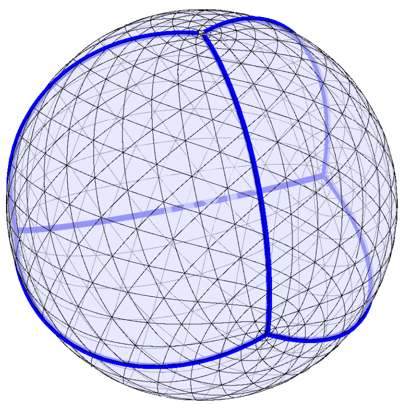
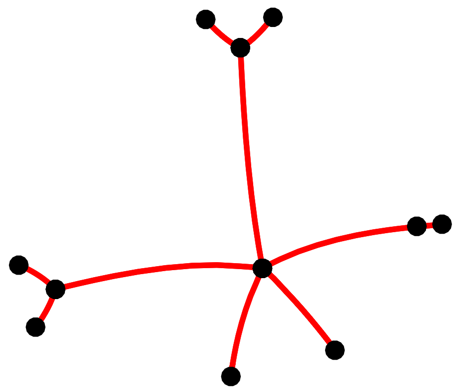
DESSINS AND DYNAMICS

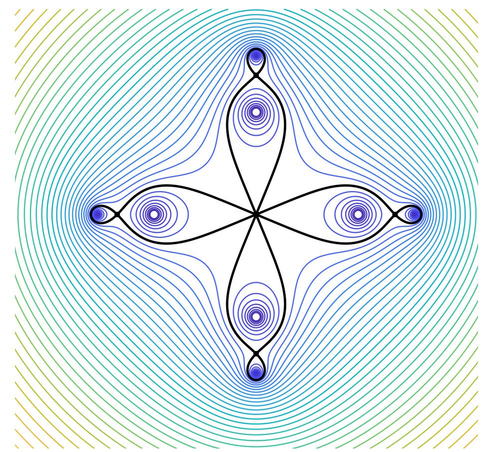
Christopher Bishop, Stony Brook

Dynamics Seminar, Imperial College
Wednesday, November 20, 2024

www.math.sunysb.edu/~bishop/lectures

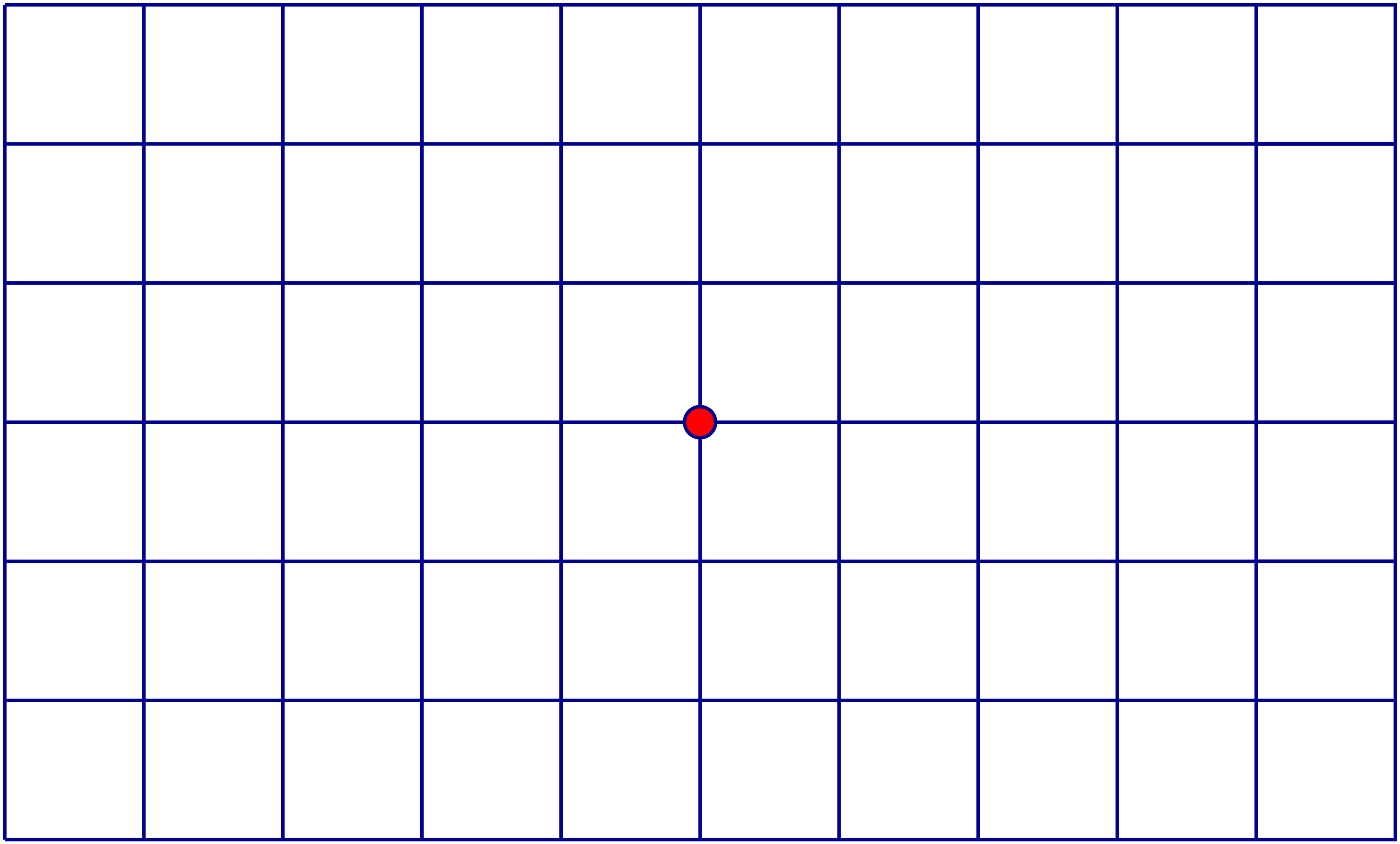


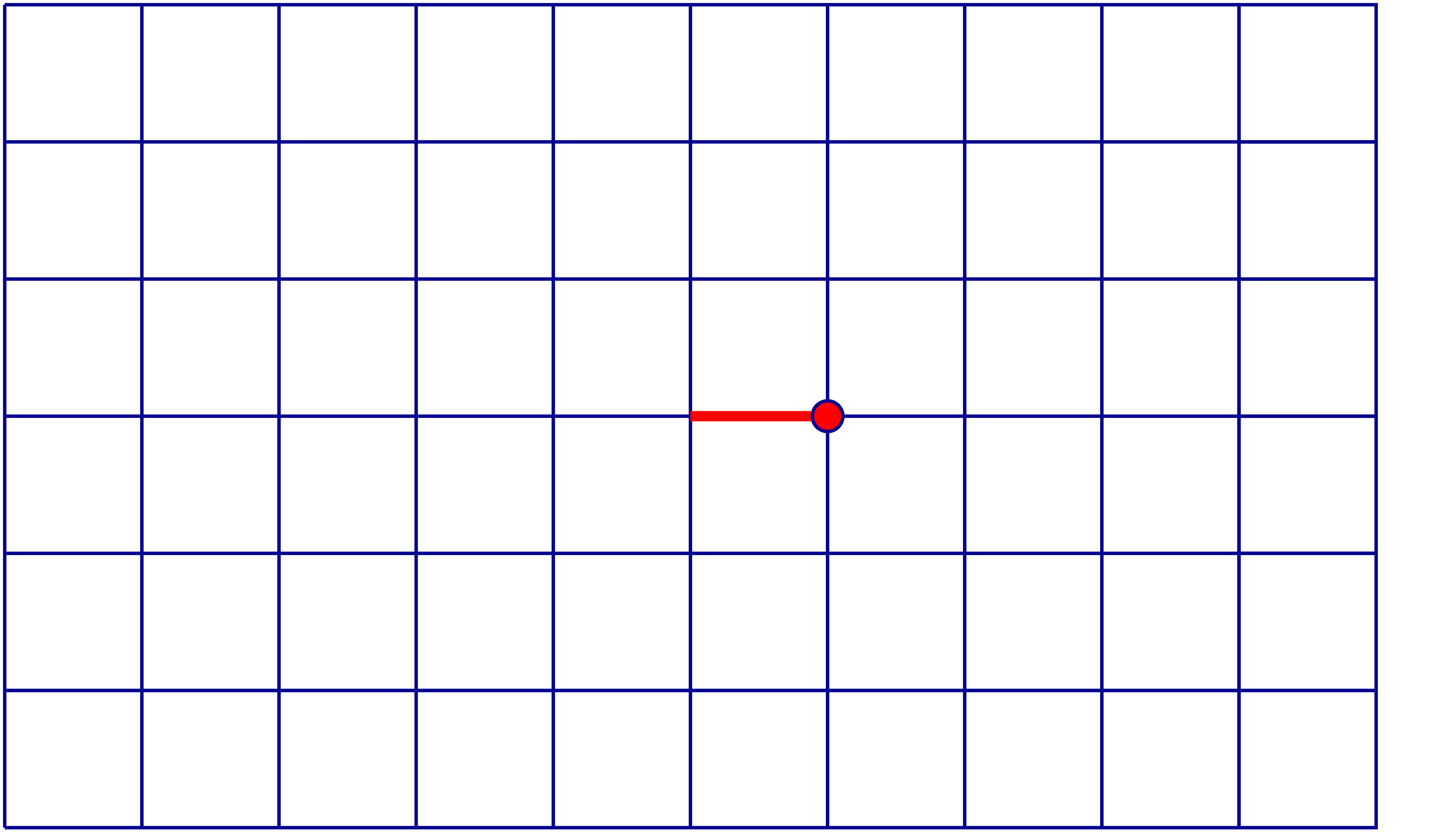


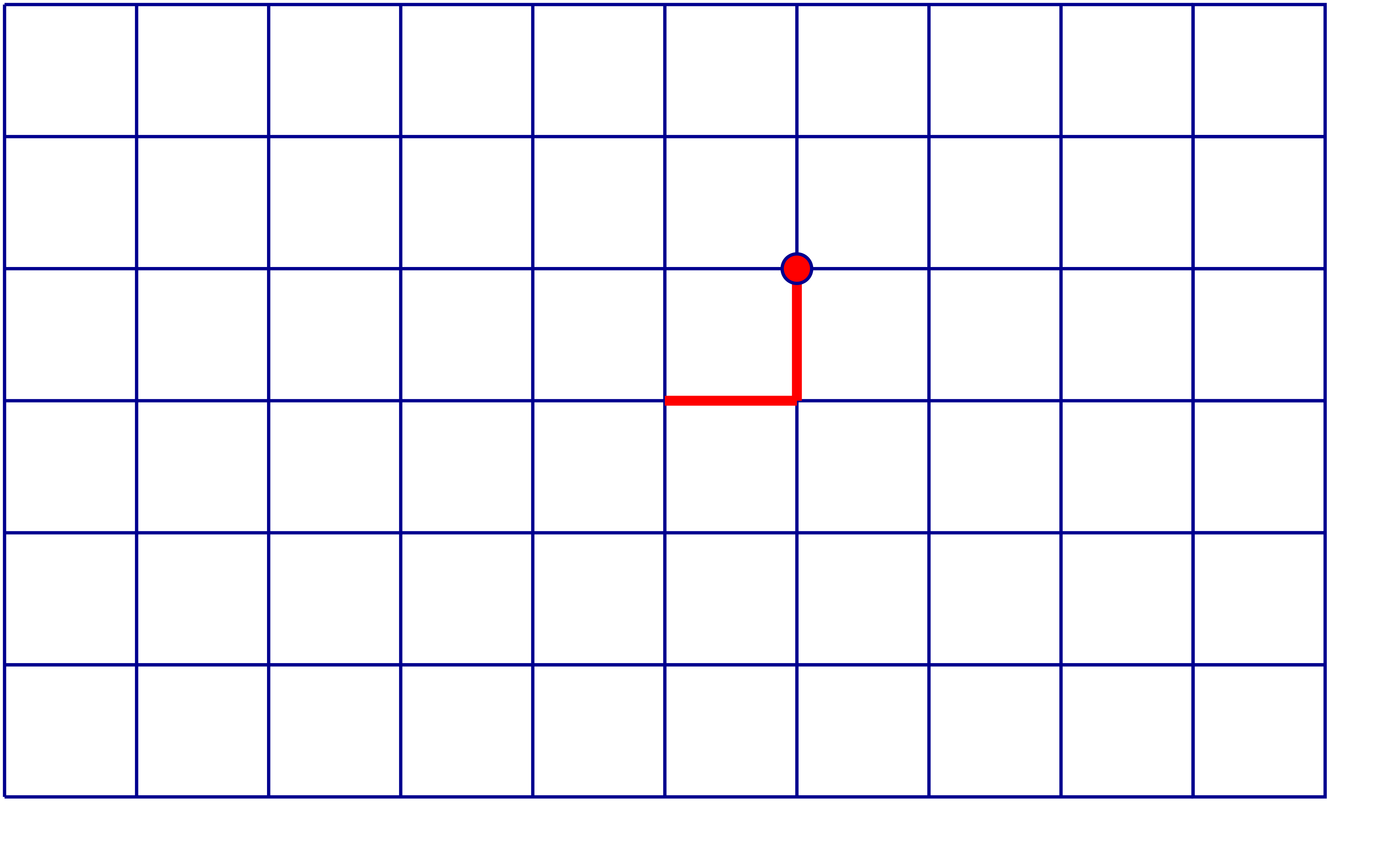


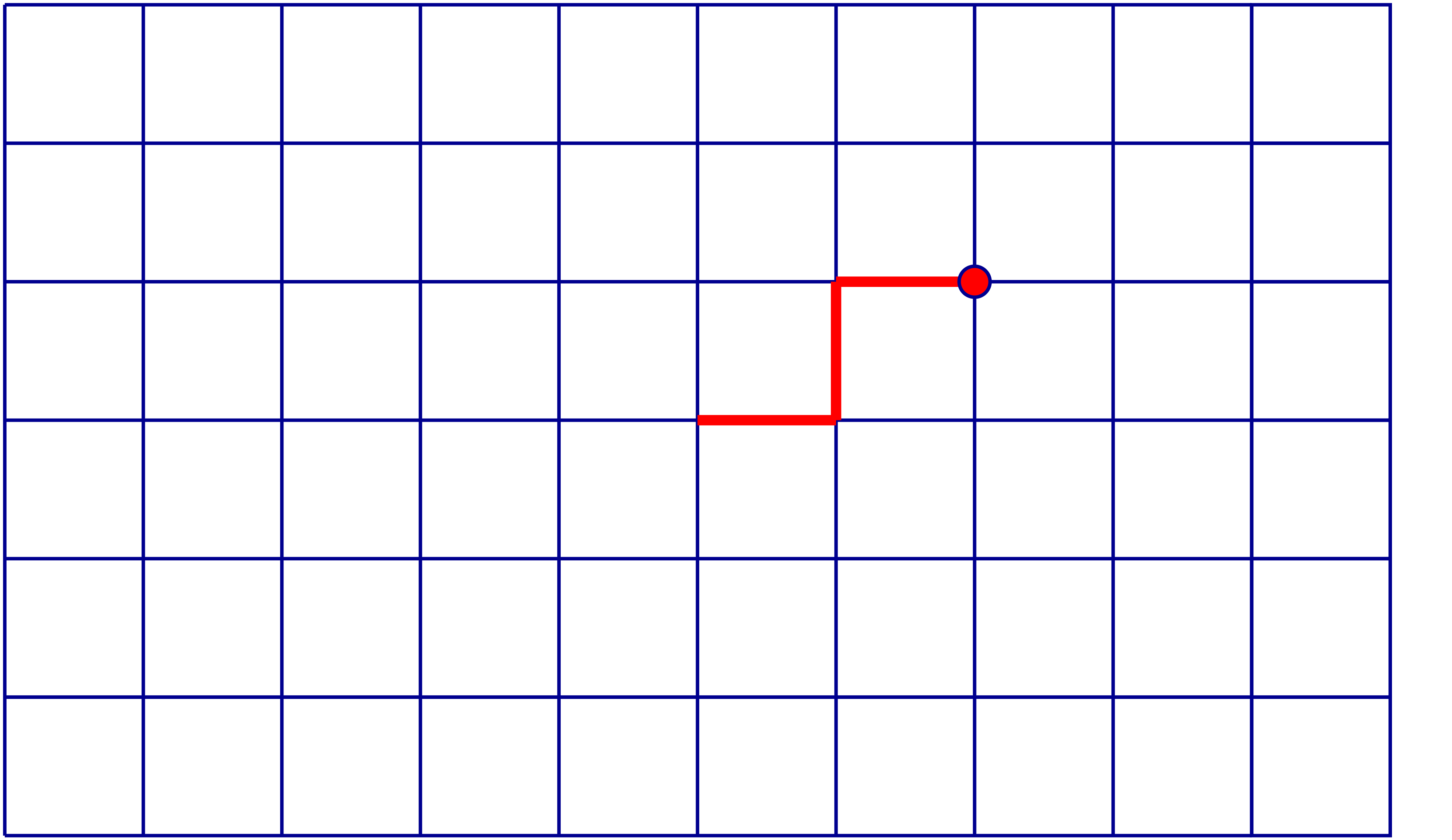
THE PLAN

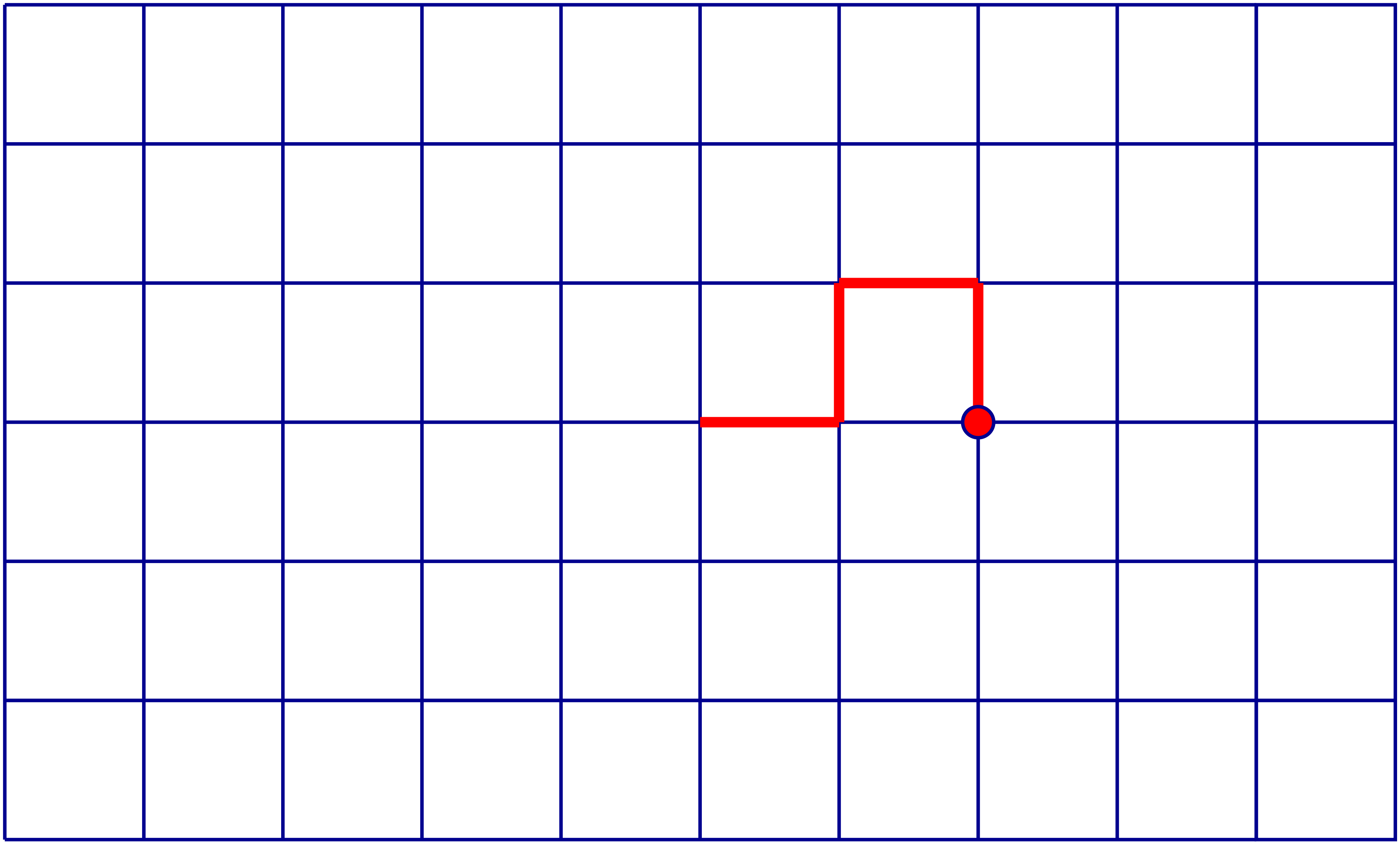
- Harmonic measure
- Finite trees and Shabat polynomials
- Infinite trees and entire functions
- Applications:
    - Wandering domains
    - Dimensions of Julia sets
    - Equilateral triangulations
    - Rational approximation
    - Prescribing critical orbits

