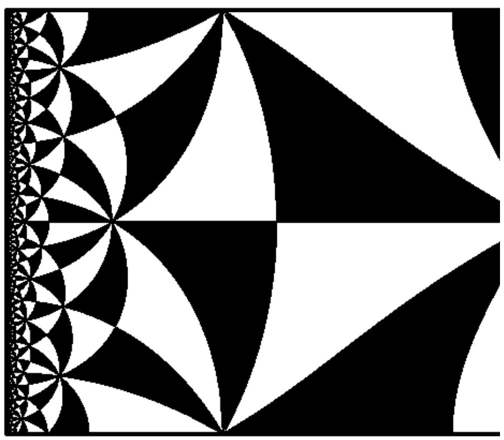
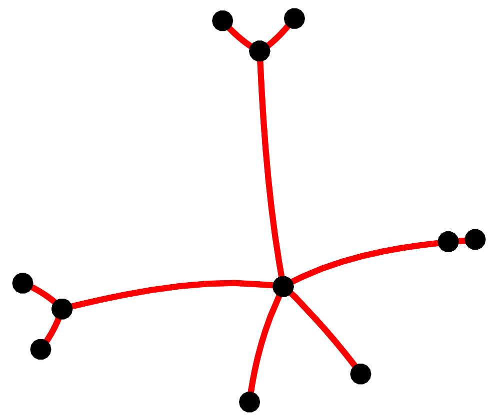
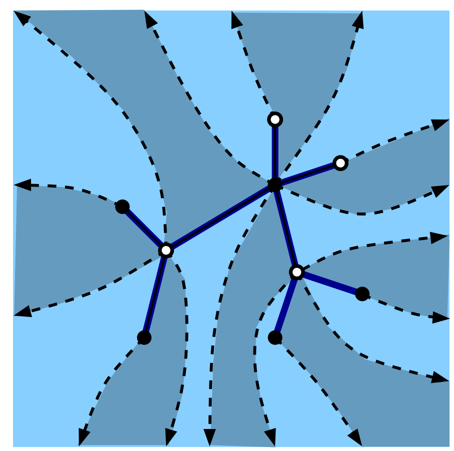
TREES AND TRIANGLES

Christopher Bishop, Stony Brook

M. Lyubich's Quasiconformal Class, May 6, 2021

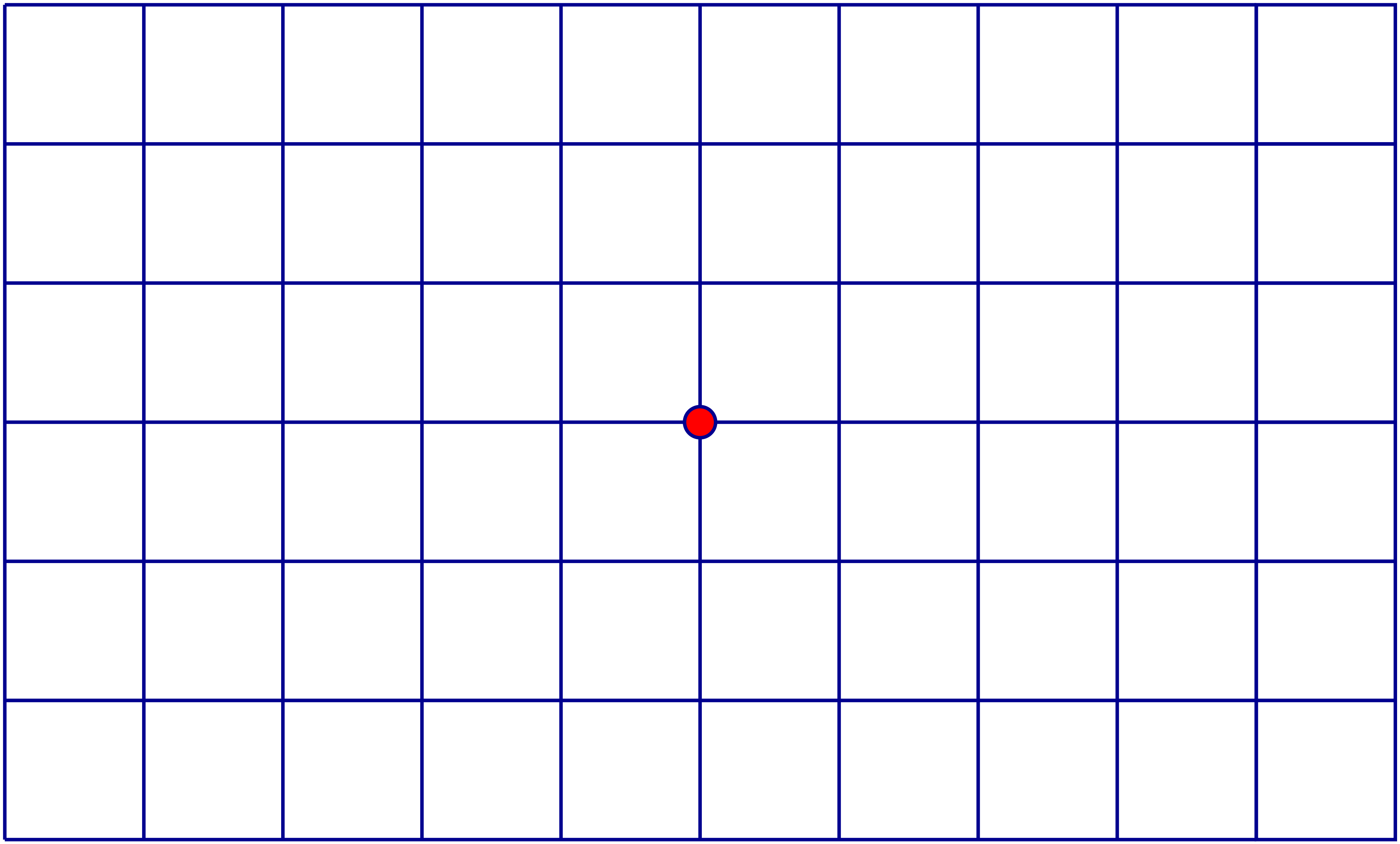


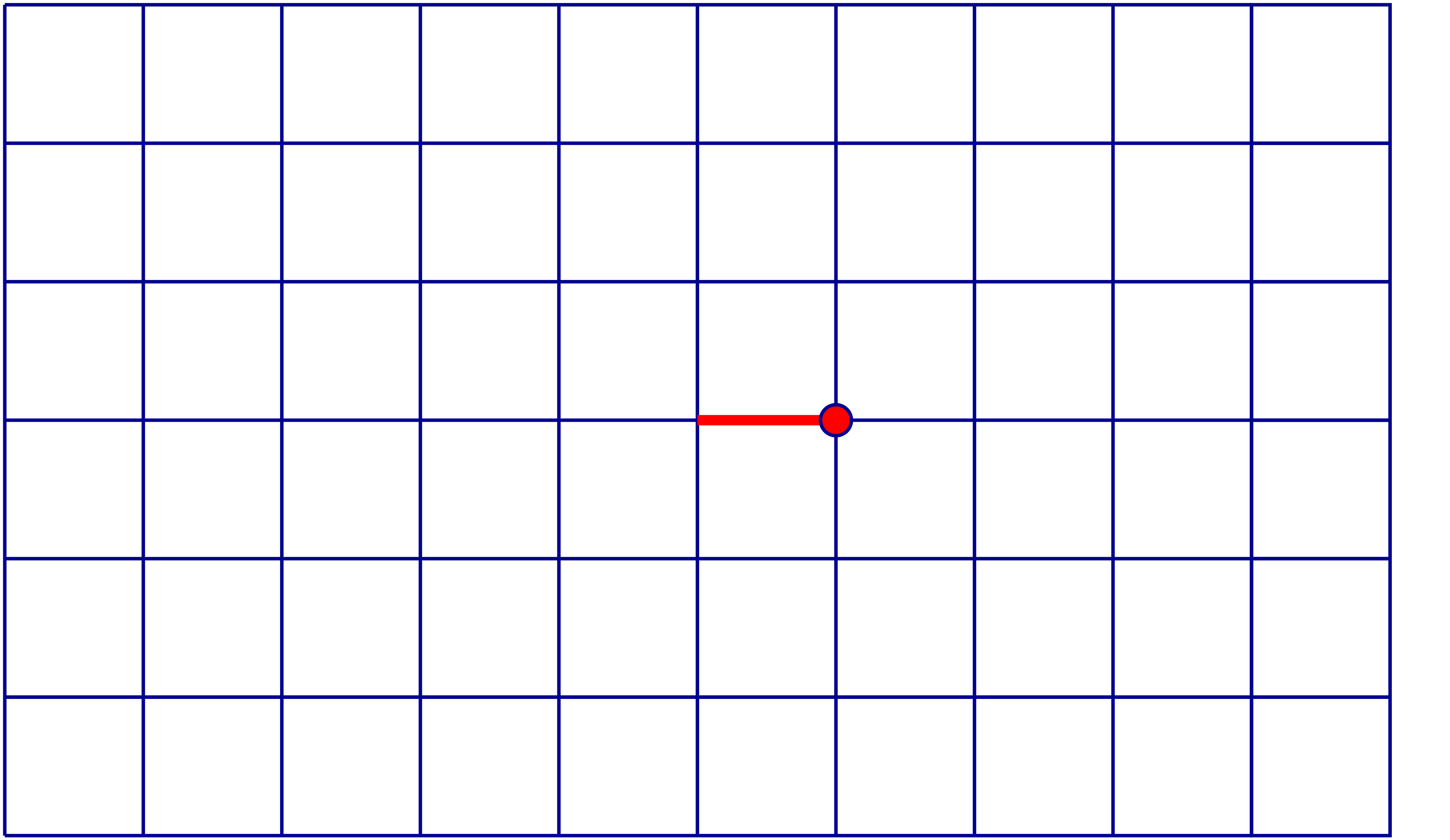


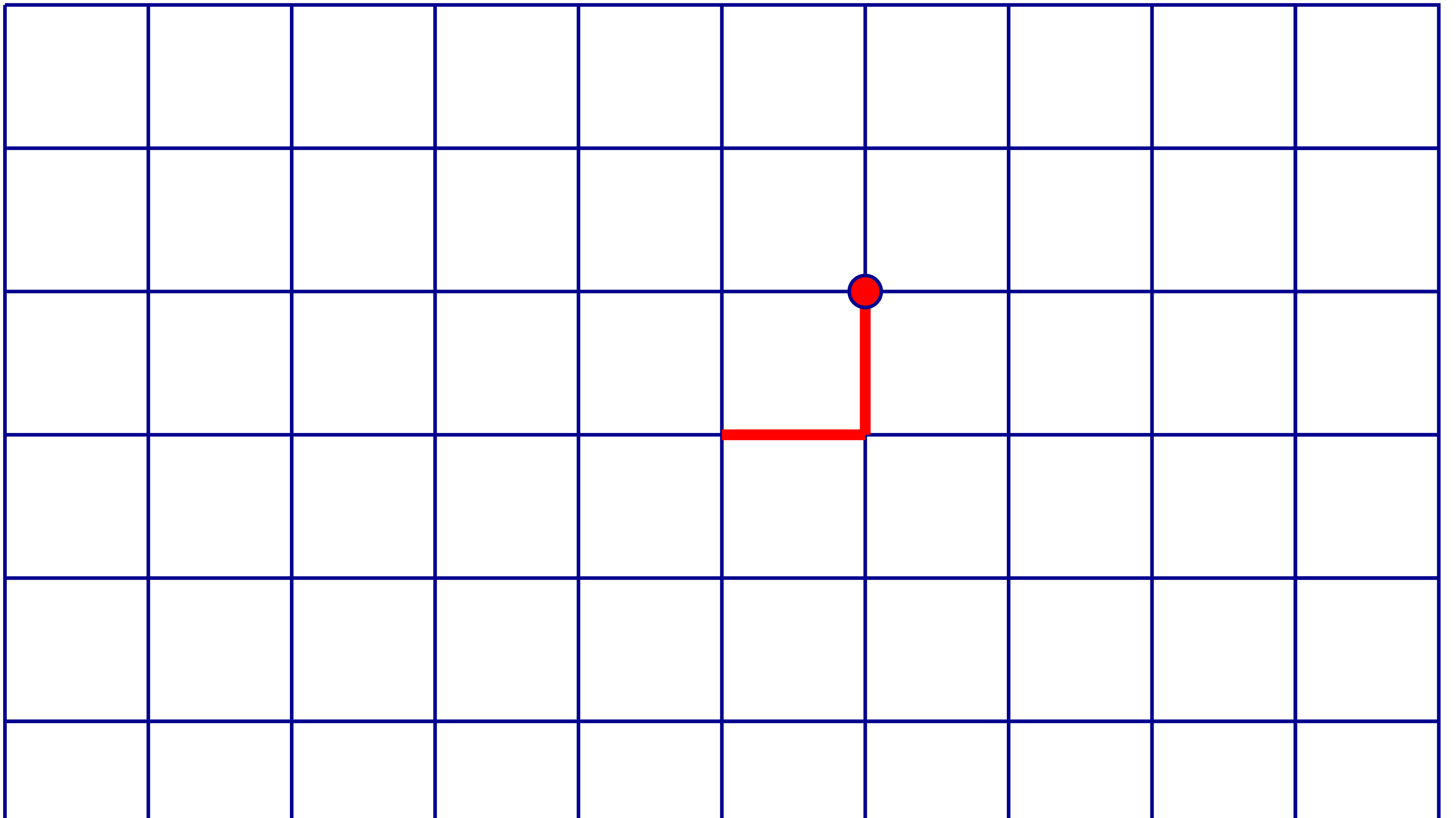


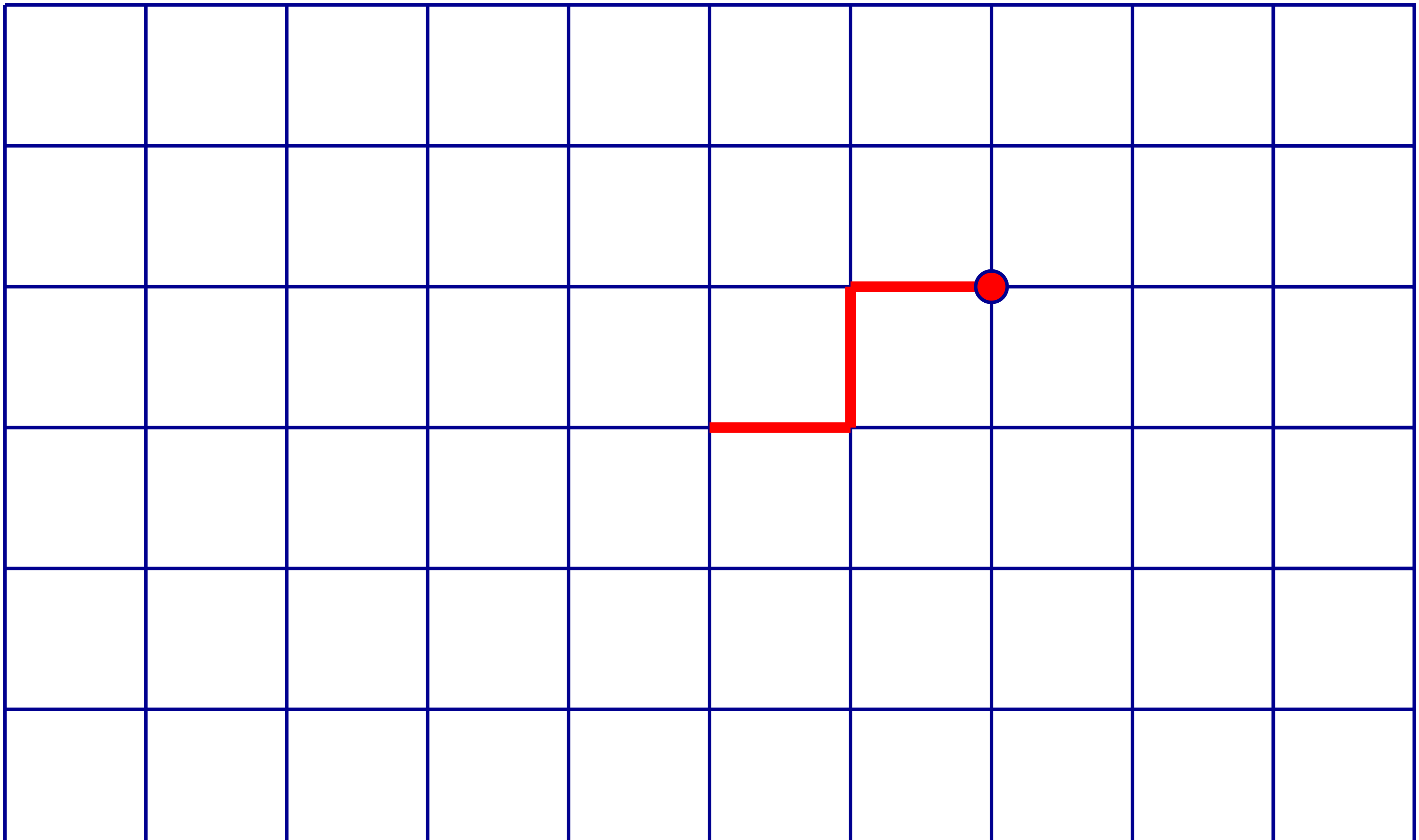
THE PLAN

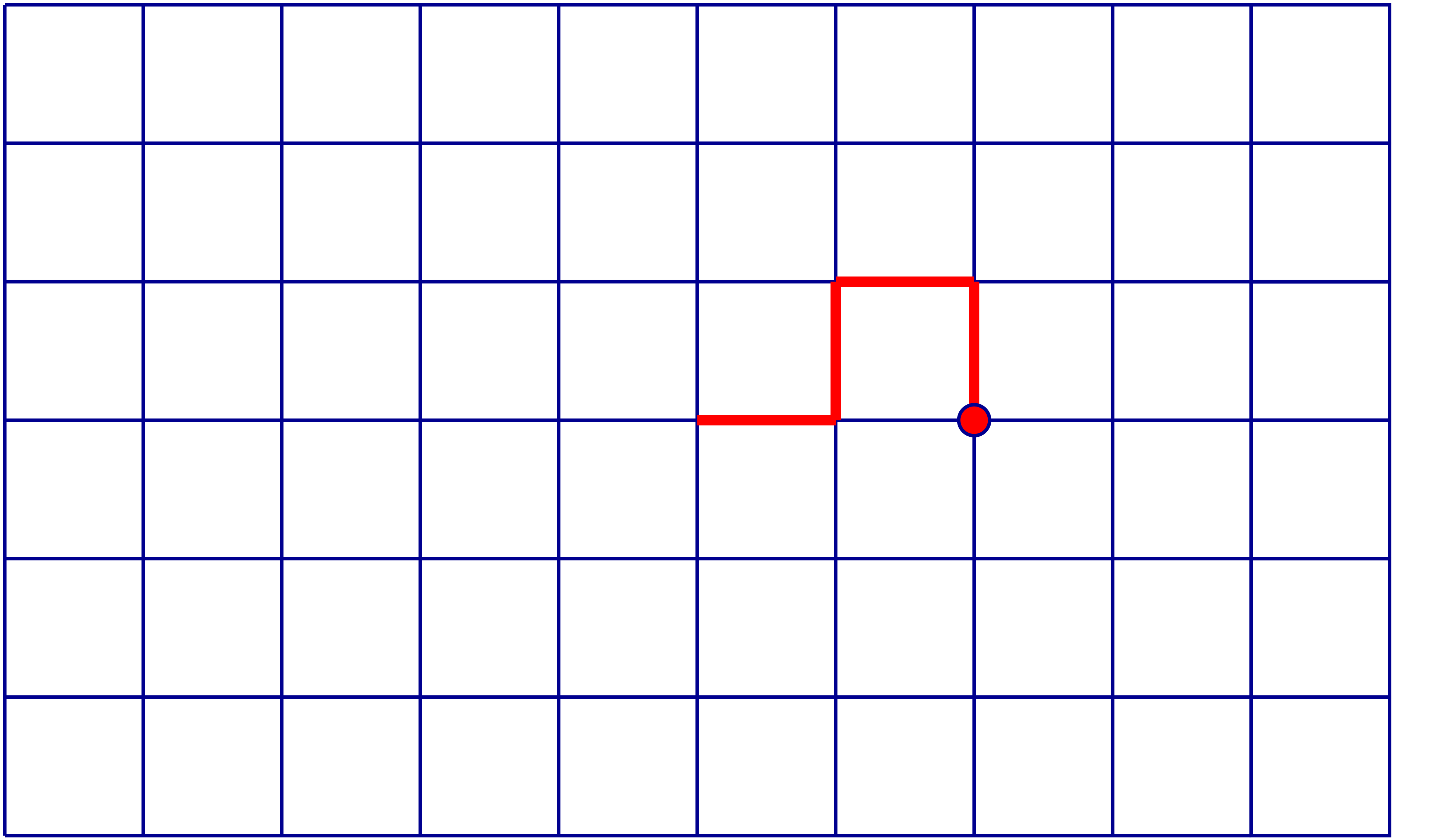
- Harmonic measure
- Finite trees and Shabat polynomials
- True trees exist: 2 proofs
- Possible shapes of a true tree.
- Surfaces with an equilateral triangulation.
- The folding theorem
- Dynamical applications

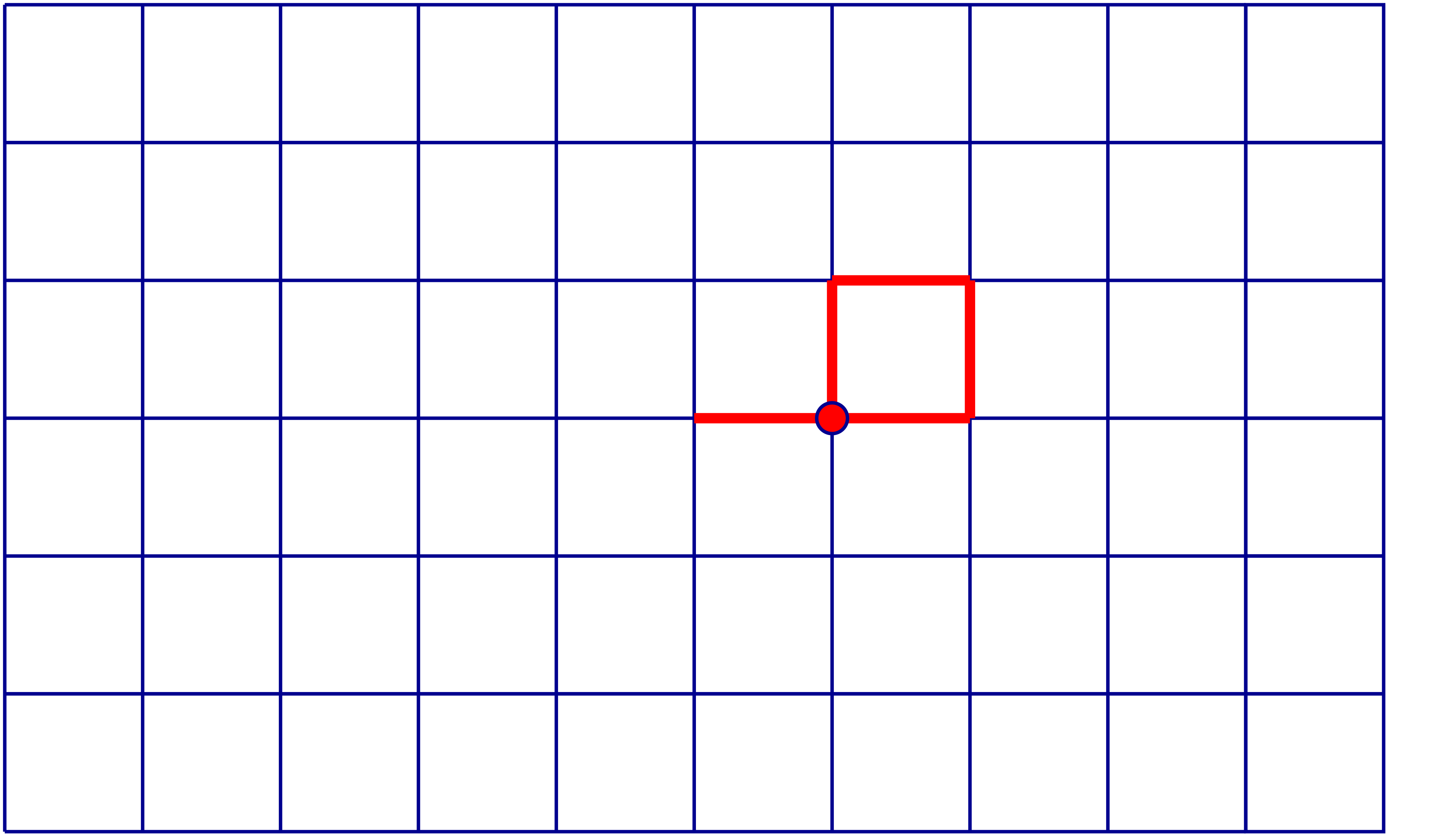


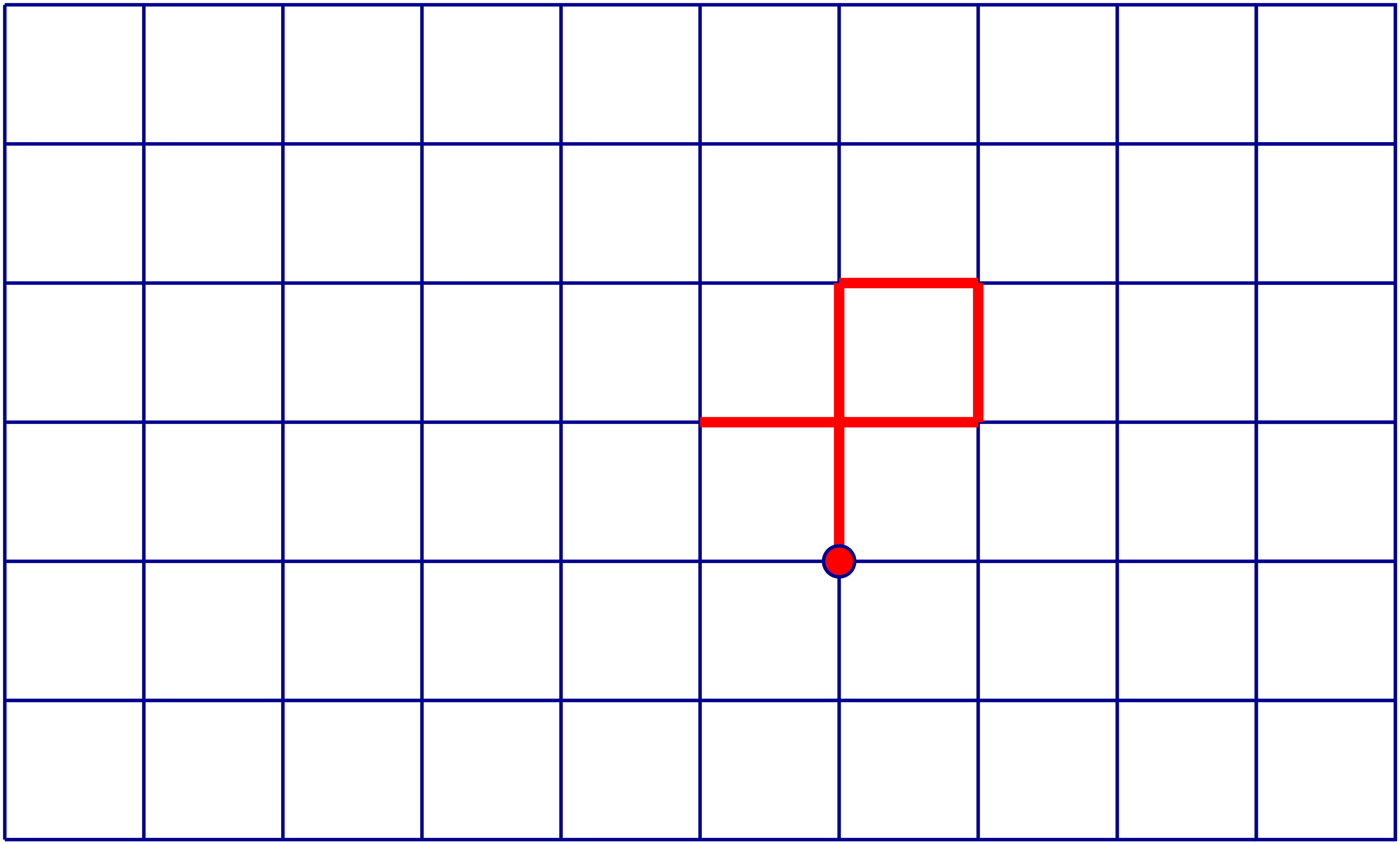


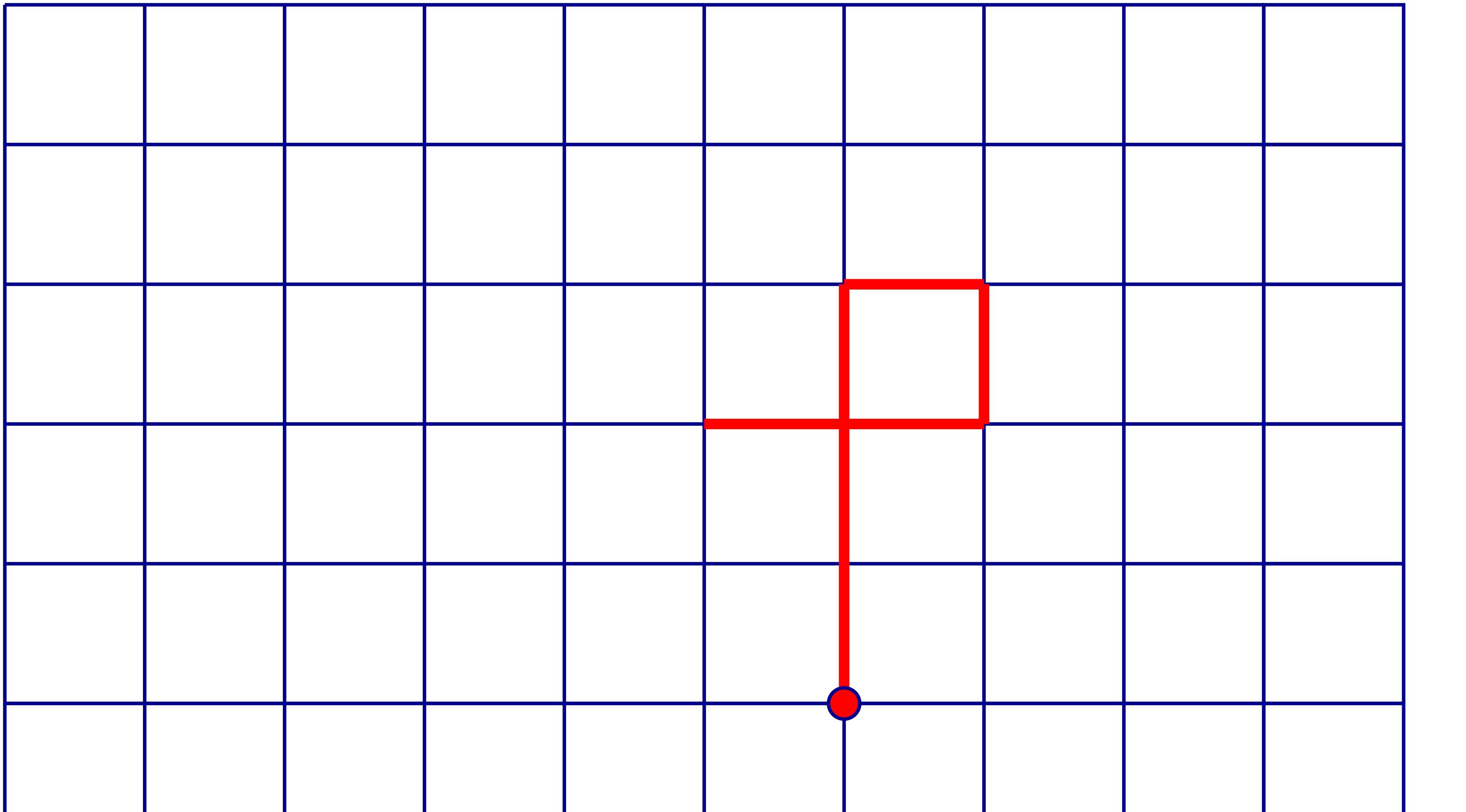


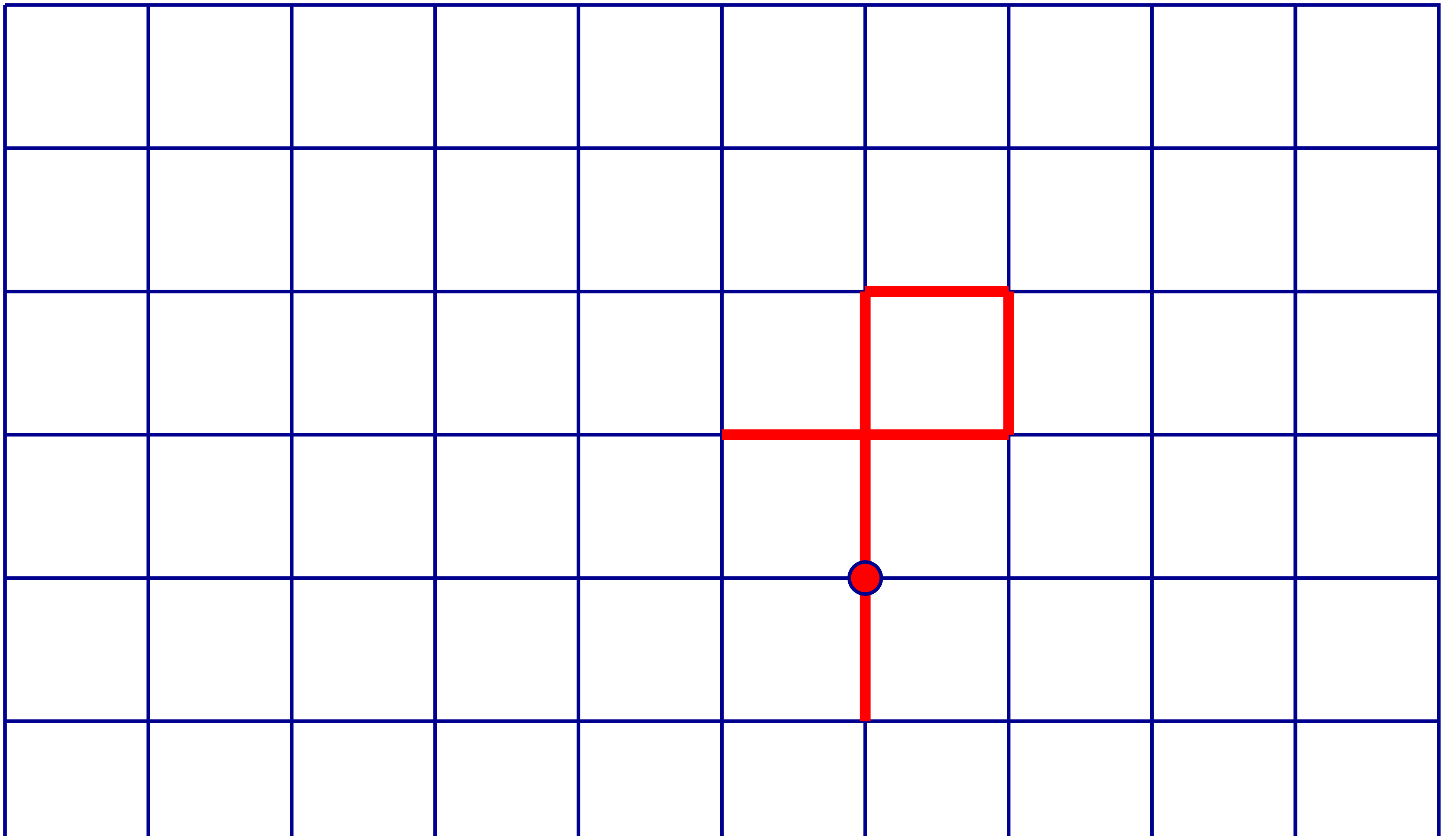


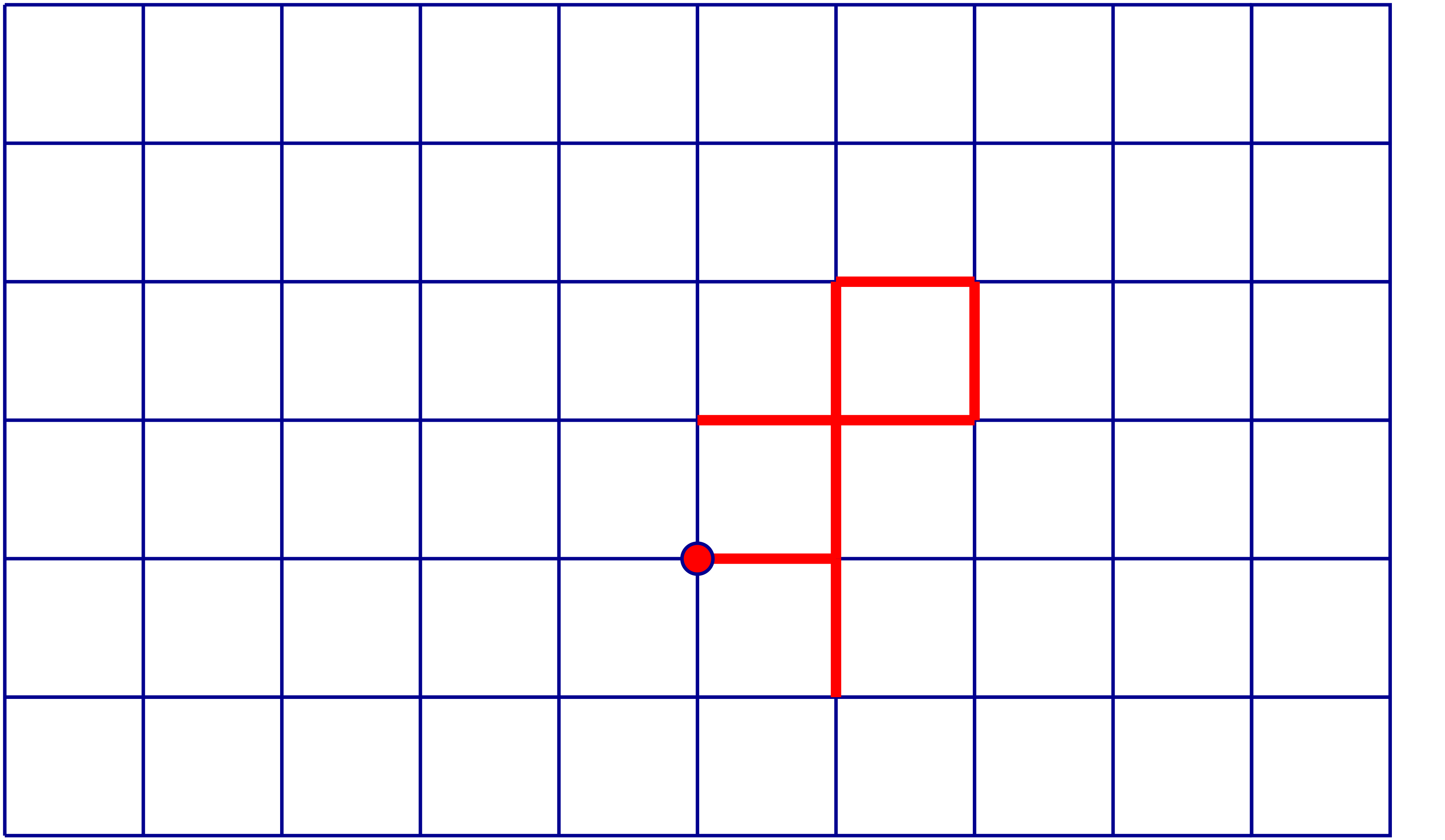


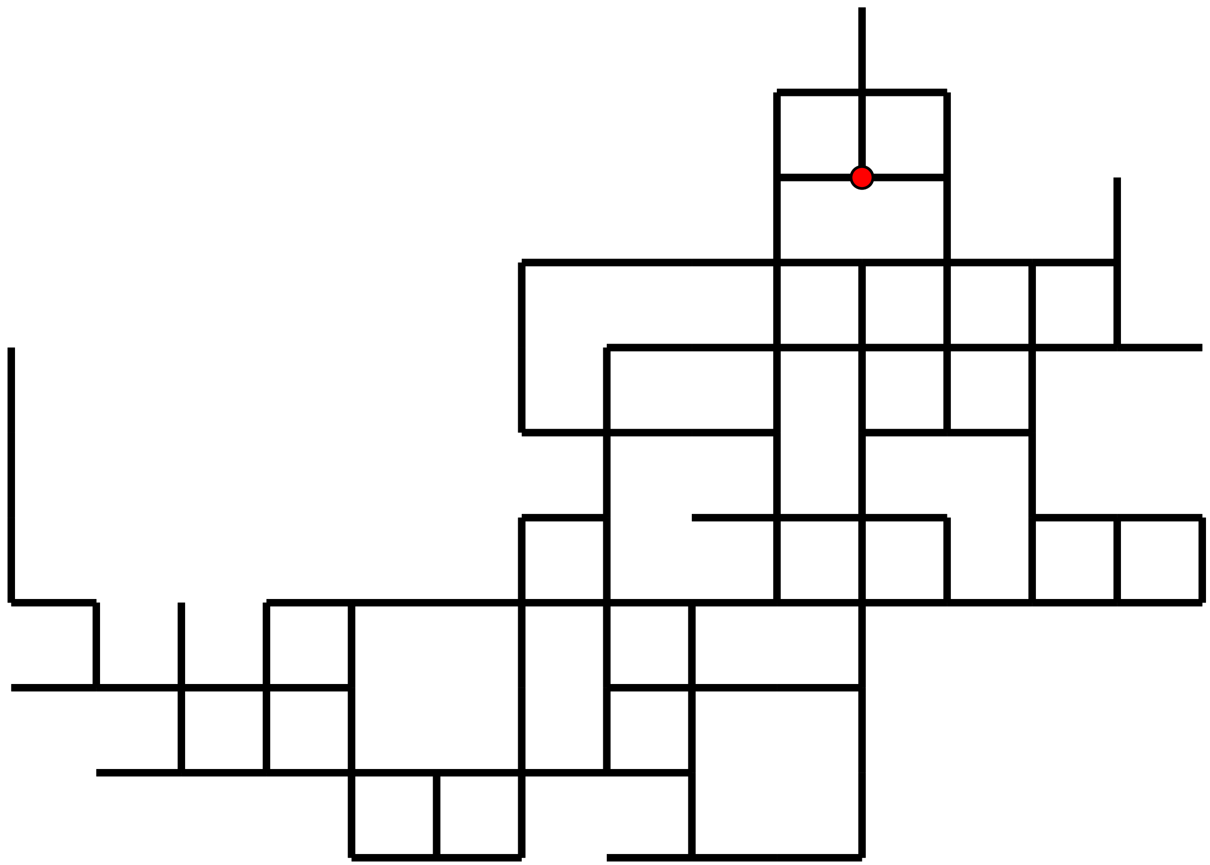




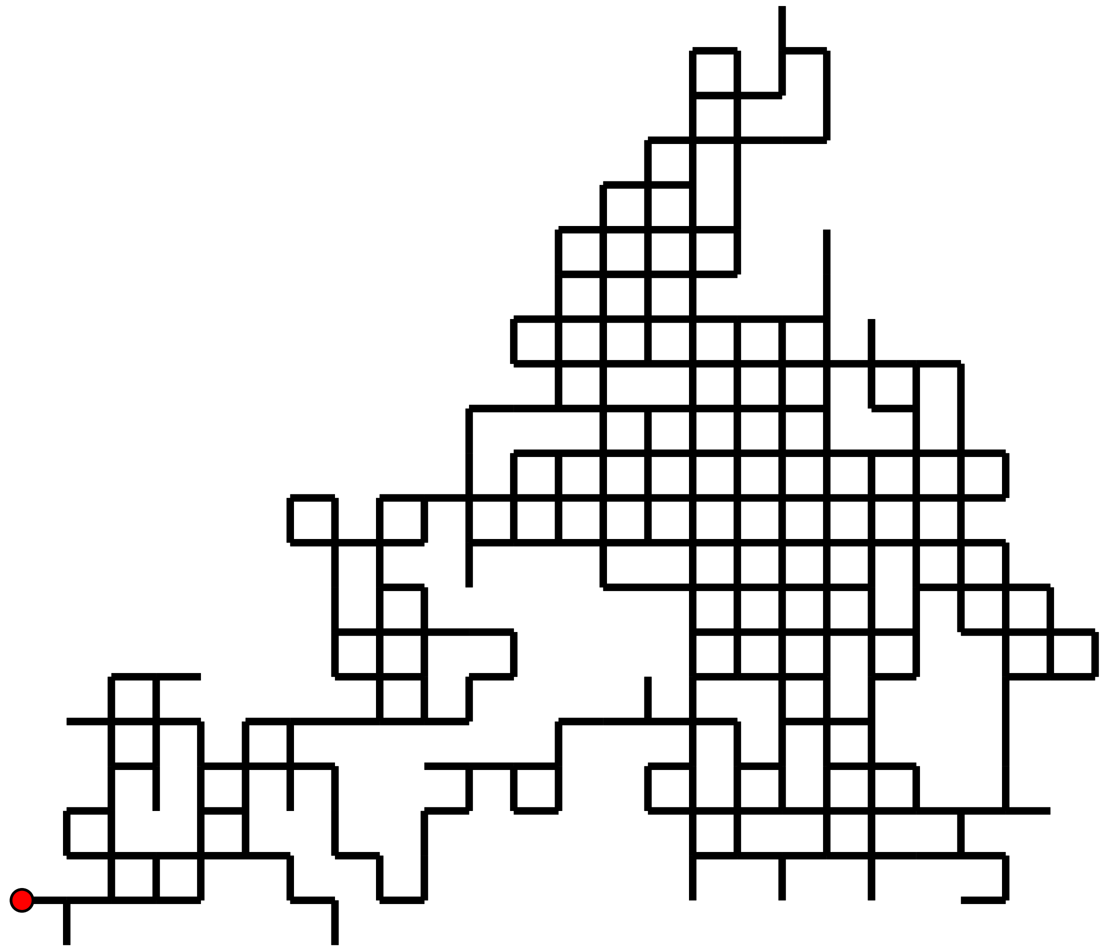




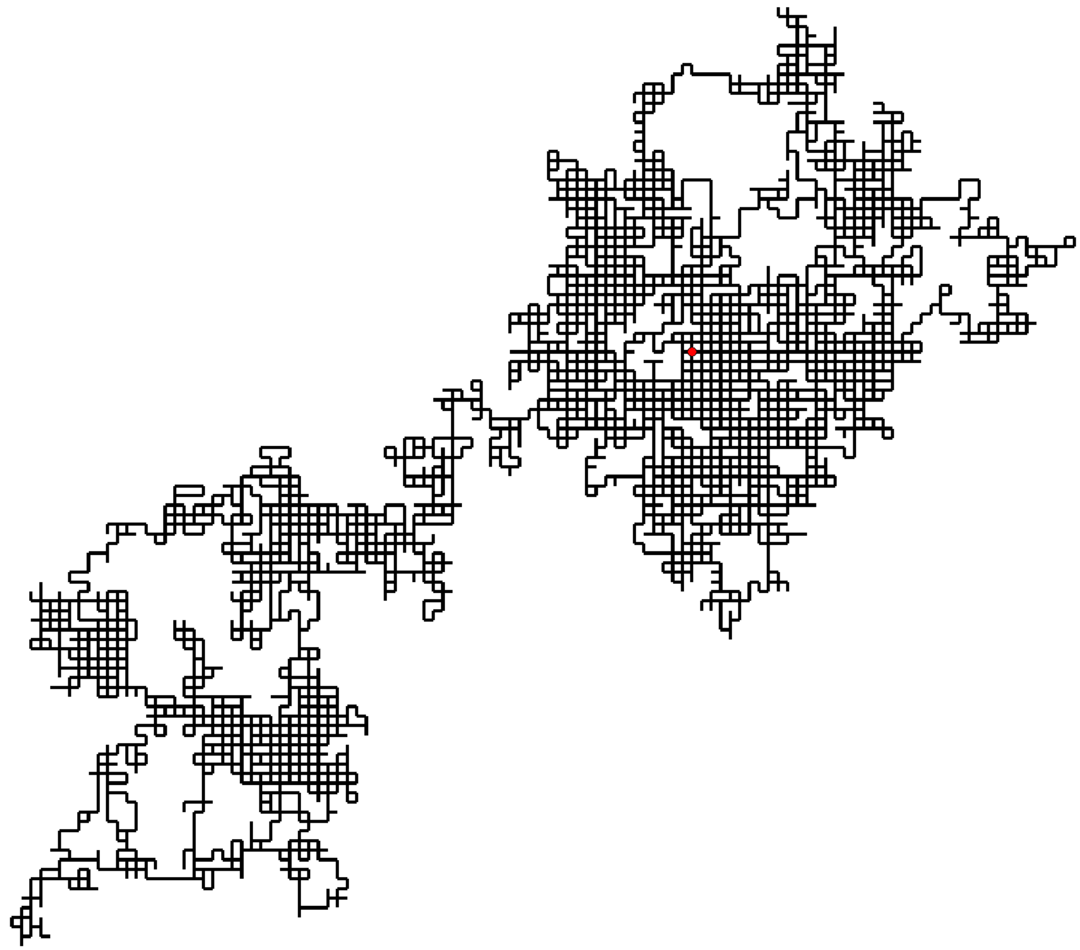




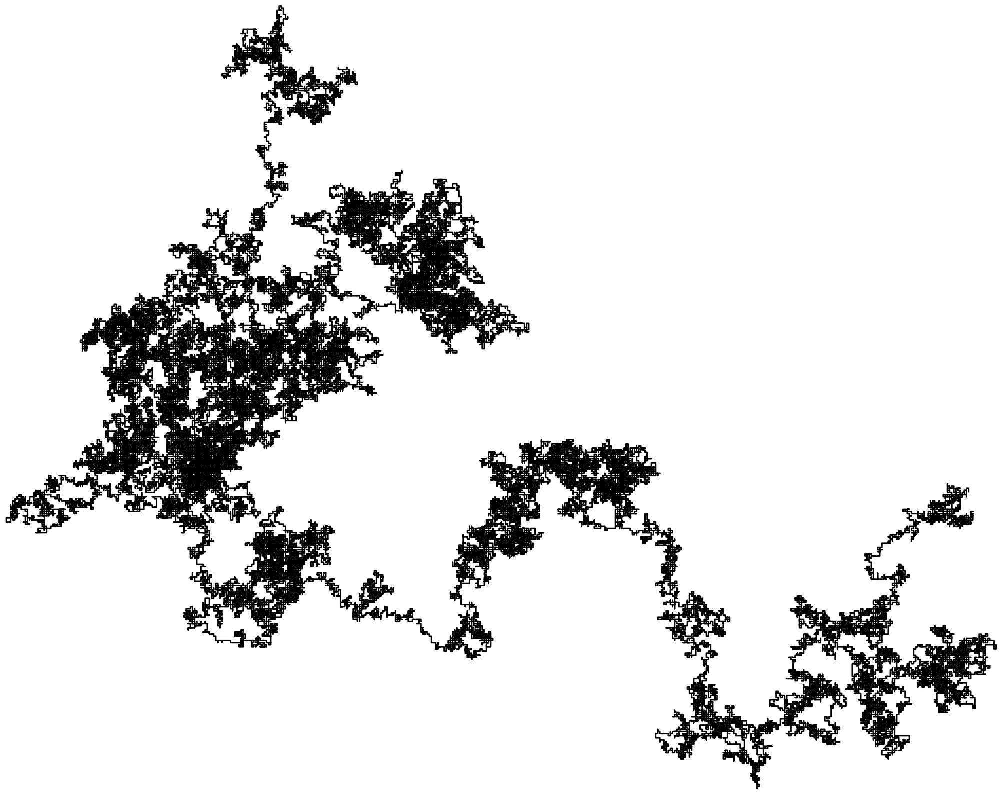
200 step random walk.



1000 step random walk.

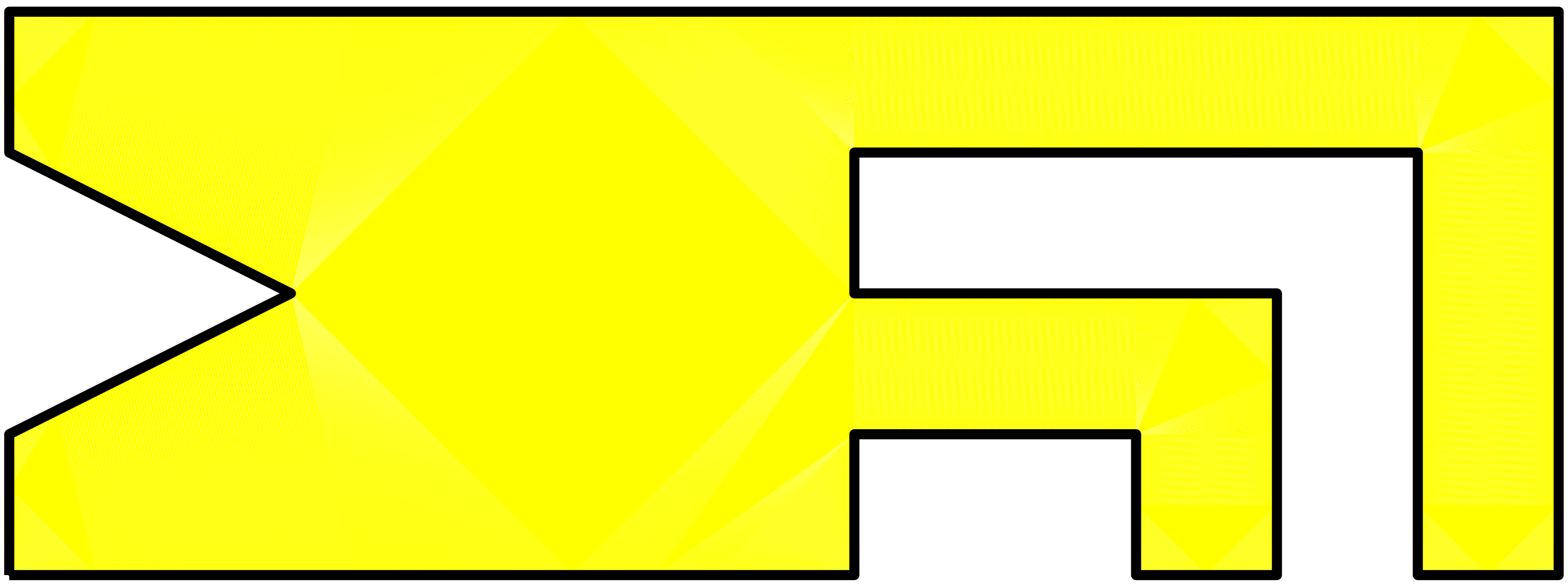


10,000 step random walk.



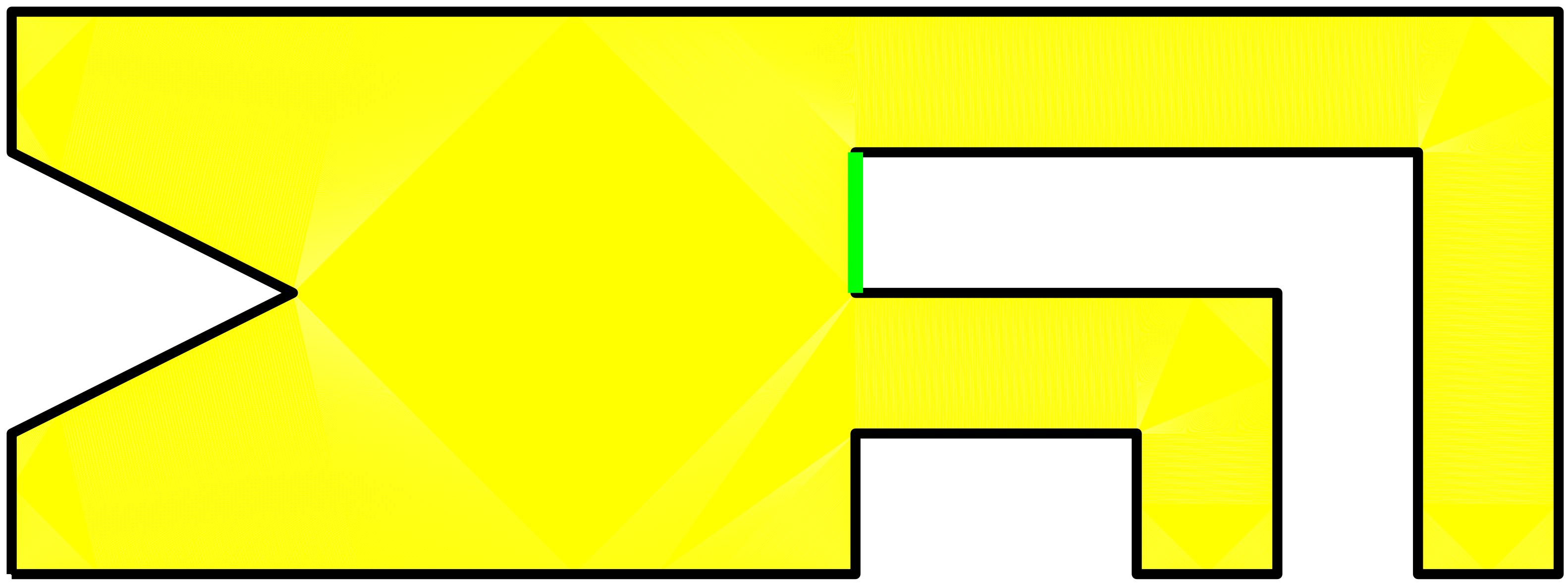
100,000 step random walk.

Harmonic measure = hitting distribution of Brownian motion



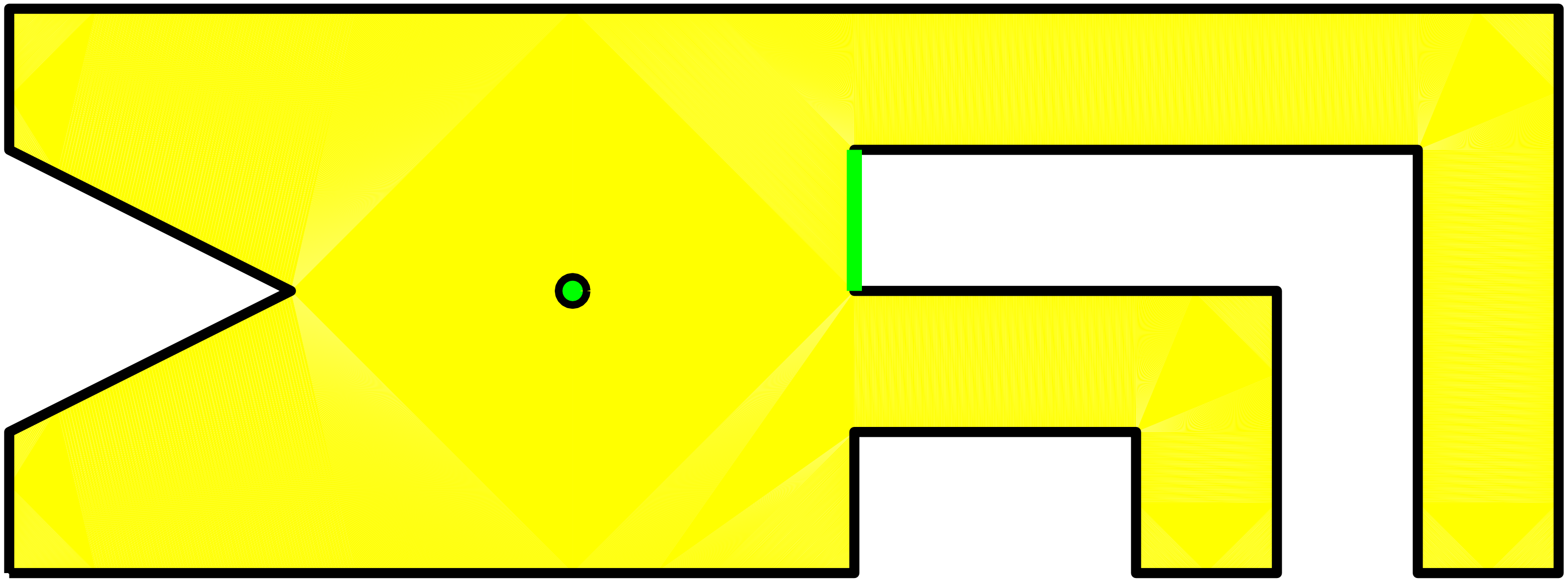
Suppose Ω is a planar Jordan domain.

