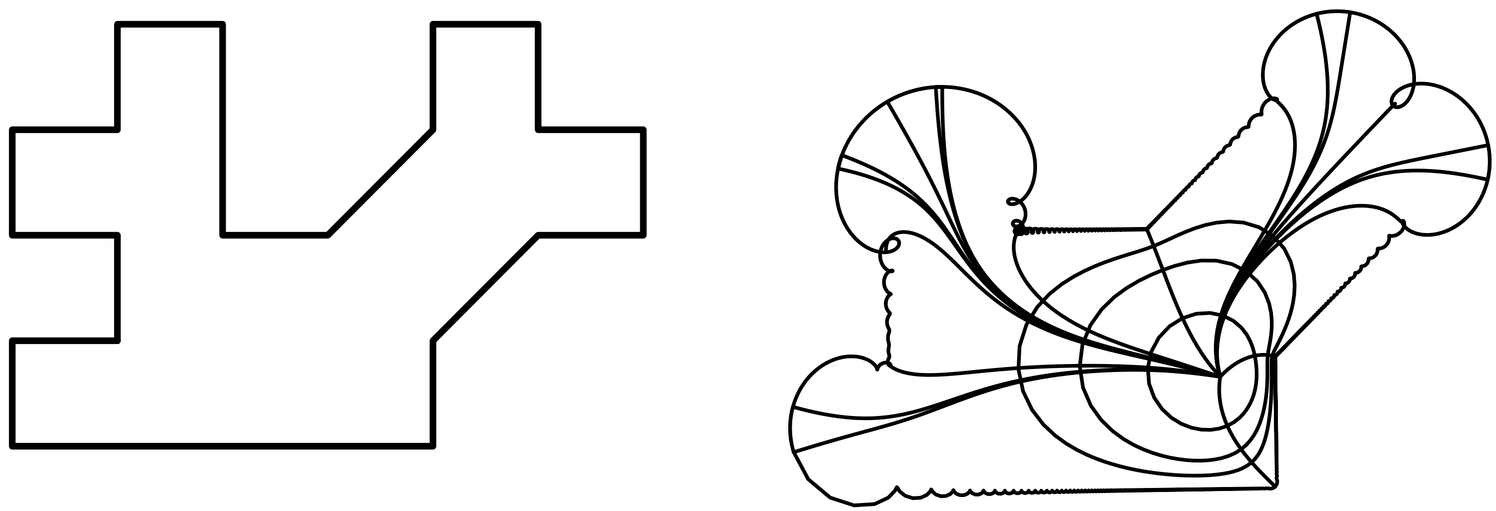
1000 terms

Conformal maps have power series, but corners of polygon create singularities on circle. Convergence is slow.



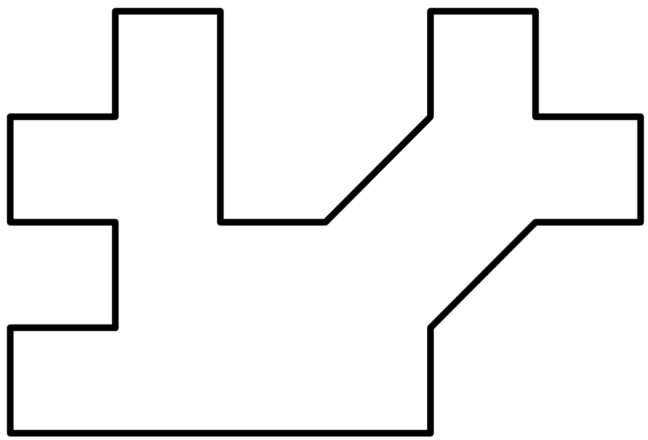
100 terms

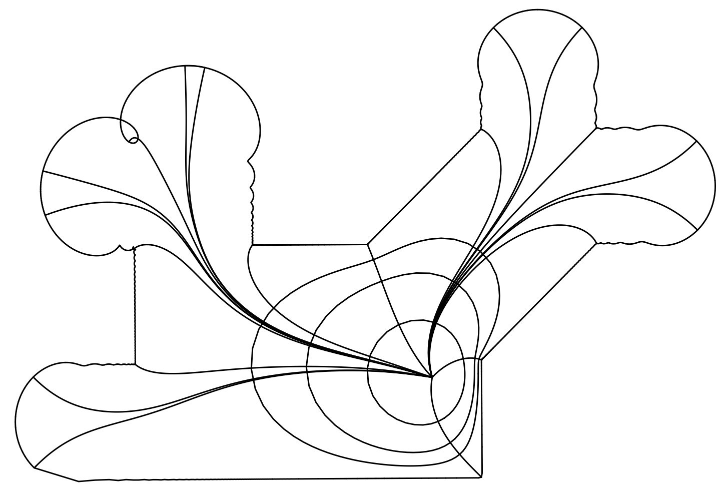
Conformal maps have power series, but corners of polygon create singularities on circle. Convergence is slow.



500 terms

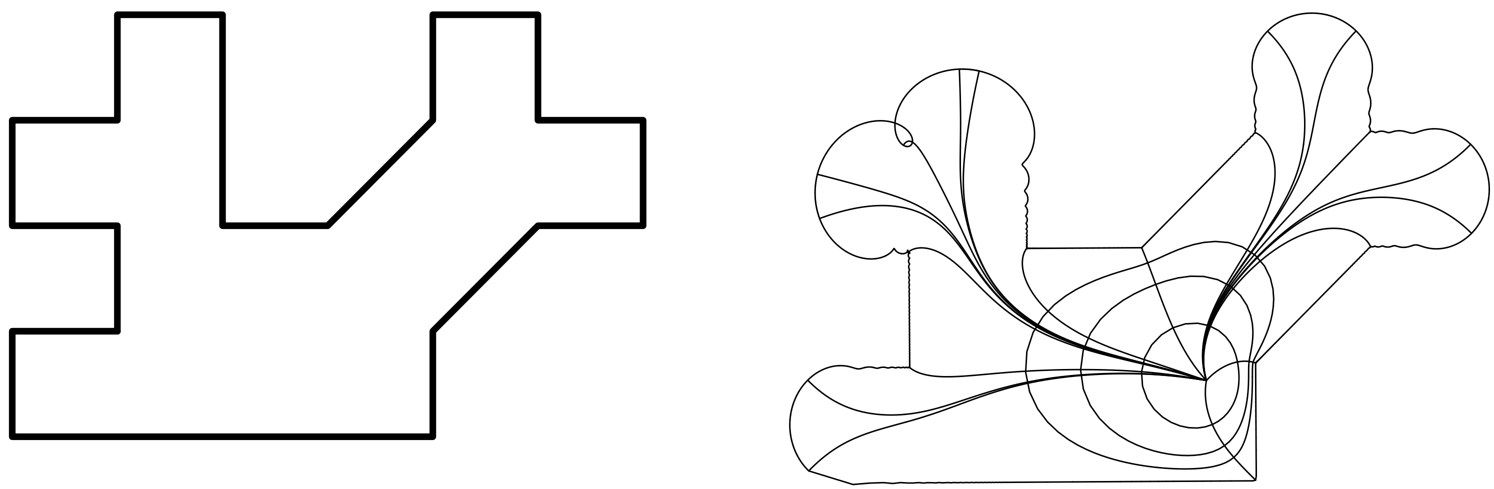
Conformal maps have power series, but corners of polygon create singularities on circle. Convergence is slow.





2500 terms

Conformal maps have power series, but corners of polygon create singularities on circle. Convergence is slow.



2500 terms

1×20 rectangle would require about 10^{15} terms.

Need more efficient representation.

Schwarz-Christoffel formula (1867):

$$f(z) = A + C \int^{z} \prod_{k=1}^{n} (1 - \frac{w}{z_k})^{\alpha_k - 1} dw,$$

Schwarz-Christoffel formula (1867):

$$f(z) = A + C \int^{z} \prod_{k=1}^{n} (1 - \frac{w}{z_k})^{\alpha_k - 1} dw,$$

Christoffel

Schwarz

Schwarz-Christoffel formula (1867):

$$f(z) = A + C \int^z \prod_{k=1}^{n} (1 - \frac{w}{z_k})^{\alpha_k - 1} dw,$$

$\{\alpha_1\pi, \ldots, \alpha_n\pi\}$, are interior angles of polygon.
$\{z_1, \ldots, z_n\}$ are points on circle mapping to vertices.

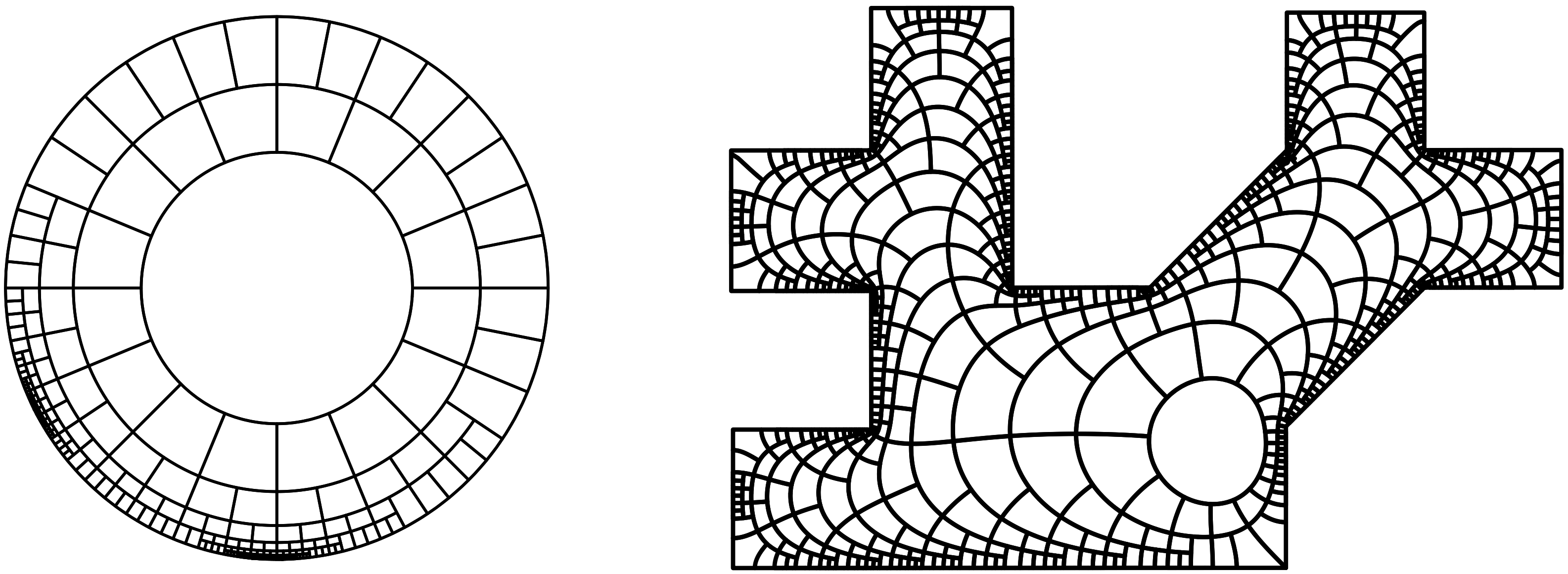
α's are known.

z's must be solved for.

Approximate z's give correct angles, wrong side lengths.

Each evaluation of integrand is n-fold product.

Local series representation: We cut disk into $O(n)$ regions and use a p-term series on each piece to approximate map with accuracy $\epsilon \approx 2^{-p}$ in hyperbolic metric.



Use partition of unity to get global map.

Easy to evaluate; just plug in.

Also more subtle advantage.

Schwarz-Christoffel always gives conformal map, but onto wrong polygon if z-parameters are wrong.

Hard to understand relationship between parameters and image domain, so **hard** to update parameters in provably correct way (OK in practice, e.g., CRDT).

Local series map is always onto correct domain, but is not conformal if series are only approximate.

Easy to improve conformality and preserve image.

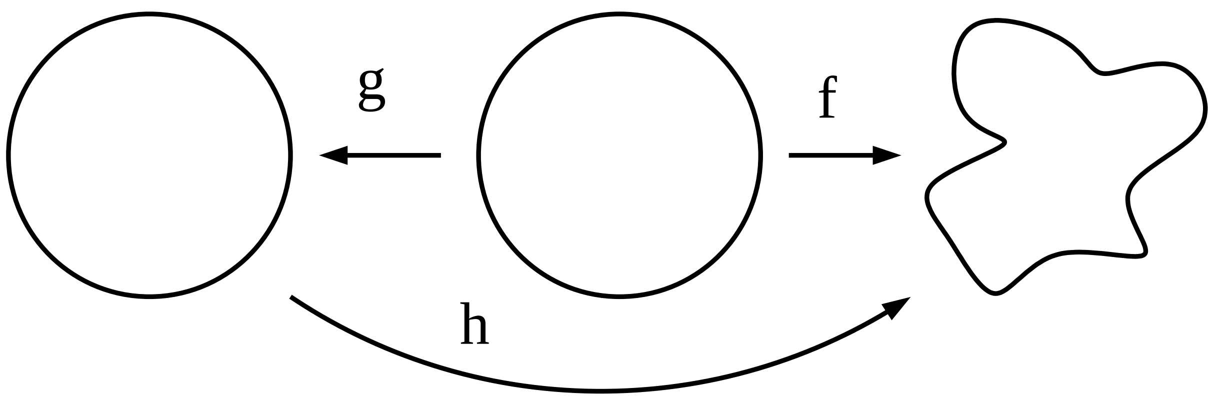
$$\partial f = \frac{1}{2}(f_x - i f_y), \quad \overline{\partial} f = \frac{1}{2i}(f_x + i f_y).$$

We want $f : \mathbb{D} \to \Omega$ with $\overline{\partial} f = 0$ (Cauchy-Riemann).