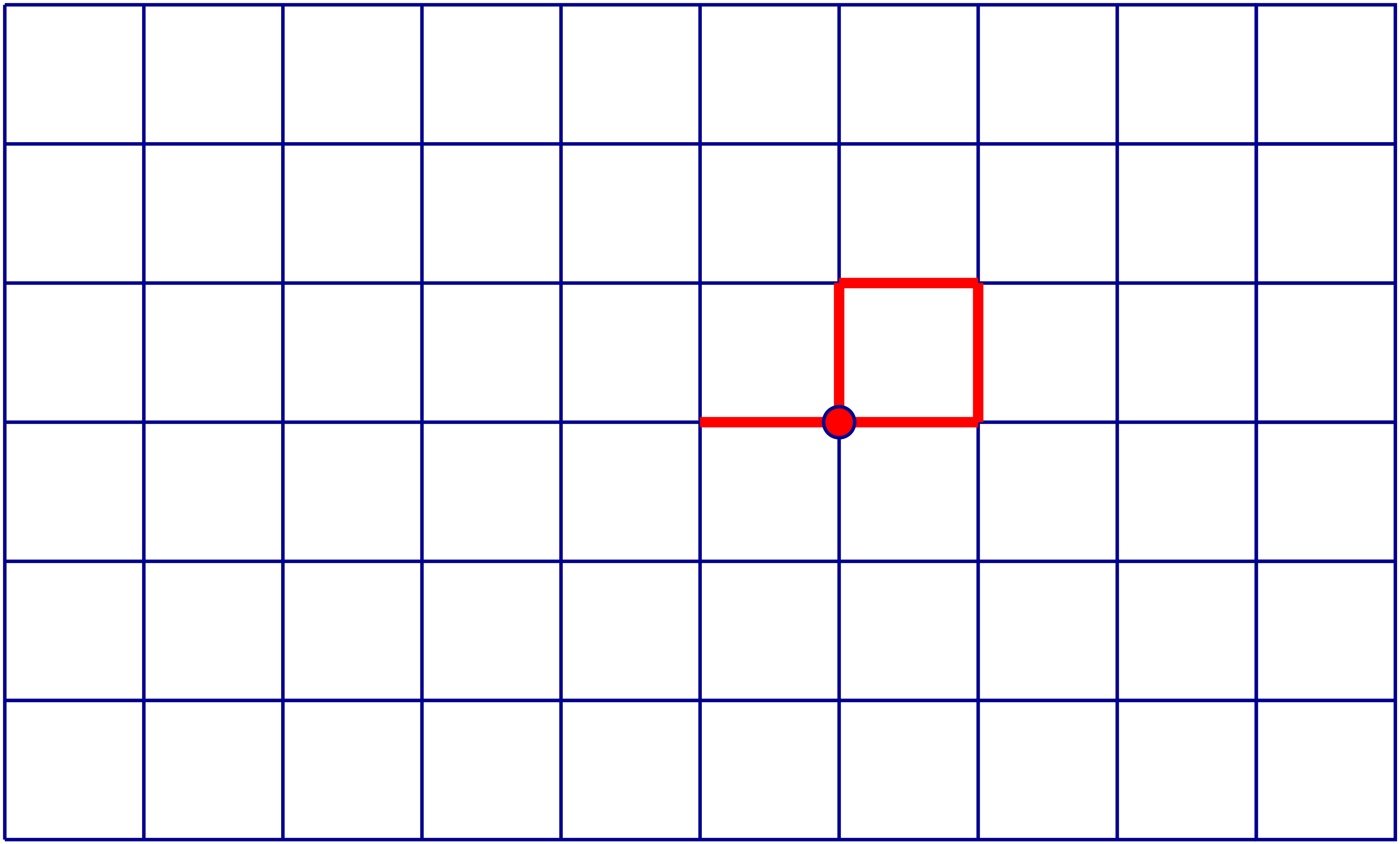


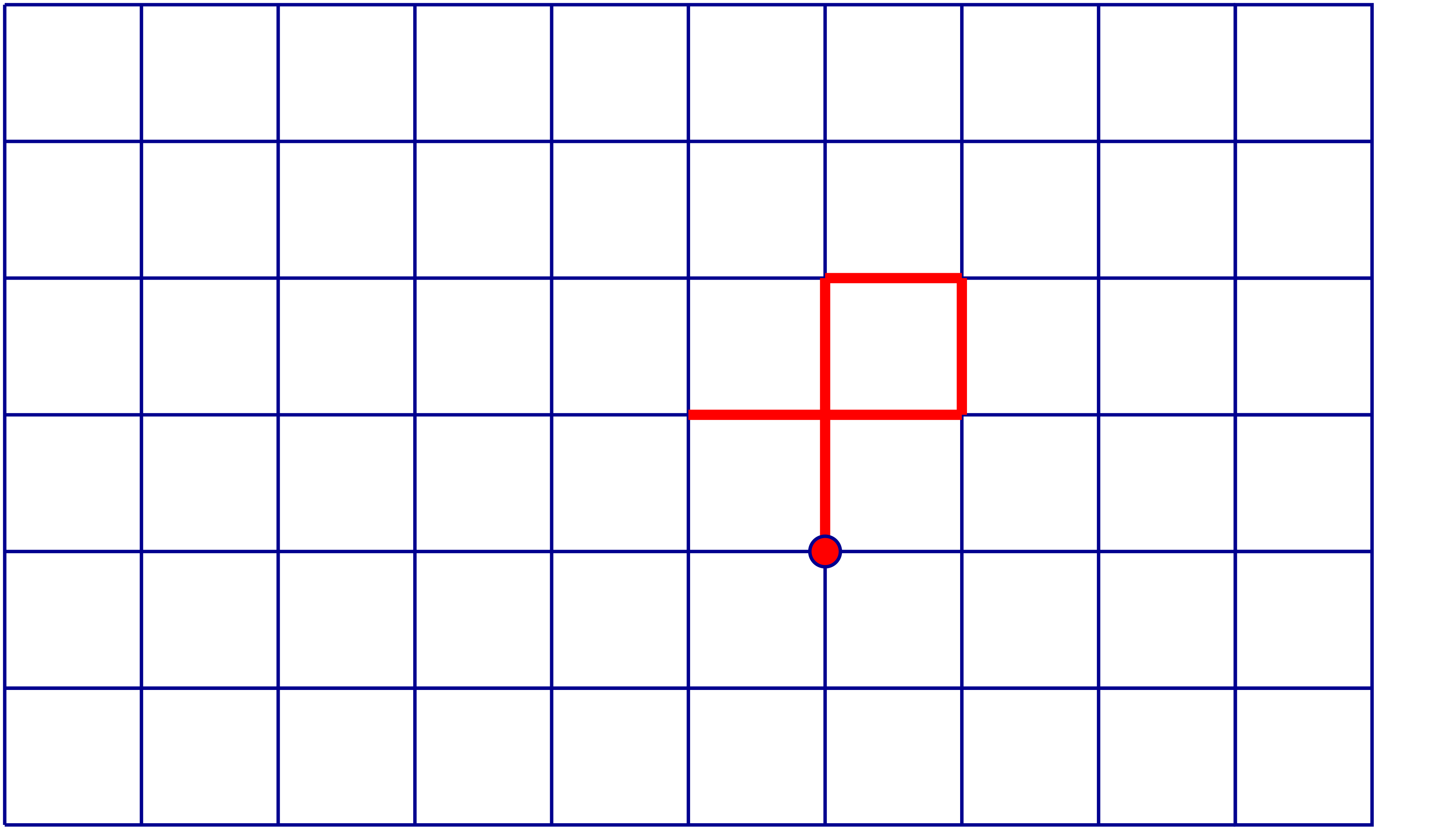


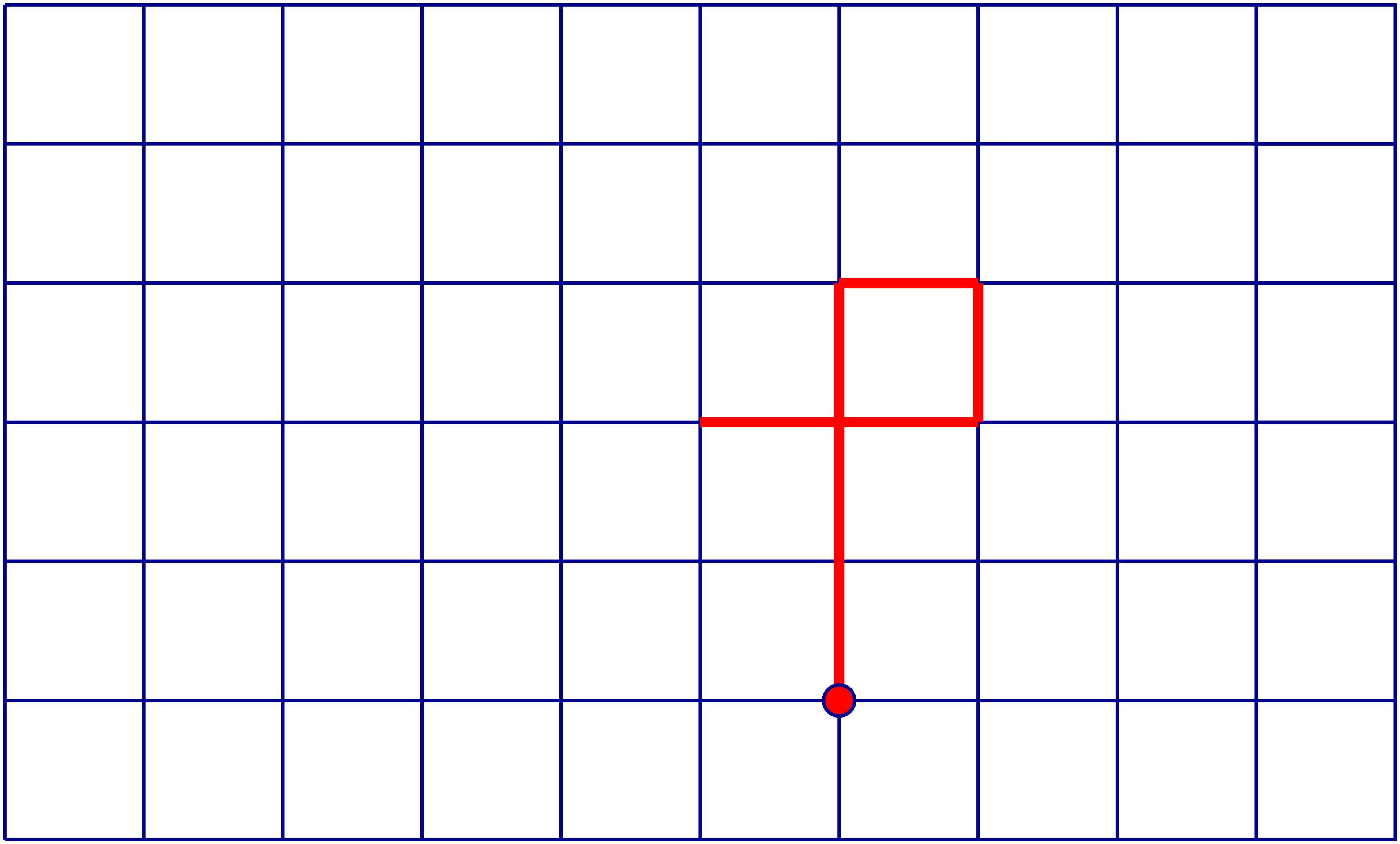


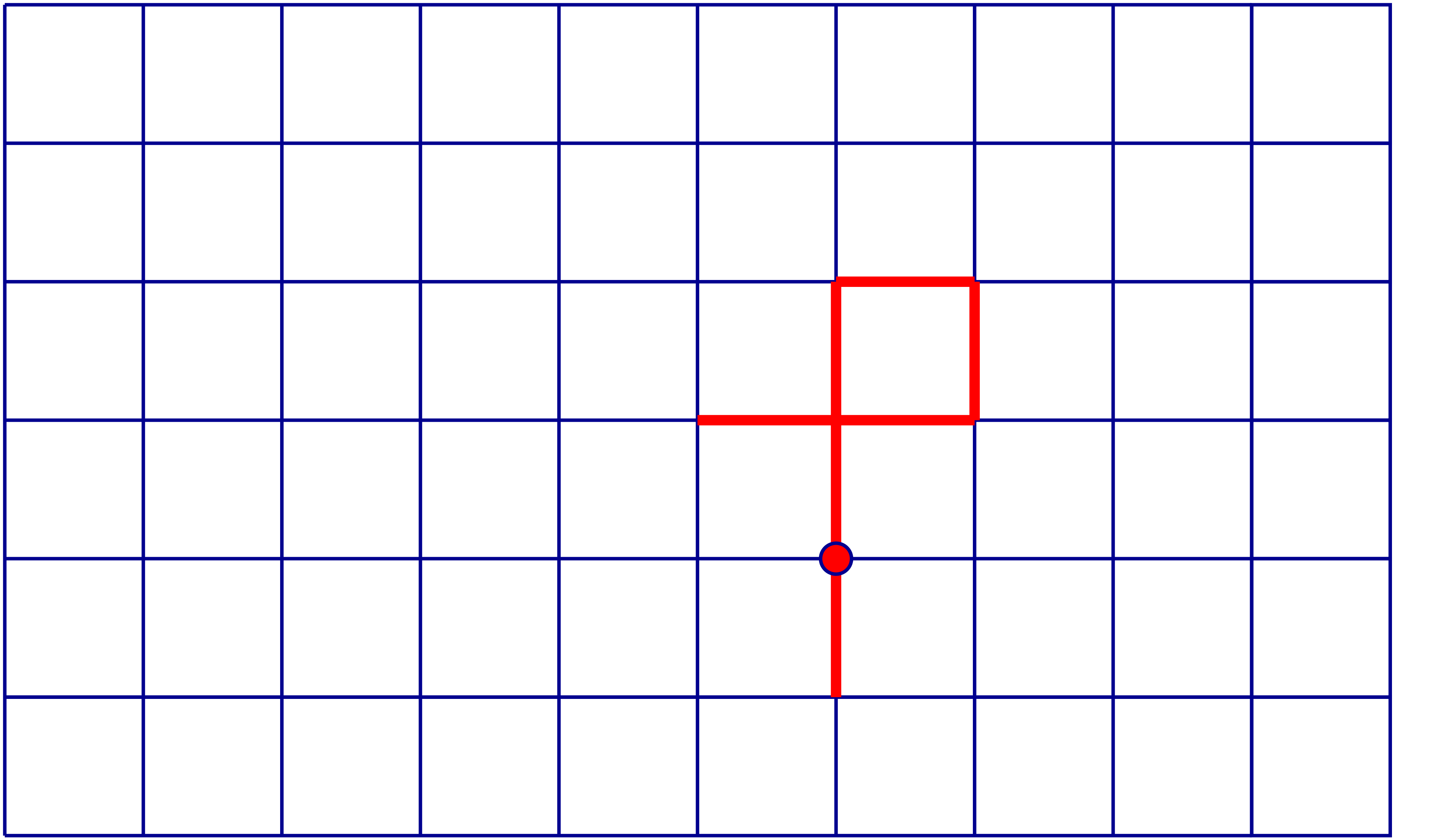


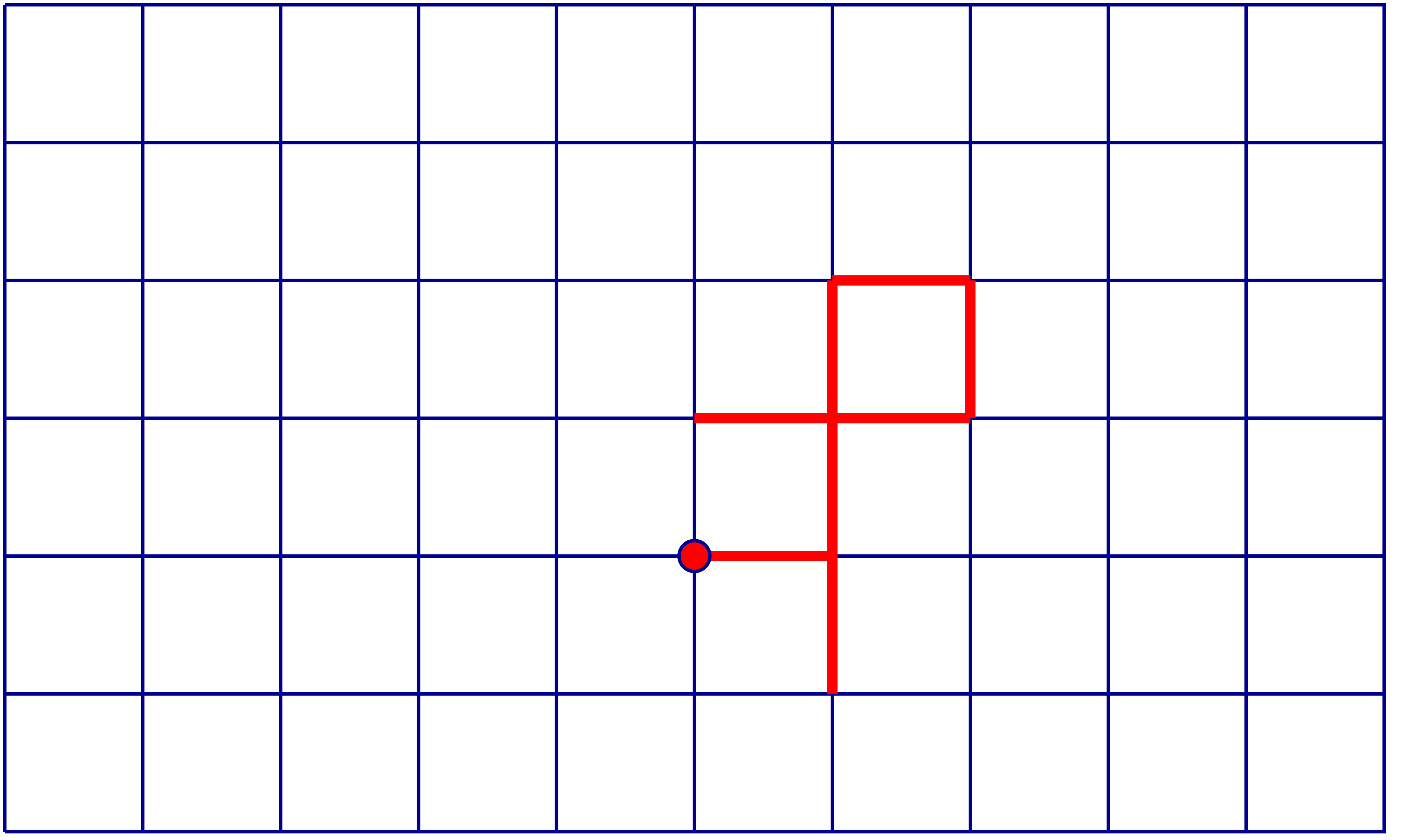


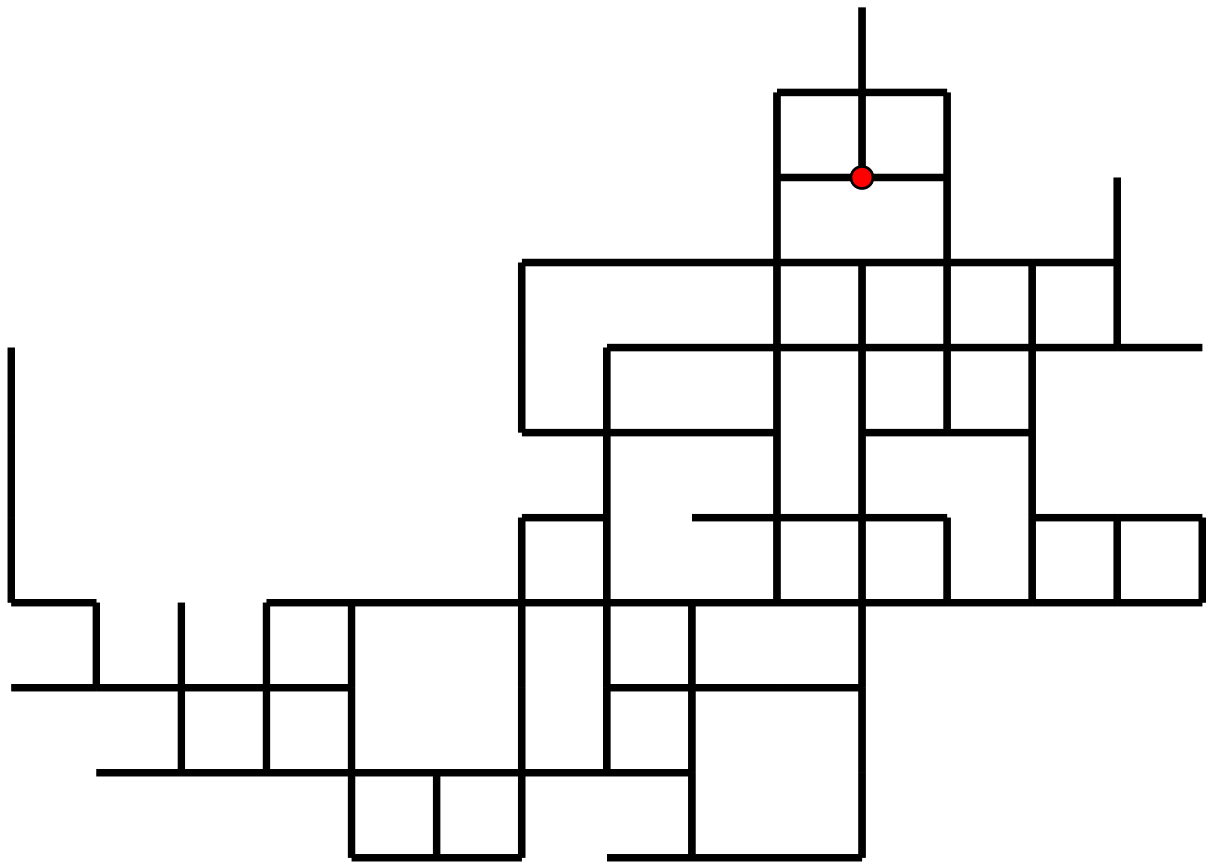




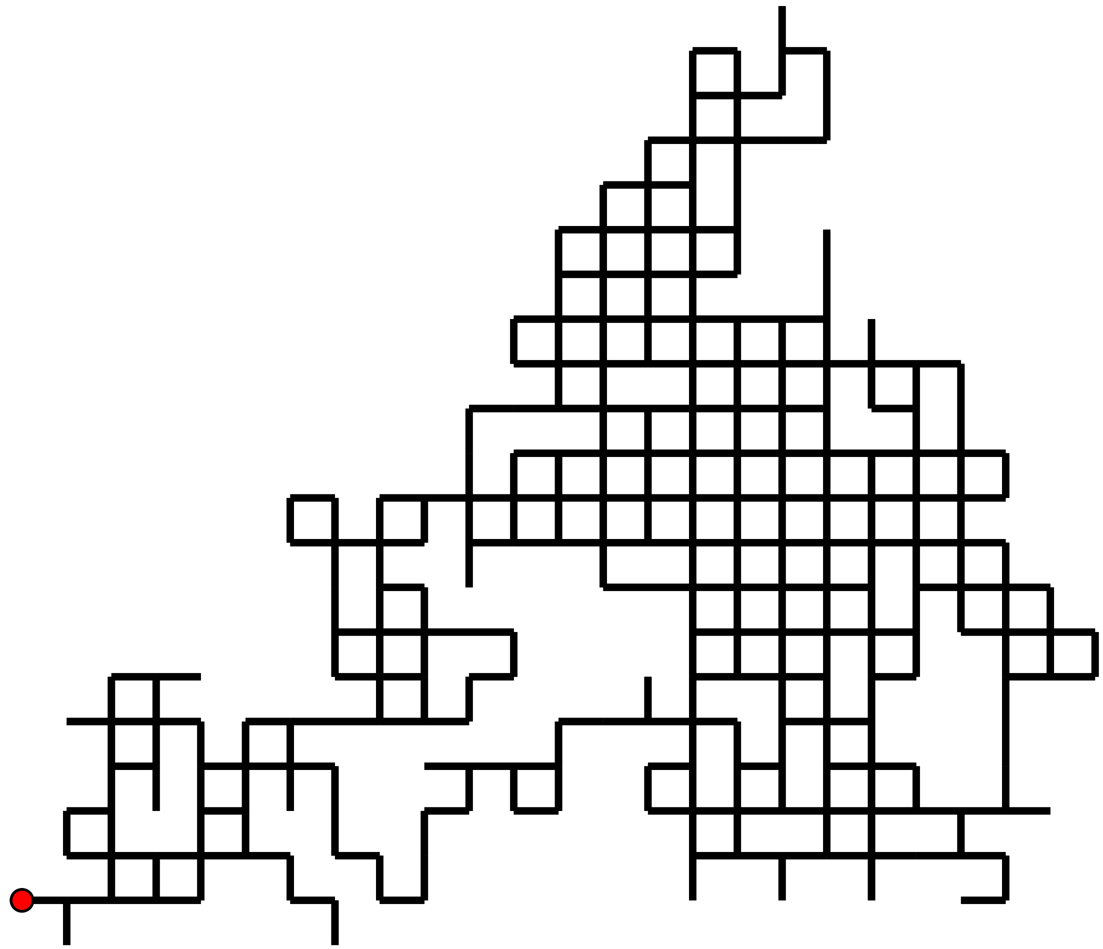




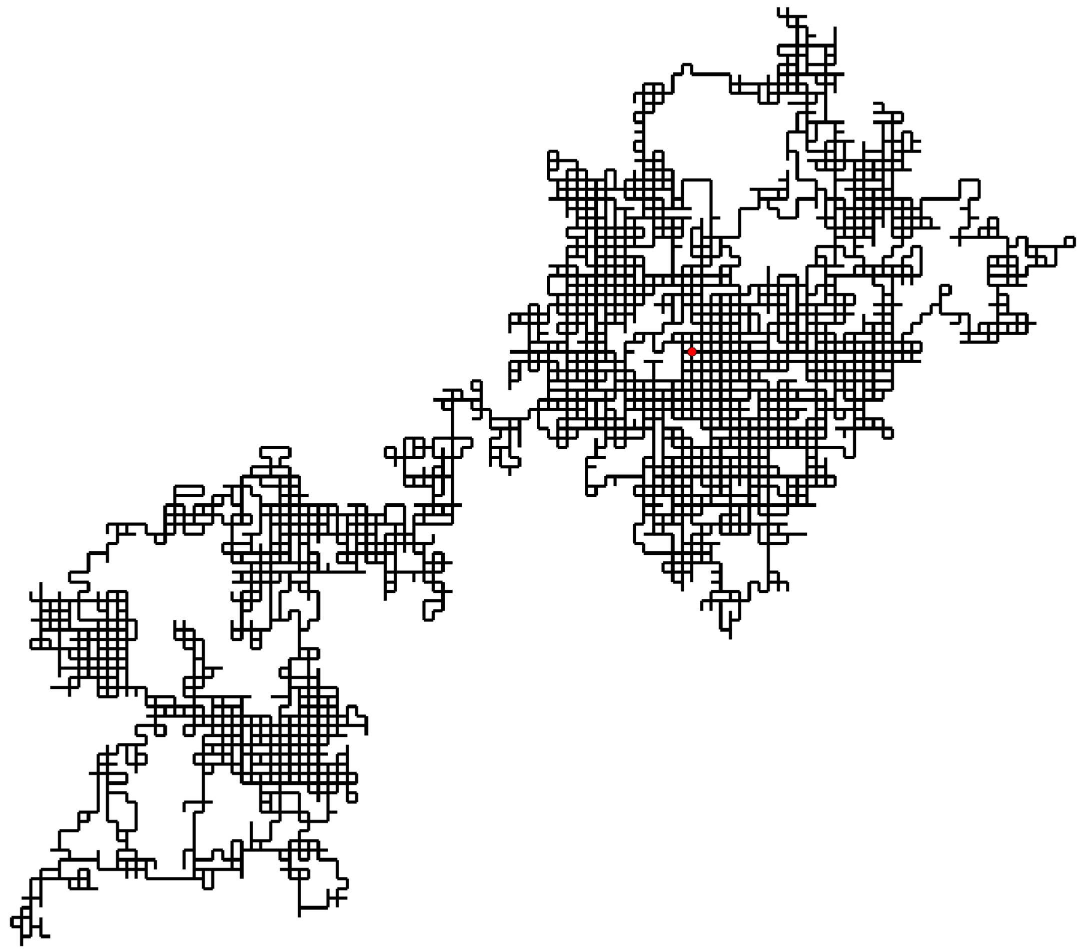




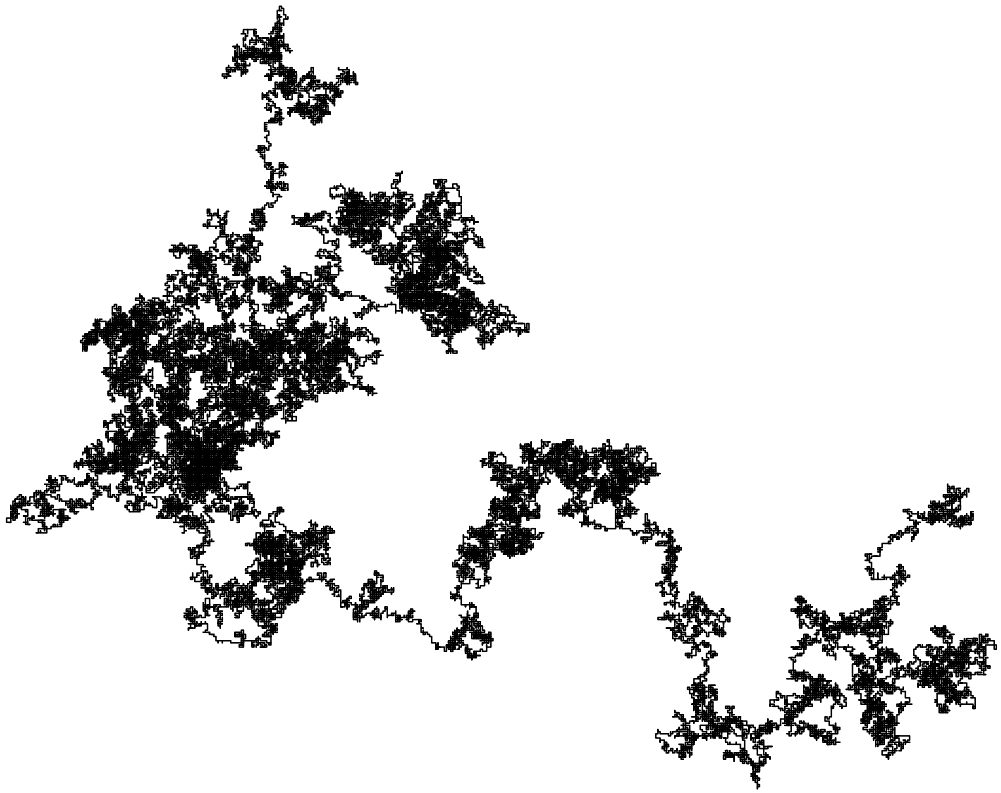
200 step random walk.



1000 step random walk.

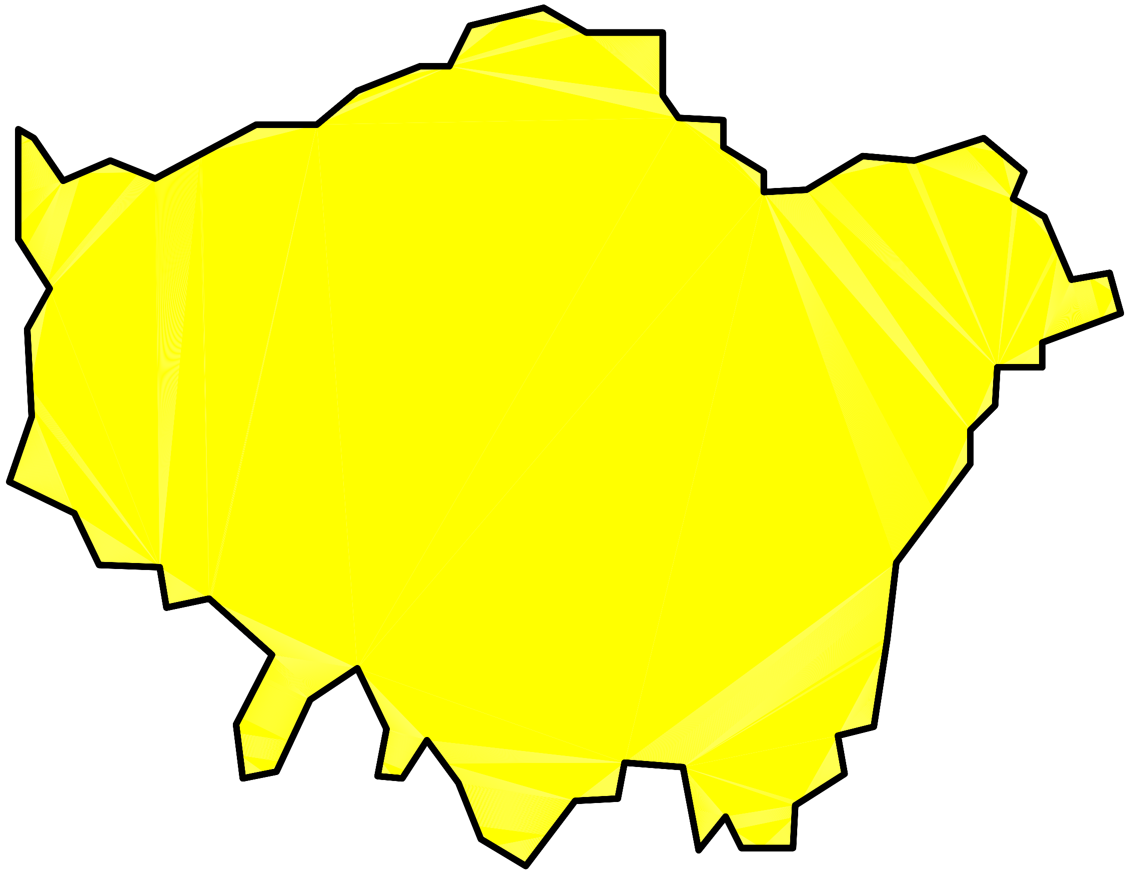


10,000 step random walk.



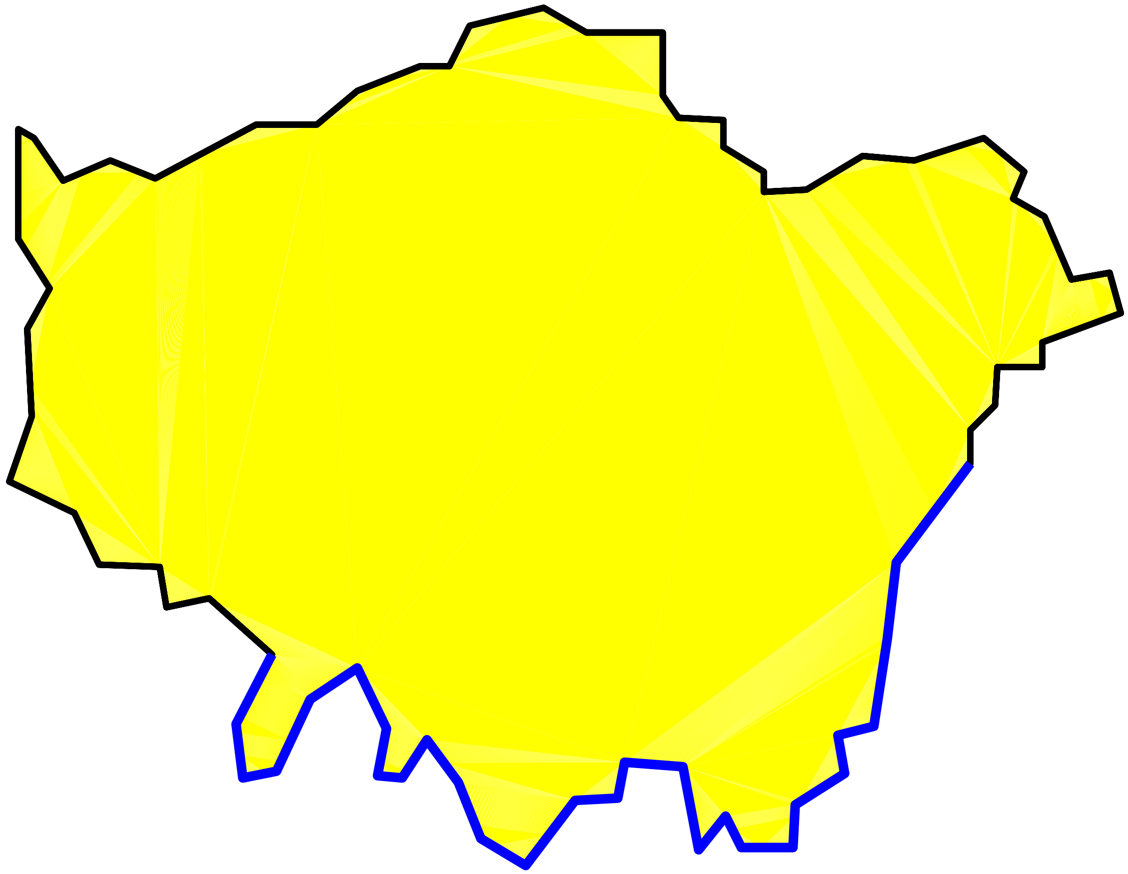
100,000 step random walk.

Harmonic measure = hitting distribution of Brownian motion



Suppose Ω is a planar Jordan domain.

Harmonic measure = hitting distribution of Brownian motion



Let E be a subset of the boundary, $\partial\Omega$.