Harmonic measure = hitting distribution of Brownian motion



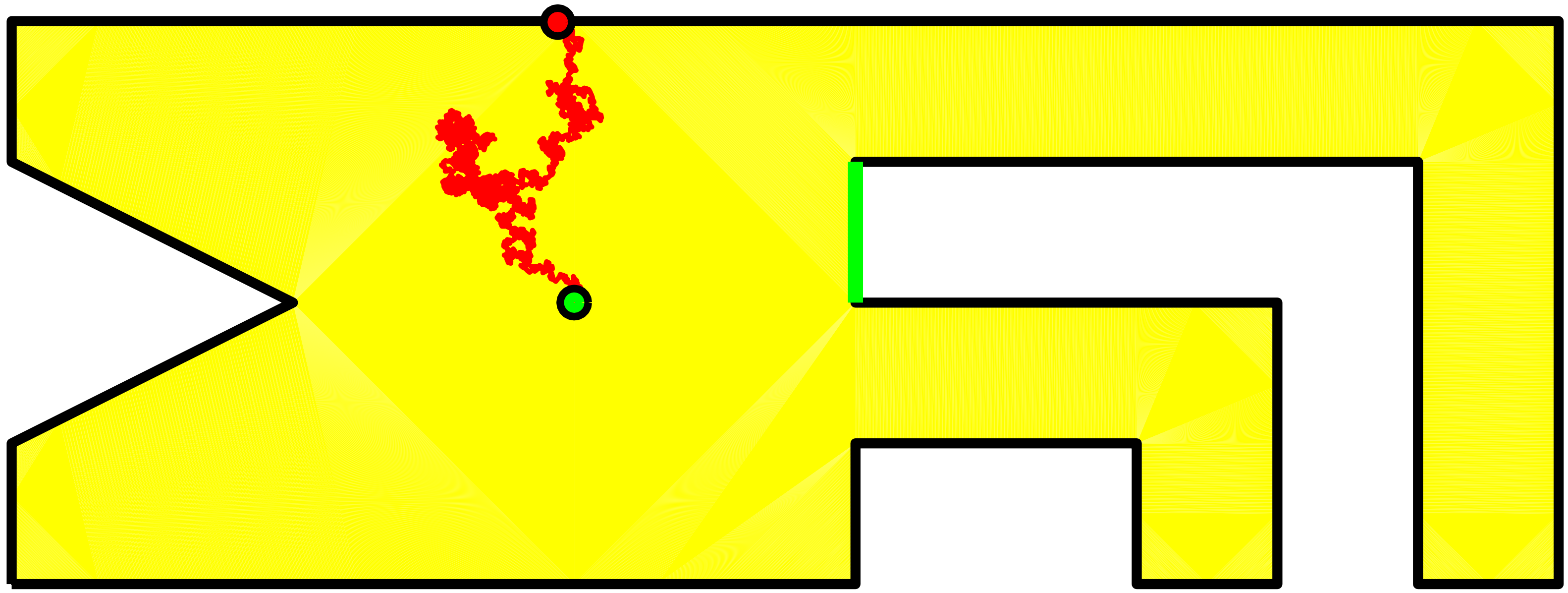
Let E be a subset of the boundary, $\partial\Omega$.

Harmonic measure = hitting distribution of Brownian motion



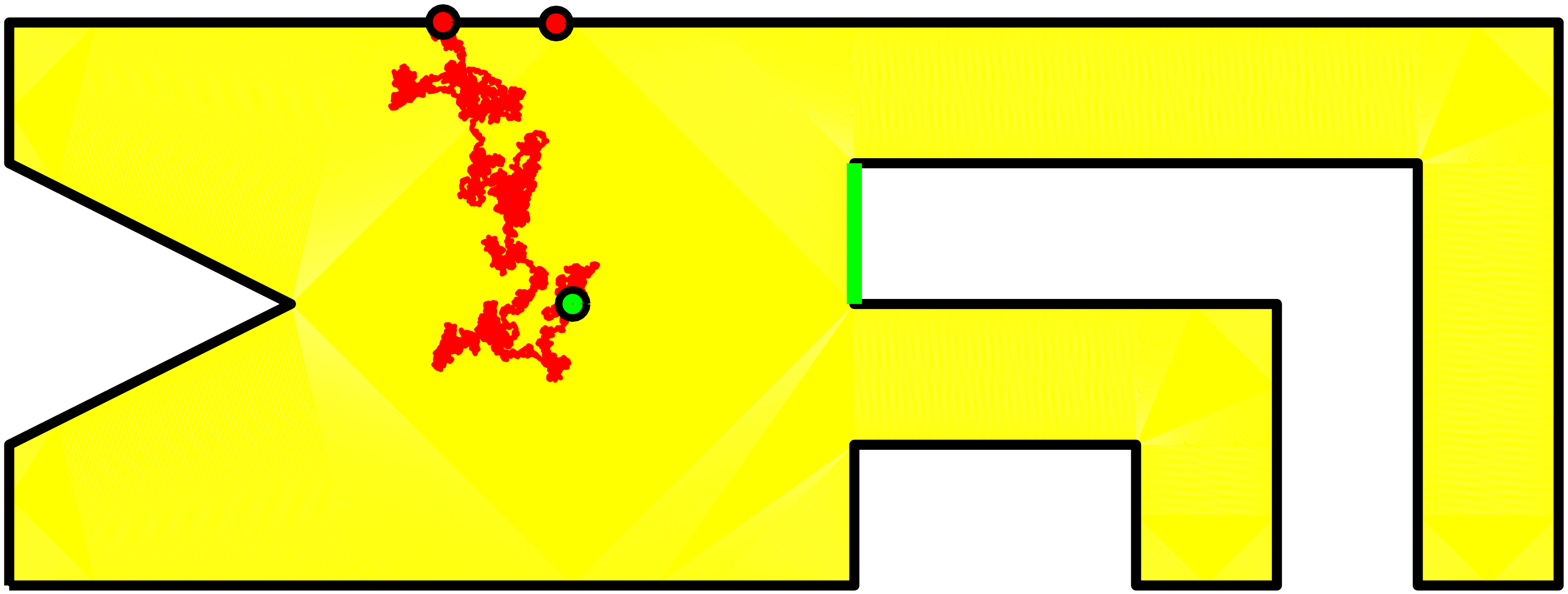
Choose an interior point $z \in \Omega$.

Harmonic measure = hitting distribution of Brownian motion



$\omega(z, E, \Omega)$ = probability a particle started at z first hits $\partial\Omega$ in E.

Harmonic measure = hitting distribution of Brownian motion



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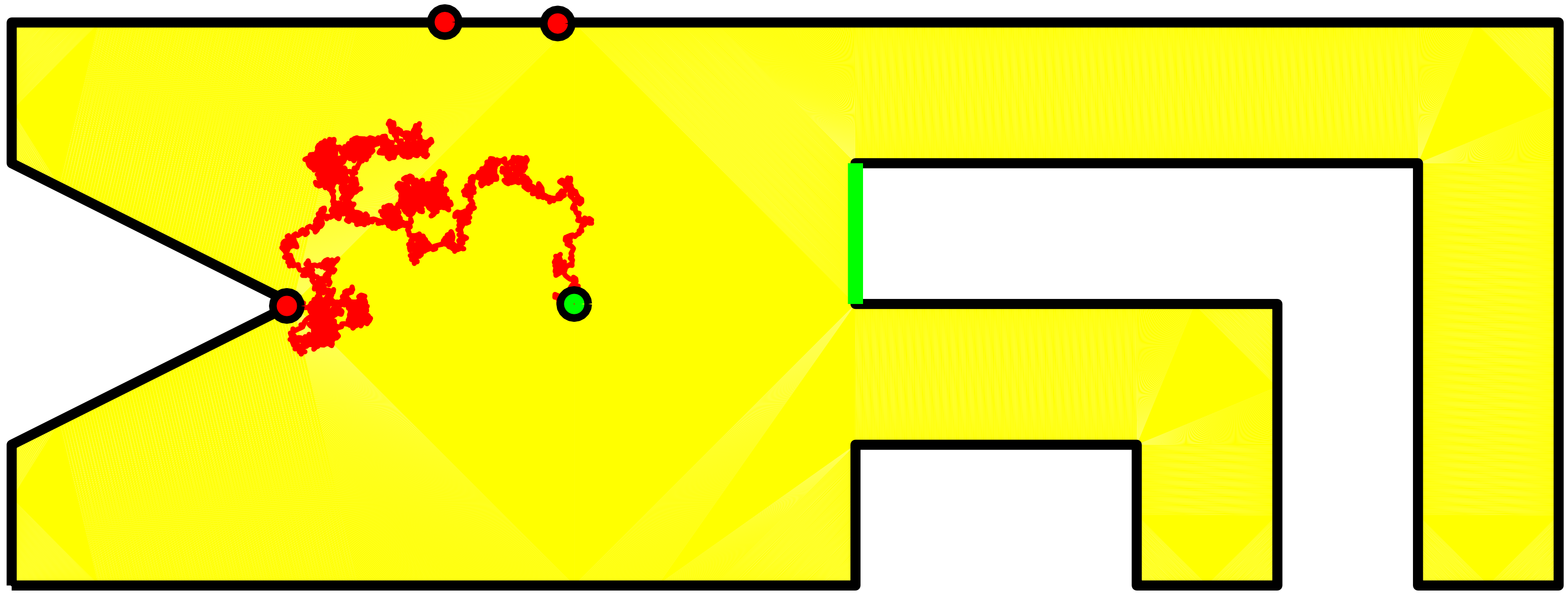

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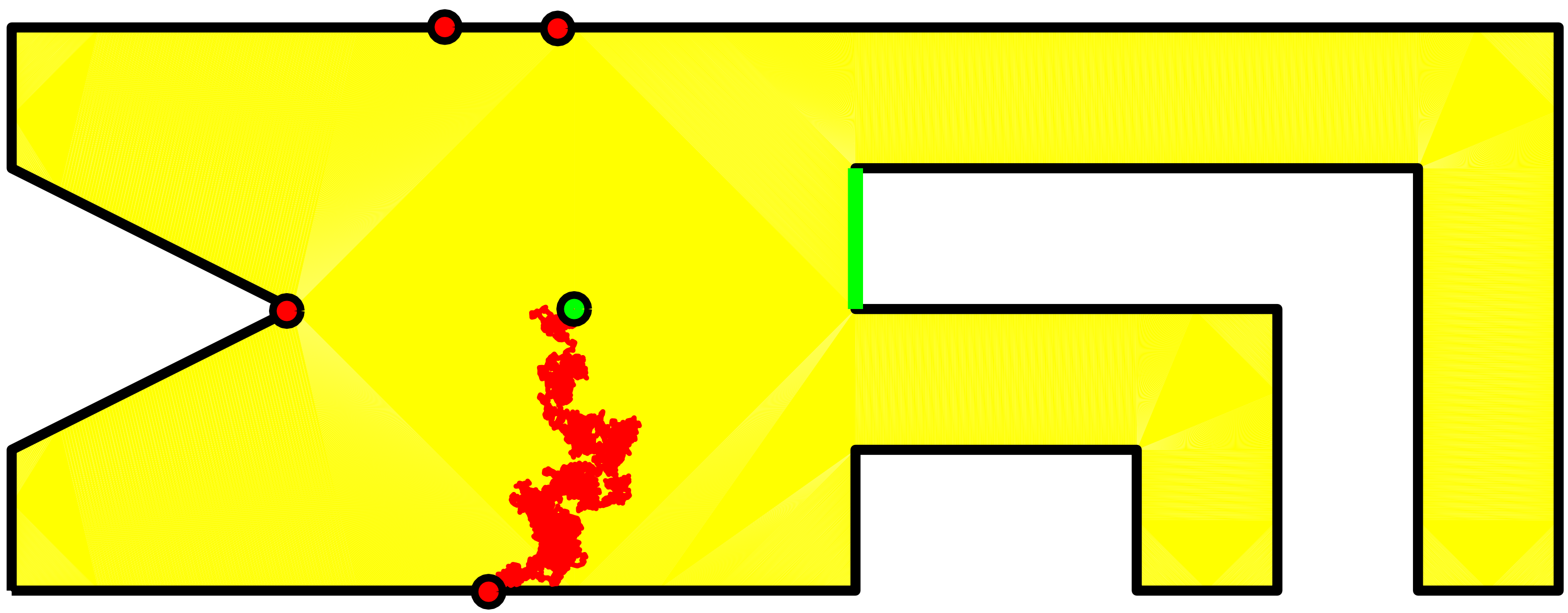

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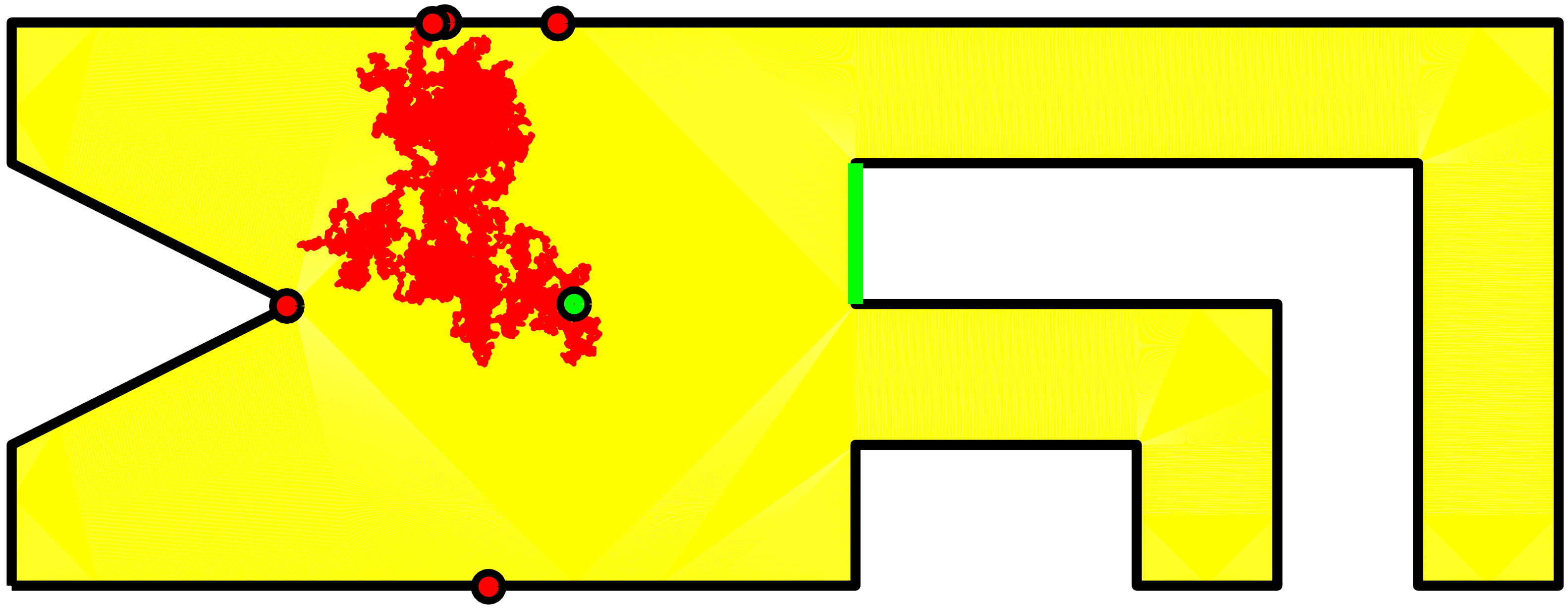

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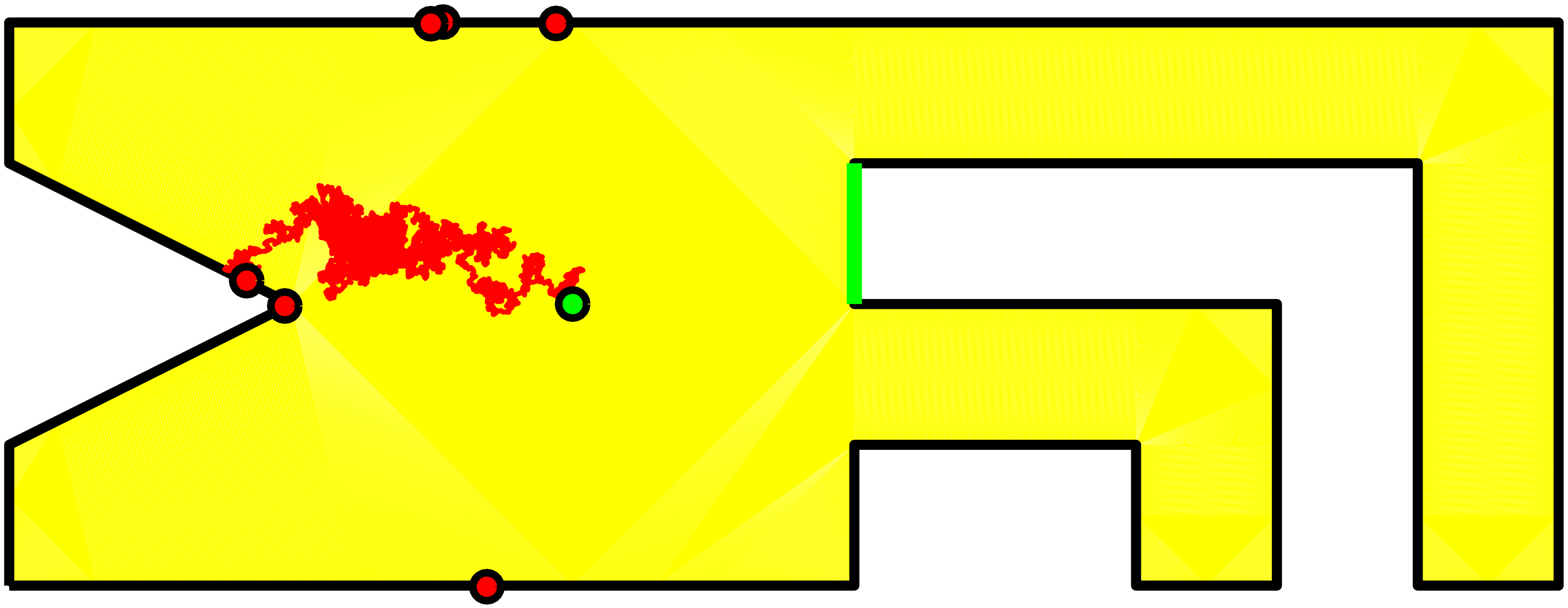

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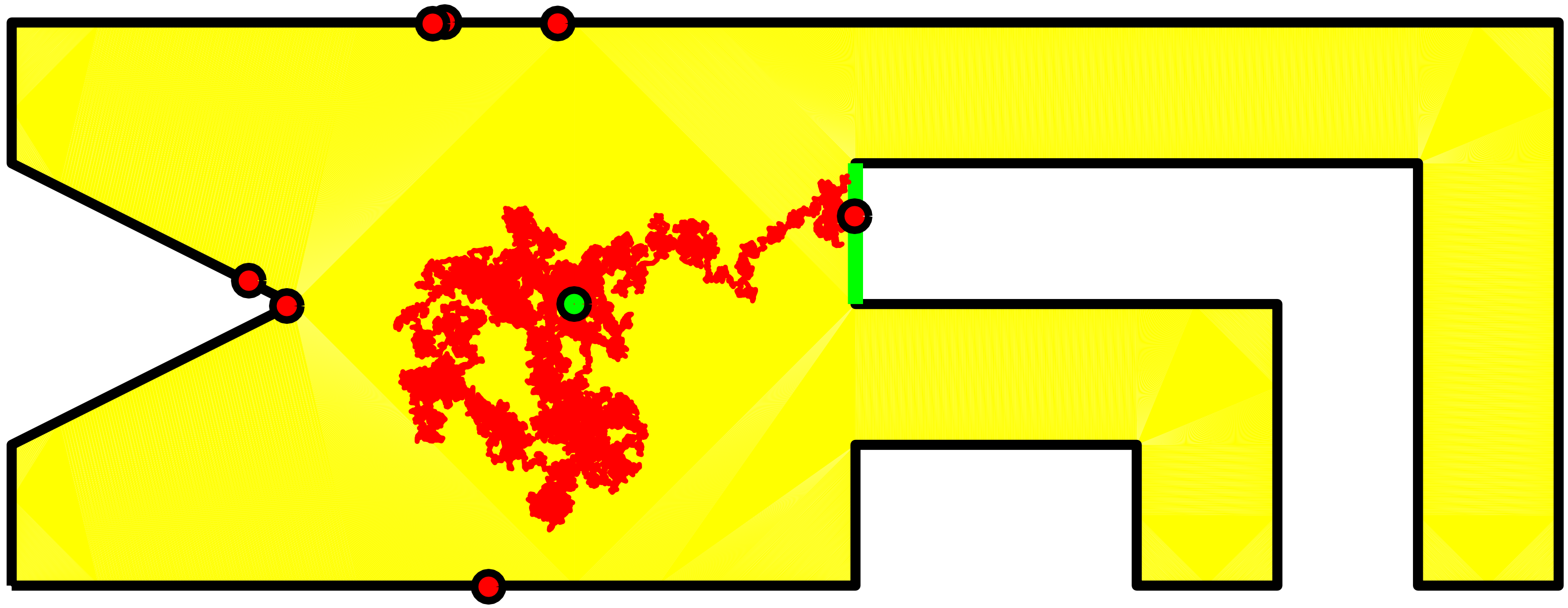

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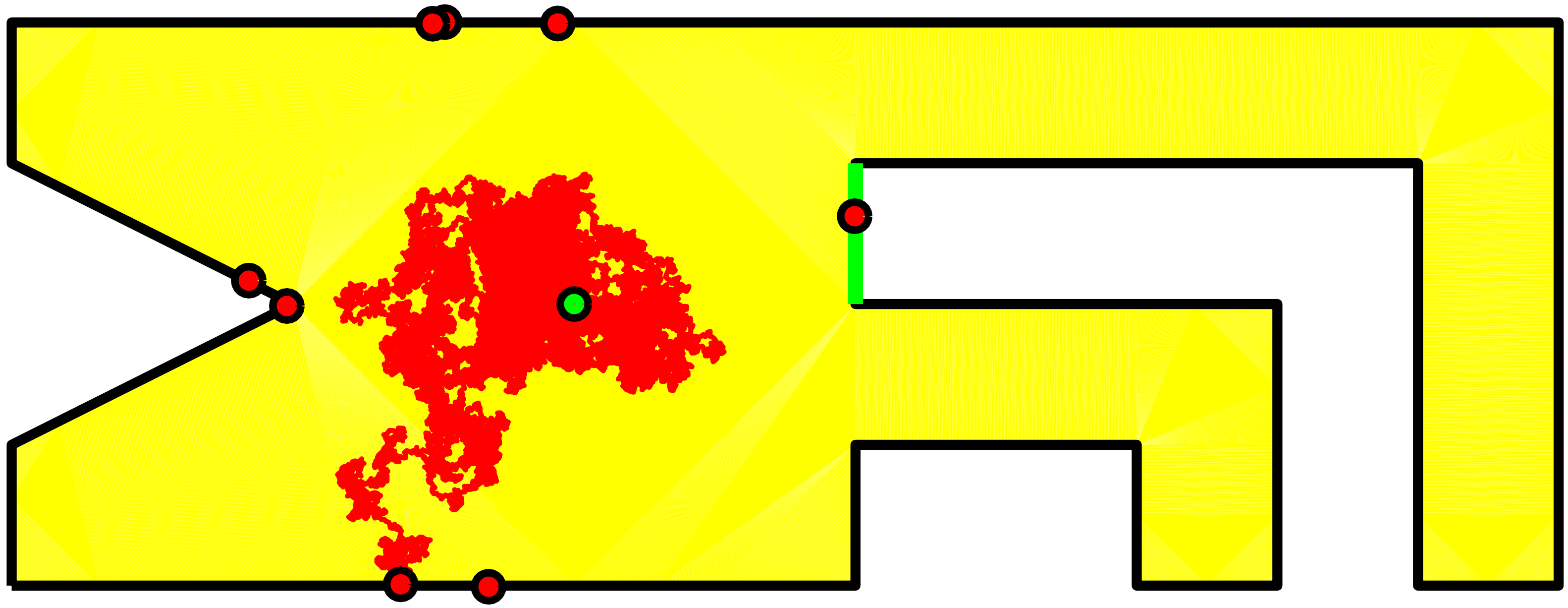

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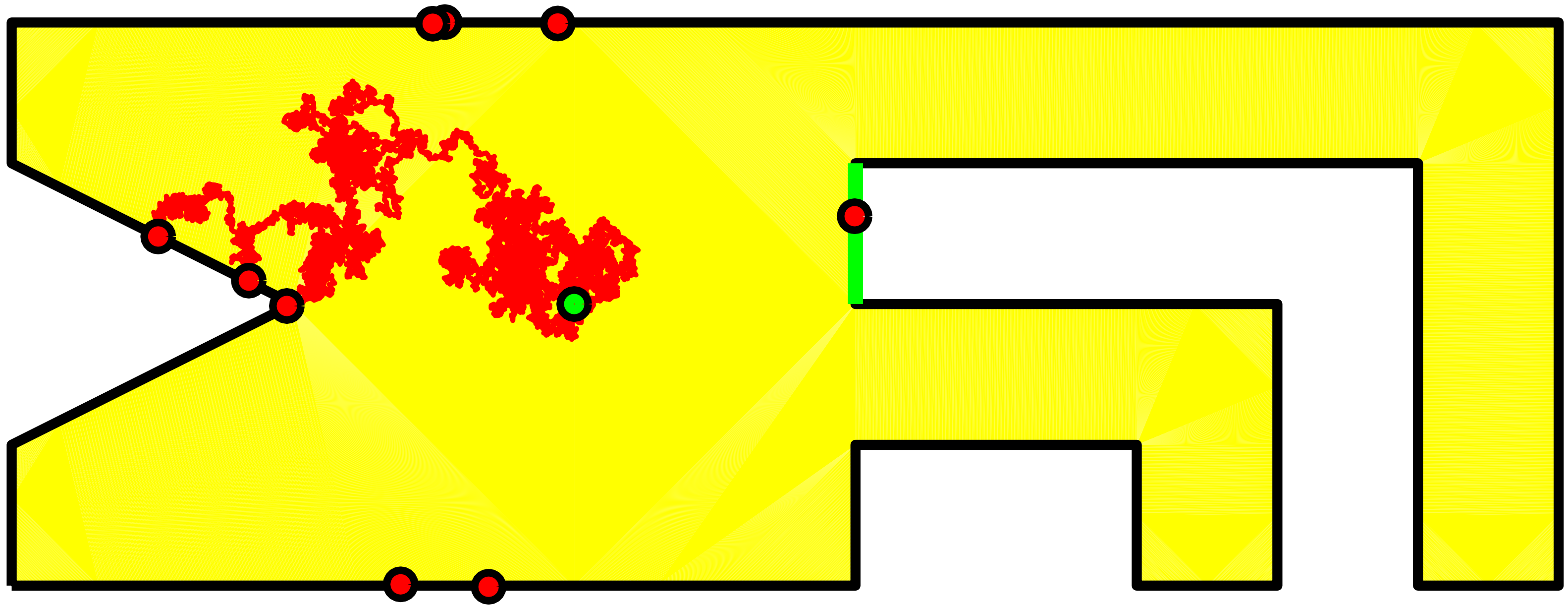

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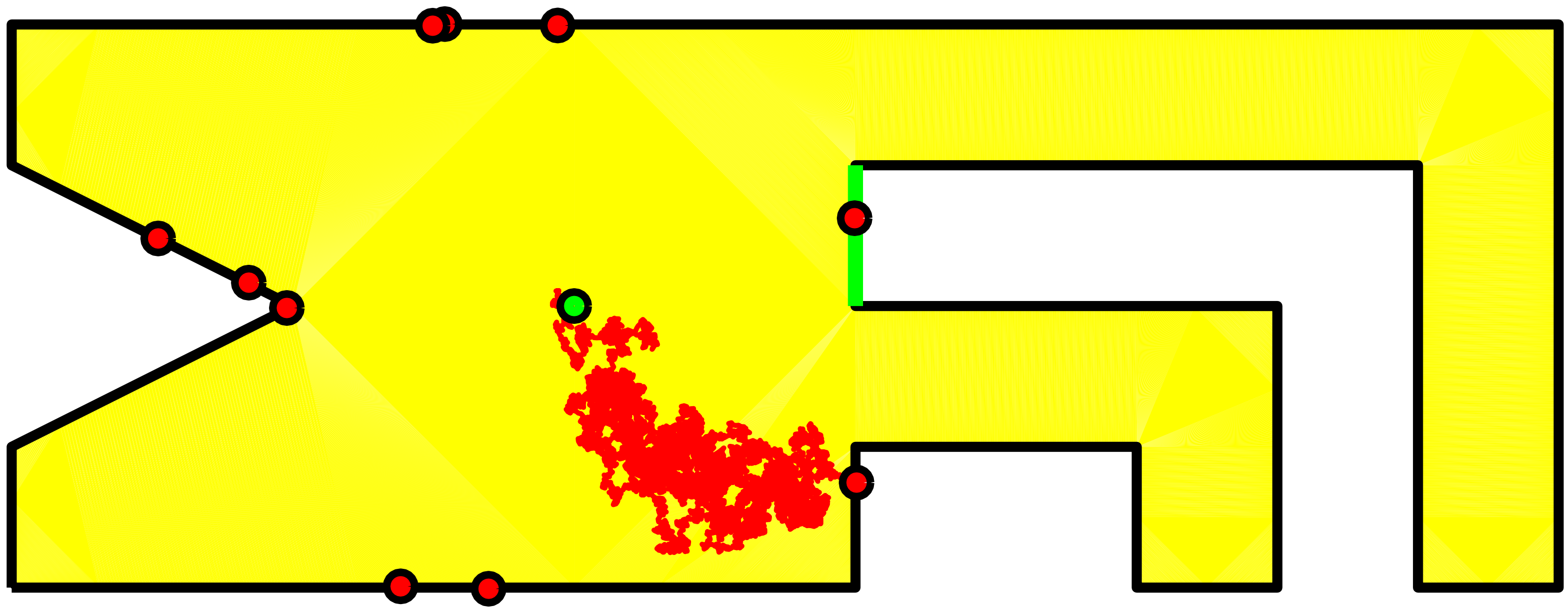
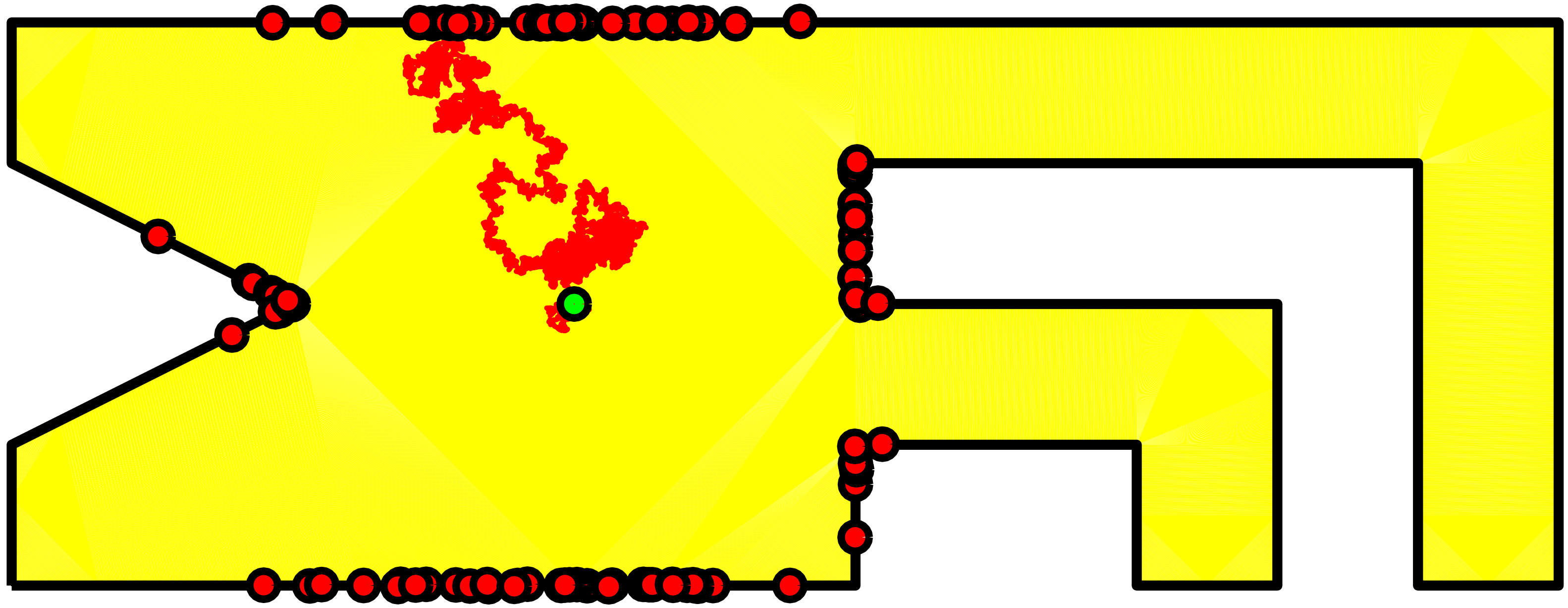


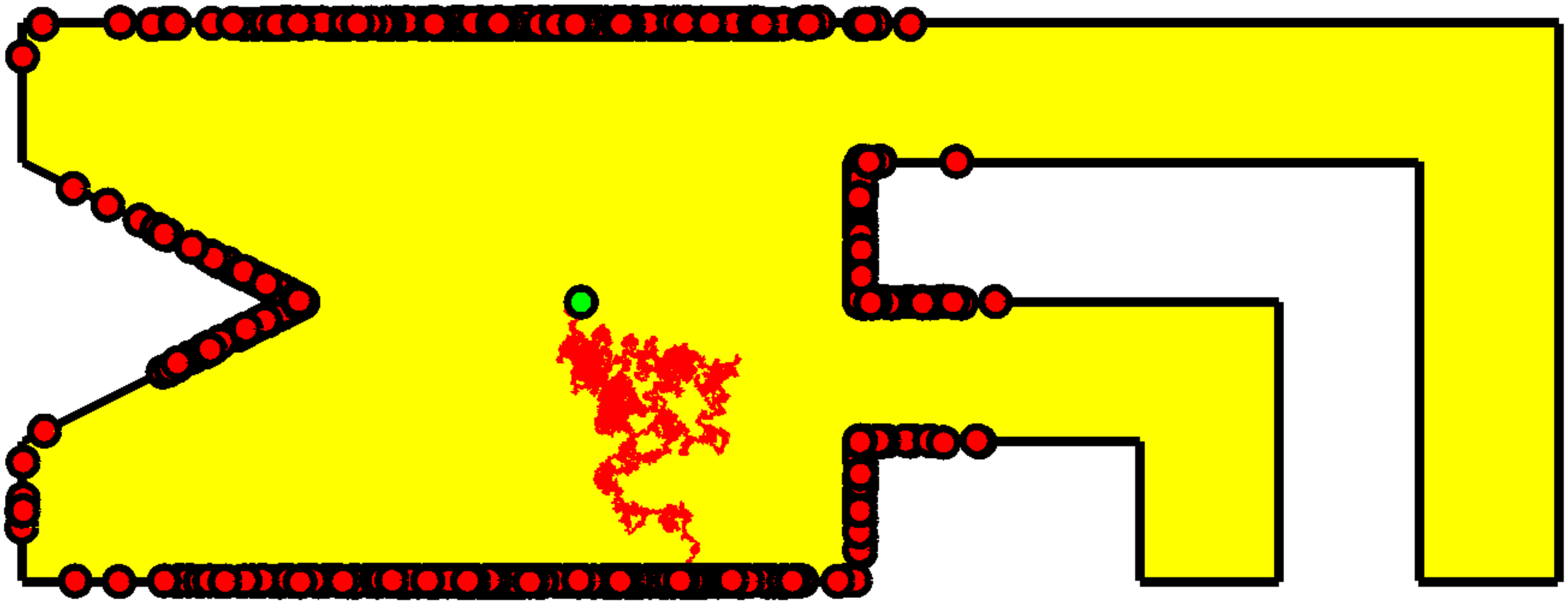
$$\omega(z, E, \Omega) \approx 1/10.$$

Harmonic measure = hitting distribution of Brownian motion



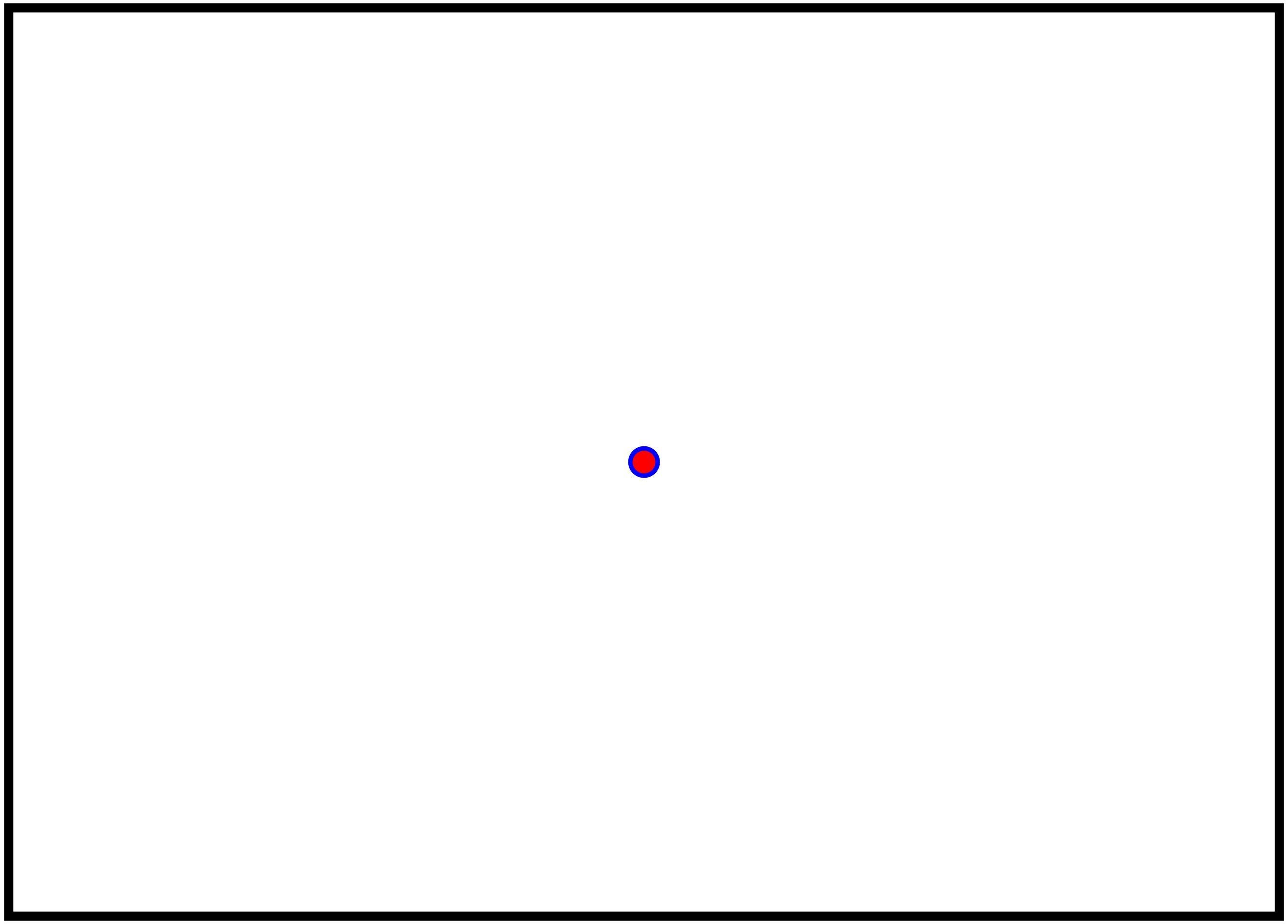
$$\omega(z, E, \Omega) \approx 13/100.$$

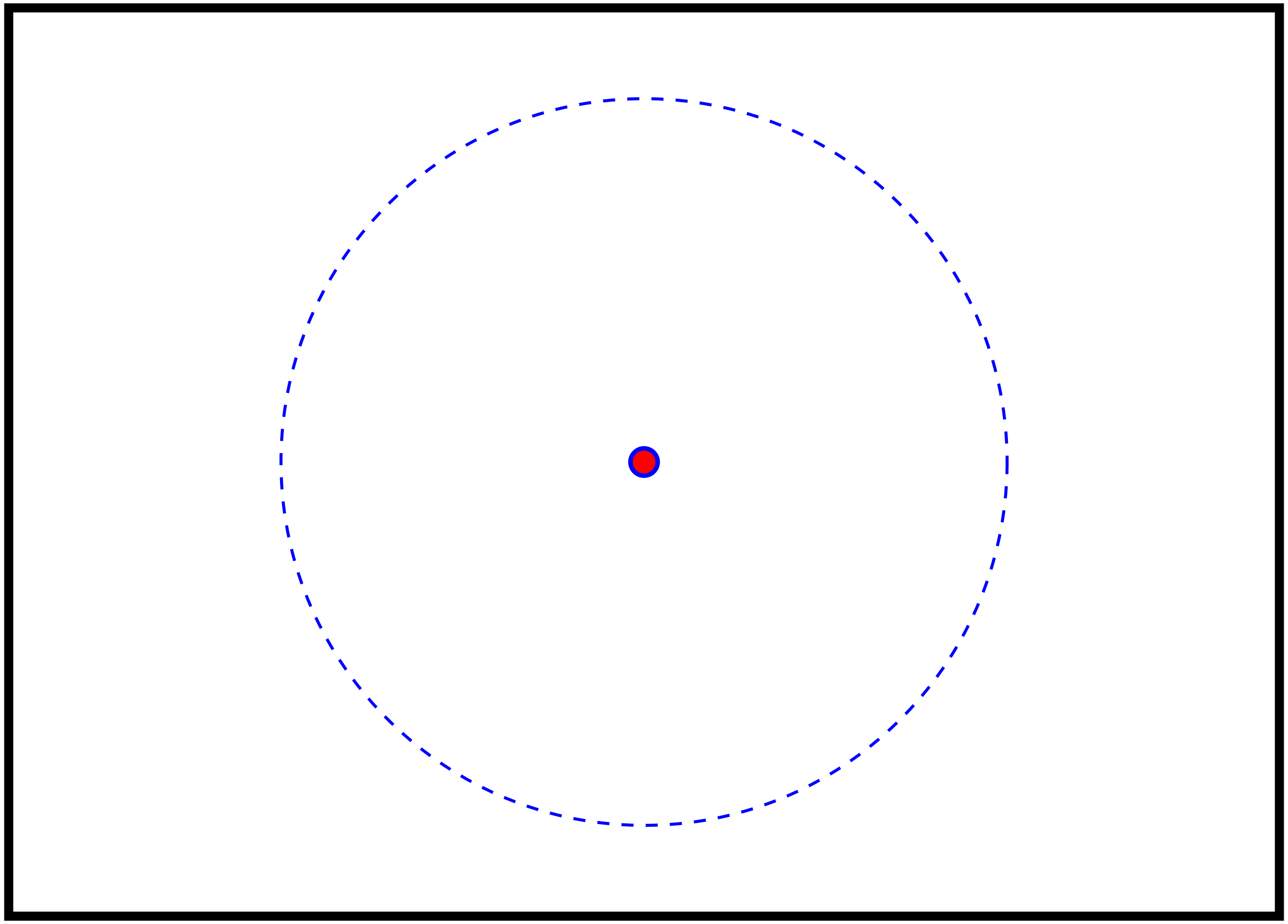
Harmonic measure = hitting distribution of Brownian motion

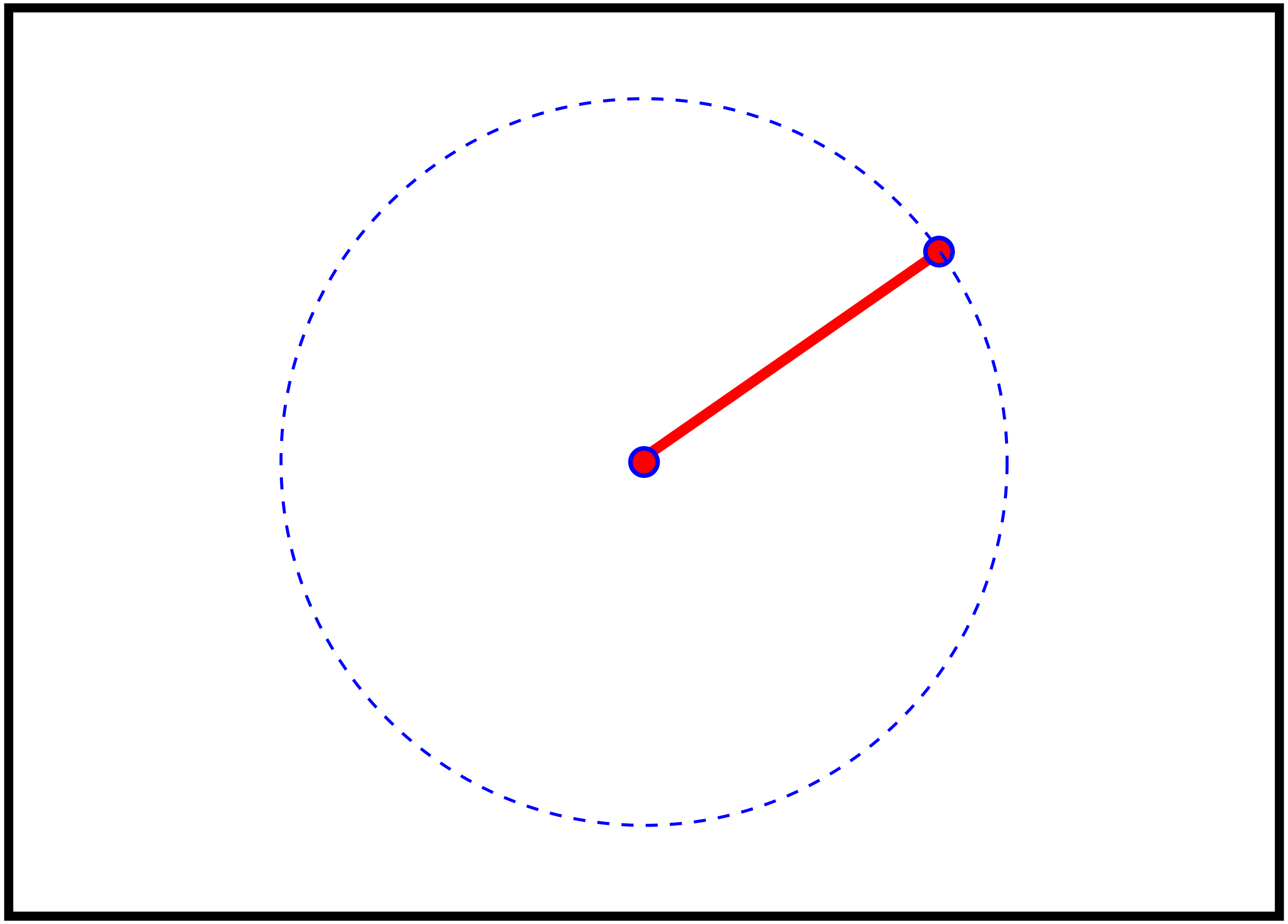


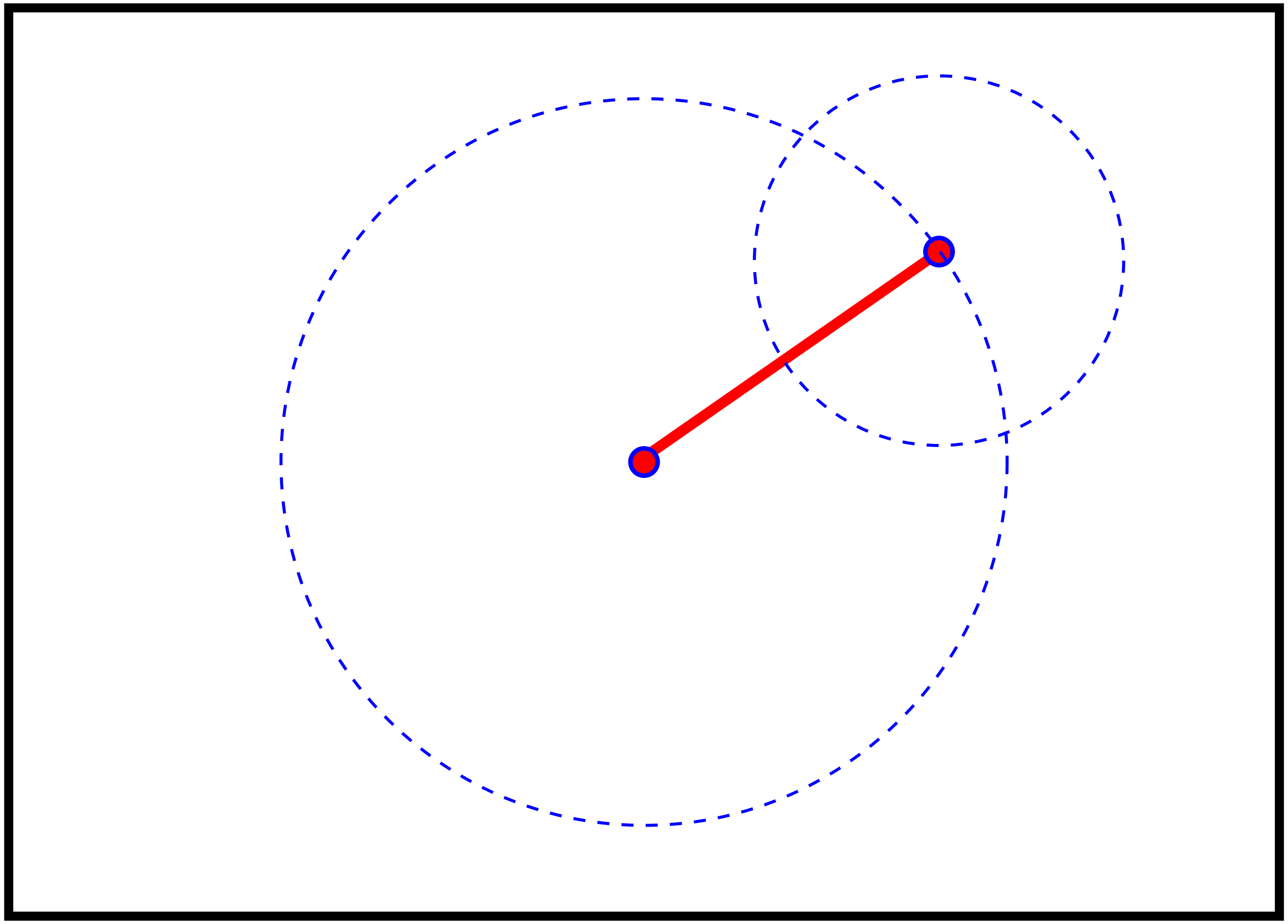
$$\omega(z, E, \Omega) \approx 126/1000.$$

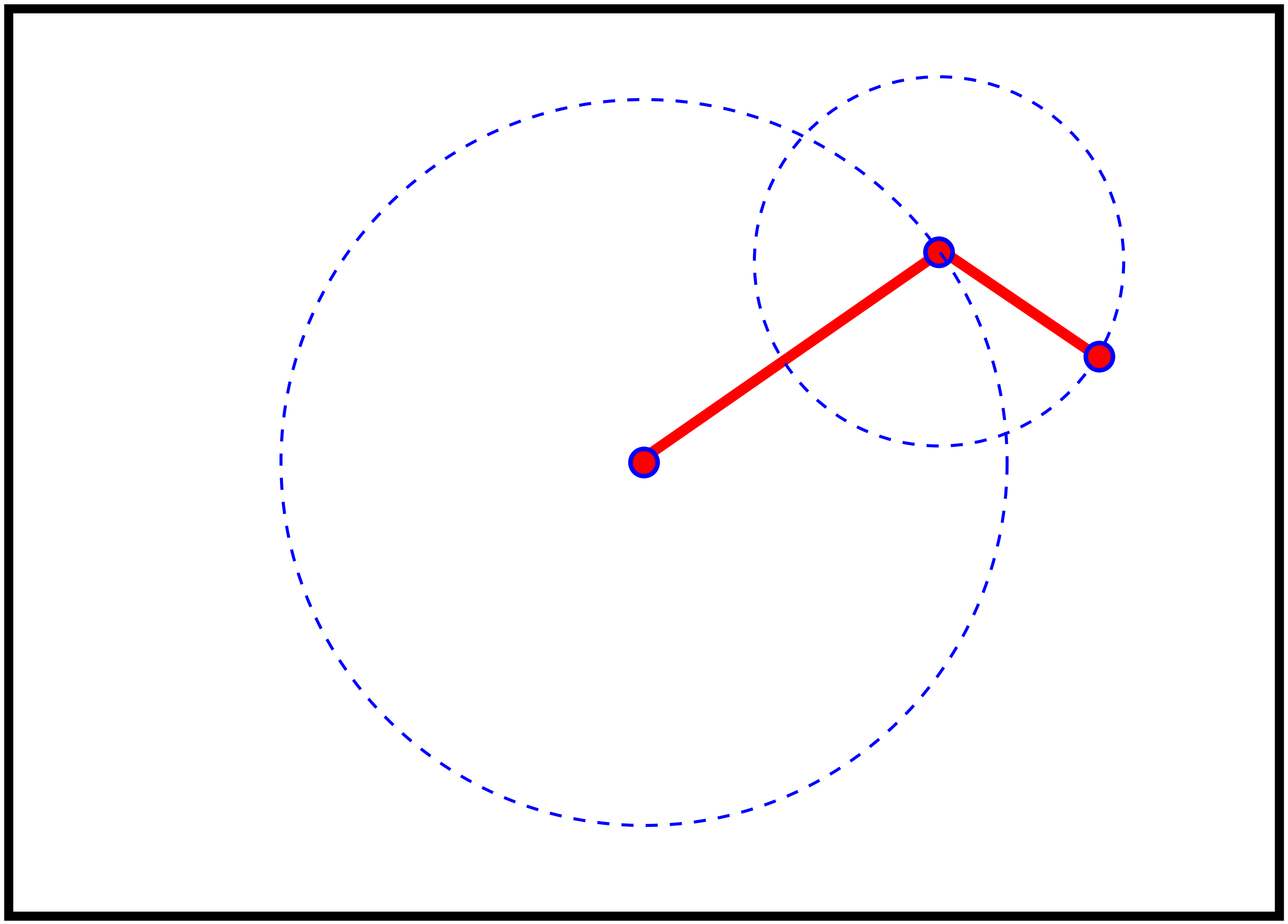
Slow to compute each path ($\simeq \epsilon^{-2}$ steps on ϵ-grid). Can we go faster?