We measure distance to conformality by dilatation

$$\|f\| = \sup |\mu_f| \equiv \sup |\overline{\partial} f / \partial f|.$$

Main point: If $f : \mathbb{D} \to \Omega$, $g : \mathbb{D} \to \mathbb{D}$ and $\mu_g = \mu_f$, then $h = f \circ g^{-1} : \mathbb{D} \to \Omega$ is conformal.



Beltrami equation: given μ find g with $\mu_g = \mu$,

Beltrami equation: given μ find g with $\mu_g = \mu$,

Set $g = P[\mu(h+1)] + z$, where

$$h = T\mu + T\mu T\mu + T\mu T\mu T\mu + \dots,$$

Beltrami equation: given μ find g with $\mu_g = \mu$,

Set $g = P[\mu(h+1)] + z$, where

$$h = T\mu + T\mu T\mu + T\mu T\mu T\mu + \dots,$$

T is the Beurling transform

$$T\varphi(w) = \lim_{r\to 0} \frac{1}{\pi} \iint_{|z-w|>r} \frac{\varphi(z)}{(z-w)^2} dxdy,$$

P is the Cauchy transform

$$P\varphi(w) = -\frac{1}{\pi} \iint \varphi(z)\left(\frac{1}{z-w} - \frac{1}{z}\right) dxdy.$$

If f has local representation by $O(n)$ p-term series, we can compute a g in time $O(np \log p)$ so that

$$\|f \circ g^{-1}\| = O(\|f\|^2).$$

If f has local representation by $O(n)$ p-term series, we can compute a g in time $O(np \log p)$ so that

$$\|f \circ g^{-1}\| = O(\|f\|^2).$$

Iteration gives quadratic convergence to conformal map.

Uses fast multipole method and FFT

For $O(n)$ bound iteration needs a starting map $\mathbb{D} \to \Omega$ that is close to conformal (independent of Ω).

If f has local representation by $O(n)$ p-term series, we can compute a g in time $O(np \log p)$ so that

$$\|f \circ g^{-1}\| = O(\|f\|^2).$$

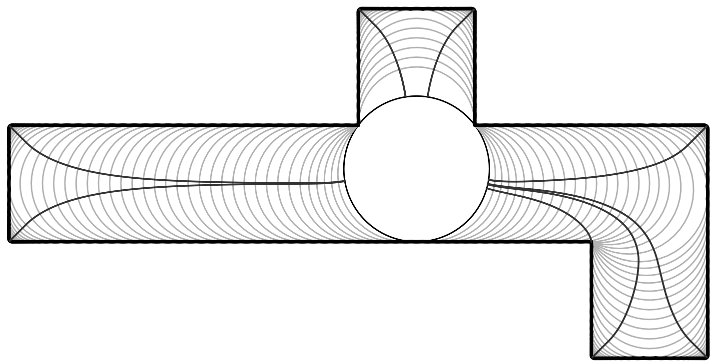
Iteration gives quadratic convergence to conformal map.

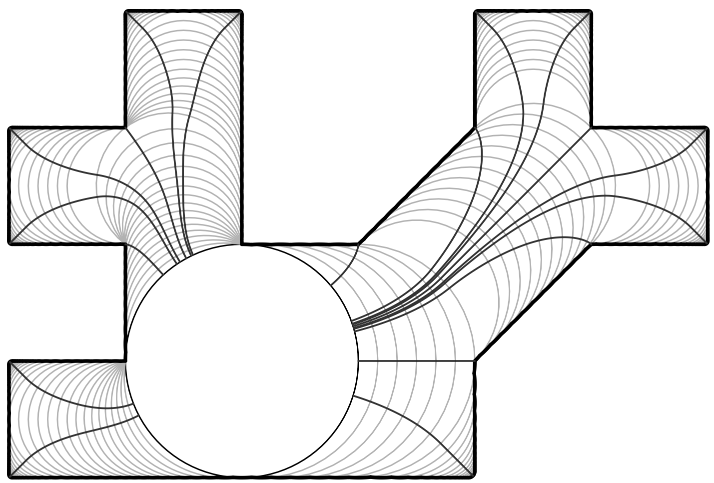
Uses fast multipole method and FFT

For $O(n)$ bound iteration needs a starting map $\mathbb{D} \to \Omega$ that is close to conformal (independent of Ω).

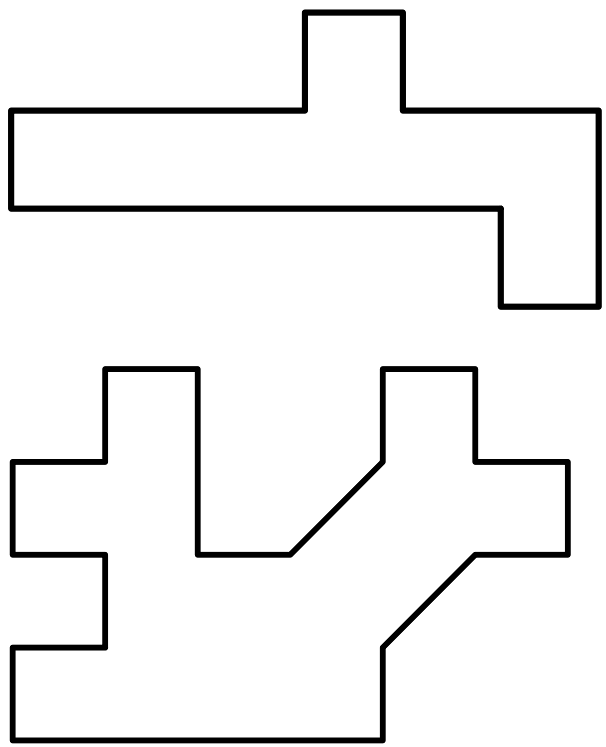
Use Iota map for initial guess. By CHT it is bounded distance to correct answer. Only requires $O(\log|\log \epsilon|)$ steps to reach accuracy ϵ.

How good is the iota approximation in practice?

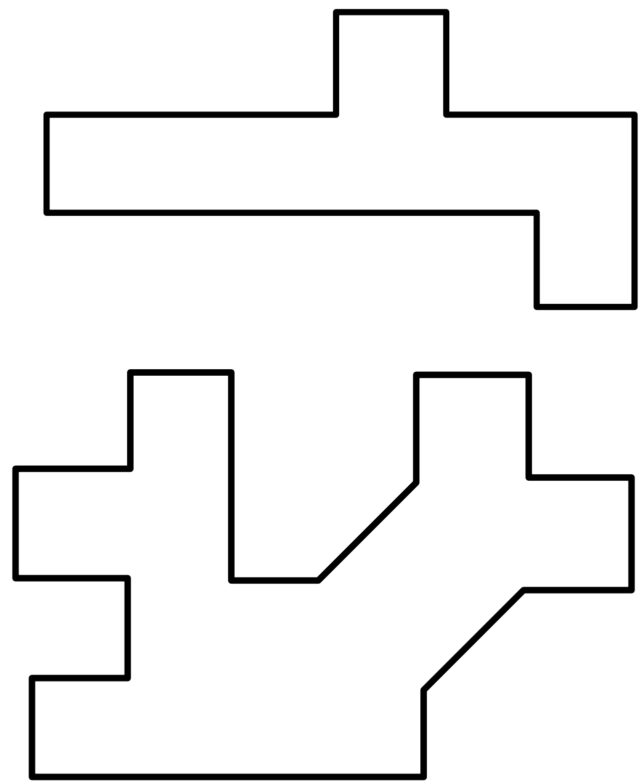




Use “iota parameters” in Schwarz-Christoffel formula.

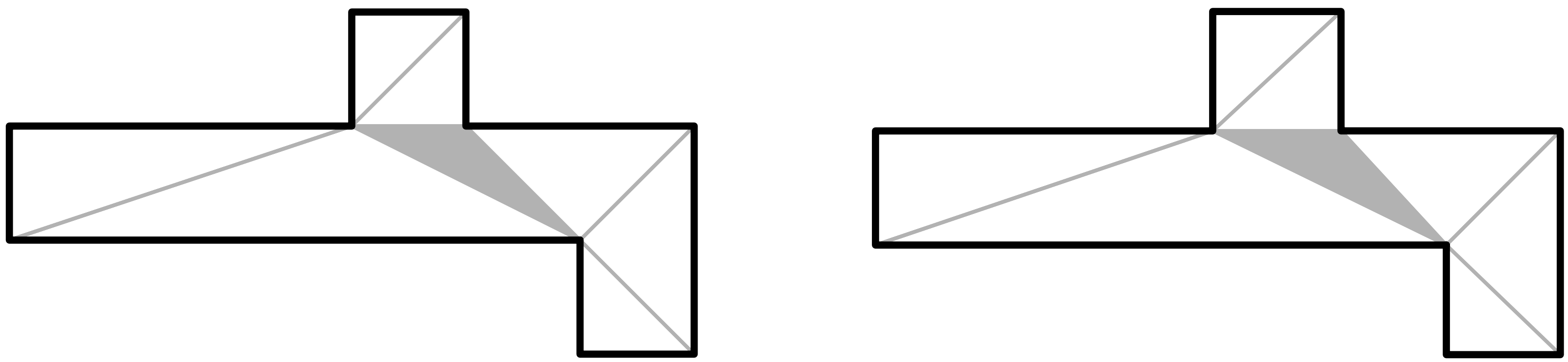


Target Polygon



Iota Parameters

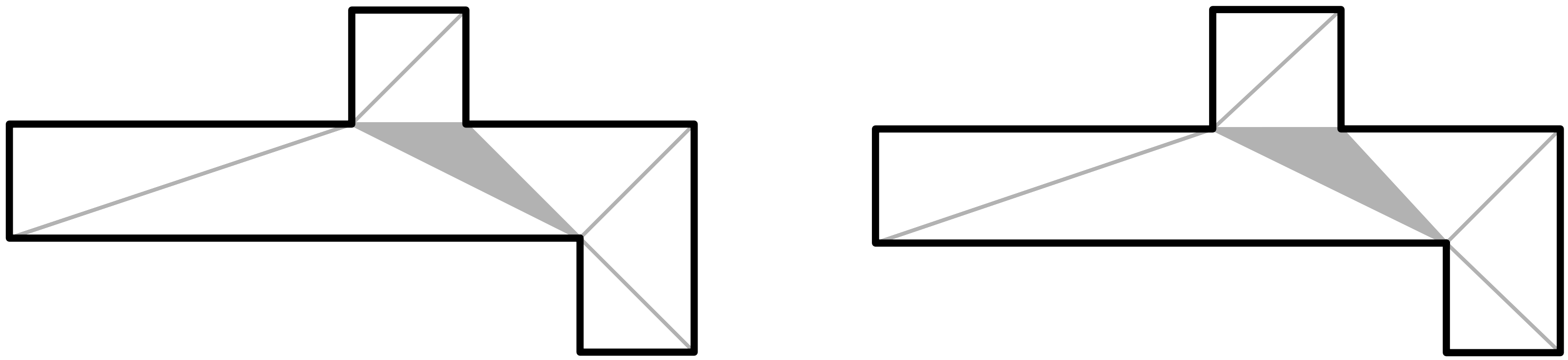
Triangulate polygons and form piecewise linear maps. Max $|\mu|$ gives upper bound for distance to conformal.



The most distorted triangle is shaded. Here $|\mu| = .108$.

We can bound conformal distance to true SC parameters even though we don't know what the are.

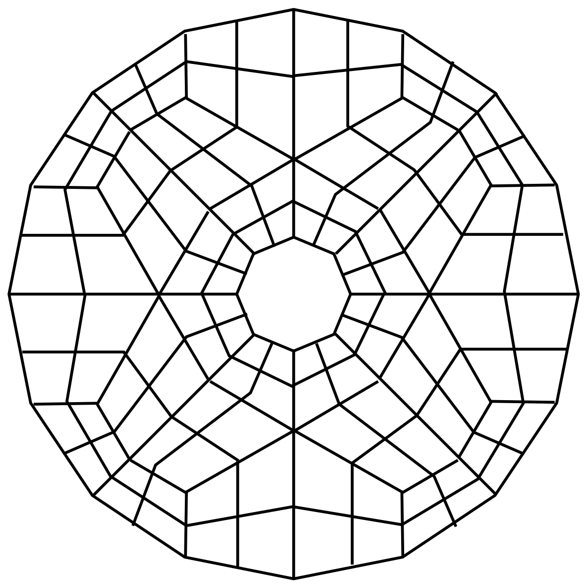
Triangulate polygons and form piecewise linear maps. Max $|\mu|$ gives upper bound for distance to conformal.