London
Hertfordshire
Essex
Buckinghamshire
Berkshire
Surrey
Kent
Enfield
Barnet
Harrow
Haringey
Waltham Forest
Redbridge
Havering
Brent
Camden
Islington
Hackney
Barking and Dagenham
Newham
Tower Hamlets
Hillingdon
Ealing
Kensington and Chelsea
Hammersmith and Fulham
City of Westminster
City of London
River Thames
Greenwich
Southwark
Hounslow
Bexley
Lambeth
Wandsworth
Richmond upon Thames
Lewisham
Kingston upon Thames
Merton
Bromley
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Croydon
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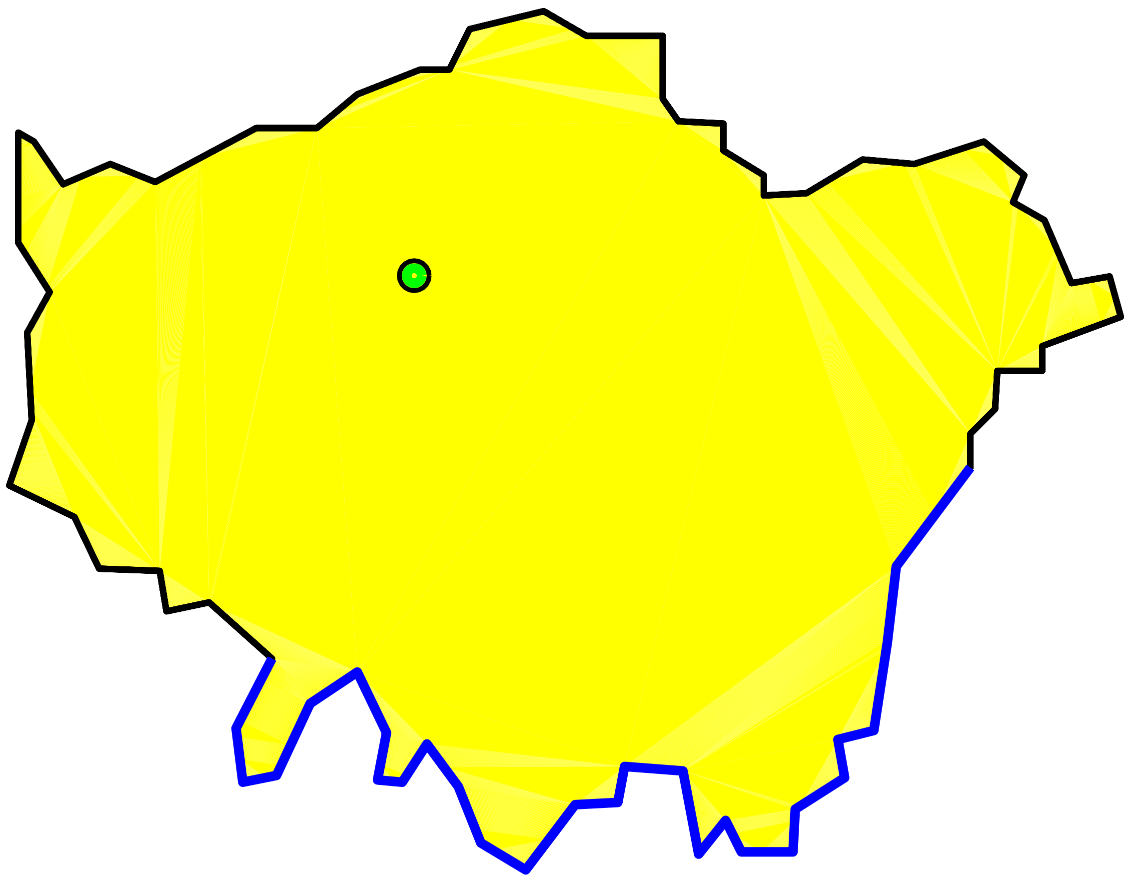
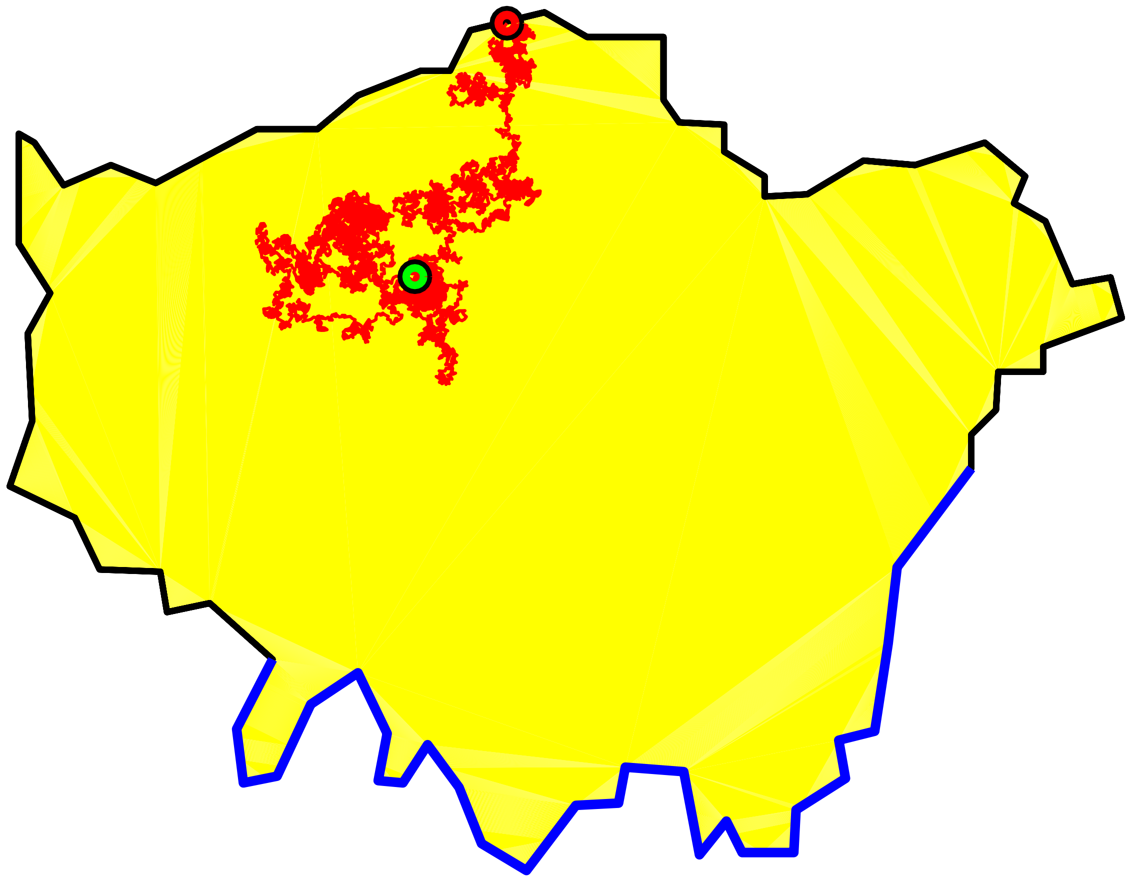
Harmonic measure = hitting distribution of Brownian motion



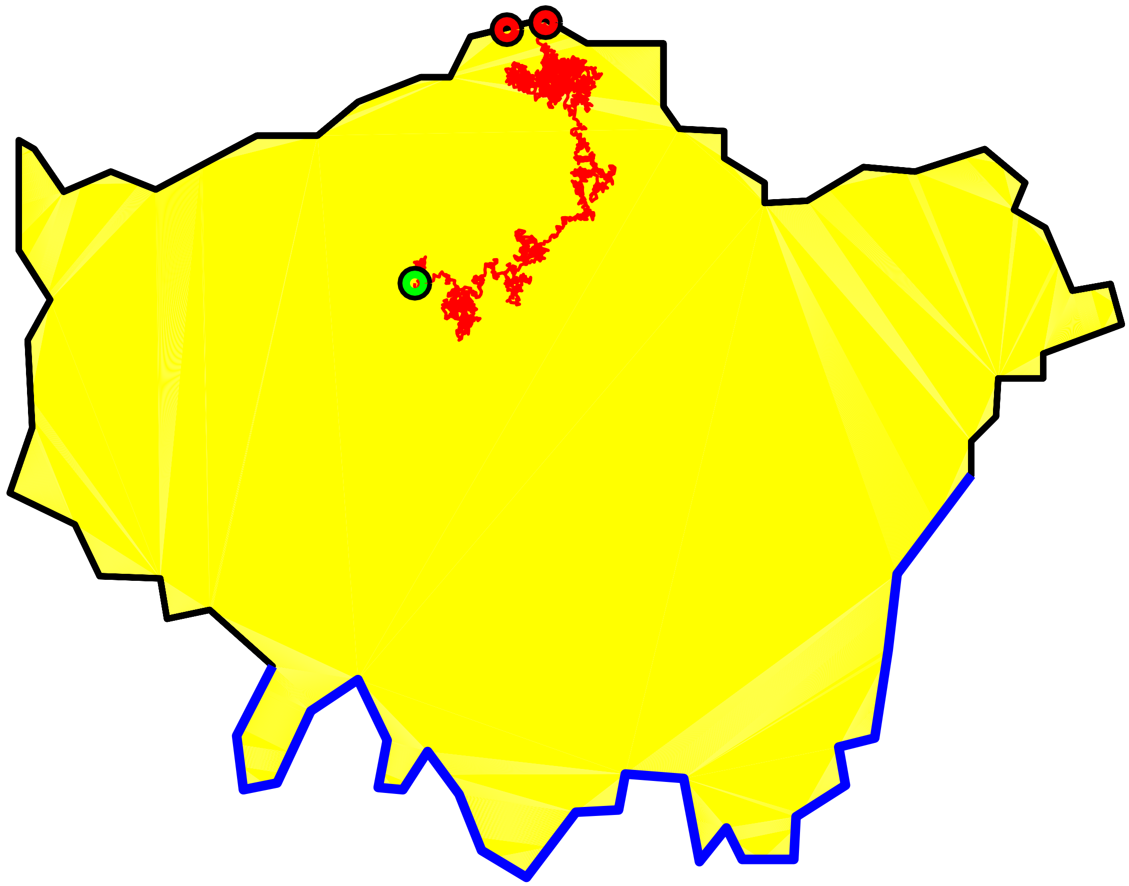
Choose a point z inside Ω.

Harmonic measure = hitting distribution of Brownian motion



$\omega(z, E, \Omega)$ = probability a particle started at z first hits $\partial\Omega$ in E.

Harmonic measure = hitting distribution of Brownian motion



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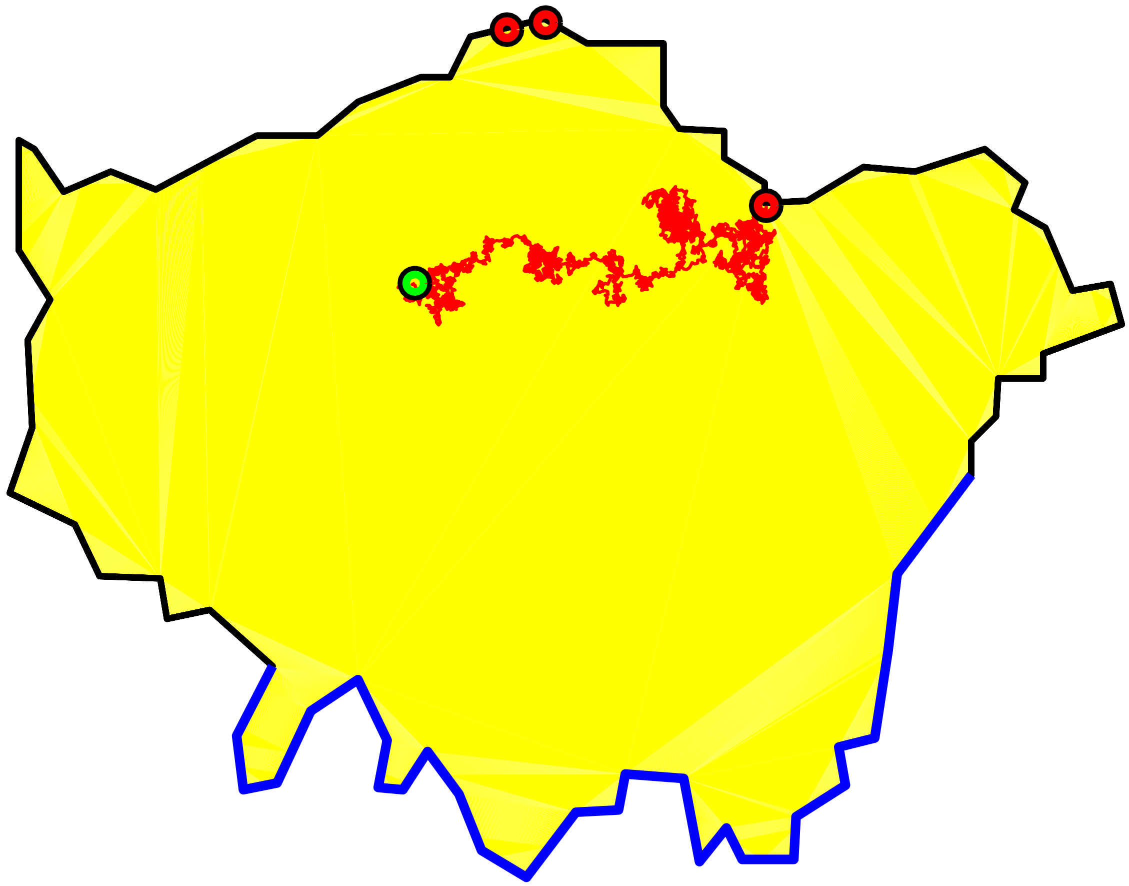

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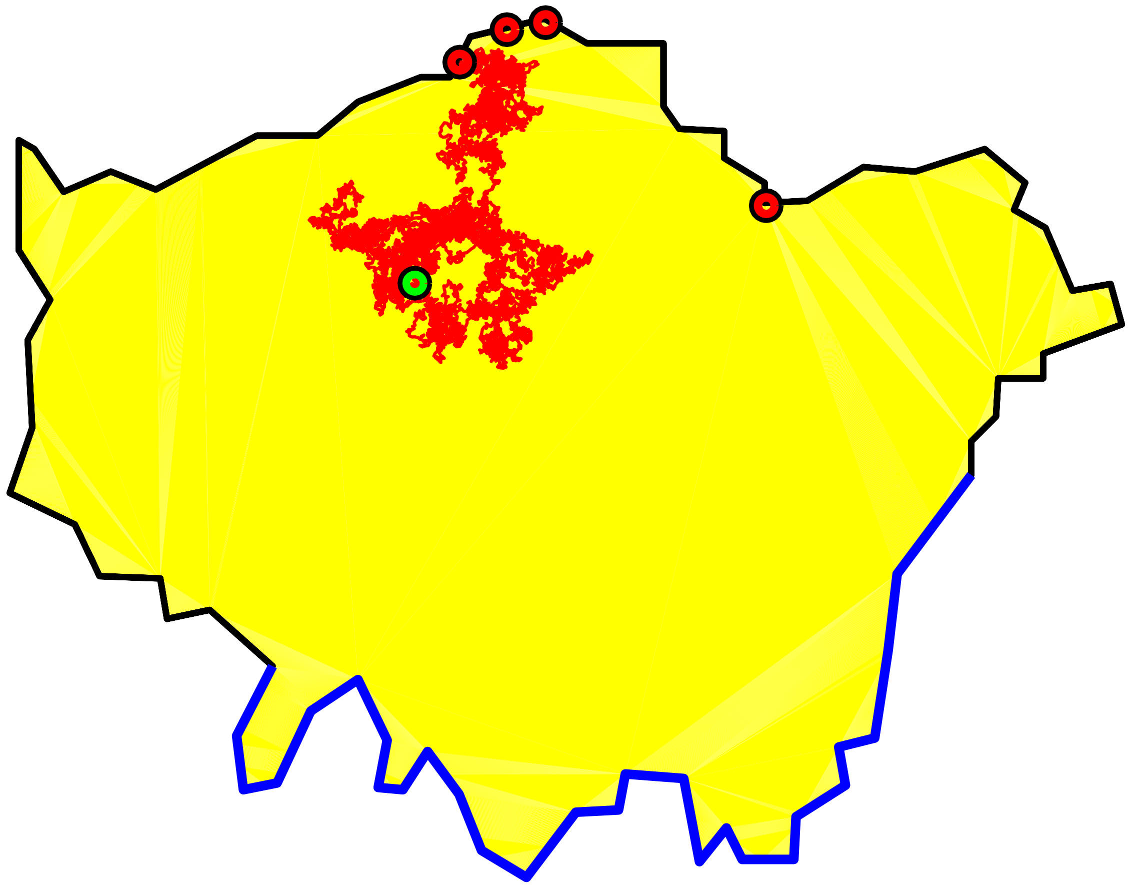

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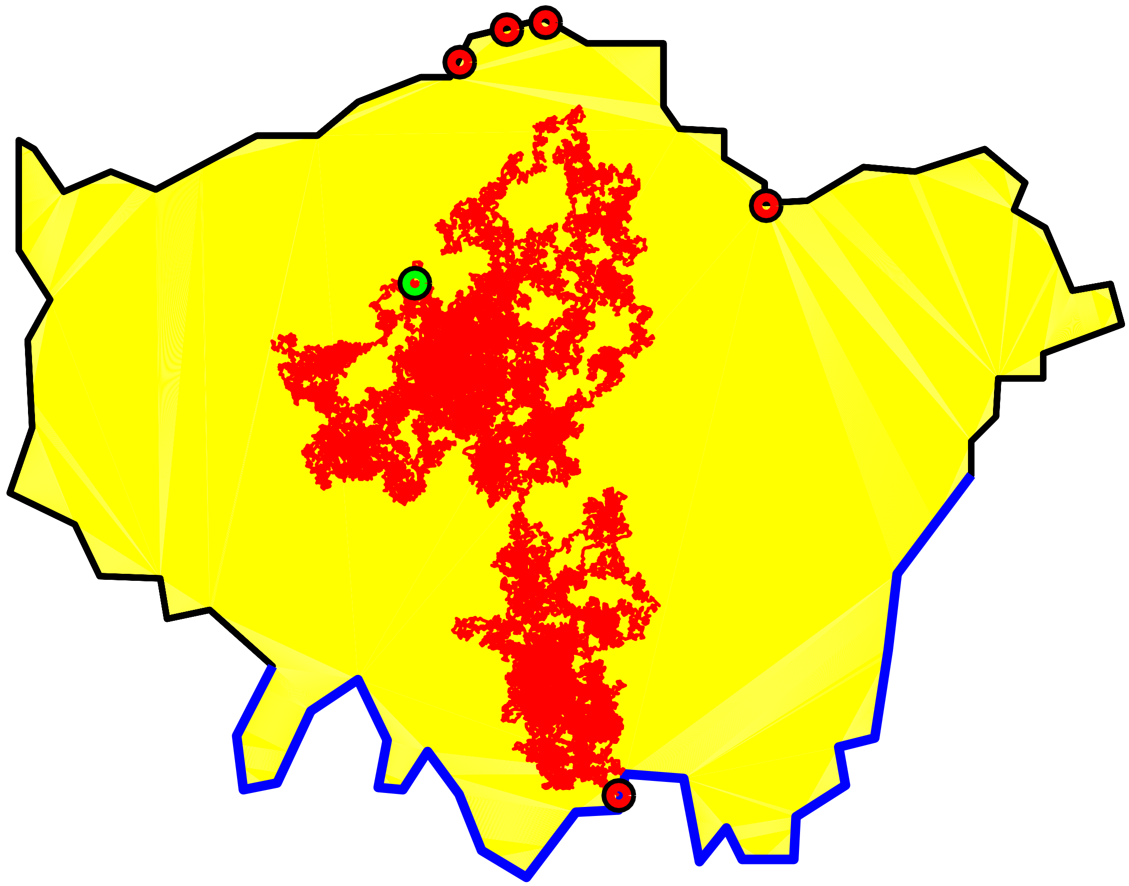

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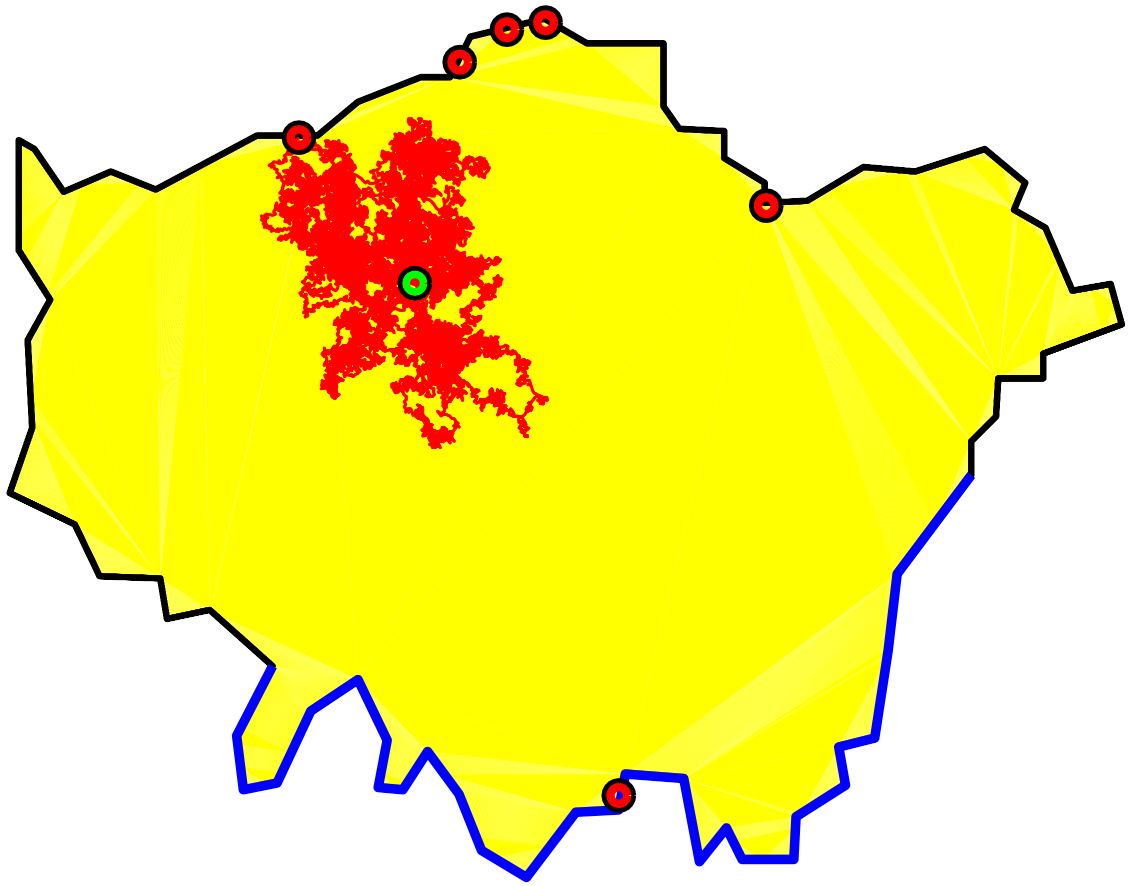

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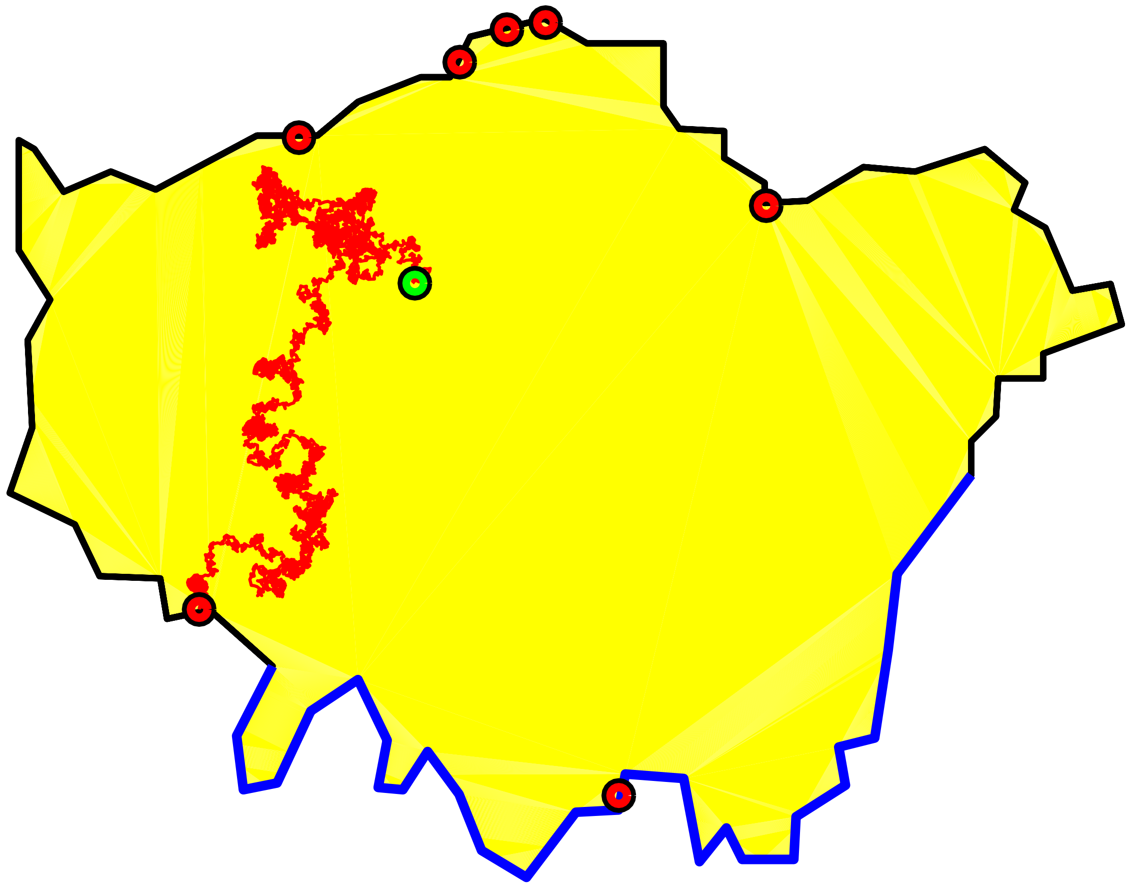

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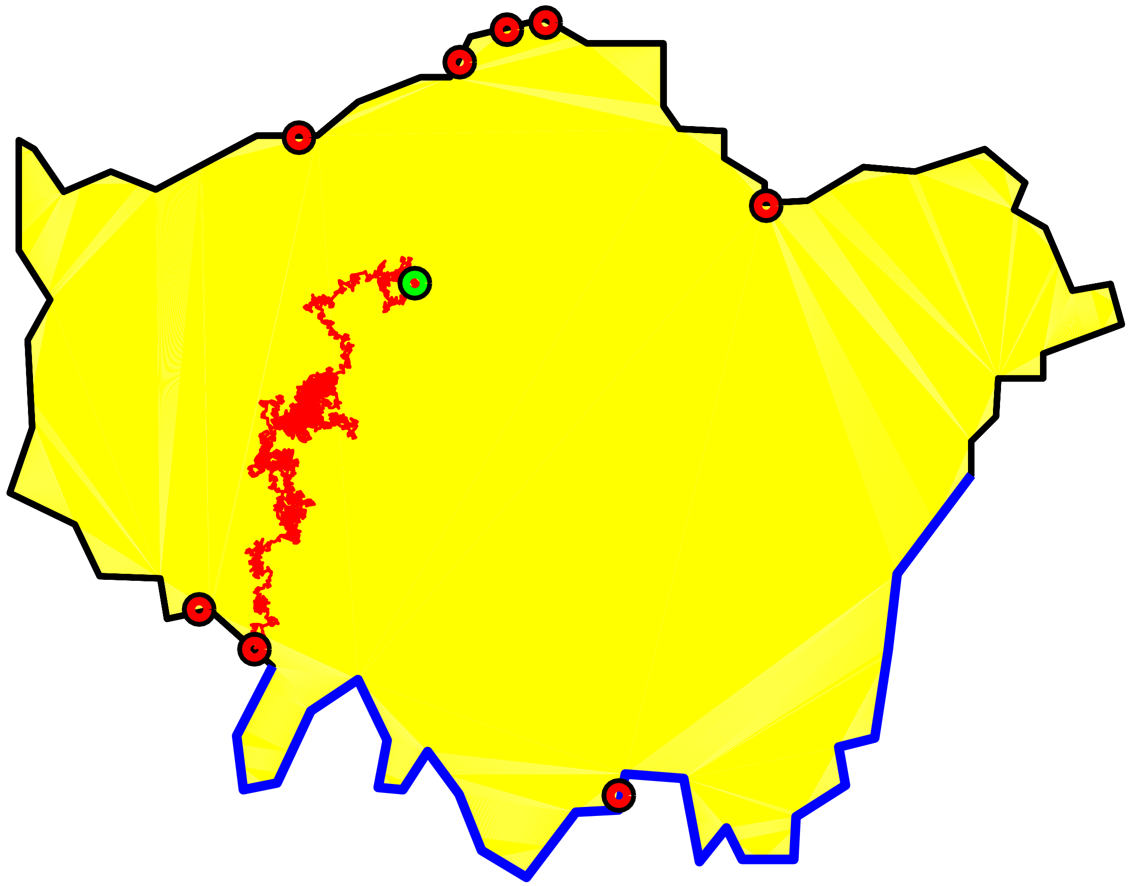