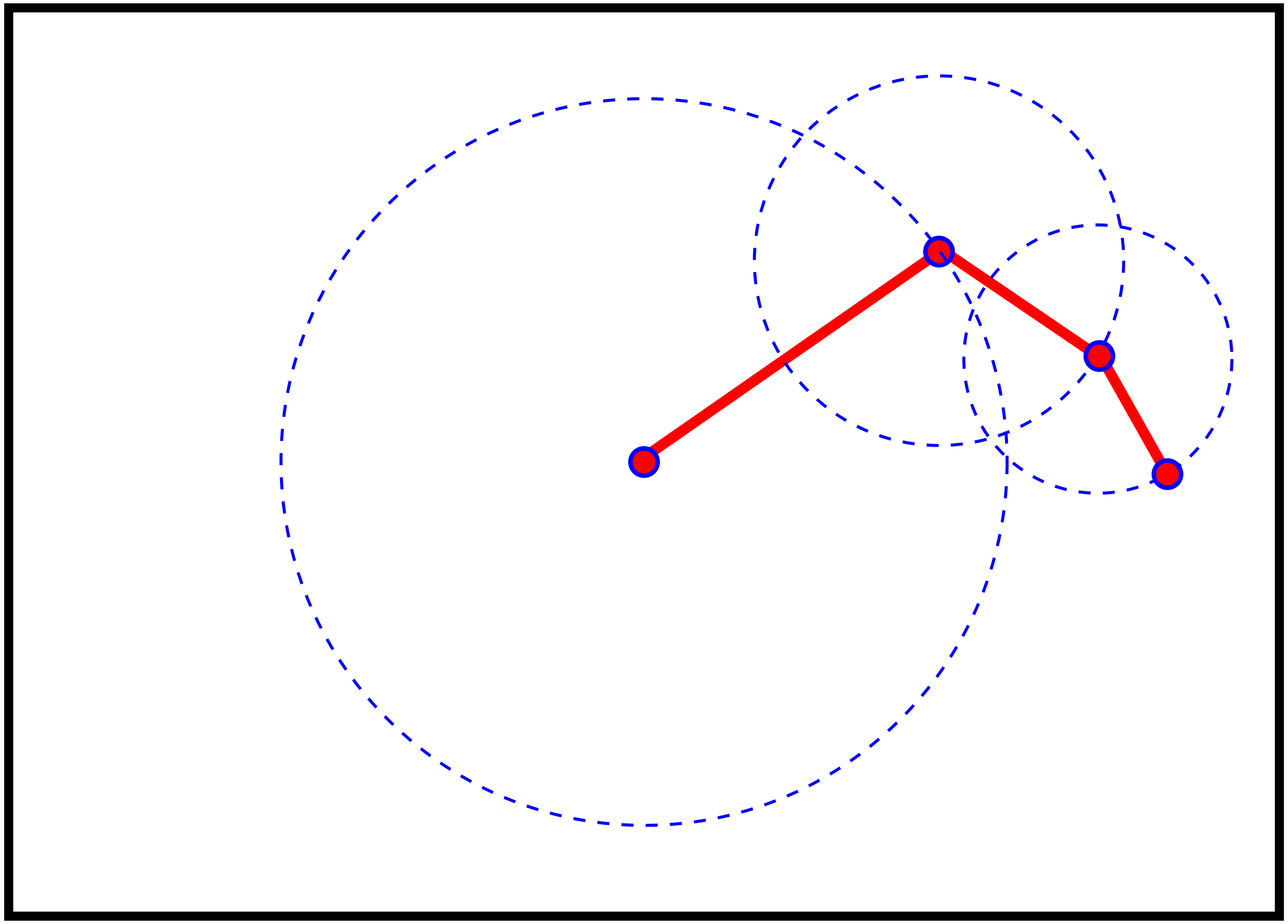


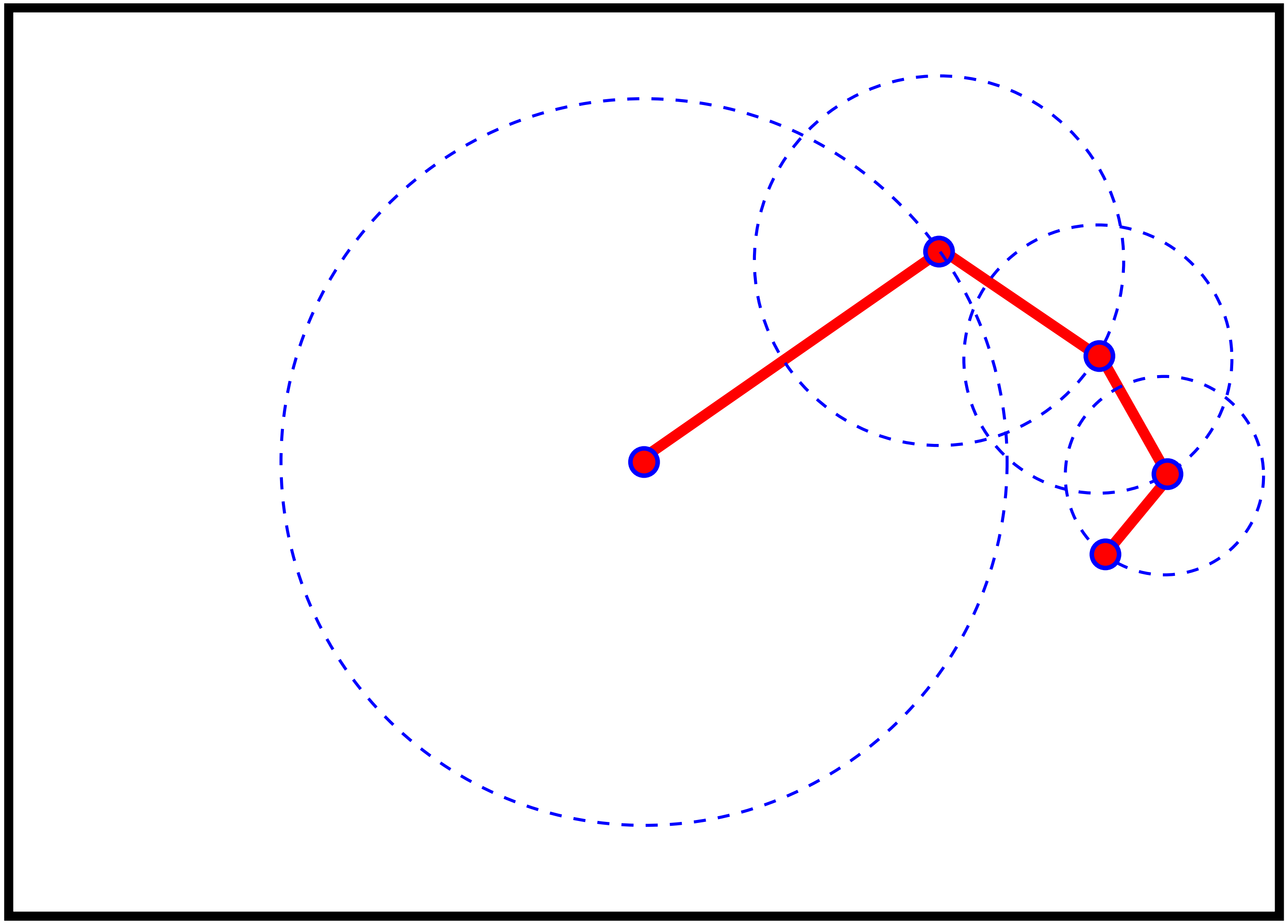


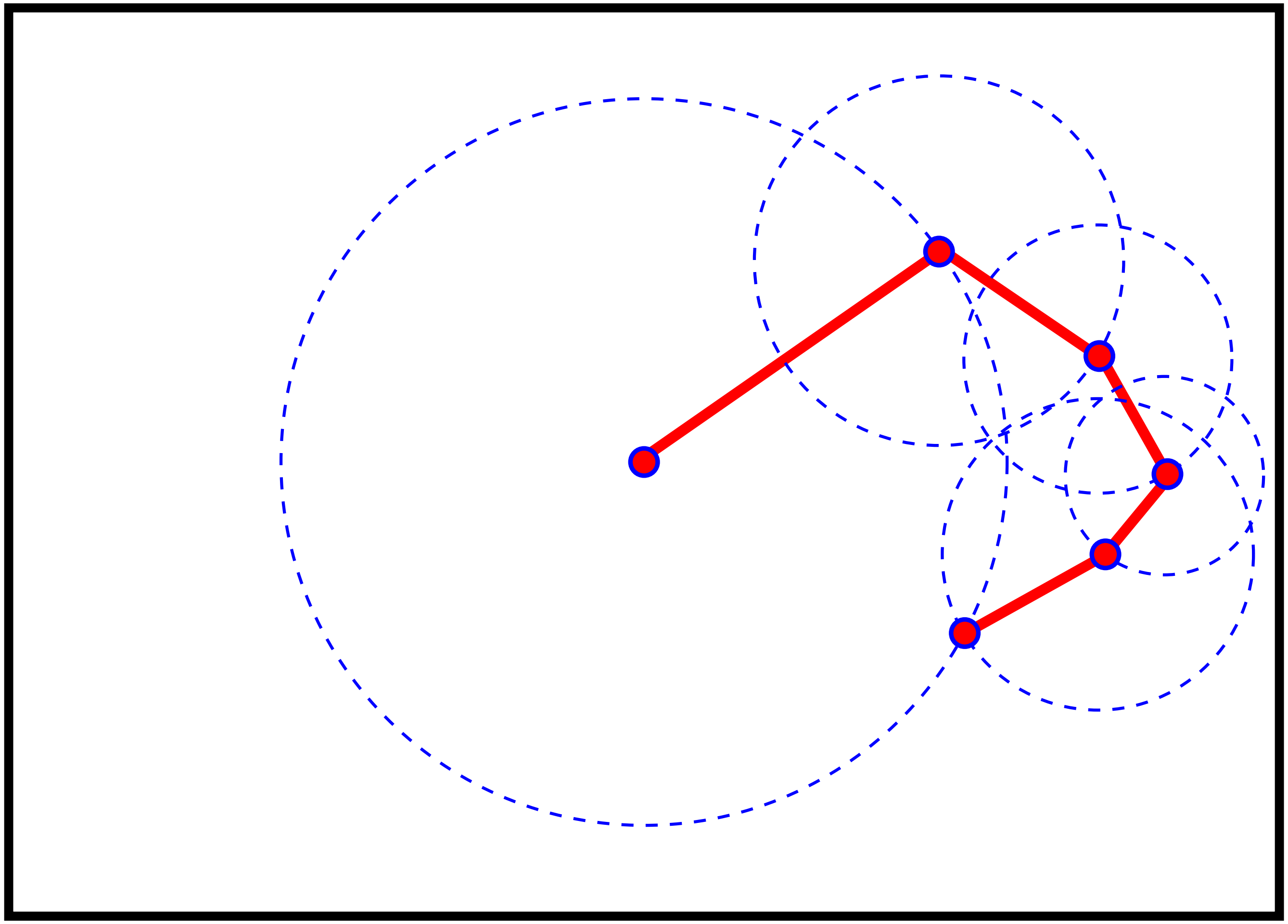


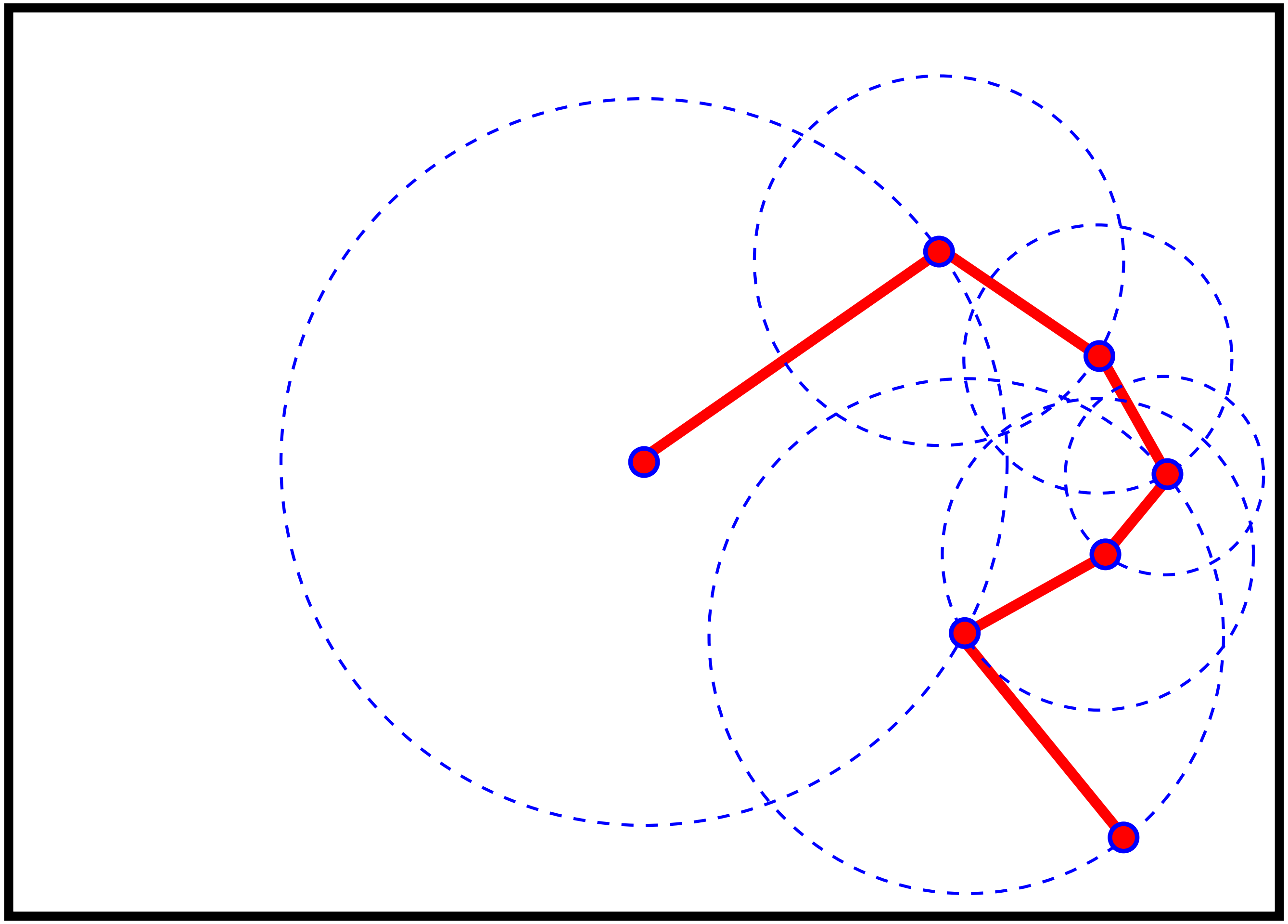


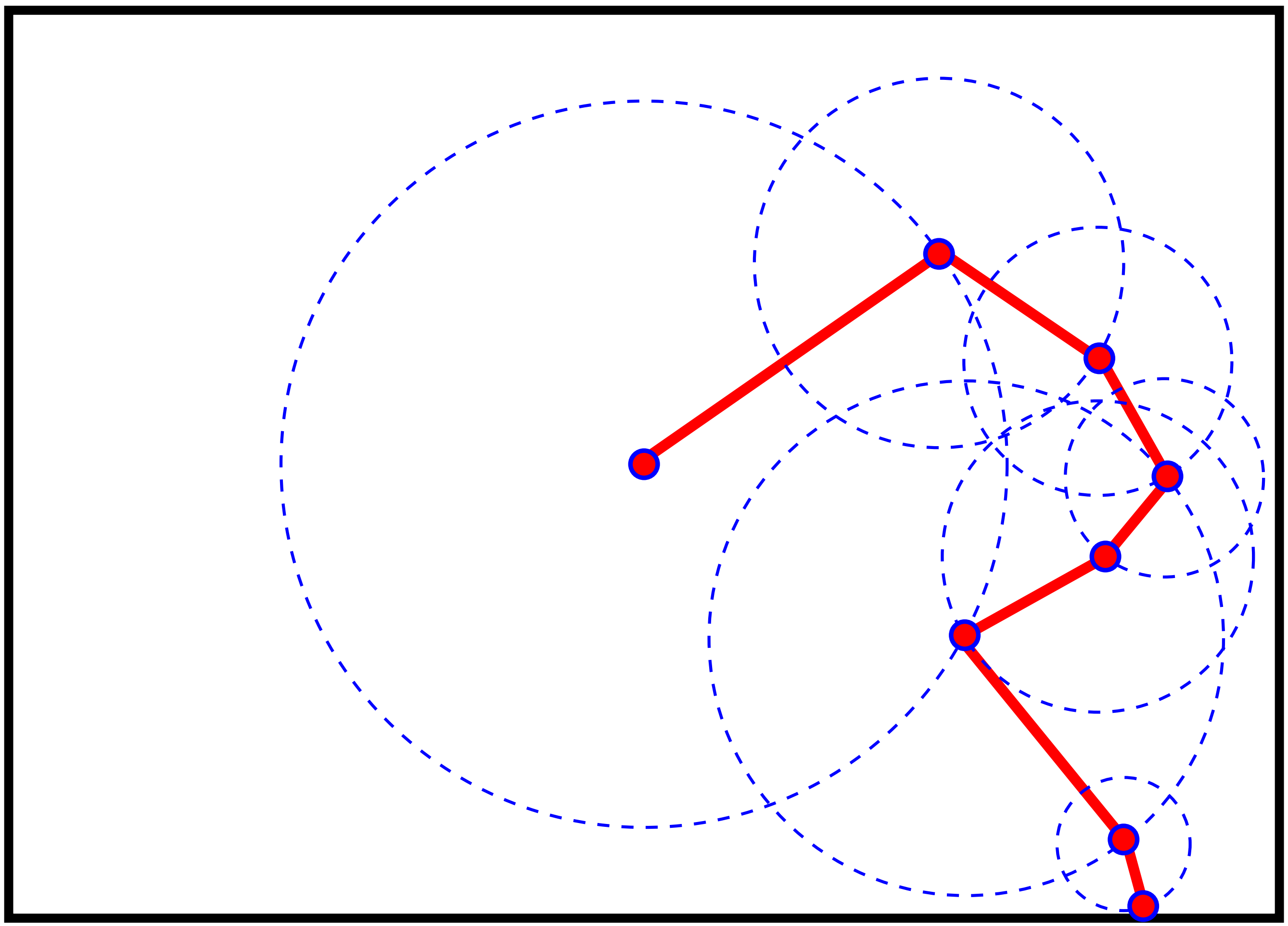






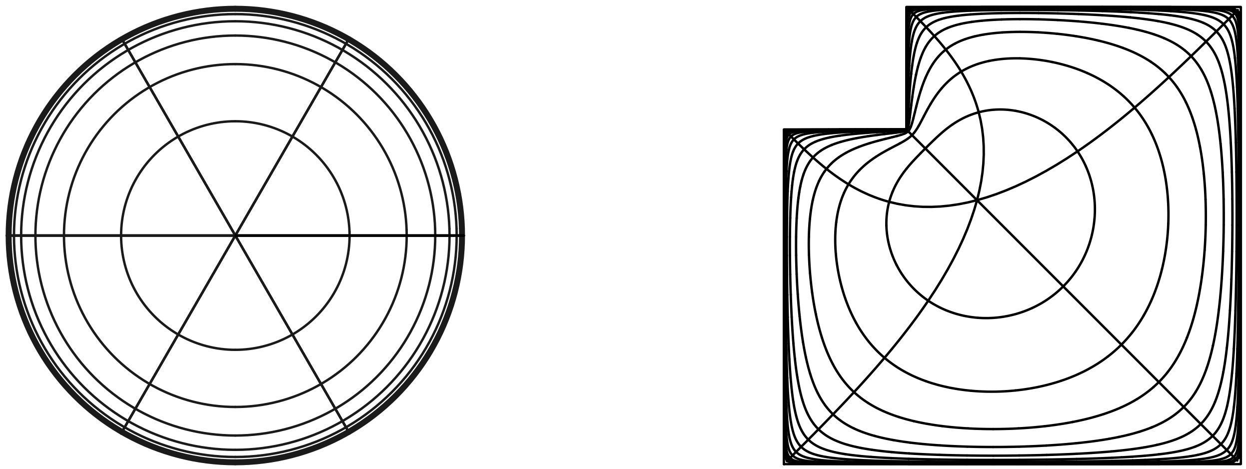




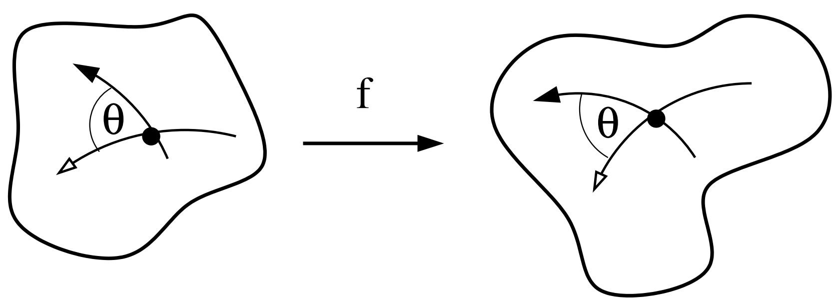


This walk will get within ϵ of boundary in about $O(\log 1/\epsilon)$ steps.
Takes about ϵ^{-2} steps on an ϵ-grid.

Riemann Mapping Theorem: If $\Omega \subsetneq \mathbb{R}^2$ is simply connected, then there is a conformal map $f : \mathbb{D} \to \Omega$.



Conformal = angle preserving



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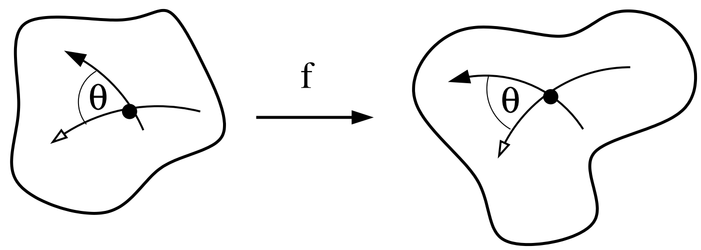
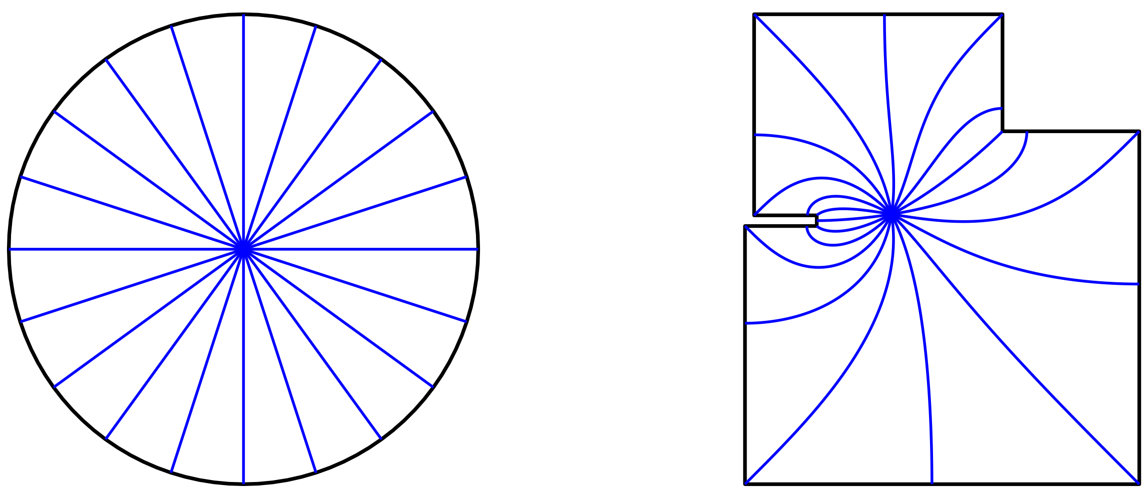


Brownian motion is conformally invariant, so normalized length measure maps to harmonic measure. Fastest way to compute harmonic measure.



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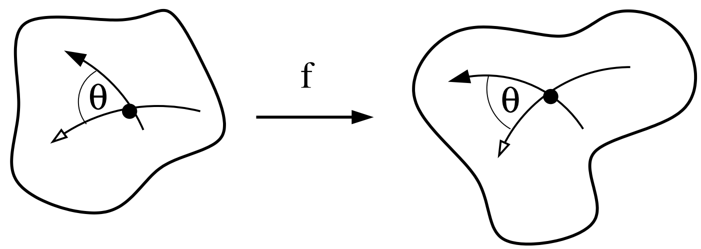

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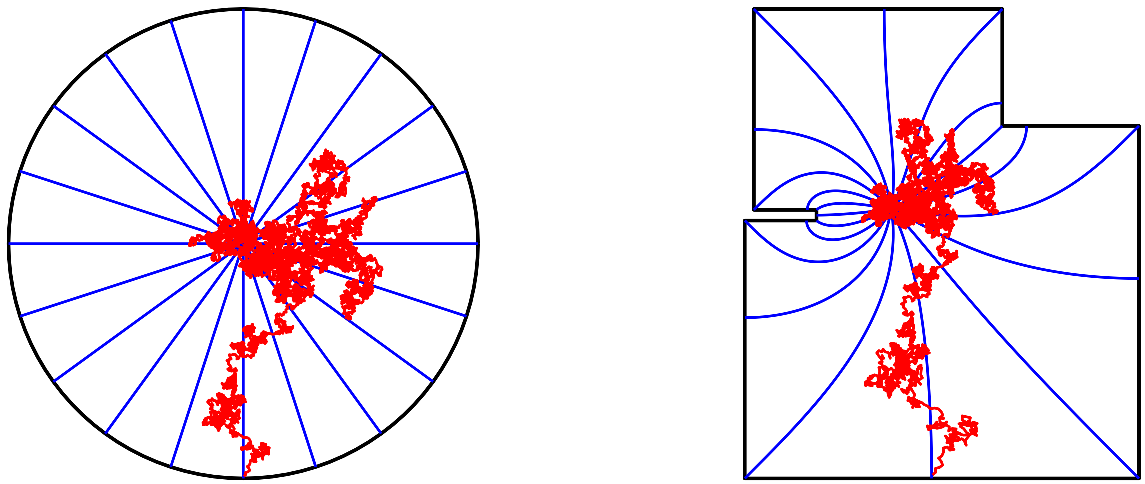
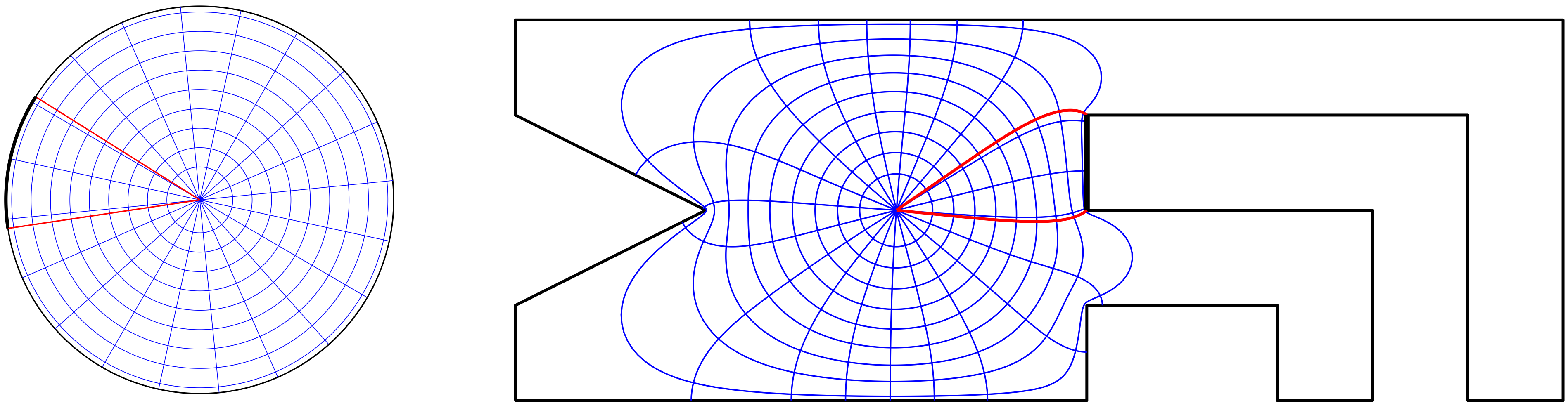
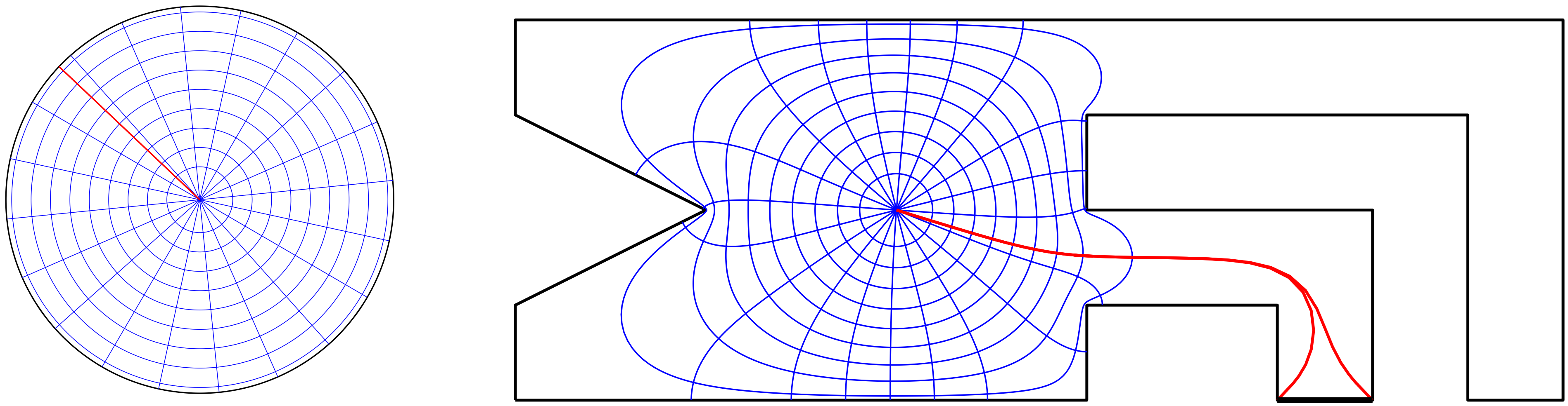


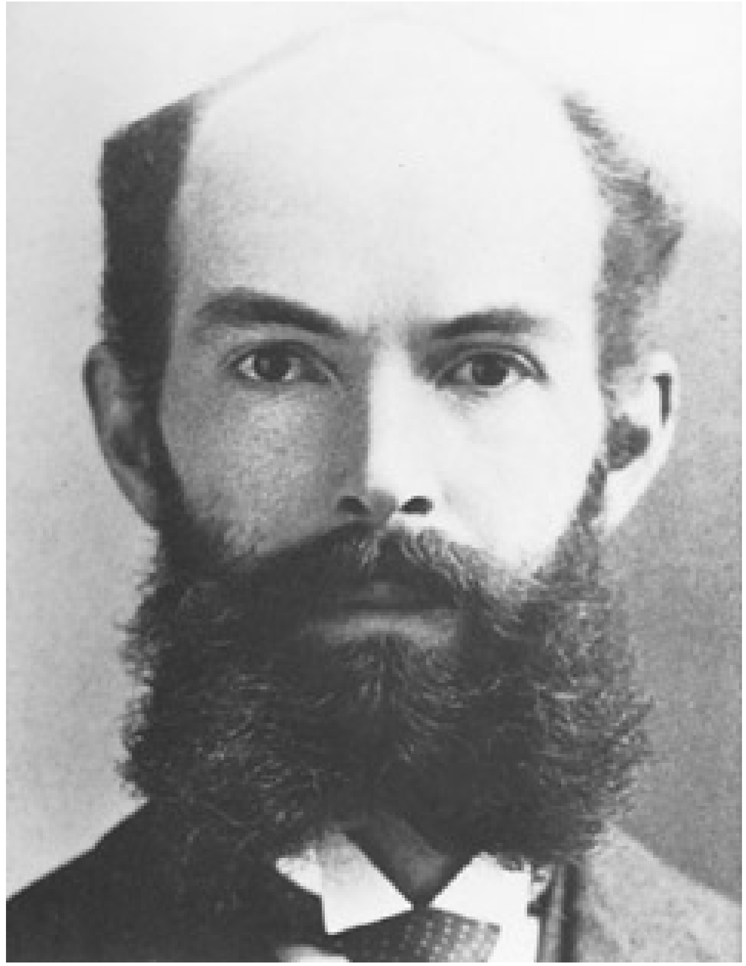


harmonic measure ≈ 0.1128027



harmonic measure $\approx 1.22155 \times 10^{-6}$

Georg Friedrich Bernhard Riemann
Stated RMT in 1851



William Fogg Osgood
First proof of RMT, Trans. AMS, vol. 1, 1900
Harvard 1866, Math Faculty 1890-1933, Chair 1918-22

The proof of Osgood represented, in my opinion, the "coming of age" of mathematics in America. Until then, ... the mathematical productivity in this country in quality lagged behind that of Europe, and no American before 1900 had reached the heights that Osgood then reached.

J.L. Walsh, "History of the Riemann mapping theorem", Amer. Math. Monthly, 1973.

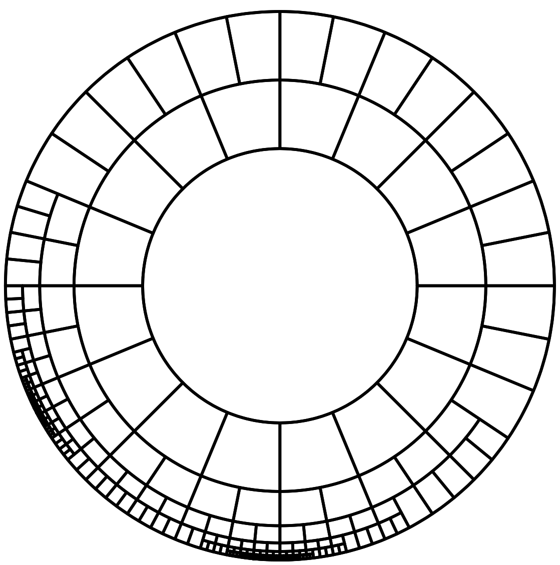
Some basic properties of harmonic measure:

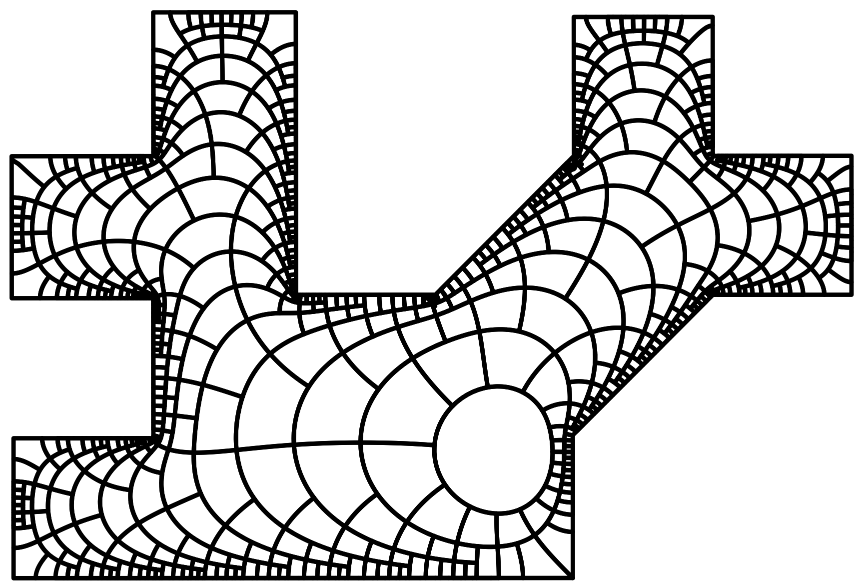
$\omega(z, E, \Omega)$ is the harmonic measure of $E \subset \partial\Omega$.

ω is harmonic in z and $0 \le \omega \le 1$

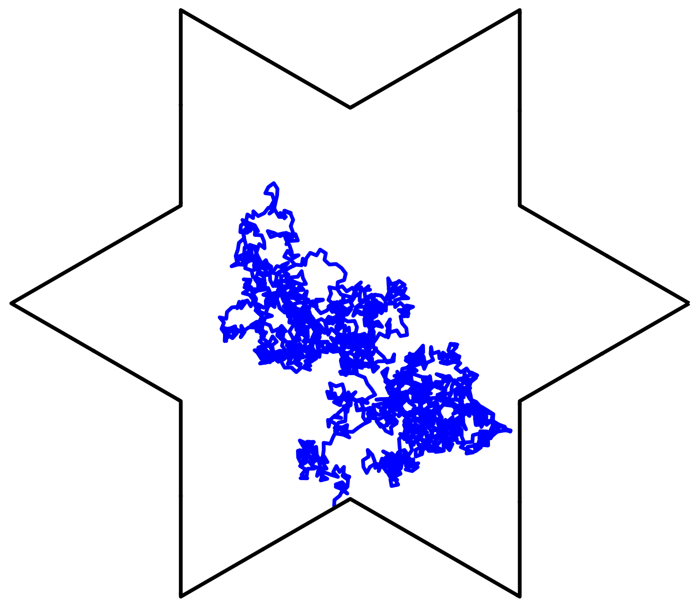
Harnack $\Rightarrow$ different base points give comparable measures.

(If base points are on same side of curve.)





Thm (F & M Riesz 1916):
For rectifiable boundaries, $\omega(E) = 0$ iff E has zero length.



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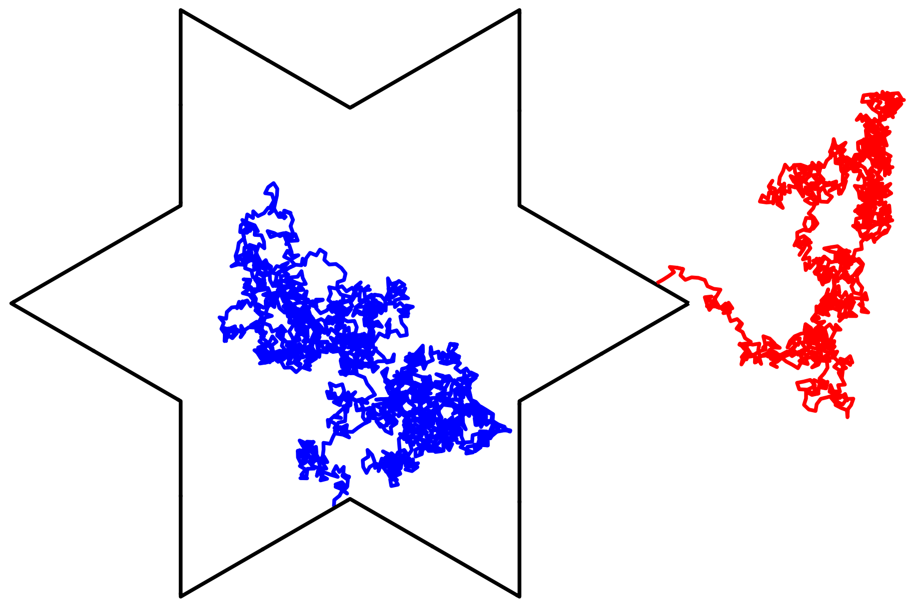
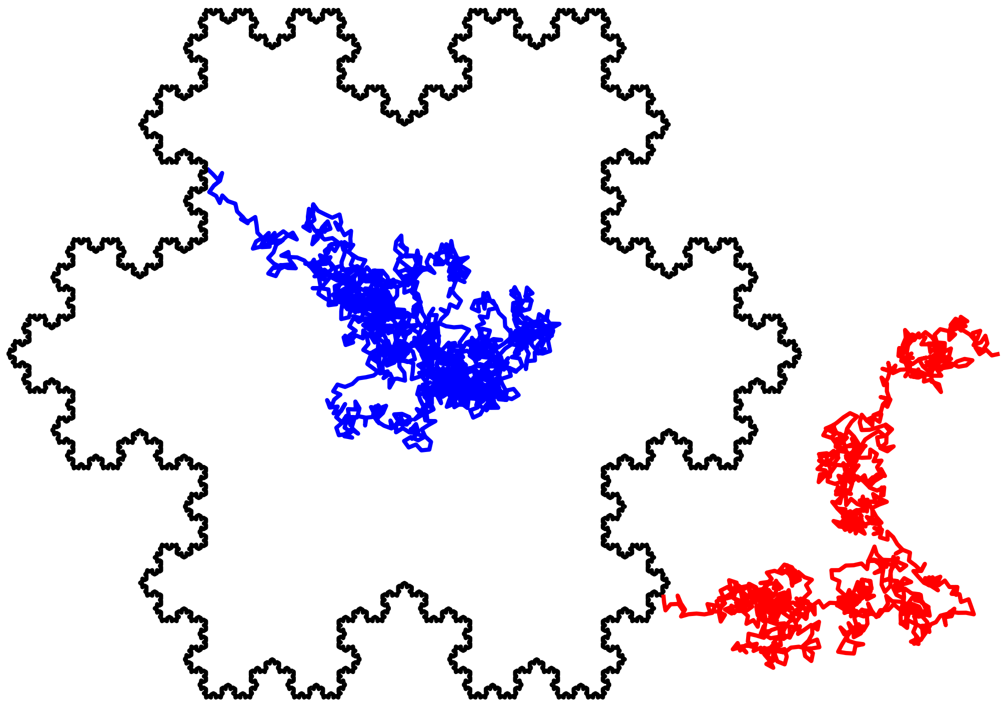


$\Rightarrow$ "Inside" and "outside" harmonic measures have same null sets.

$\Rightarrow \omega_1, \omega_2$ are mutually absolutely continuous.

Recent deep generalizations to $\mathbb{R}^n$ by Tolsa and others.

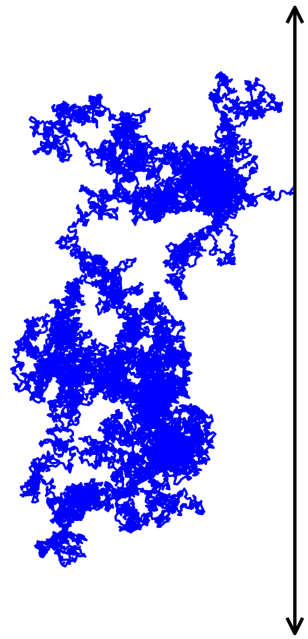
For a fractal curve, inside and outside harmonic measures are singular.



$\omega_1 \perp \omega_2$ iff tangents points have zero length.

For which curves is $\omega_1 = \omega_2$?

For which curves is $\omega_1 = \omega_2$? True for lines:



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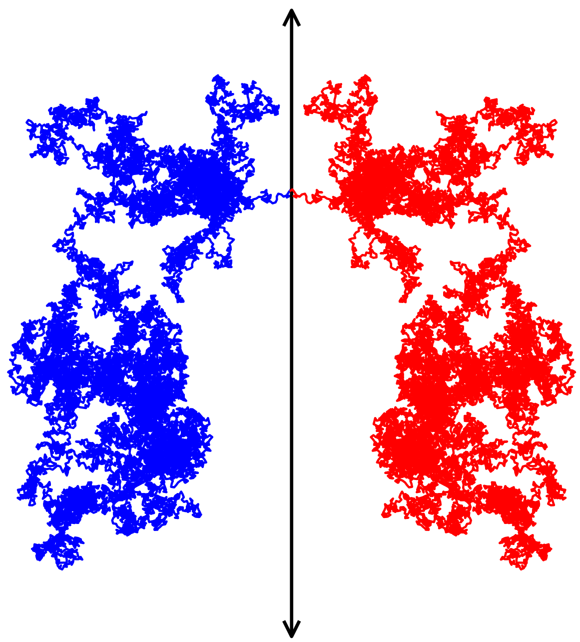
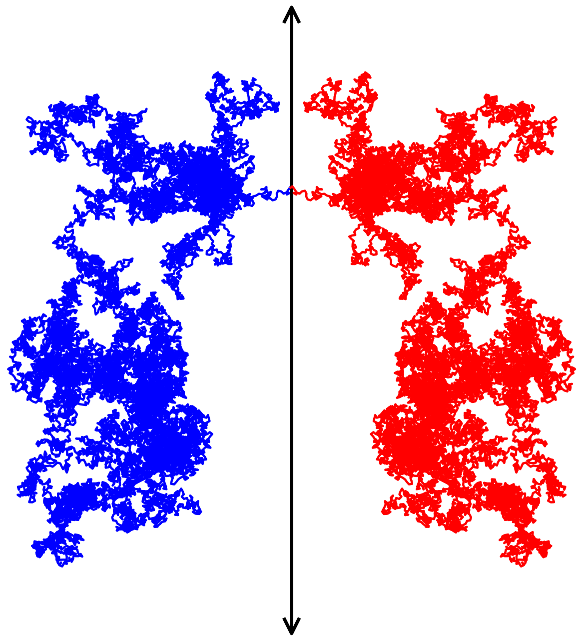


Also for circles ($=$ lines conformally).

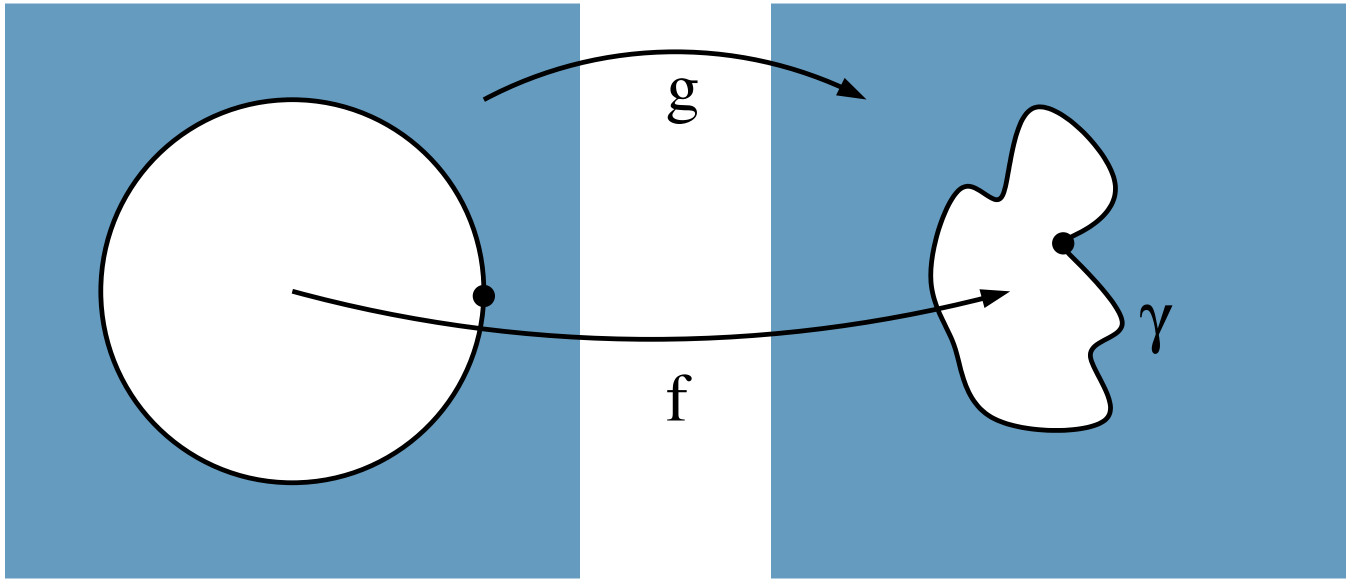
Converse? If $\omega_1 = \omega_2$ must Γ be a circle/line?

For which curves is $\omega_1 = \omega_2$? True for lines:



Also for circles (= lines conformally).

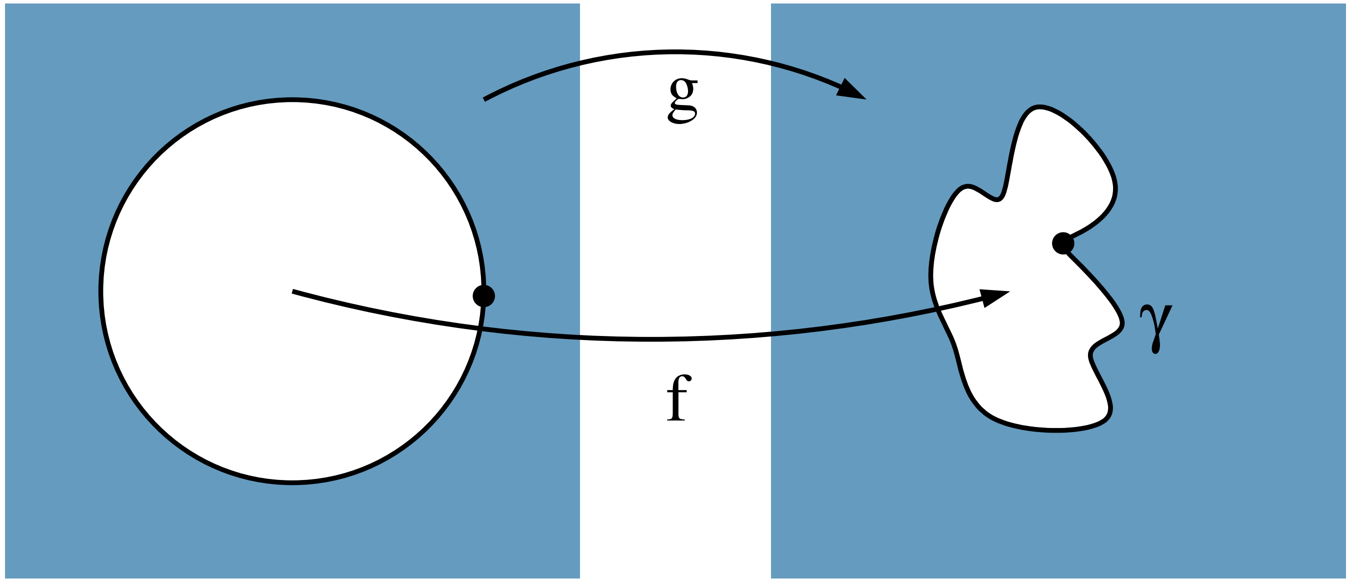
Converse? If $\omega_1 = \omega_2$ must Γ be a circle/line? **Yes**



Suppose $\omega_1 = \omega_2$ for a curve γ.

Conformally map two sides of circle to two sides of γ so $f(1) = g(1)$.

$\omega_1 = \omega_2$ implies maps agree on whole boundary.



Suppose $\omega_1 = \omega_2$ for a curve γ.

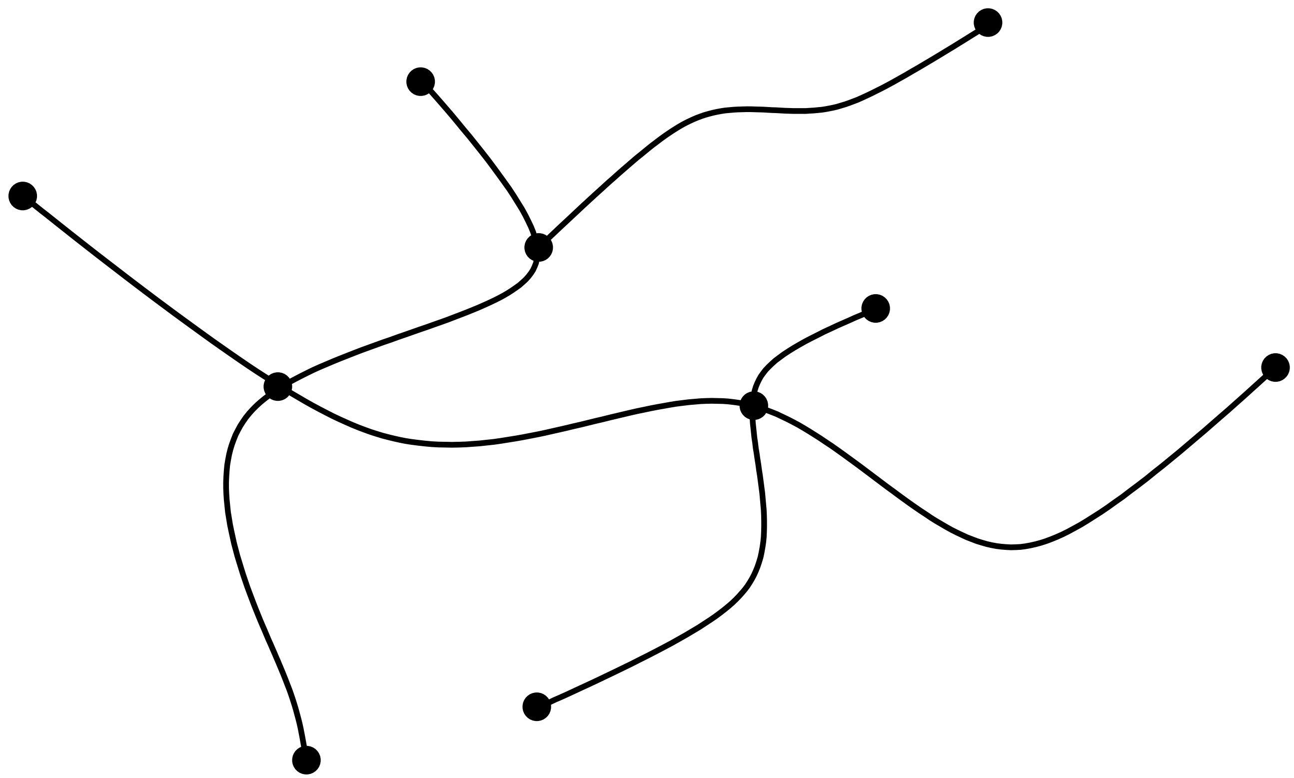
Conformally map two sides of circle to two sides of γ so $f(1) = g(1)$.

$\omega_1 = \omega_2$ implies maps agree on whole boundary.

So f, g define homeomorphism h of plane holomorphic off circle.

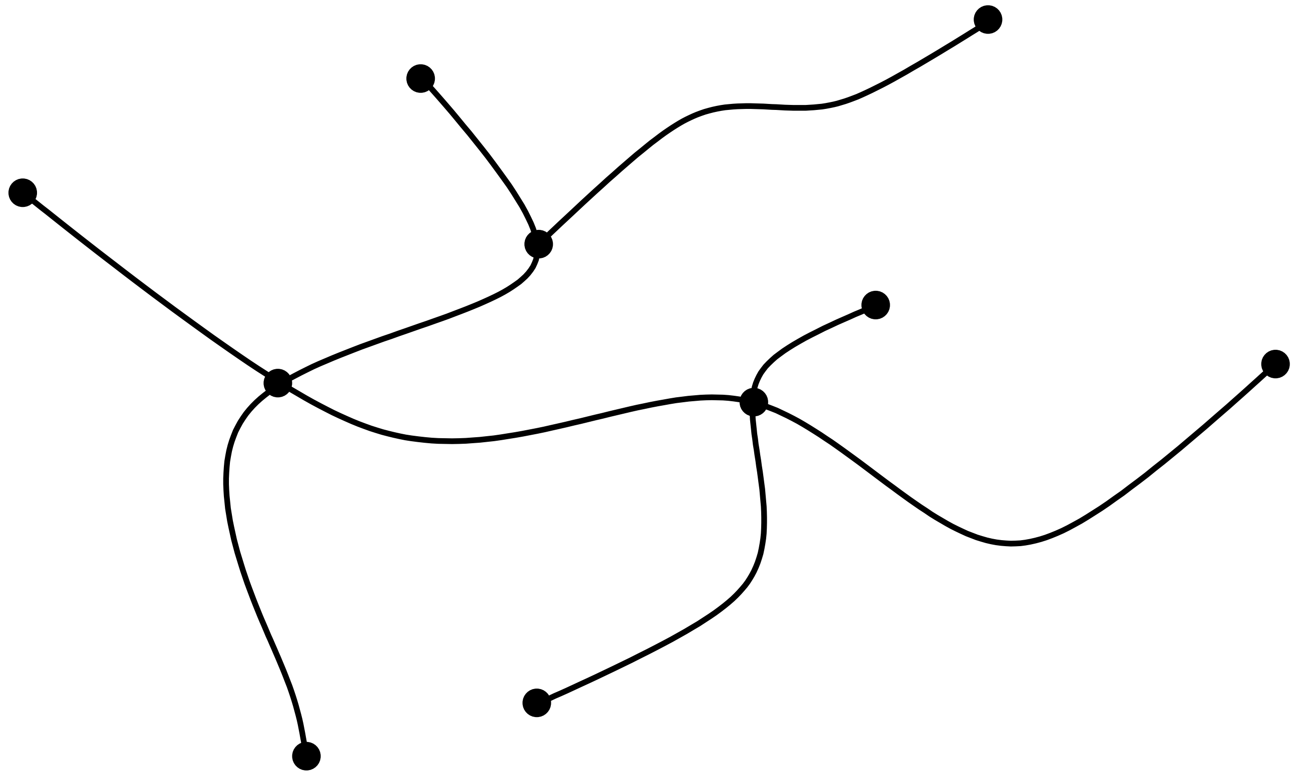
Then h is entire by Morera's theorem.

Entire and 1-1 implies h is linear (Liouville's thm), so γ is a circle.



A planar graph is a finite set of points connected by non-crossing edges.

It is a tree if there are no closed loops.



A planar tree is **conformally balanced** if

- every edge has equal harmonic measure from ∞
- edge subsets have same measure from both sides

This is also called a "**true tree**" (true form of a tree).

A line segment is an example.

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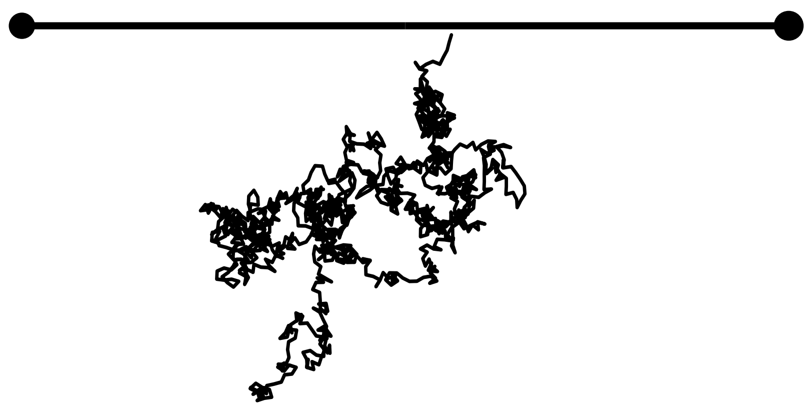
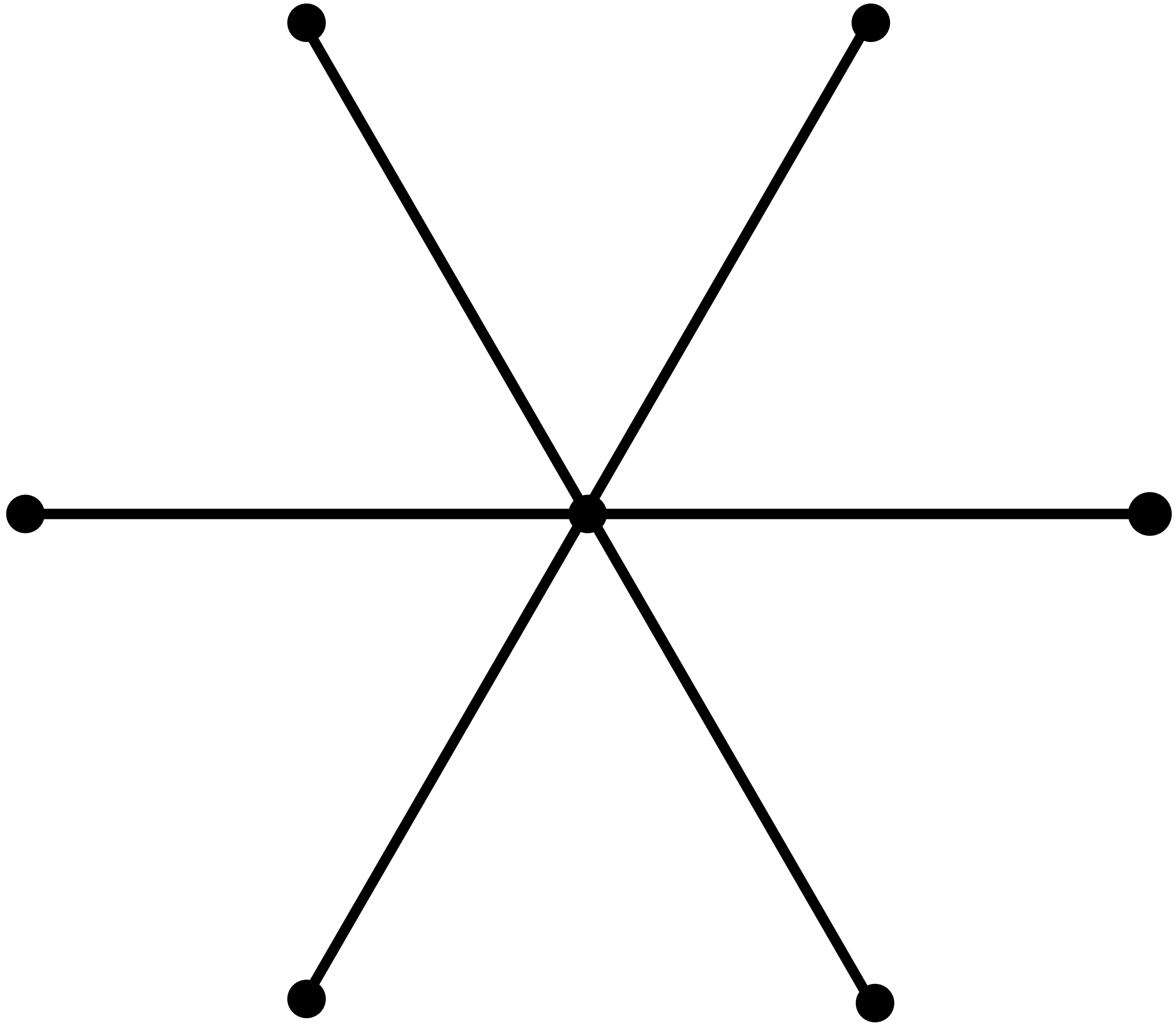
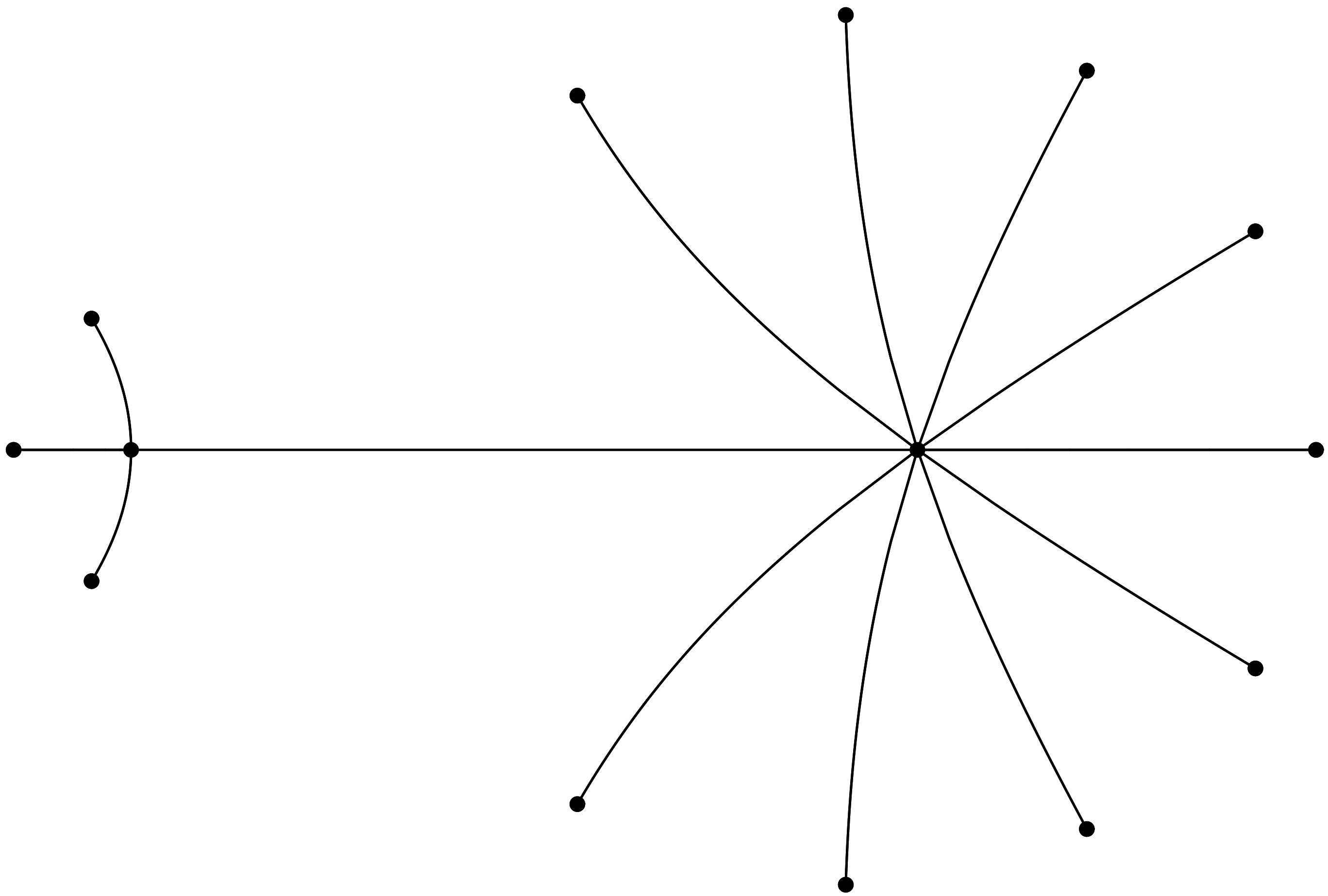




Trivially true by symmetry

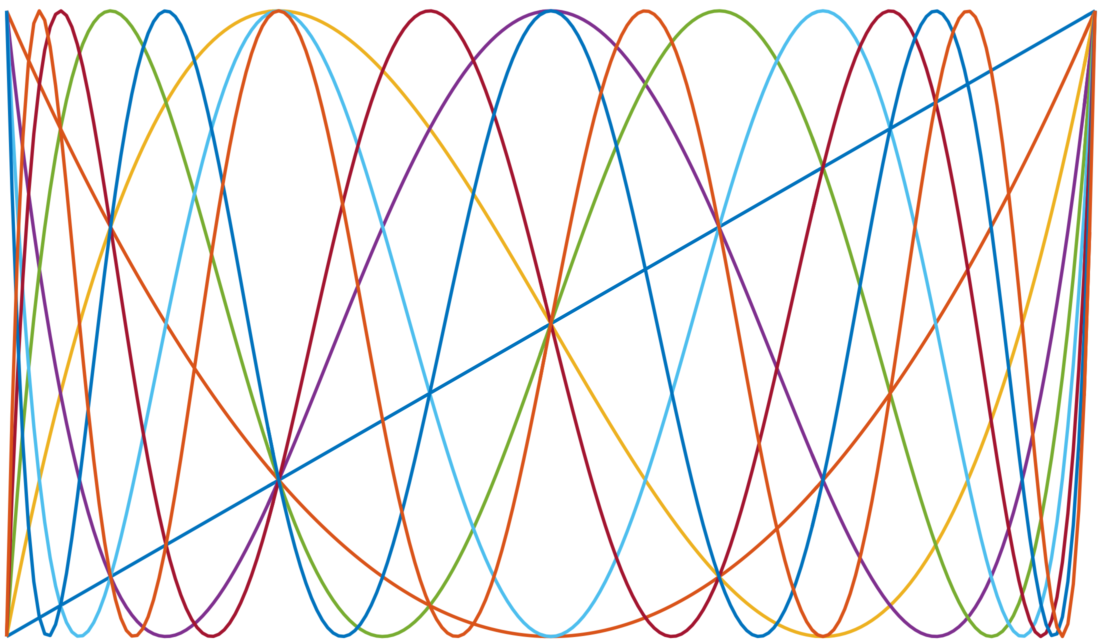


Non-obvious true tree

Definition of critical value: if p = polynomial, then

$$\mathrm{CV}(p) = \{p(z) : p'(z) = 0\} = \text{critical values}$$

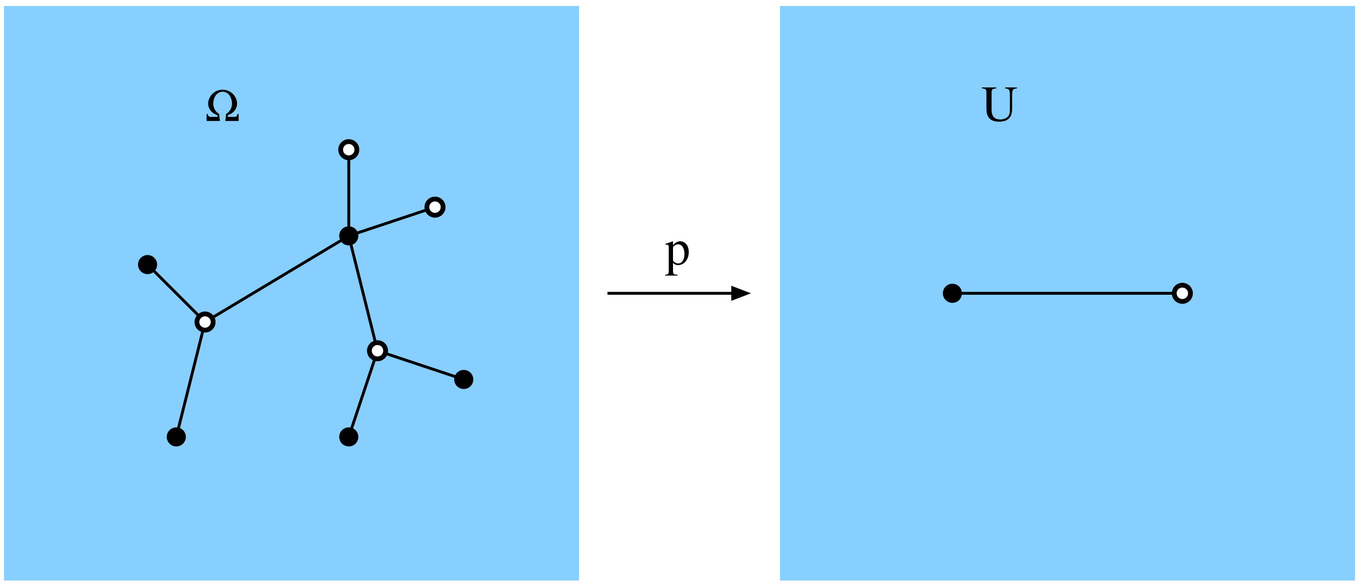
If $\mathrm{CV}(p) = \pm 1$, p is called **generalized Chebyshev** or **Shabat**.