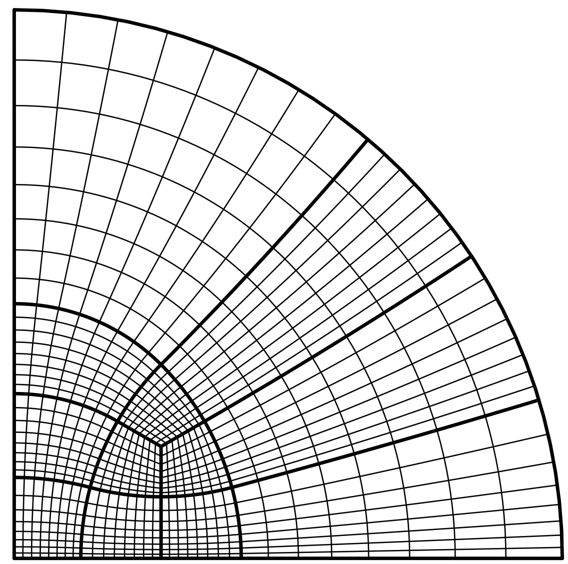


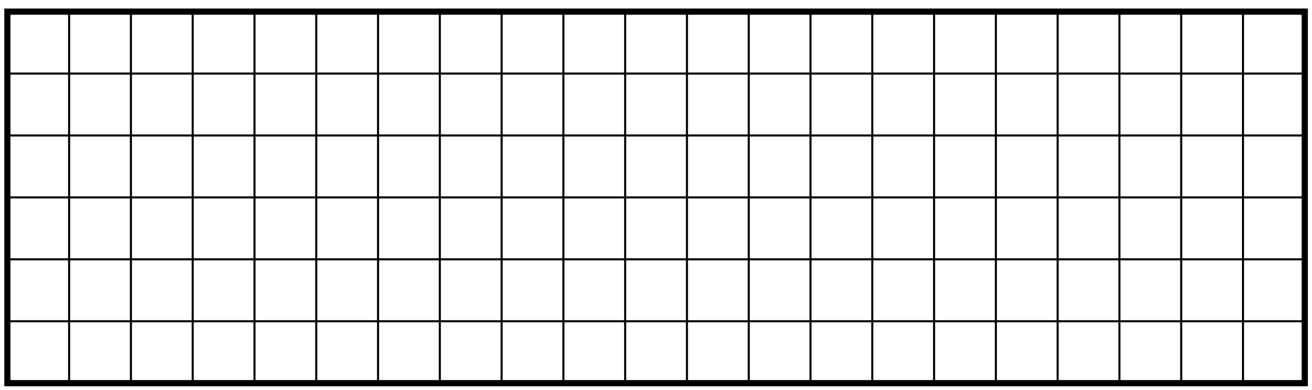
The most distorted triangle is shaded. Here $|\mu| = .108$.

Chris Green: Use μ from piecewise linear map and linear approximation to Beltrami equation to recompute SC parameters. Seems to work well in theory and practice.

Quadrilateral meshes







Does every n-gon have a good mesh?

Does every n-gon have a good mesh?

Good geometry = angles bounded away from 0, 180.

Good complexity = # of elements polynomial in n.

Common element types = triangles, quadrilaterals.

Many more results for triangulations.

Marshall Bern, David Eppstein (2000):

Theorem: Every simple n-gon has $O(n)$ quad mesh with angles $\leq 120°$.

Sharp: Any quad mesh of hexagon has angle $\geq 120°$.

Marshall Bern, David Eppstein (2000):

Theorem: Every simple n-gon has $O(n)$ quad mesh with angles $\leq 120°$.

P

P^2

P = hyperbolic geometry, University of Warwick

P^2 = computational geometry, UC Irvine

Marshall Bern, David Eppstein (2000):

Theorem: Every simple n-gon has $O(n)$ quad mesh with angles $\leq 120°$.

Theorem (DCG 2010): Every n-gon has $O(n)$ quad mesh with all angles $\leq 120°$ and new angles $\geq 60°$.

Angles bounds and complexity are sharp.

Marshall Bern, David Eppstein (2000):

Theorem: Every simple n-gon has $O(n)$ quad mesh with angles $\leq 120°$.

Theorem (DCG 2010): Every n-gon has $O(n)$ quad mesh with all angles $\leq 120°$ and new angles $\geq 60°$.

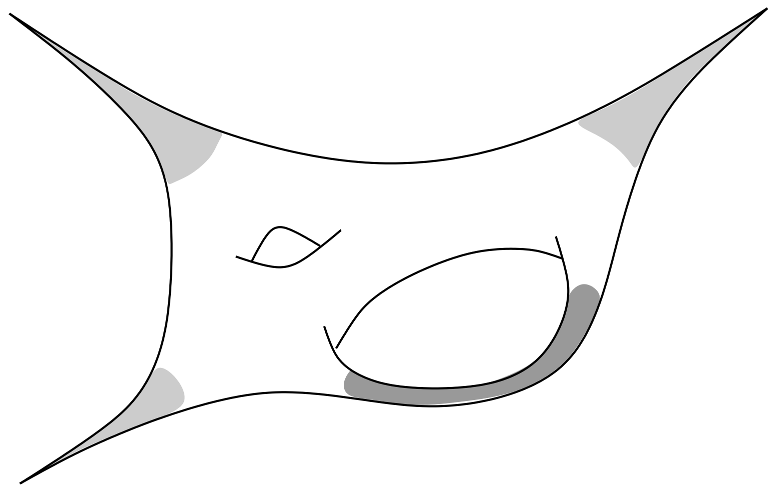
Angles bounds and complexity are sharp.

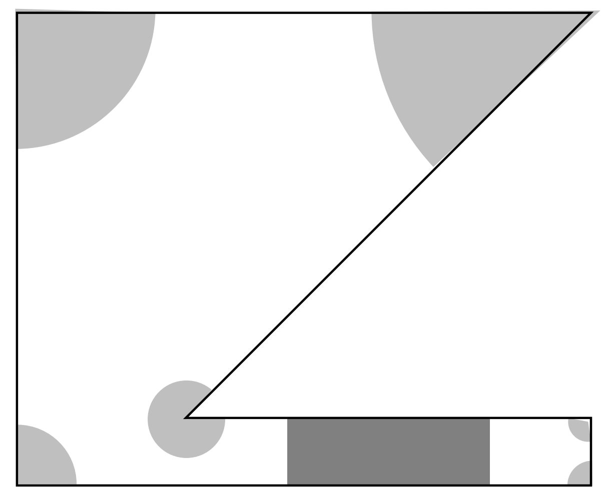
Idea of proof:

- Decompose polygon into thick and thin parts.
- Mesh thin parts by explicit construction.
- Thick parts use hyperbolic geometry.

Thick and Thin parts

An ϵ**-thin part** of a surface is a union of non-trivial loops of length $\leq \epsilon$ (parabolic/hyperbolic).

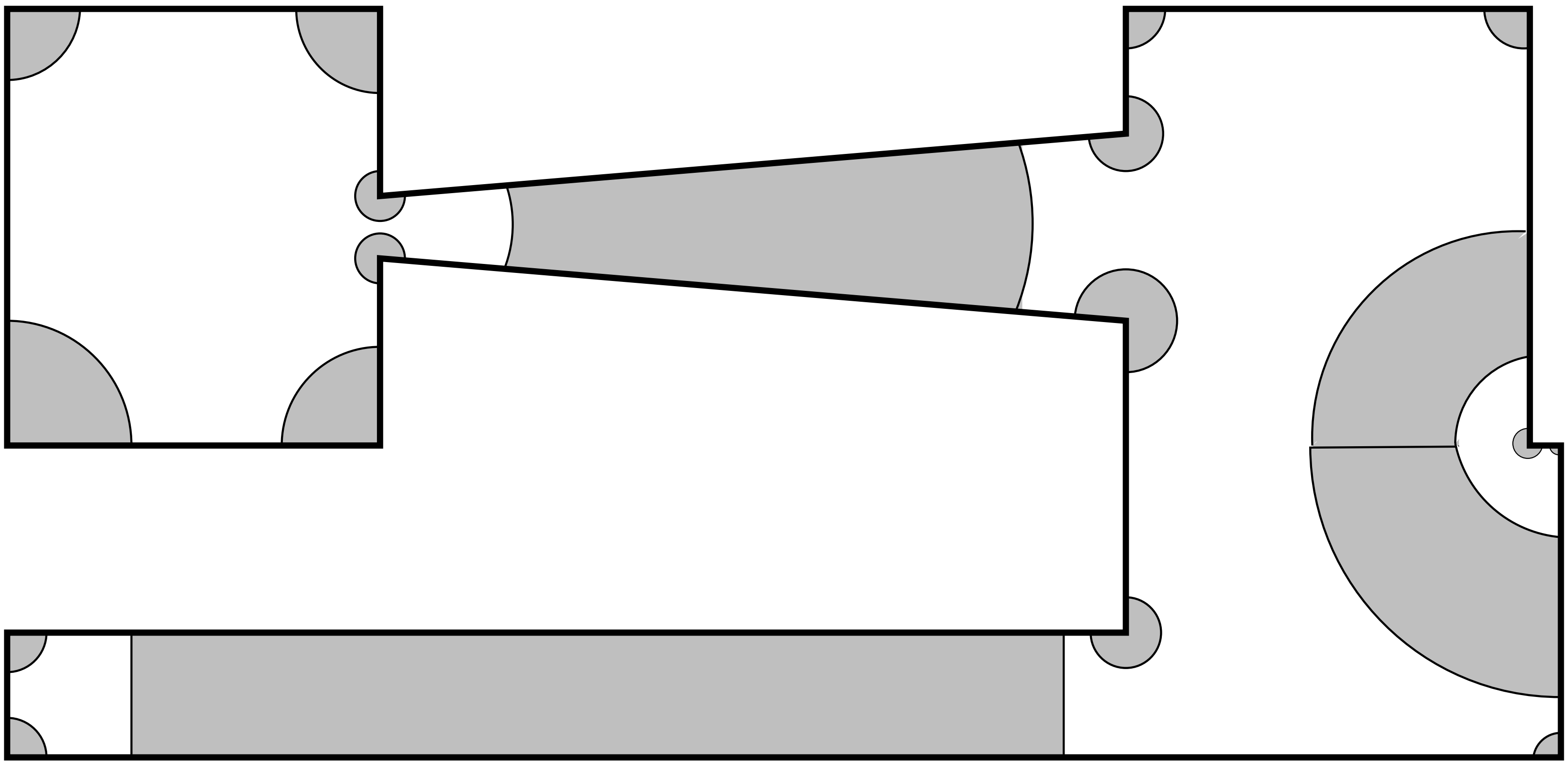




Thin piece is a sector whose two straight sides satisfy

$$\text{dist}(I, J) \ll \min(|I|, |J|).$$

Precise definition: extremal distance in Ω is $< \epsilon$.

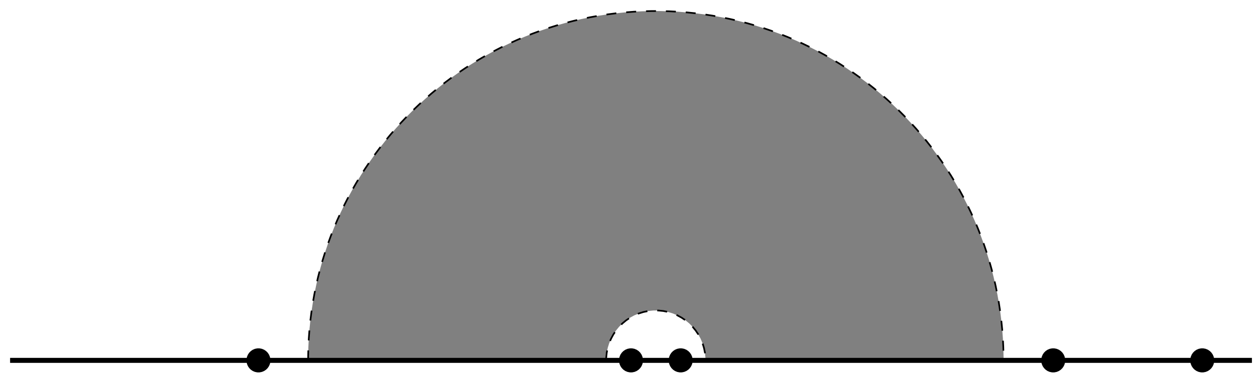


Parabolic and hyperbolic thin parts correspond to thin parts of “doubled” polygon = Riemann surface.

Thin parts computable in $O(n)$ using conformal map.