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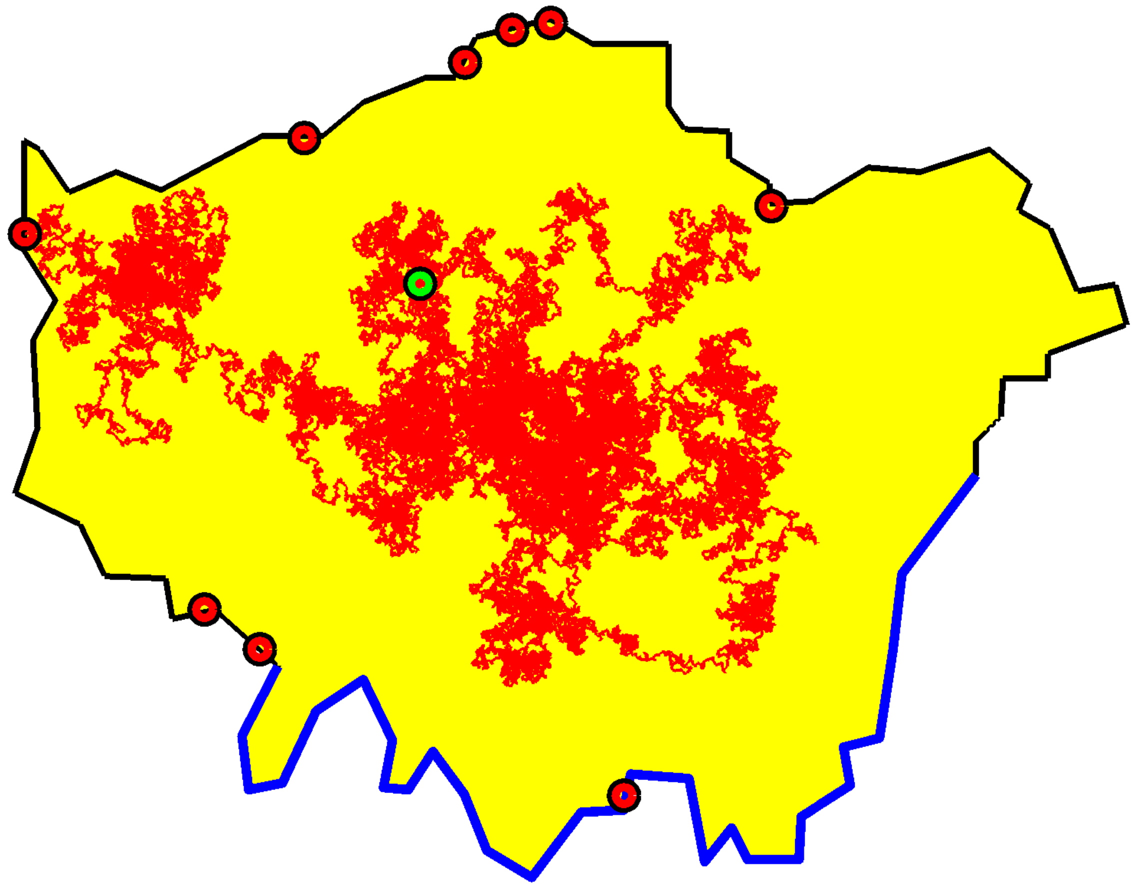

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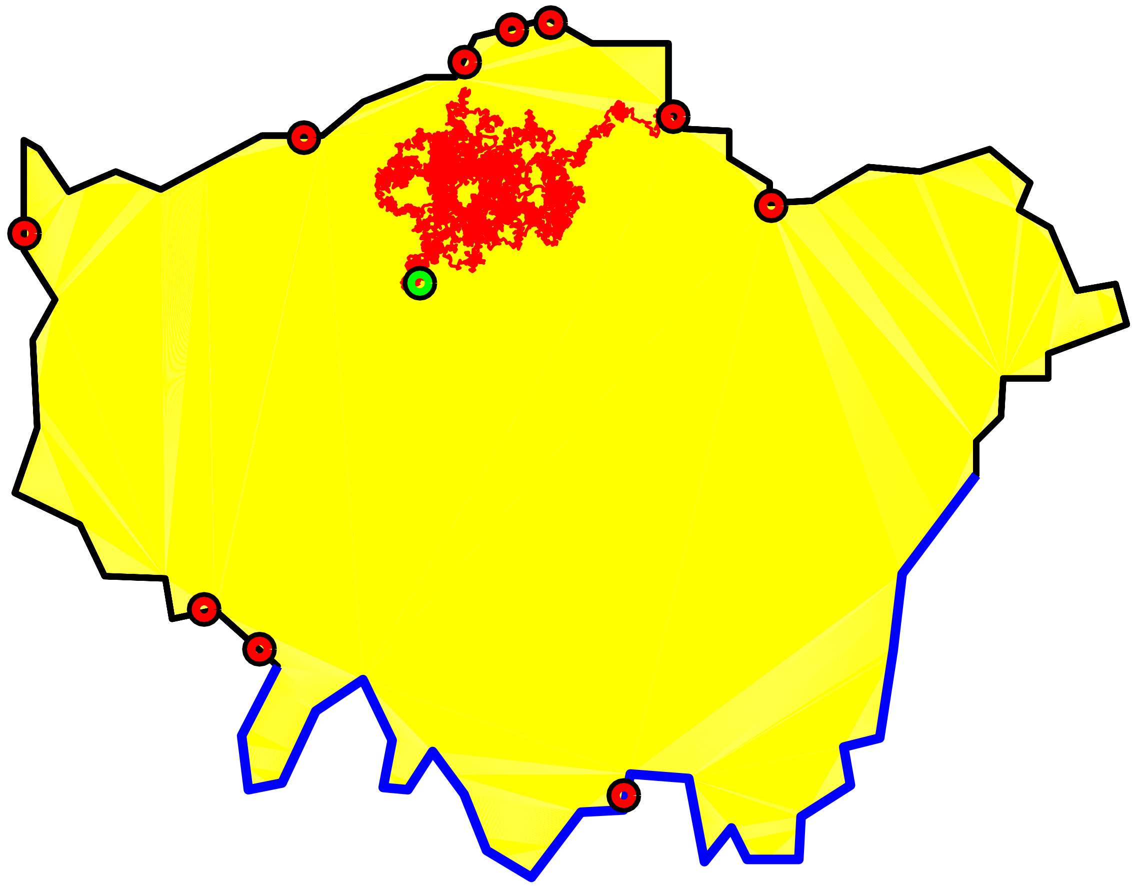
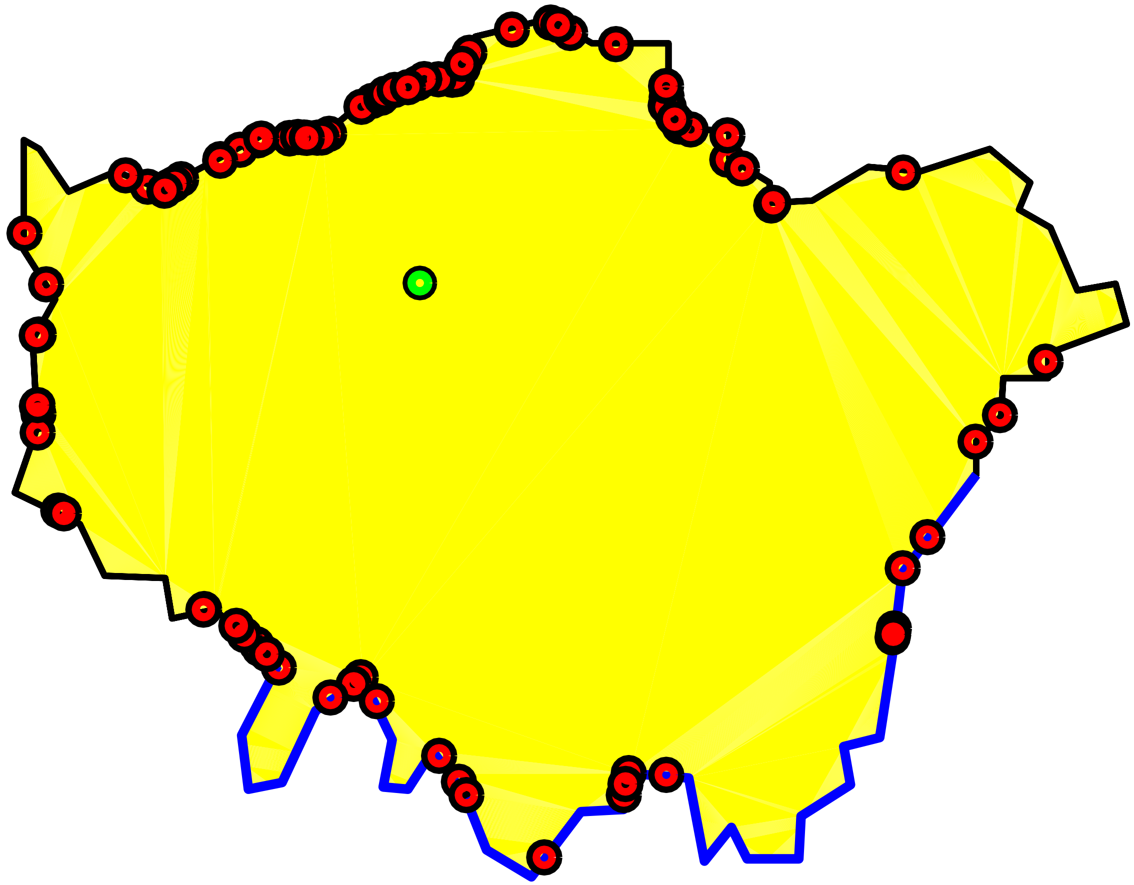


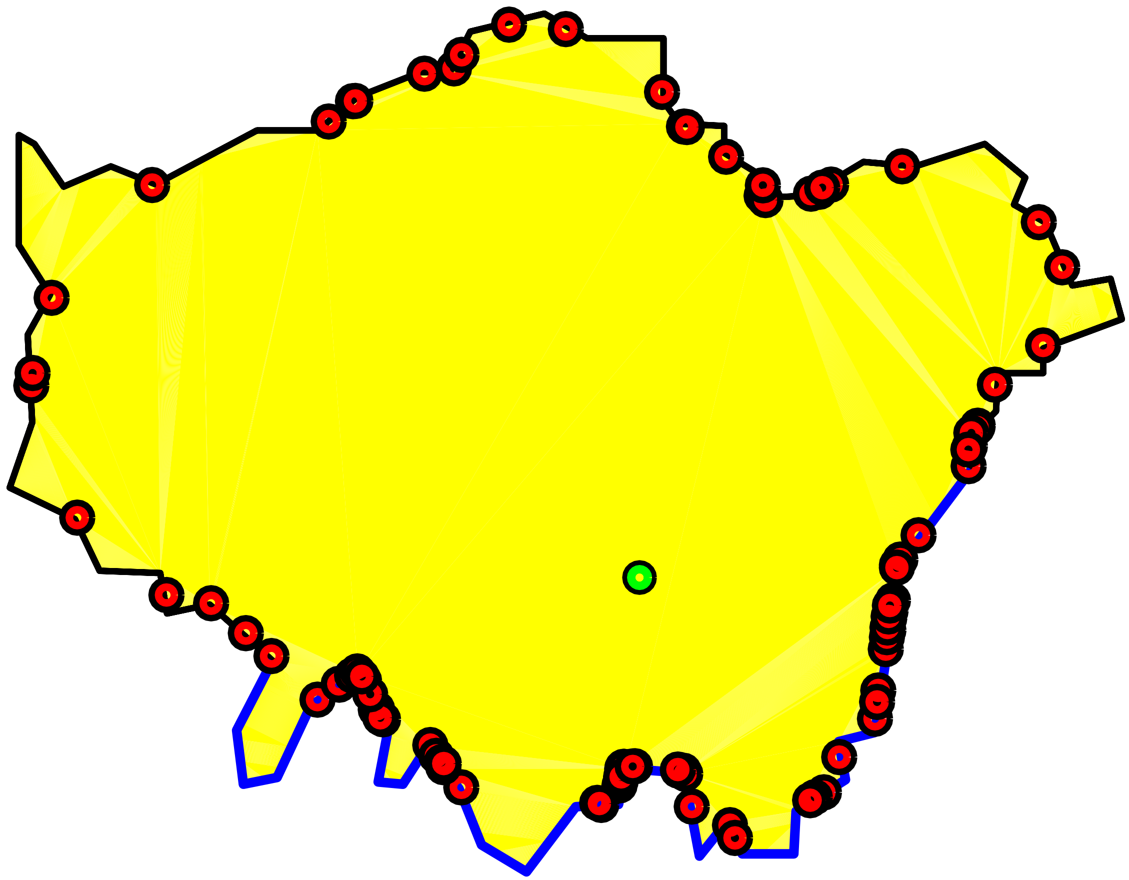
$$\omega(z, E, \Omega) \approx 2/10$$

Harmonic measure = hitting distribution of Brownian motion



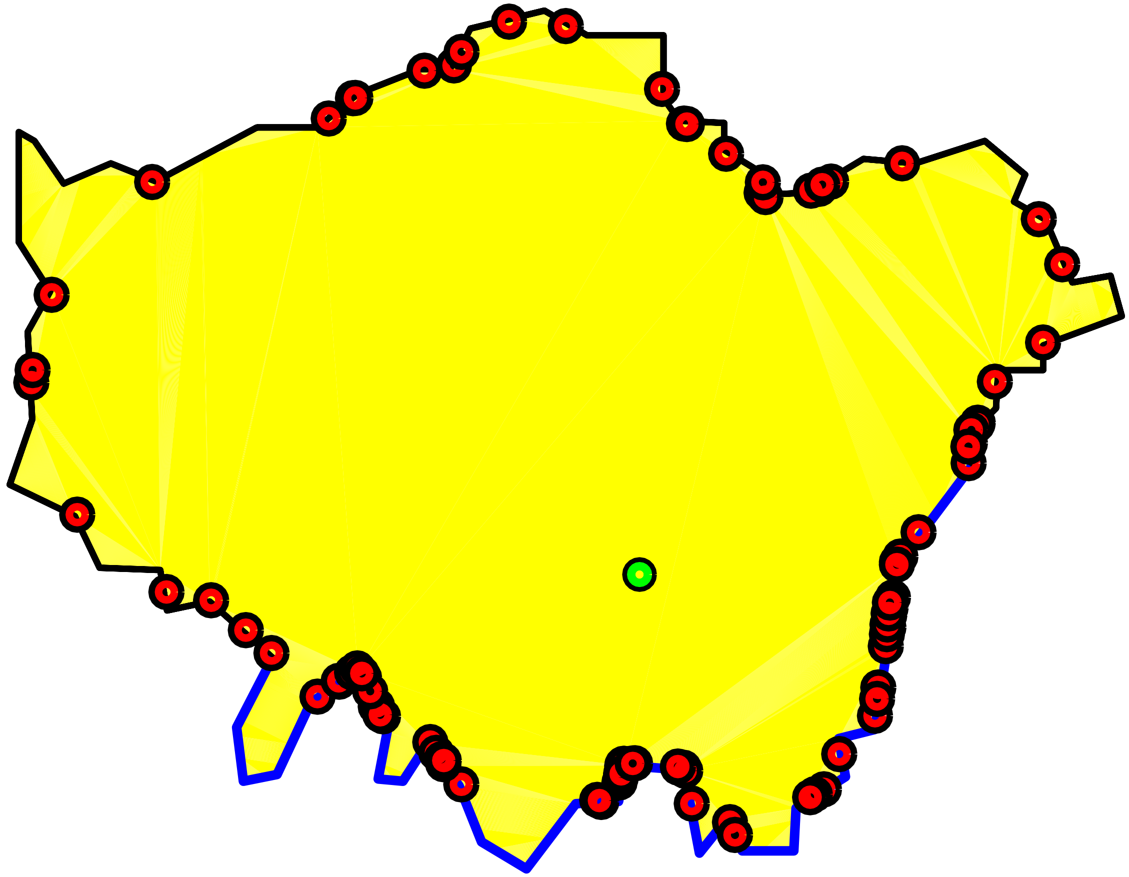
$$\omega(z, E, \Omega) \approx 21/100.$$

Harmonic measure = hitting distribution of Brownian motion



$$\omega(z, E, \Omega) \approx 64/100.$$

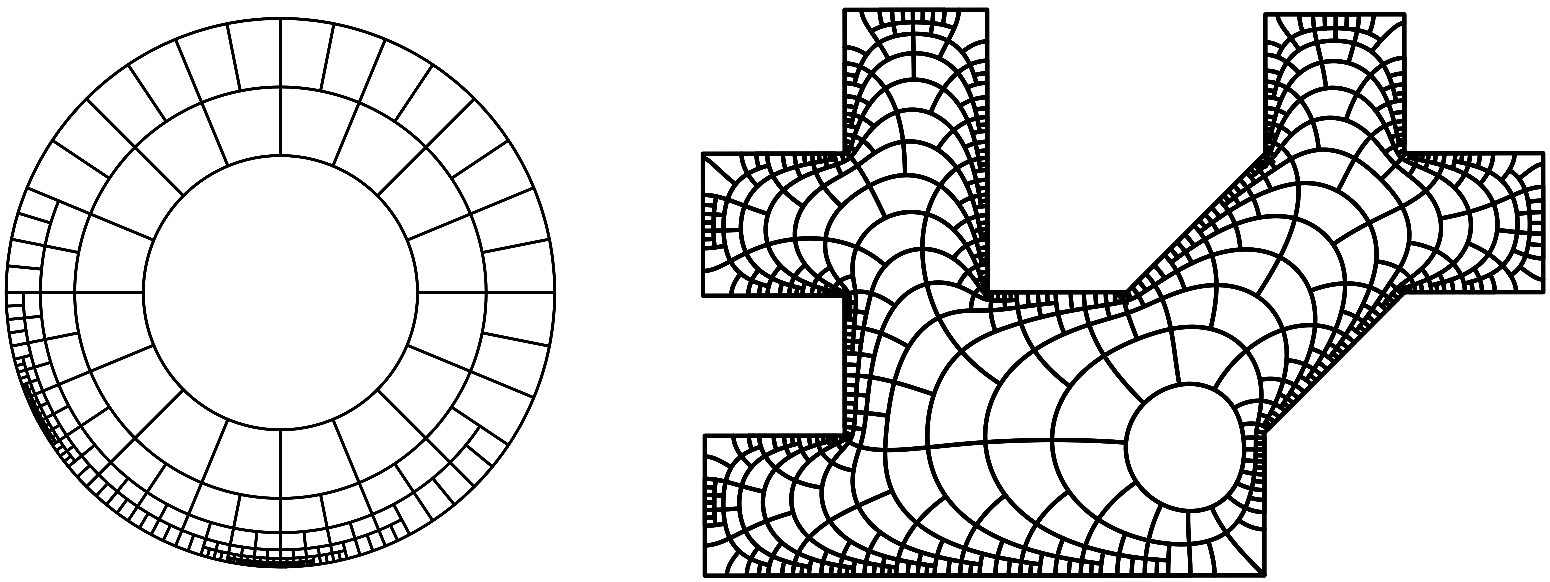
Harmonic measure = hitting distribution of Brownian motion



$\omega(z, E, \Omega)$ is a harmonic function of z on Ω.
Boundary values are 1 on E and 0 elsewhere.
Easy to solve this PDE on the disk: Poisson kernel.

Riemann Mapping Theorem: If $\Omega \subsetneq \mathbb{R}^2$ is simply connected and $z \in \Omega$, then there is a conformal map $f : \mathbb{D} \to \Omega$ with $f(0) = z$.

conformal = 1-1, holomorphic= angle and orientation preserving



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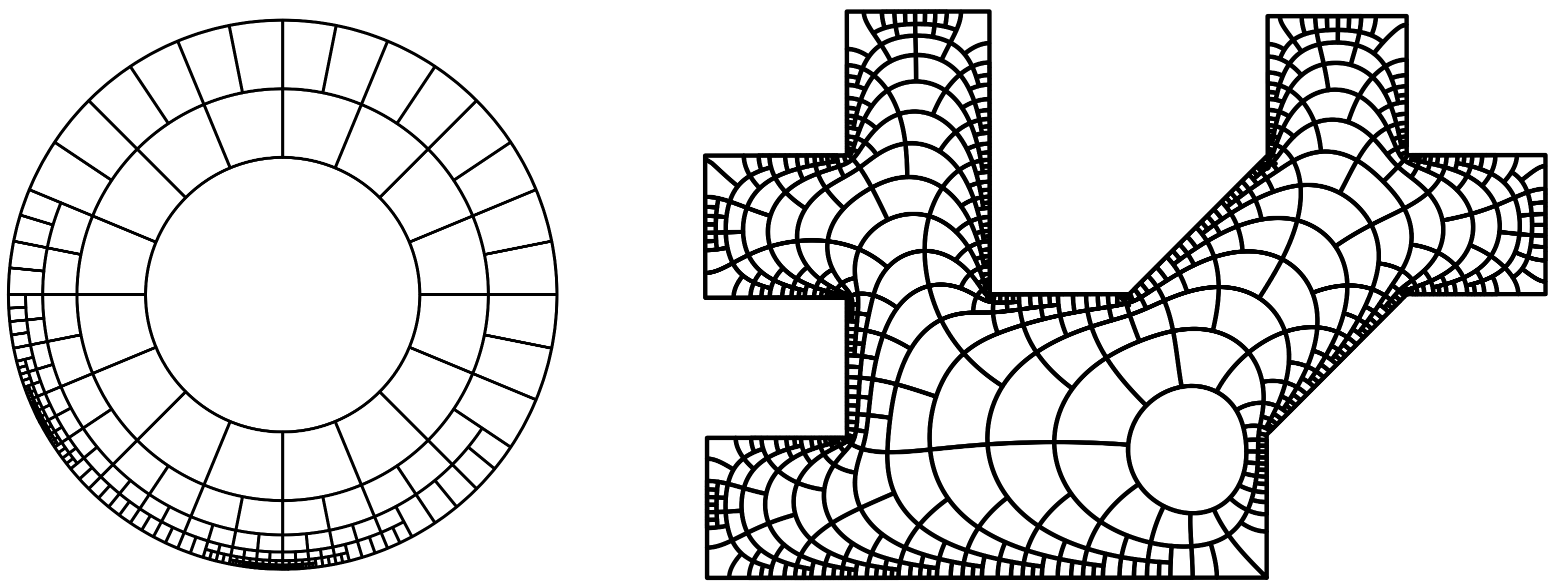
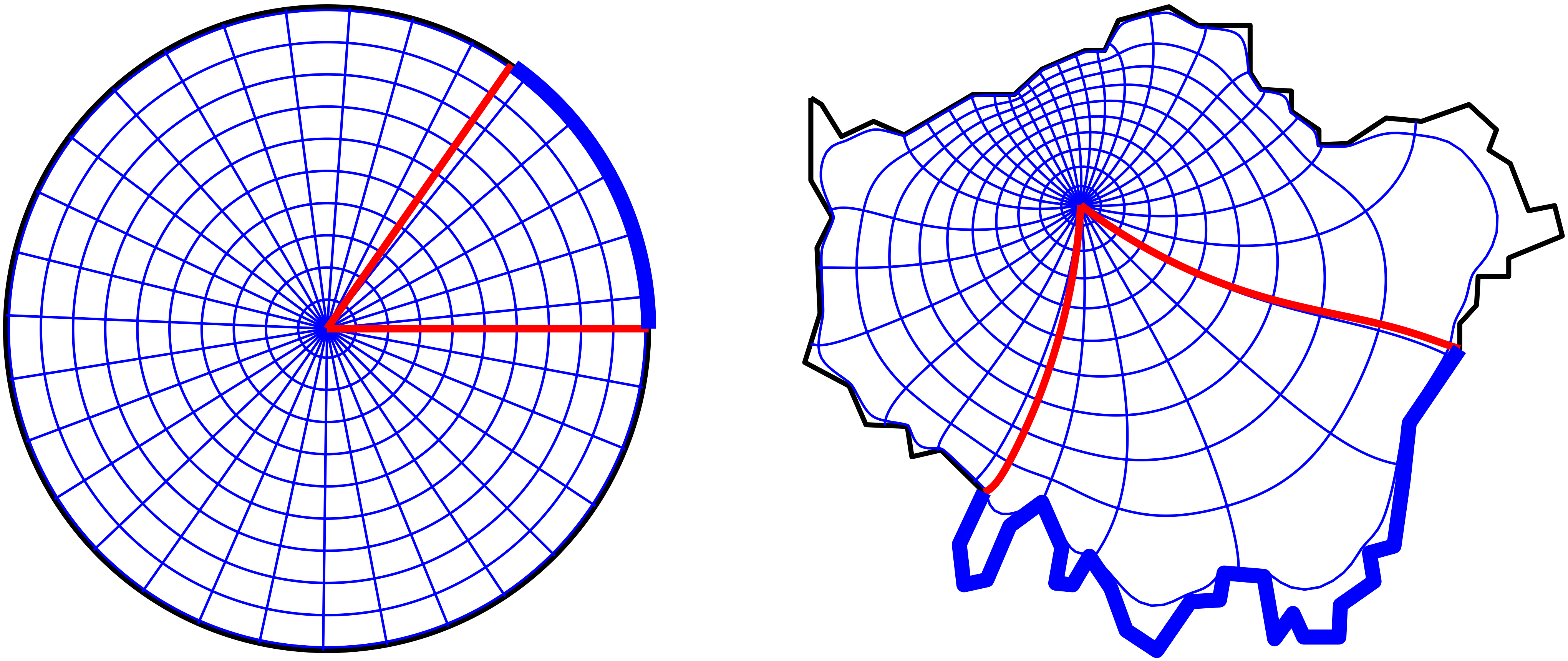


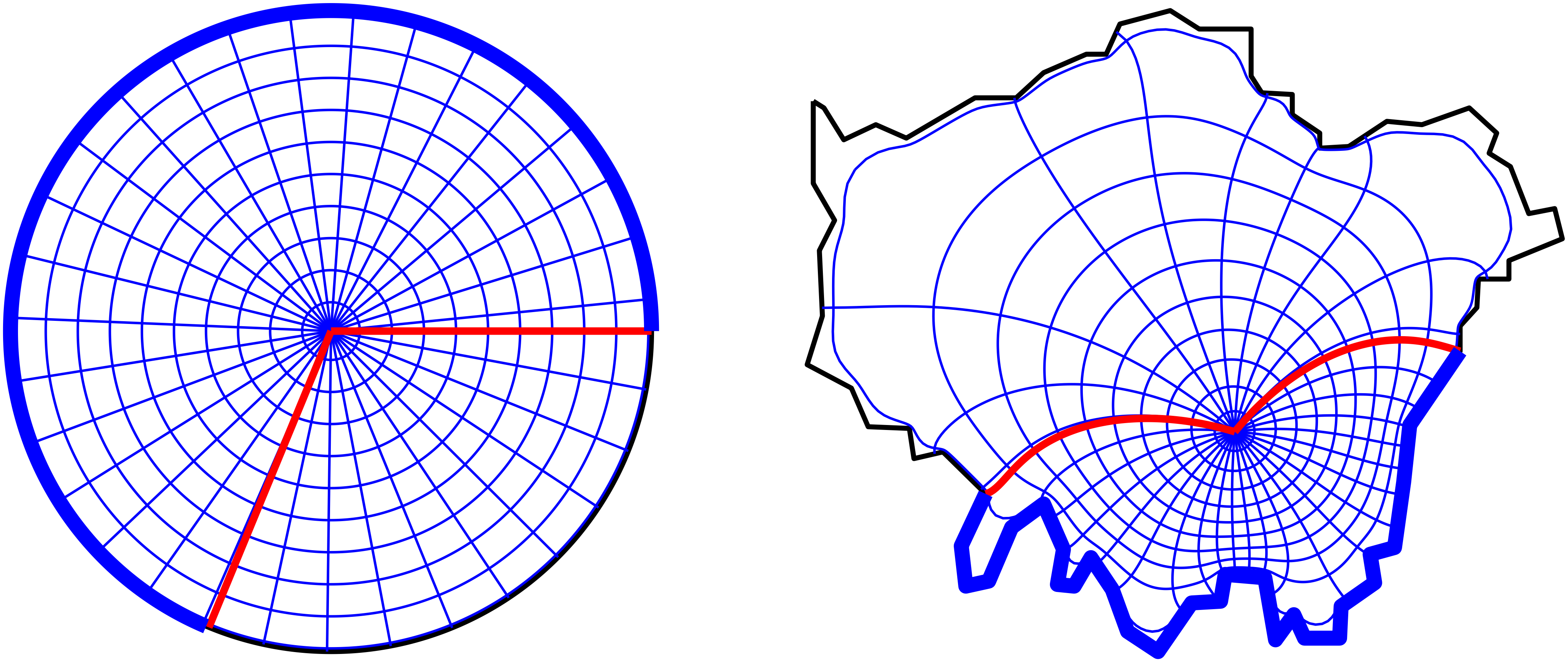
If ω is harmonic on Ω, then $\omega \circ f$ is harmonic on $\mathbb{D}$.

$\Rightarrow \omega(z, E, \Omega) = \omega(0, f^{-1}(E), \mathbb{D}) = |E|/2\pi$.