10 classical Chebyshev polynomials

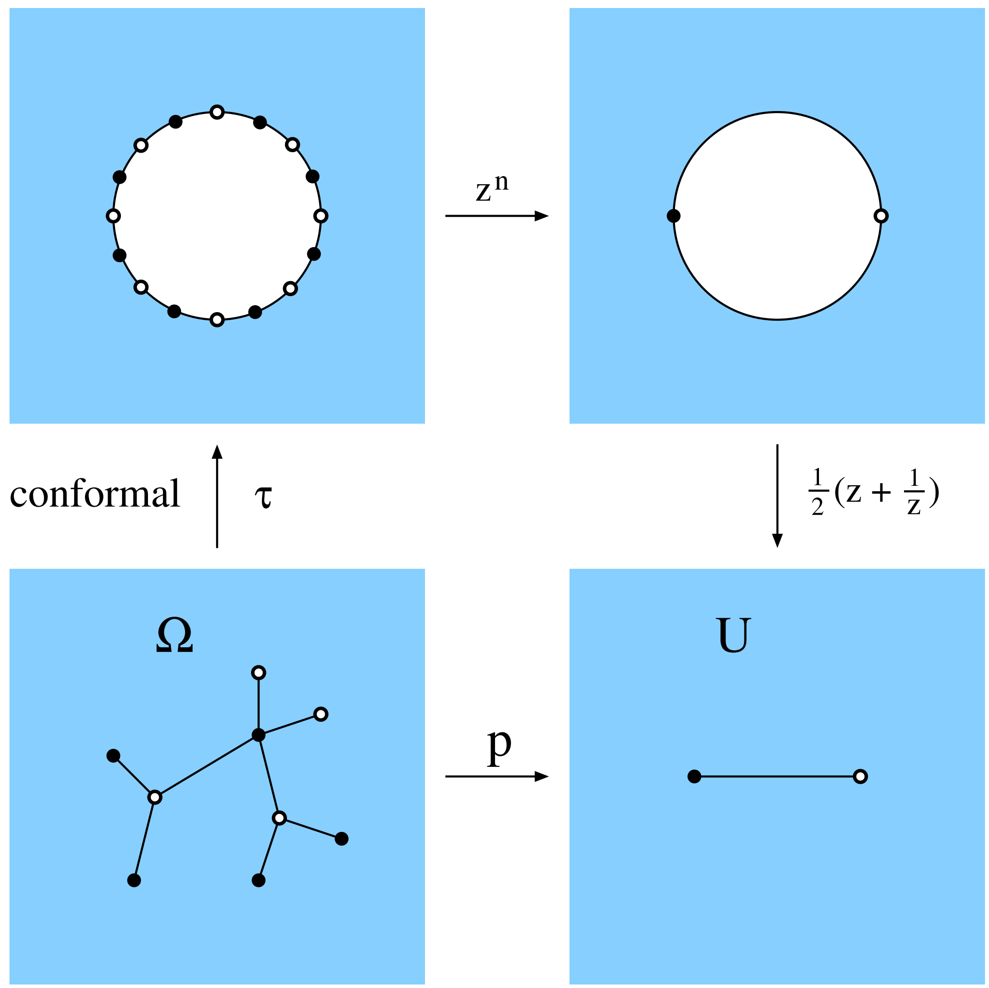
Balanced trees $\leftrightarrow$ Shabat polynomials

Fact: T is balanced iff $T = p^{-1}([-1,1])$, $p =$ Shabat.



$\Omega = \mathbb{C} \setminus T$ $\qquad$ $U = \mathbb{C} \setminus [-1,1]$

T conformally balanced $\Leftrightarrow$ p Shabat.



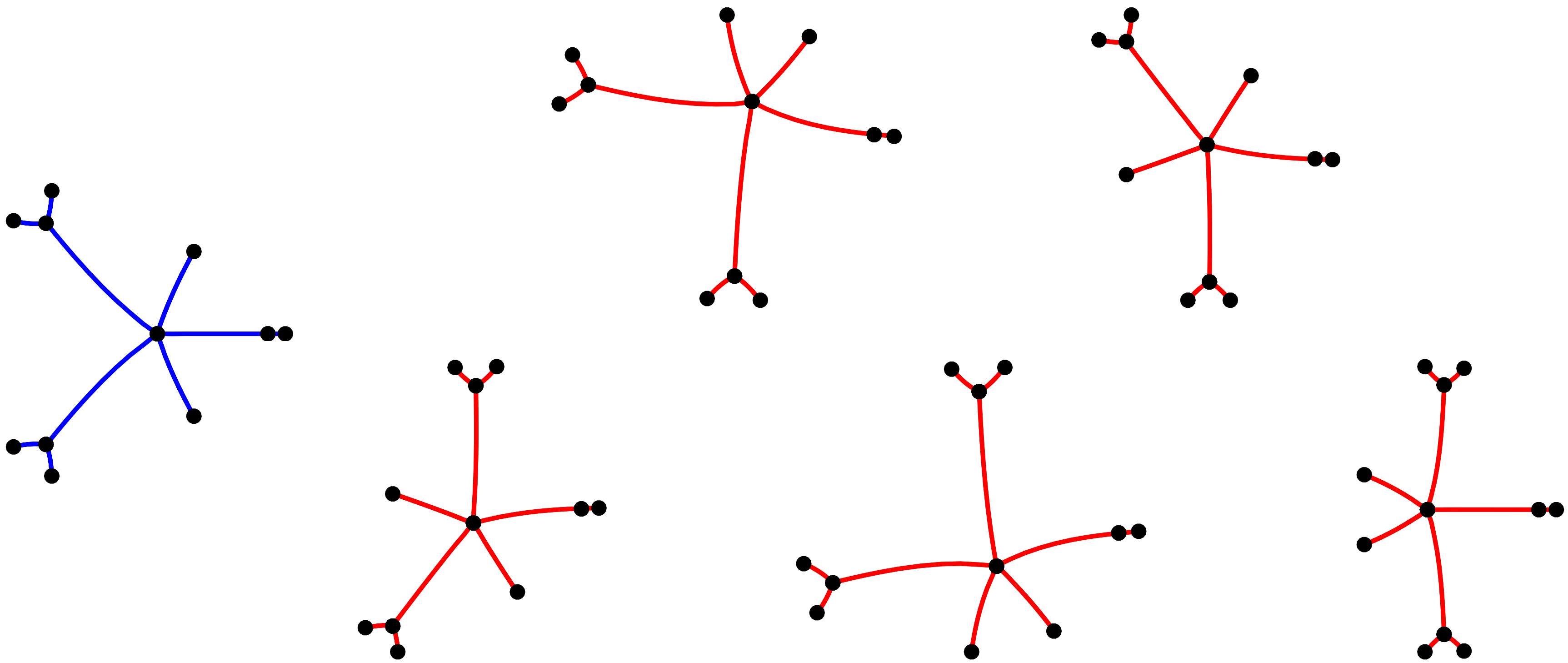
p is entire and n-to-1 $\Leftrightarrow$ p = polynomial.

$\mathrm{CV}(p) \not\in U \Leftrightarrow p : \Omega \to U$ is covering map.

Algebraic aside:

True trees are examples of Grothendieck's *dessins d'enfants* on sphere.

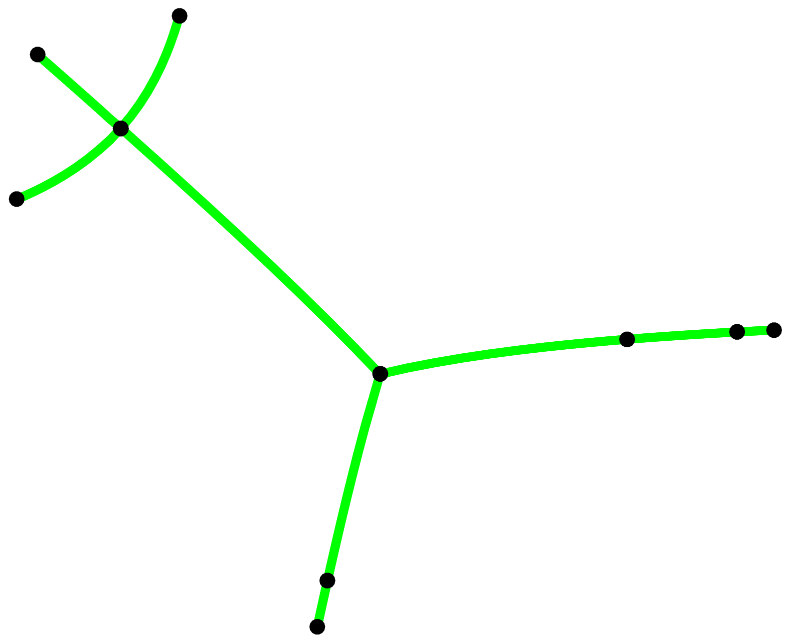
Normalized polynomials are algebraic, so planar trees correspond to number fields. Absolute Galois group acts on trees, but orbits unknown.



Six graphs of type 5 1 1 1 1 1 - 3 3 2 1 1, two orbits.

Even computing number field from tree is difficult.

Kochetkov (2009, 2014): did all trees with 9 and 10 edges.



For example, the polynomial for this 9-edge tree is

$$p(z) = z^4(z^2 + az + b)^2(z - 1),$$

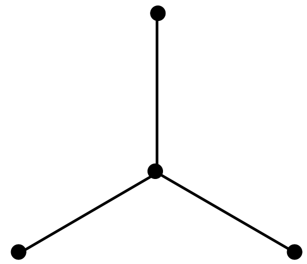
where a is a root of ...

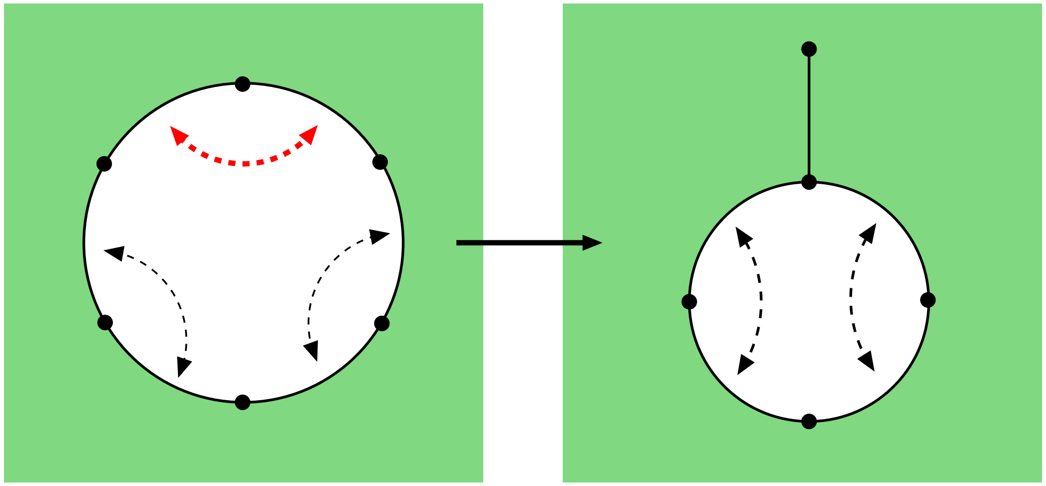
$$
\begin{aligned}
0 = \quad & 126105021875\, a^{15} + 873367351500\, a^{14} \\
& +2340460381665\, a^{13} + 2877817869766\, a^{12} \\
& +3181427453757\, a^{11} - 68622755391456\, a^{10} \\
& -680918281137097\, a^{9} - 2851406436711330\, a^{8} \\
& -7139130404618520\, a^{7} - 12051656256571792\, a^{6} \\
& -14350515598839120\, a^{5} - 12058311779508768\, a^{4} \\
& -6916678783373312\, a^{3} - 2556853615656960\, a^{2} \\
& -561846360735744\, a - 65703906377728
\end{aligned}
$$

This is **not** the most complicated formula in Kochetkov's paper.

However, true form can be drawn without knowing the polynomial.

Don Marshall's `ZIPPER` uses conformal mapping to draw true trees.





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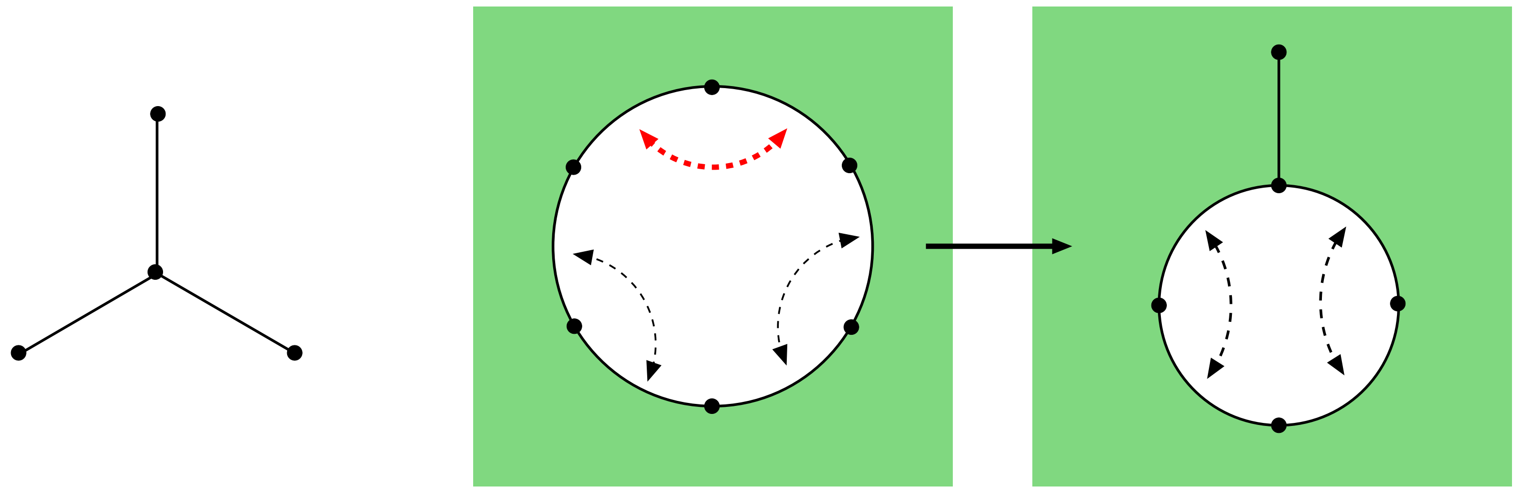
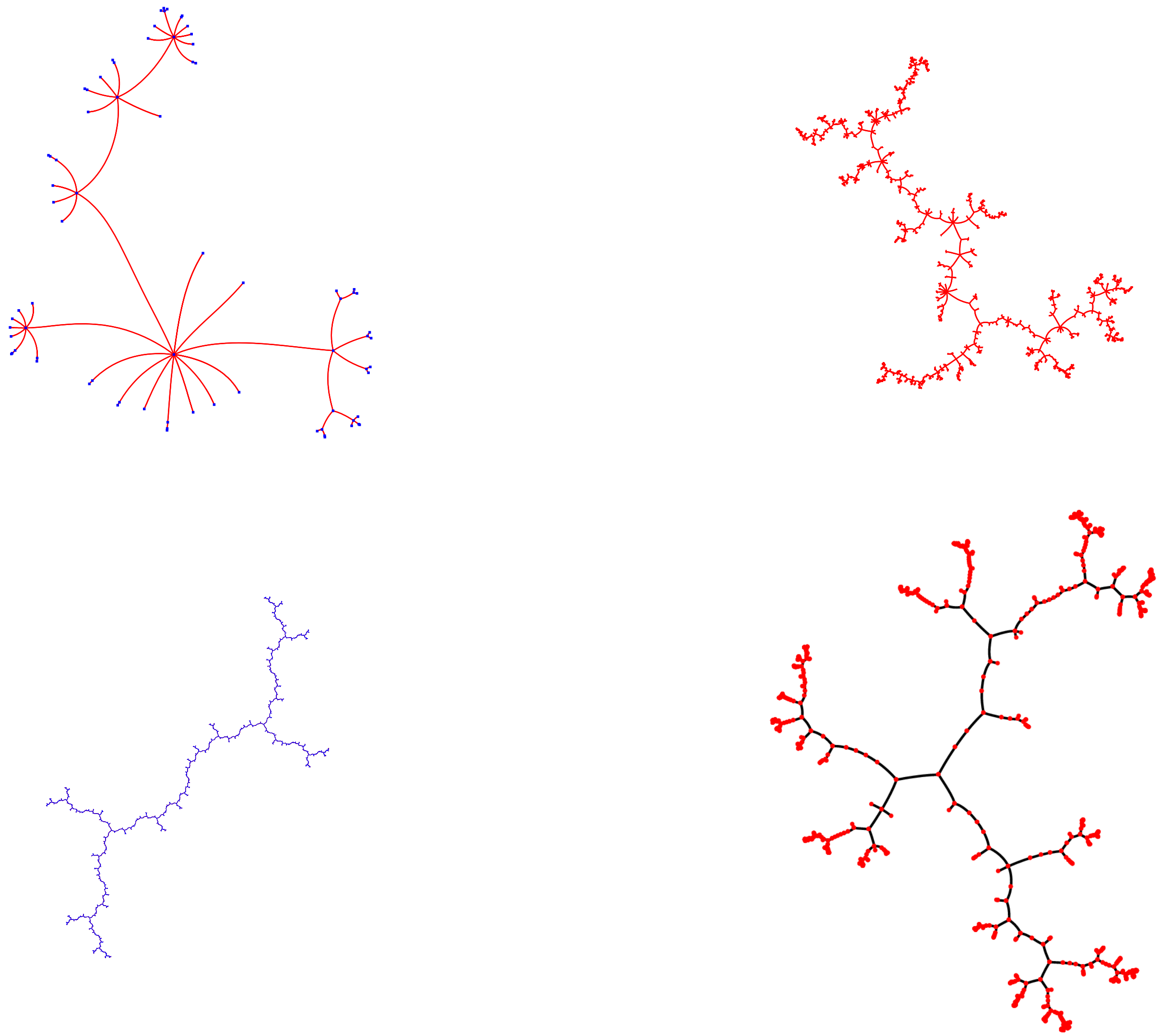


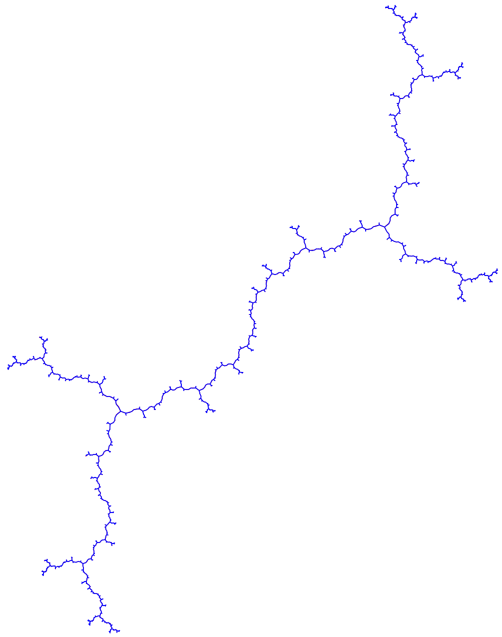
Marshall and Rohde approximated all true trees with ≤ 14 edges.

They can compute vertices to 1000's of digits of accuracy.

Test if $\alpha \in \mathbb{C}$ is algebraic by seeking integer relationships between $1, \alpha, \alpha^2, \dots$ using lattice reduction or PSLQ algorithm.



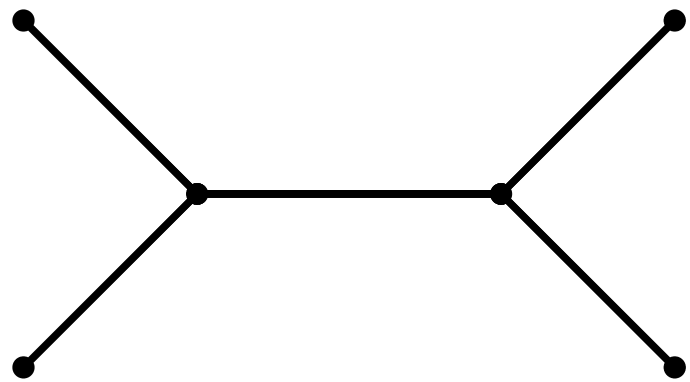
Some true trees, courtesy of Marshall and Rohde



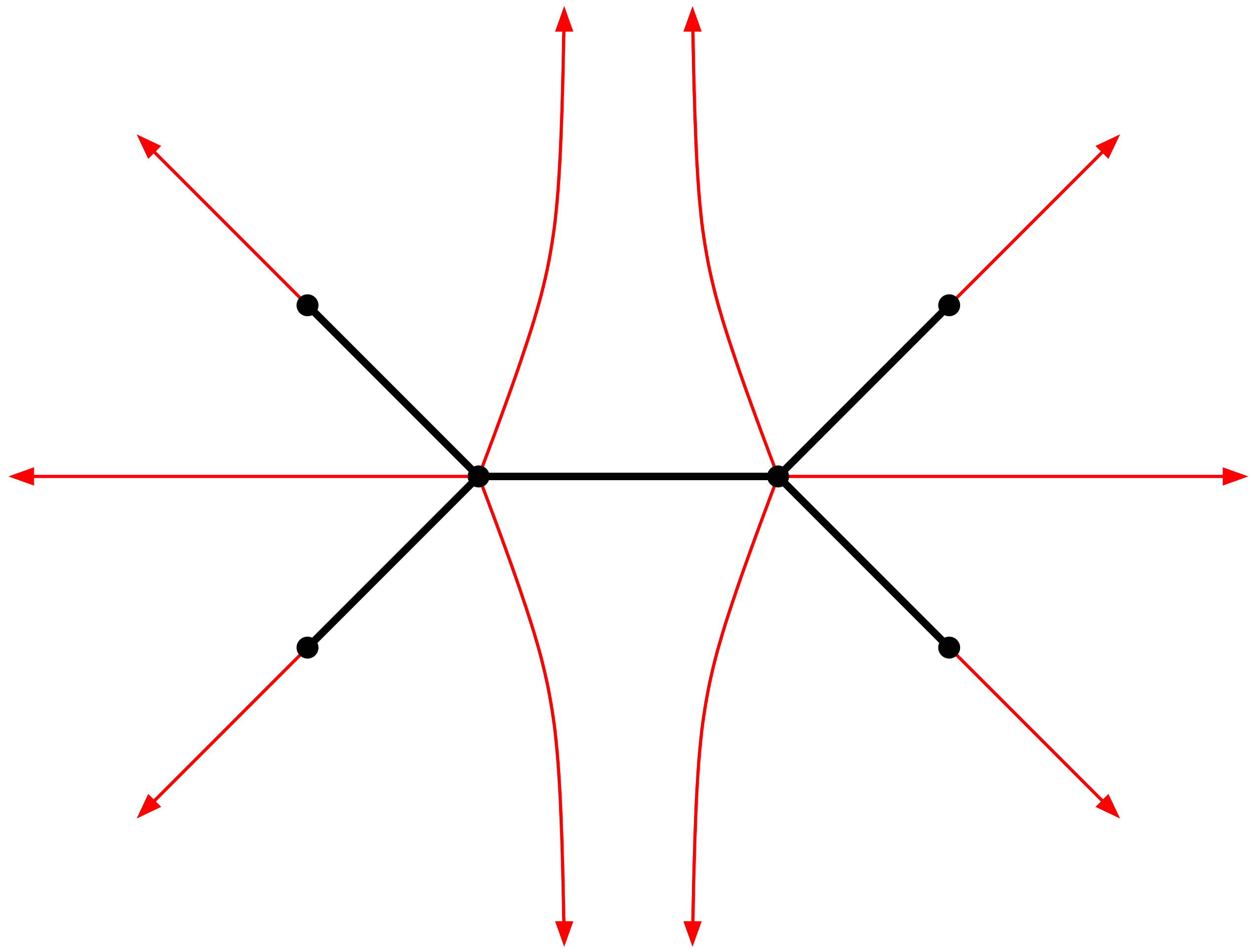
Ceci n'est pas un ensemble de Julia.

Theorem: Every finite tree has a true form.

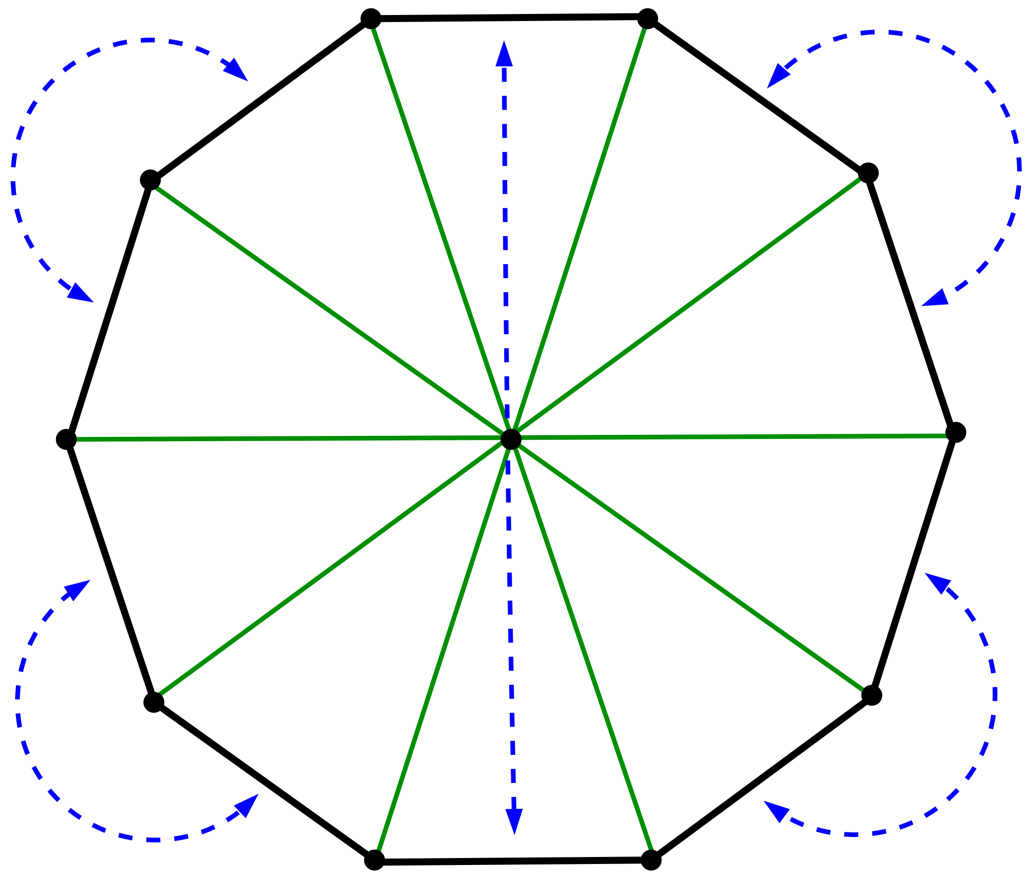
Standard proof uses the uniformization theorem.



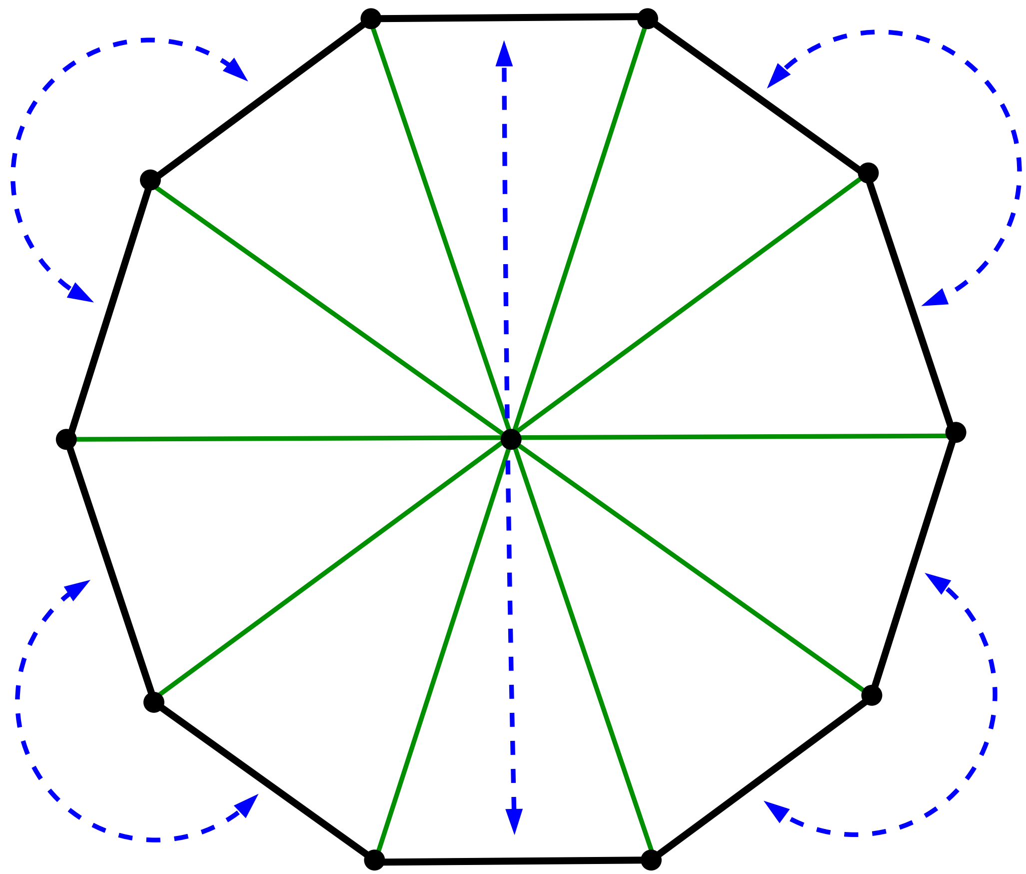
- start with a finite tree.



- connect vertices of T to infinity; gives finite triangulation of sphere.
- Defines adjacencies between triangles.



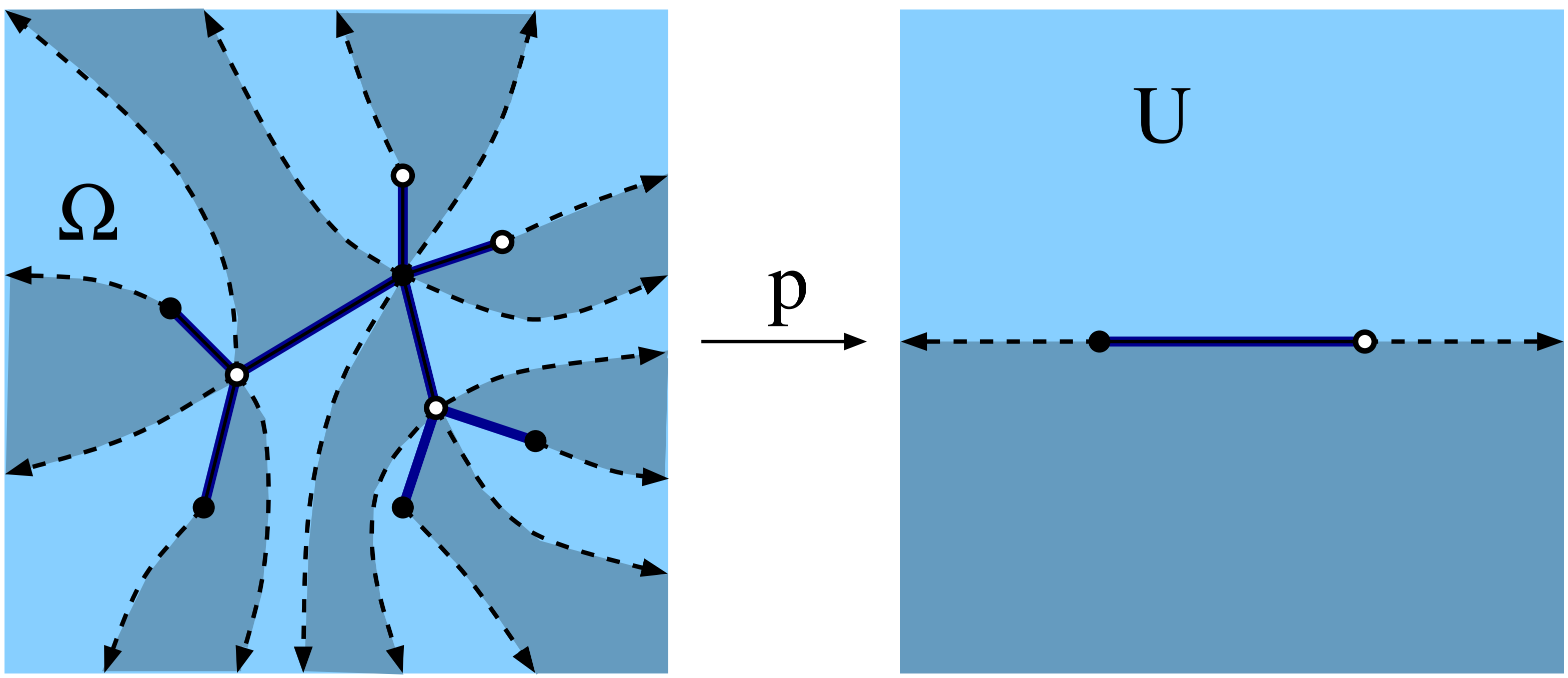
- Glue equilateral triangles using adjacencies. Get conformal 2-sphere.
- By uniformization theorem, equivalent to Riemann sphere.
- Tree maps to balanced tree. Shape unclear.



- Note: conformal 2-sphere has many “equilateral triangulations”.
- Which other Riemann surfaces can be obtained in this way?

Belyi function = holomorphic map from Riemann surface with 3 critical values (usually $0, 1, \infty$).

Theorem: A Riemann surface can constructed from equilateral triangles iff it has a Belyi function.



Triangles are inverse images of upper and lower half-planes.

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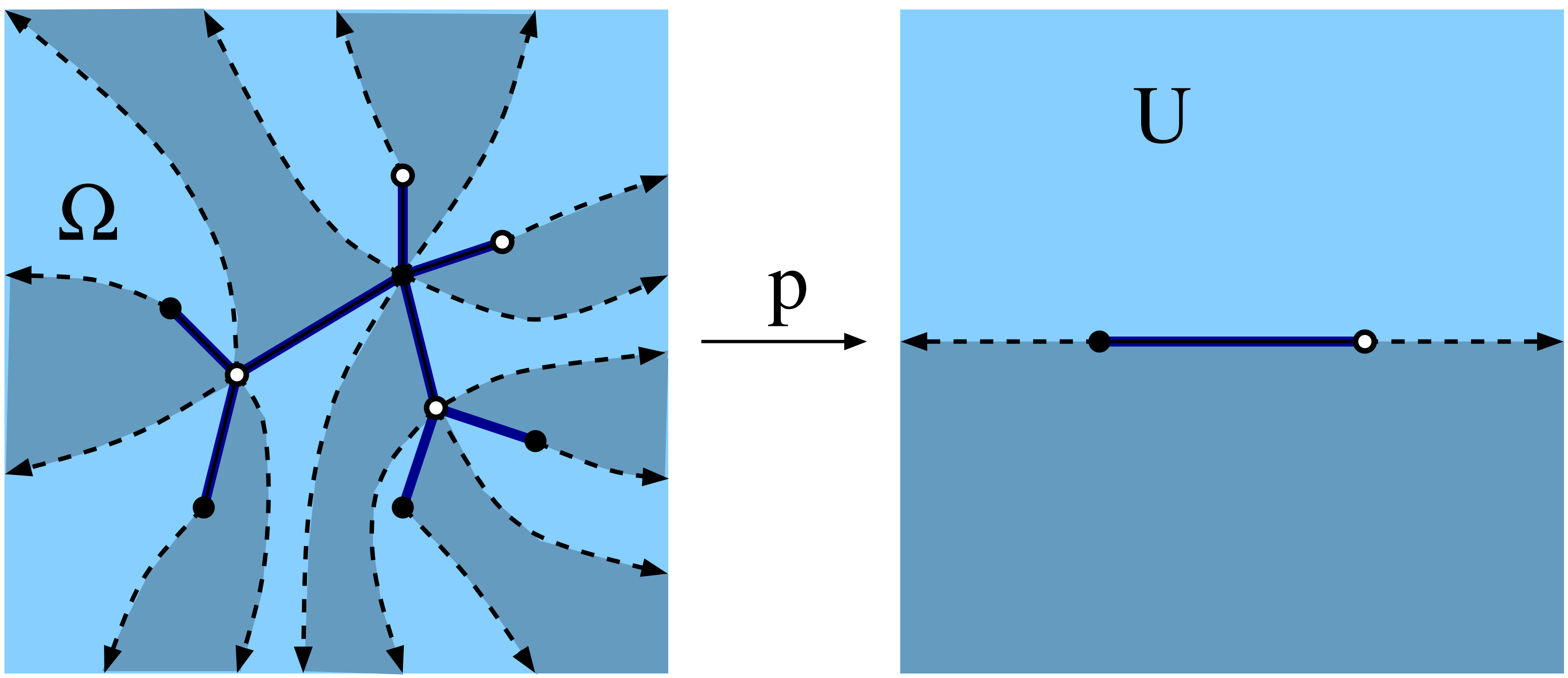

A triangulation is equilateral iff every edge has an anti-holomorphic reflection fixing it pointwise and swapping the two adjacent triangles.

Belyi function = holomorphic map from Riemann surface with 3 critical values (usually $0, 1, \infty$).

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There are only finitely many ways to glue together n triangles.

$\Rightarrow$ only countably many compact surfaces have a Belyi function.

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There are only finitely many ways to glue together n triangles.

$\Rightarrow$ only countably many compact surfaces have a Belyi function.

Belyi's Thm: A compact surface has a Belyi function iff it is algebraic.

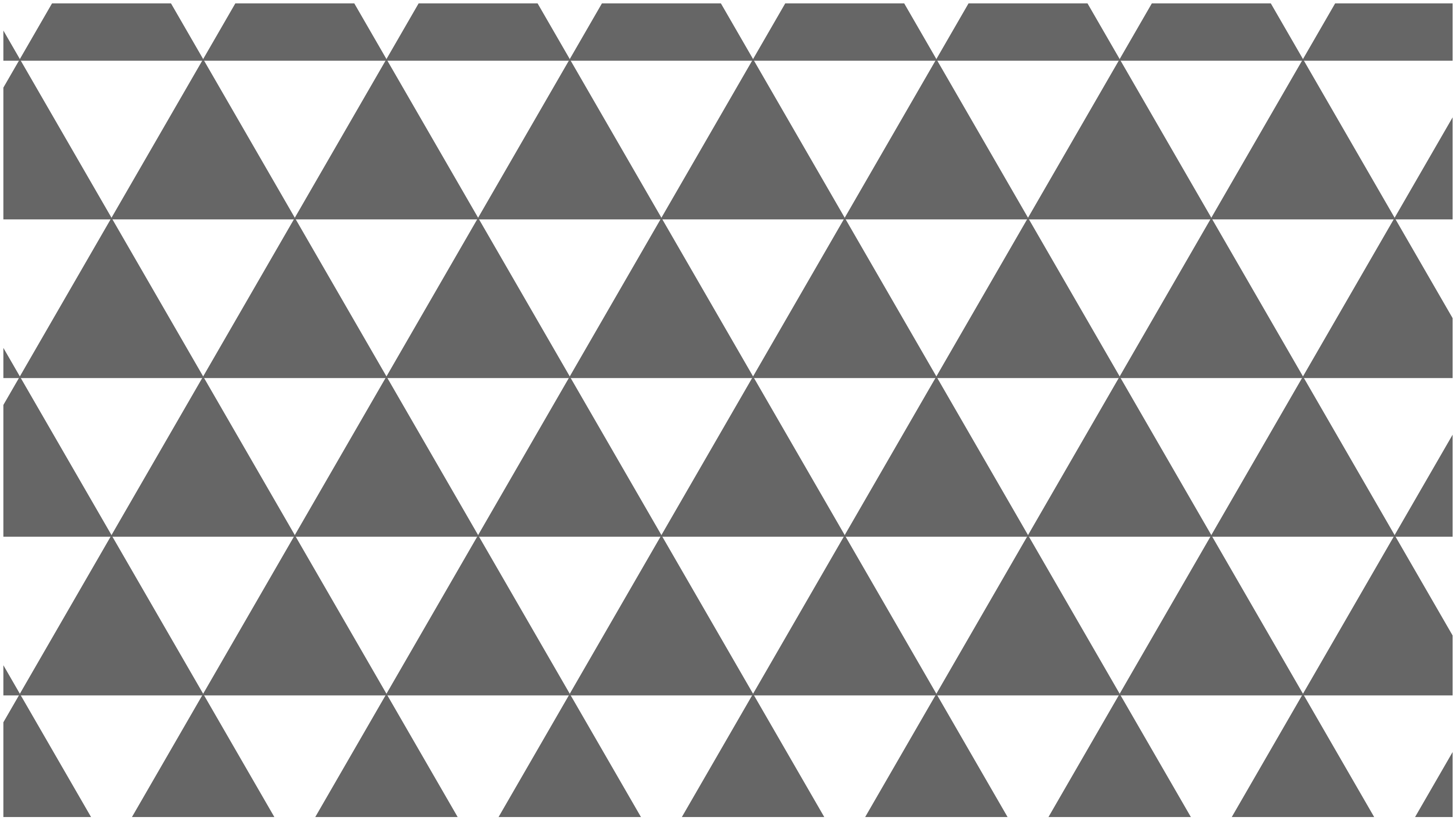
(Zero set of $P(z, w)$ with algebraic integers as coefficients).

Motivated Grothendieck's theory of *dessins d'enfant*.

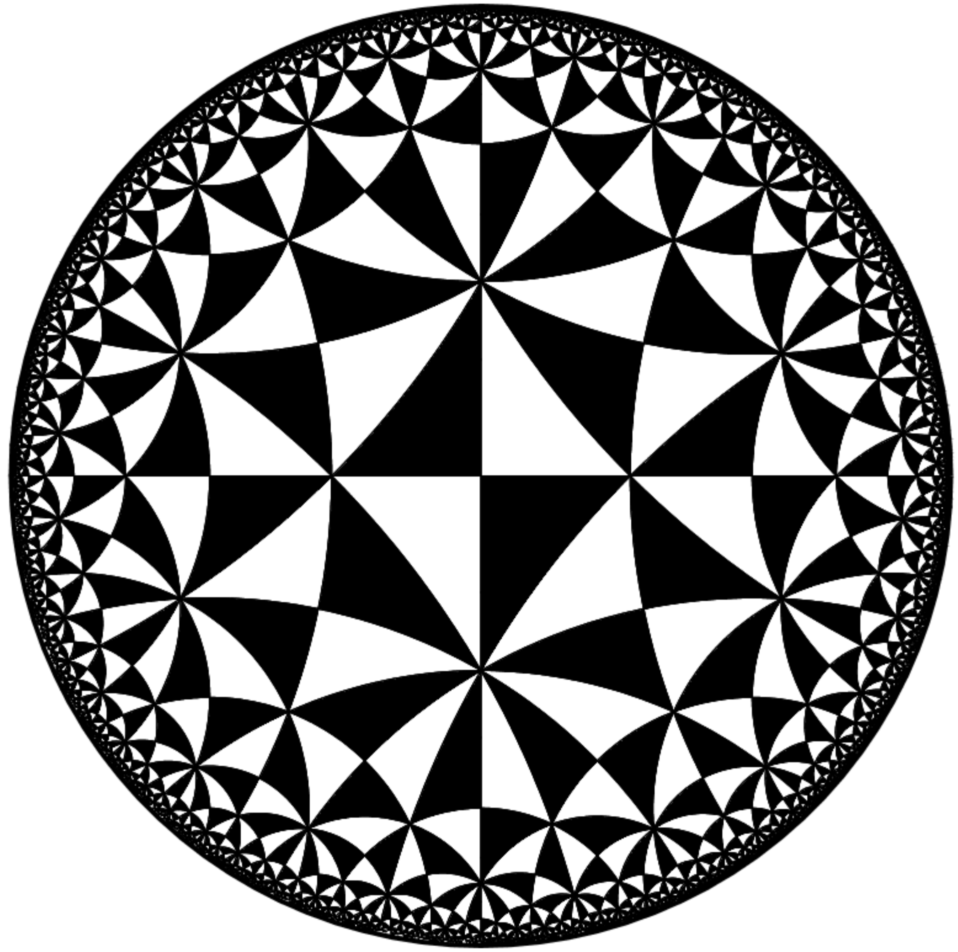
Which non-compact Riemann surfaces are Belyi?

$=$ has a Belyi function

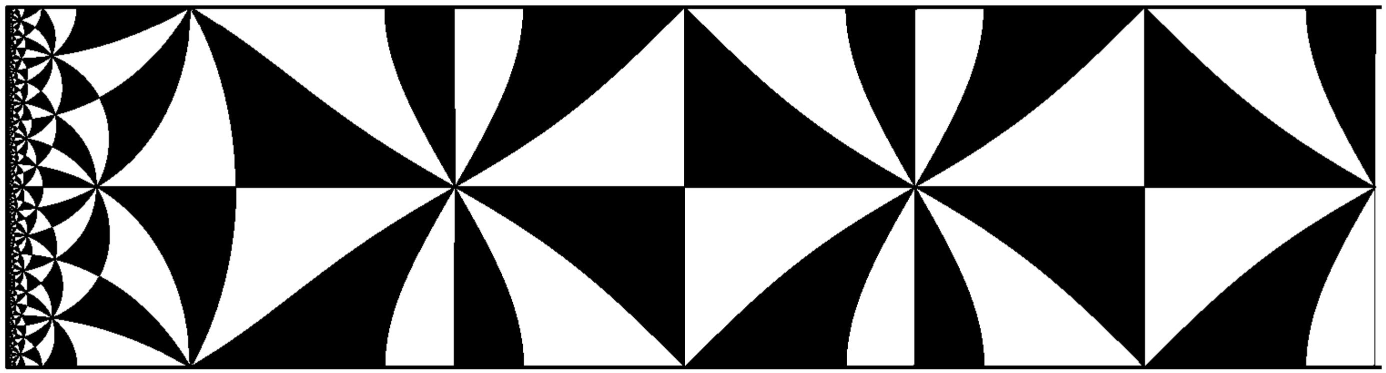
$=$ has an equilateral triangulation



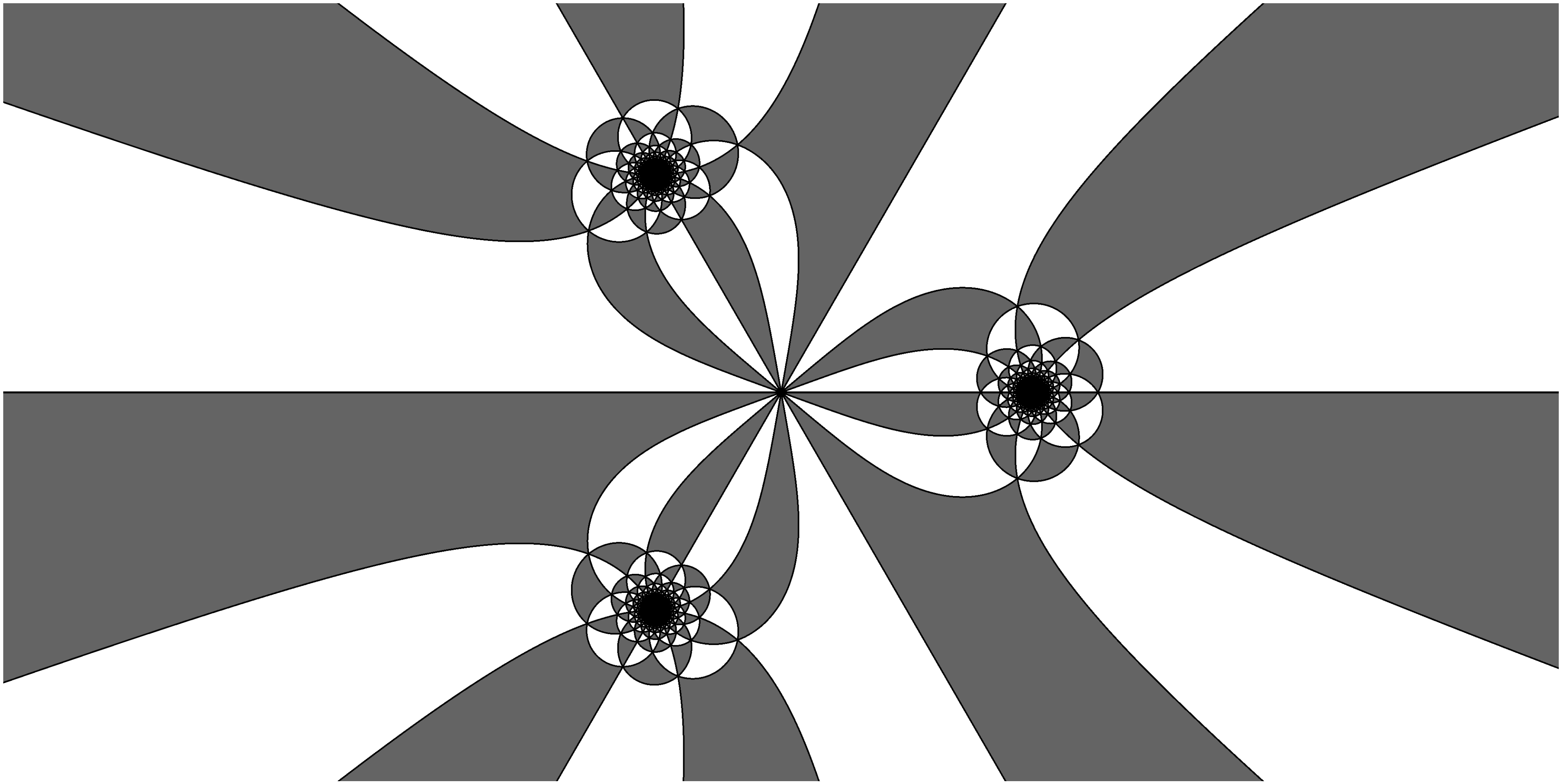
The plane



The disk



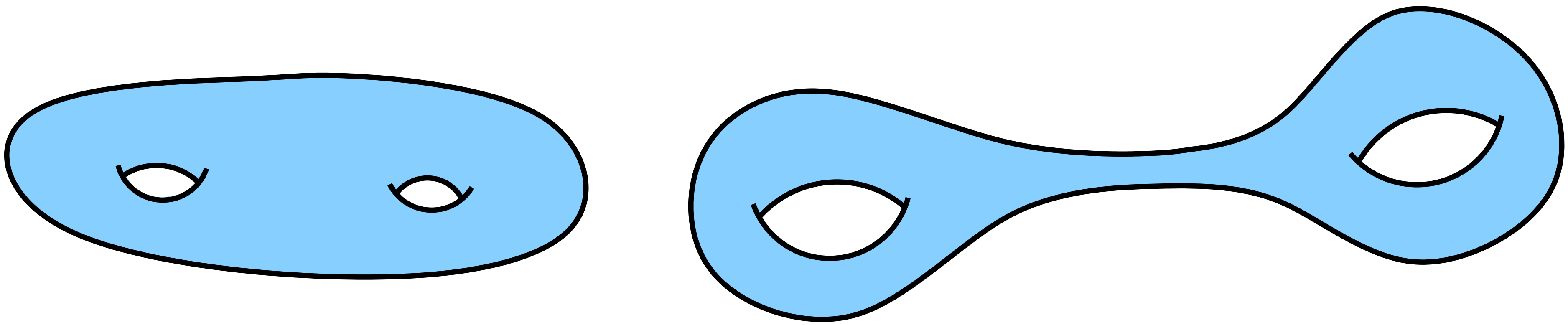
The punctured disk (identity top and bottom sides)



The trice punctured sphere

Most surfaces have many conformal structures, e.g., disk $\neq$ plane.

Higher genus surfaces have multi-dimensional moduli spaces.

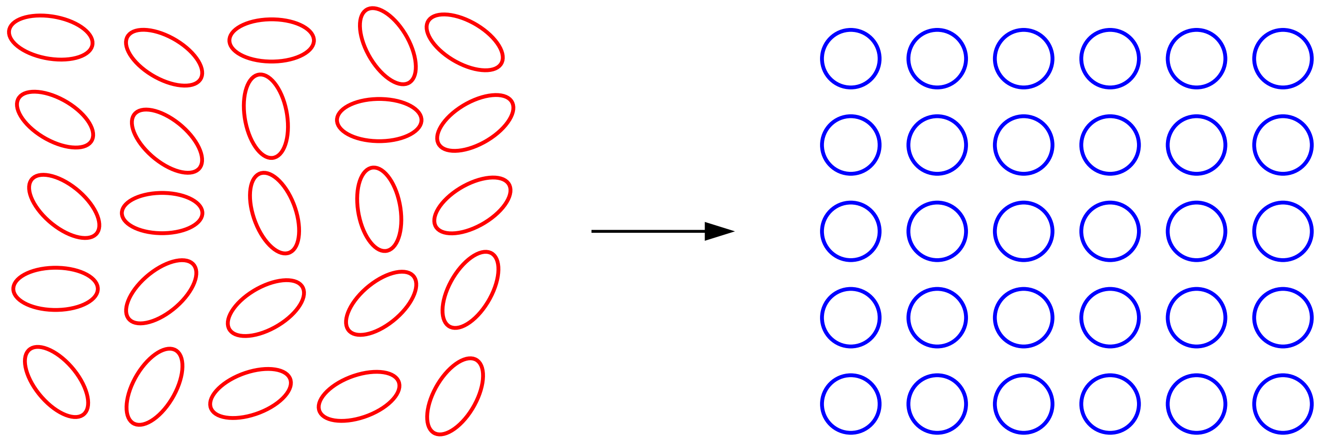


Every Riemann surface can be triangulated.

$\Rightarrow$ every surface is homeomorphic a Belyi surface.

Which conformal structures occur? Need QC maps to answer this.

Quasiconformal (QC) maps send infinitesimal ellipses to circles.



Eccentricity = ratio of major to minor axis of ellipse.

For K-QC maps, ellipses have eccentricity $\leq K$

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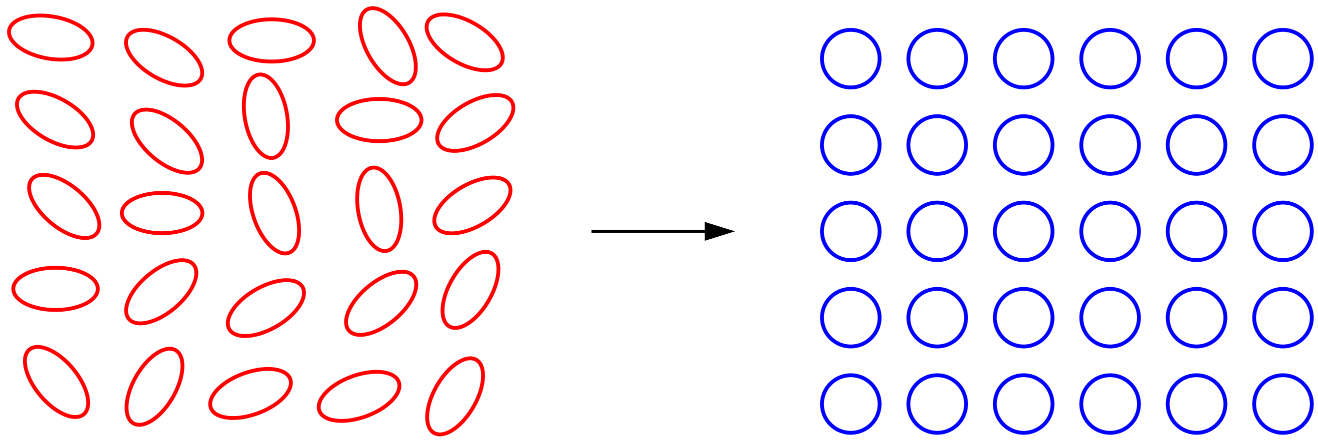


Eccentricity = ratio of major to minor axis of ellipse.

For K-QC maps, ellipses have eccentricity $\leq K$

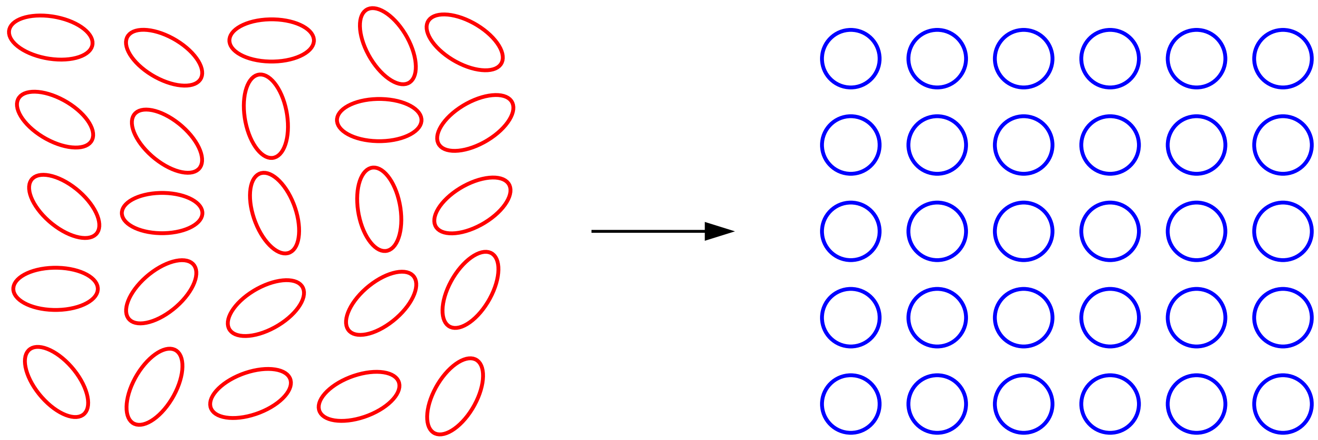
Ellipses determined a.e. by measurable dilatation $\mu = f_{\overline{z}}/f_z$ with

$$f_z = f_x - i f_y \qquad f_{\overline{z}} = f_x + i f_y.$$

$$|\mu| \leq \frac{K-1}{K+1} < 1.$$

Conversely, ...