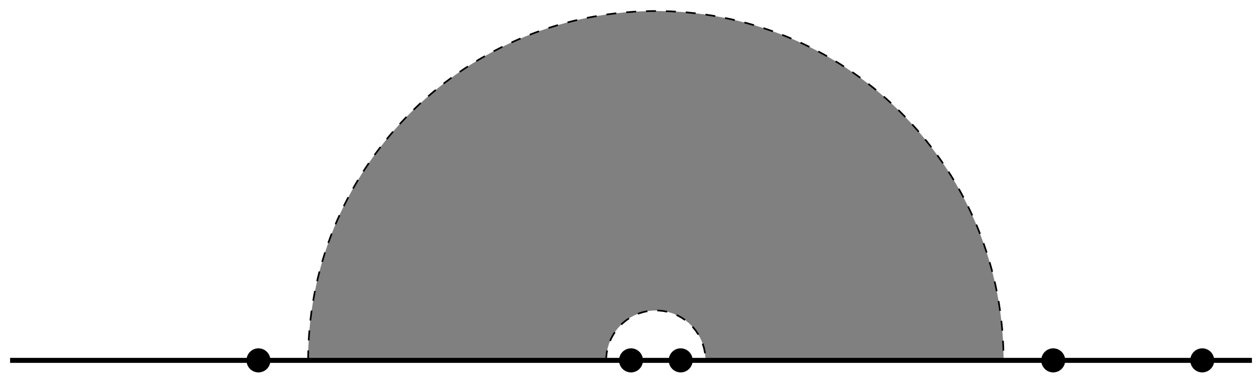
- Map polygon conformally to half-plane.
  Vertices map to points on line.

Thin parts computable in $O(n)$ using conformal map.



- Map polygon conformally to half-plane. Vertices map to points on line.
- Thin parts = wide annuli separating vertices.

Thin parts computable in $O(n)$ using conformal map.

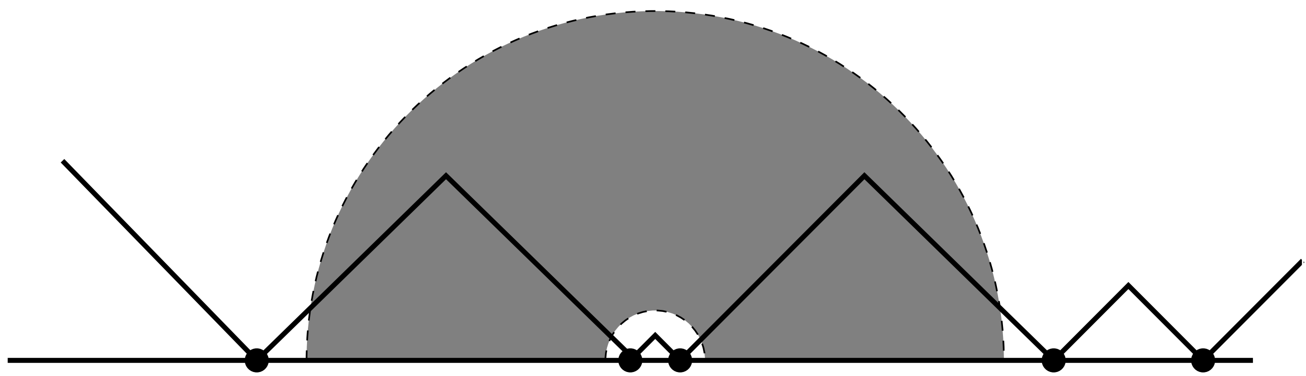


- Map polygon conformally to half-plane.
  Vertices map to points on line.
- Thin parts = wide annuli separating vertices.

Find clusters $T \subset S$ so distances inside cluster are much smaller than connecting cluster to complement.

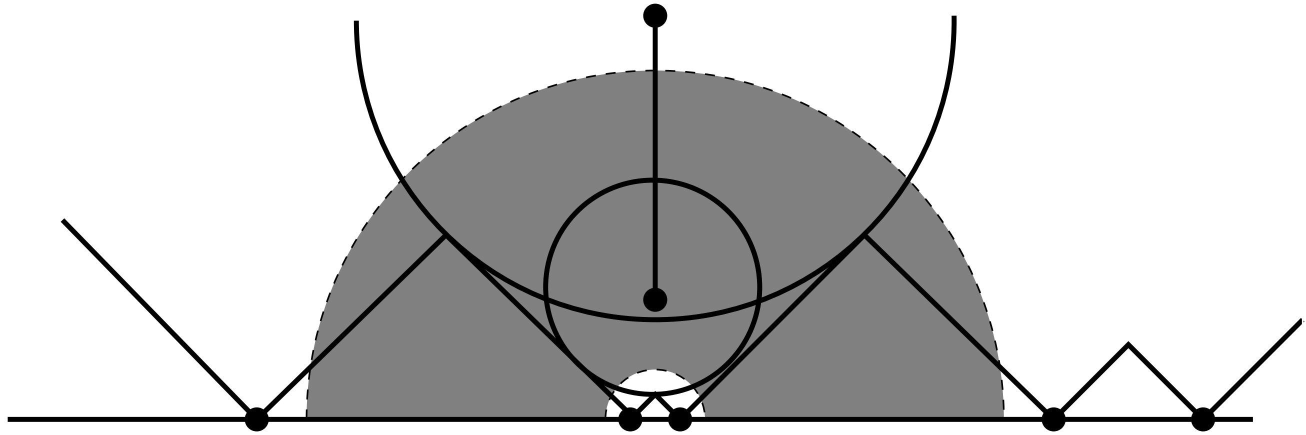
$$\max\{|x-y| : x, y \in T\} \leq \delta \min\{|x-z| : x \in T, z \in S \backslash T\}$$

Thin parts computable in $O(n)$ using conformal map.



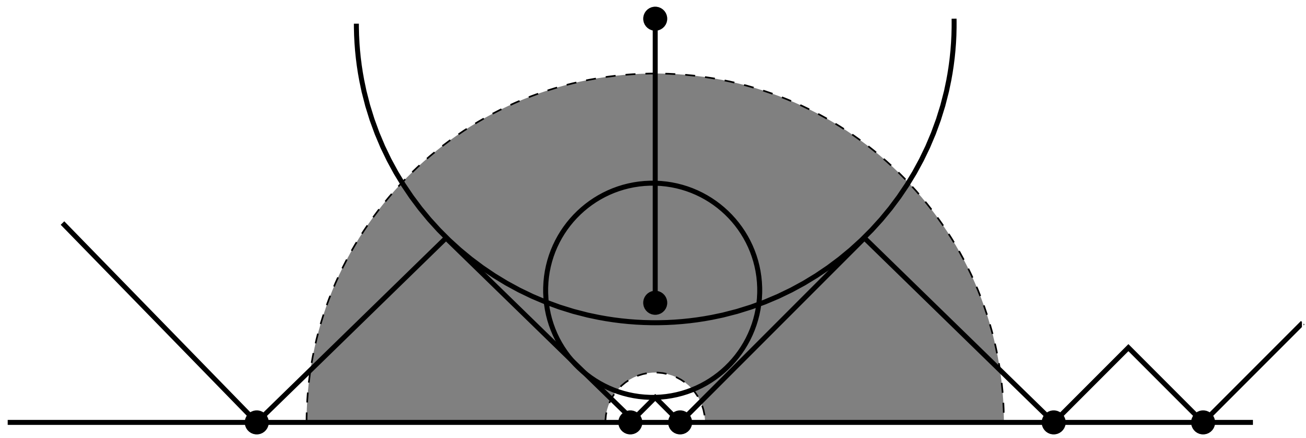
- Map polygon conformally to half-plane.
  Vertices map to points on line.
- Thin parts = wide annuli separating vertices.
- Draw sawtooth domain.

Thin parts computable in $O(n)$ using conformal map.



- Map polygon conformally to half-plane.
  Vertices map to points on line.
- Thin parts = wide annuli separating vertices.
- Draw sawtooth domain.
- Compute medial axis, find long vertical segments

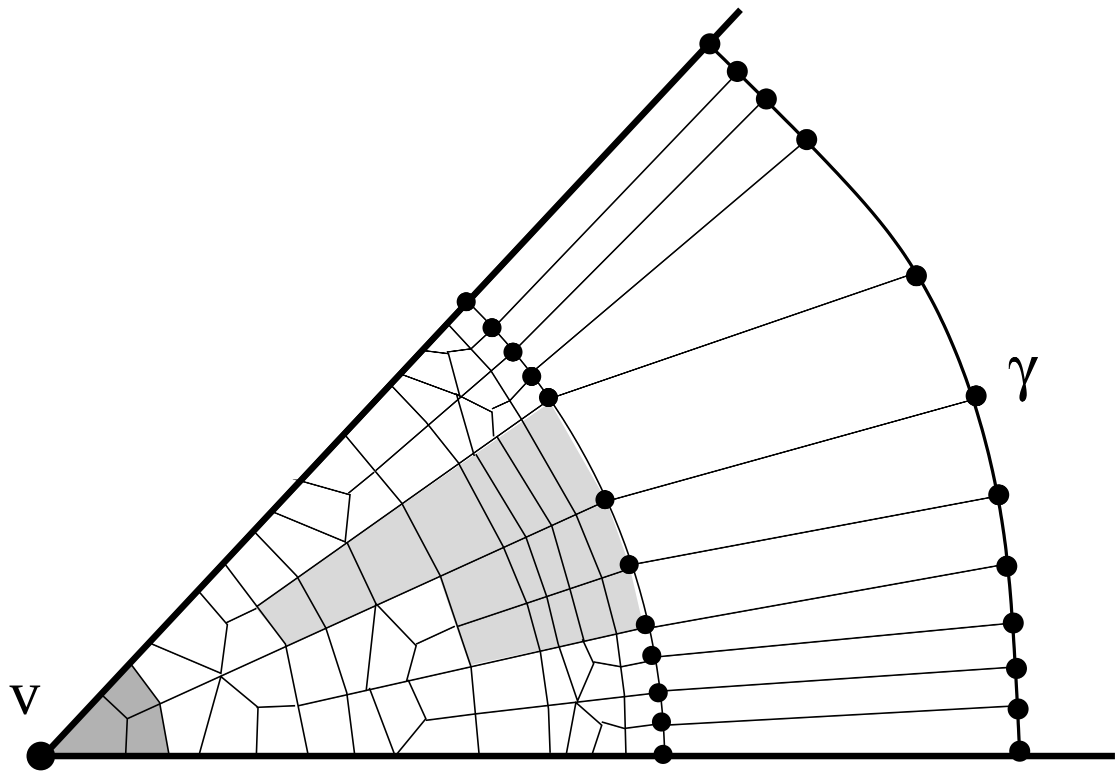
Thin parts computable in $O(n)$ using conformal map.



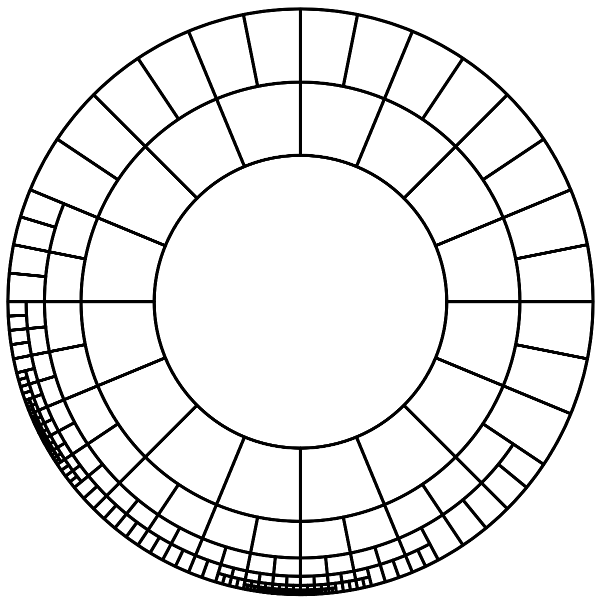
- Map polygon conformally to half-plane.
  Vertices map to points on line.
- Thin parts = wide annuli separating vertices.
- Draw sawtooth domain.
- Compute medial axis, find long vertical segments

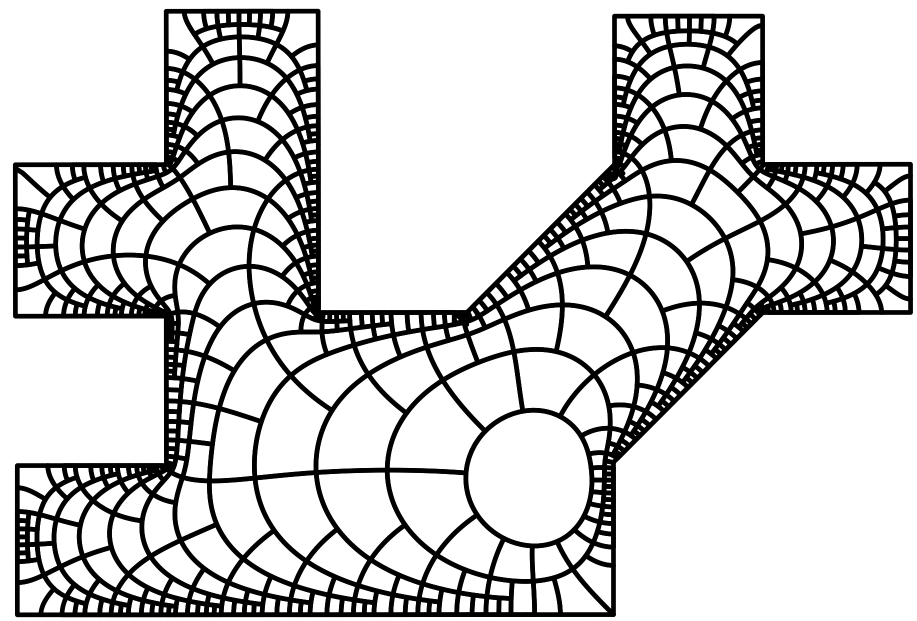
- Finds all thin parts in linear time.
- By CHT enough to use Iota instead of conformal map.