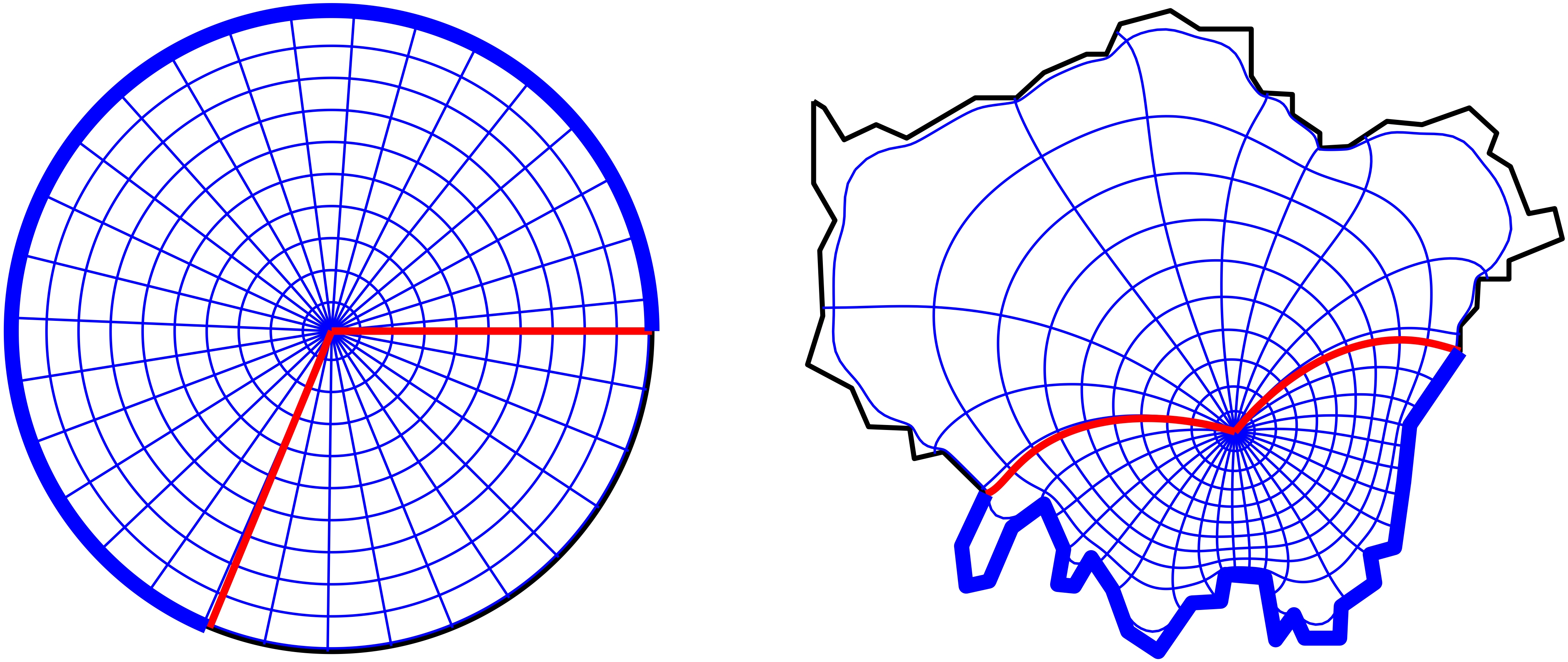
$\Rightarrow \omega$ on $\partial\Omega$ = conformal image of (normalized) length on $\mathbb{T}$



Harmonic measure ≈ .15287



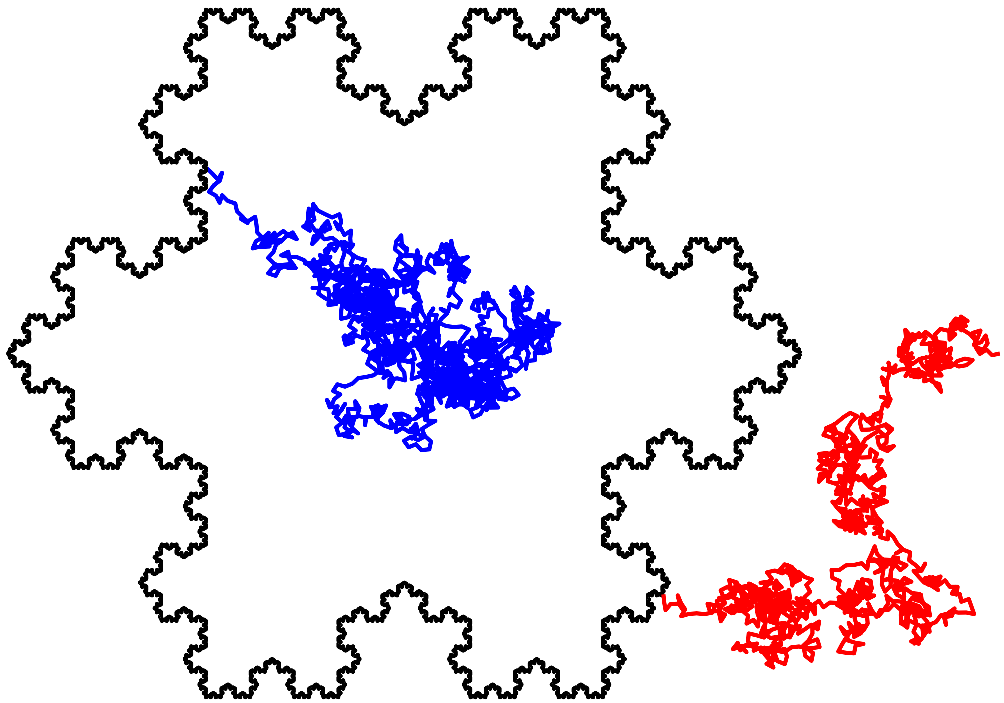
Harmonic measure ≈ .68775



- different base points $z \in \Omega$ give different distributions on $\partial\Omega$,
- but all points in Ω give mutually continuous distributions.

What about points on different sides of $\partial\Omega$?

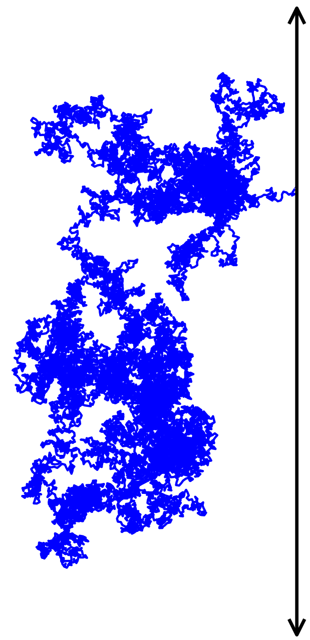
For a fractal curve, inside and outside harmonic measures are singular.



$\omega_1 \perp \omega_2$ iff tangents points have zero length.

(B. 1987, based on results of N.G. Makarov)

For which curves is $\omega_1 = \omega_2$? (inside measure = outside measure)



True for lines and circles.

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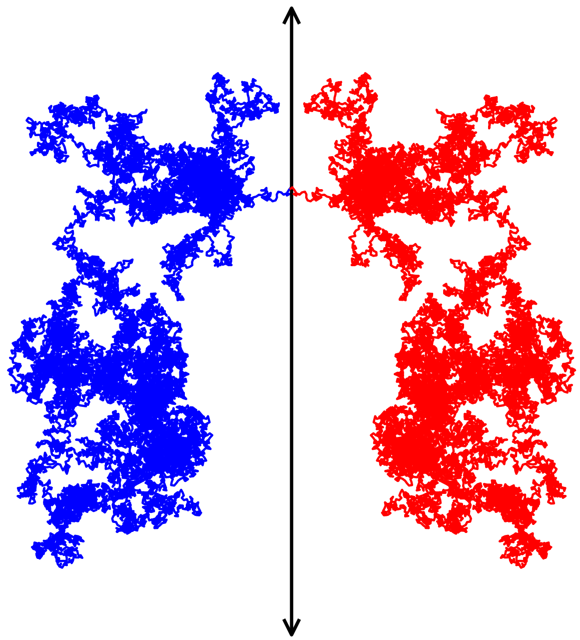
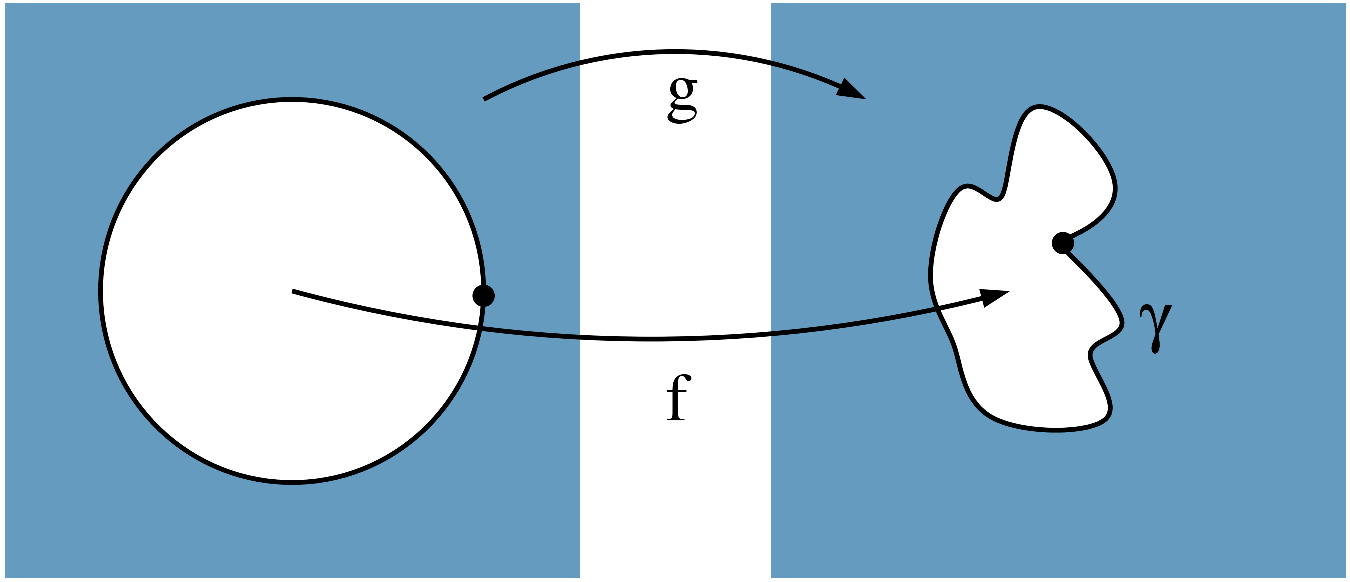


True for lines and circles.

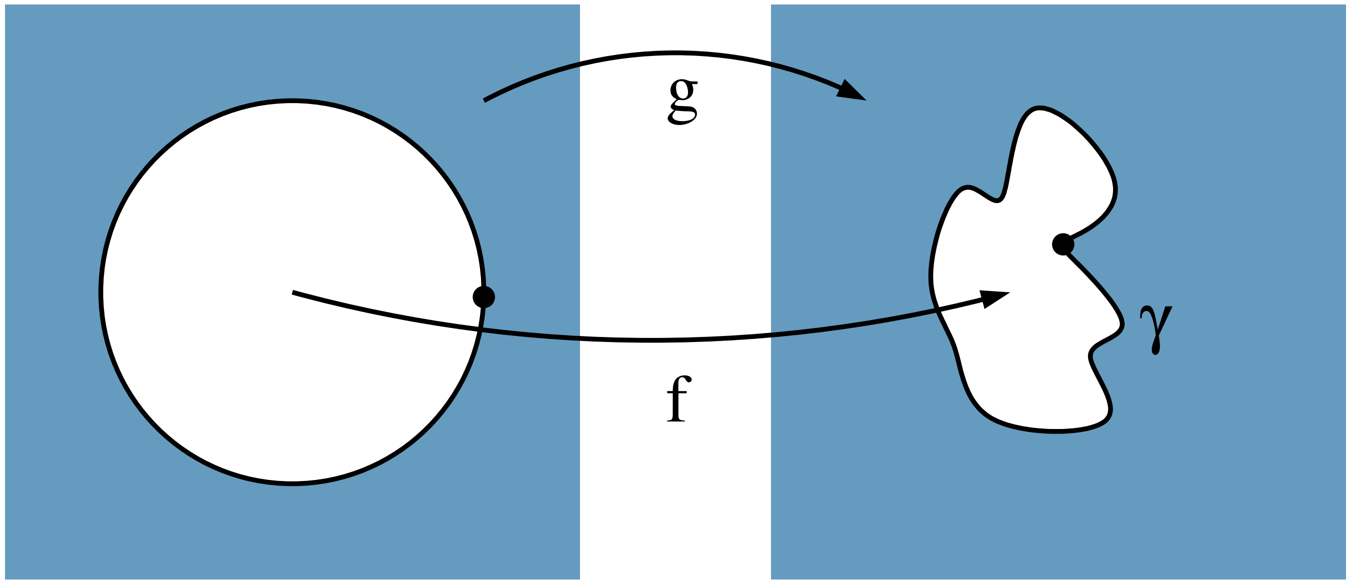
Is converse true? Equal harmonic measures $\Rightarrow$ circle?



Suppose $\omega_1 = \omega_2$ for a curve γ.

Conformally map two sides of circle to two sides of γ so $f(1) = g(1)$.

$\omega_1 = \omega_2$ implies maps agree on whole boundary.



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Conformally map two sides of circle to two sides of γ so $f(1) = g(1)$.

$\omega_1 = \omega_2$ implies maps agree on whole boundary.

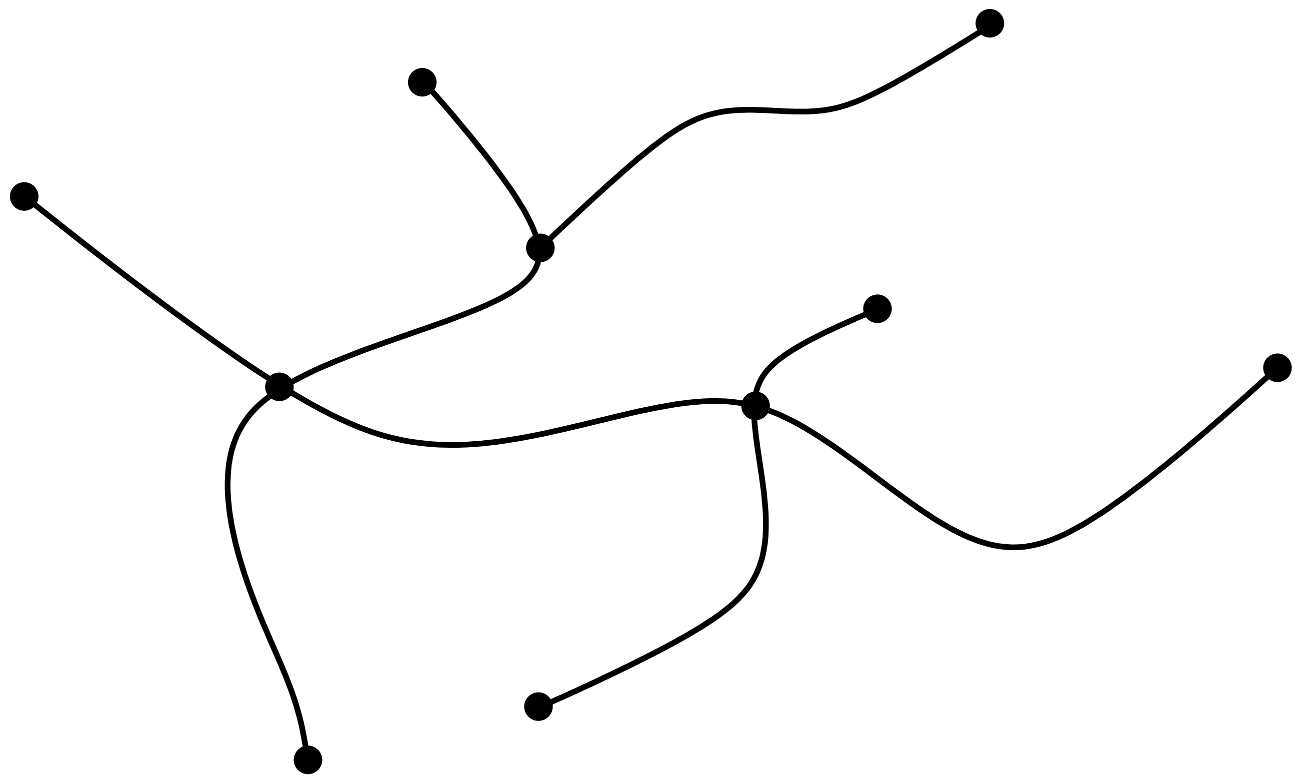
So f, g define homeomorphism h of plane holomorphic off circle.

Then h is entire by Morera's theorem.

Entire and 1-1 implies h is linear (Liouville's thm), so γ is a circle.

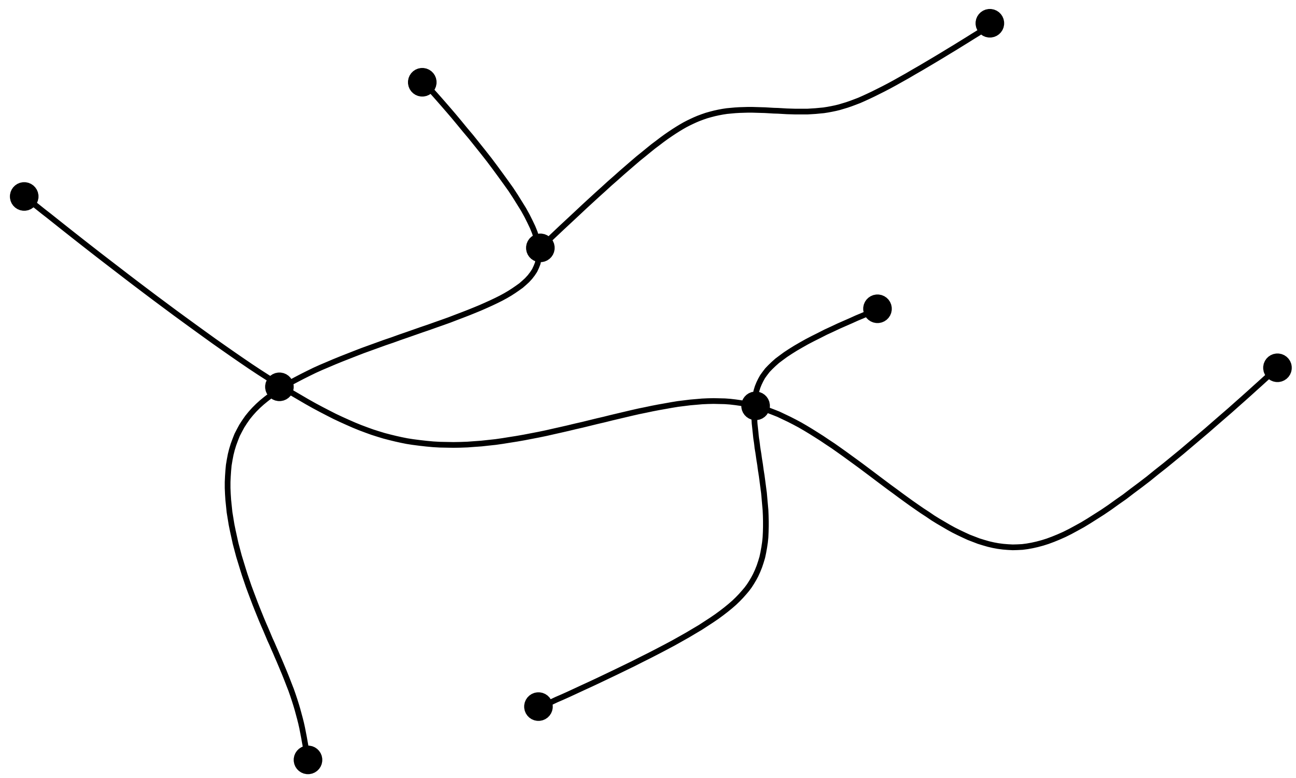
Circles are only closed curves with equal harmonic measures on both sides.

What about other kinds of 2-sided objects?



A planar graph is a finite set of points connected by non-crossing edges.

It is a tree if there are no closed loops.



A planar tree is **conformally balanced** if

- every edge has equal harmonic measure from ∞
- edge subsets have same measure from both sides

This is also called a "**true tree**". A line segment is an example.

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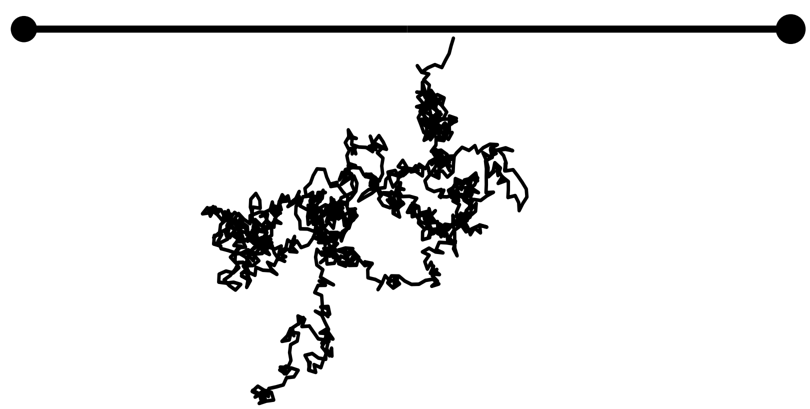
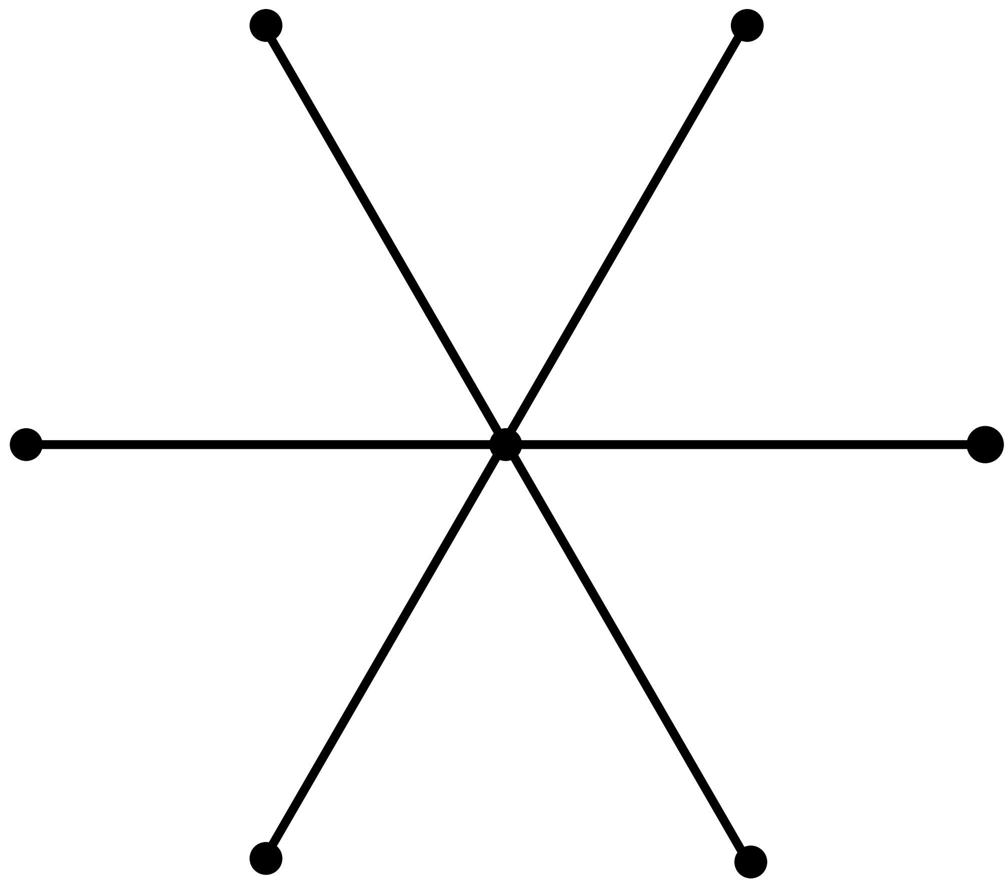
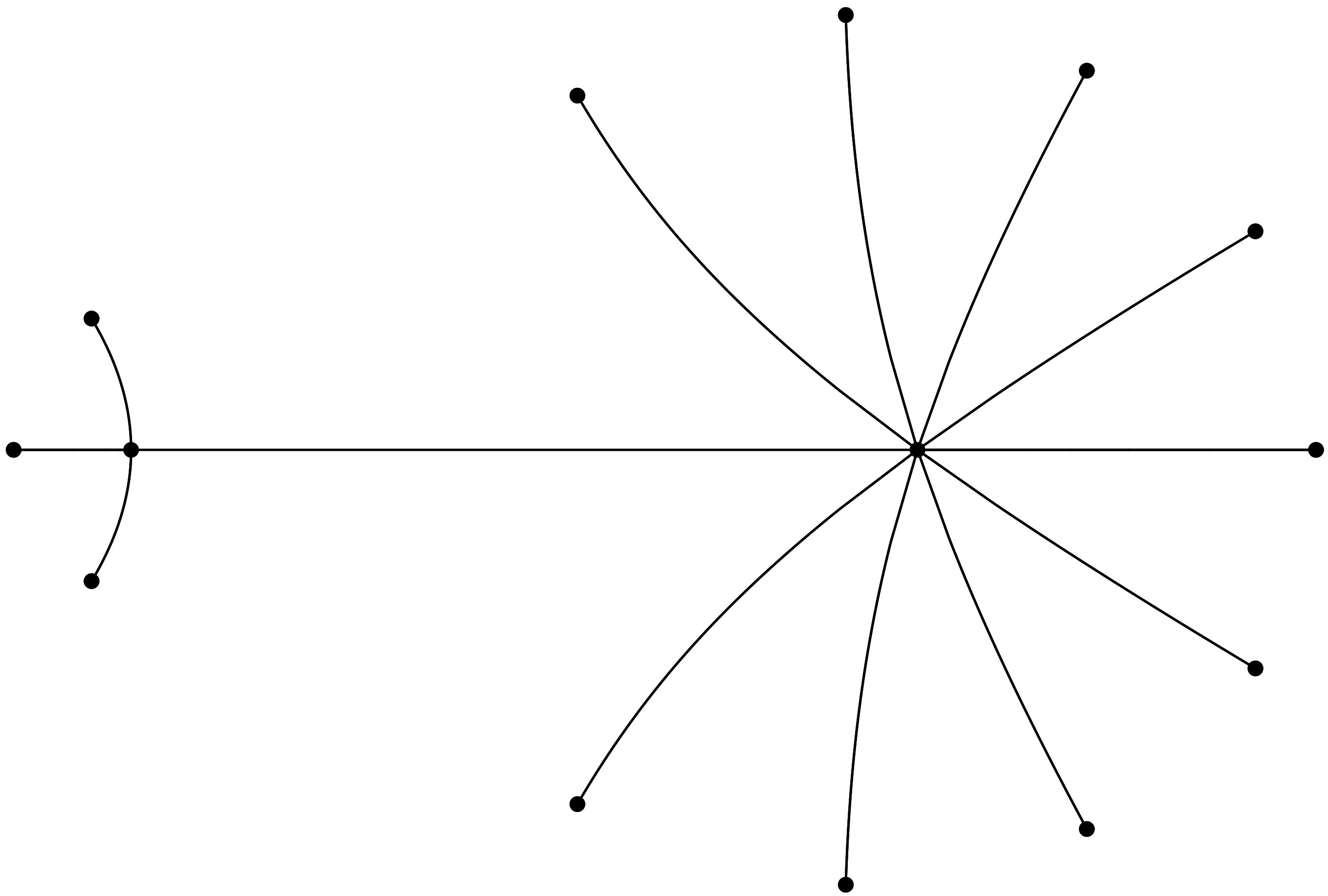




Trivially true by symmetry



Non-obvious true tree

Definition of critical value: if $p =$ polynomial, then

$$\mathrm{CV}(p) = \{p(z) : p'(z) = 0\} = \text{critical values}$$

If $\mathrm{CV}(p) = \pm 1$, p is called **generalized Chebyshev** or **Shabat**.