Quasiconformal (QC) maps send infinitesimal ellipses to circles.



Mapping theorem: any such μ comes from some QC map f.

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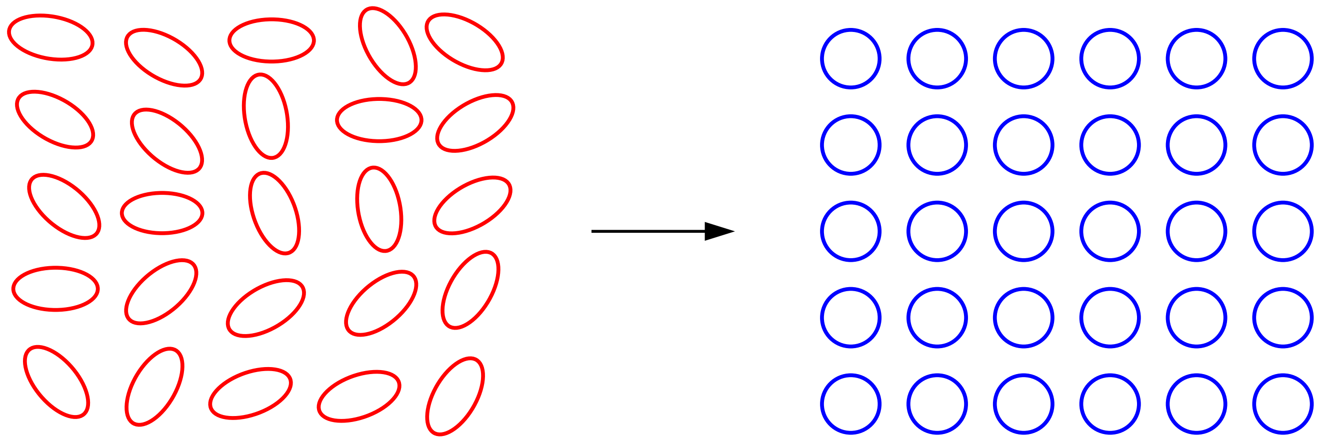
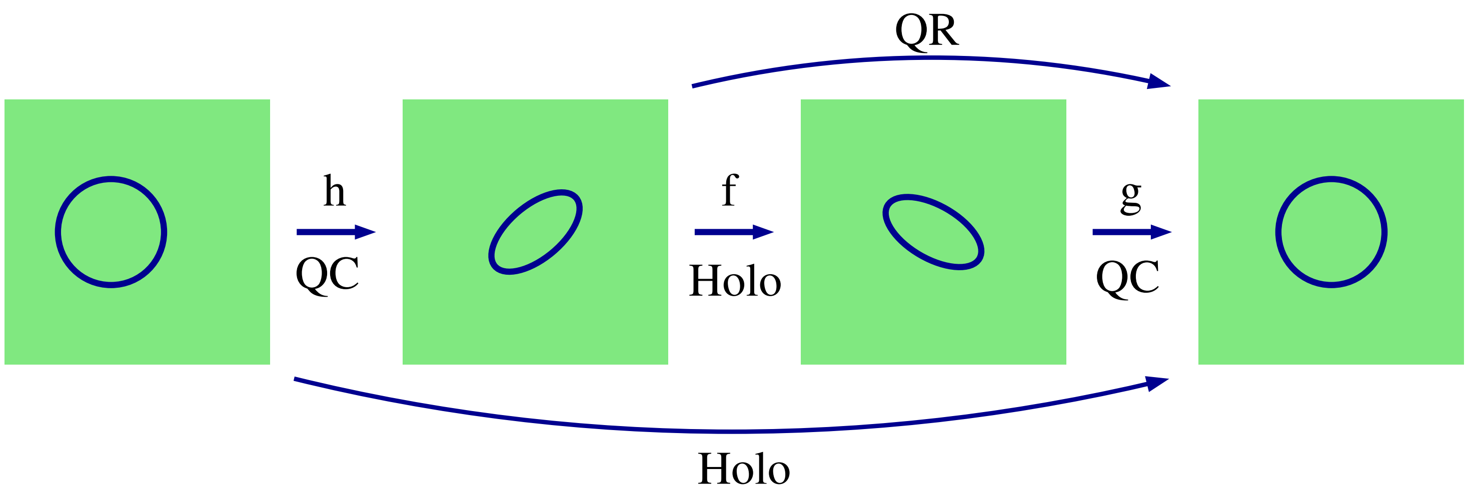
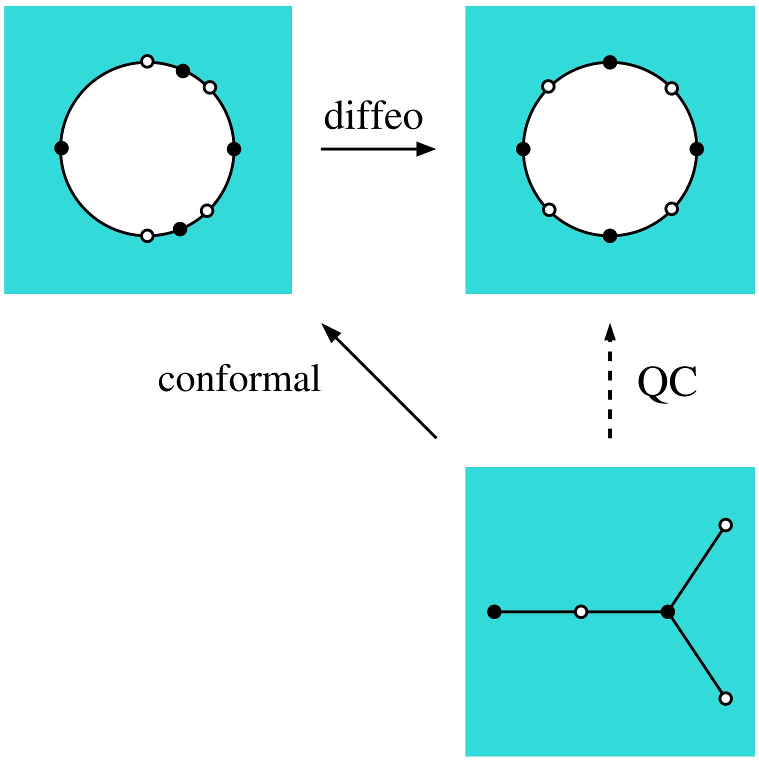


Mapping theorem: any such μ comes from some QC map f.

Cor: If f is holomorphic and g is QC, then there is a QC map h so that $F = g \circ f \circ h$ is also holomorphic. ($g \circ f$ is quasiregular.)



QC proof that every finite tree has a true form:



“Equalize intervals” by diffeomorphism. Composition is quasiconformal.

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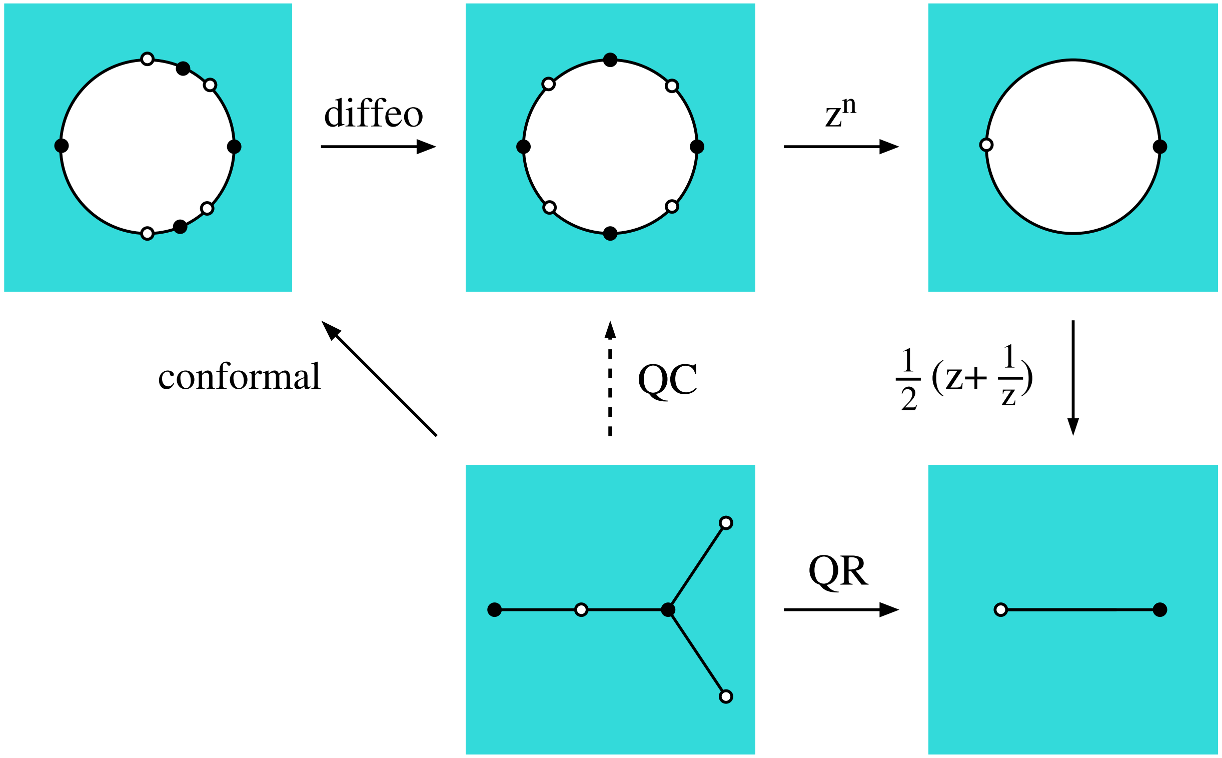

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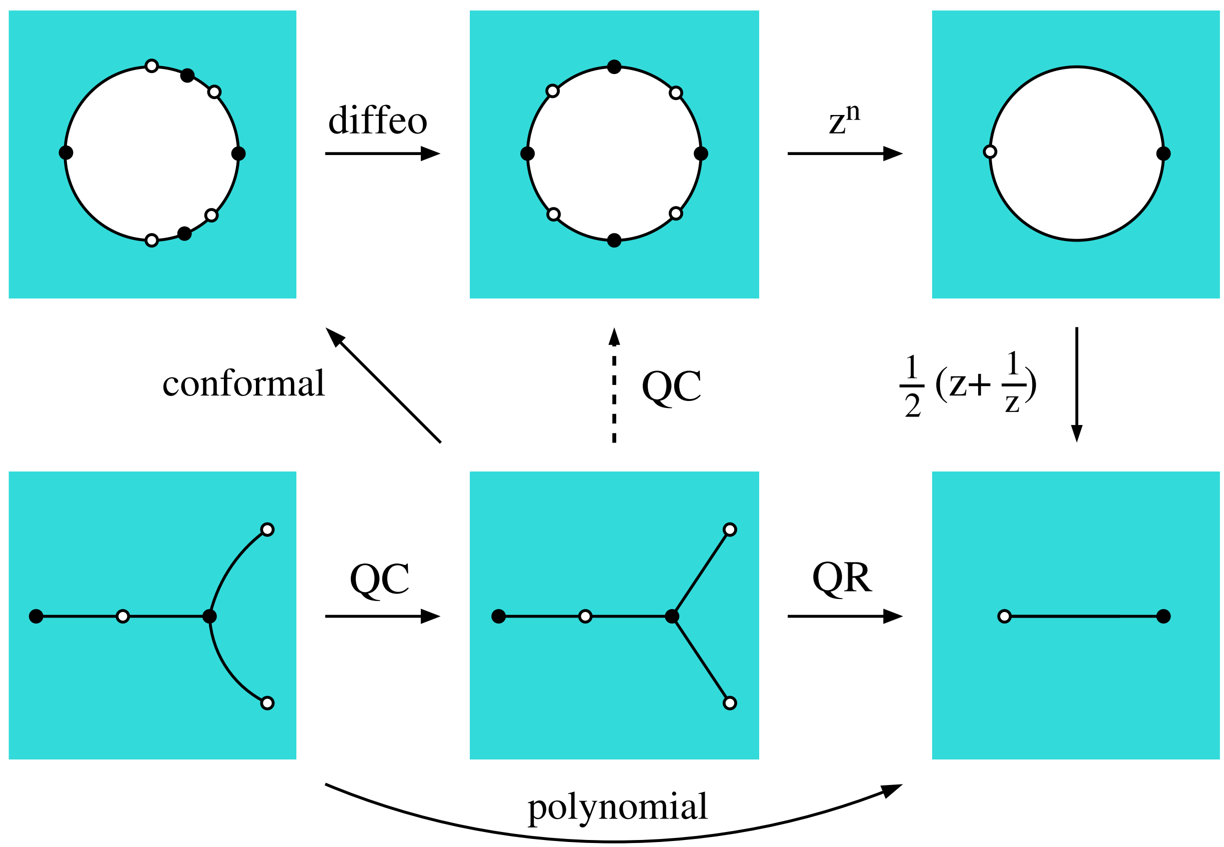
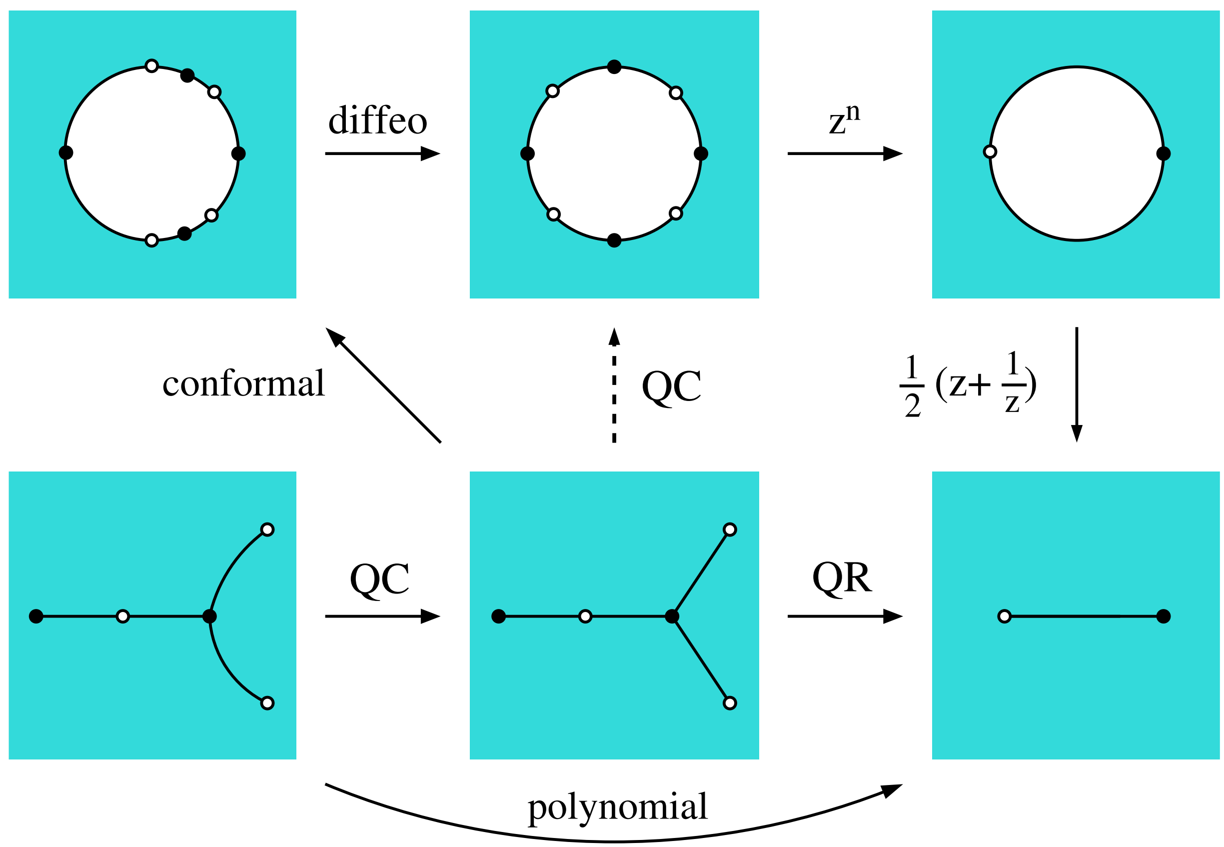


Mapping theorem implies there is a QC φ so $p = q \circ \varphi$ is a polynomial.

Only possible critical points are vertices of tree; these map to ± 1.

QC proof that every finite tree has a true form:



Constructive: can estimate QC correction, get shape of true tree.

Eremenko: do true trees approximate all possible shapes?

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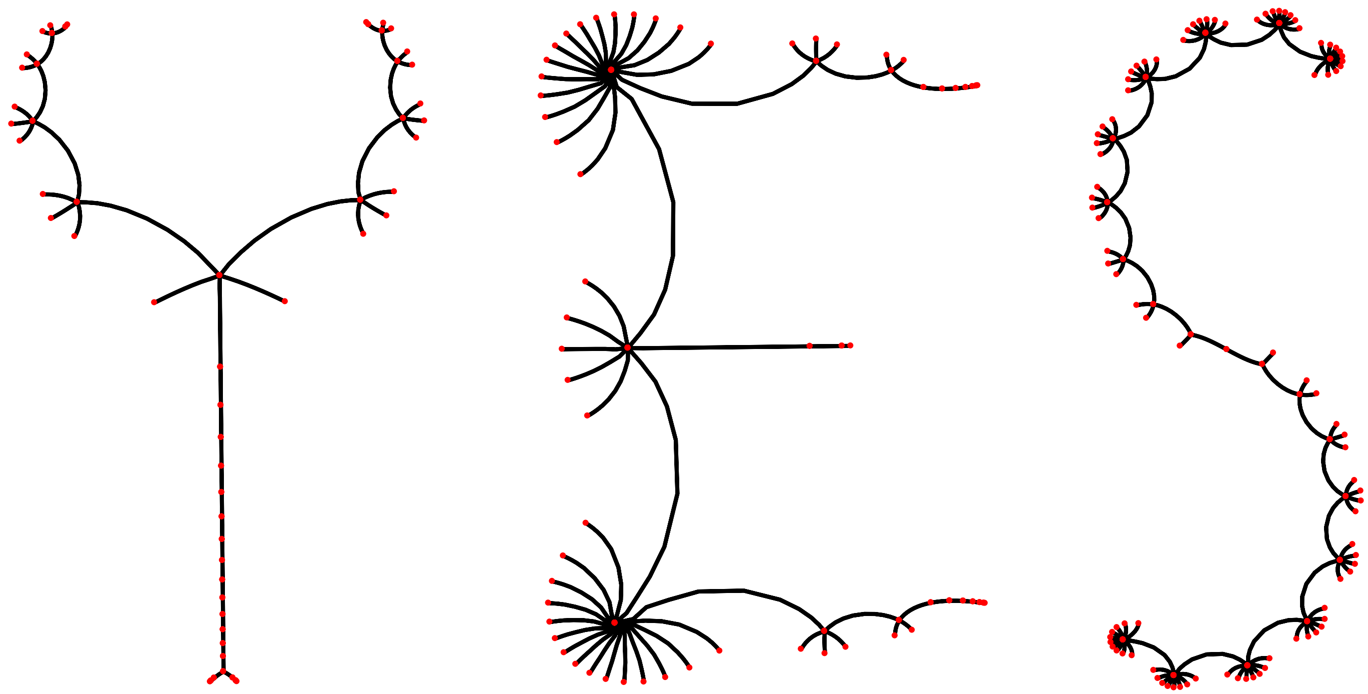

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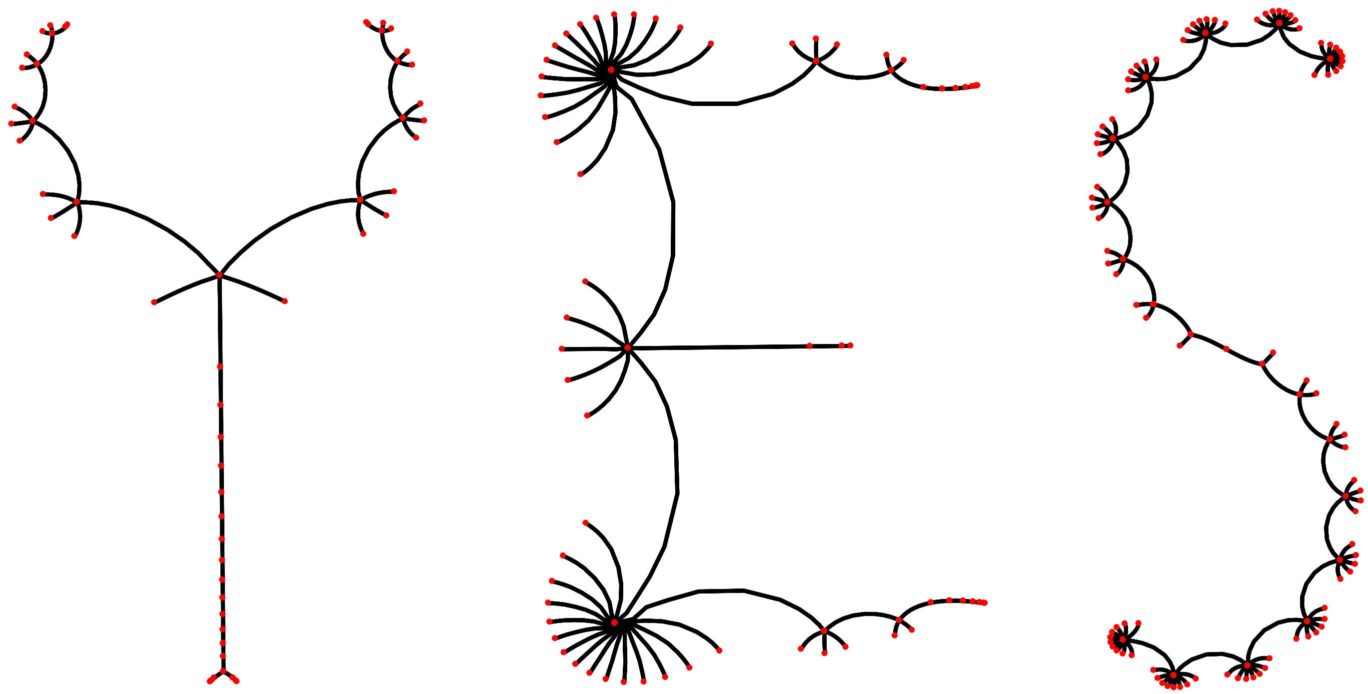
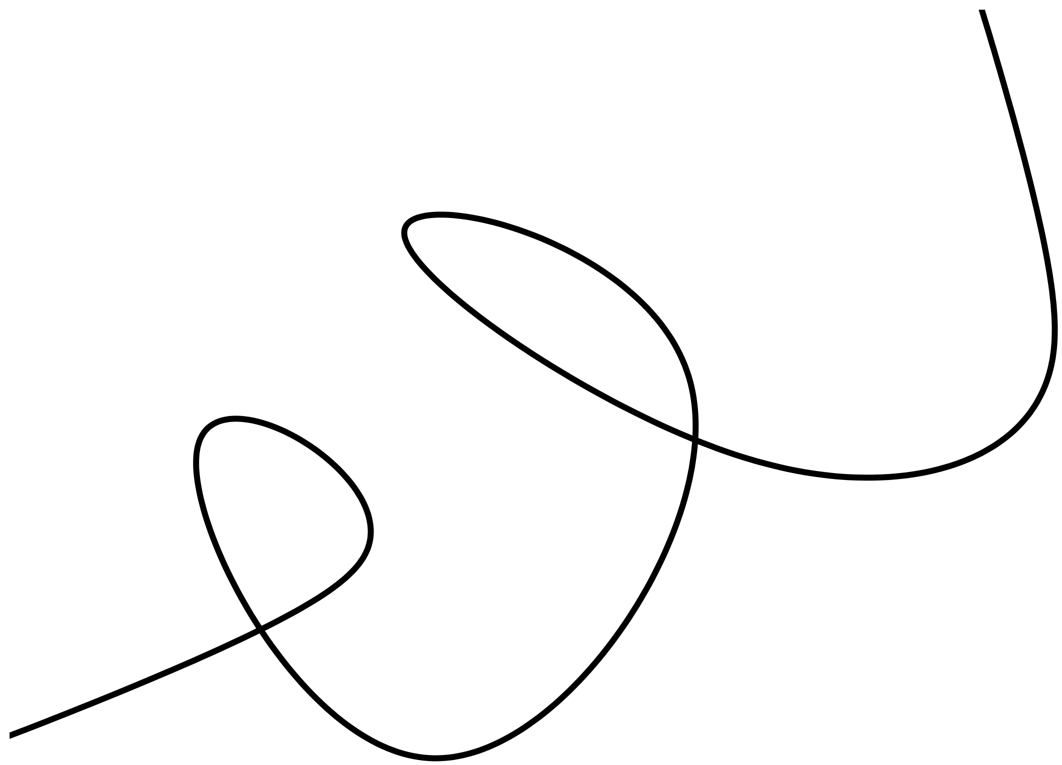
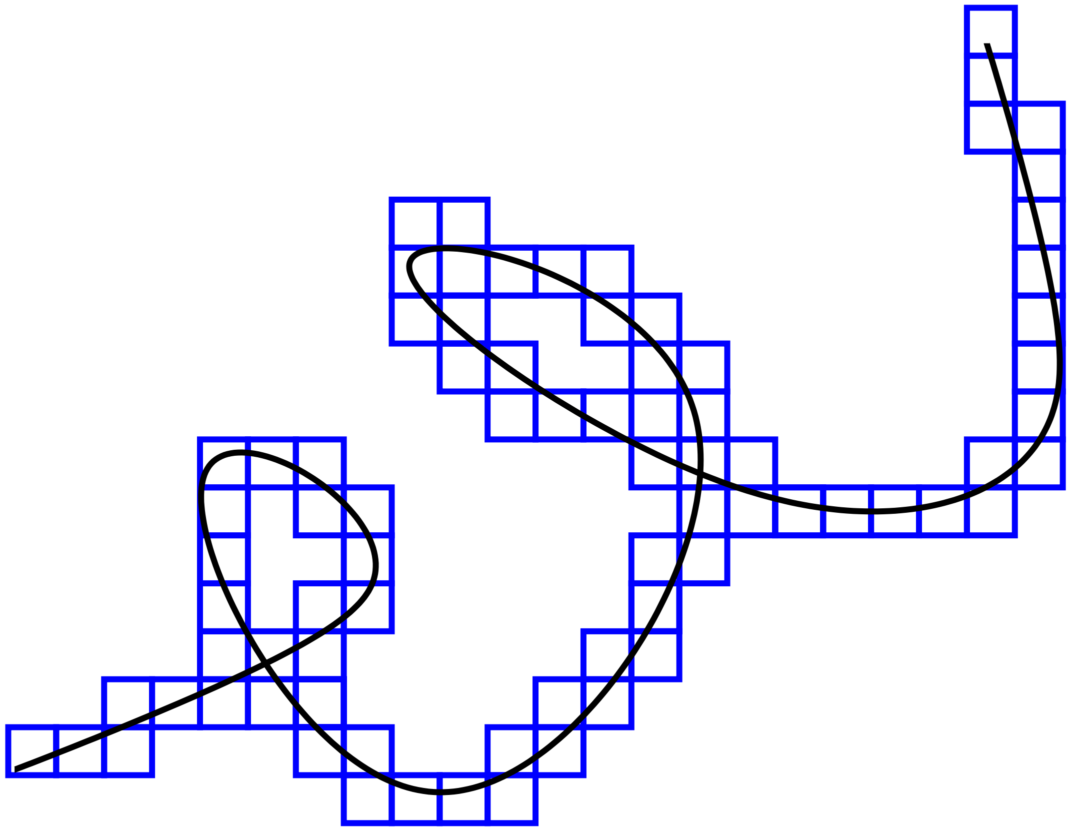


Thm: Every planar continuum is Hausdorff limit of true trees.

E, F are distance $\leq \epsilon$ apart if each is in an ϵ-neighborhood of the other.



Suffices to approximate subtrees of a grid.



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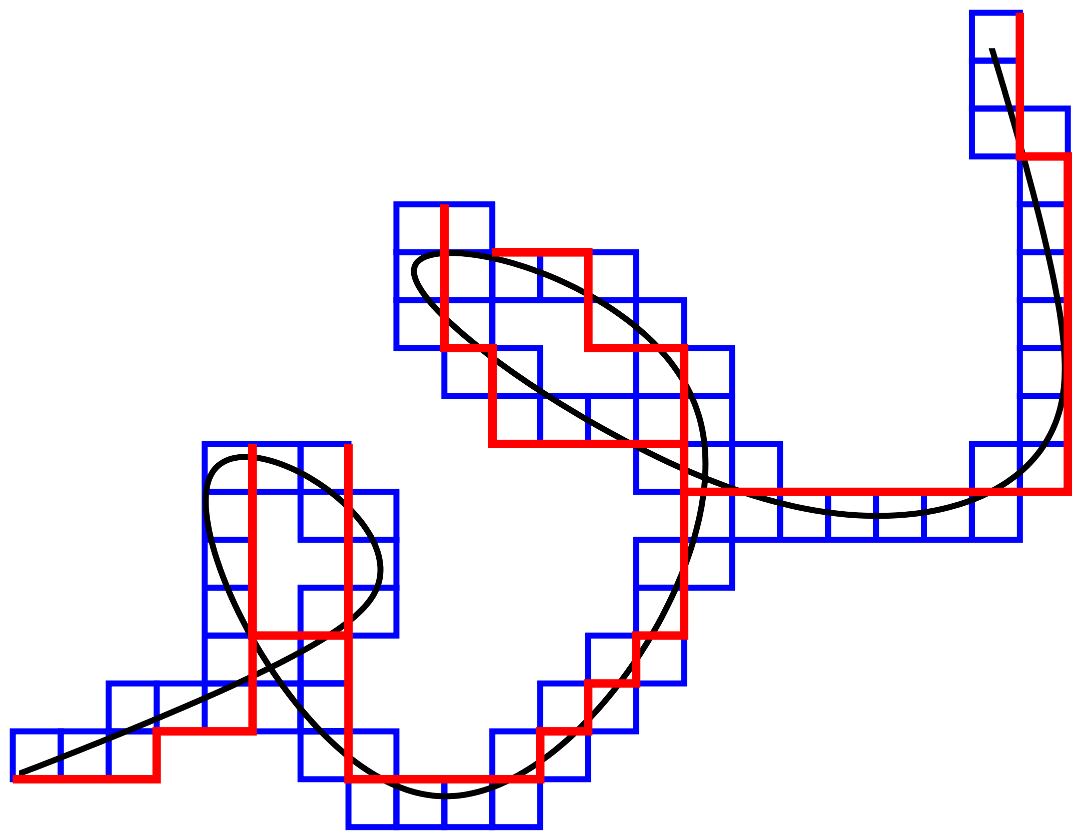

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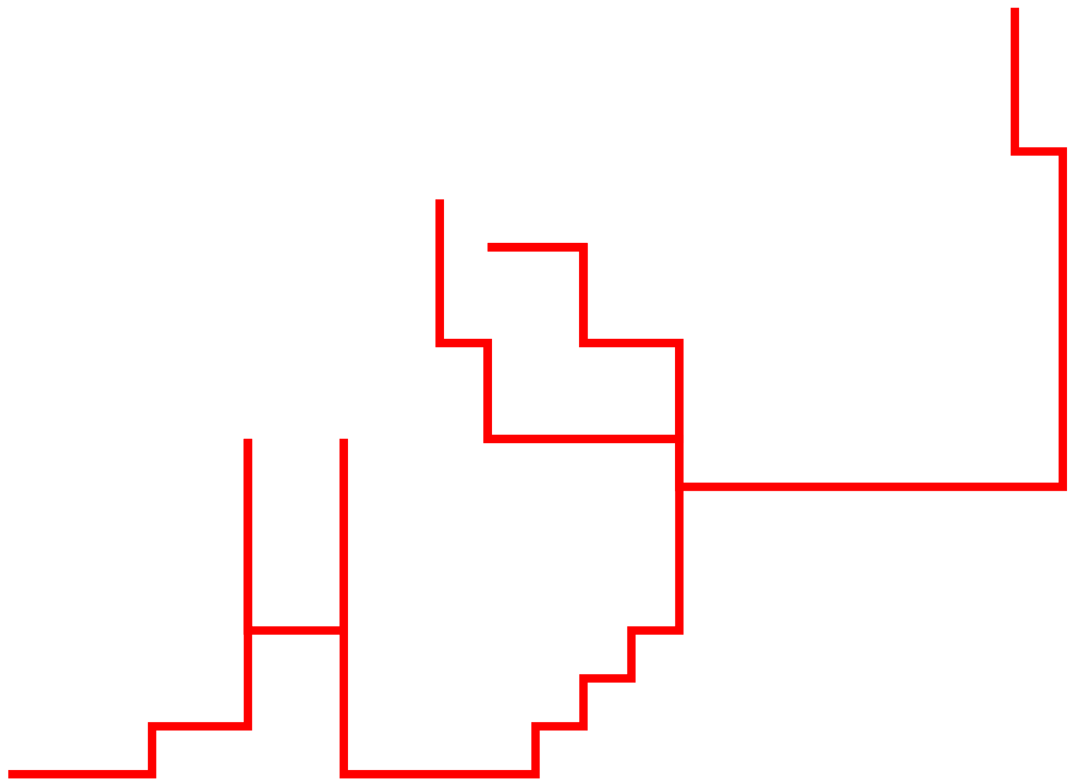

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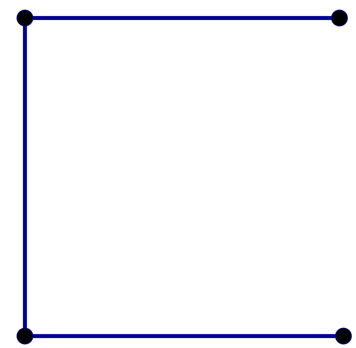
Theorem: Every planar continuum is a limit of true trees.

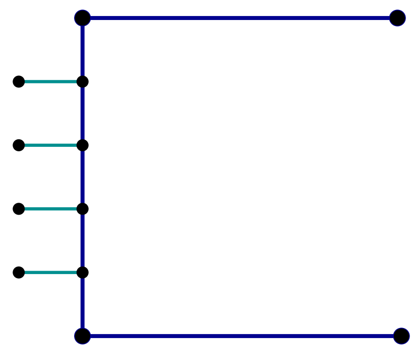
Idea of Proof: reduce harmonic measure ratio by adding edges.



Vertical side has much larger harmonic measure from left.

Theorem: Every planar continuum is a limit of true trees.

Idea of Proof: reduce harmonic measure ratio by adding edges.



“Left” harmonic measure is reduced (roughly 3-to-1).

New edges are approximately balanced (universal constant).

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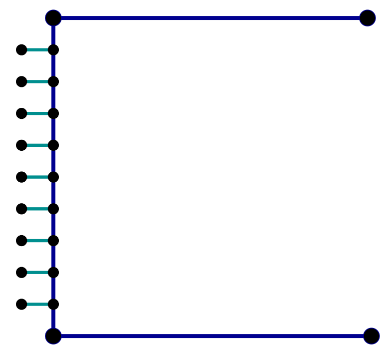


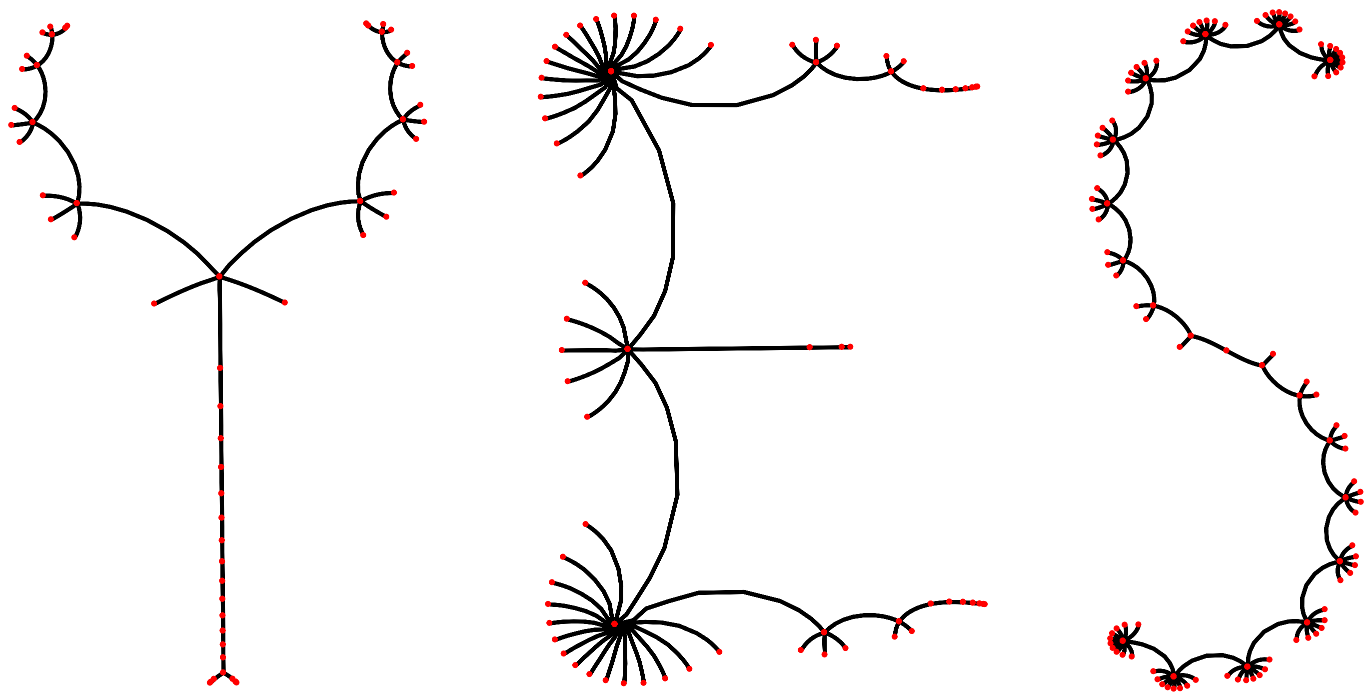
“Left” harmonic measure is reduced (roughly 3-to-1).

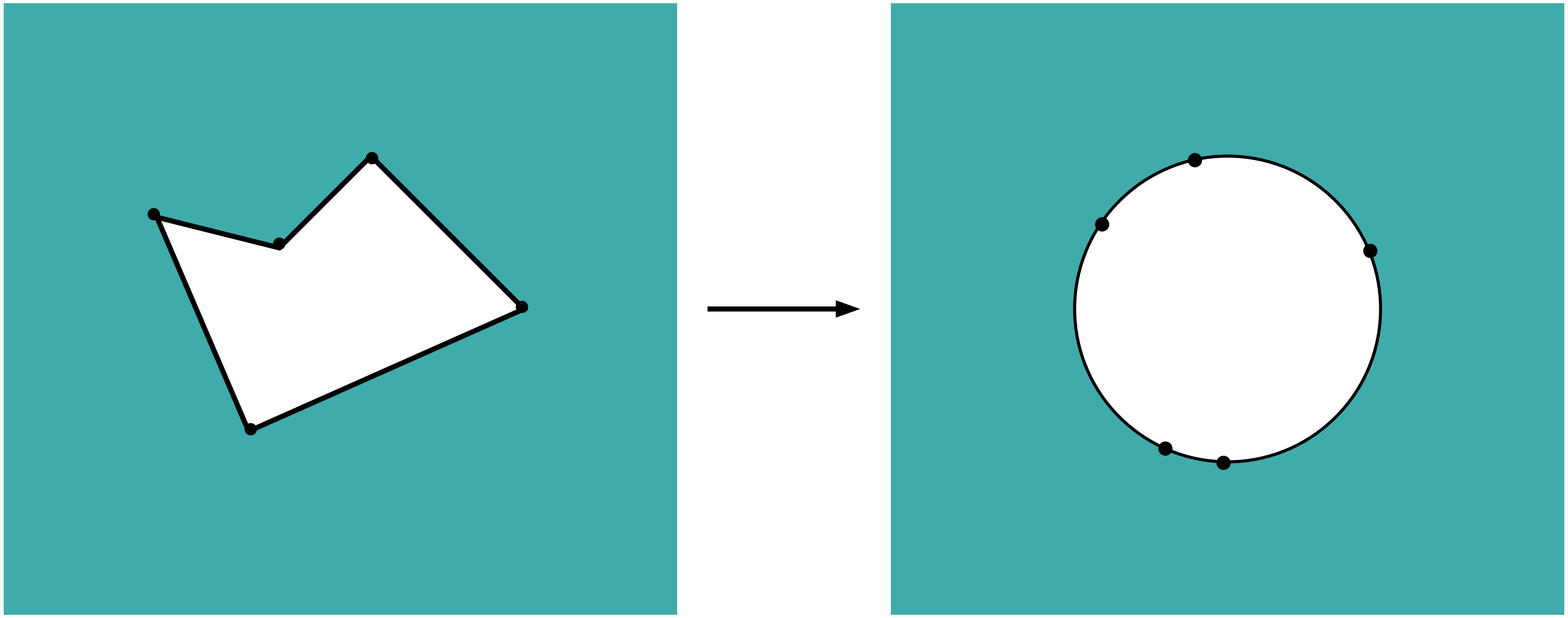
New edges are approximately balanced (universal constant).

Mapping theorem gives exactly balanced.

QC correction map is near identity if “spikes” are short.

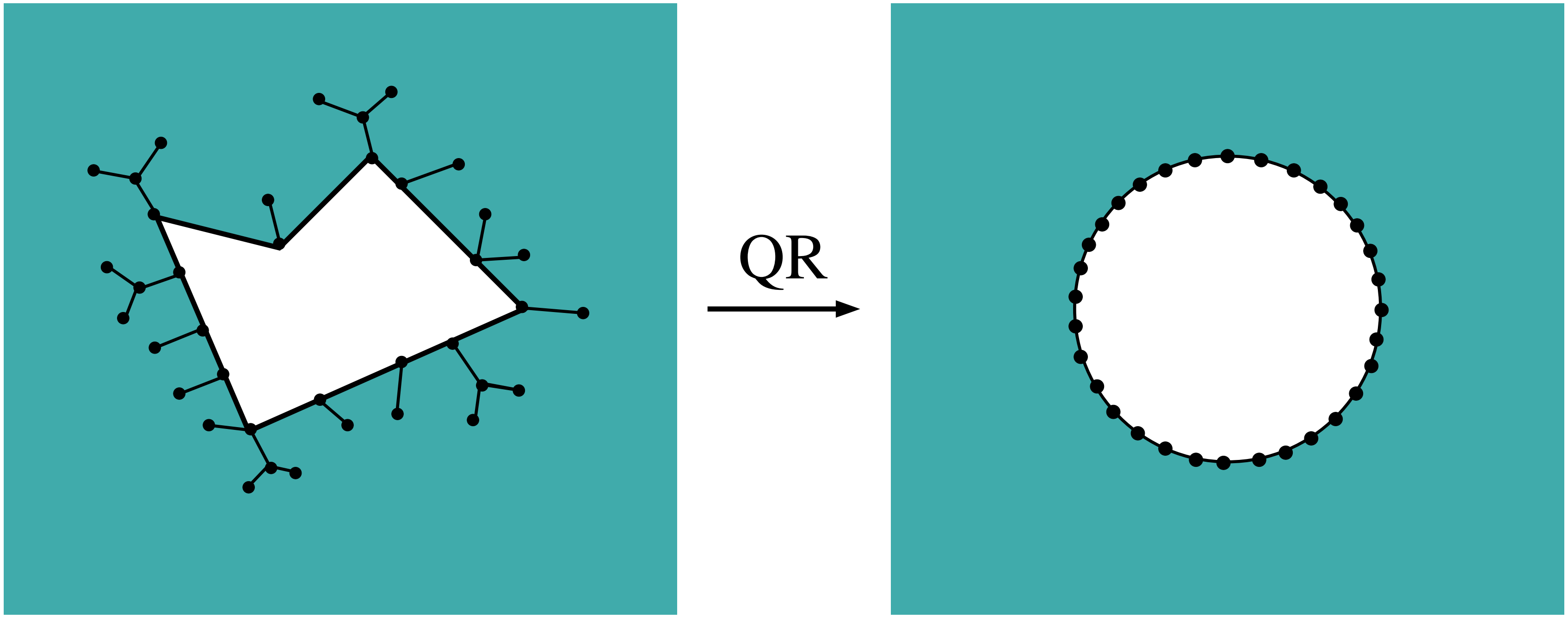
New tree approximates shape of old tree; different combinatorics.


YES



This works on other shapes besides trees.

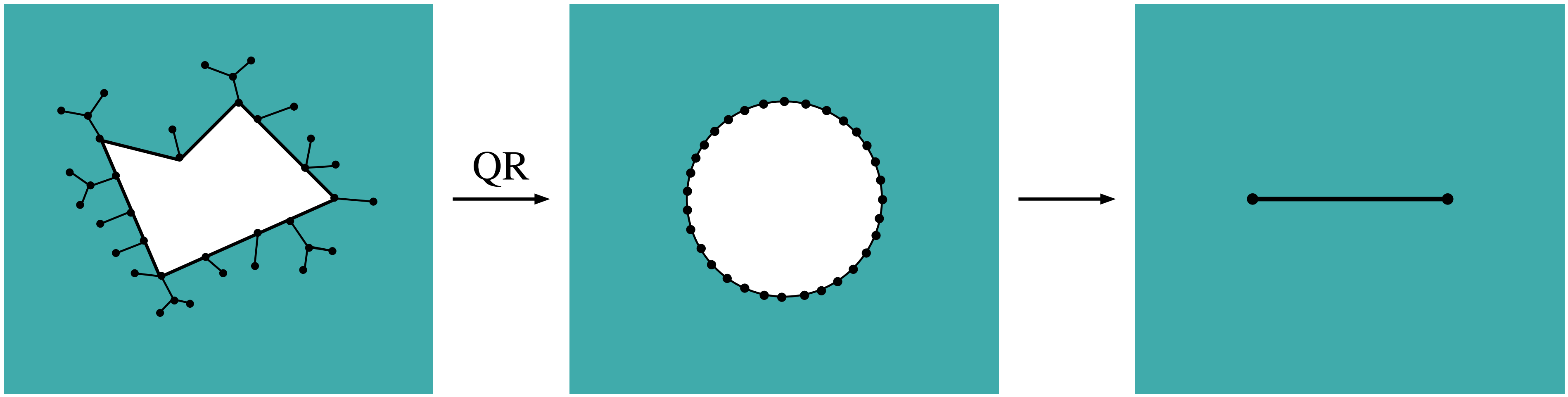
Suppose we are given “unbalanced polygon”.



We can add short branches to make it almost balanced.

Exterior is QR mapped to evenly divided circle.

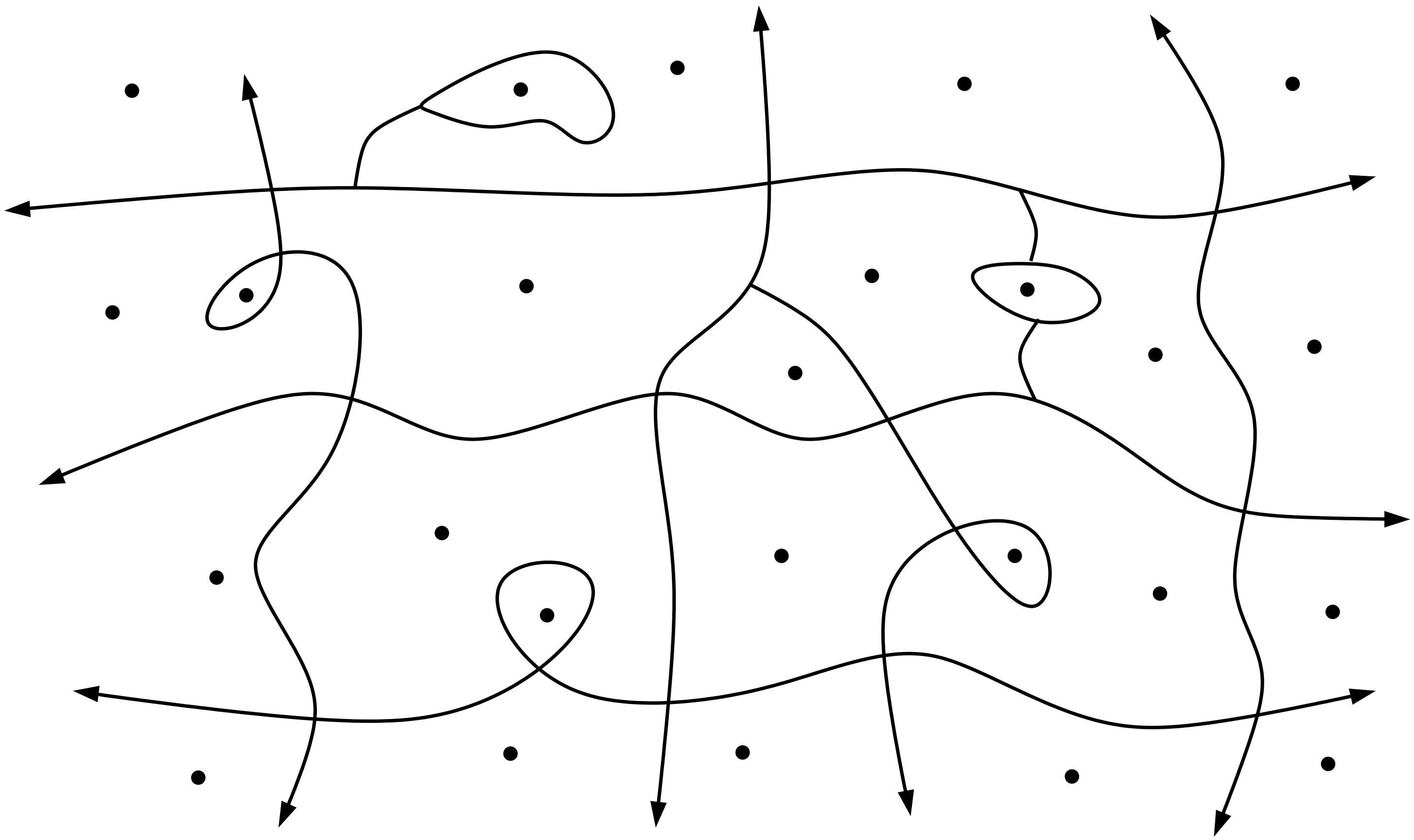
Again, QR distortion bounded, supported on tiny area.

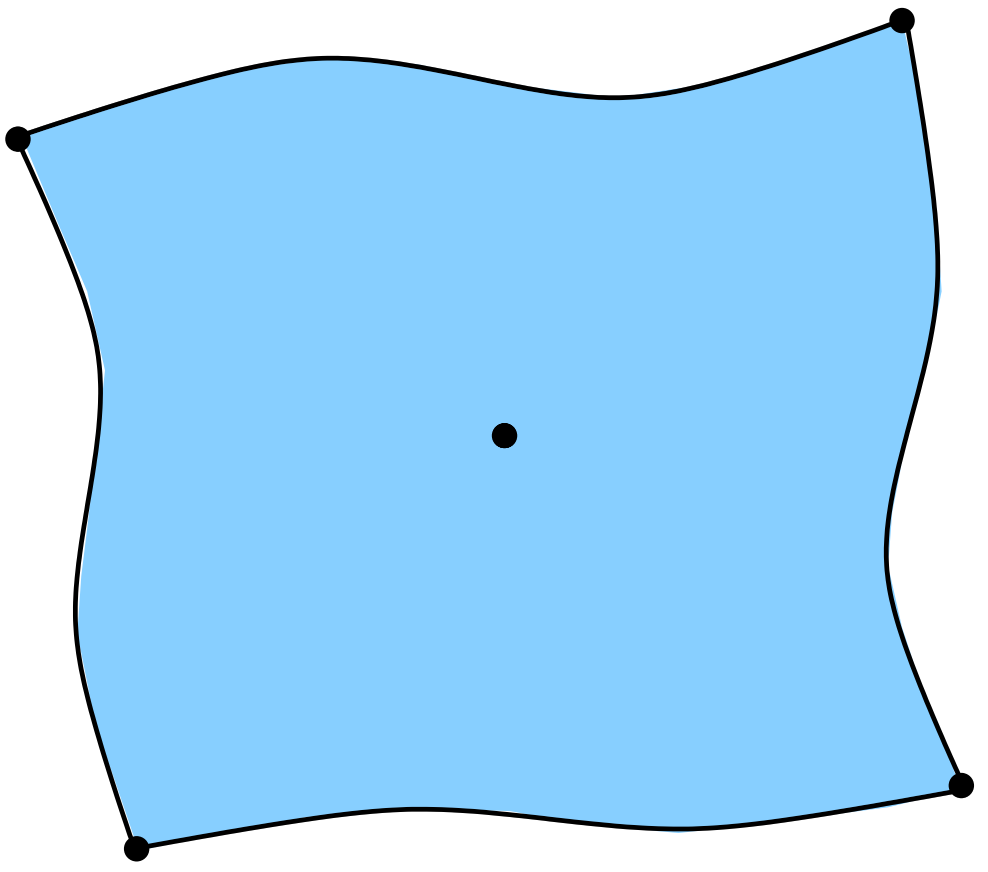


Important detail: QR map multiples arclength on each boundary segment.

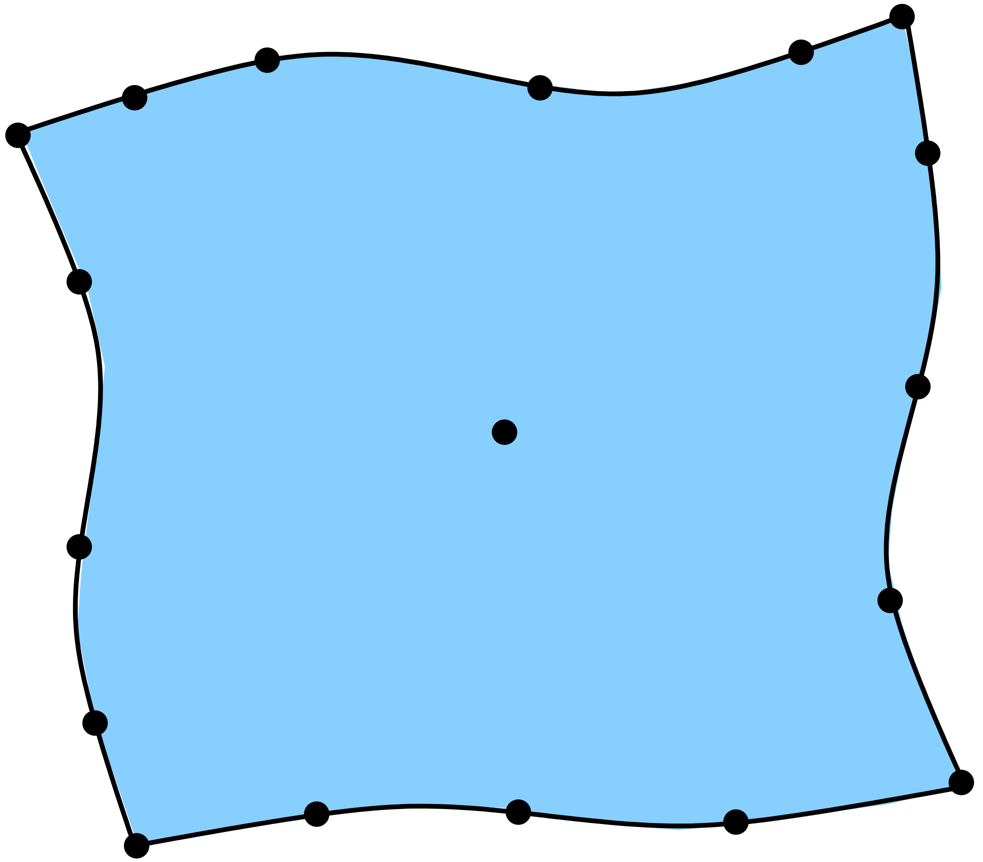
This allows us to "glue together" local maps to get a global QR map.

Can replace trees by meshes. Can replace plane by other surfaces. Get rational or meromorphic functions.

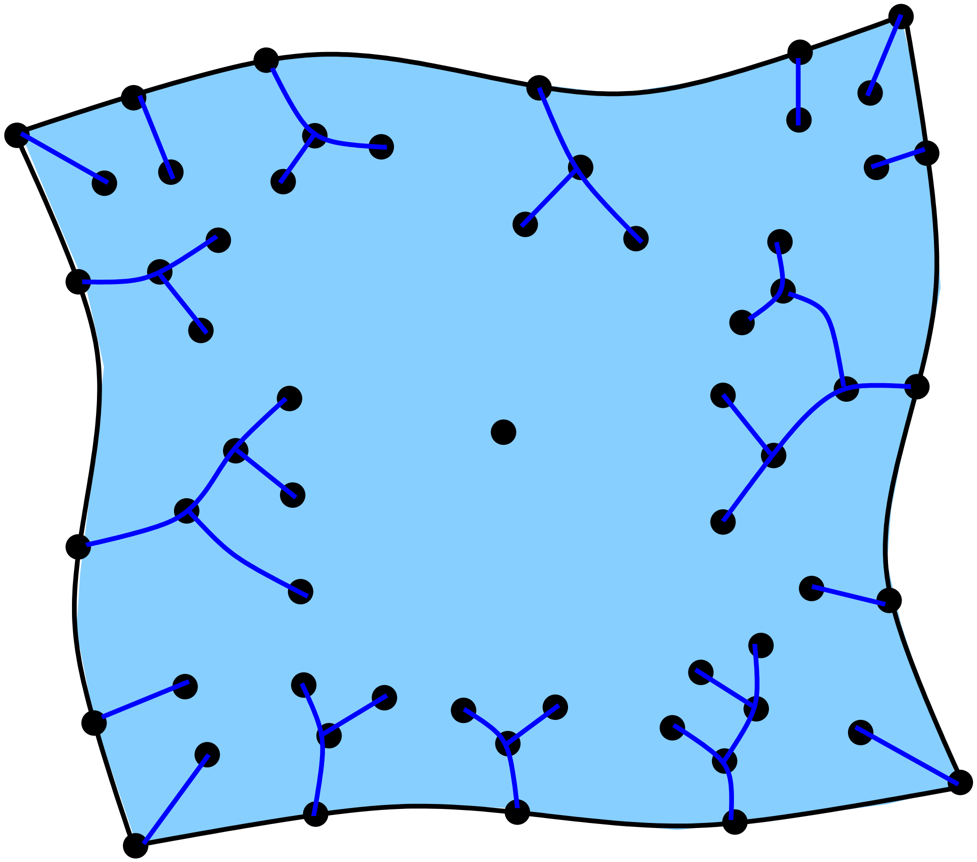




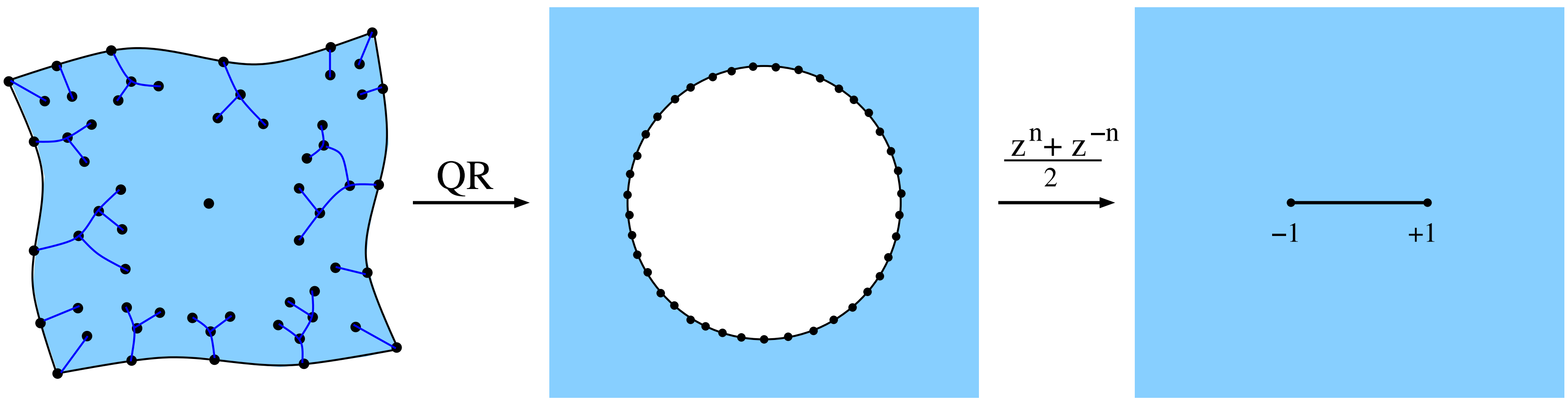
Using inversion we can work in a bounded domain.
Start with a piecewise smooth domain.