Thin parts are meshed by explicit construction (easy).



Basic idea for meshing thick parts: Conformal map from disk preserves angles except near vertices.

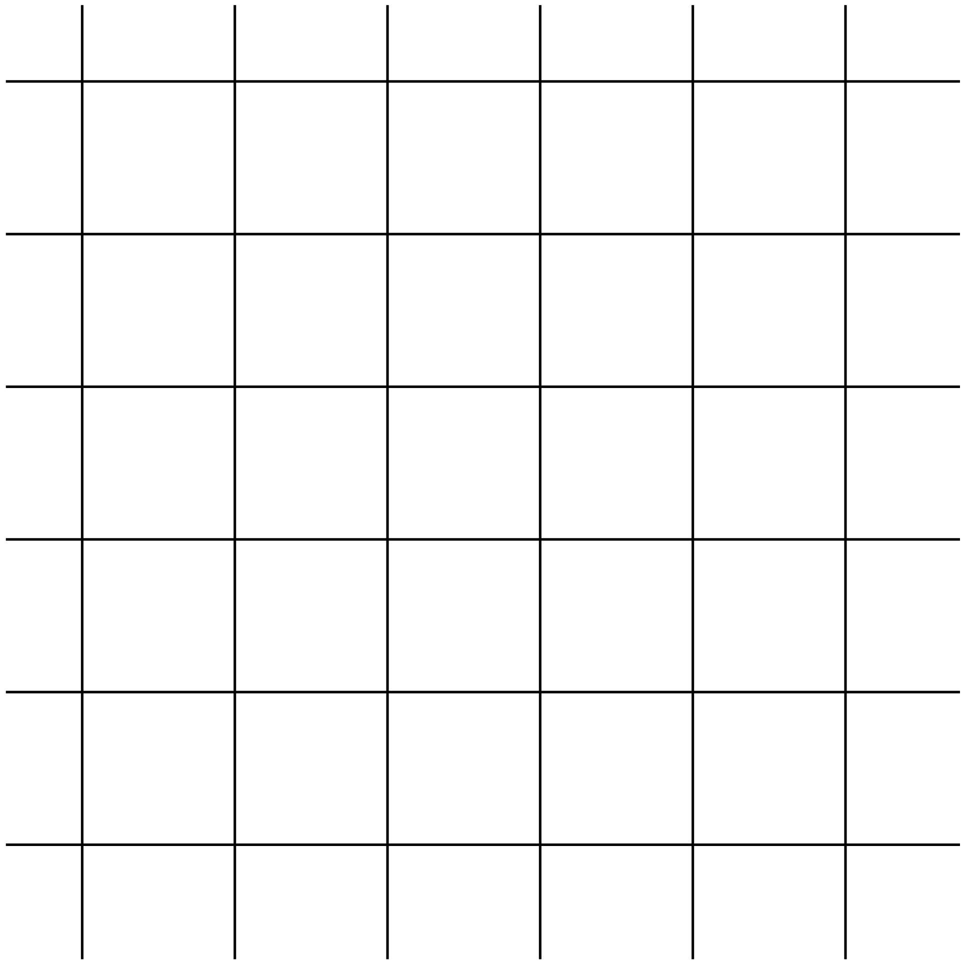




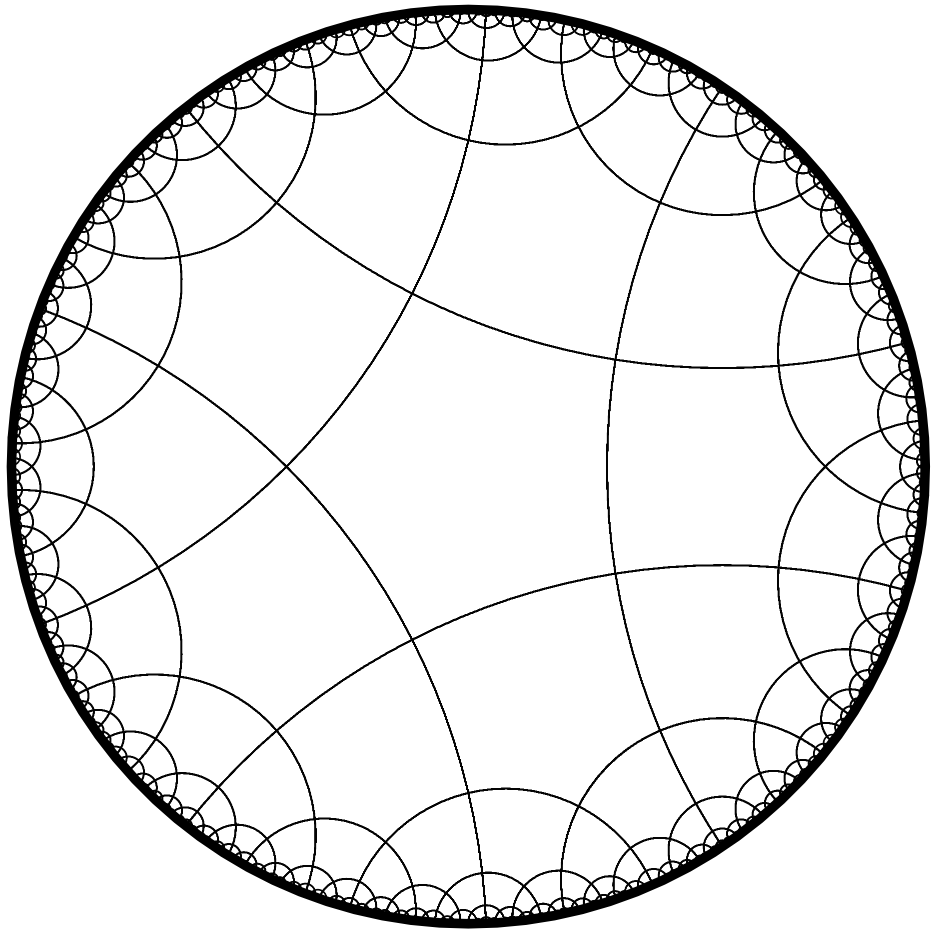
Transfer mesh on disk to mesh of polygon.

Need to be careful with tiles and timing.

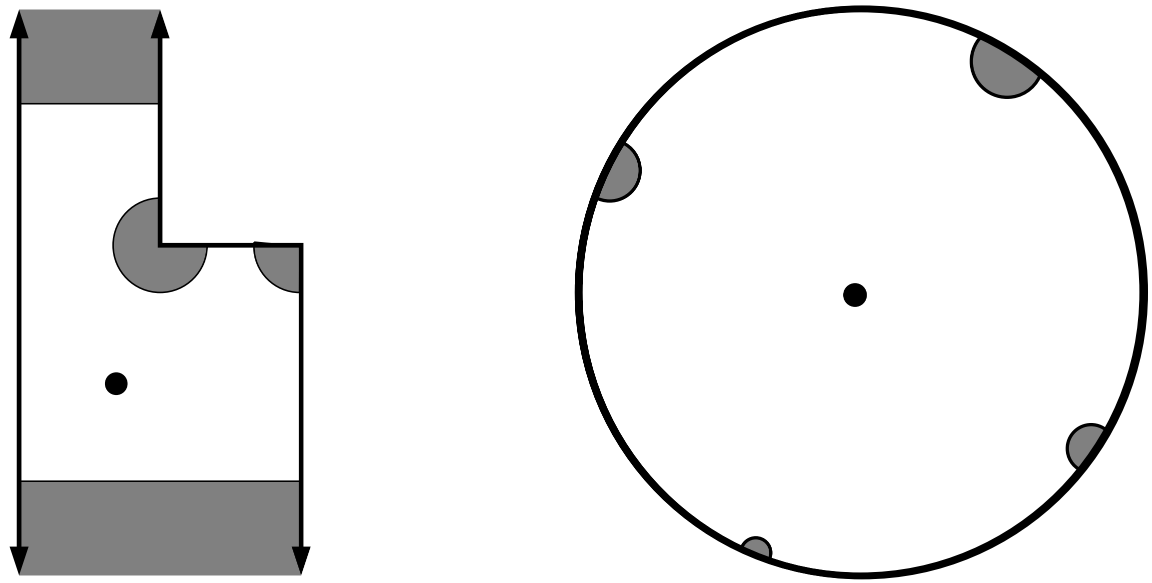
Euclidean plane can be tesselated by squares



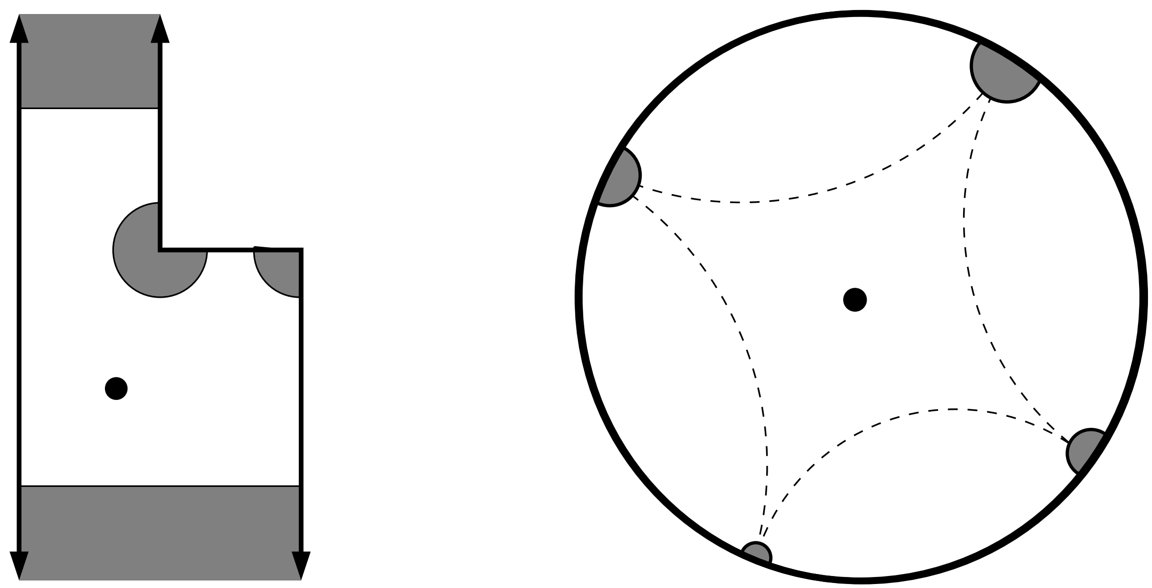
Hyperbolic disk can be tesselated by right pentagons.



Conformal map from polygon to disk takes thick and thin parts to disk as shown.

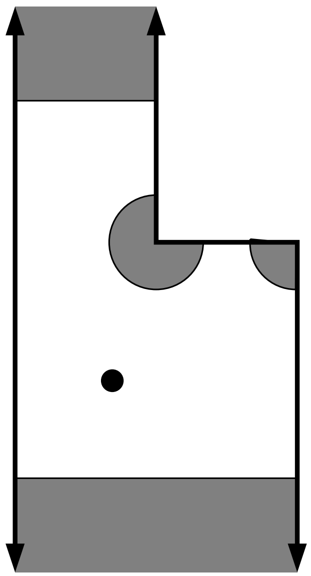


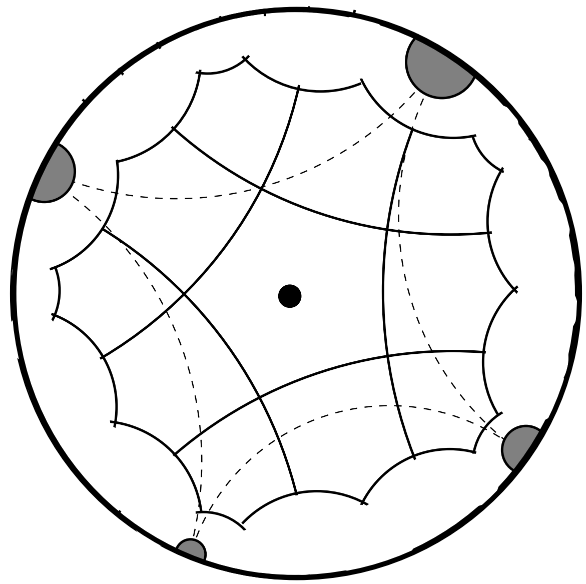
Conformal map from polygon to disk takes thick and thin parts to disk as shown.



Draw (hyperbolic) convex hull of thin regions.

Conformal map from polygon to disk takes thick and thin parts to disk as shown.