Definition of critical value: if p = polynomial, then

$$\mathrm{CV}(p) = \{p(z) : p'(z) = 0\} = \text{critical values}$$

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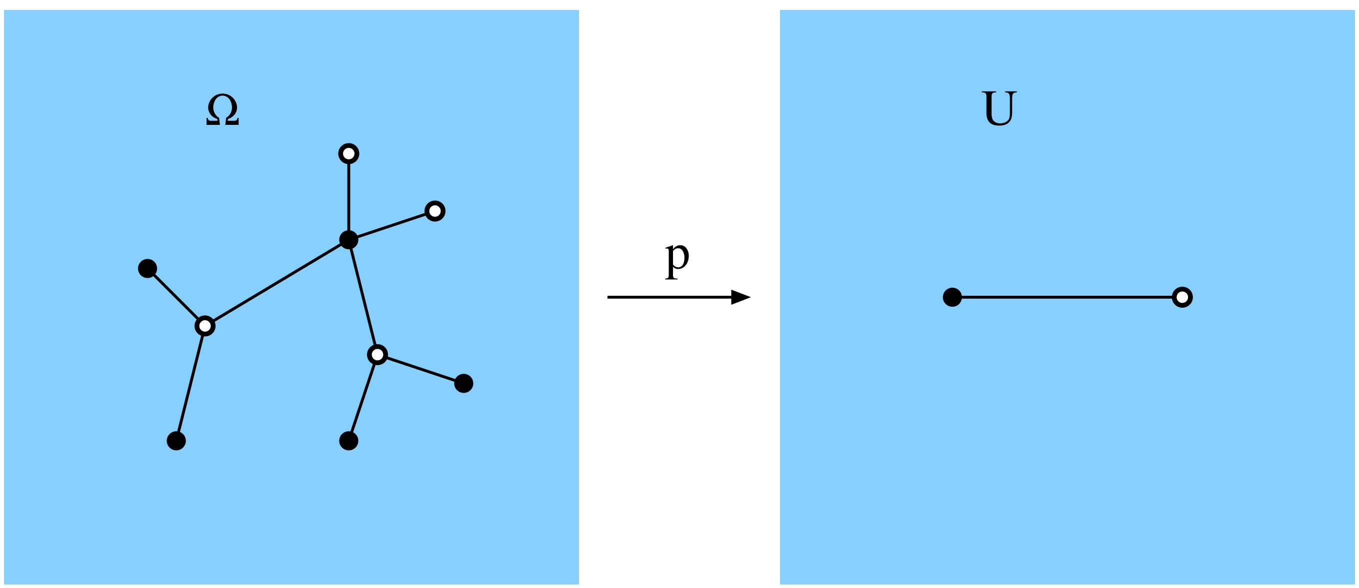
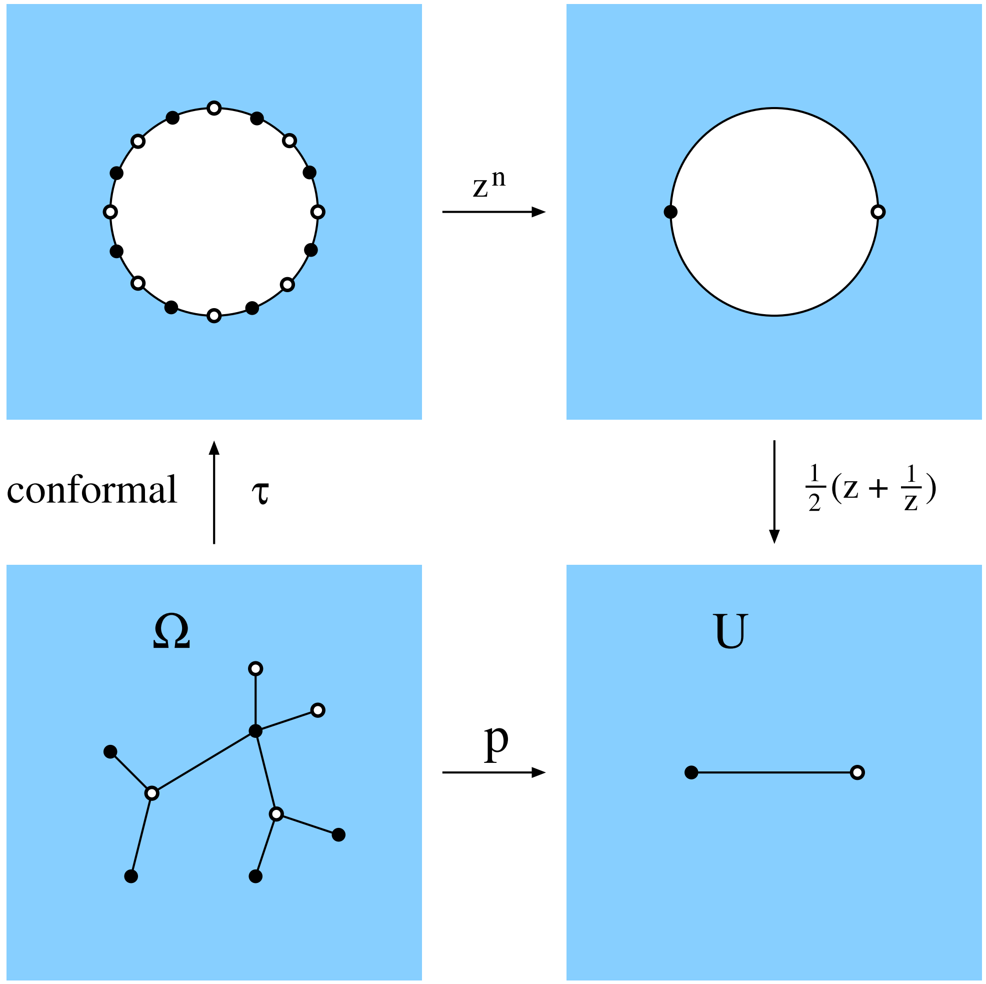
Thm: T is balanced iff $T = p^{-1}([-1, 1])$, p = Shabat.



$\Omega = \mathbb{C} \setminus T$ $\qquad$ $U = \mathbb{C} \setminus [-1, 1]$

T conformally balanced $\Leftrightarrow$ p Shabat.



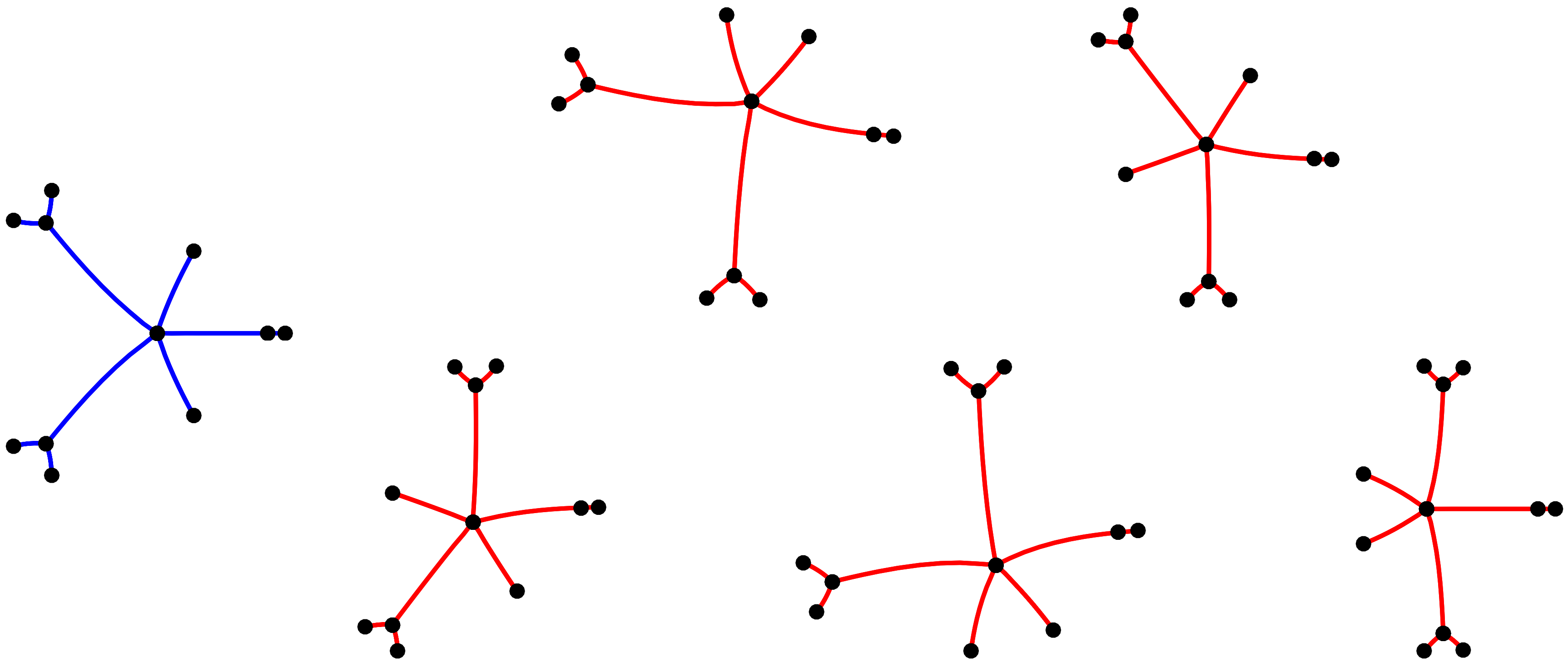
p is entire and n-to-1 $\Leftrightarrow$ p = polynomial.

$\mathrm{CV}(p) \not\in U \leftrightarrow p : \Omega \to U$ is covering map.

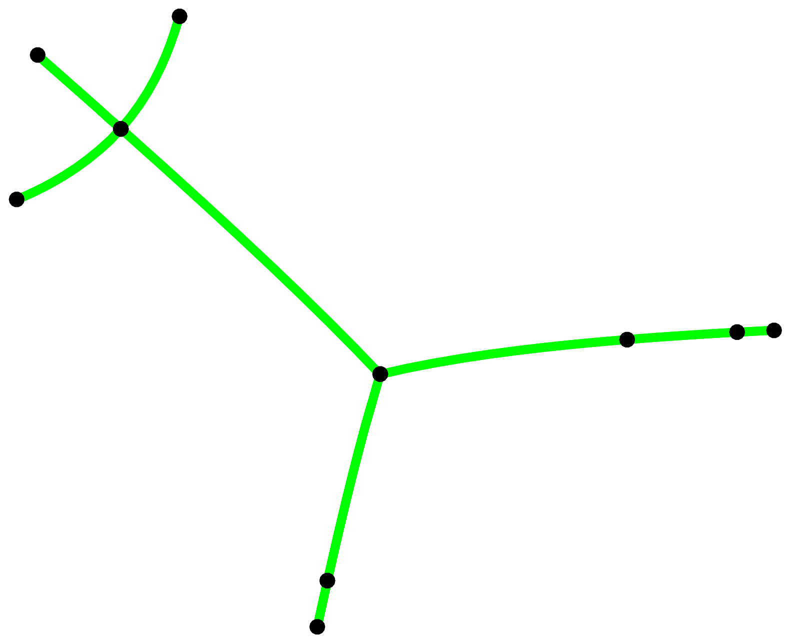
Algebraic aside:

True trees are examples of Grothendieck's *dessins d'enfants* on sphere.

Normalized polynomials are algebraic, so trees correspond to number fields.
Computing number field from tree is difficult.



Kochetkov (2009, 2014): cataloged all trees with 9 and 10 edges.



For example, the polynomial for this 9-edge tree is

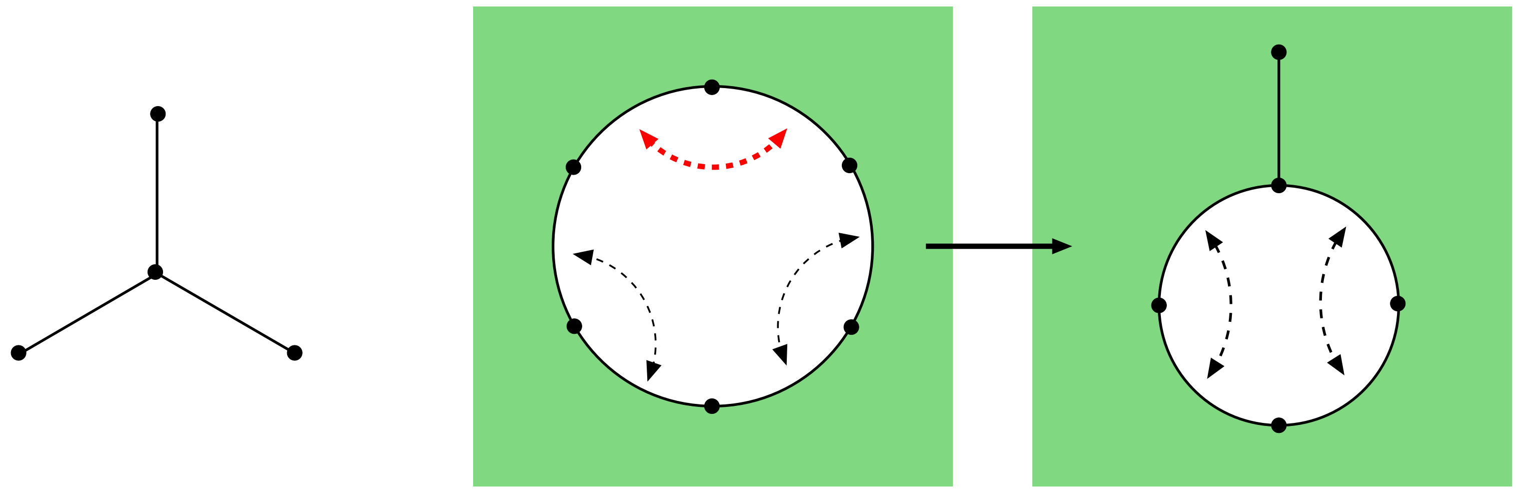
$$p(z) = z^4(z^2 + az + b)^2(z - 1),$$

where a is a root of ...

$$
\begin{aligned}
0 = \quad & 126105021875\, a^{15} + 873367351500\, a^{14} \\
& +2340460381665\, a^{13} + 2877817869766\, a^{12} \\
& +3181427453757\, a^{11} - 68622755391456\, a^{10} \\
& -680918281137097\, a^{9} - 2851406436711330\, a^{8} \\
& -7139130404618520\, a^{7} - 12051656256571792\, a^{6} \\
& -14350515598839120\, a^{5} - 12058311779508768\, a^{4} \\
& -6916678783373312\, a^{3} - 2556853615656960\, a^{2} \\
& -561846360735744\, a - 65703906377728
\end{aligned}
$$

I did **not** use this to draw tree on previous slide.

Don Marshall's `ZIPPER` uses conformal mapping to draw true trees.



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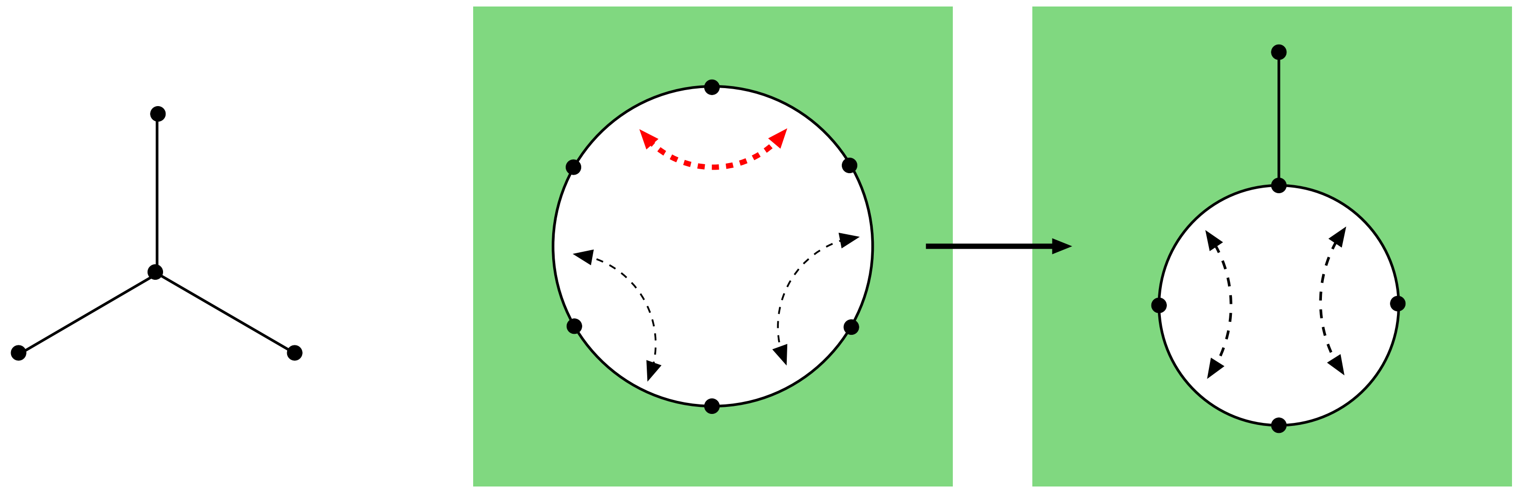
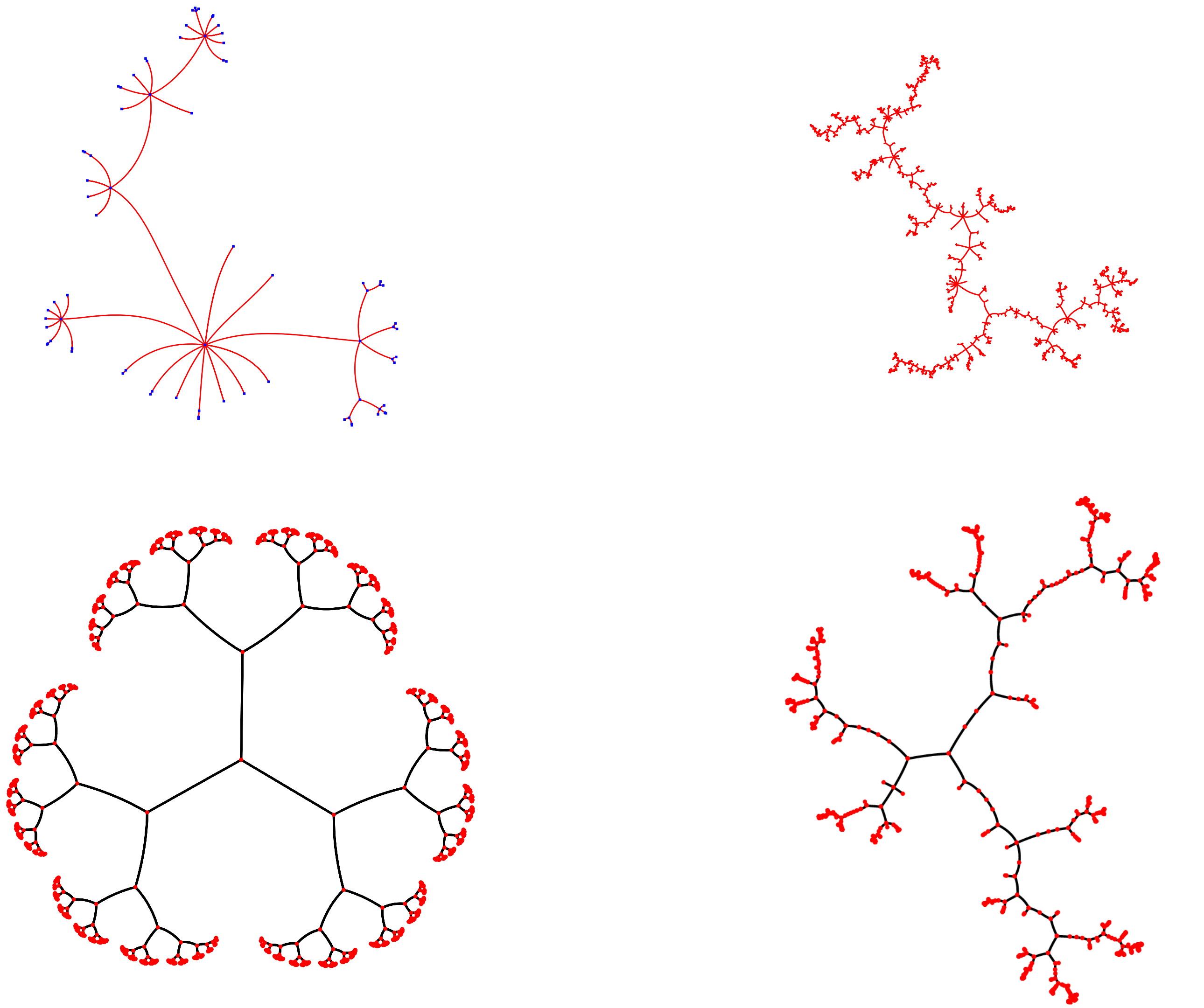


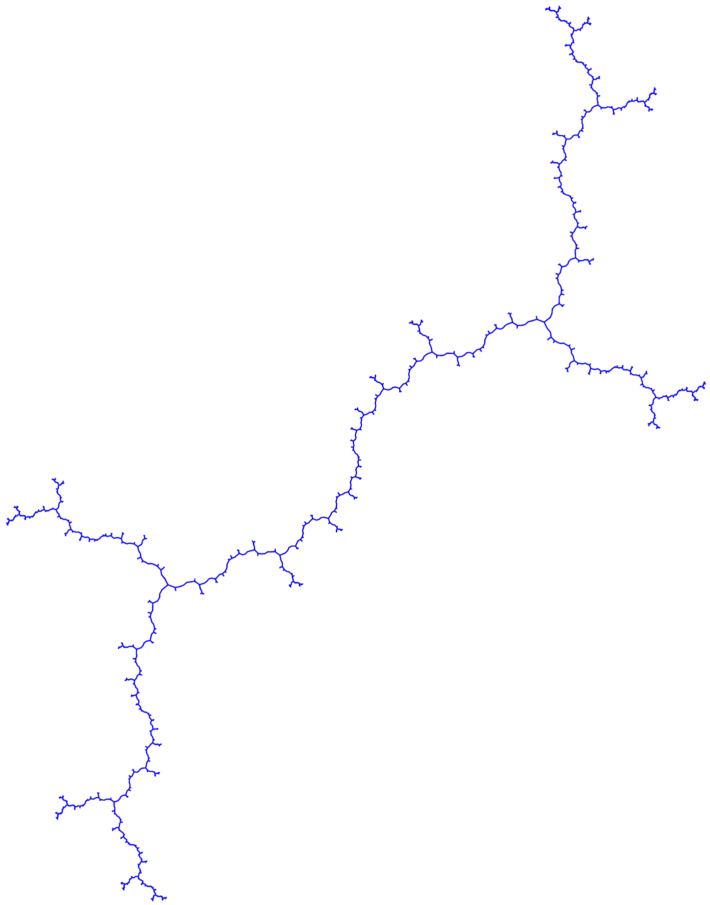
Conformal mapping can handle trees with thousands of edges.

Can obtain polynomial roots α with thousands of digits of accuracy.

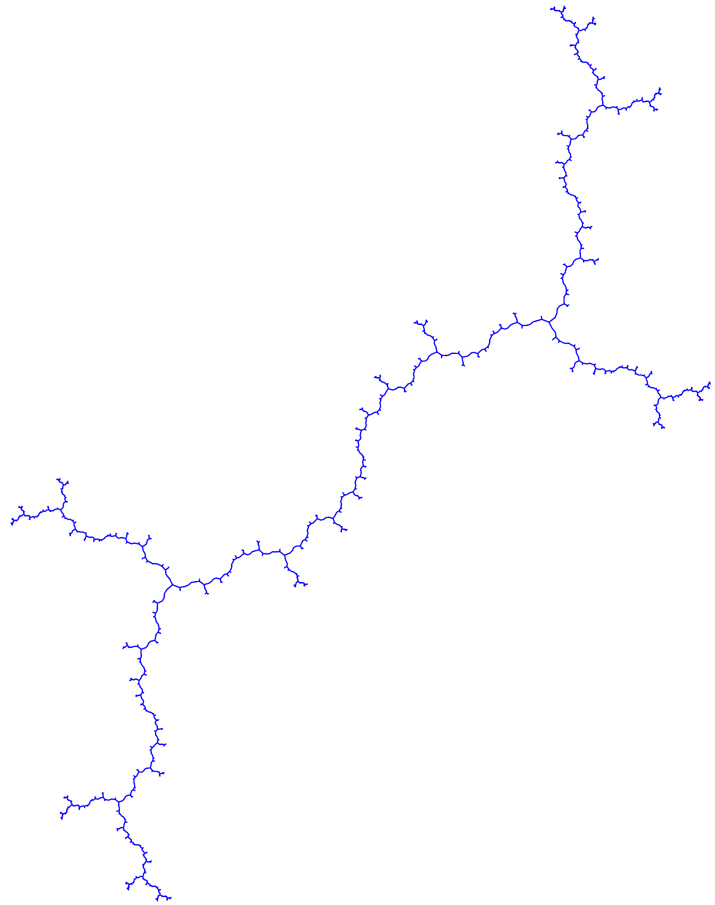
Marshall and Rohde compute number field by finding integer relation between $1, \alpha, \alpha^2, \ldots$ using Ferguson's PSLQ algorithm.



Some true trees, courtesy of Marshall and Rohde



Ceci n'est pas un ensemble de Julia.



True tree approximating combinatorics of the Julia set.
Rigidity: combinatorial data determines geometry.
Julia set is the limit of true trees.

Which planar trees have a true form?

What can that true form look like?

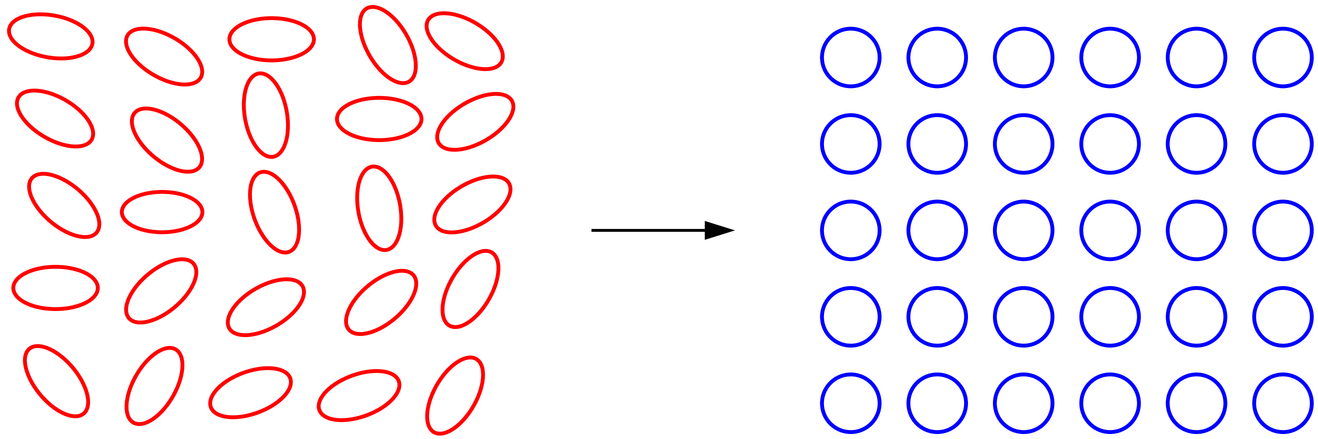
Theorem: Every finite planar tree has a true form.

Unique up to similarities (Morera + Liouville).

Standard proof uses the uniformization theorem.

I will describe an alternate proof using quasiconformal maps.

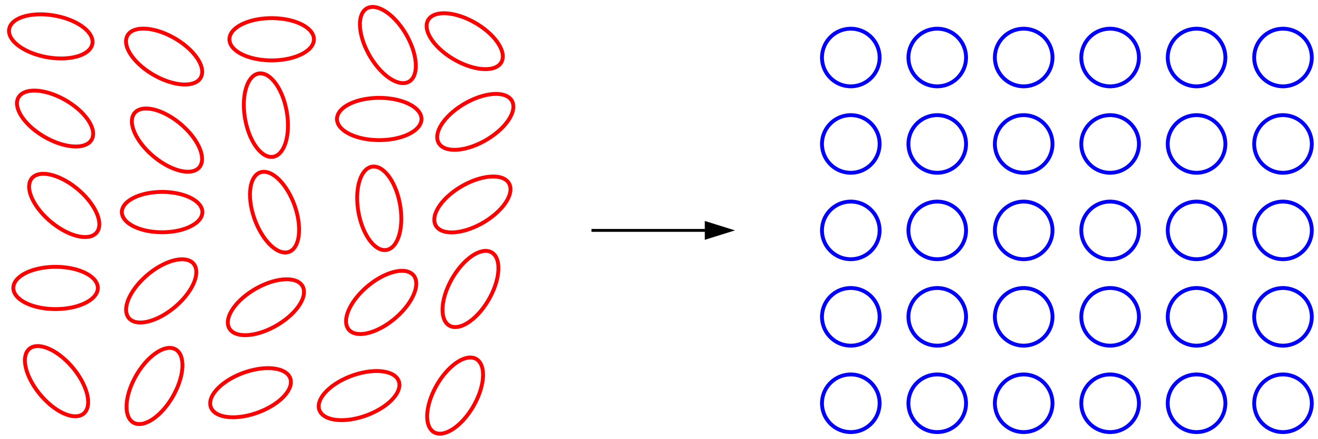
Diffeomorphisms send infinitesimal ellipses to circles.



Eccentricity = ratio of major to minor axis of ellipse.

K-quasiconformal = ellipses have eccentricity $\leq K$ almost everywhere

Diffeomorphisms send infinitesimal ellipses to circles.



Eccentricity = ratio of major to minor axis of ellipse.