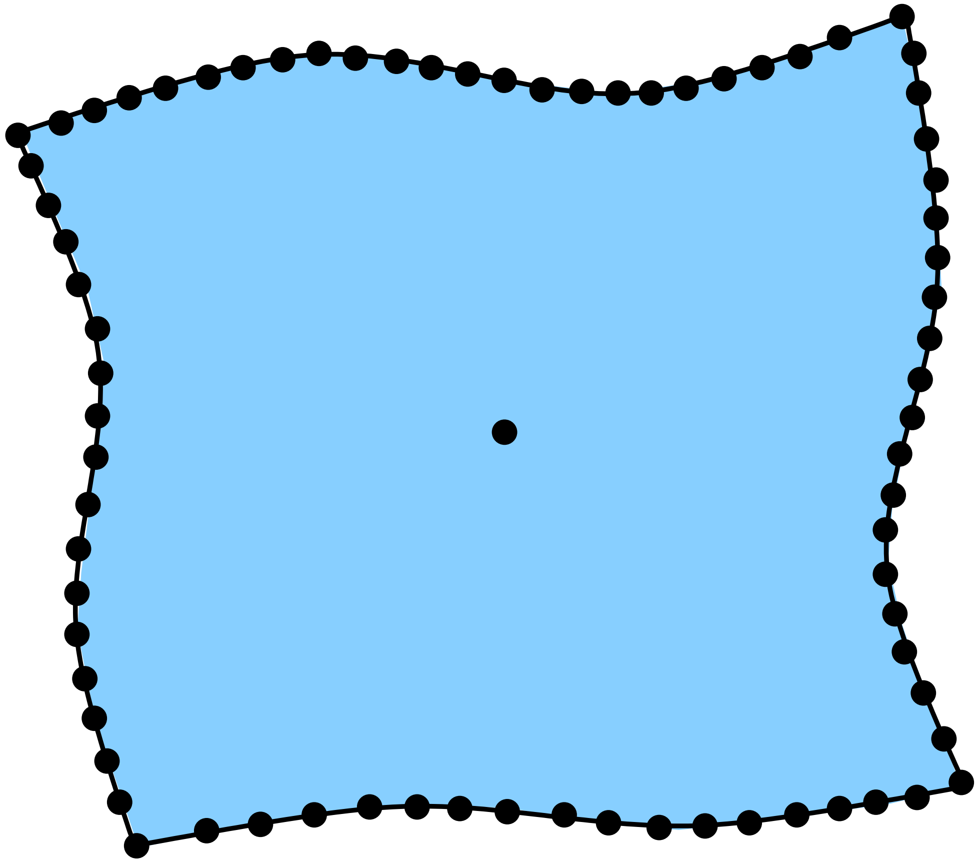
Add boundary points. Roughly evenly spaced.
Add interior point. Harmonic measures not all the same.



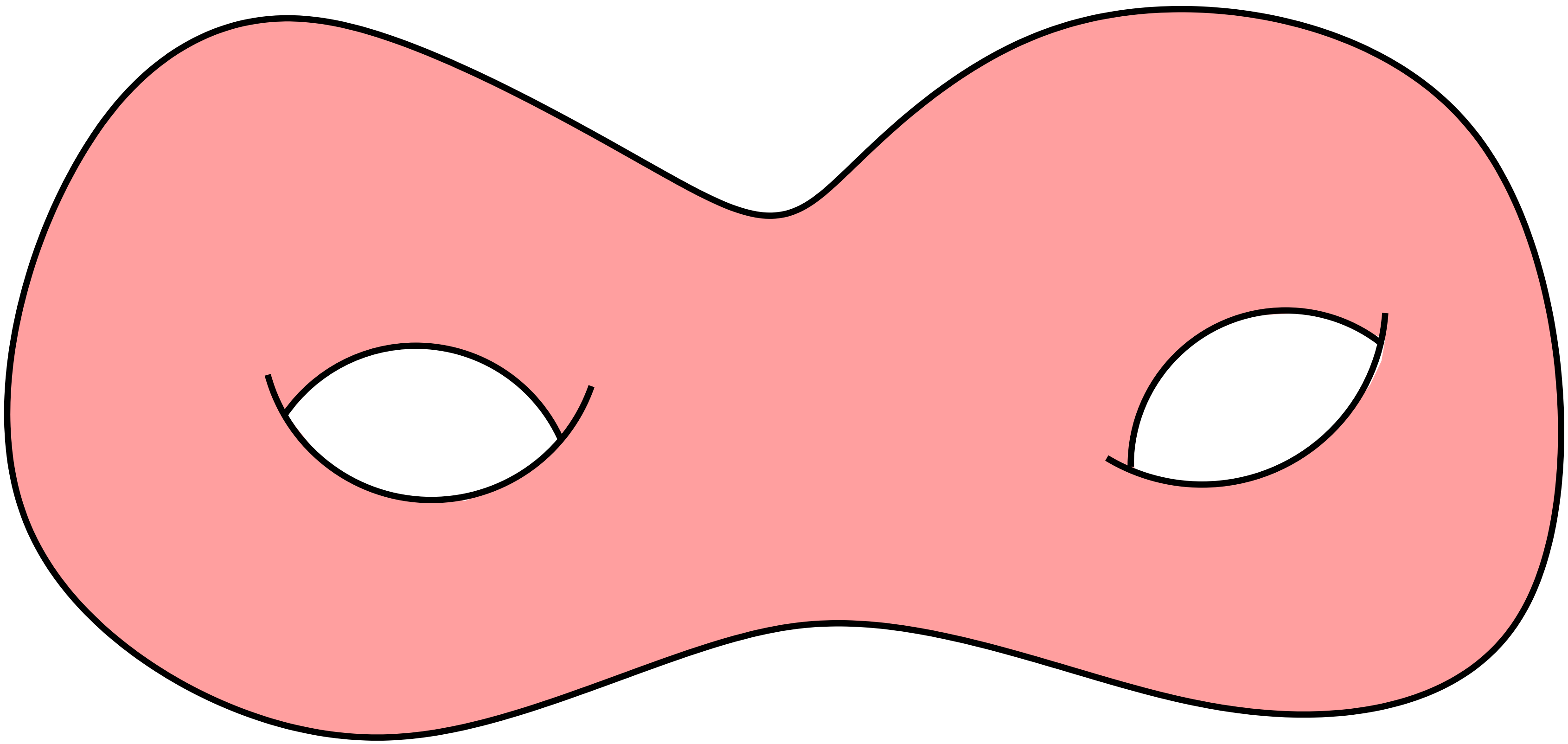
Add slits and trees so that all boundary arcs
have roughly the same harmonic measure.



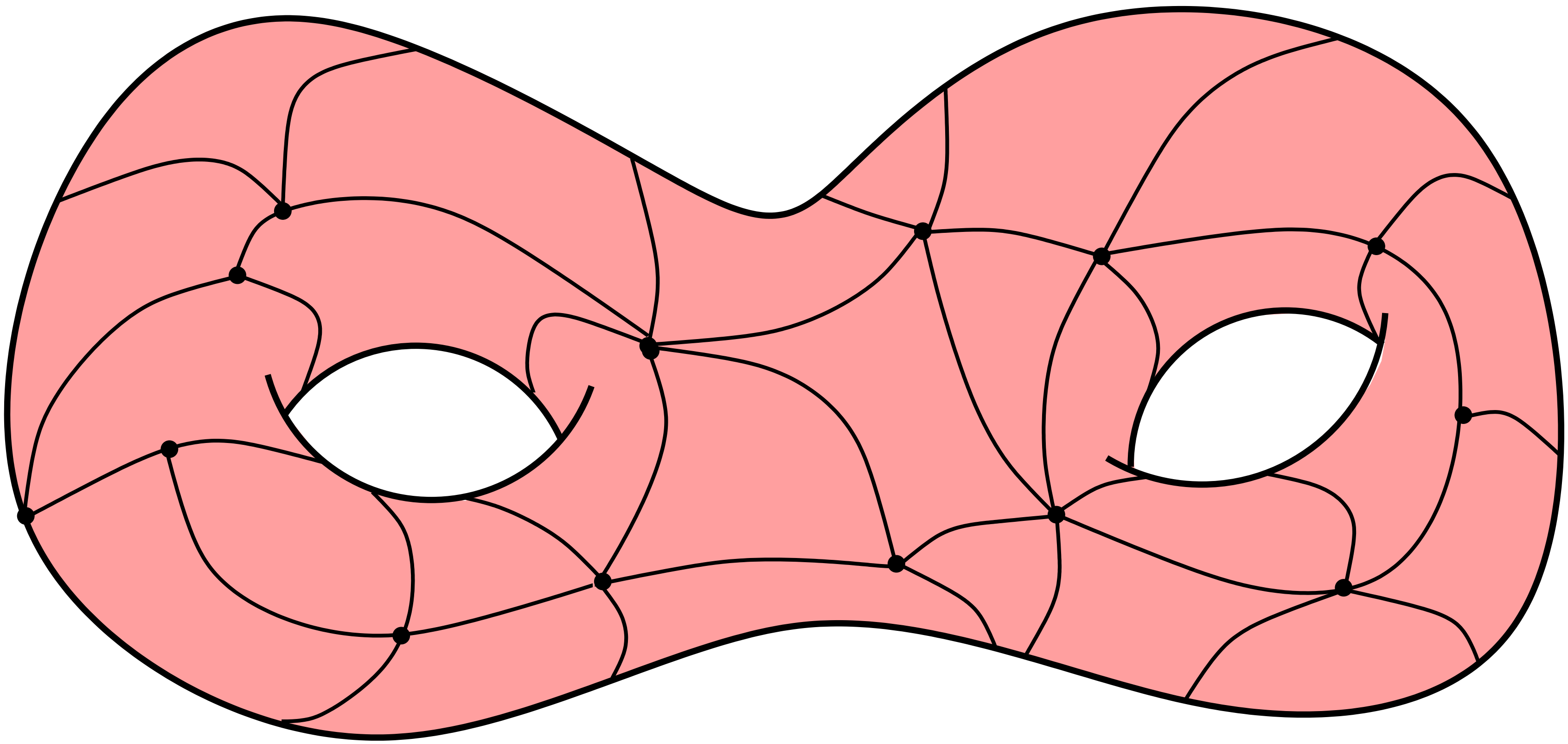
Find quasiregular map taking base point to ∞,
sending boundary arcs to equal intervals on circle.
Then each arc is mapped to $[-1, 1]$ via arclength.



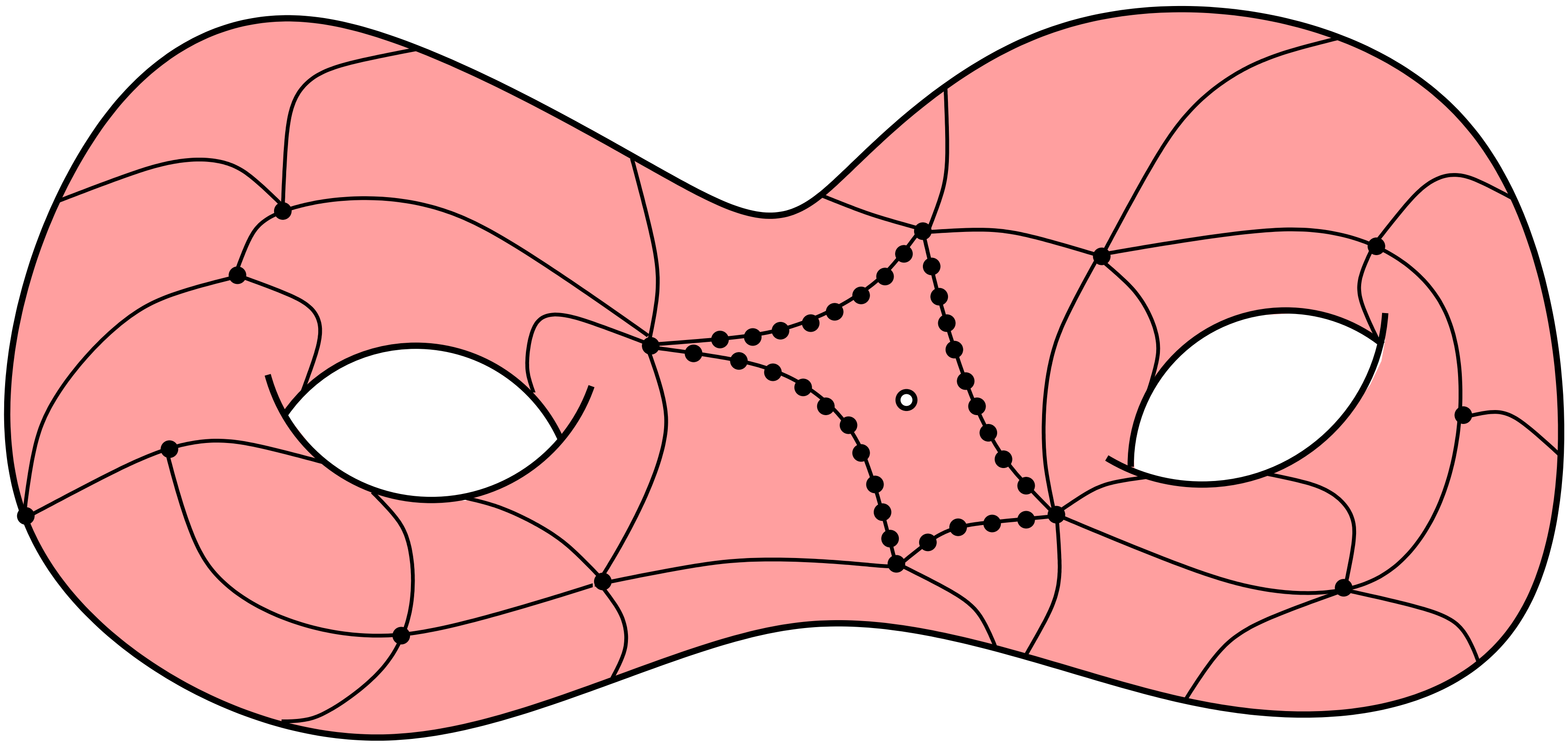
Repeat with more closely spaced points. Get QR map holomorphic except on thin neighborhood of boundary.



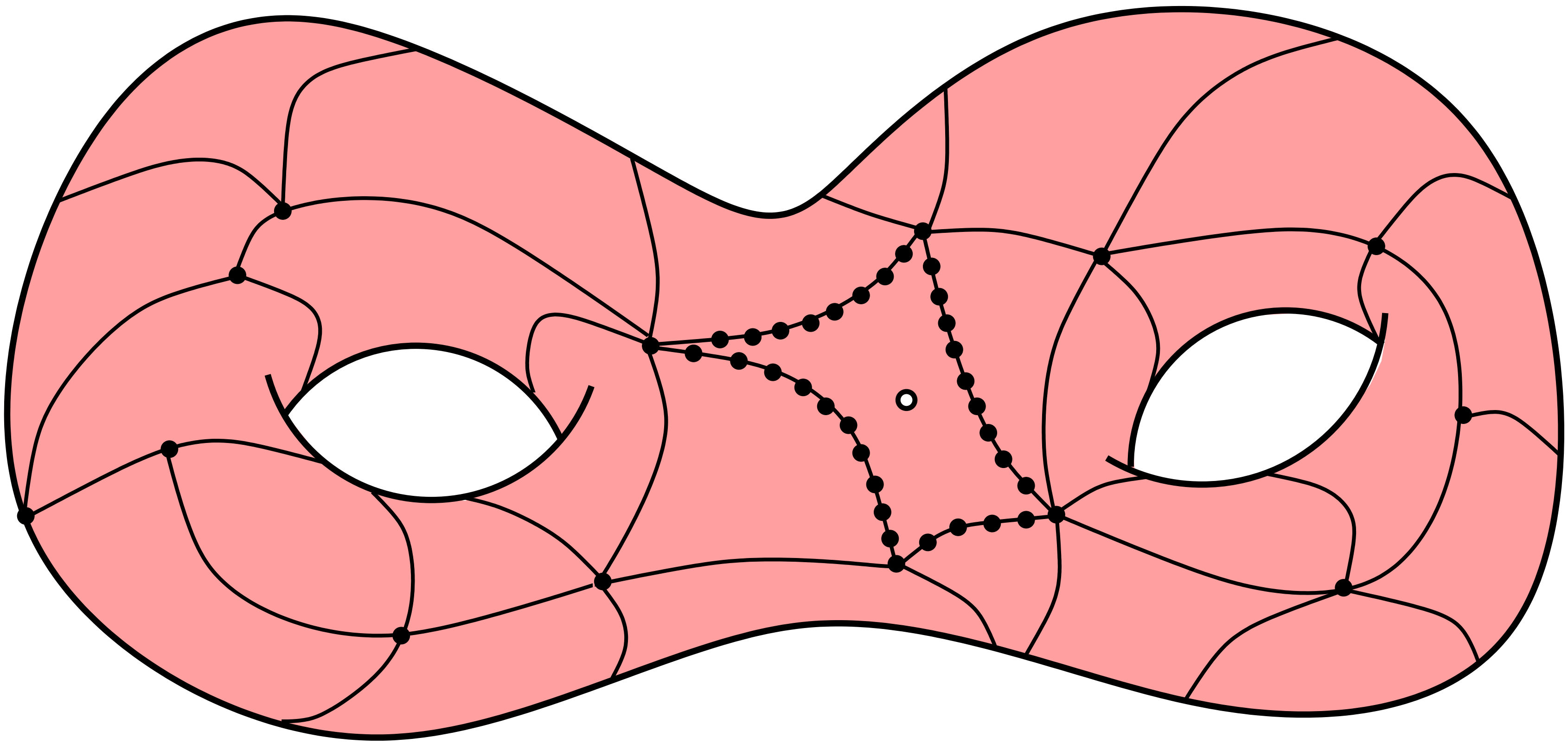
Now do this on a Riemann surface.



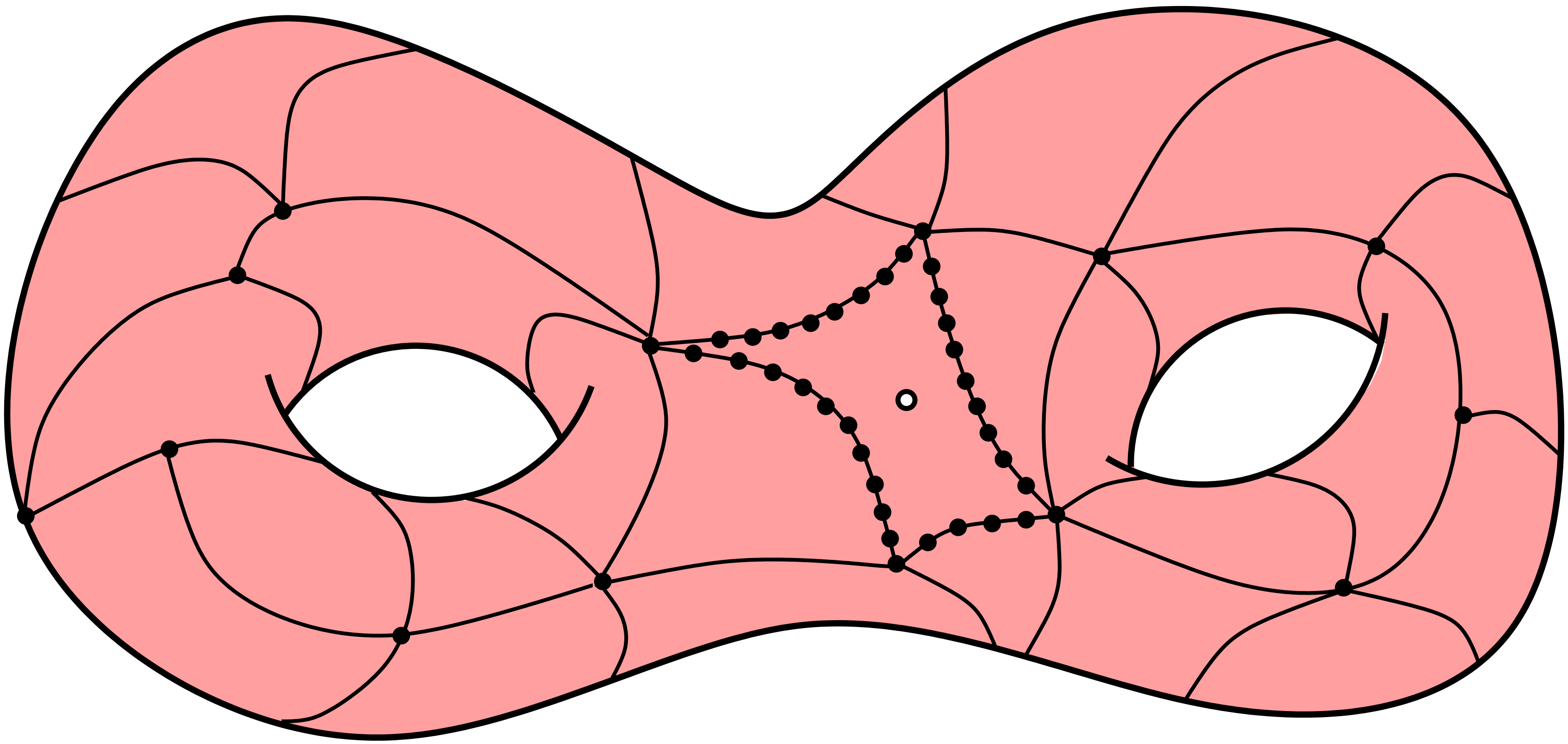
Cut the surface into nice pieces.



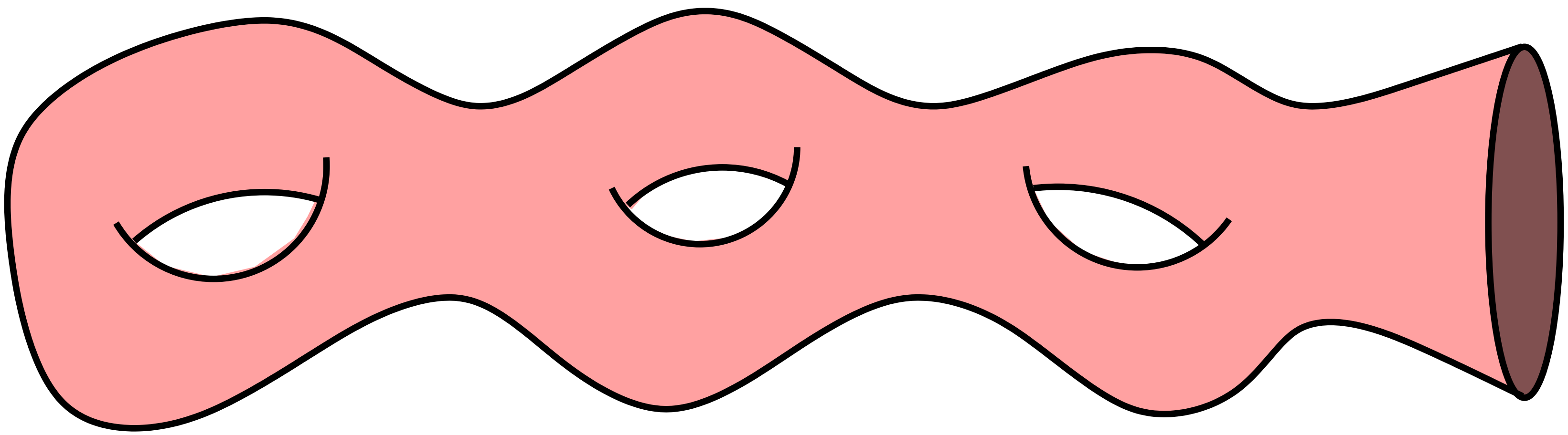
Subdivide edges into many small arcs.
Our construction gives QR maps on each piece.
Continuous across boundaries. QR on whole surface.



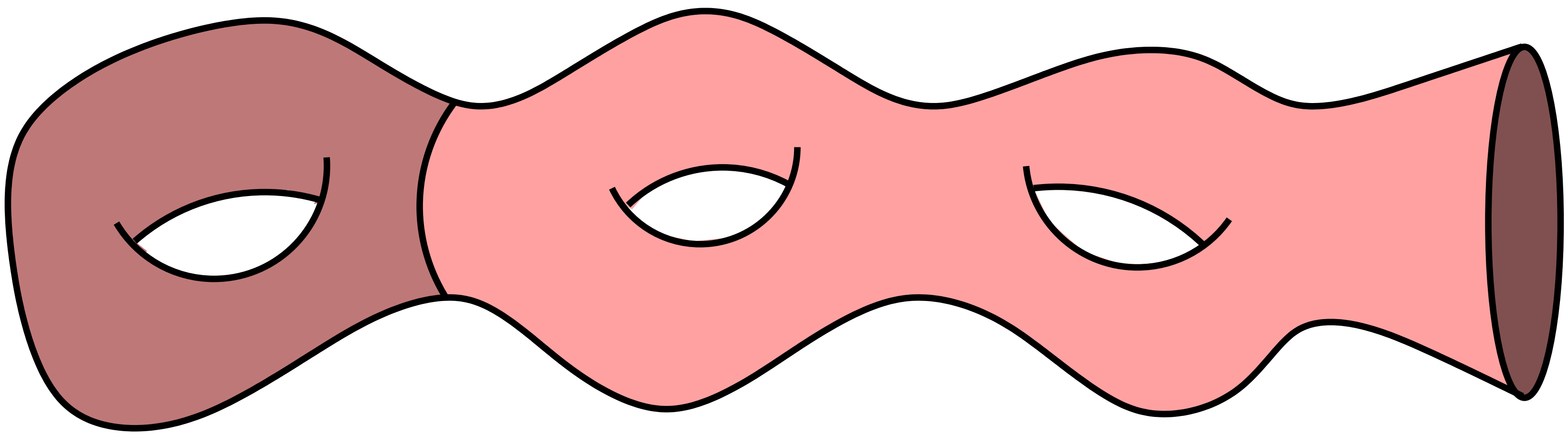
Change conformal structure to make QR map holomorphic.
Closely spaced dots $\Rightarrow$ small conformal change.
$\Rightarrow$ Belyi surfaces are dense in all surfaces.



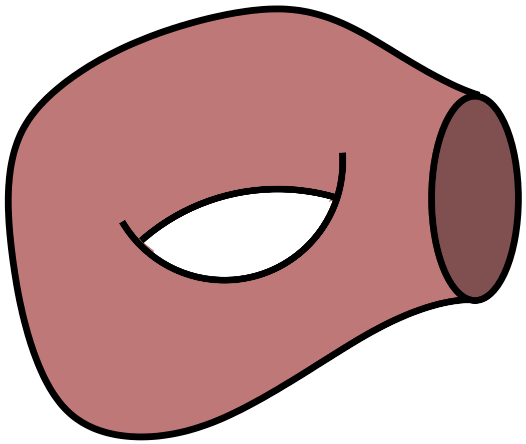
Every Riemann surface has a "QR Belyi" function.
Solving Beltrami gives countably, dense set of compact surfaces.
What about non-compact surfaces? .



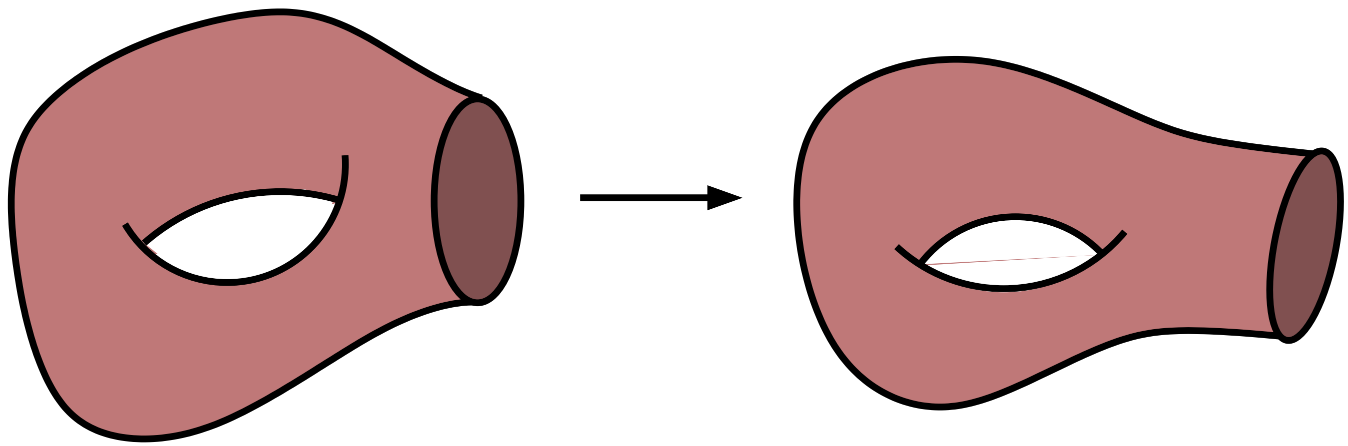
Consider a non-compact surface R.



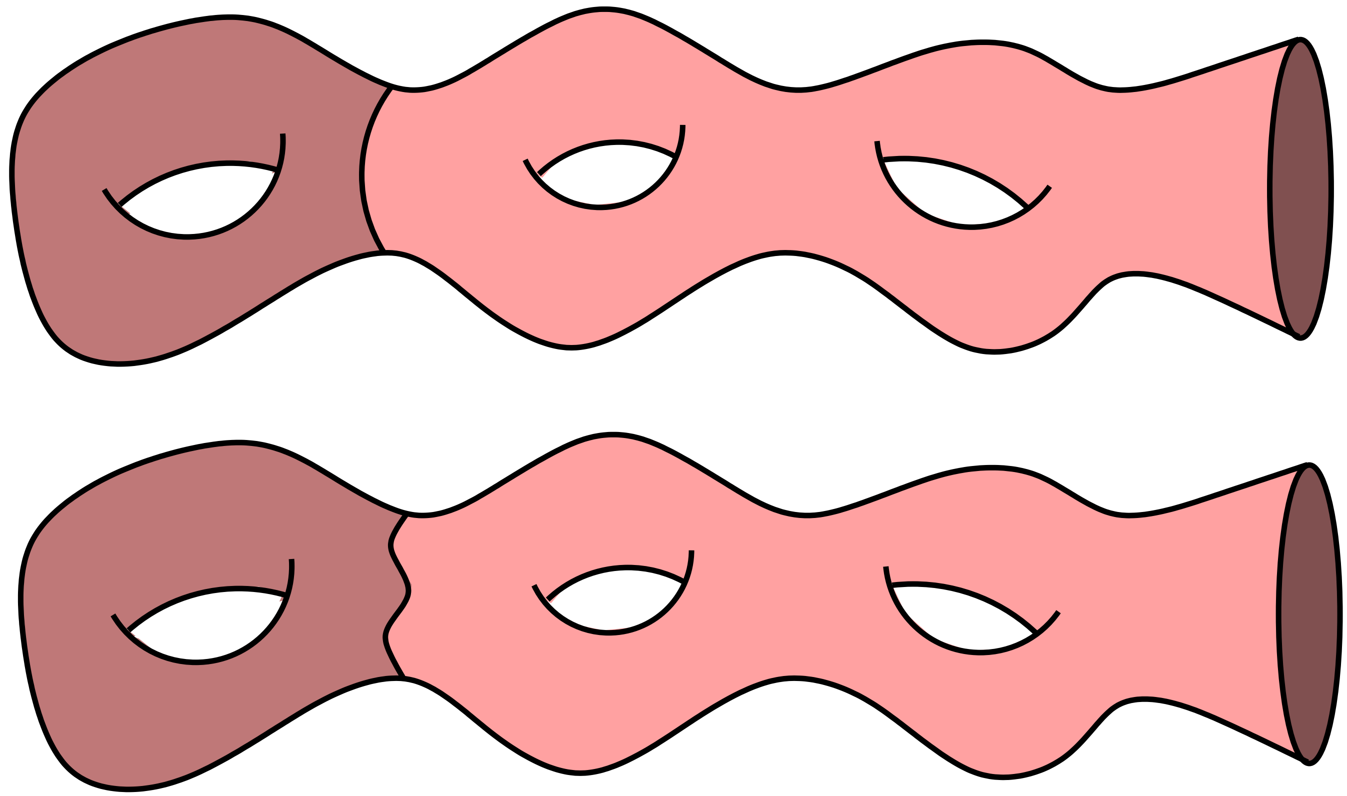
Take a compact, smoothly bordered piece Y_1.



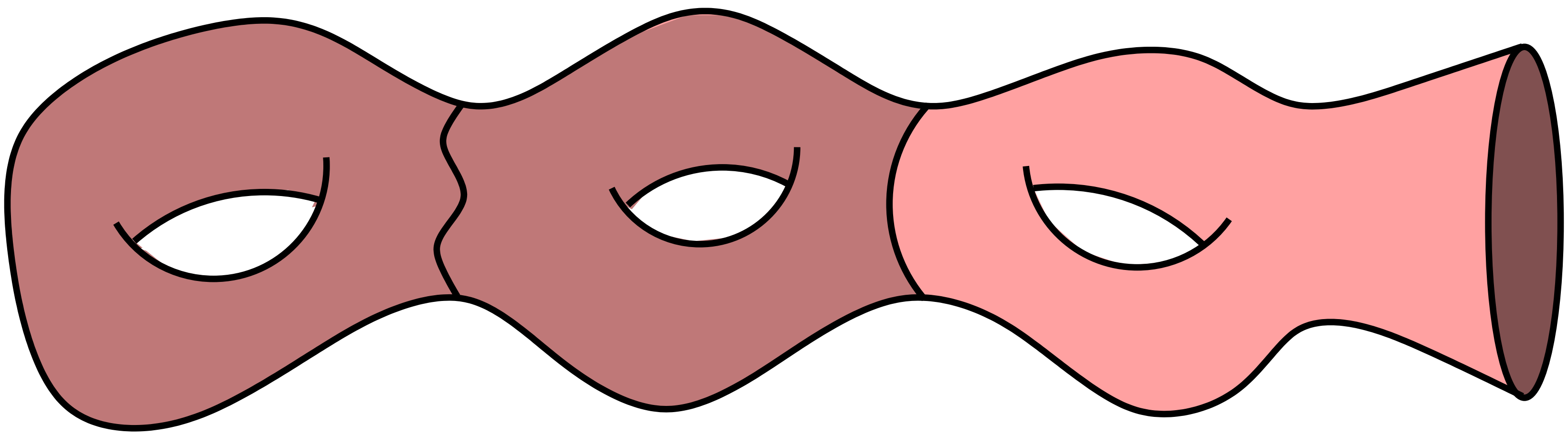
Take a compact, smoothly bordered piece Y_1.
Build a QR covering map on this piece as before.



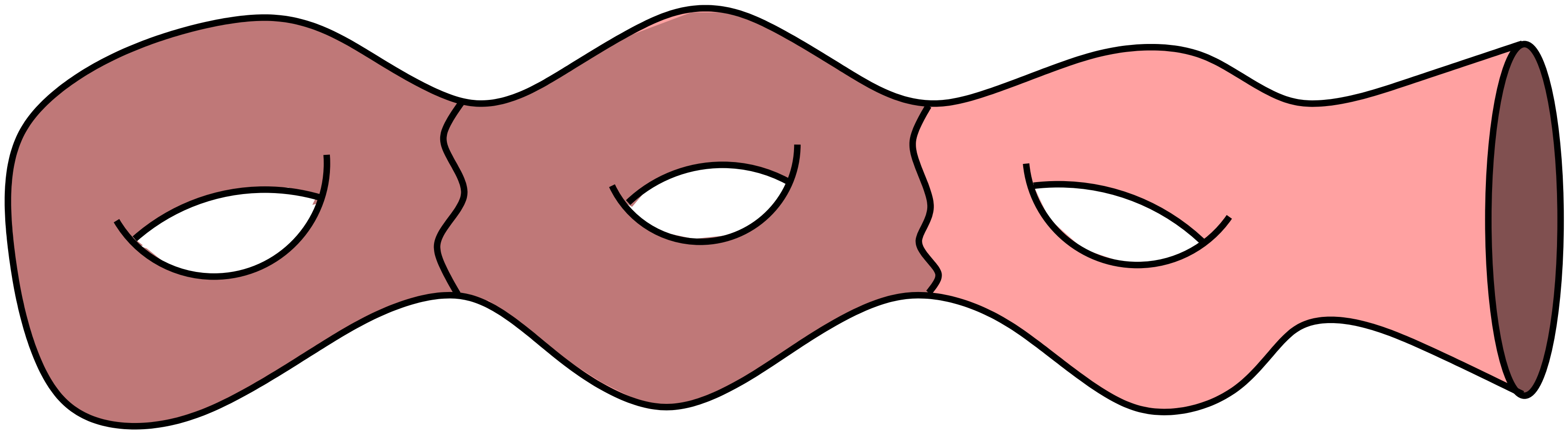
Take a compact, smoothly bordered piece Y_1.
Build a QR covering map on this piece as before.
Solve Beltrami on the compact piece. Get new surface $\widetilde{Y}_1 \neq Y_1$.



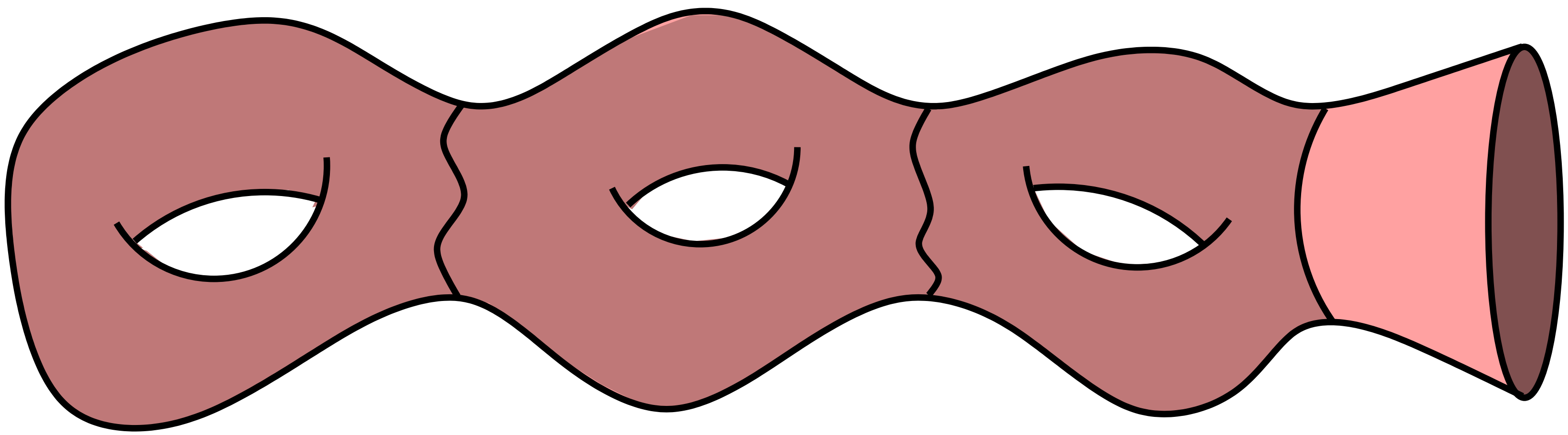
Key idea: small dilatation implies $\widetilde{Y}_1$ embeds in R.



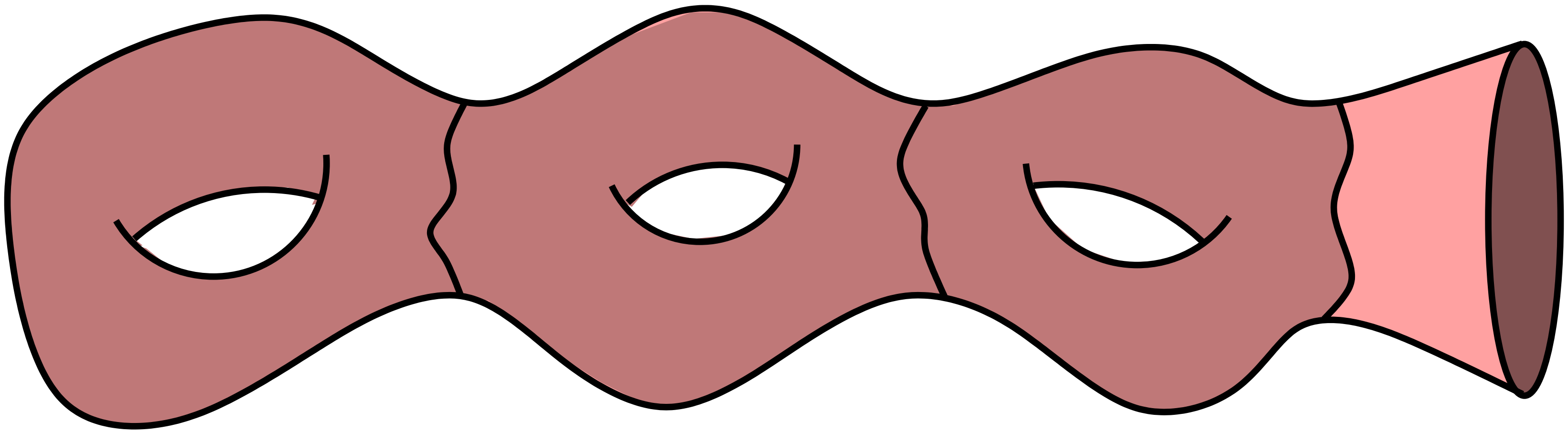
Take a larger compact, bordered piece Y_2.



Construct QR, solve Beltrami, embed $\widetilde{Y}_2$ in R.



Induction: Construct, Beltrami, re-embed in R,...



Induction: Construct, Beltrami, re-embed in R,...
Eventually build Belyi function on R.

Thm (B-Rempe): Every non-compact surface has a Belyi function.

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Corollary: Every non-compact surface can be obtained by gluing together equilateral triangles.

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Corollary: Every non-compact surface can be obtained by gluing together equilateral triangles.

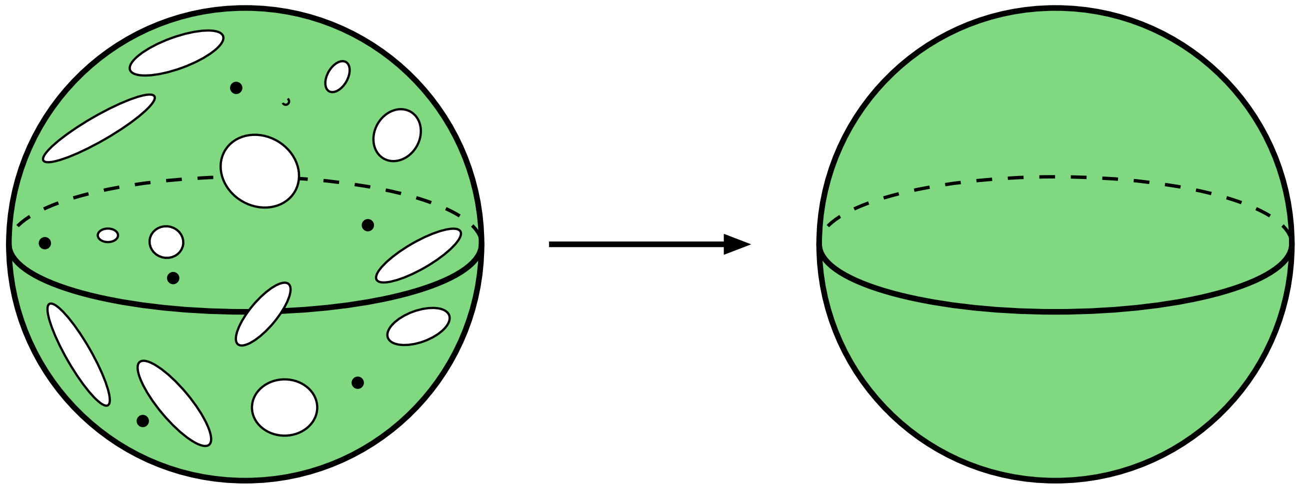
Corollary: Every Riemann surface is a branched cover of the sphere, branched over finitely many points.

For compact surfaces, this is Riemann-Roch.

Compact, genus g sometimes needs $3g$ branch points.

3 branch points suffice for all non-compact surfaces.

Corollary: Any open $U \subset \mathbb{C}$ is 3-branched cover of the sphere.



A. Epstein: a holomorphic map $X \to Y$ is finite type if

- Y is compact,
- f is open,
- f has no isolated removable singularities,
- the set of singular values is finite.

Gives many new dynamical systems of finite type.

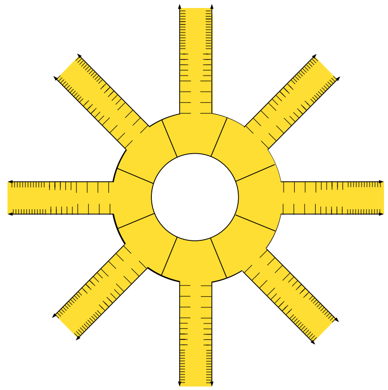
E.g., escaping points have more than one point they can escape to.

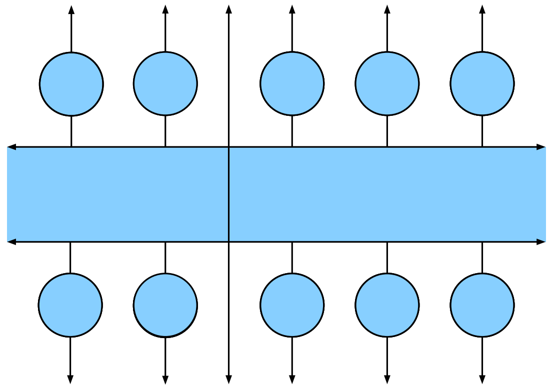
Connection between finite planar trees and polynomials can be extended to infinite trees and entire functions.

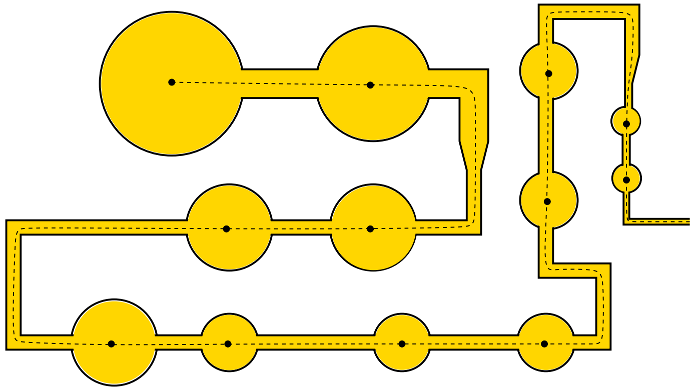
Folding theorem: given an infinite planar tree T with mild geometric assumptions, there is an entire function f with two singular values so that $f^{-1}([-1,1])$ approximates T.

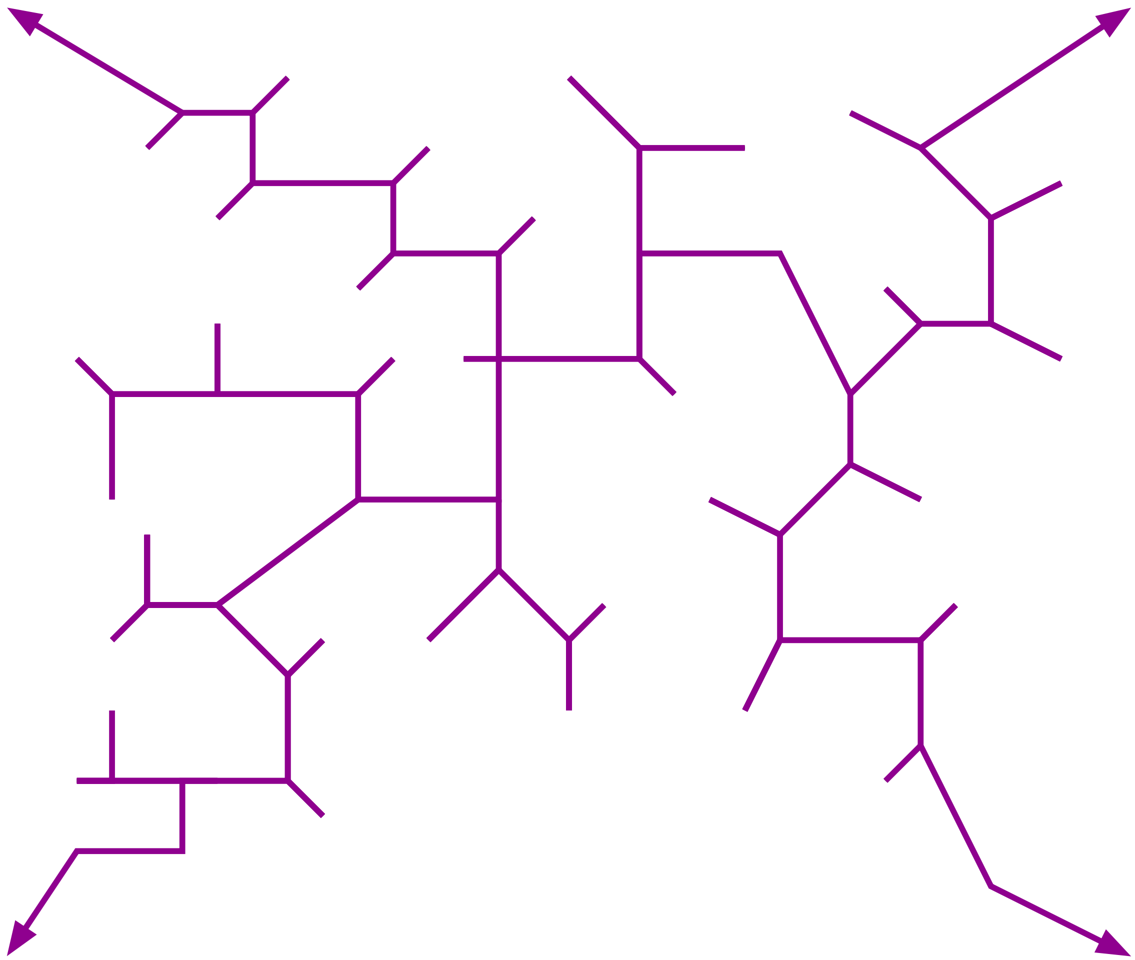
This has various applications, e.g.,

- existence of certain "small" Julia sets,
- new wandering domains for certain entire functions,
- meromorphic functions with prescribed postcritical dynamics,



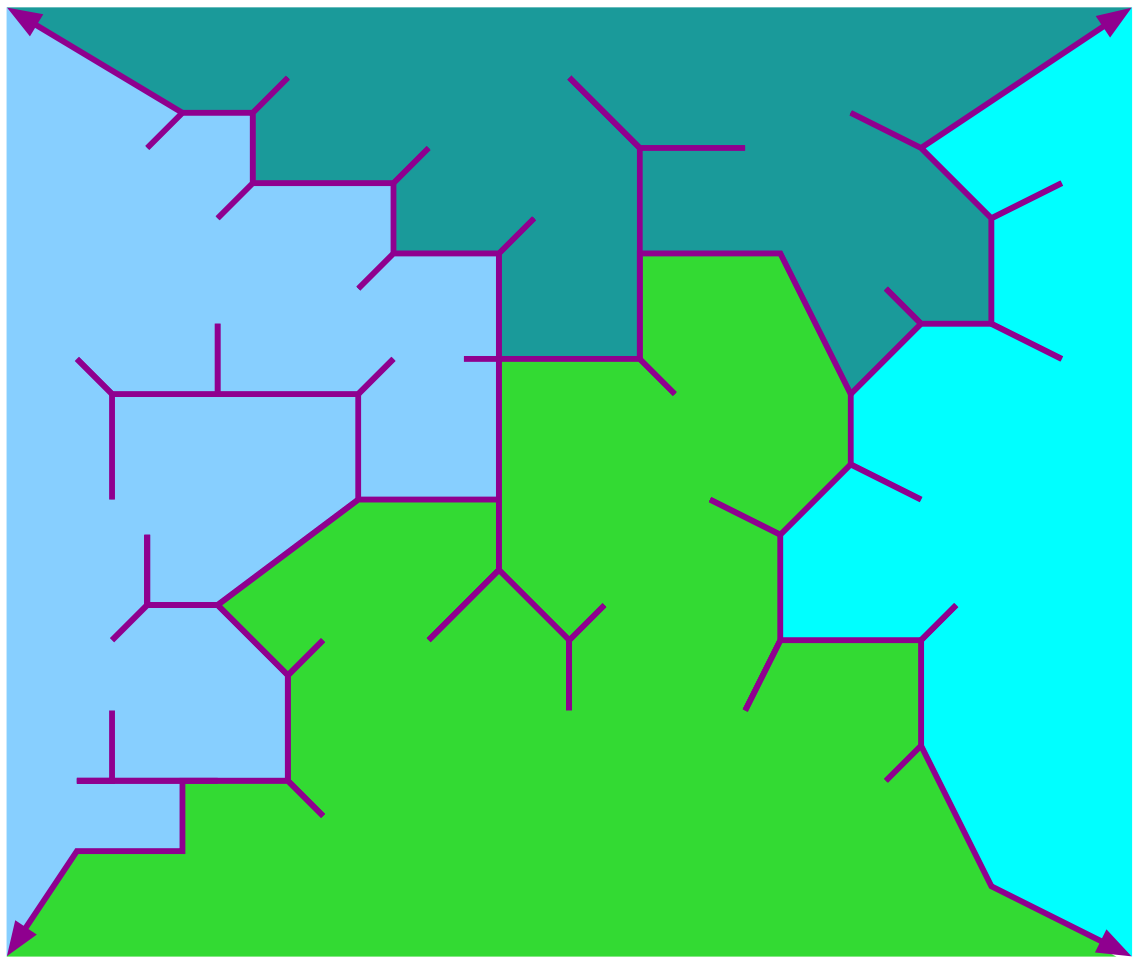






Finite trees correspond to polynomials with 2 critical values.

Do infinite trees correspond to entire functions with 2 critical values?

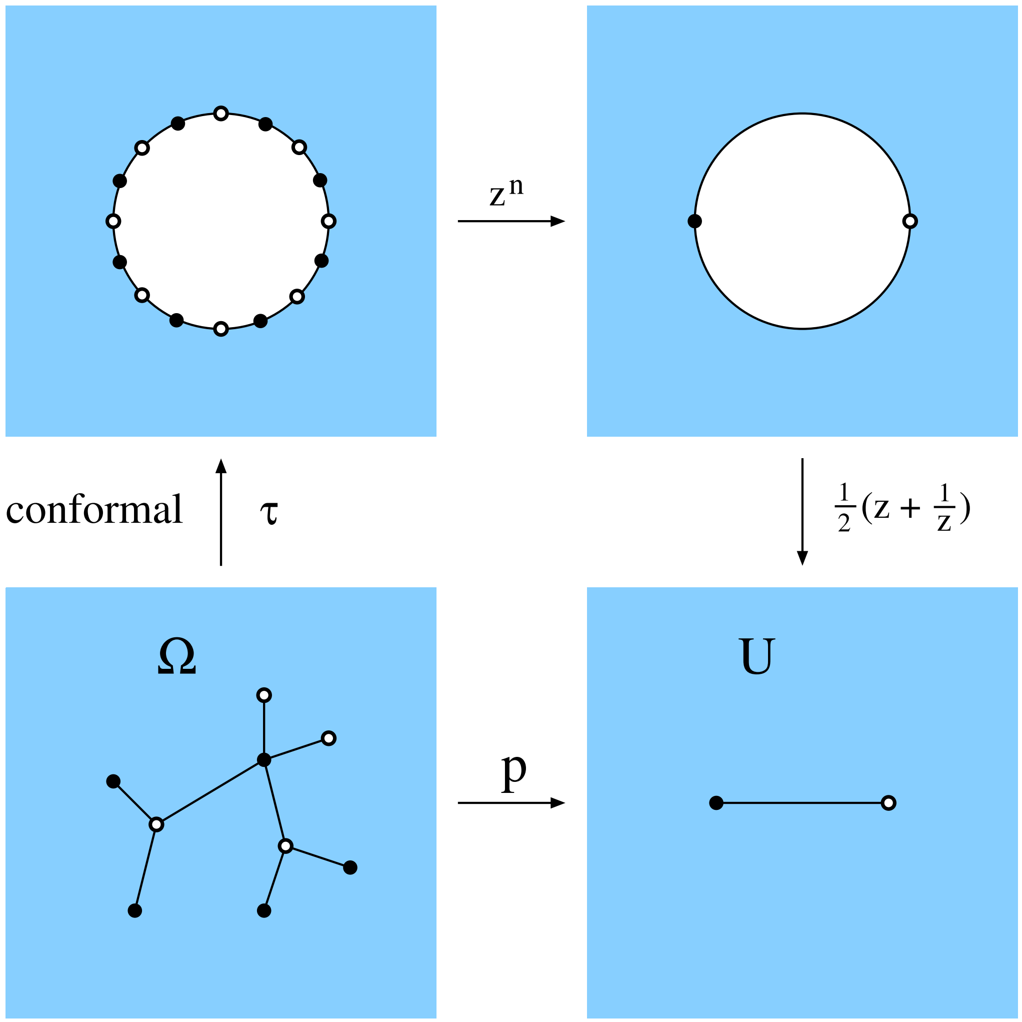


Main difference:

$\mathbb{C}\backslash$ finite tree = one topological annulus

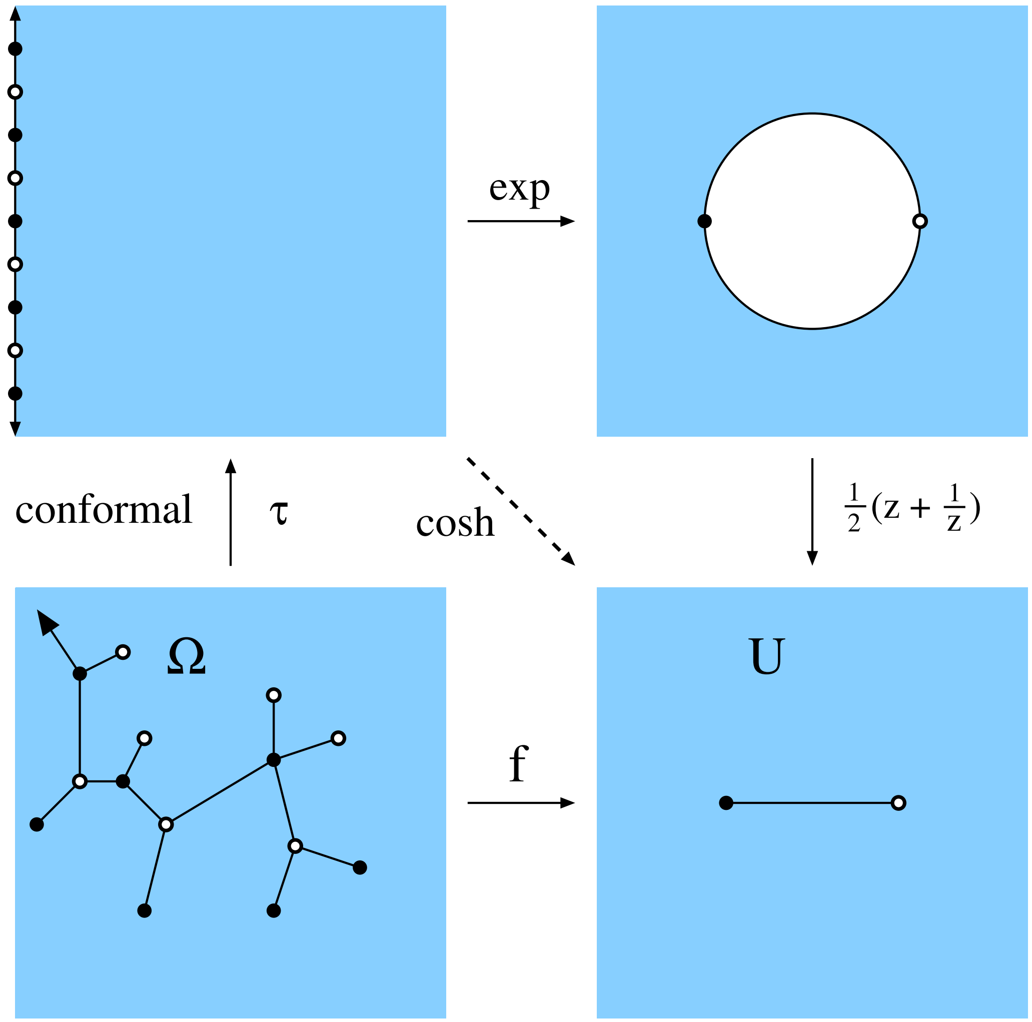
$\mathbb{C}\backslash$ infinite tree = many simply connected components

Recall finite case

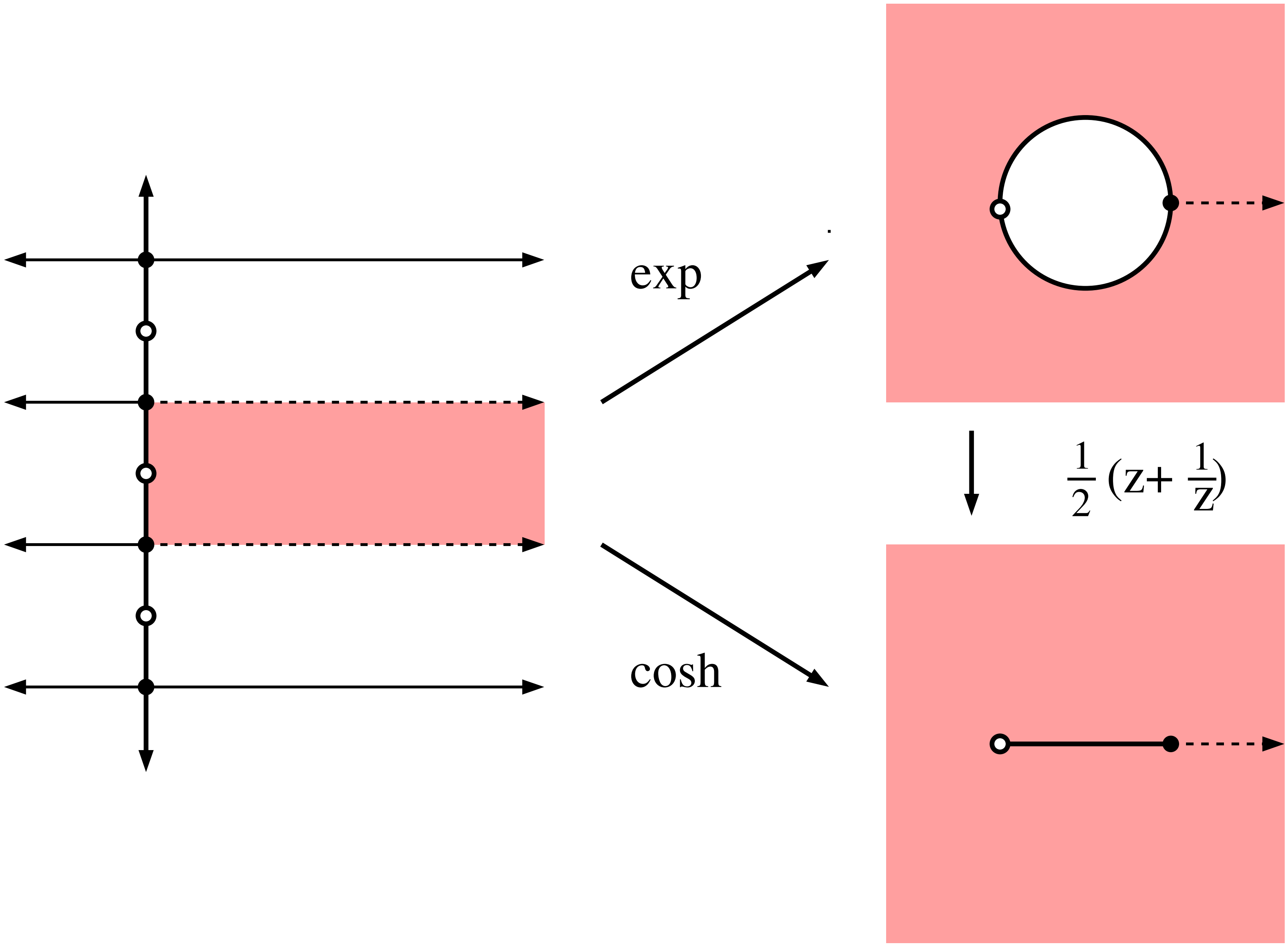


T is true tree $\Leftrightarrow p = \frac{1}{2}(\tau^n + 1/\tau^n)$ is continuous across T.