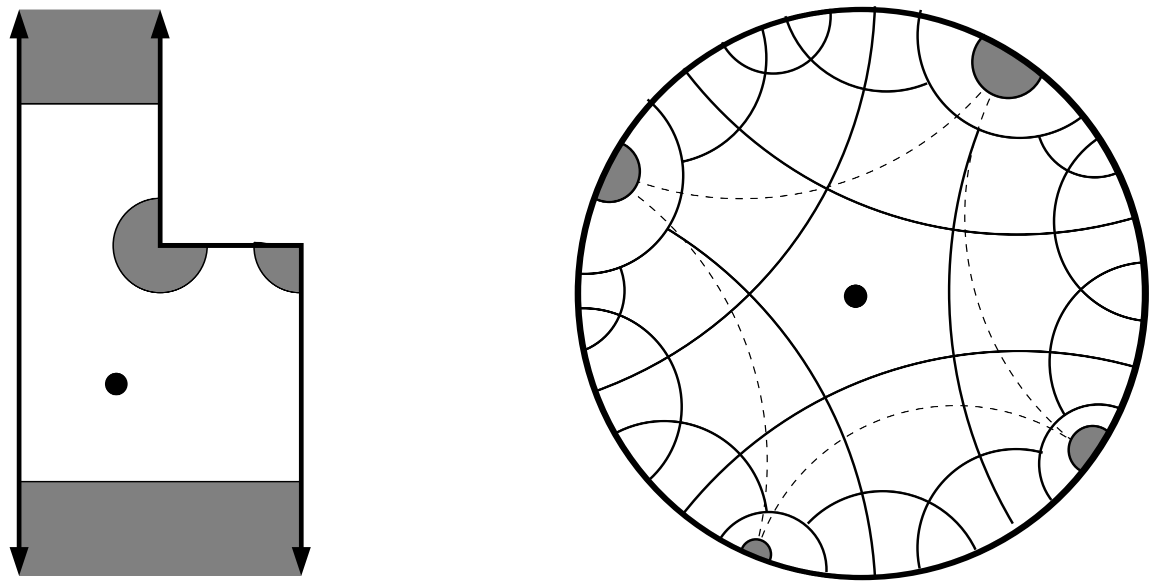




Draw (hyperbolic) convex hull of thin regions.

Take pentagons from tesselation hitting convex hull but missing thin parts.

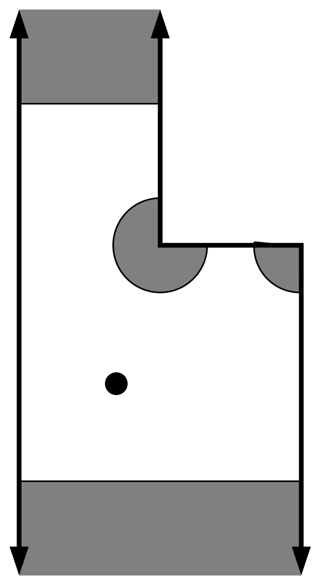
Conformal map from polygon to disk takes thick and thin parts to disk as shown.

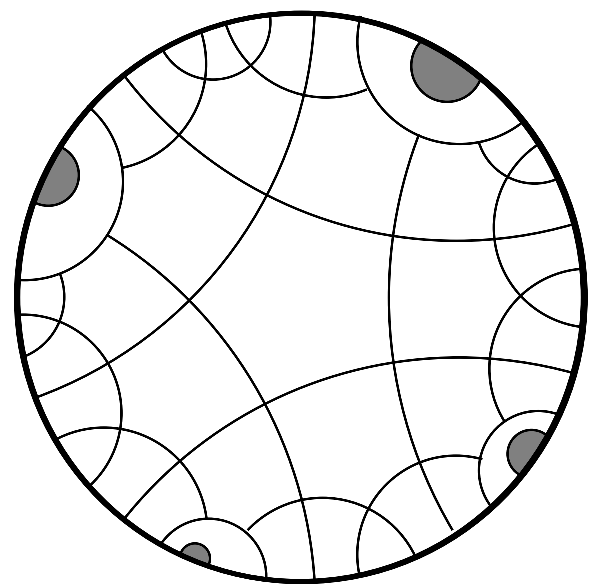


Draw (hyperbolic) convex hull of thin regions.

Take pentagons from tesselation hitting convex hull but missing thin parts. Extend pentagon edges to boundary.

Conformal map from polygon to disk takes thick and thin parts to disk as shown.



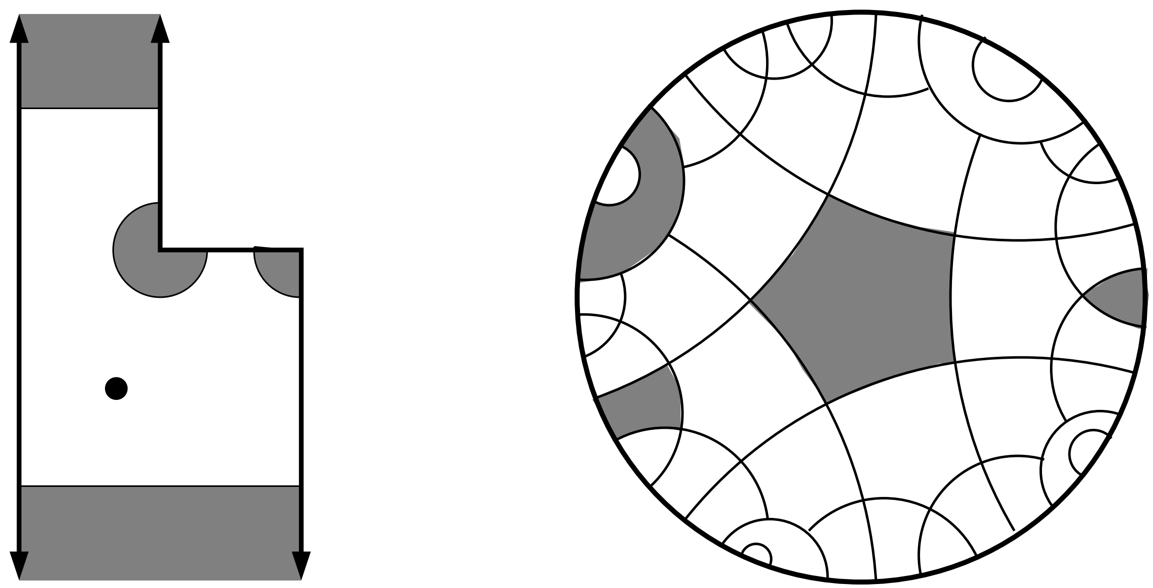


Draw (hyperbolic) convex hull of thin regions.

Take pentagons from tesselation hitting convex hull but missing thin parts. Extend pentagon edges to boundary.

Analog of Whitney or quadtreee construction.

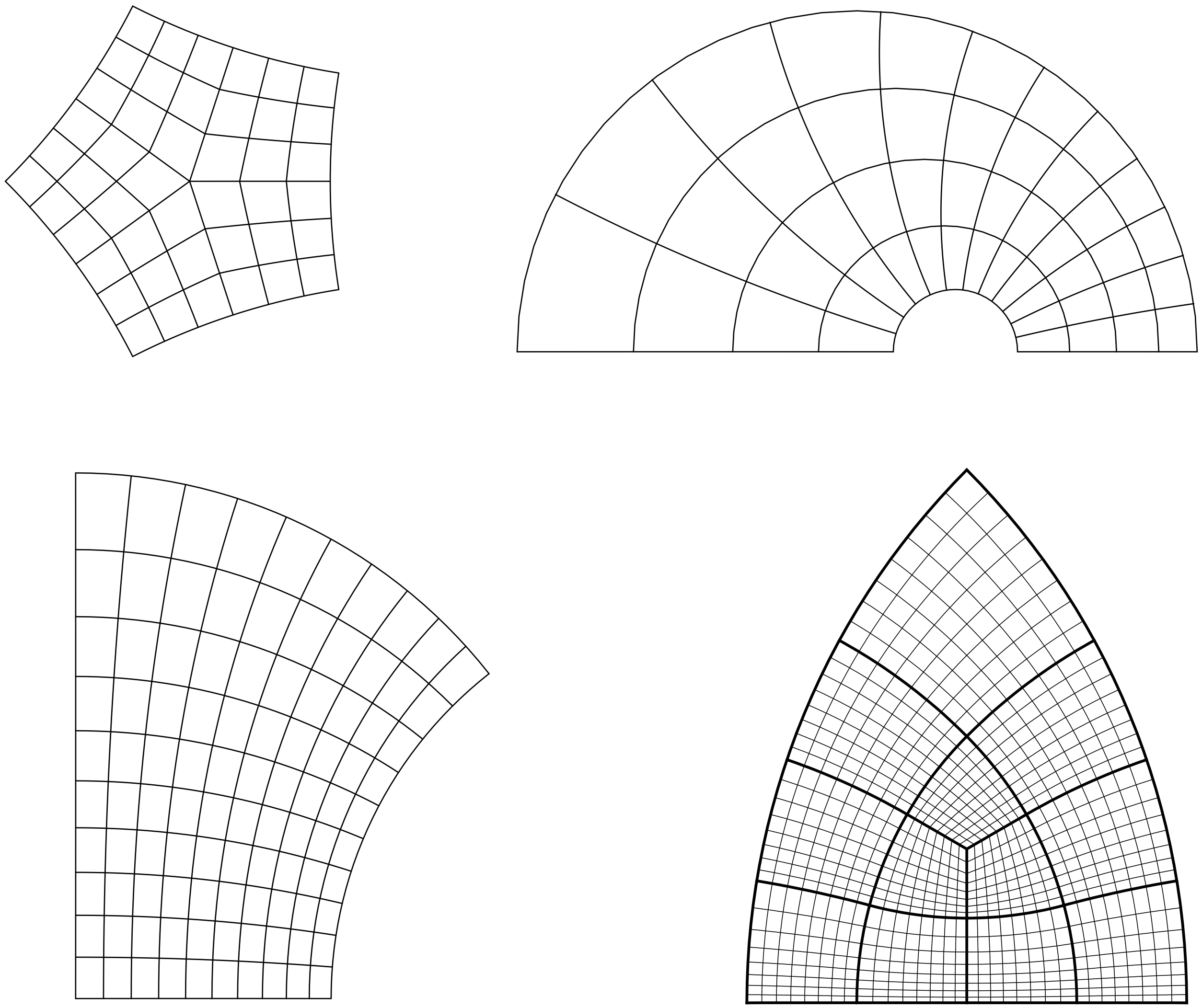
Conformal map from polygon to disk takes thick and thin parts to disk as shown.



Draw (hyperbolic) convex hull of thin regions.

Take pentagons from tesselation hitting convex hull but missing thin parts. Extend pentagon edges to boundary.

Pentagons, quadrilaterals, triangles and half-annuli.

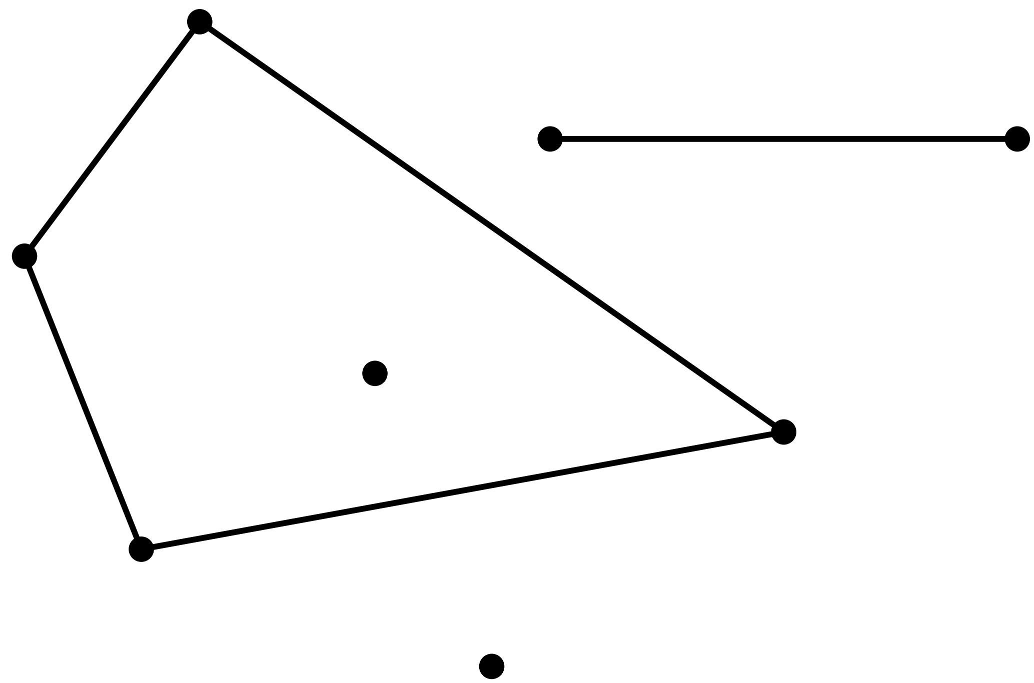


Shapes can be meshed to match along common edges.

