An A_1 weight not comparable to any QC Jacobian

Christopher J. Bishop
SUNY Stony Brook

copies of lecture slides available at
`www.math.sunysb.edu/~bishop/lectures/lec.html`

Theorem: There is an A_1 weight w on $\mathbb{R}^2$ which is not comparable to any QC Jacobian.

area$(\{x : w(x) > \lambda\}) \to 0$ as fast as we wish.

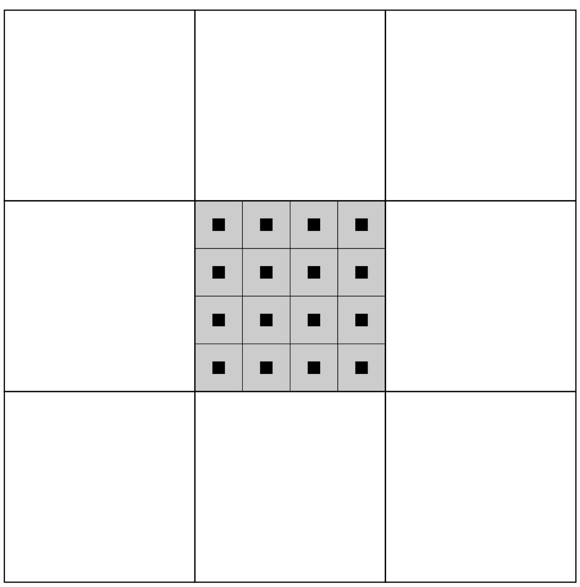
Theorem: There is a set E with zero area and a $K > 1$ so that no function that tends to ∞ on E is comparable to the Jacobian of a K-QC map.

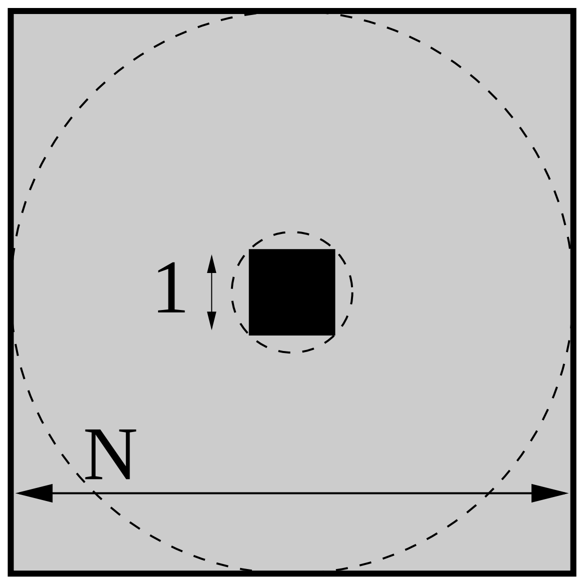
$w \in A_1$ if $\frac{1}{\text{area}(B)} \int_B w = O(\mathrm{essinf_B w})$

Idea of proof: Build a set E and a weight w so $w \nearrow \infty$ as $z \to E$. Show that if $J_f \sim w$, then $f(E)$ contains rectifiable curve γ. Then f^{-1} has Jacobian 0 on γ, so $f^{-1}(E)$ is a point. This is contradiction, so no such f exists.

E is a Sierpinski carpet:

- Given a square Q, divide it into 9 subsquares.
- Divide center square into M^2 equal subsquares.
- Divide each of these into N^2 equal subsquares.
- Remove the M^2 center squares of size $1/LMN$.





At nth stage use:

$$\sum_n N_n^{-2} = \infty, \tag{1}$$

$$\sum_n N_n^{-3} < \infty, \quad \sum_n M_n^{-2} < \infty. \tag{2}$$

E.g., $M_n = n$, $N_n = \lfloor \sqrt{n} \rfloor$.

Define E_n inductively using these sequences.

E has zero area:

$$\text{area}(E_n) \le C \prod_{k=1}^{n} (1 - \frac{1}{k}) \to 0$$

The definition of the weight w:
Let F_n be s_n-neighborhood of E_n, s_n is side length of nth generation squares.
Set $w(x) = 1$ for $x \notin F_1$ and set

$$w(x) = 1 + \sum_{k=1}^{\infty} a_k \chi_{F_k}(x), \quad \sum_{k=1}^{\infty} a_k = \infty$$

otherwise. This is integrable as long as

$$\sum_{k=1}^{\infty} a_k \cdot \mathrm{area}(F_k) < \infty$$

Easy to check $w \in A_1$.