Infinite case



Infinite balanced tree $\Leftrightarrow f = \cosh \circ \tau$ is continuous across T.



Definitions of exp and cosh.

We need three definitions before stating our result.

These adapt “obvious” properties of finite trees to infinite case.

(1) Tree neighborhoods: replaces Hausdorff metric ϵ-neighborhoods.

(2) Bounded geometry: nearby edges have comparable sizes.

(3) τ-lower bound: lower bound for measure of edges.

If e is an edge of T and $r > 0$ let

$$e(r) = \{z : \mathrm{dist}(z, e) \leq r \cdot \mathrm{diam}(e)\}$$

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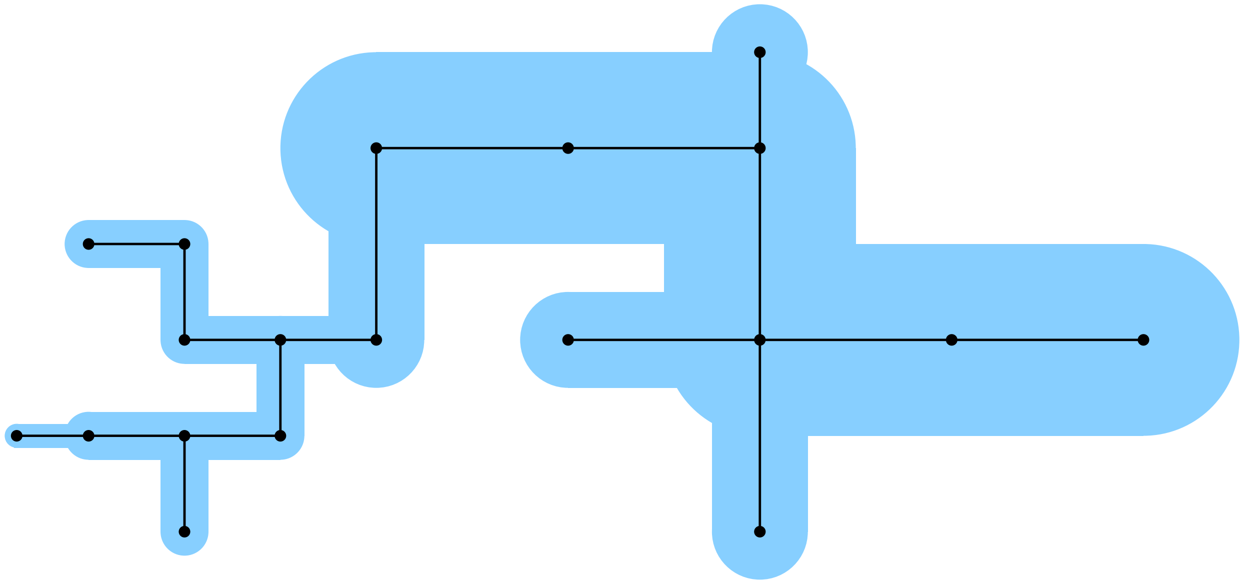
Define neighborhood of T: $T(r) = \cup\{e(r) : e \in T\}$.



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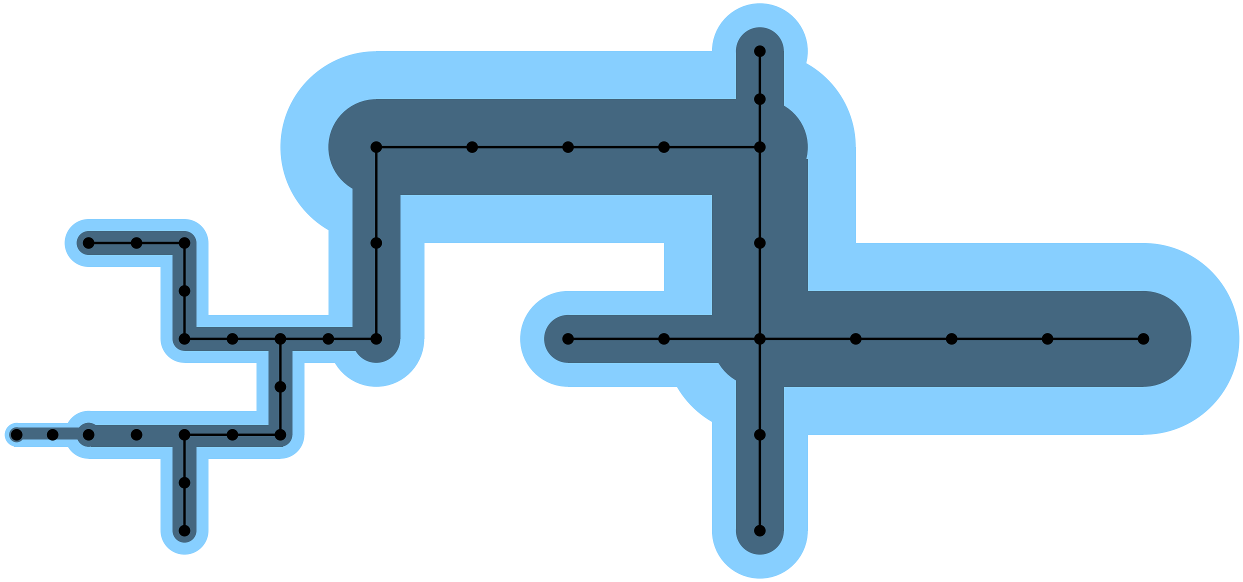
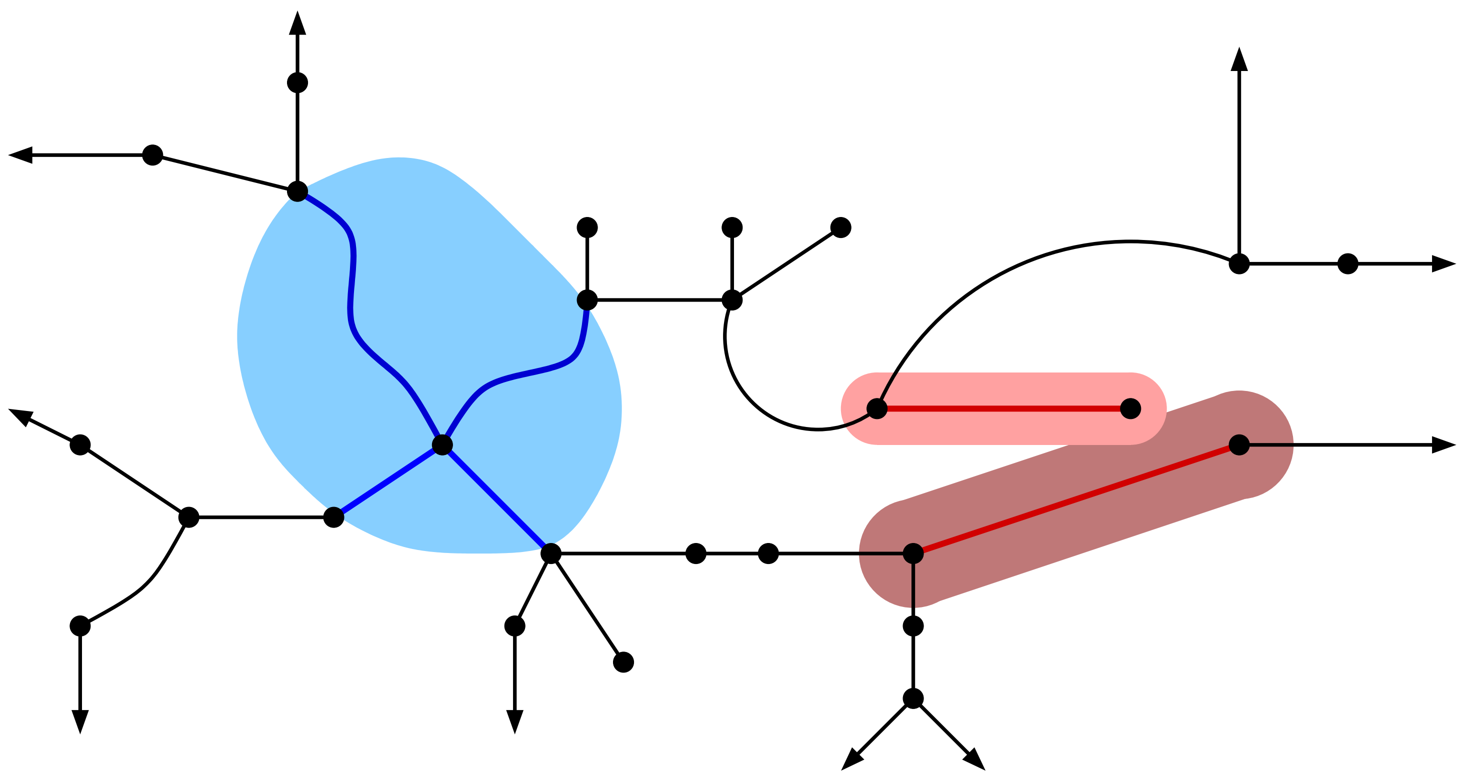


Adding vertices reduces $T(r)$. Useful scaling property.

Bounded Geometry (local condition; easy to verify):

- edges are uniformly smooth.
- adjacent edges form bi-Lipschitz image of a star $= \{z^n \in [0, r]\}$
- non-adjacent edges are well separated,

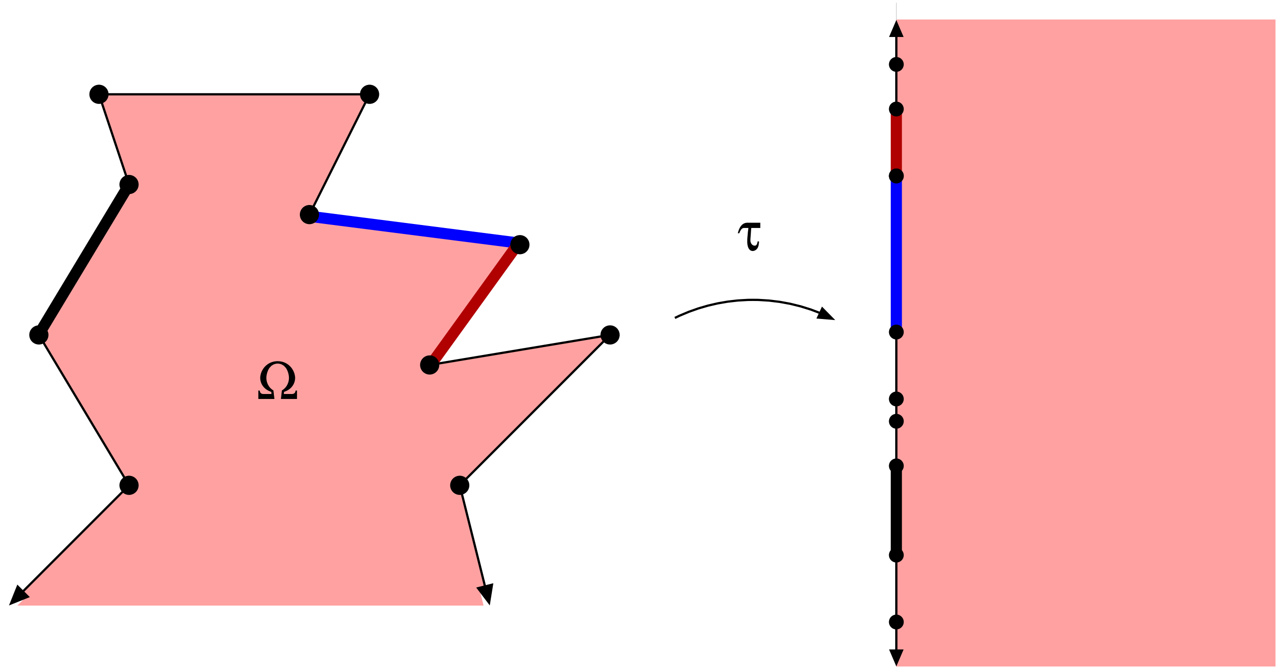
$$\mathrm{dist}(e, f) \geq \epsilon \cdot \min(\mathrm{diam}(e), \mathrm{diam}(f)).$$



τ-Lower Bound (global condition; harder to check):

Complementary components of tree are simply connected.

Each can be conformally mapped to right half-plane. Call map τ.

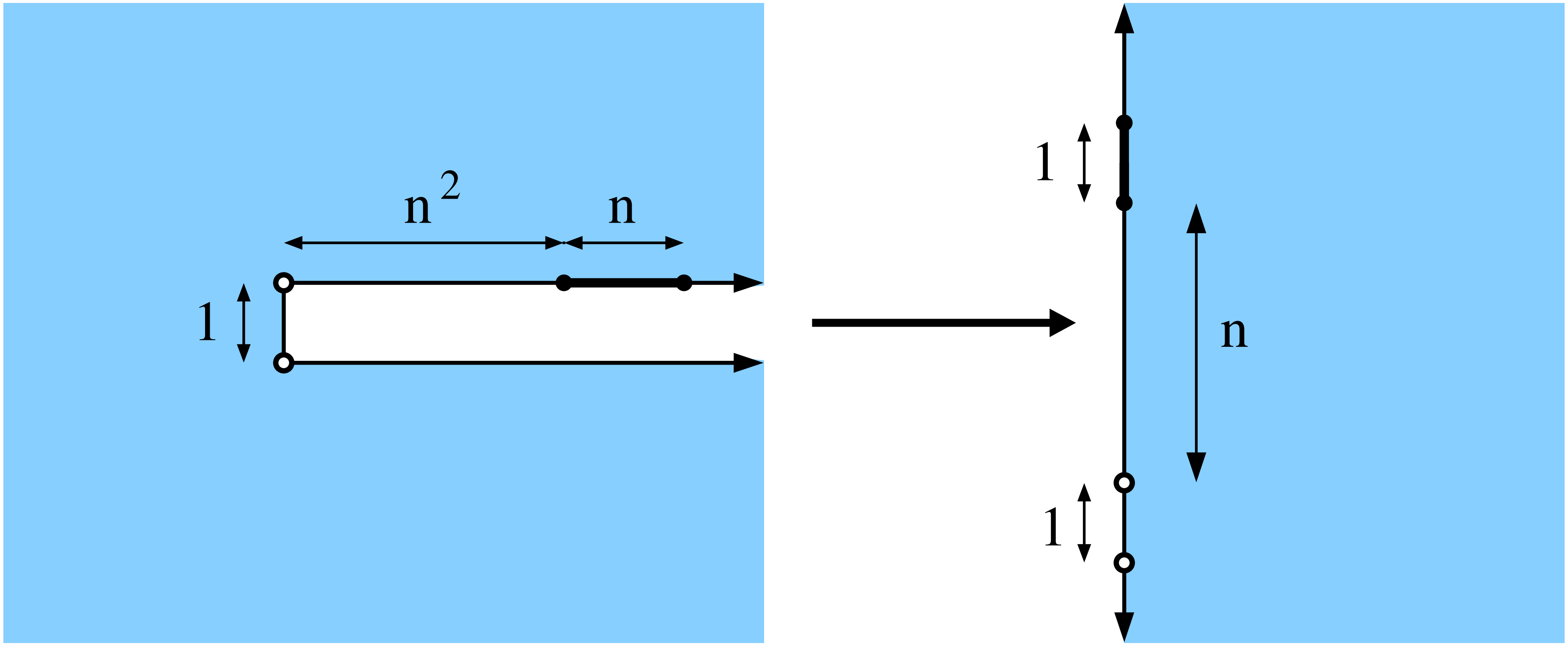


We assume all images have length $\geq \pi$.

Need positive lower bound; actual value usually not important.

Components are "thinner" than half-plane near ∞.

Non-example: half-strip. “Inside” is OK, but ...

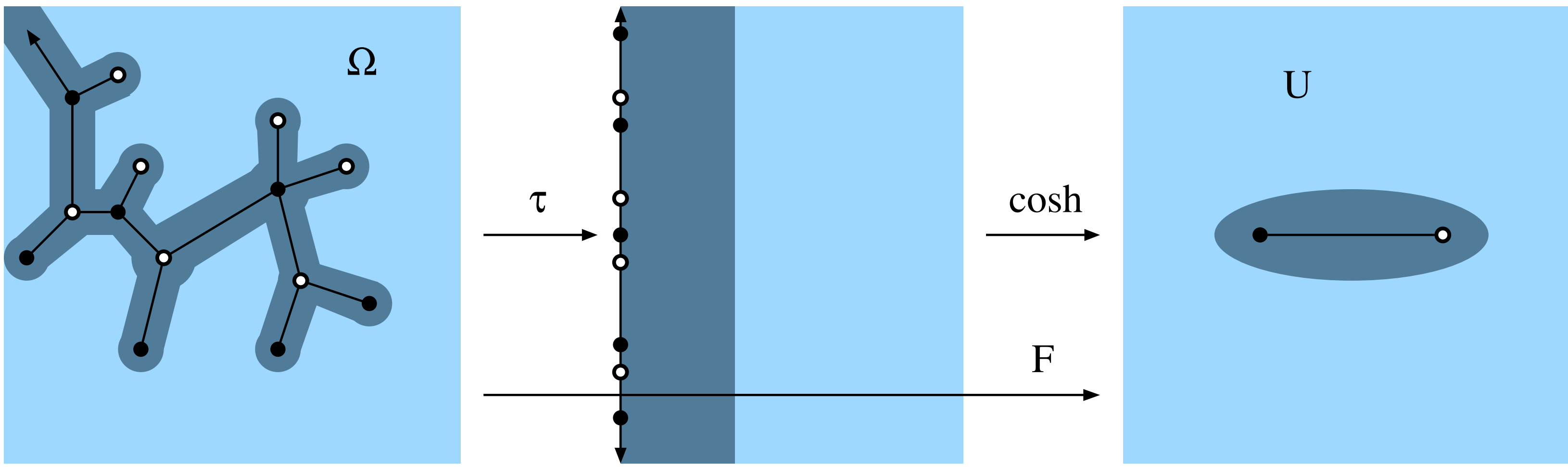


Conformal map of outside to half-plane is $\tau(z) \approx \sqrt{z}$.

Unit intervals on half-plane have pre-images $\simeq n$.

$\Rightarrow$ Bounded geometry and τ-condition can't both hold.

QC-Folding Theorem: Suppose T has bounded geometry and the τ-lower bound. Let $F = \cosh \circ \tau$. There is a K-quasiregular g and $r > 0$ such that $g = F$ off $T(r)$ (shaded) and $\mathrm{CV}(g) = \pm 1$.



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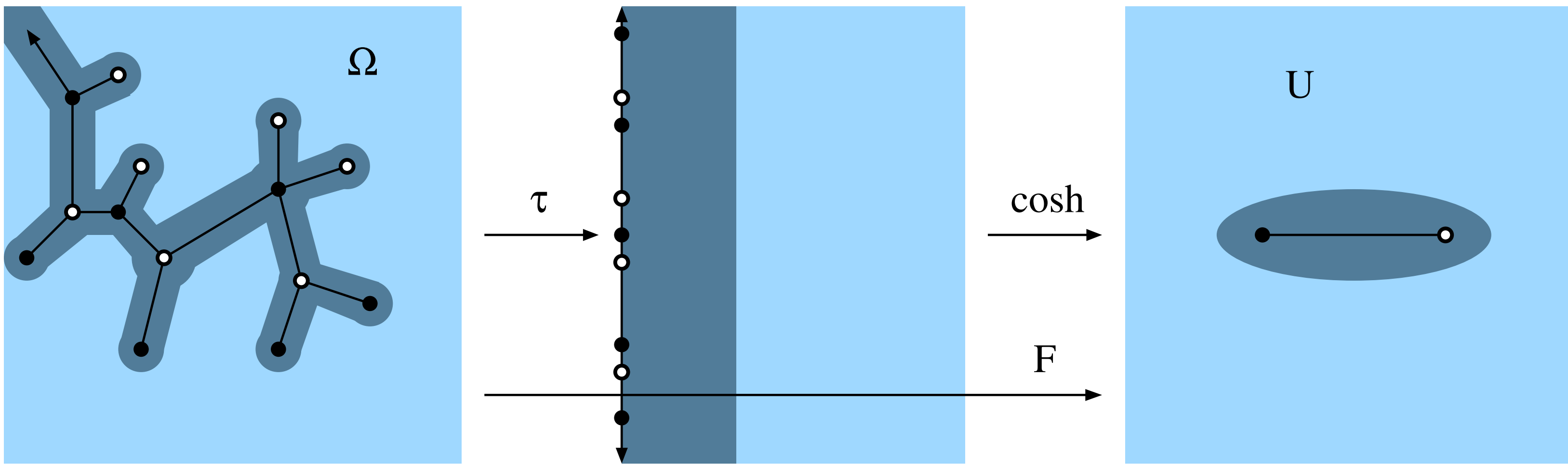
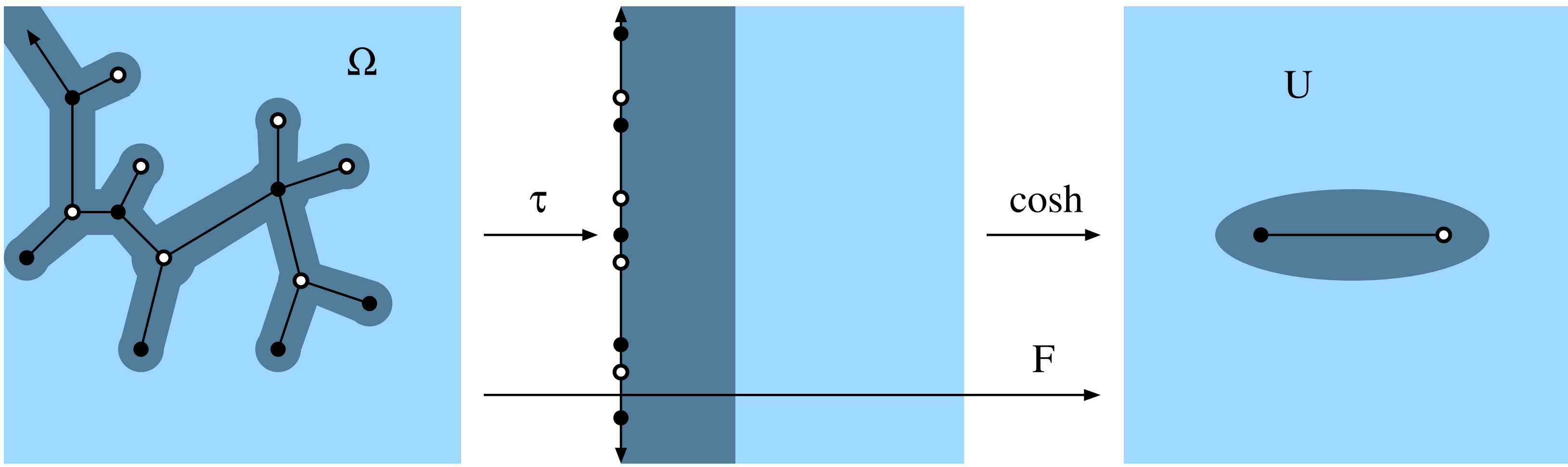


K and r only depend on the bounded geometry constants.

$g = F$ on light blue.

F may be discontinuous across T. g is continuous everywhere.

QC-Folding Theorem: Suppose T has bounded geometry and the τ-lower bound. Let $F = \cosh \circ \tau$. There is a K-quasiregular g and $r > 0$ such that $g = F$ off $T(r)$ (shaded) and $\mathrm{CV}(g) = \pm 1$.

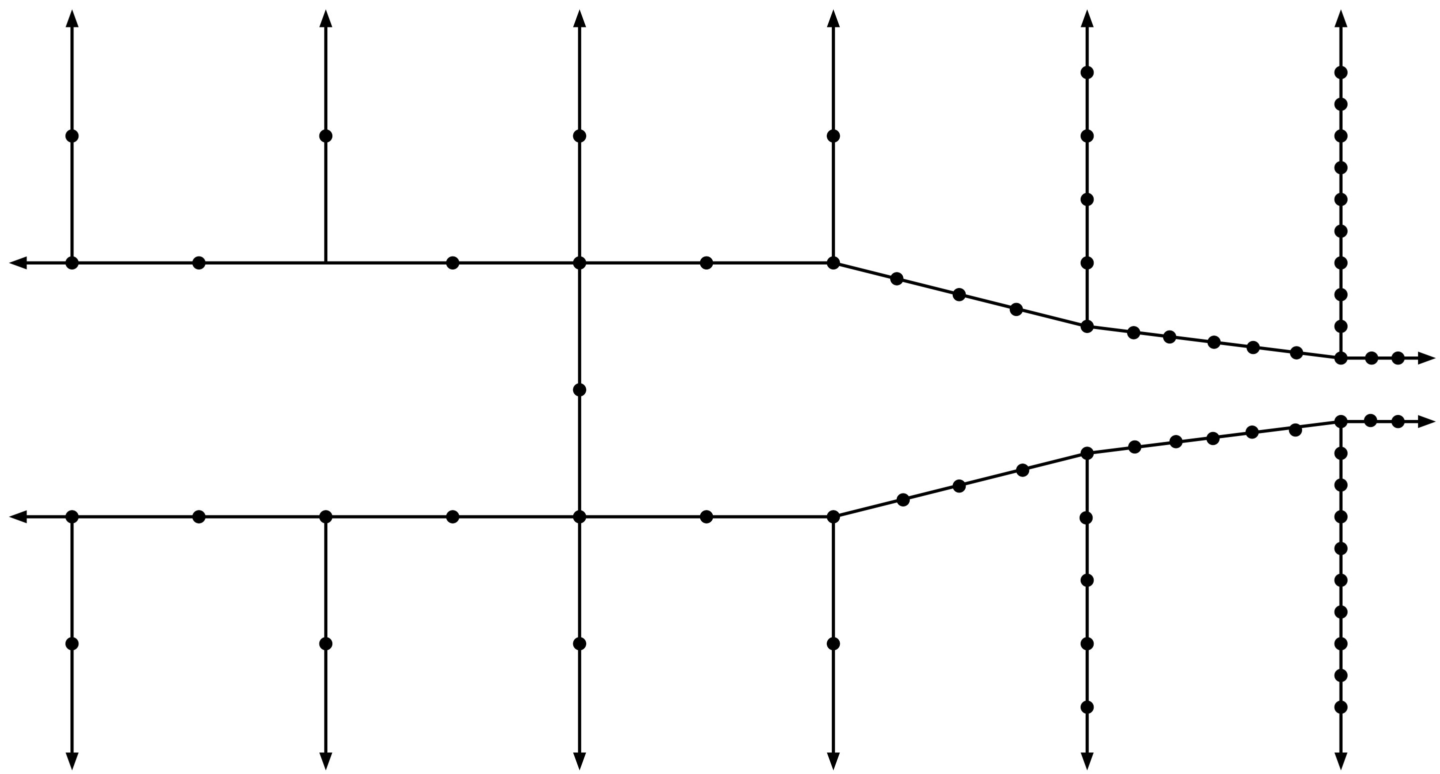


K and r only depend on the bounded geometry constants.

Cor: Any T as above is approximated by $\varphi(T')$ where $T' = f^{-1}([-1, 1])$, f is entire with $\mathrm{CV}(f) = \pm 1$, φ is QC and conformal off $T(r)$.

In many applications, φ is close to identity, so $T \approx T'$.

Example:

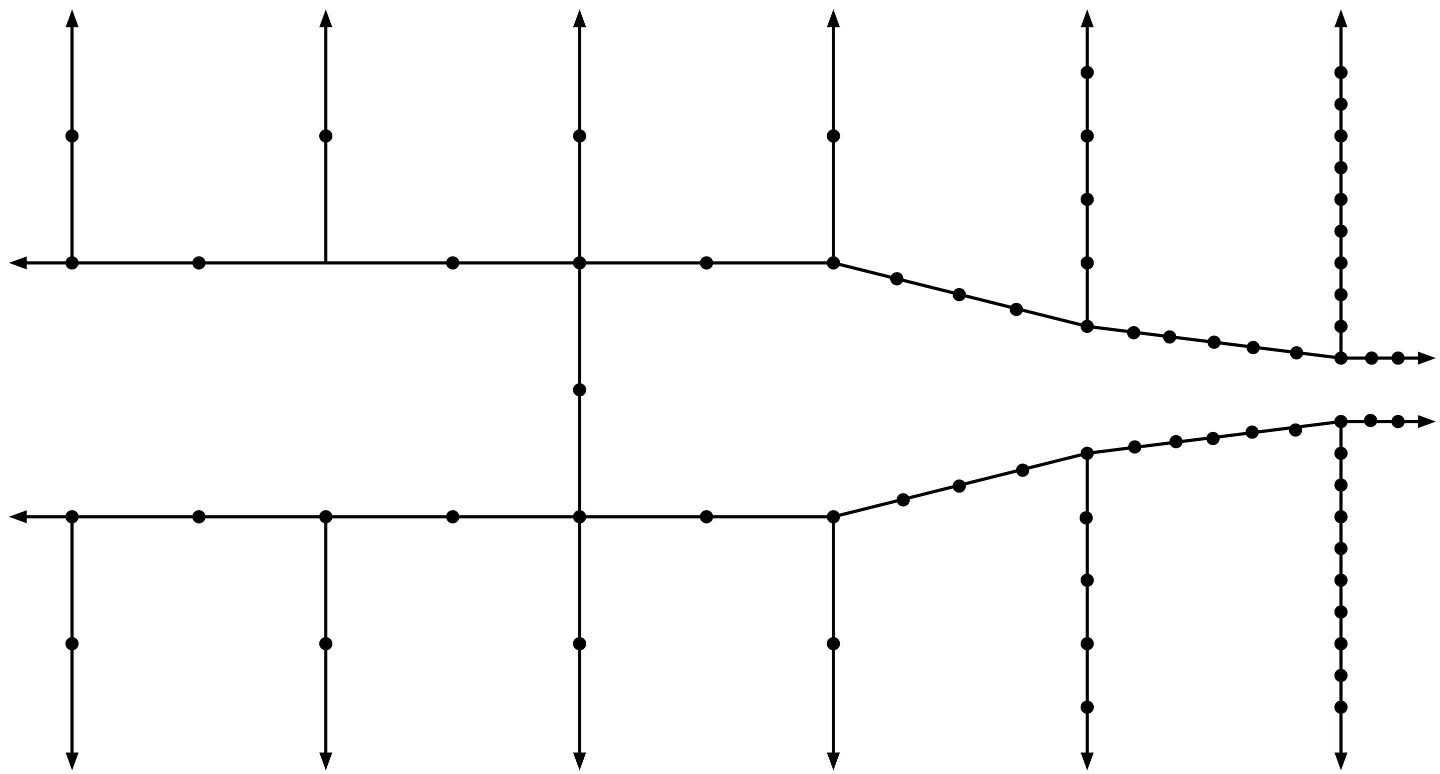


Check that this tree has:

(1) bounded geometry,

(2) the τ-lower bound.

Rapid increase in Speiser class

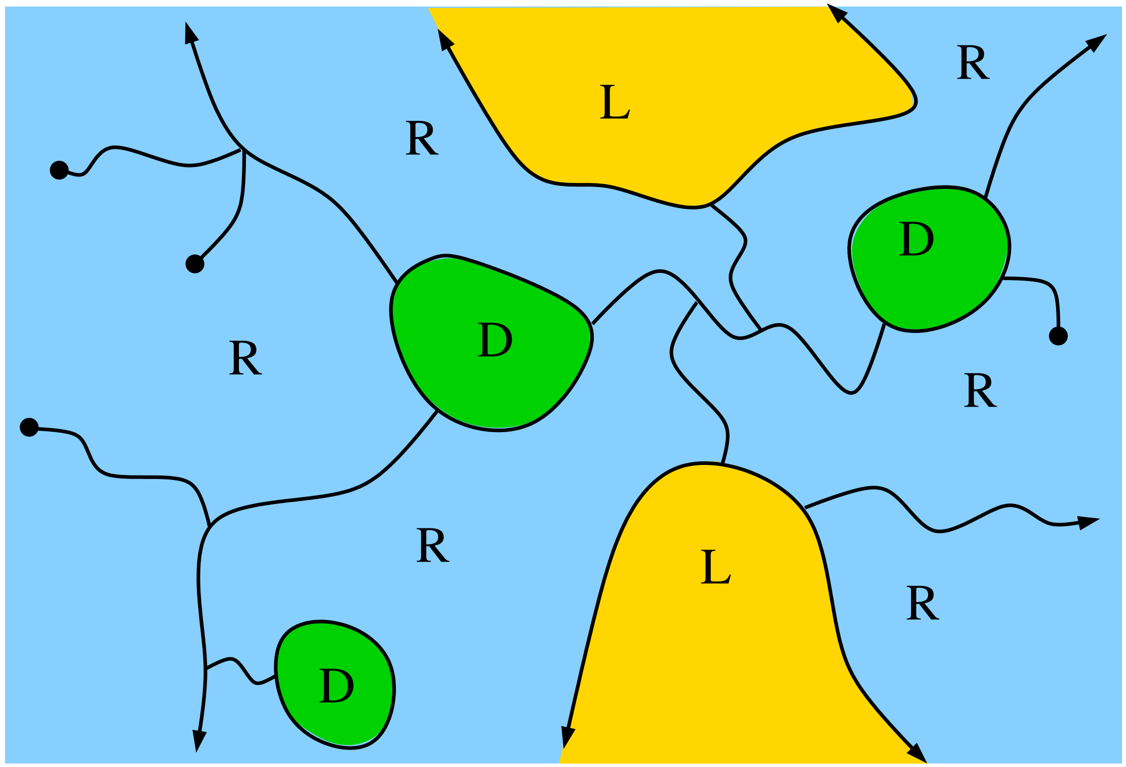


We get f with 2 singular values, $f(x) \nearrow \infty$ as fast as we wish.

Correction map φ is Hölder, only slows growth a little.

Similar examples due to Sergei Merenkov (2008) (3 singular values).

More general folding theorem: replace tree by graph, faces labeled D,L,R.



D = bounded Jordan domains (z^n, high degree critical points)

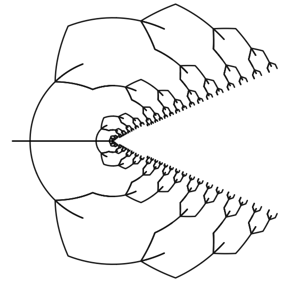
L = unbounded Jordan domains ($e^{-\tau(z)}$, finite asymptotic values)

RR-edges map to $[-1,1]$, other edges map to $\mathbb{T}$.

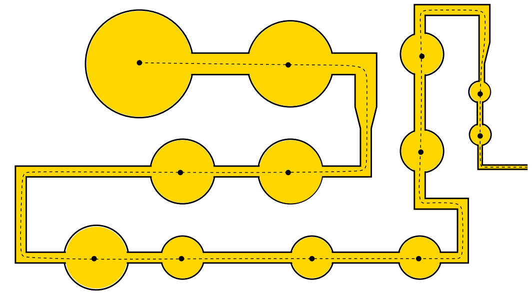
Another variation inverts the D-components, introduce poles.

APPLICATIONS OF QC-FOLDING

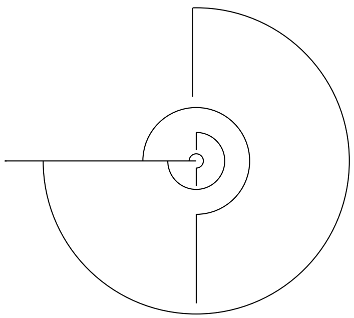
EL Models



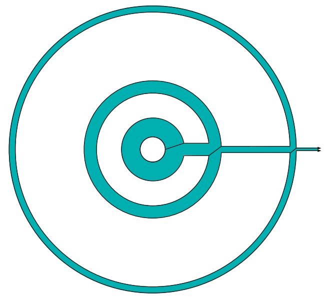
Order Conjecture



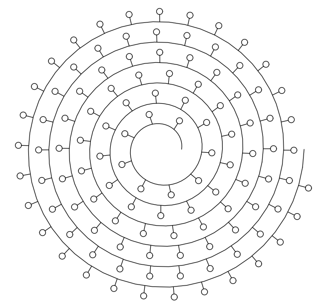
Post-singular dynamics



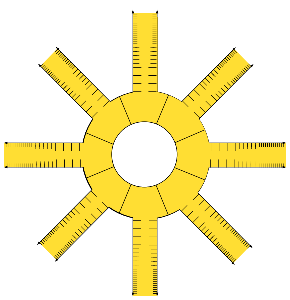
Eremenko Conjecture



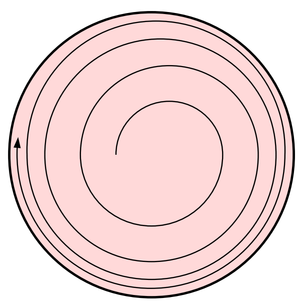
Near zero



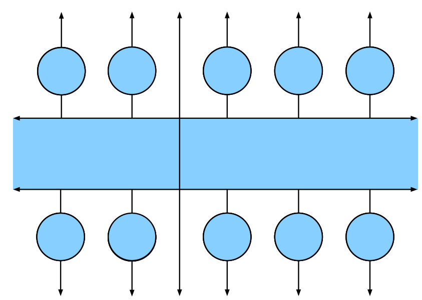
Wiman's Conjecture



Dimension near 1



Folding in disk



Wandering domain

$$\begin{aligned}\textbf{Singular set} \;&=\; \text{closure of critical values and finite asymptotic values}\\ &=\; \text{smallest set so that } f \text{ is a covering map onto } \mathbb{C}\setminus S\end{aligned}$$

Eremenko-Lyubich class = bounded singular set $= \mathcal{B}$

Speiser class = finite singular set $= \mathcal{S} \subset \mathcal{B}$

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Eremenko-Lyubich class = bounded singular set $= \mathcal{B}$

Speiser class = finite singular set $= \mathcal{S} \subset \mathcal{B}$

"Plain" folding produces functions in the smaller Speiser class.

Using D and L components, we can create singular values other than ± 1.

There is an "easier" version of folding for the Eremenko-Lyubich class.

Application: wandering domains

Given an entire function f,

Fatou set $= \mathcal{F}(f) =$ open set where iterates are normal family.

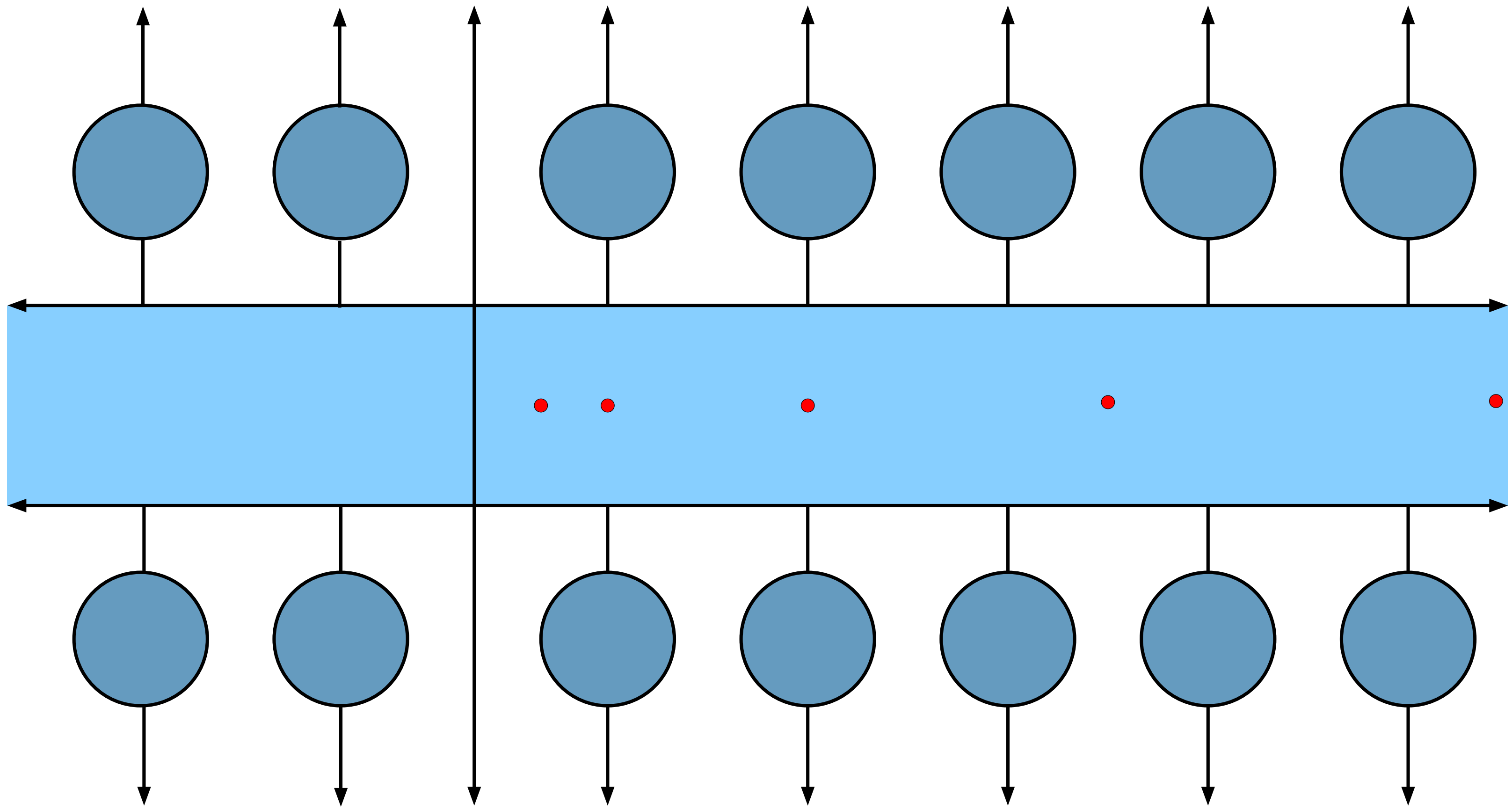
Julia set $= \mathcal{J}(f) =$ complement of Fatou set.

f permutes components of its Fatou set.

Wandering domain = Fatou component with infinite orbit.

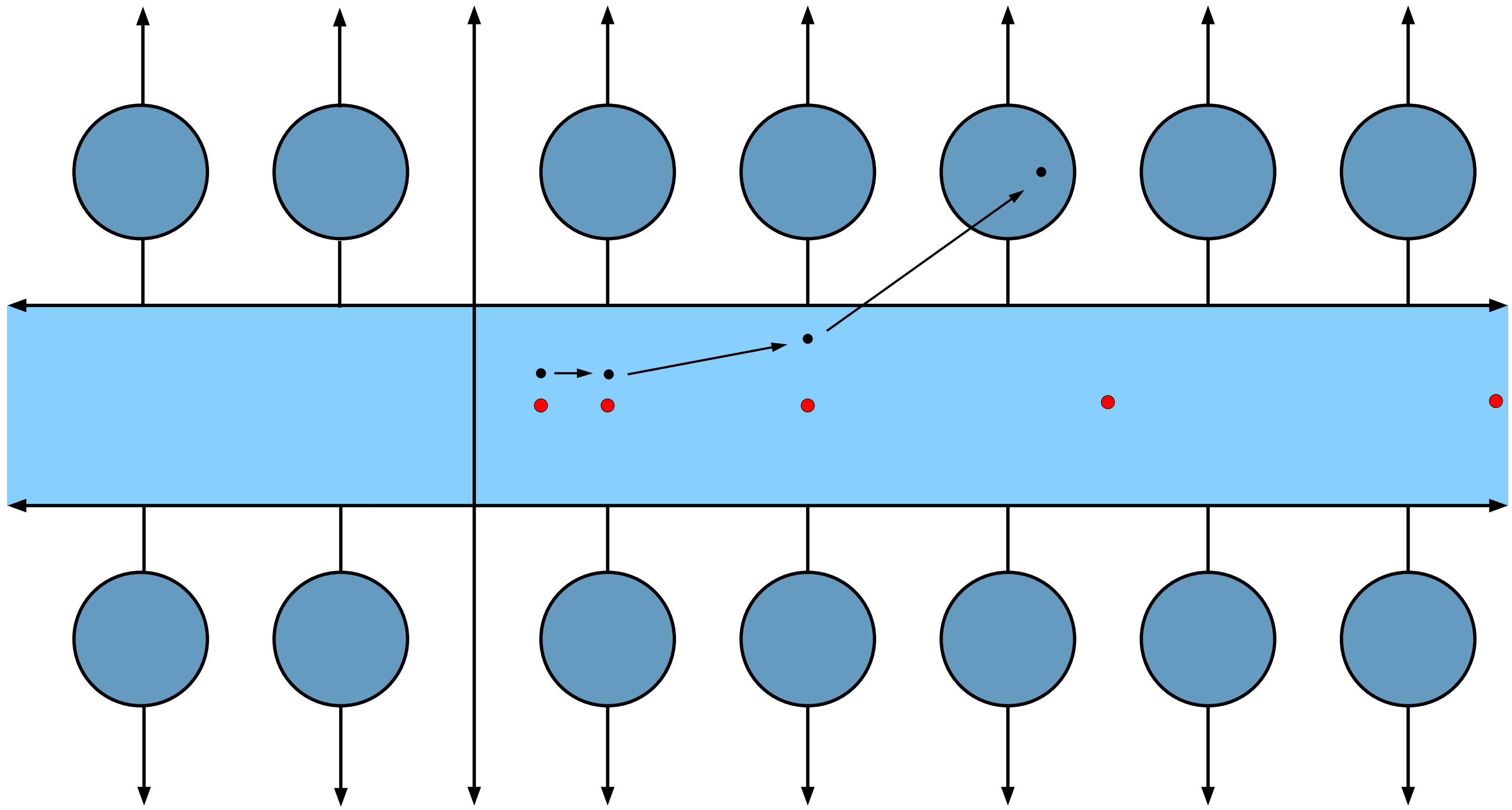
- Entire functions can have wandering domains (Baker 1975).
- No wandering domains for rational functions (Sullivan 1985).
- Also none in Speiser class (Eremenko-Lyubich, Goldberg-Keen).
- More generally, none for finite type maps (Epstein).

Are there wandering domains in Eremenko-Lyubich class?



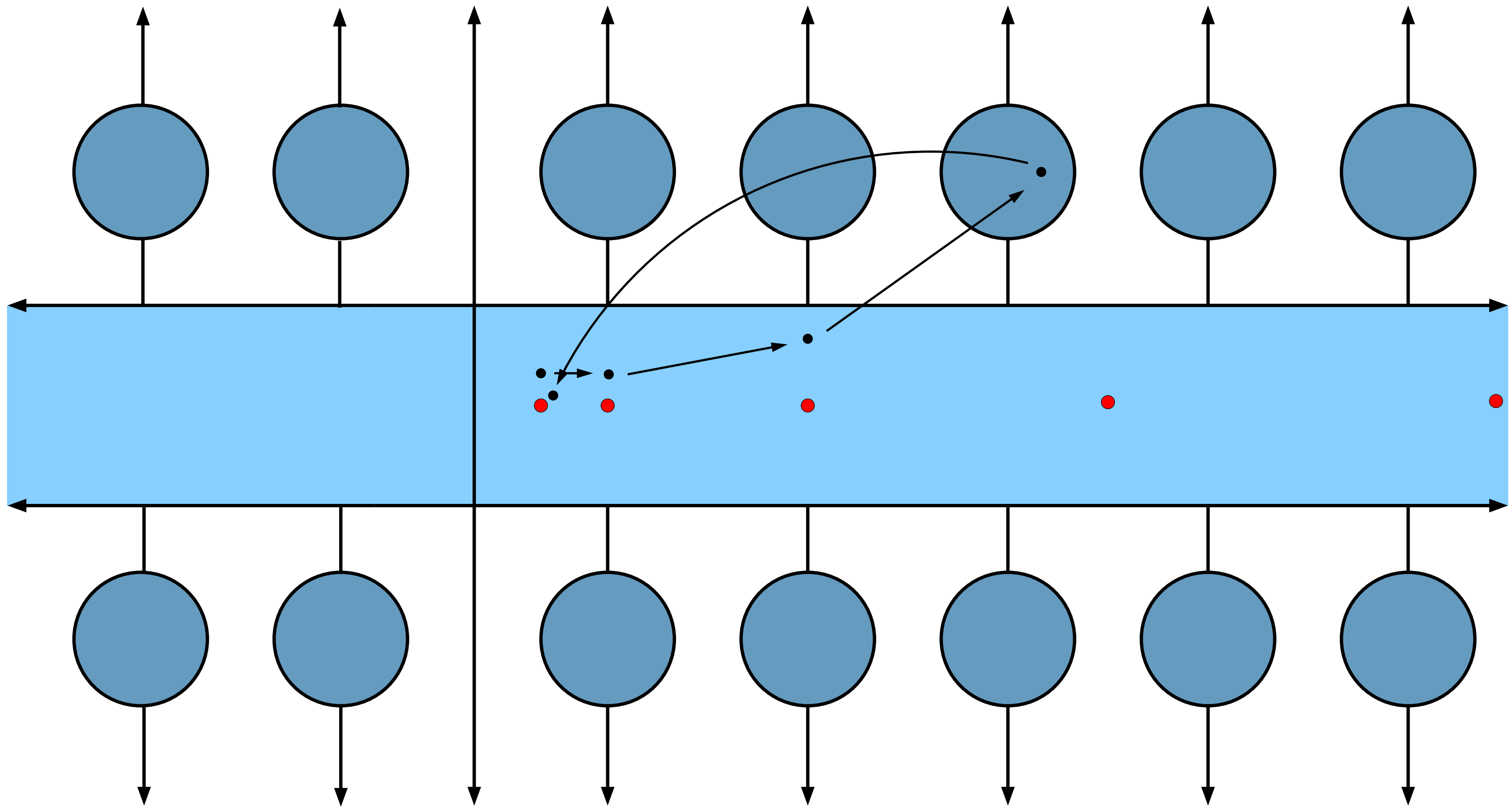
Graph giving wandering domain in Eremenko-Lyubich class.

Original proof corrected by Marti-Pete and Shishikura,
who also give alternate construction.



Graph giving wandering domain in Eremenko-Lyubich class.

Variations by Lazebnik, Fagella-Godillon-Jarque, Osborne-Sixsmith.



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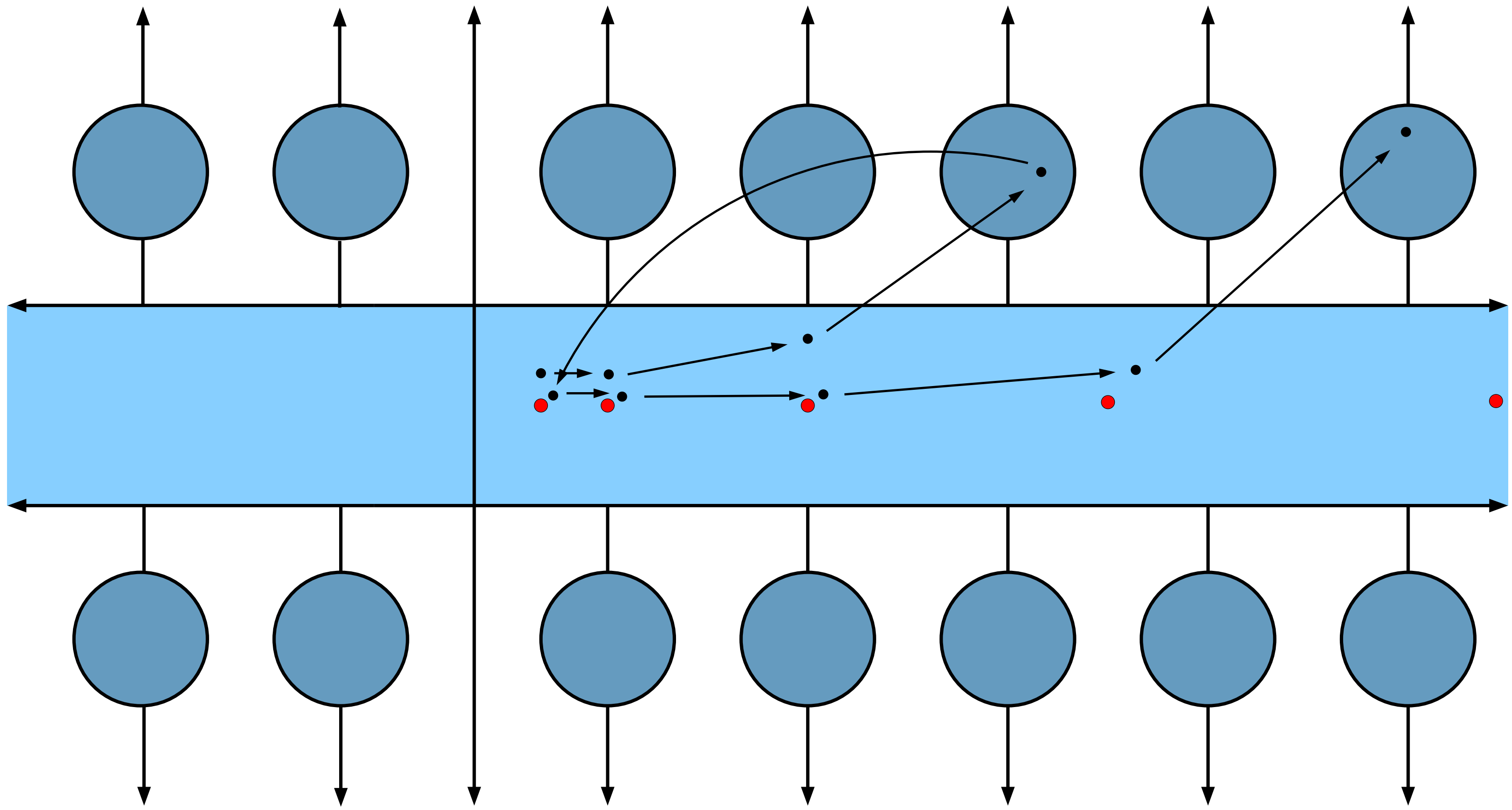

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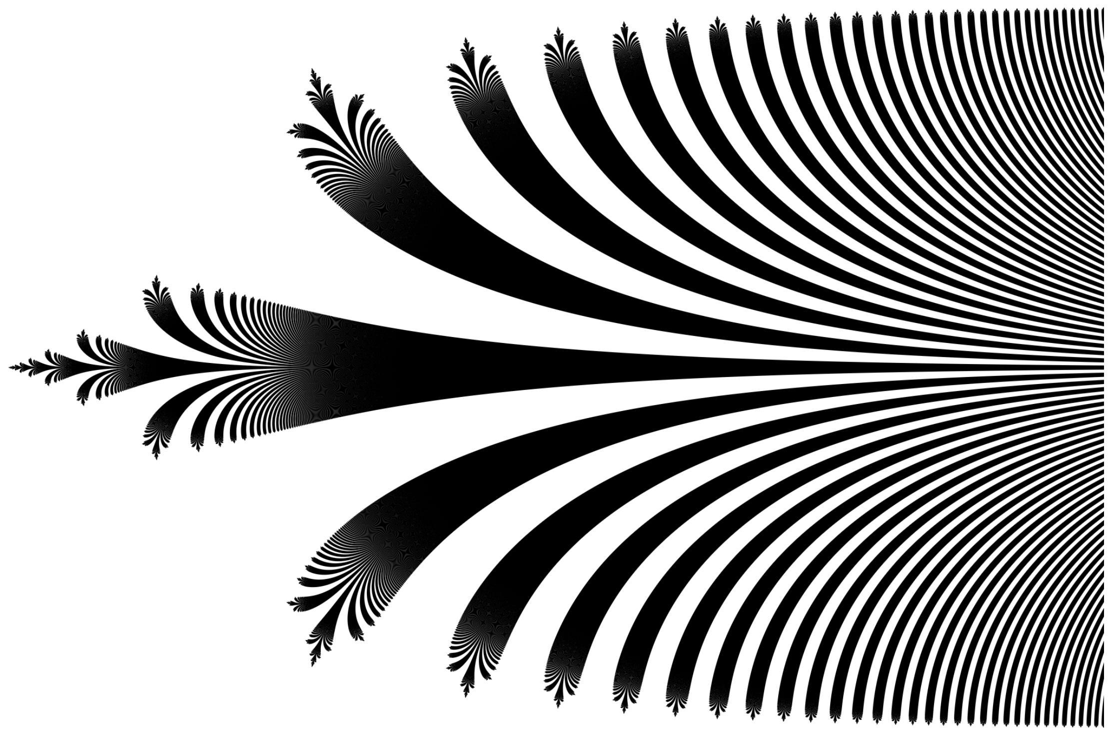
Application: dimensions of Julia sets Julia set is usually fractal. What is its (Hausdorff) dimension?

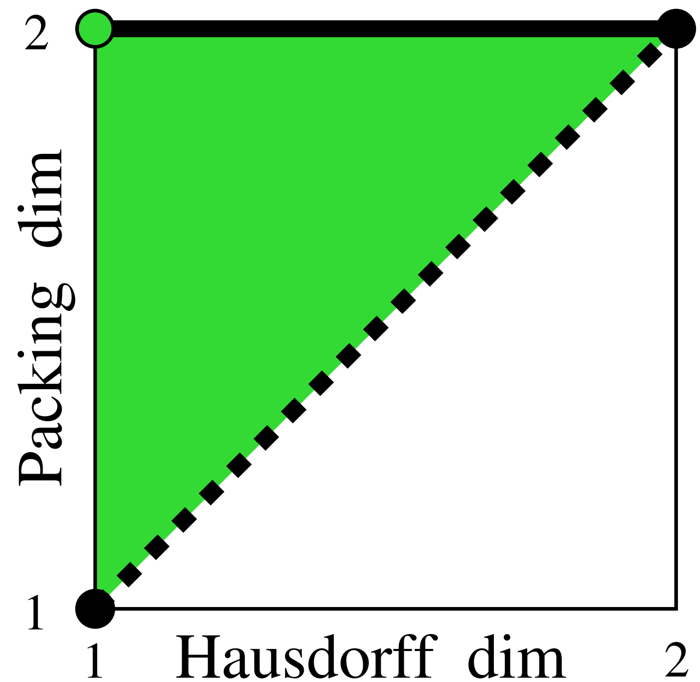
Packing dimension?