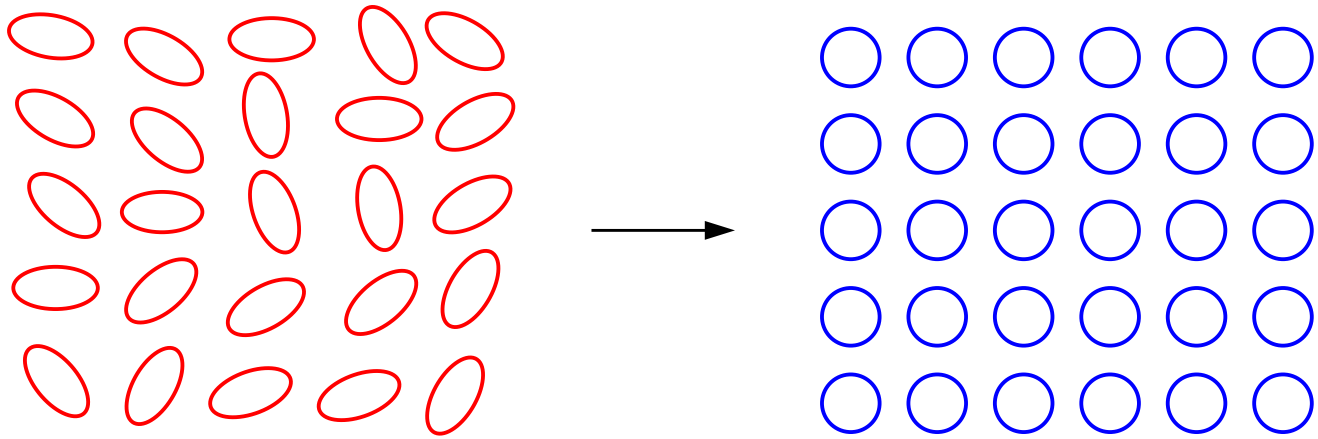
K-quasiconformal = ellipses have eccentricity $\leq K$ almost everywhere

Ellipses determined a.e. by measurable dilatation $\mu = f_{\overline{z}}/f_z$ with

$$|\mu| \leq \frac{K-1}{K+1} < 1.$$

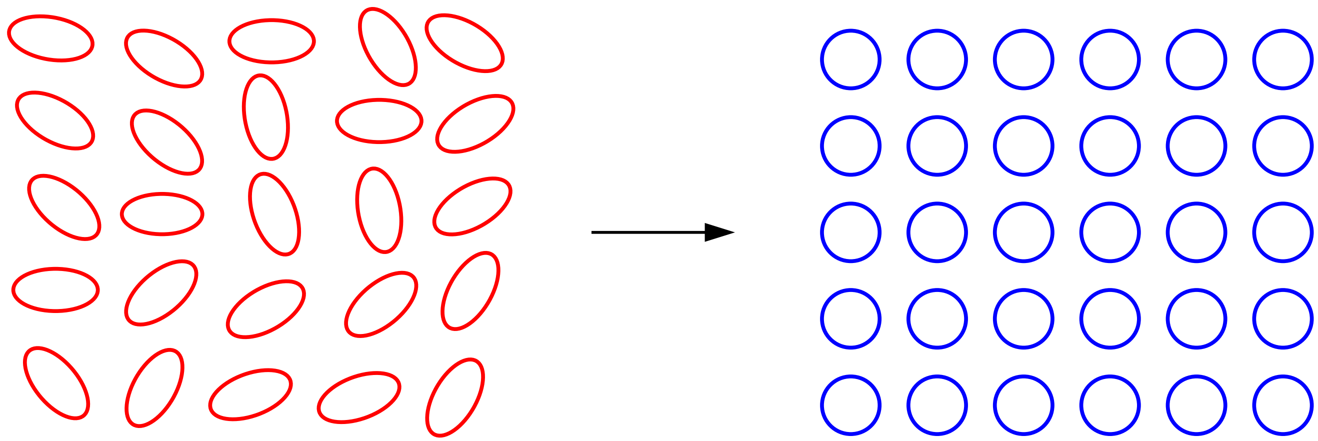
Conversely, ...

Diffeomorphisms send infinitesimal ellipses to circles.



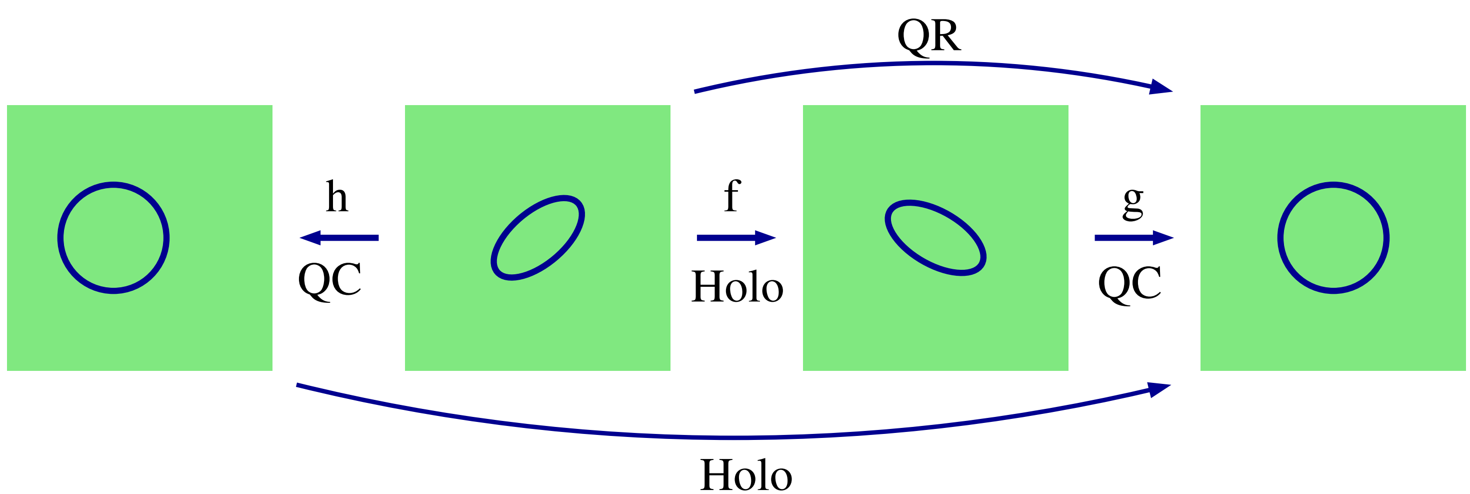
Mapping theorem: any such μ comes from some QC map f.

Diffeomorphisms send infinitesimal ellipses to circles.



Mapping theorem: any such μ comes from some QC map f.

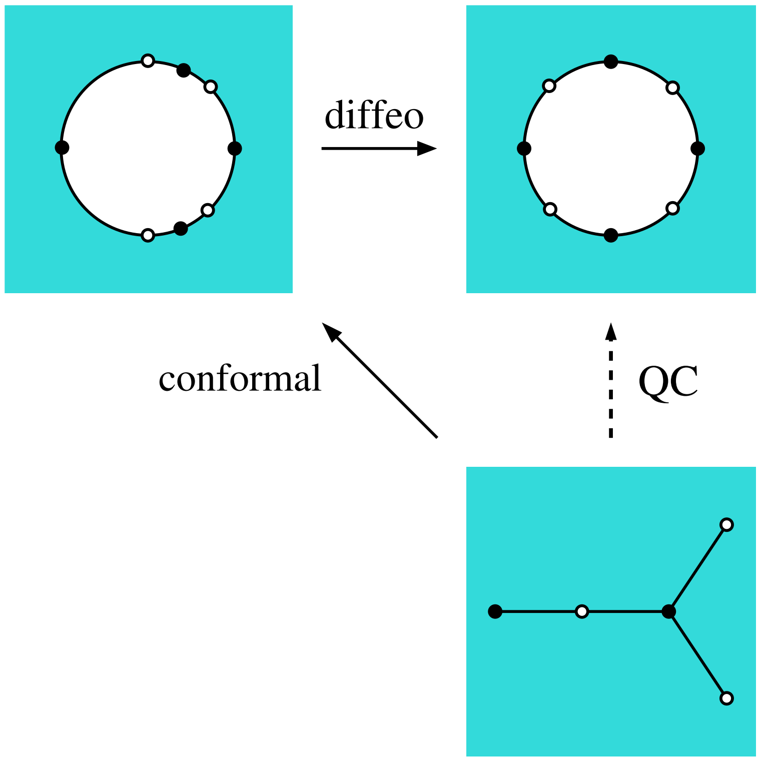
Cor: If f is holomorphic and g is QC, then there is a QC map h so that $F = g \circ f \circ h^{-1}$ is also holomorphic. ($g \circ f$ is called QR = quasiregular)



QC proof that every finite tree has a true form:

Map $\Omega = \mathbb{C} \setminus T$ to $\{|z| > 1\}$ conformally.

"Equalize intervals" by diffeomorphism. Composition is quasiconformal.



Dilatation of QC map depends degree of "imbalance" of harmonic measure.

QC proof that every finite tree has a true form:

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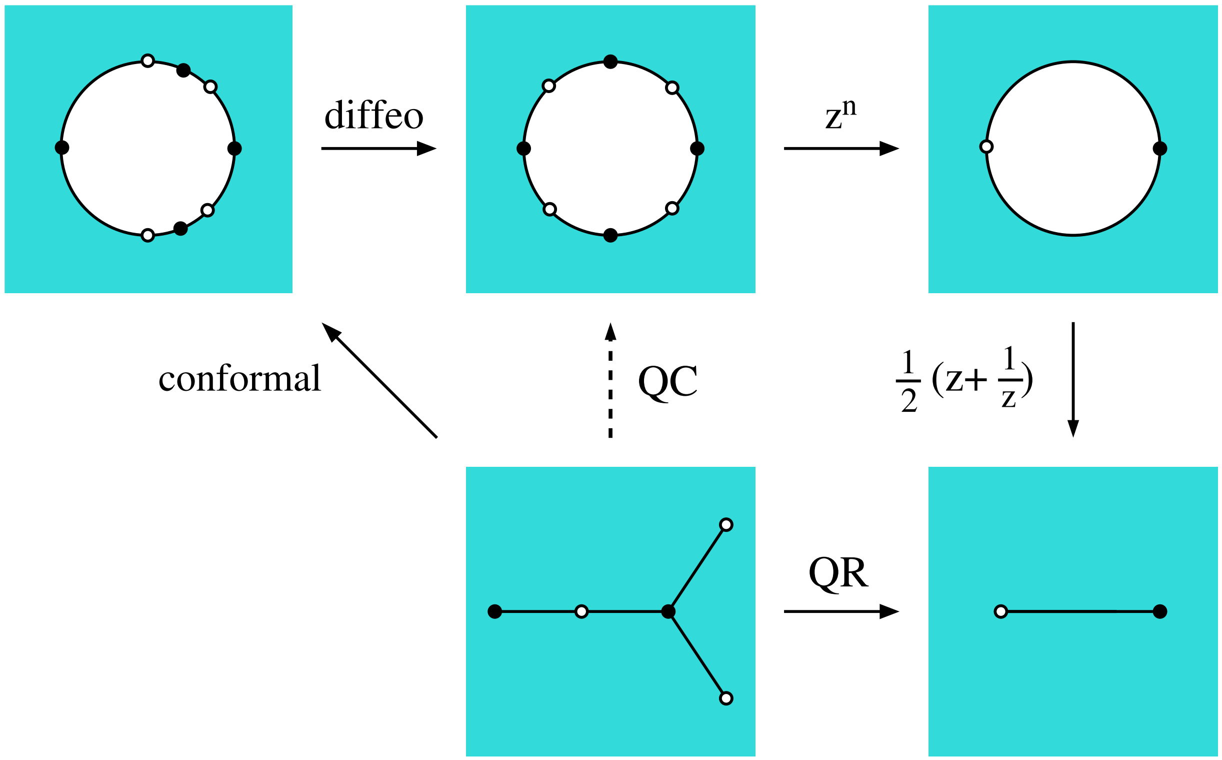

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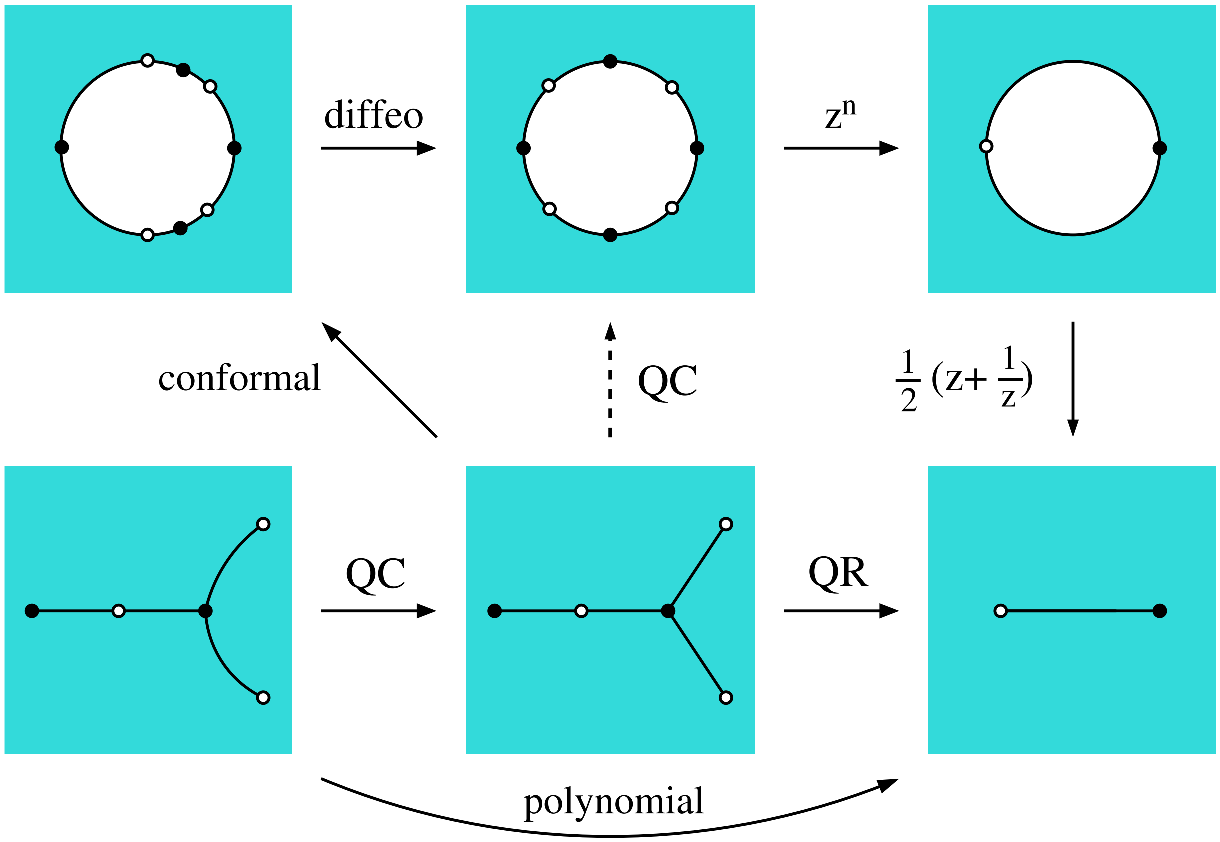


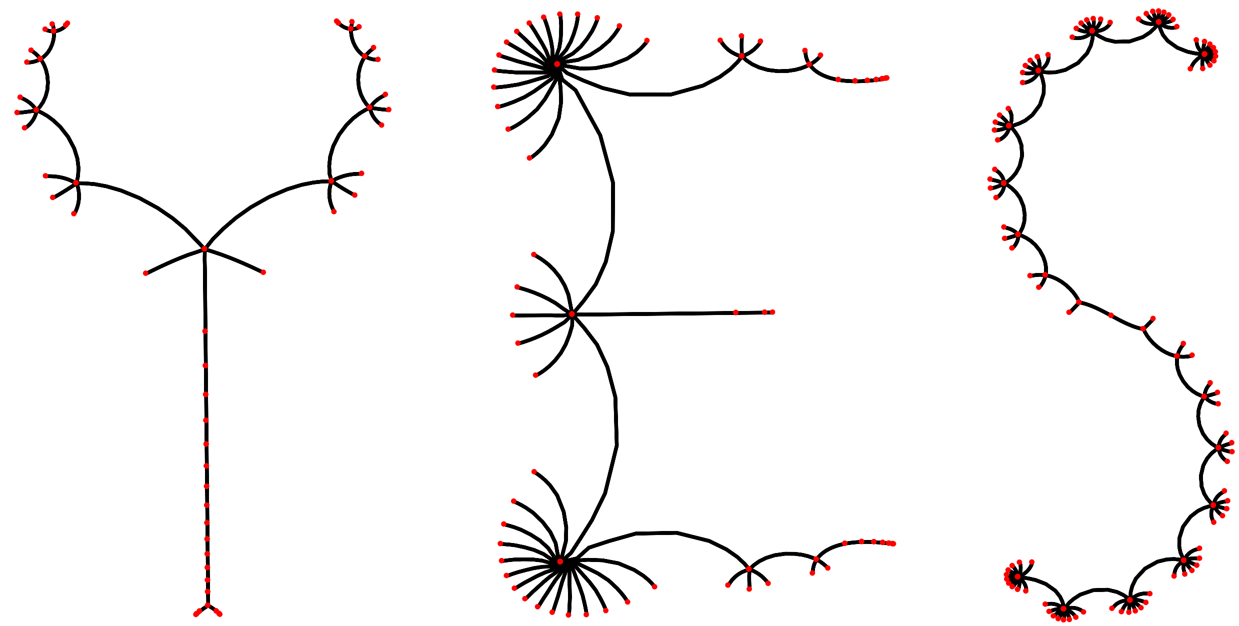
Mapping theorem implies there is a QC φ so $p = q \circ \varphi$ is a polynomial.

Thus true trees can have any combinatorics.

Alex Eremenko: can they have any shape?

Thus true trees can have any combinatorics.

Alex Eremenko: can they have any shape?



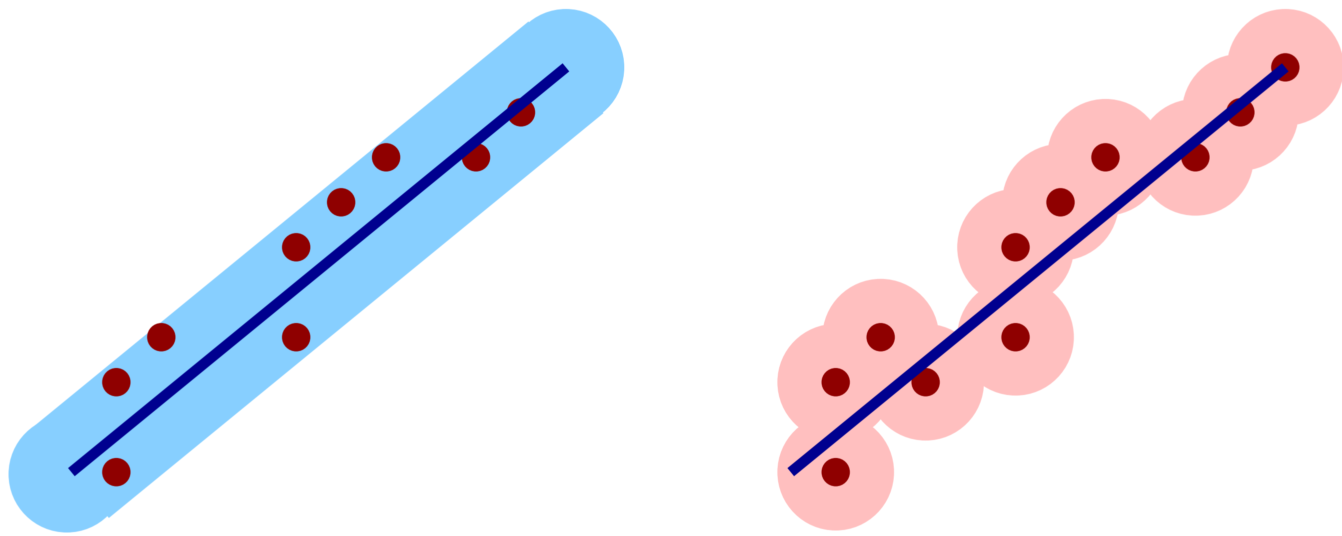
Thm: Every planar continuum is Hausdorff limit of true trees.

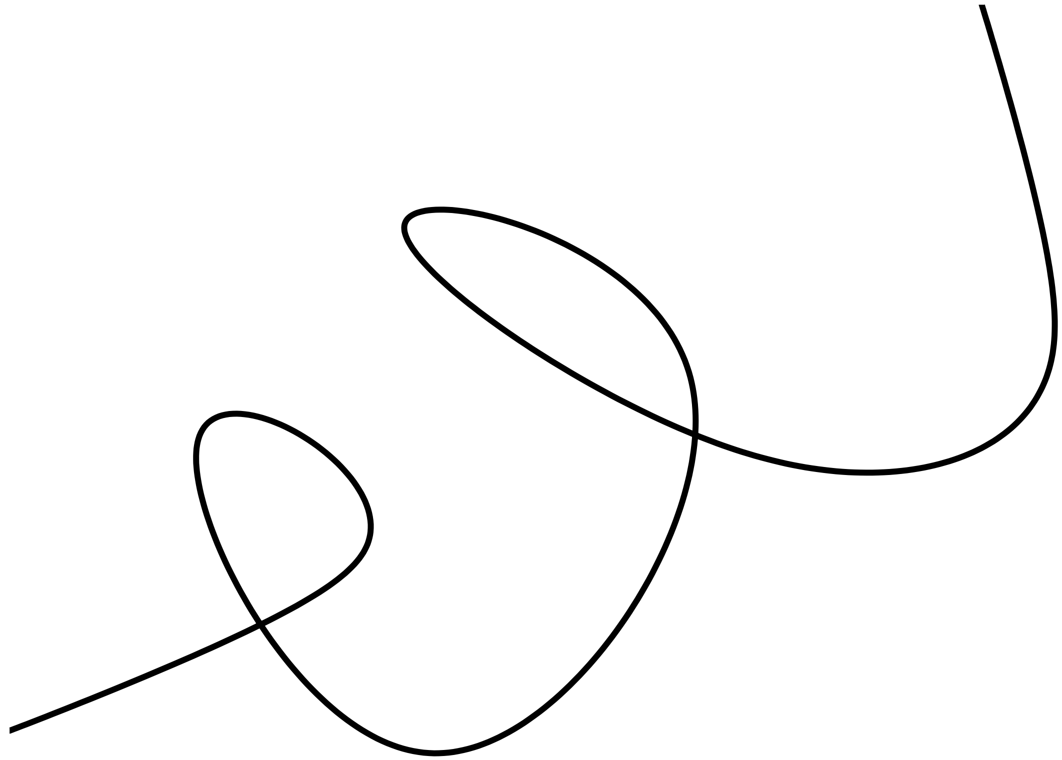
For a set E and $\epsilon > 0$ define ϵ-neighborhood of E:

$$E(\epsilon) = \{z : \text{dist}(z, E) < \epsilon\}.$$

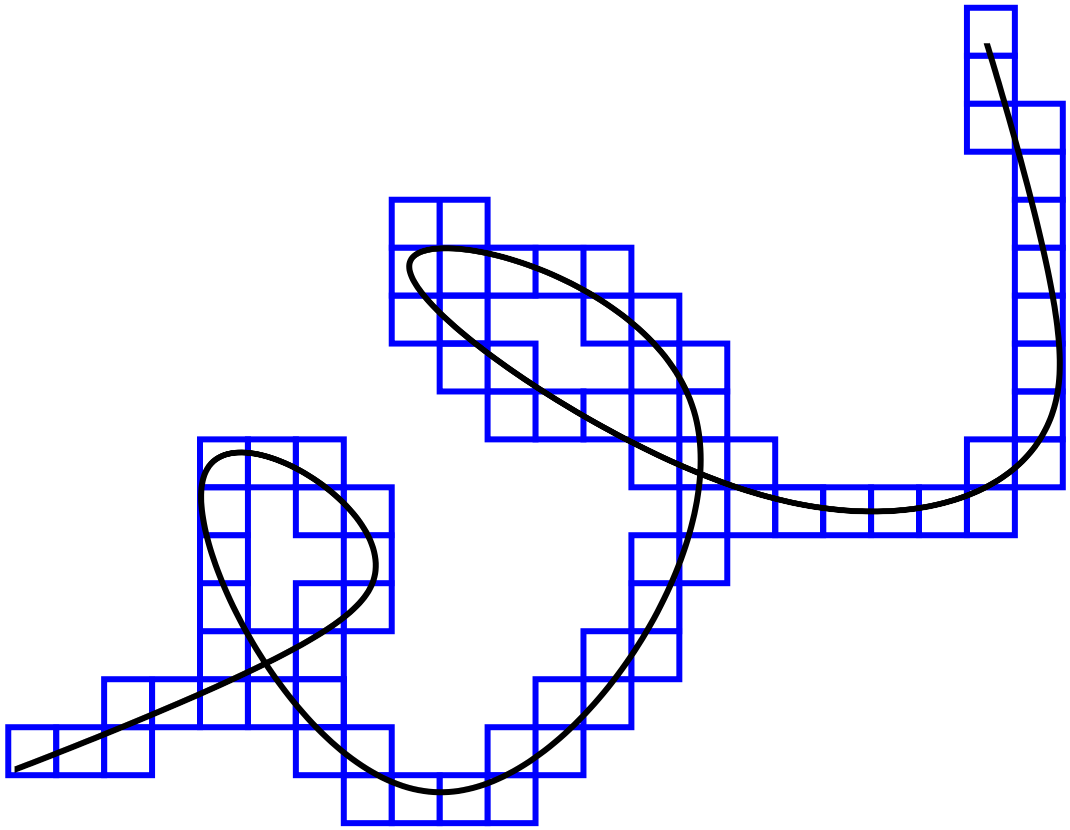
Defn: Hausdorff distance between sets is

$$d(E, F) = \inf\{\epsilon > 0 : E \subset F(\epsilon), F \subset E(\epsilon)\}.$$





Suffices to approximate subtrees of a grid.



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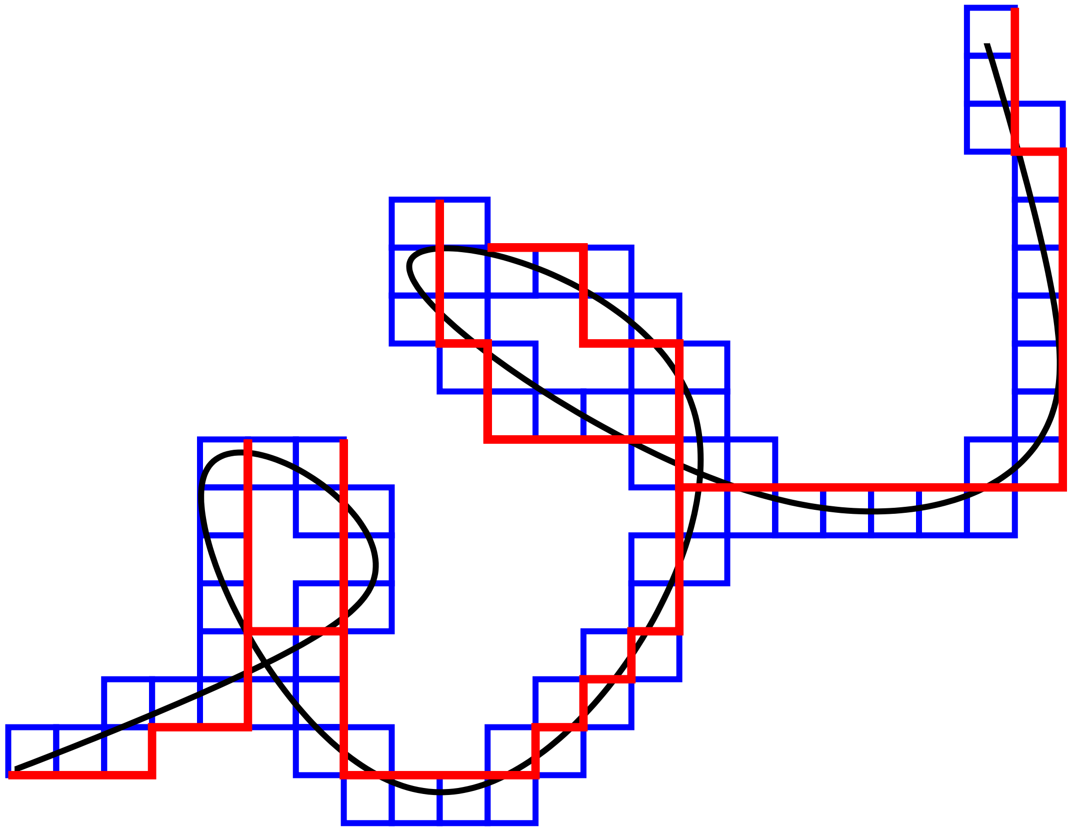

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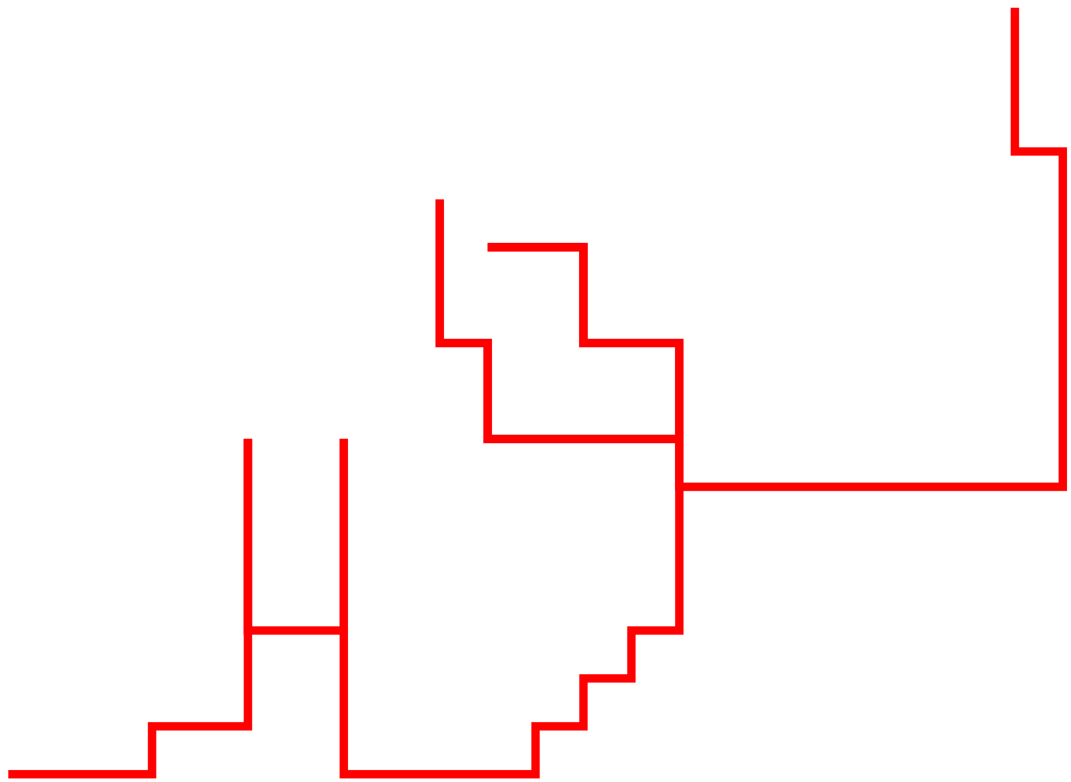
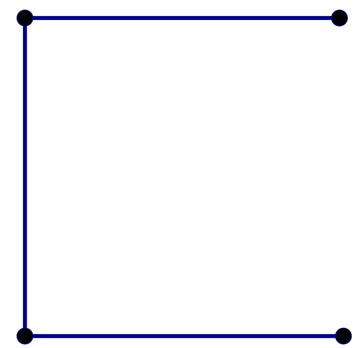


Suffices to approximate subtrees of a grid.