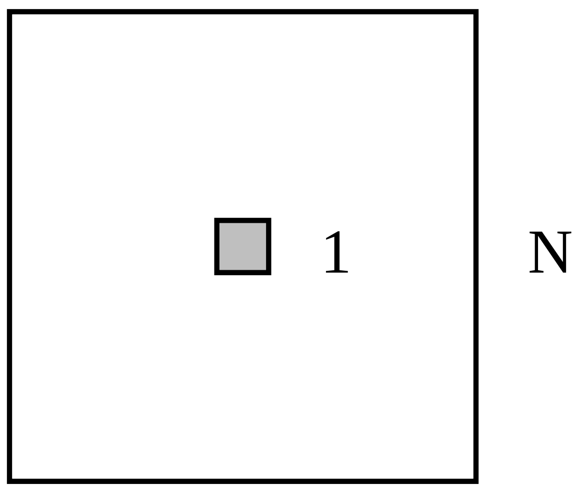
We can take $\mathrm{area}(\{x : w(x) > \lambda\}) \to 0$ as quickly as we wish, as claimed earlier.

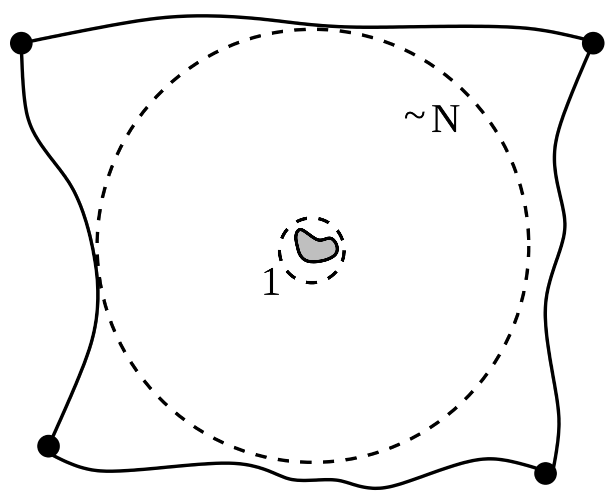
A “good path” for $f(E_n)$ is a polygonal path which is contained in $f(E_n)$ and all of whose vertices lie in sets of the form $f(\partial Q)$ where Q is a nth generation square.

Good Path Lemma: Suppose f is K-QC and $J_f \simeq w$. If γ is a good path for $f(E_{n-1})$, then there is a good path γ' for $f(E_n)$ such that

1. every vertex of γ is also a vertex of γ',
2. $\ell(\gamma') \leq \ell(\gamma)(1 + O(M_n^{-2}) + O(N_n^{-3}))$.

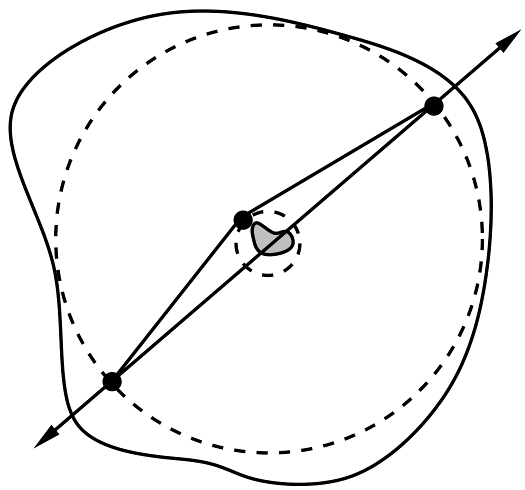
Lemma: if f is QC and $J_f \simeq w$ then $f(E_n)$ has annuli around its holes with appoximately the same modulus that E_n does.