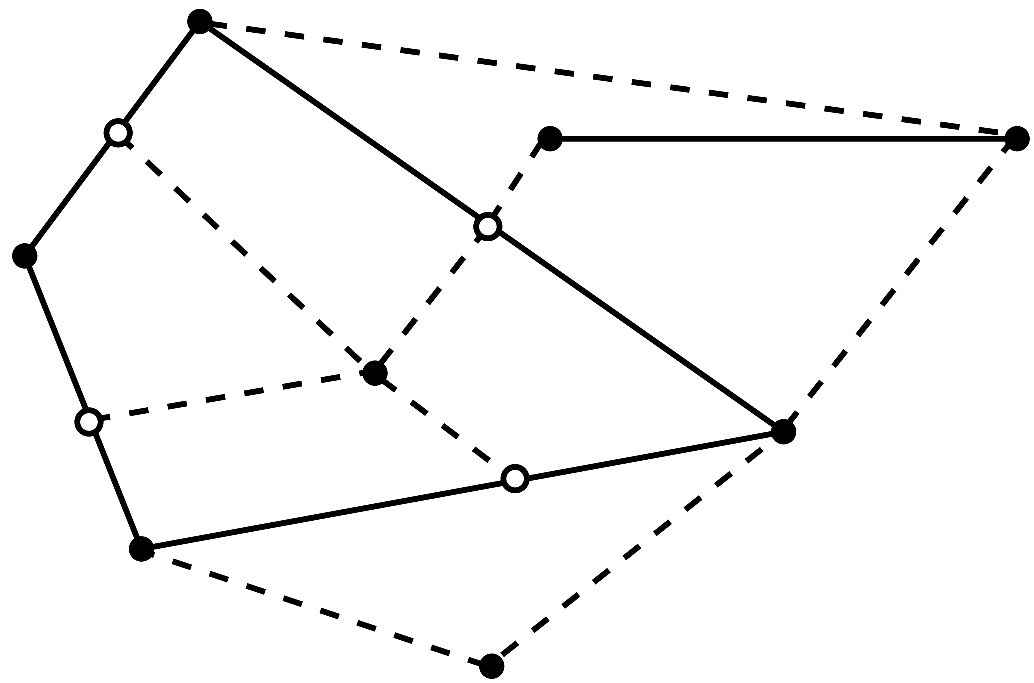
This completes sketch of quad meshing of polygons.

Theorem can be extended from polygons to PSLGs.

A **Planar Straight Line Graph** (PSLG) is a finite point set plus a set of disjoint edges between them.



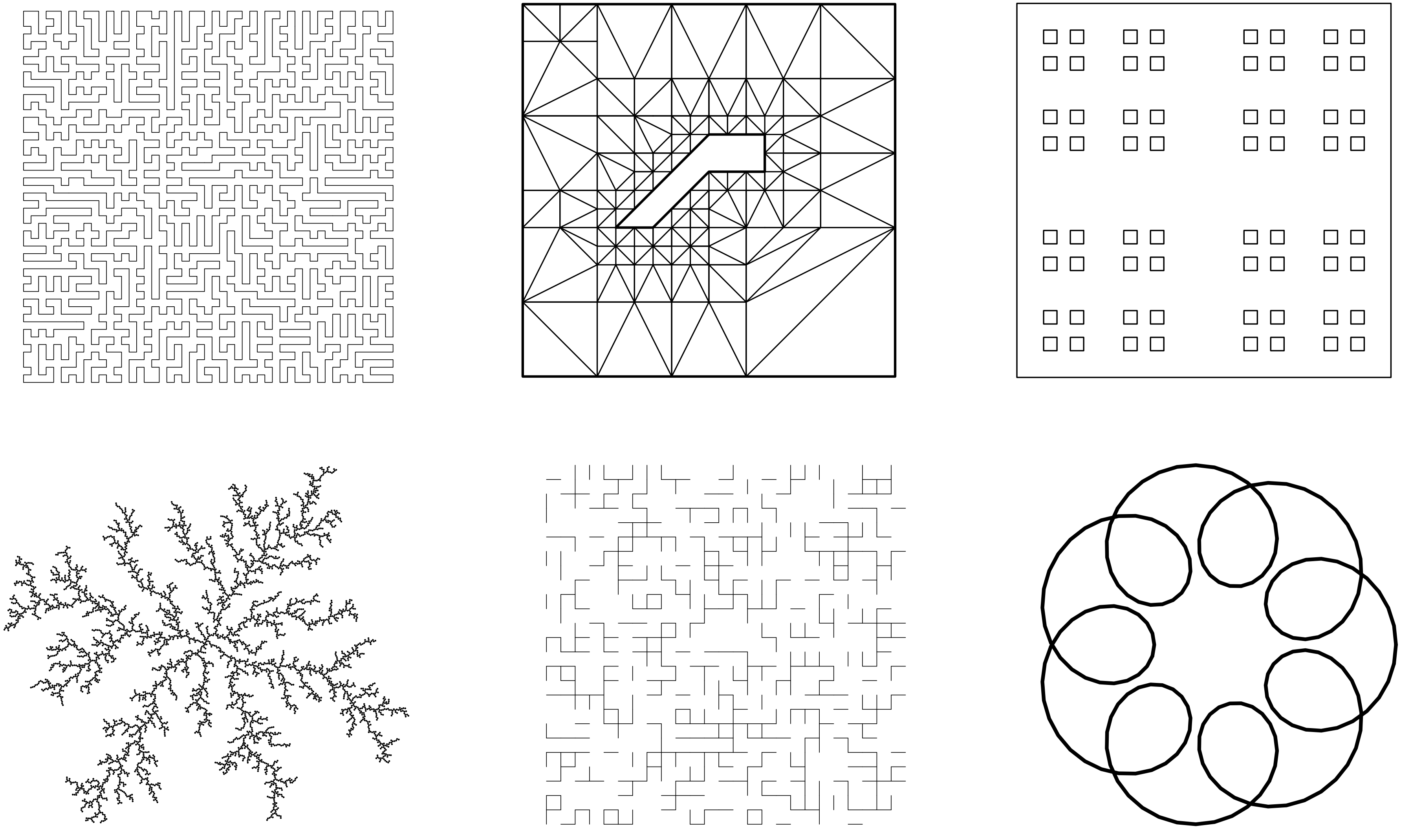
A **Planar Straight Line Graph** (PSLG) is a finite point set plus a set of disjoint edges between them.



Mesh must **cover** the edges of the PSLG.

May be necessary to add Steiner points.

Fills convex hull.



More PSLGs

Theorem (B, 2011): Every PSLG has a quadrilateral mesh with $O(n^2)$ elements, all angles less than $120°$ and all new angles greater than $60°$.

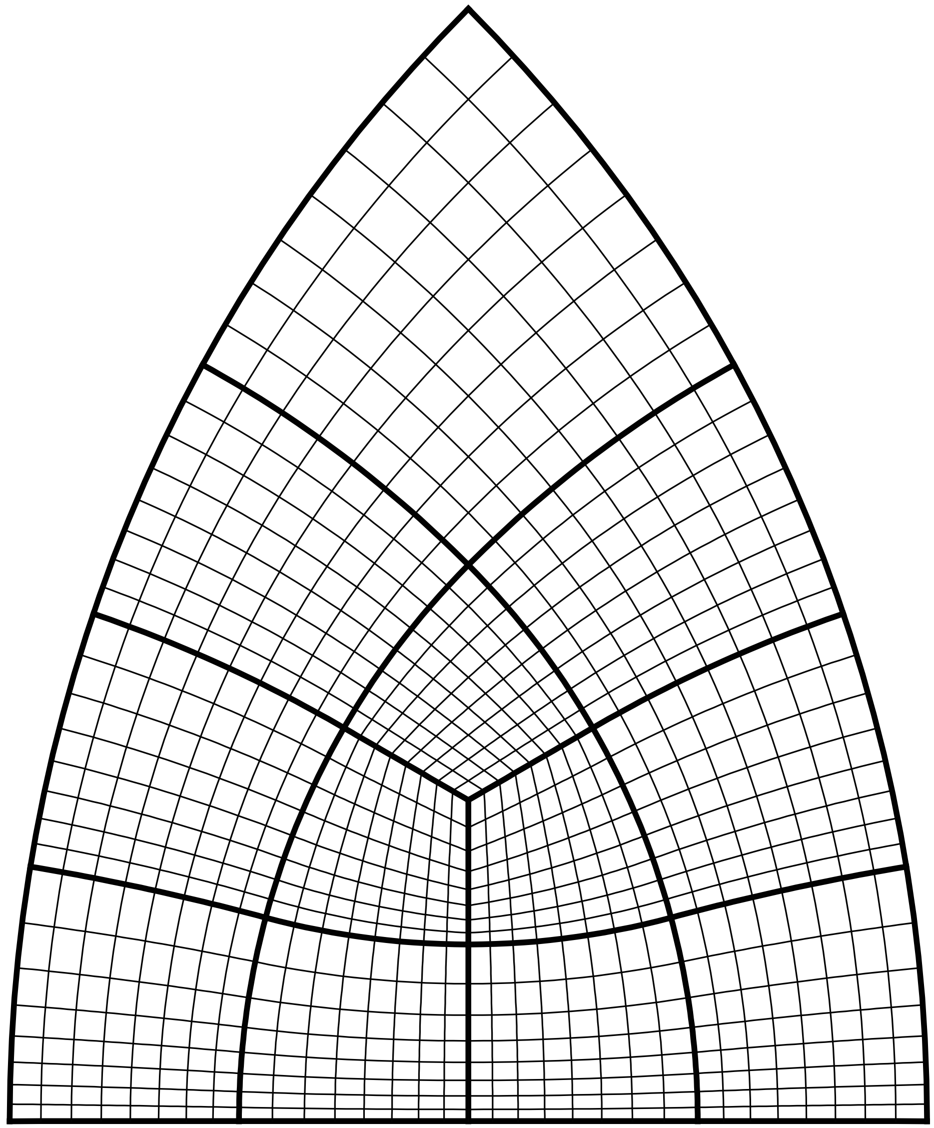
Theorem (B, 2011): Every PSLG has a quadrilateral mesh with $O(n^2)$ elements, all angles less than $120°$ and all new angles greater than $60°$.

Angles and complexity sharp.

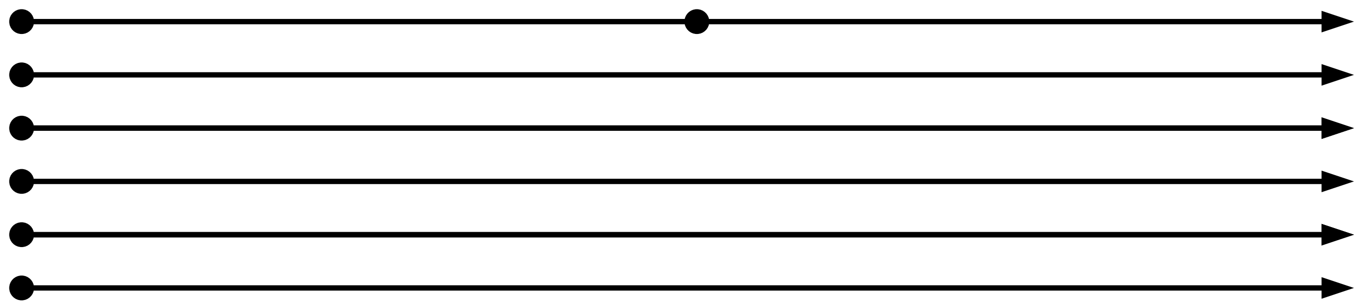
All but $O(n)$ vertices have angles in $[89°, 91°]$.

All but $O(n)$ vertices are degree 4.

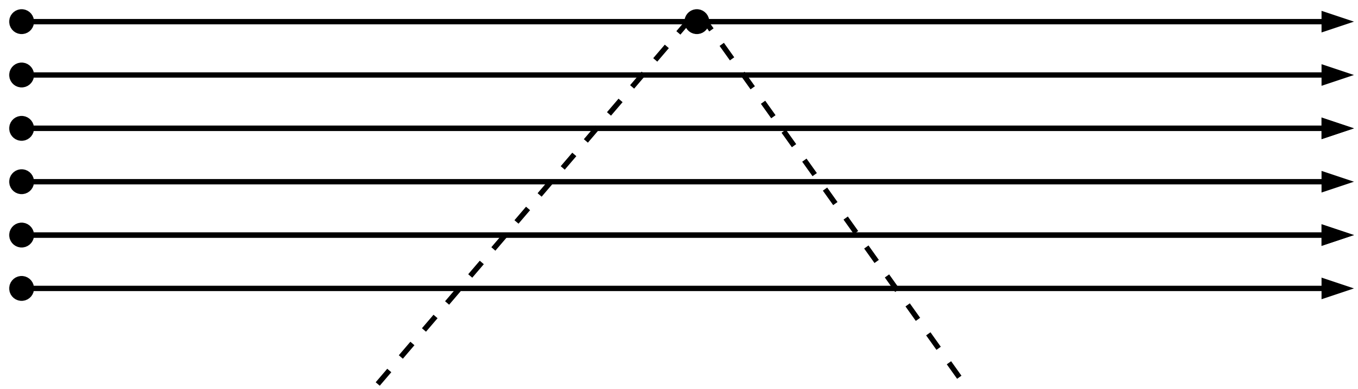
Mesh has $O(n)$ sub-meshes, each a rectangular grid.