$\mathcal{J}((e^z-1)/2)$, courtesy of Arnaud Chéritat

A short (and incomplete) history:

- Baker (1975): f transcendental $\Rightarrow \mathcal{J}$ contains a continuum $\Rightarrow$ dim ≥ 1.
- Misiurewicz (1981), McMullen (1987) dim =2 occurs (is common)
- Stallard (1997, 2000): $\{\dim(\mathcal{J}(f)) : f \in \mathcal{B}\} = (1,2]$.
- Rippon-Stallard (2005) Packing dim $= 2$ for $f \in \mathcal{B}$.
- Albrecht-B. (2020) $1 <$ H-dim < 2 can occur in Speiser class
- B (2018) H-dim $=$ P-dim $= 1$ can occur (outside $\mathcal{B}$).
- Burkart (2019) $1 <$ P-dim < 2 can occur (outside $\mathcal{B}$).
- Many other results known, e.g., relating growth rate to dimension.

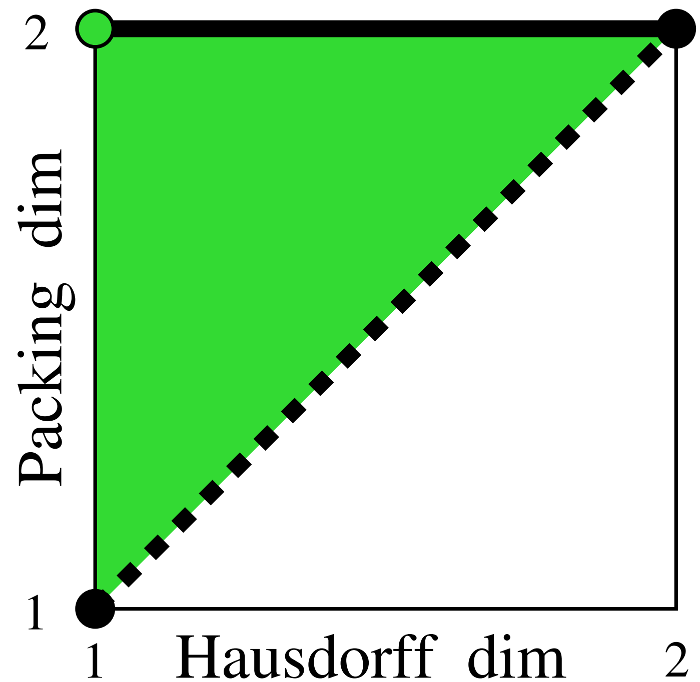


Hausdorff, upper Minkowski and packing dimension are defined as

$$\mathrm{Hdim}(K) = \inf\{s : \inf\{\textstyle\sum_j r_j^s : K \subset \cup_j D(x_j, r_j)\} = 0\},$$

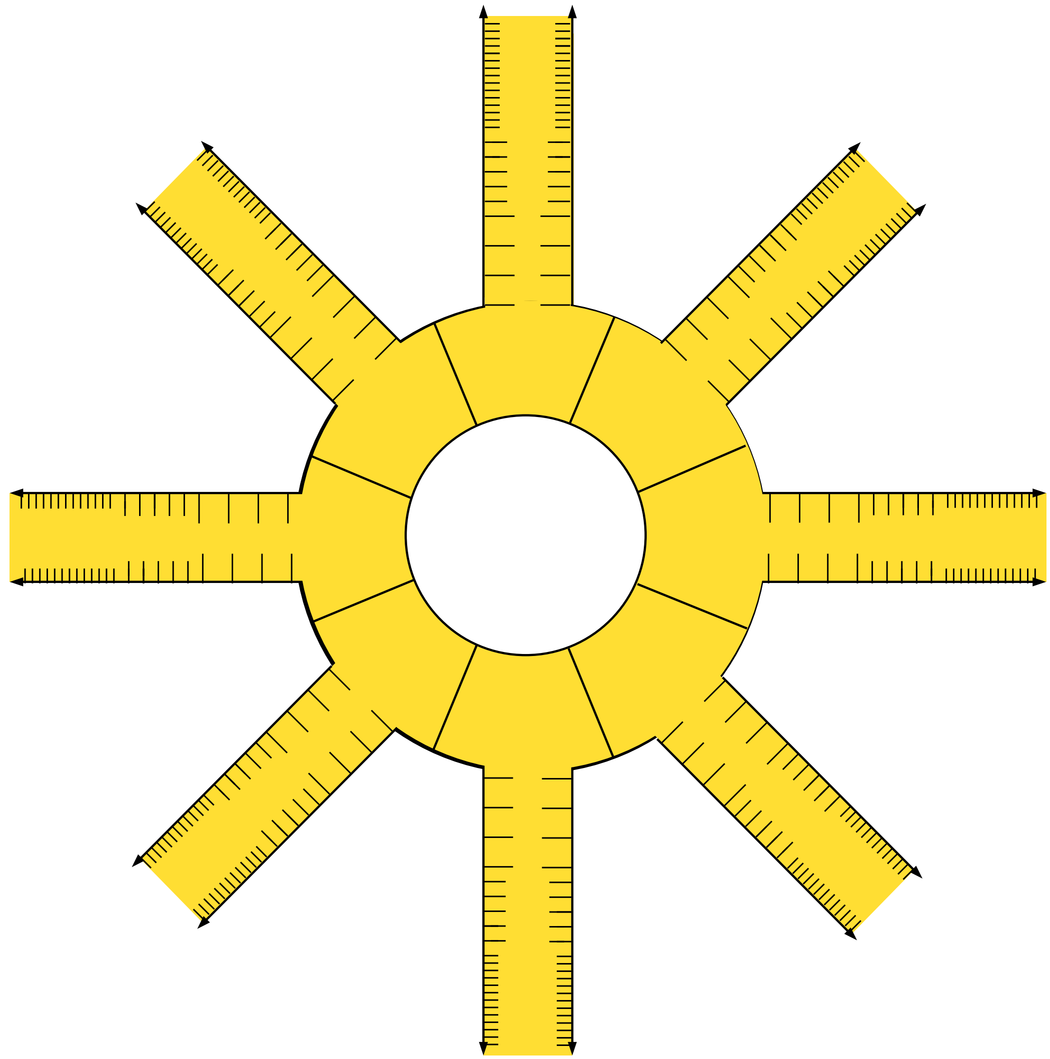
$$\overline{\mathrm{Mdim}}(K) = \inf\{s : \limsup_{r\to 0} \inf N(r) r^s = 0 : K \subset \cup_{j=1}^{N} D(x_j, r)\},$$

$$\mathrm{Pdim}(K) = \inf\{s : K \subset \cup_{j=1}^{\infty} K_j : \overline{\mathrm{Mdim}}(K_j) \le s \text{ for all } j\},$$



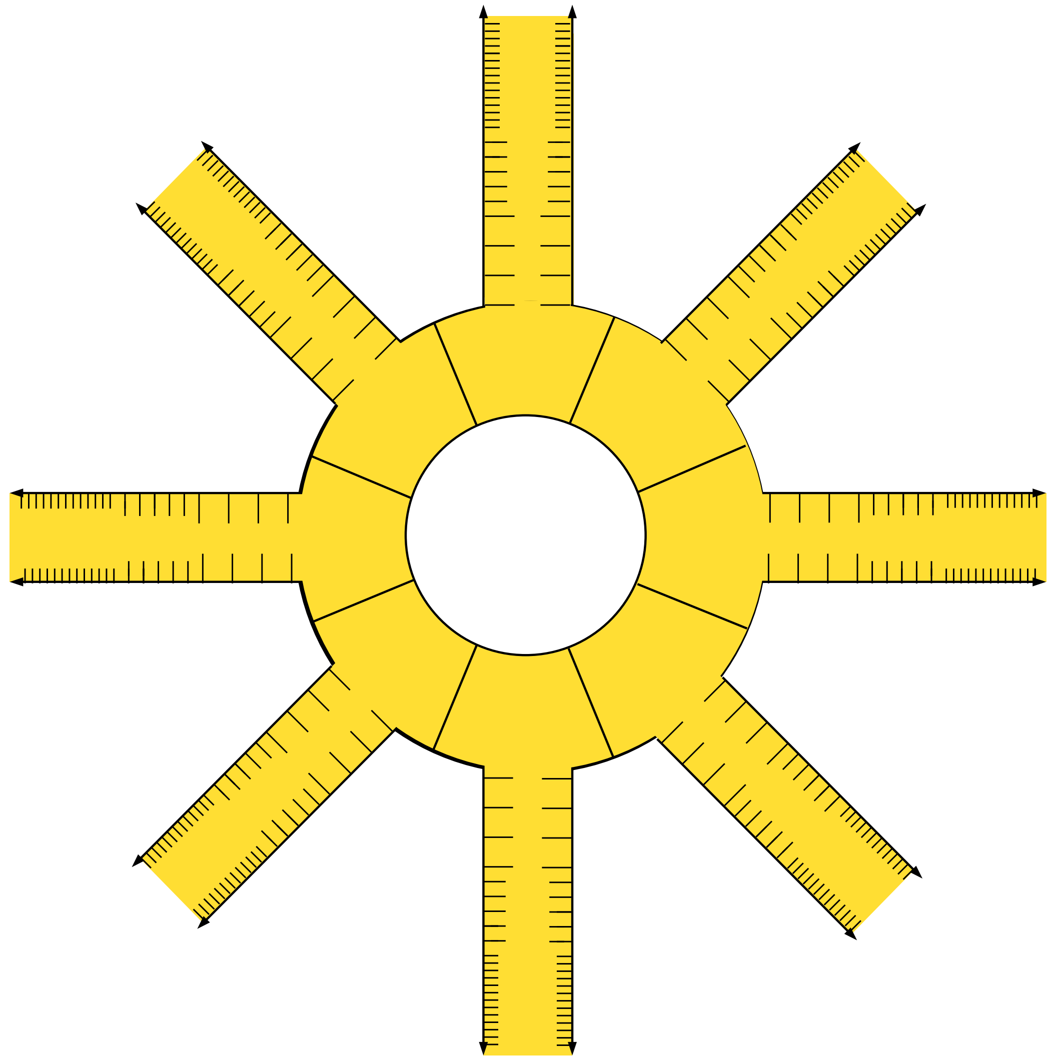
In polynomial dynamics it is difficult to construct examples with large dimension or positive area (Shishikura, Buff, Cheritat).

For entire functions, it is harder to find small Julia sets.



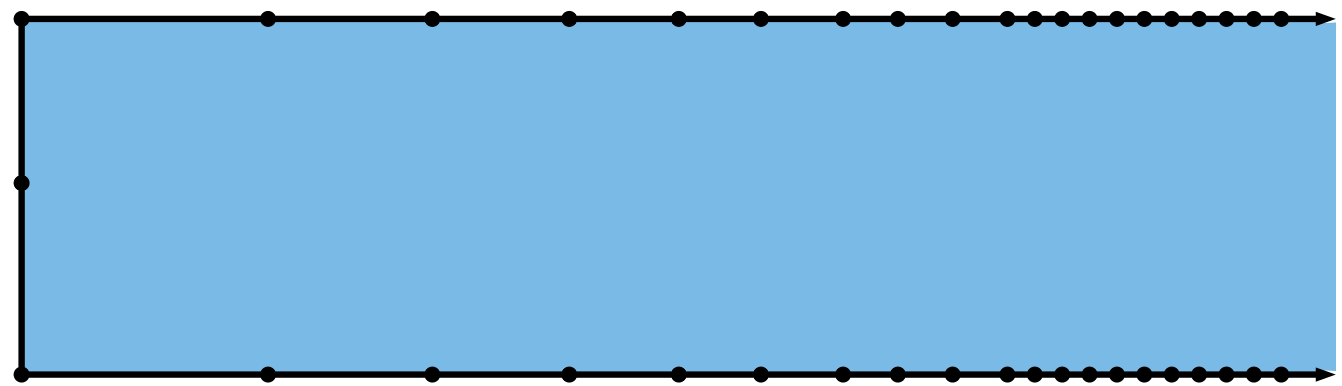
$$\inf\{\dim(\mathcal{J}(f)) : f \in \mathcal{S}\} = 1 \qquad \text{(B.-Albrecht, 2018).}$$

Uses upper and lower τ-bounds to control dimension.



Exponential imbalance. “Spikes” are chosen to give bounded imbalance.
Do this to control number and sizes of preimages of a disk.

Theorem (Stallard): There are Eremenko-Lyubich functions whose Julia sets have Hausdorff dimension close to 1.



There is EL function with tract $\{z : |f(z)| > 1\} \approx$ half-strip.

Use "EL-folding" to define QR g so that:

- $g(0) = 0$ and $|g(z)| < 1$ outside $S =$ half-strip.
- Inside S, $g(z) = \exp(\exp(z - K))$ (conformal $S \to \mathbb{H}_r$, then exp).

Fixed point $g(0) = 0$ attracts everything in complement of S

Thus the Julia set is inside S. More precisely,

$$\mathcal{J}(g) \subset \bigcap X_n,$$

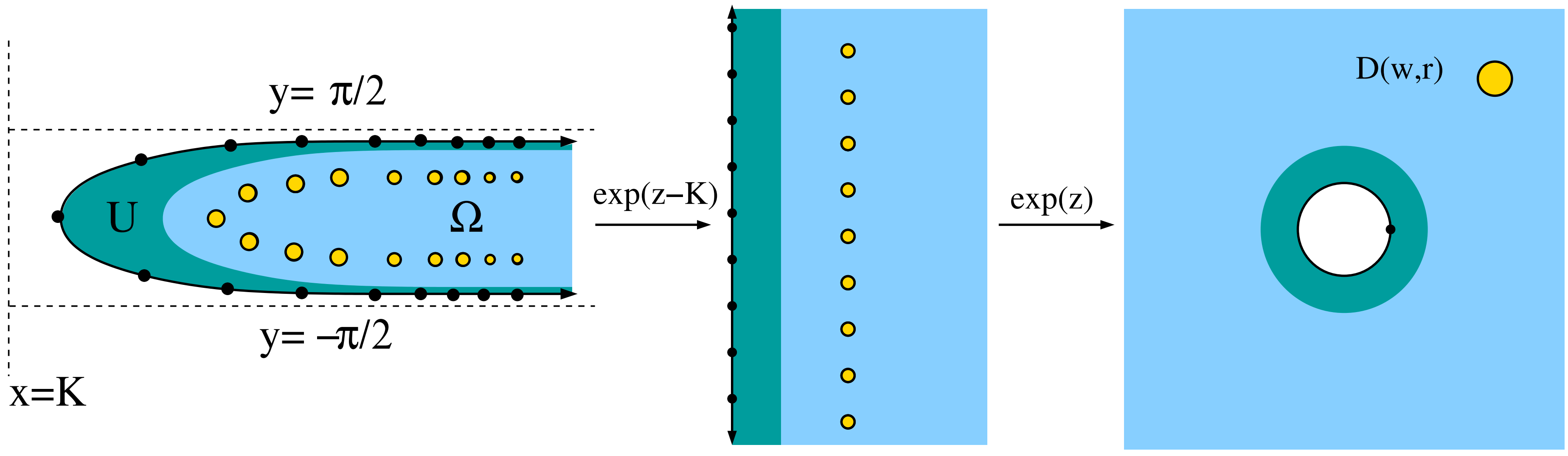
$$X_n = \{z : |g^k(z)| \geq K, k = 1, \ldots, n\}.$$

To prove $\dim(\mathcal{J}) \leq 1+\delta$, it suffices to show:

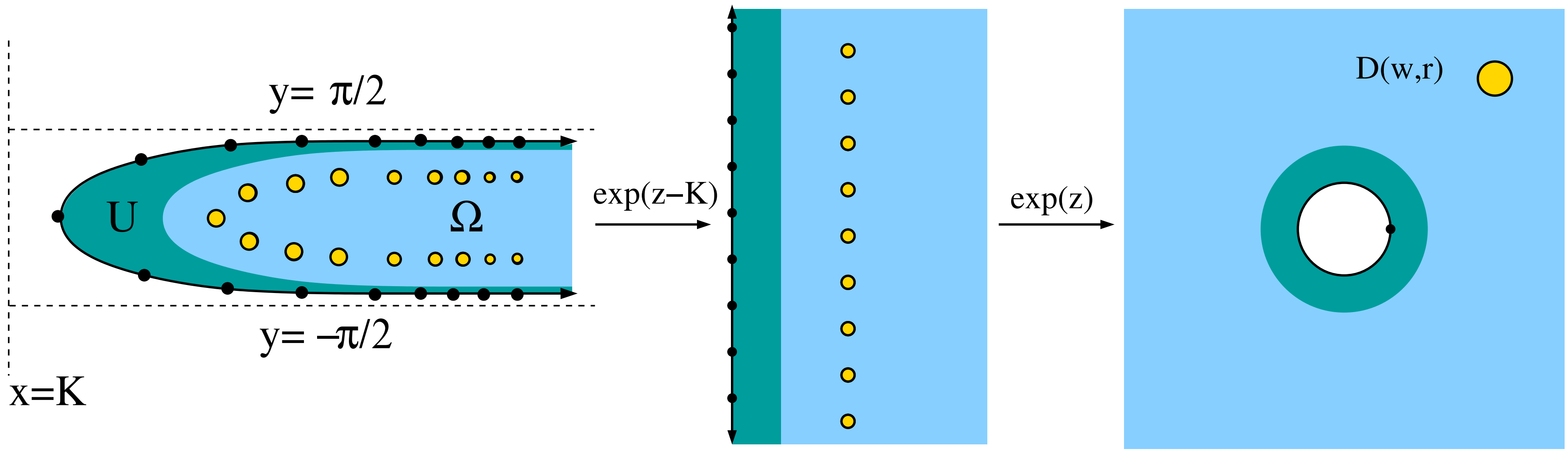
(1) X_1 can be covered by disks $\{D_j\}$ so that $\sum_j \mathrm{diam}(D_j)^{1+\delta} < \infty$,

(2) if D hits $\mathcal{J} \cap \{|z| > K\}$, then its preimages satisfy

$$\sum_{W_j \in f^{-1}(D)} \mathrm{diam}(W_j)^{1+\delta} \leq \epsilon \cdot \mathrm{diam}(D)^{1+\delta}.$$



This is what preimages of one disk look like.

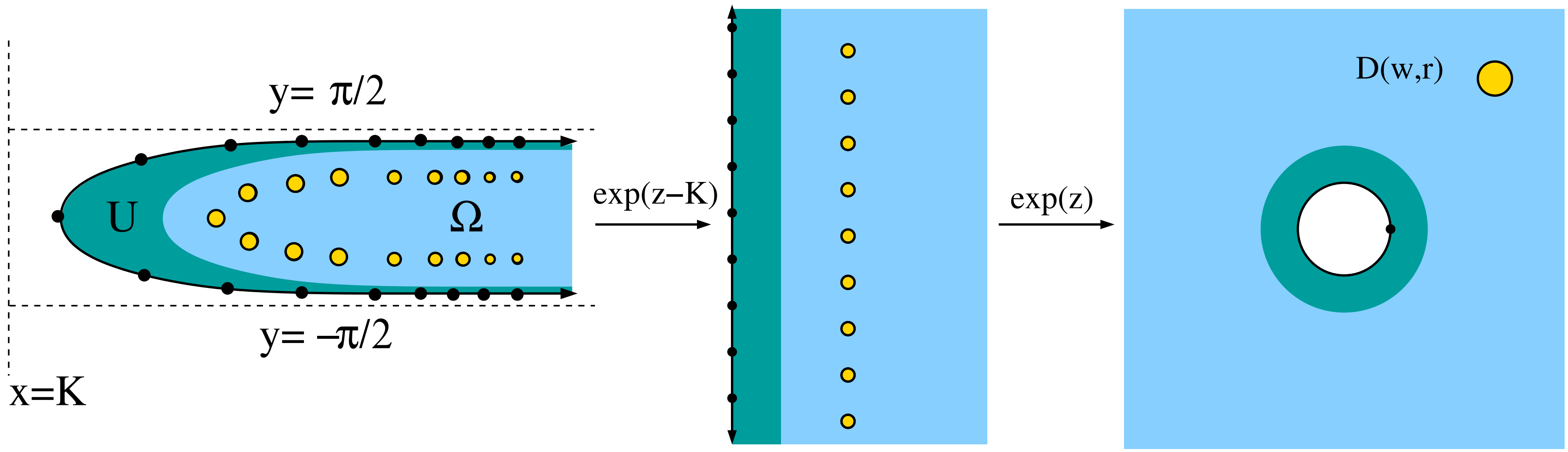


Preimage of gold disk $D = D(w,r)$ defined in two steps:

- stack of regions of diameter $O(r/|w|)$ on line $\{x = \log|w|\}$.
- region at height $2\pi k$ in stack has single preimage U_k of diameter

$$O\left(\frac{r}{|w|(\log|w| + 2\pi|k|)}\right).$$

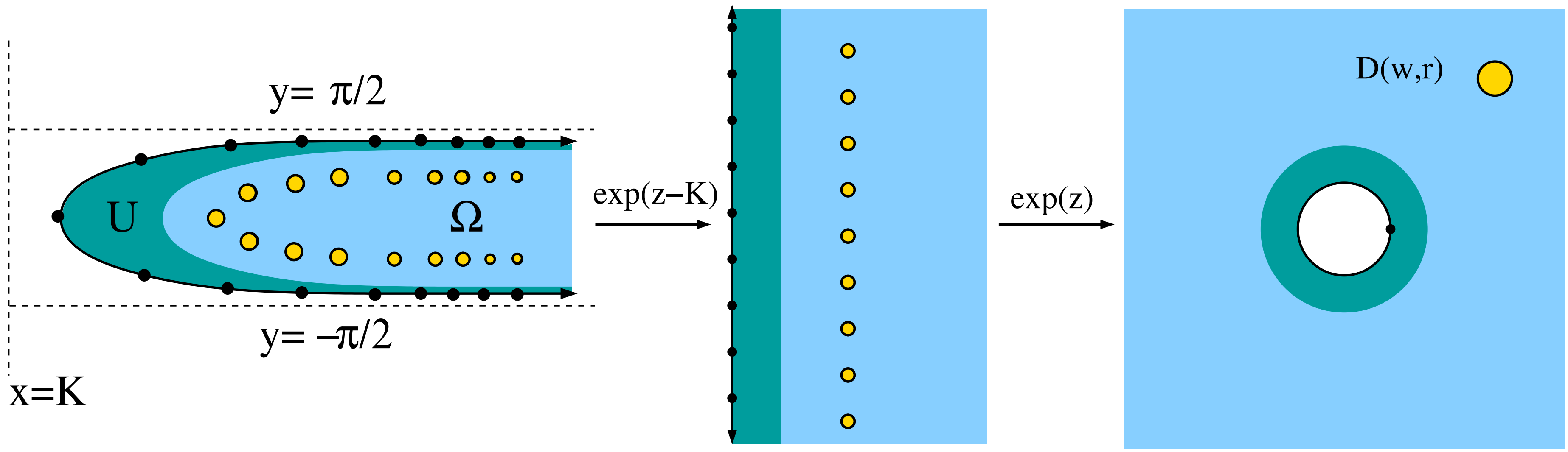
These estimates only use $(\log z)' = 1/z$.



If $\delta > 0$ is fixed and R is large enough, then

$$\sum_k \operatorname{diam}(U_k)^{1+\delta} \lesssim \left(\frac{r}{|w|}\right)^{1+\delta} \sum_k \frac{1}{(\log|w| + 2\pi|k|)^{1+\delta}}$$

$$\lesssim \left|\frac{r}{w}\right|^{1+\delta} \frac{1}{\delta \log^{1+\delta}|w|} \ll \left|\frac{r}{w}\right|^{1+\delta} \ll r^{1+\delta}$$



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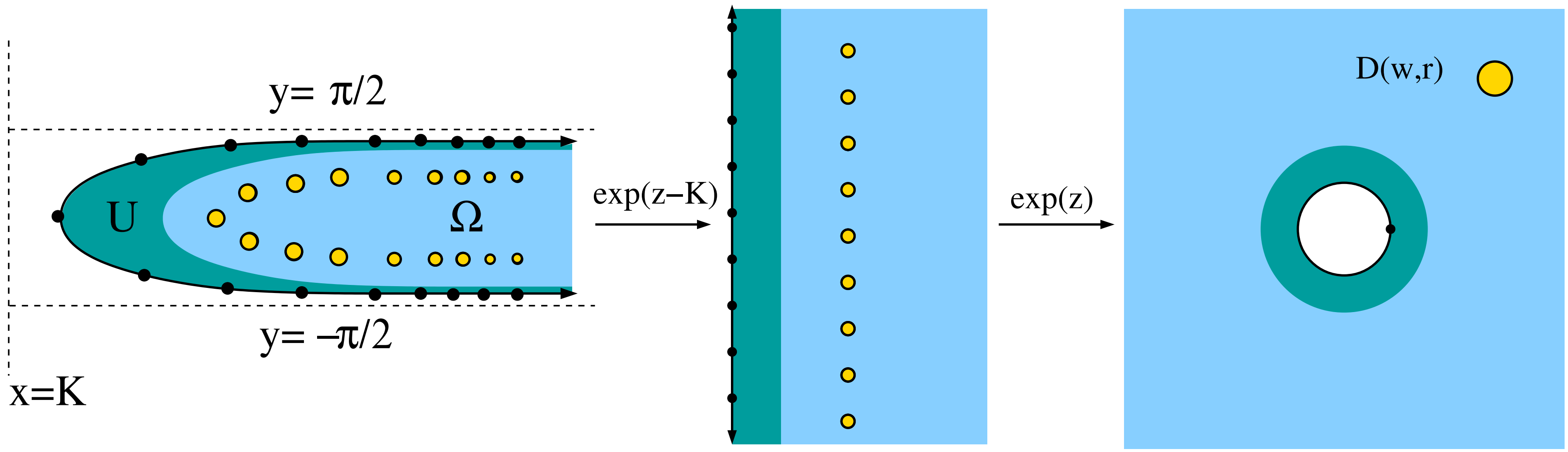
This proves $\dim(\mathcal{J}(g)) \leq 1 + \delta$.

QC correction φ is bi-Lipschitz on $\mathcal{J}$ so $f = g \circ \varphi$ works too.

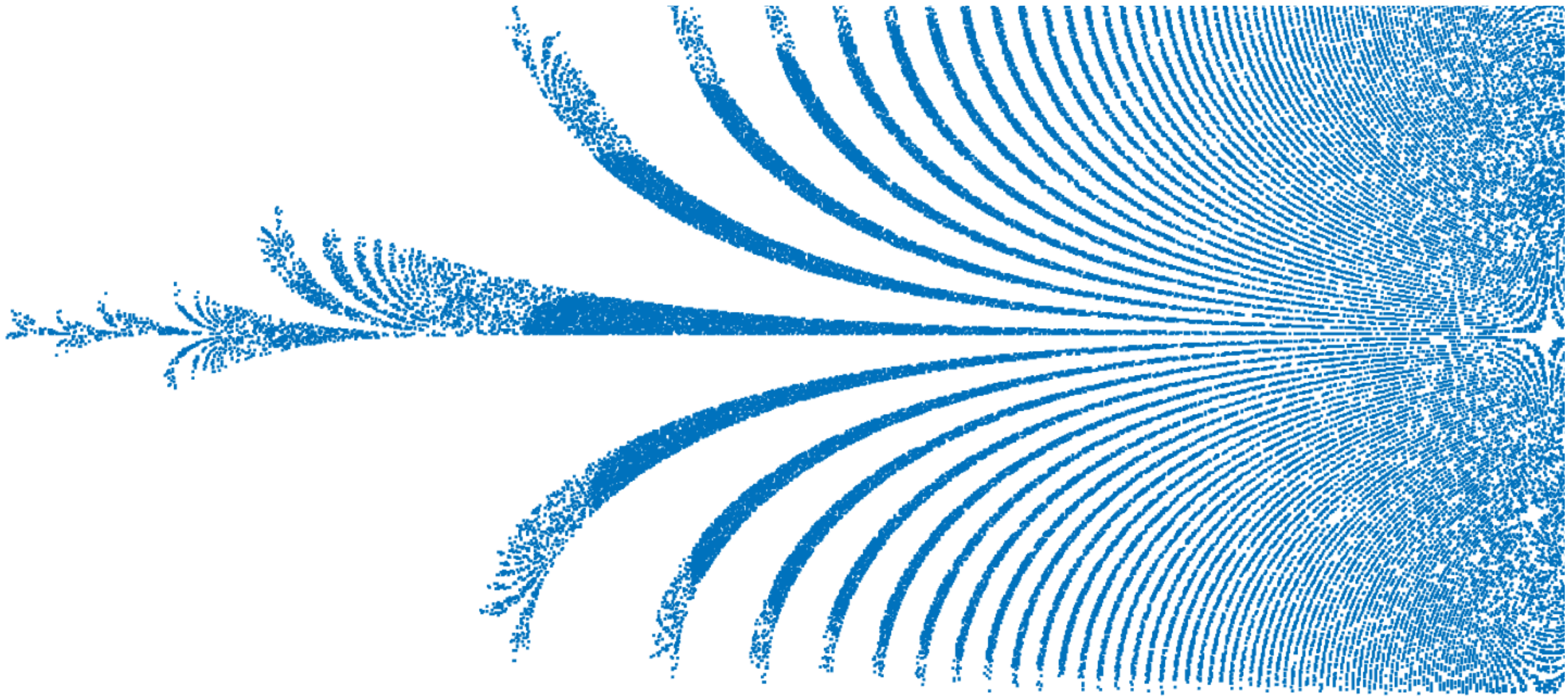


Lemma: If A, B are disjoint, planar sets and $|\mu_\varphi| \le k\chi_A$ where

$$\int_A \frac{dxdy}{|z-w|^2} \le C < \infty, \quad \forall w \in B,$$

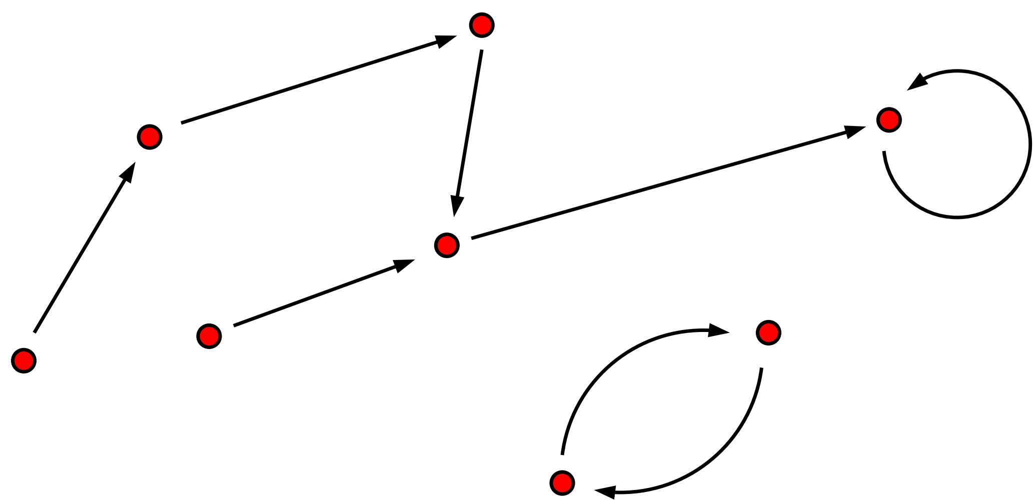
then φ is biLipschitz on B with constant $M(C, K)$.

Follows from work of Bojarski, Lehto, Techihmüller and Wittich.



Application: prescribing postcritical dynamics

DeMarco, Koch and McMullen proved that a post-critically finite rational map can have **any** given dynamics.



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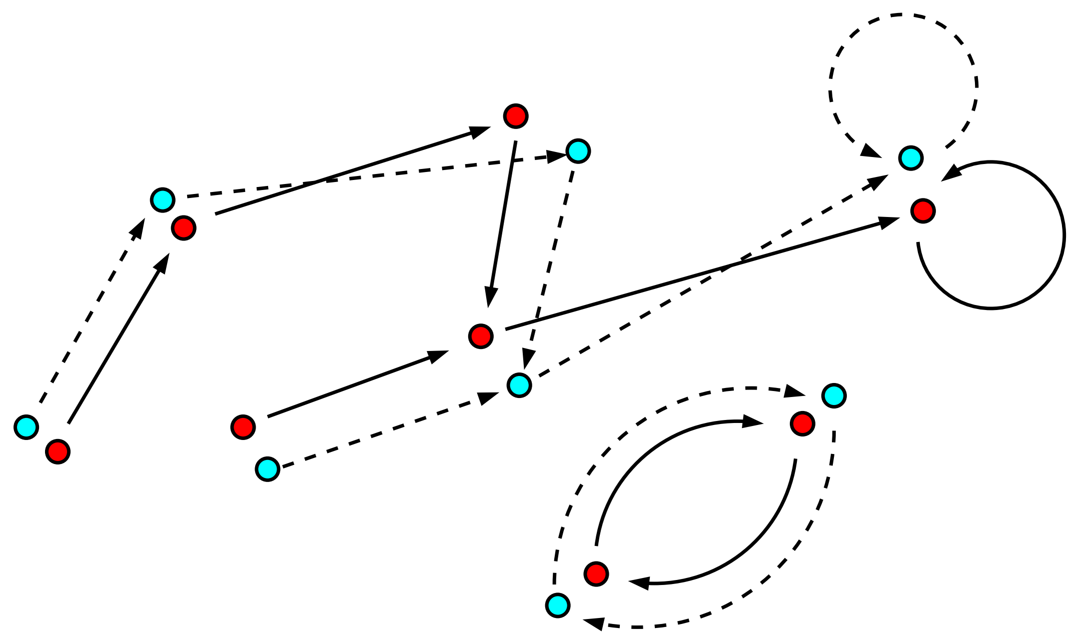
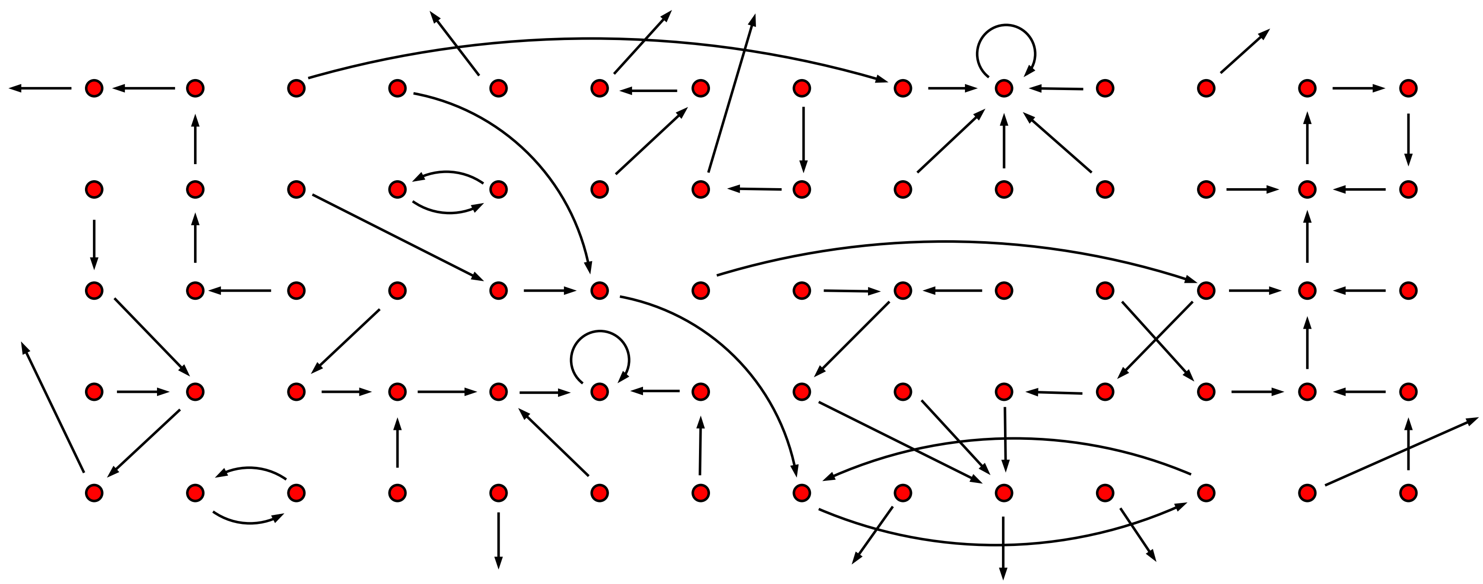


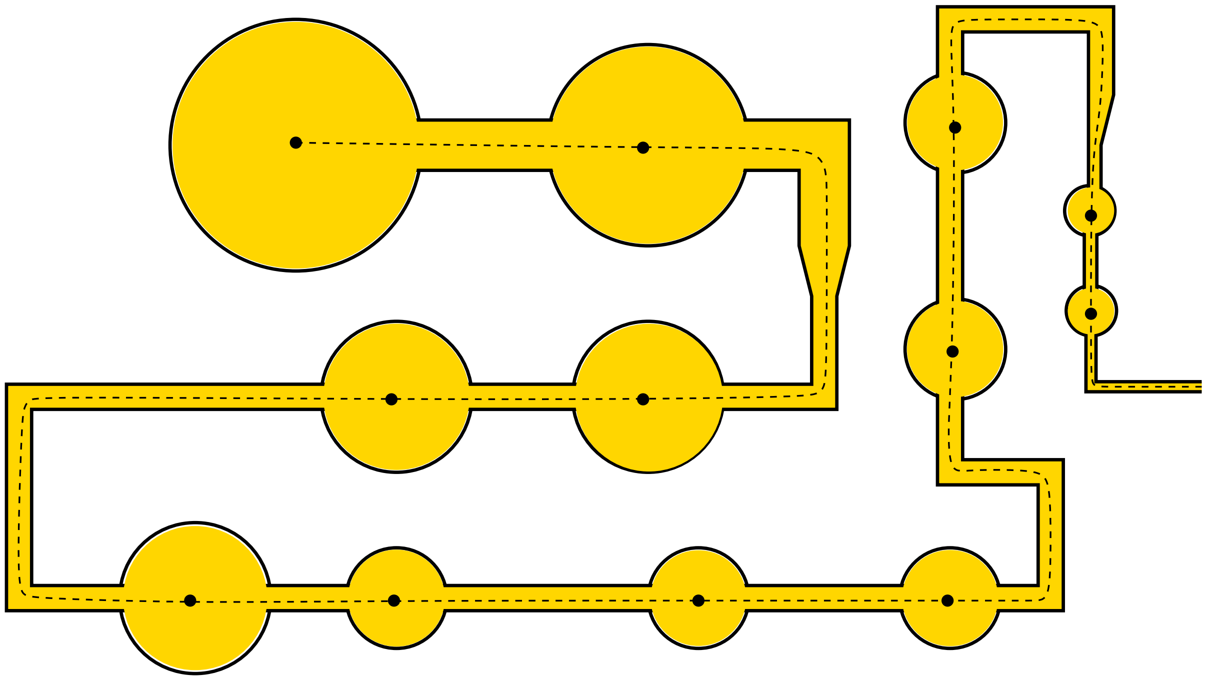
More precisely, given any $\epsilon > 0$, any finite set X and any map $h : X \to X$ there is a rational r whose post-singular set $P(r)$ is the same size as X, $P(r)$ ϵ-approximates X in Hausdorff metric, and $|r - h| < \epsilon$ on P.



Theorem (Lazebnik, B.): Let X be discrete ($\geq$ 4 points), let $h : X \to X$ be any map, and let $\epsilon > 0$. Then there is a transcendental meromorphic function f and a bijection $\psi : X \to P(f)$ so that

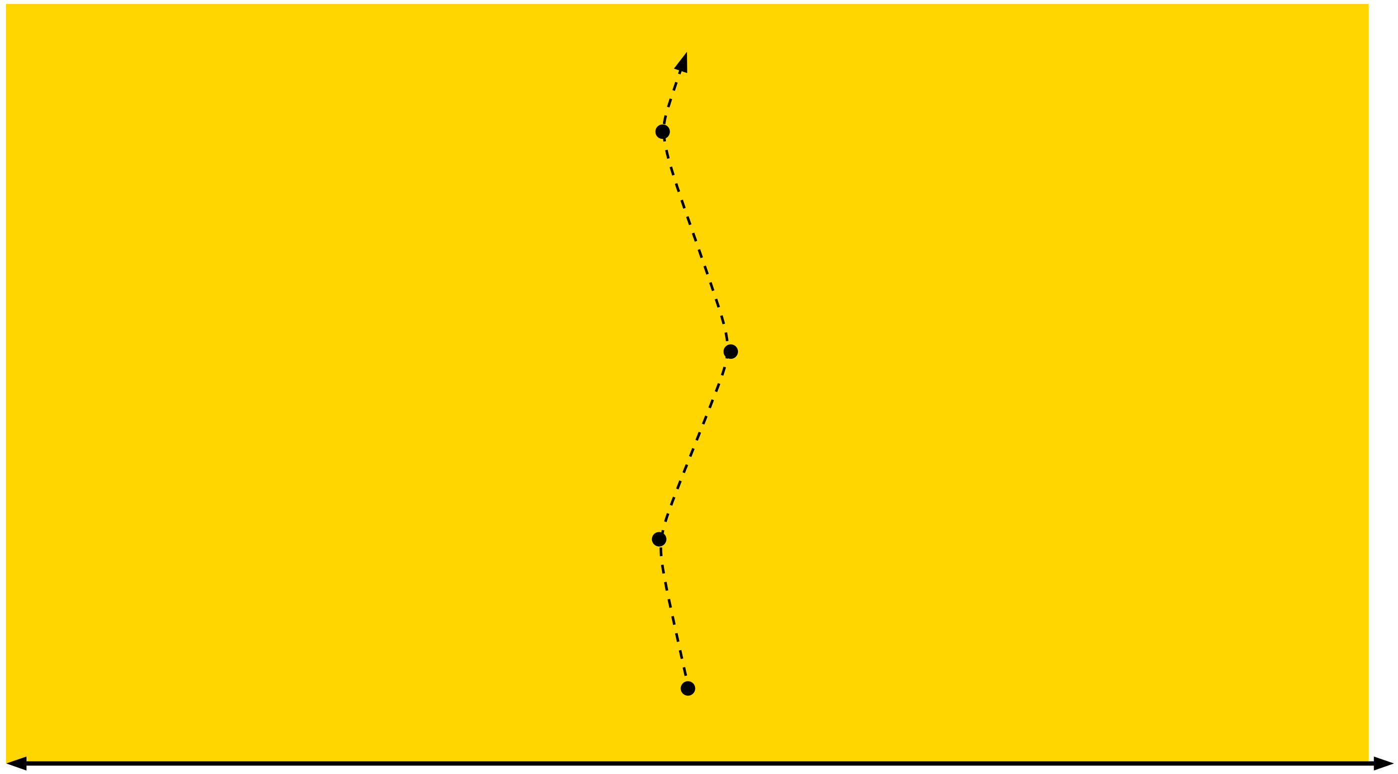
$$|\psi(z) - z| \leq \max(\epsilon, o(1)) \quad \text{and} \quad f = \psi \circ h \circ \psi^{-1} \text{ on } P(f).$$

Main argument (folding + fixed point thm) due to Kirill.

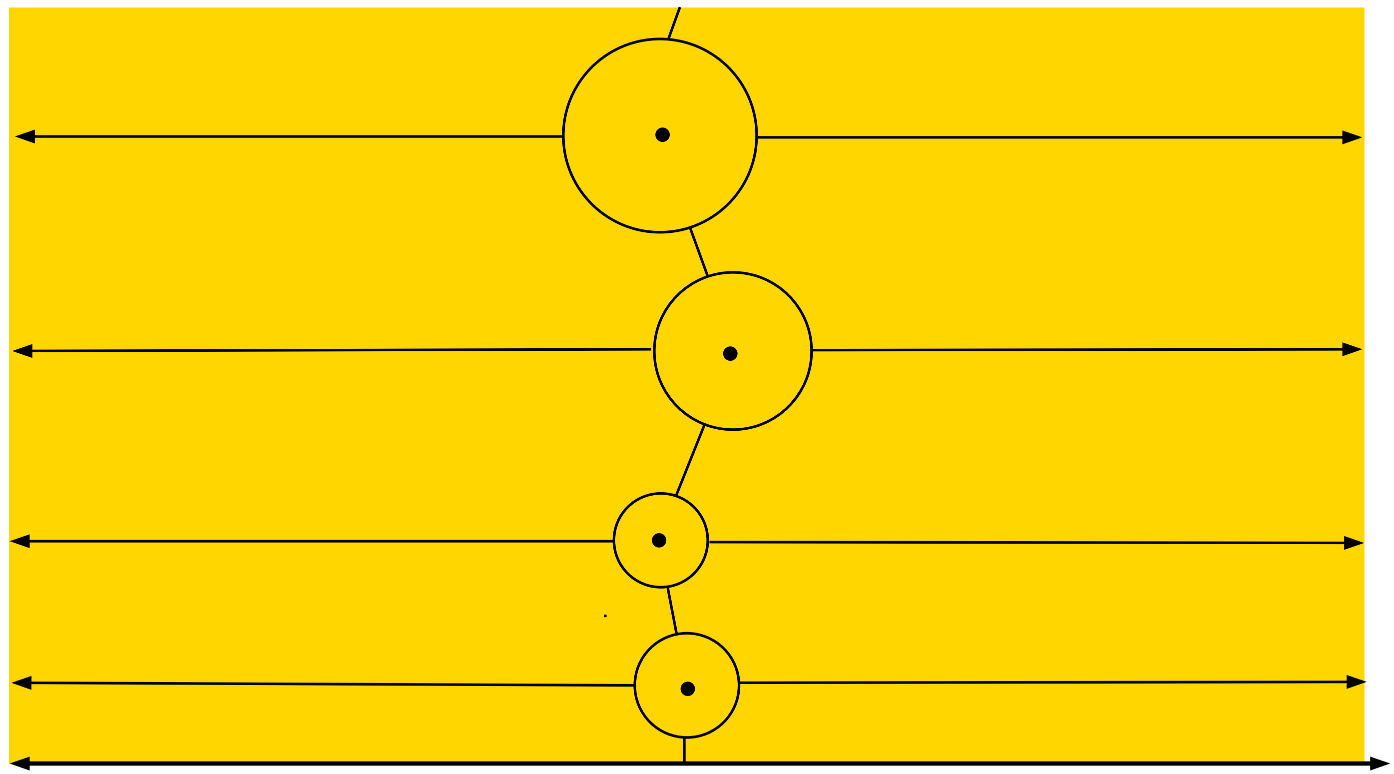


Reduce to case where ± 1 are in post-singular set and all other post-singular points are off the unit circle.

Construct a domain W that contains the given set, and so that these points lie very close to hyperbolic geodesic to ∞.



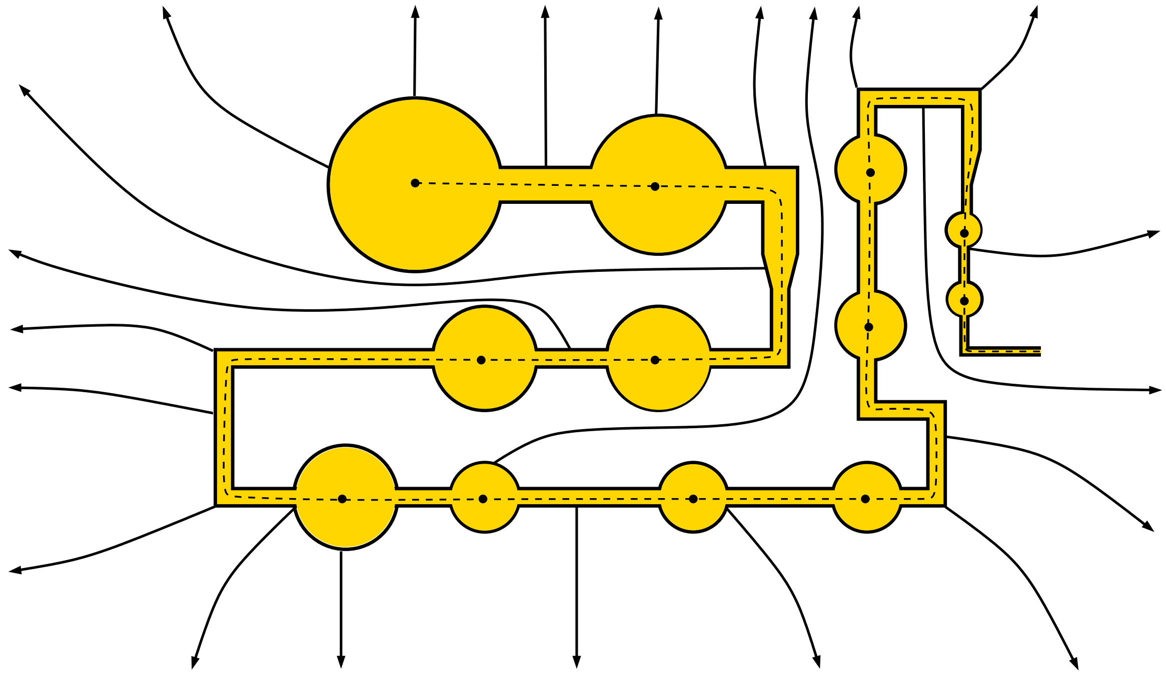
Conformally map W to upper half-plane; points almost on vertical ray.



Divide upper half-plane into R and D components.

Folding map sends centers anywhere we want in $\mathbb{D}$.

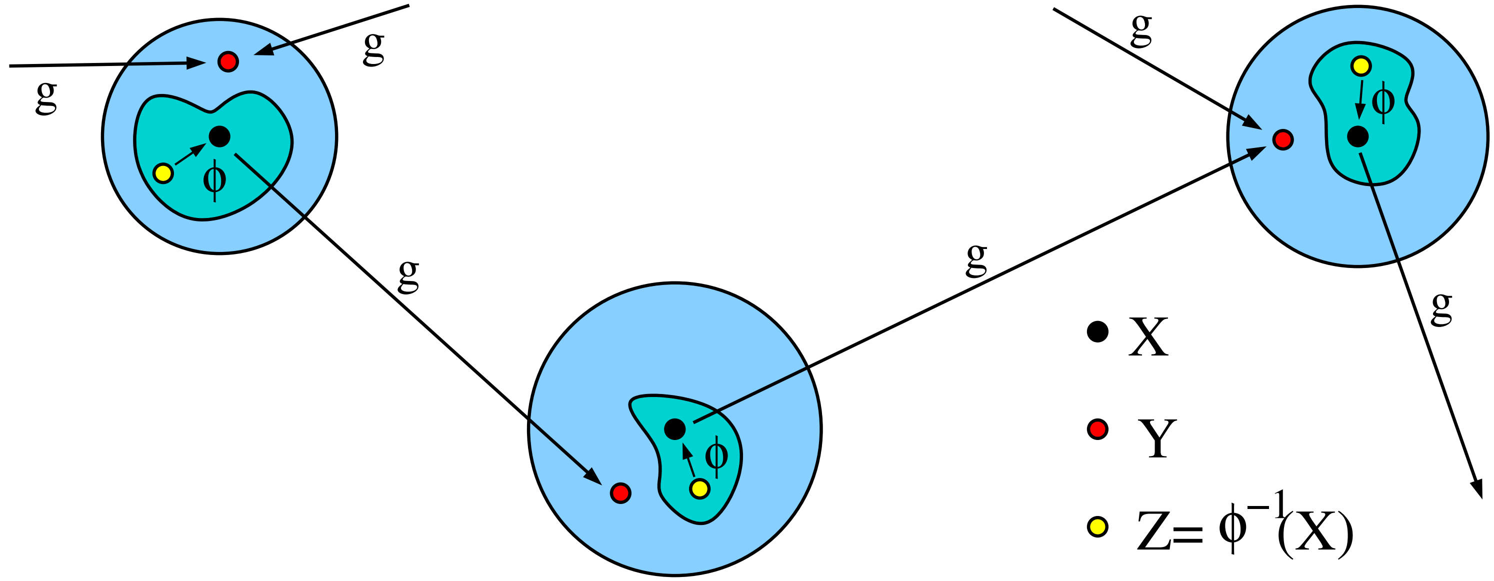
Analogous construction maps centers outside $\mathbb{D}$; introduces poles.



Transfer graph back to W. Decompose complement of W into R-components (not hard) and define map on whole plane by folding.

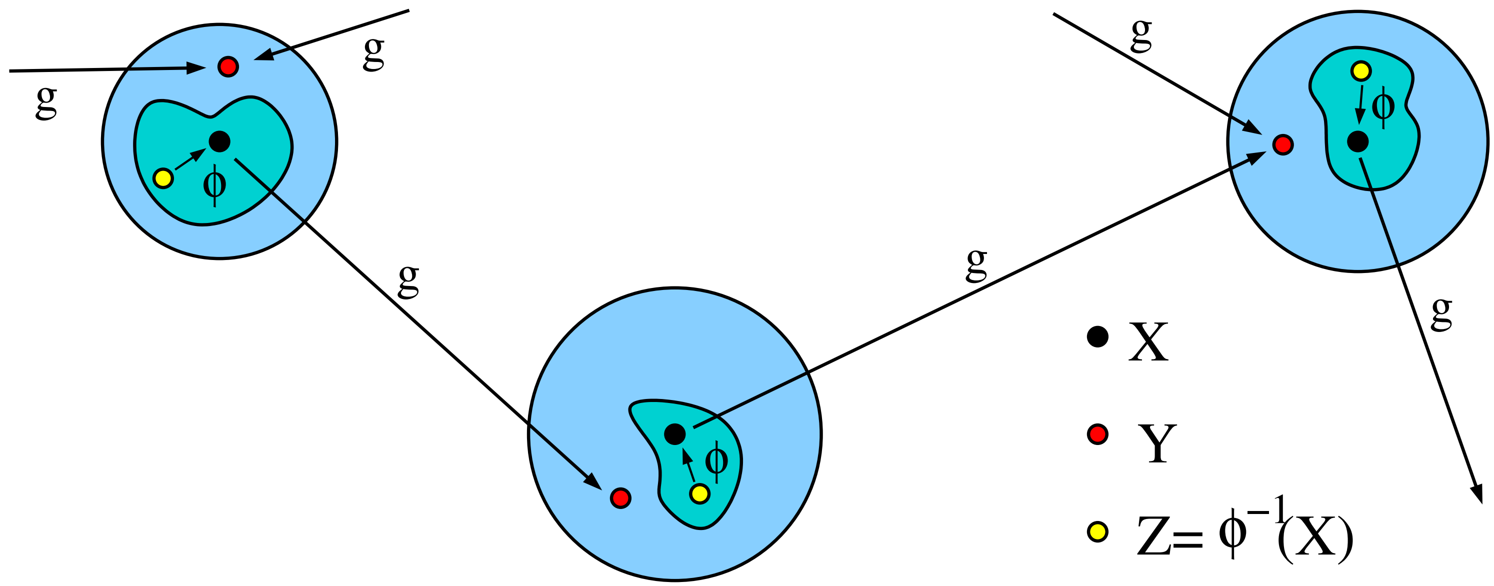
Gives quasi-meromorphic map g with desired post-singular behavior.

Mapping theorem gives a meromorphic $f = g \circ \phi$. However, because of ϕ, the post-singular set might not be invariant under f. How to fix this?



Replace X (black) by a union of disjoint disks (blue), and let $Y \approx X$ be one point from each disk (red). As above, construct g so $X \to Y$. Get meromorphic $f = g \circ \phi$ that has singular points at $Z = \phi^{-1}(X)$ (yellow).

We want $Z = Y$ (yellow points = red points).



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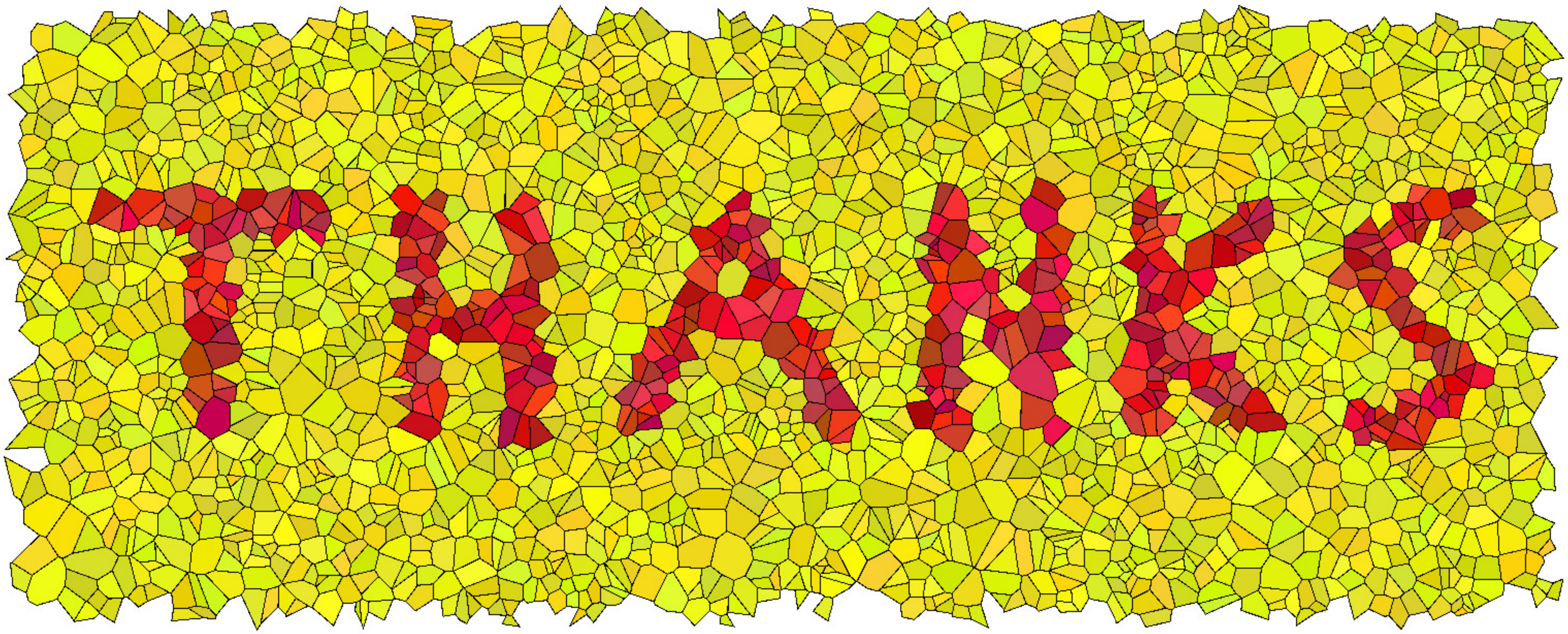
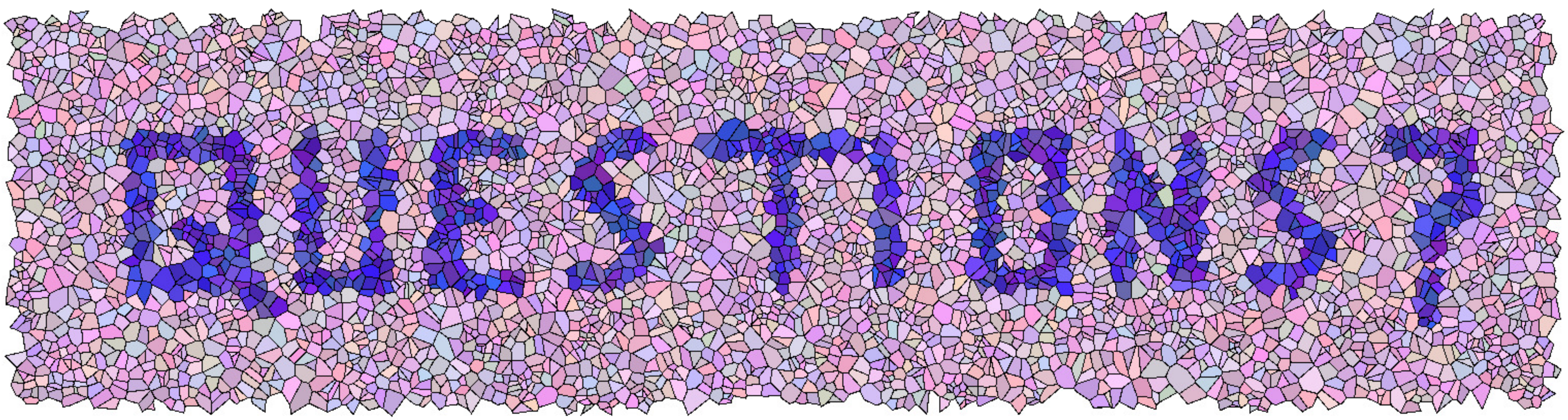
Z (yellow) is continuous function of Y (red). Because ϕ is close to the identity, Z remain inside the disks. An infinite-dimensional fixed point theorem gives choice of Y so that $Z = Y$, as desired.


THANKS


QUESTIONS?

Coming Attractions

"Mappings and Meshes: connections between continuous and discrete geometry"

2 lectures: 11am Fri May 7, and 11am Sat May 8 (NY time)

FRG Workshop on Geometric Methods for Analyzing Discrete Shapes

https://cmsa.fas.harvard.edu/frg-2021/

Abstract: I will give two lectures about some interactions between conformal, hyperbolic and computational geometry. The first lecture shows how ideas from discrete and computational geometry can help compute conformal mappings, and the second lecture reverses the direction and shows how conformal maps can give meshes of polygonal domains with optimal geometry.