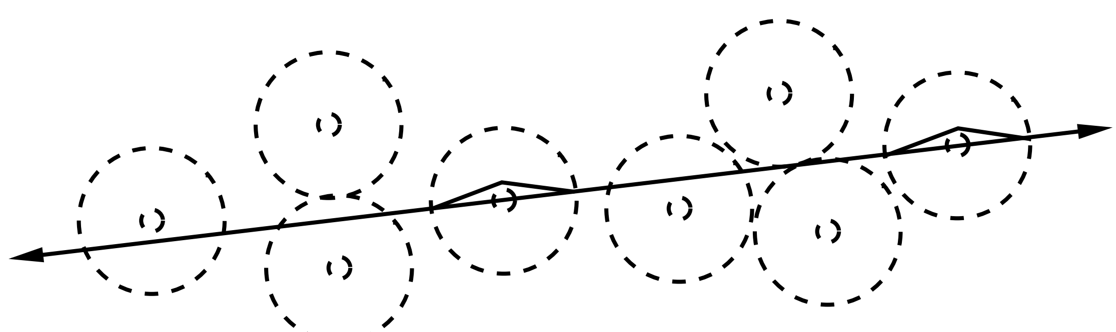




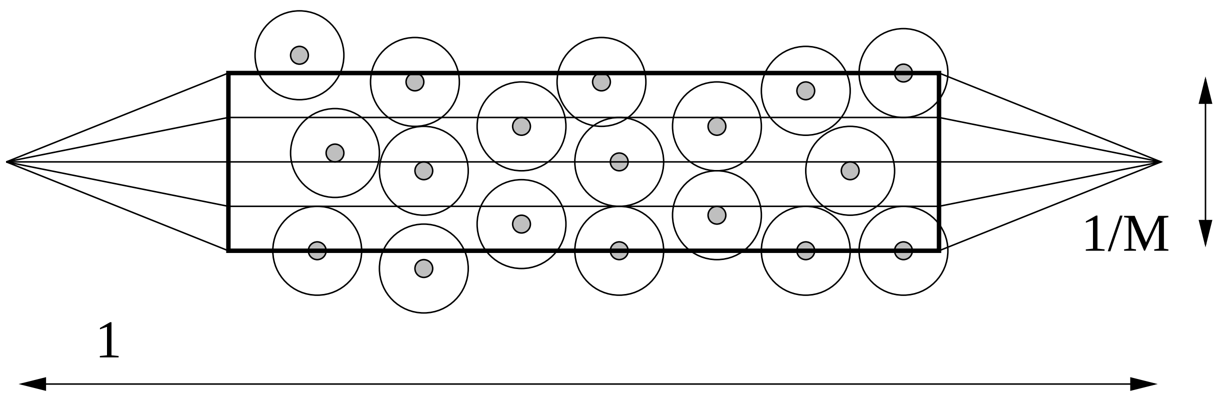
Suppose γ is segment in $f(E_{n-1})$. Assume length 1. Only have to change γ if it hits "hole" S of $f(E_n)$, which is surrounded by annulus A of size $N\text{diam}(S)$. Replacement longer by

$$N^{-2}\text{diam}(A) \simeq N^{-1}\text{diam}(S)$$





Make this precise by defining family of paths of length $1 + O(M^{-2})$.



$$\begin{aligned}
&\text{average extra length} \\
&\quad \simeq M \int_0^\delta \sum_j \chi_{S_j^*} \operatorname{diam}(S_j)/N \\
&\quad \simeq M \sum_j \operatorname{diam}(A_j)^2/N^3 \\
&\quad \simeq N^{-3}.
\end{aligned}$$

Some path with length $1 + O(M^{-2}) + O(N^{-3})$.

Corollary: There is a surface $S \subset \mathbb{R}^3$ that is 2-regular and LLC, but not bi-Lipschitz to plane.

X is Ahlfors 2-regular if $\mathcal{H}_2(B(x,r)) \simeq r^2$

X is LLC if $B(x,r)$ is contractible in $B(x,Cr)$.

Bonk and Kleiner showed: LLC and 2-regular implies quasisymmetric image of $\mathbb{R}^2$. Laakso gave surface which is LLC and Ahlfors 2-regular, but not BL equivalent to $\mathbb{R}^2$. His example is not embeddable in R^n.

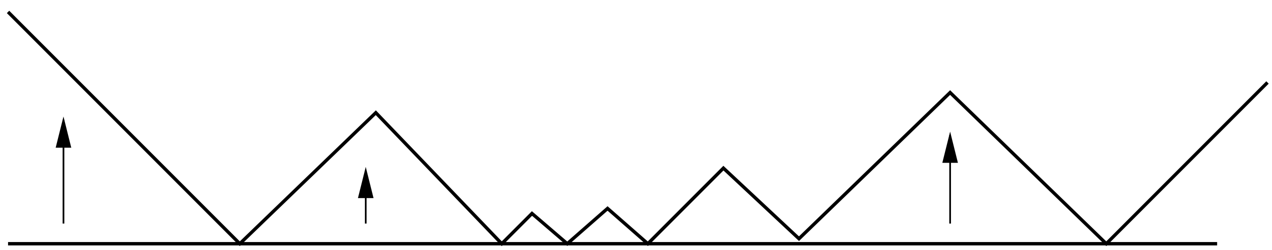
Stephen Semmes proved that any A_1 weight is the Jacobian of an embedding of $\mathbb{R}^2$ into some R^n and the image is LLC and 2-regular.

Bonk, Heinonen and Saksman observed that this embedding is biLipschitz to $\mathbb{R}^2$ iff the weight w is comparable to a QC Jacobian.

Surface in R^3 not BL equivalent to plane

First project to a Lipschitz graph,

$$P(x) = (x, \min(1, \mathrm{dist}(x, E))).$$



Follow by "snowflake" map $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$

$$J_\Phi(x, y, t) \simeq f(t)^3$$

where $f(t) \nearrow \infty$ as $t \searrow 0$ and

$$w(z) \simeq f(\mathrm{dist}(z, E))^2.$$

(Φ exists by Bishop, David-Toro).

If $\Psi : S = \Phi(P(\mathbb{R}^2)) \to \mathbb{R}^2$ is biLipschitz then $\Psi \circ \Phi$ is QC with Jacobian $\simeq w$, $\Rightarrow\Leftarrow$.

Question: Is there a compact set E of measure zero so that no weight w that blows up on E can be comparable to a quasiconformal Jacobian?

Question: Is there a compact set E of measure zero so that every quasiconformal image of E contains a rectifiable curve?

Problem: Characterize surfaces in R^n which are BL images of plane.

Question: BL image of plane iff LLC + 2-regular + TST holds?

Question: TST holds iff curvature is a Carleson measure?