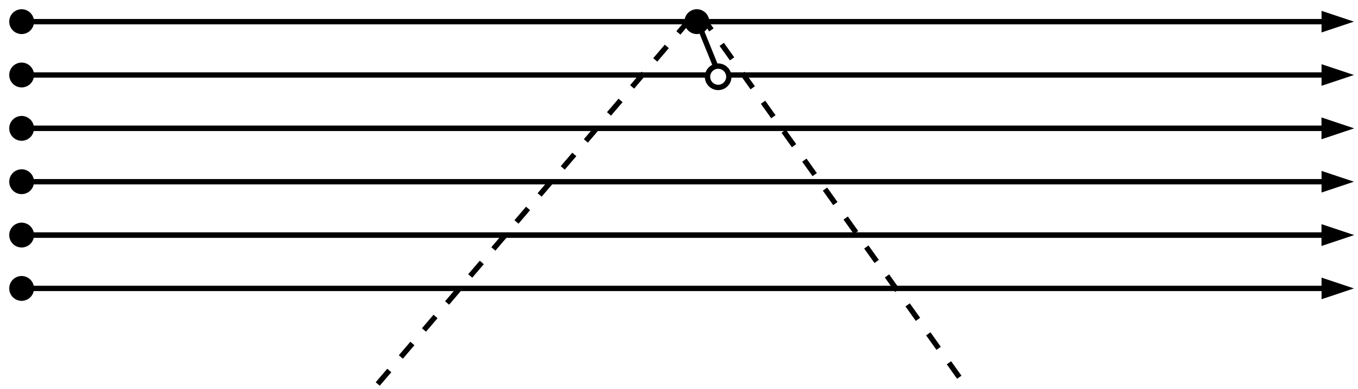
Any bound $\theta < 180^\circ$ sometimes requires n^2 vertices.



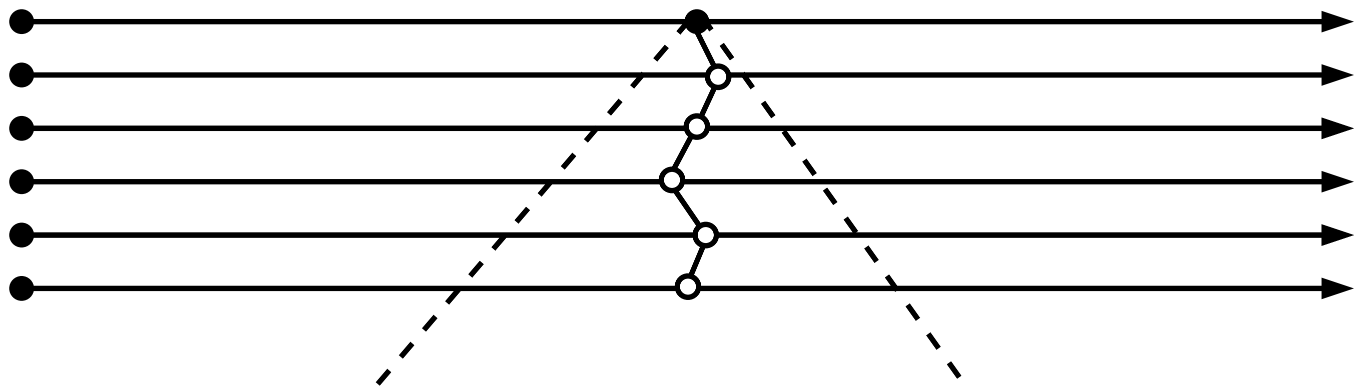
Any bound $\theta < 180^\circ$ sometimes requires n^2 vertices.



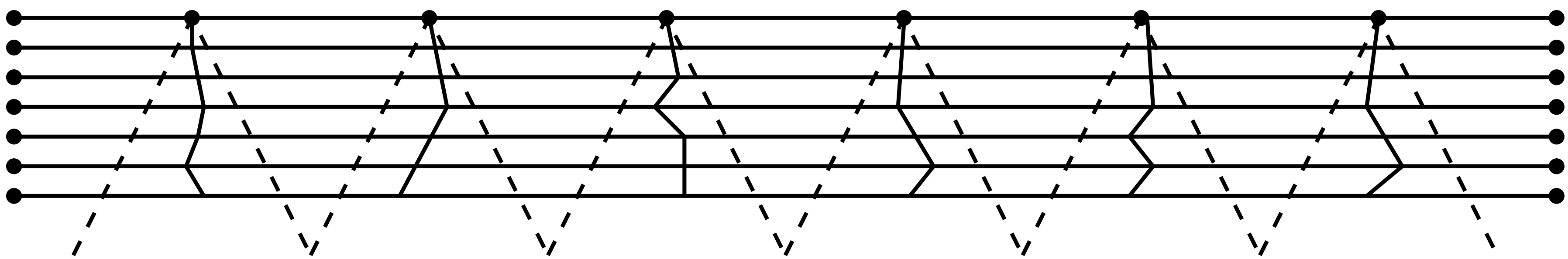
Any bound $\theta < 180^\circ$ sometimes requires n^2 vertices.



Any bound $\theta < 180^\circ$ sometimes requires n^2 vertices.



Any bound $\theta < 180°$ sometimes requires n^2 vertices.



Proof quad meshing for PSLGs requires new ideas:

- connect PSLG without small angles
- thick/thin decompose complementary components
- thick parts meshed by polygon method
- foliation of thin parts
- bending foliation paths
- traps, sinks

Convert quadrilaterals to triangles by adding diagonals.

Corollary: Every PSLG has a $O(n^2)$ triangulation with maximum angle $\leq 120°$.

Compare S. Mitchell 1993 $(157.5°)$ and Tan 1996 $(132°)$.

Convert quadrilaterals to triangles by adding diagonals.

Corollary: Every PSLG has a $O(n^2)$ triangulation with maximum angle $\leq 120^\circ$.

Compare S. Mitchell 1993 (157.5°) and Tan 1996 (132°).

Can the 120° upper bound be improved?

Convert quadrilaterals to triangles by adding diagonals.

Corollary: Every PSLG has a $O(n^2)$ triangulation with maximum angle $\leq 120^\circ$.

Compare S. Mitchell 1993 (157.5°) and Tan 1996 (132°).

Can the 120° upper bound be improved?

Can we get a positive lower angle bound?

Convert quadrilaterals to triangles by adding diagonals.

Corollary: Every PSLG has a $O(n^2)$ triangulation with maximum angle $\leq 120°$.

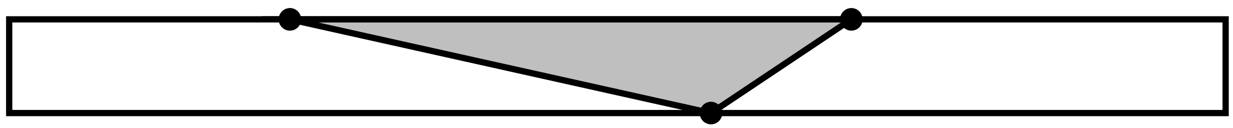
Compare S. Mitchell 1993 $(157.5°)$ and Tan 1996 $(132°)$.

Can the $120°$ upper bound be improved? **Yes**

Can we get a positive lower angle bound? **No**

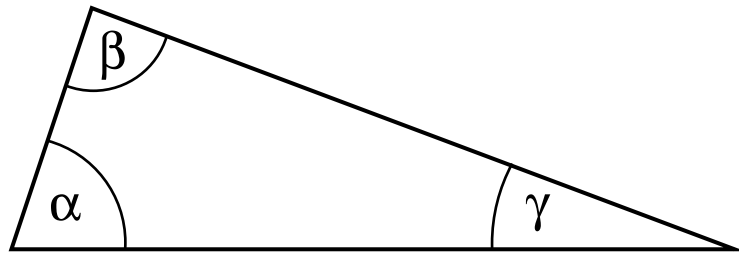
No lower angle bound. For $1 \times R$ rectangle

$$\text{number of triangles} \gtrsim R \times (\text{smallest angle})$$



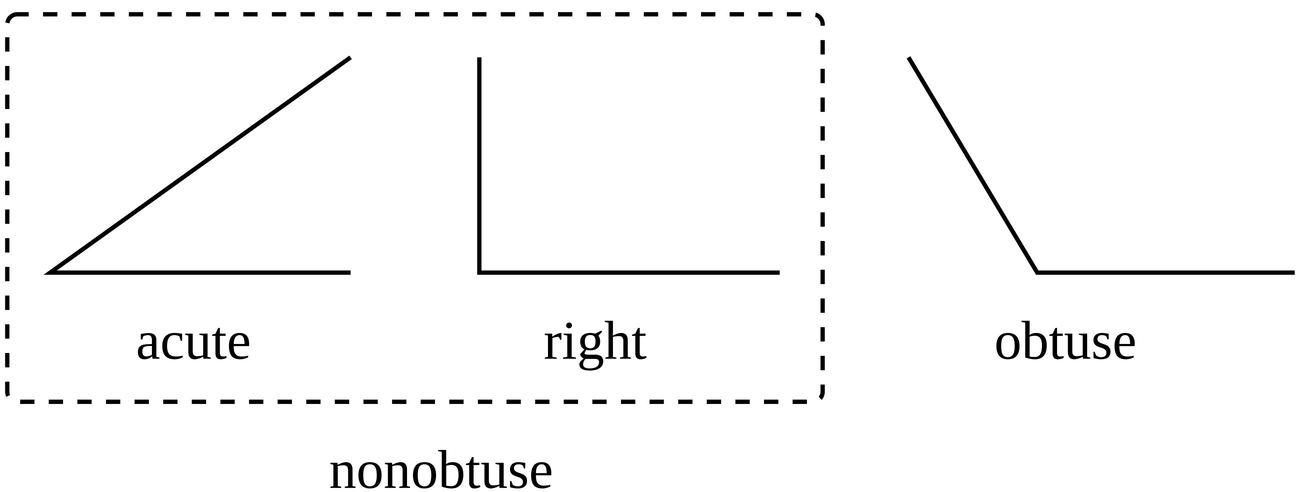
So uniform complexity $\Rightarrow$ no lower angle bound.

No upper bound $< 90°$:



If angles are $\leq 90° - \epsilon$ then all angles are $\geq 2\epsilon$.
$\gamma = 180 - \alpha - \beta \geq 180 - (90 - \epsilon) - (90 - \epsilon) \geq 2\epsilon$.

So **nonobtuse** triangulation is best we can hope for.



Brief history of nonobtuse triangulation:

- Always possible: Burago, Zalgaller 1960.
- Rediscovered: Baker, Grosse, Rafferty, 1988.
- $O(n)$ for points sets: Bern, Eppstein, Gilbert 1990
- $O(n^2)$ for polygons: Bern, Eppstein 1991
- $O(n)$ for polygons: Bern, Mitchell, Ruppert 1994

Numerous applications, heuristics: discrete maximum priniciple, condition numbers for finite element method, fast marching method, computer learning, . . .

Open problem: does every PSLG have a polynomial sized nonobtuse triangulation?

Theorem (B, 2011): Every PSLG has a nonobtuse triangulation with $O(n^{2.5})$ elements.

Theorem (B, 2011): Every PSLG has a nonobtuse triangulation with $O(n^{2.5})$ elements.

Cor (of proof): Every PSLG has a triangulation with all angles $\leq 90^\circ + \epsilon$ and $O(n^2/\epsilon^2)$ elements.

Theorem (B, 2011): Every PSLG has a nonobtuse triangulation with $O(n^{2.5})$ elements.

Cor (of proof): Every PSLG has a triangulation with all angles $\leq 90° + \epsilon$ and $O(n^2/\epsilon^2)$ elements.

Cor: Any triangulation of a simple n-gon has an acute refinement of size $O(n^2)$. (see Bern, Eppstein 1992)

Theorem (B, 2011): Every PSLG has a nonobtuse triangulation with $O(n^{2.5})$ elements.

Cor (of proof): Every PSLG has a triangulation with all angles $\leq 90° + \epsilon$ and $O(n^2/\epsilon^2)$ elements.

Cor: Any triangulation of a simple n-gon has an acute refinement of size $O(n^2)$. (see Bern, Eppstein 1992)

Cor: Any PSLG has a conforming Delaunay triangulation of size $O(n^{2.5})$. (see Edelsbrunner, Tan 1993)

Theorem (B, 2011): Every PSLG has a nonobtuse triangulation with $O(n^{2.5})$ elements.

Cor (of proof): Every PSLG has a triangulation with all angles $\leq 90° + \epsilon$ and $O(n^2/\epsilon^2)$ elements.

Cor: Any triangulation of a simple n-gon has an acute refinement of size $O(n^2)$. (see Bern, Eppstein 1992)

Cor: Any PSLG has a conforming Delaunay triangulation of size $O(n^{2.5})$. (see Edelsbrunner, Tan 1993)

Cor: For any PSLG there is a set of size $O(n^{2.5})$ whose Voronoi diagram covers the PSLG.

Proofs have no conformal maps or hyperbolic geometry.

CHT motivates proof, but is not explicitly used.

Applications of Sullivan's CHT:

- Dimension of geometrically infinite limit sets.
- Conformal factorization into Lipschitz maps.
- Bowen's dichotomy for divergence type groups.
- Linear time algorithm for conformal mapping.
- Optimal quadrilateral meshing of polygons and PSLGs.
- Polynomial algorithm for nonobtuse triangulation.

Questions:

- Can we replace 2.5 by 2?
- Best QC constant for iota map $(2.1 < K < 7.82)$?
- Can we do better than iota?
- 2-QC + Lipschitz $\Rightarrow$ Brennan's Conj.
- 3-D meshes? Thick/Thin? Convex hull? Ricci flow?
- Applications of Mumford-Bers compactness?
- Applications of Kahn-Markovic results?