Theorem: Every planar continuum is a limit of true trees.

Idea of Proof: reduce harmonic measure ratio by adding edges.

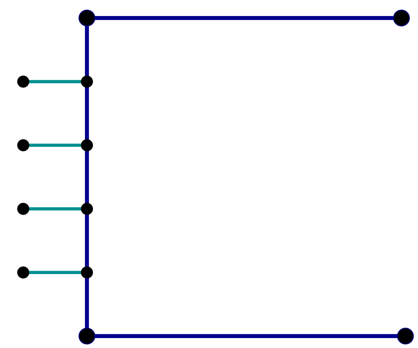


Vertical side has much larger harmonic measure from left.

Add edges ($\Rightarrow$ change combinatorics) to "balance" harmonic measure.

Theorem: Every planar continuum is a limit of true trees.

Idea of Proof: reduce harmonic measure ratio by adding edges.



"Left" harmonic measure is reduced (roughly 3-to-1).

New edges are approximately balanced (universal constant).

Theorem: Every planar continuum is a limit of true trees.

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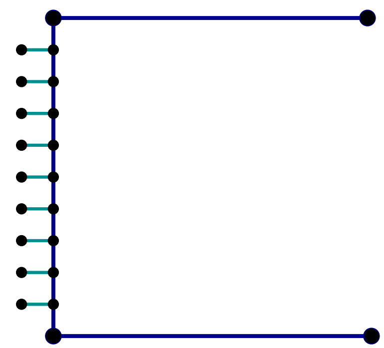


"Left" harmonic measure is reduced (roughly 3-to-1).

New edges are approximately balanced (universal constant).

Mapping theorem gives exactly balanced.

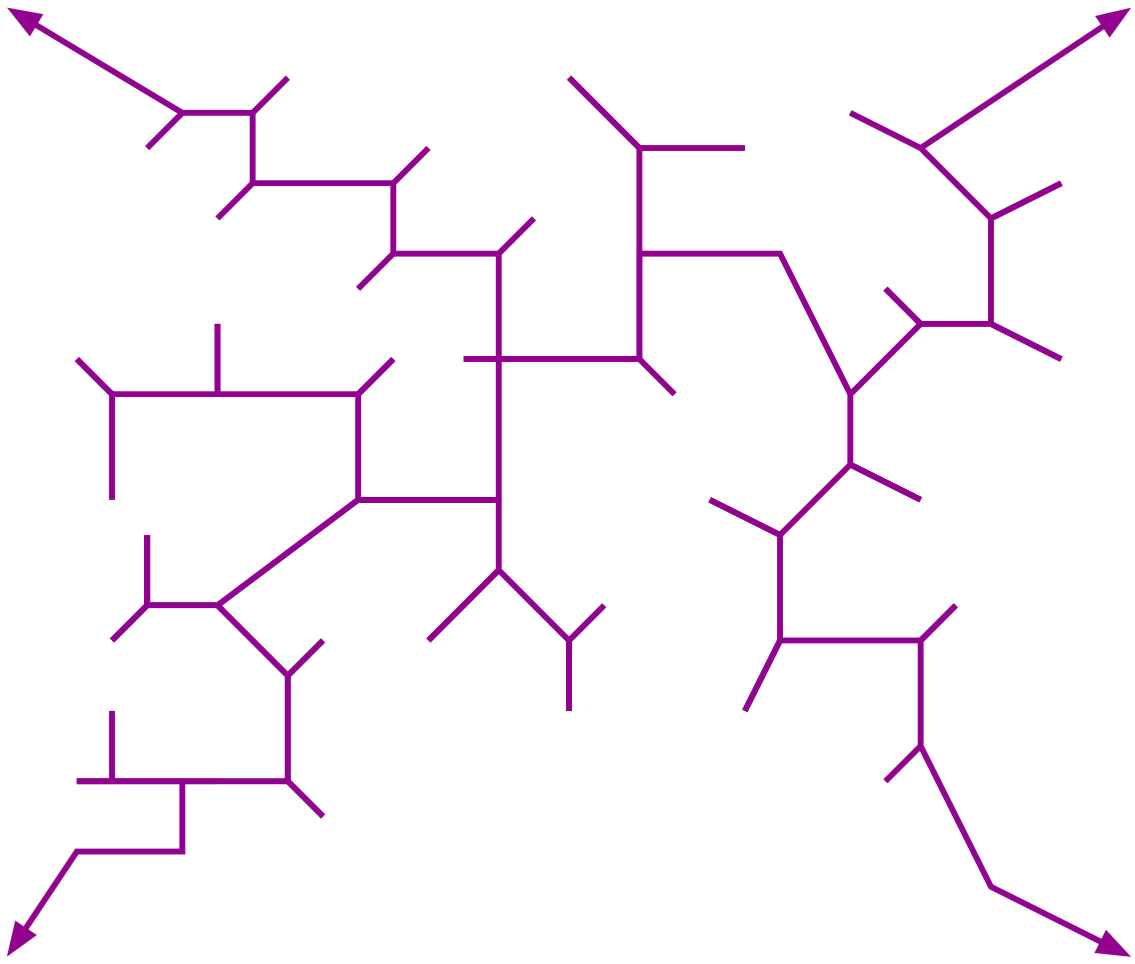
QC correction map is near identity if "spikes" are short.

New tree approximates shape of old tree; different combinatorics.

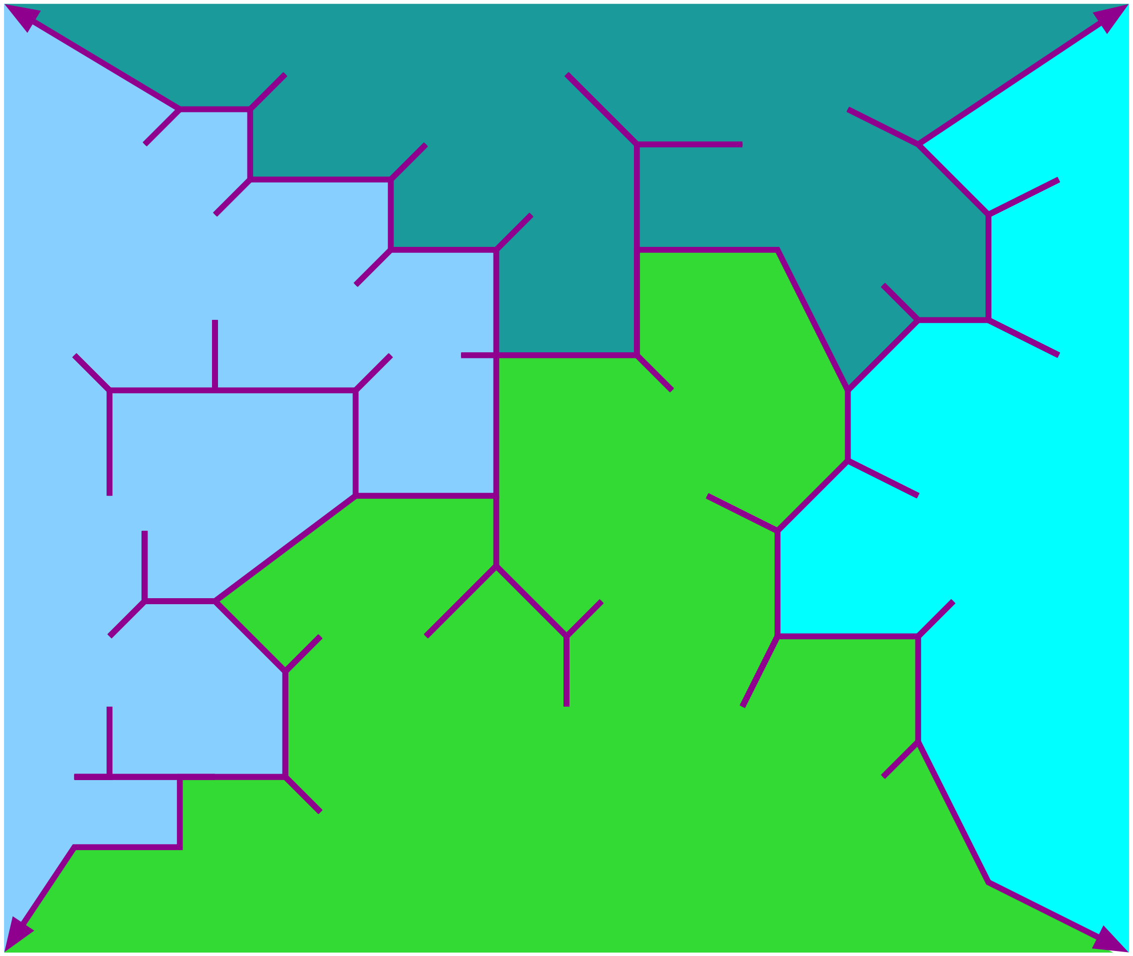
Dessins d'adolescents

Finite planar trees $\Leftrightarrow$ polynomials with 2 critical values.

What about infinite planar trees? Shabat entire functions?



Do infinite trees correspond to entire functions with 2 critical values?

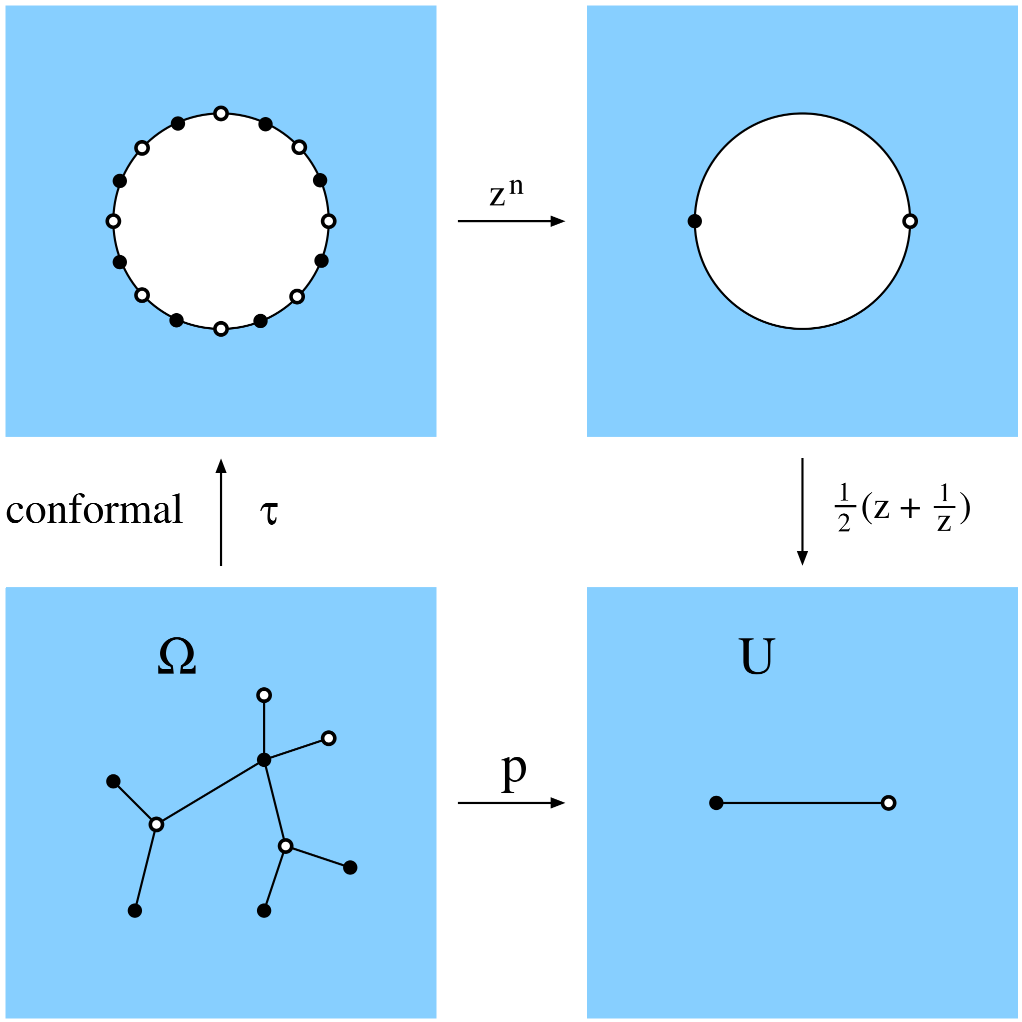


Main difference:

$\mathbb{C}\setminus$ finite tree = one topological annulus

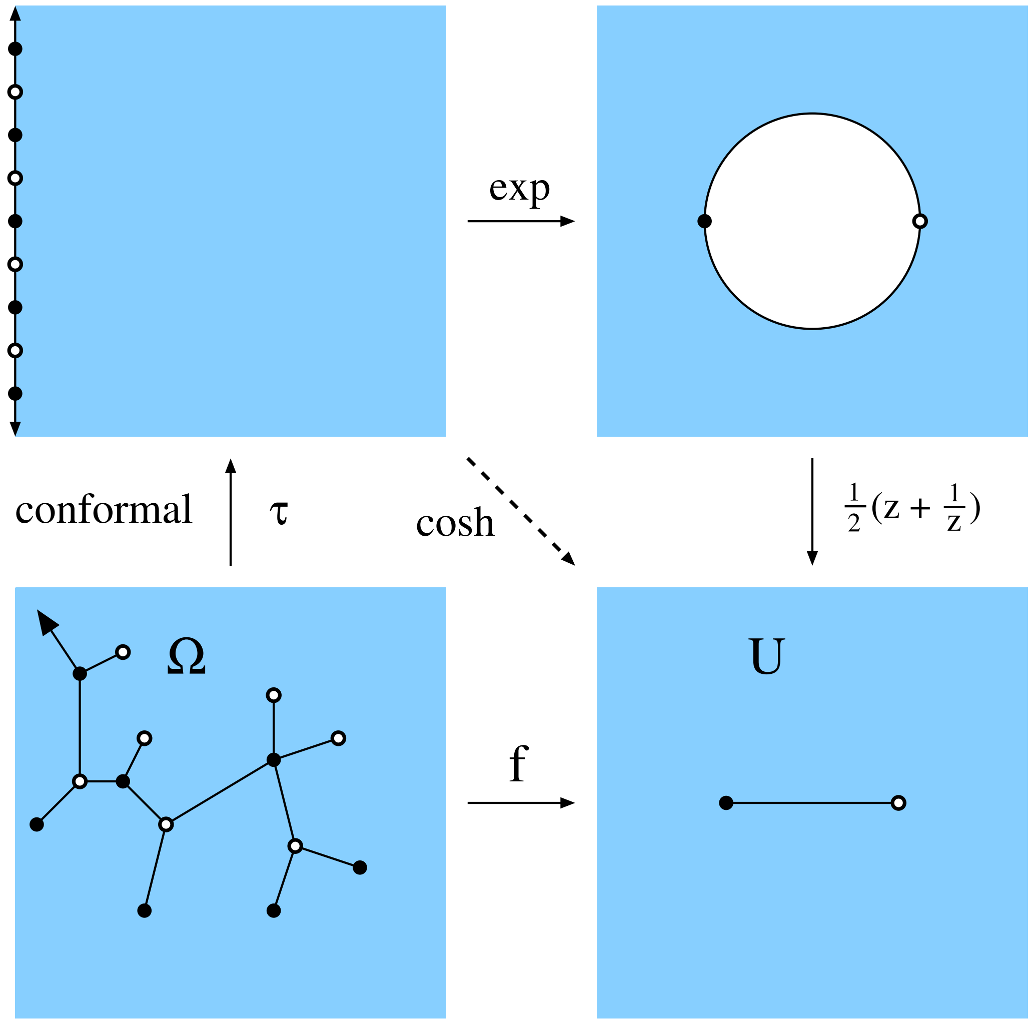
$\mathbb{C}\setminus$ infinite tree = many simply connected components

Recall finite case



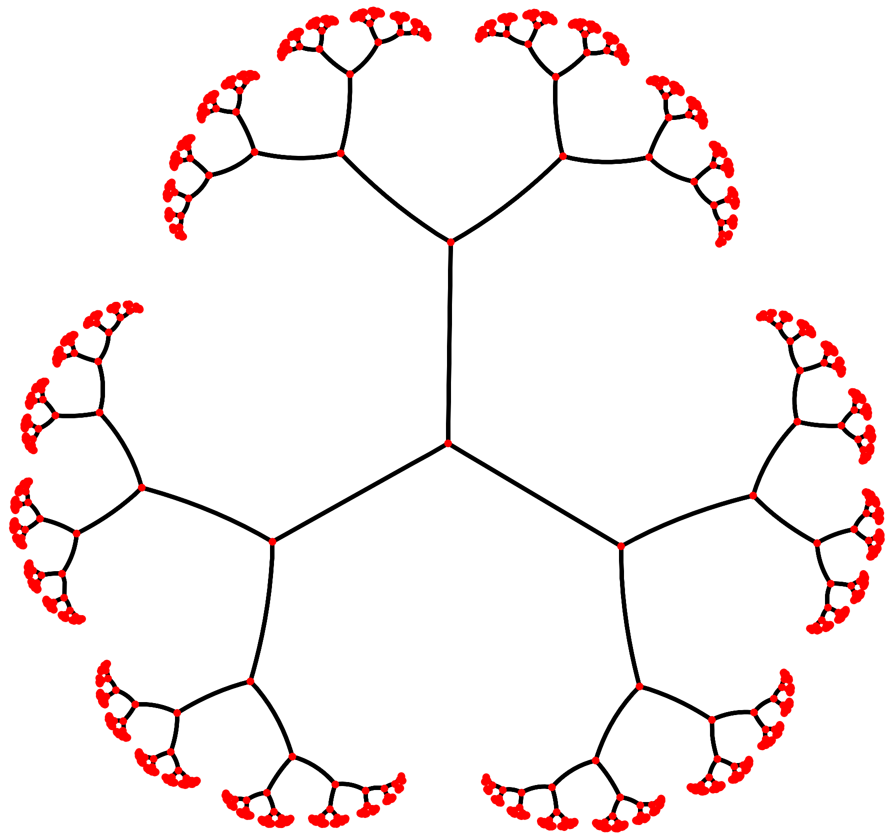
T is true tree $\Leftrightarrow p = \frac{1}{2}(\tau^n + 1/\tau^n)$ is continuous across T.

Infinite case



Infinite balanced tree $\Leftrightarrow f = \cosh \circ \tau$ is continuous across T.

Infinite 3-regular tree does not have a true form in the plane.



Instead it has a “true embedding” into the hyperbolic disk.

Studied by Oleg Ivrii, Peter Lin, Steffen Rohde, Emanuel Sygal.

Links to rational and group dynamics.

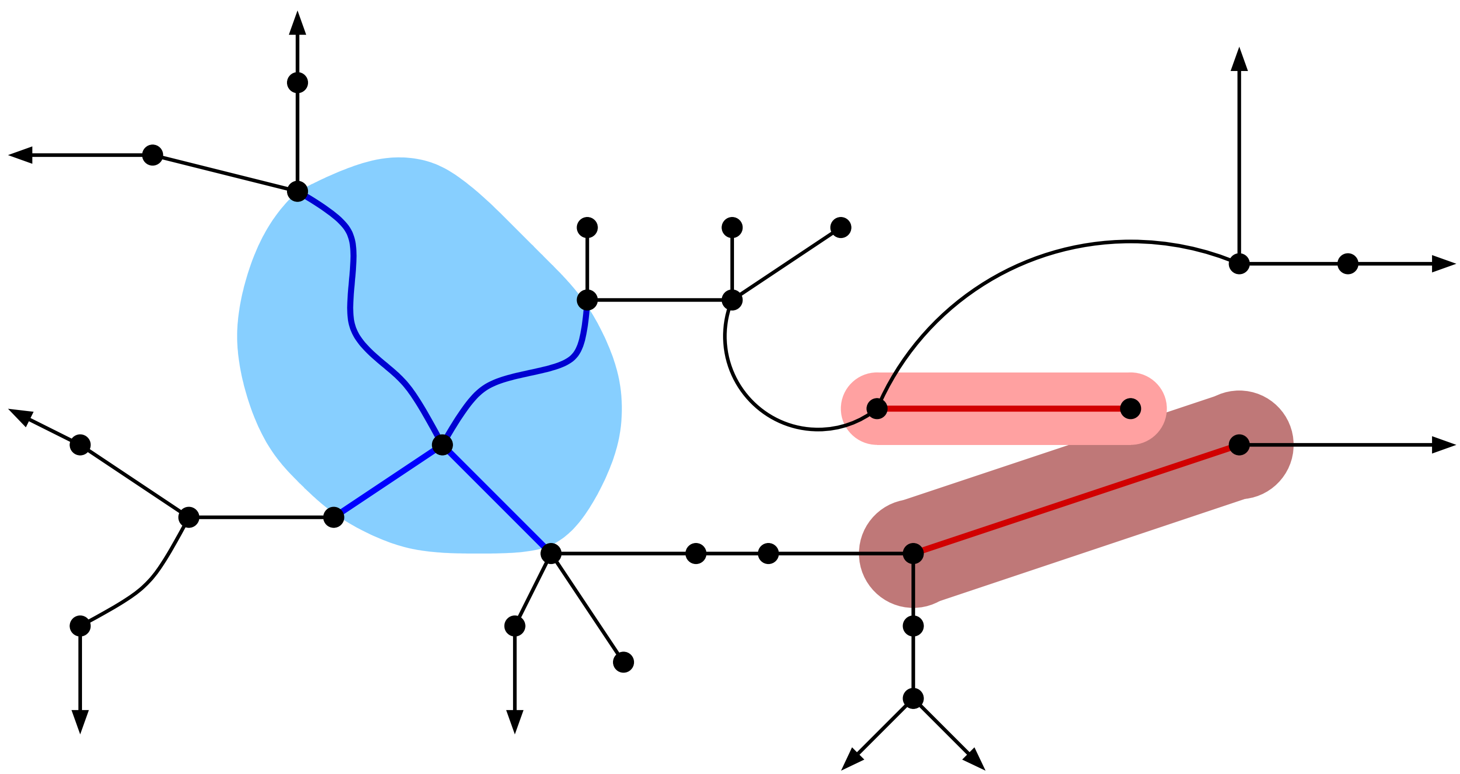
Thm: every "nice" infinite planar tree T is approximated by $f^{-1}([-1,1])$ for some entire function f with CV $= \{\pm 1\}$.

"nice" $=$ two conditions that are automatic for finite trees.

(1) Bounded Geometry (local condition; easy to verify):

- edges are uniformly smooth.
- adjacent edges form bi-Lipschitz image of a star $= \{z^n \in [0, r]\}$
- non-adjacent edges are well separated,

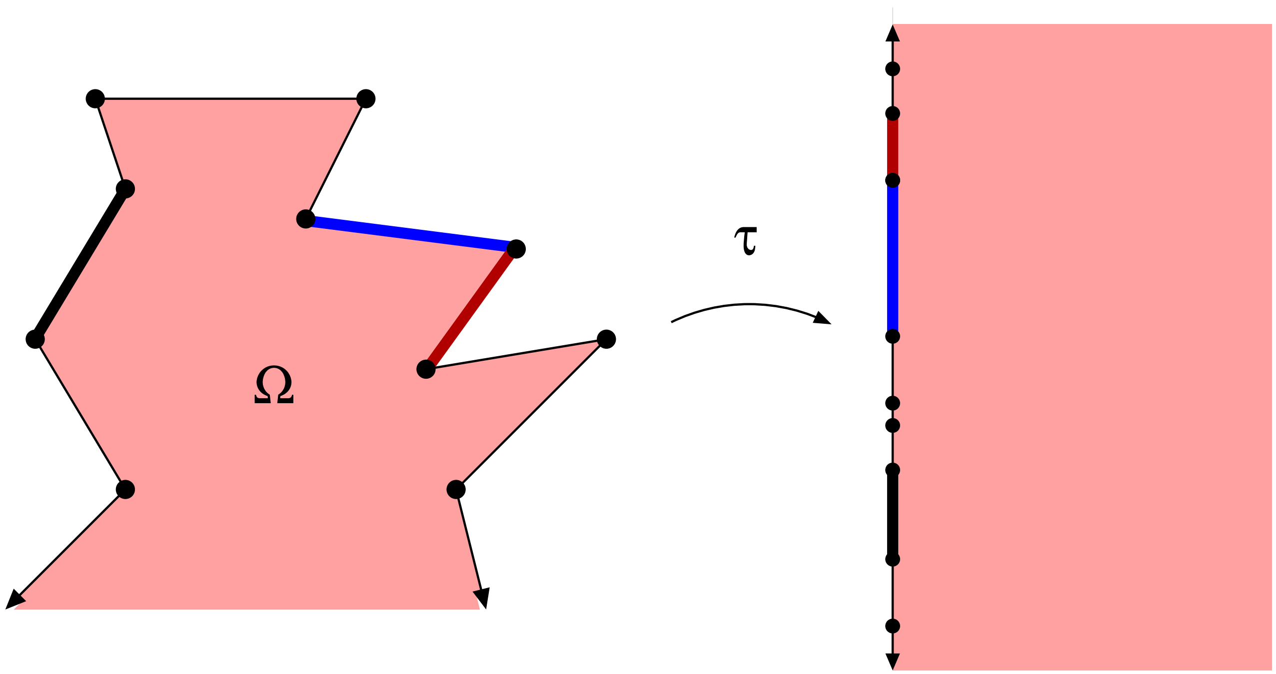
$$\mathrm{dist}(e, f) \geq \epsilon \cdot \min(\mathrm{diam}(e), \mathrm{diam}(f)).$$



(2) τ-Lower Bound (global condition; harder to check):

Complementary components of tree are simply connected.

Each can be conformally mapped to right half-plane. Call map τ.



We assume all images have length $\geq \pi$.

Need positive lower bound; actual value usually not important.

Components are "thinner" than half-plane near ∞.

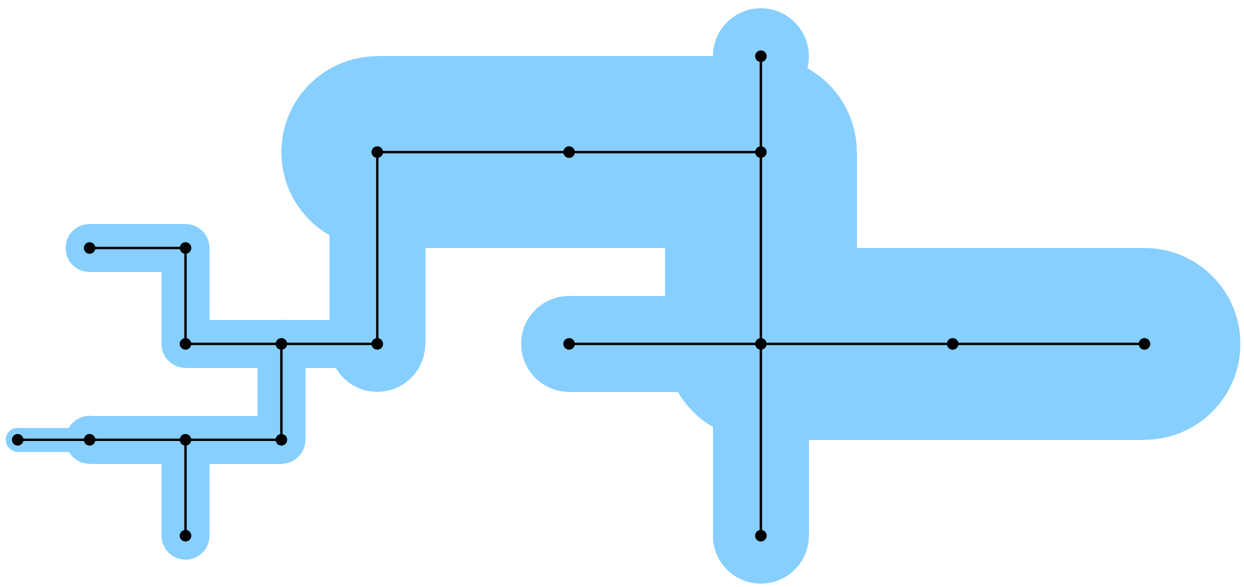
If e is an edge of T and $r > 0$ let

$$e(r) = \{z : \mathrm{dist}(z, e) \leq r \cdot \mathrm{diam}(e)\}$$

If e is an edge of T and $r > 0$ let

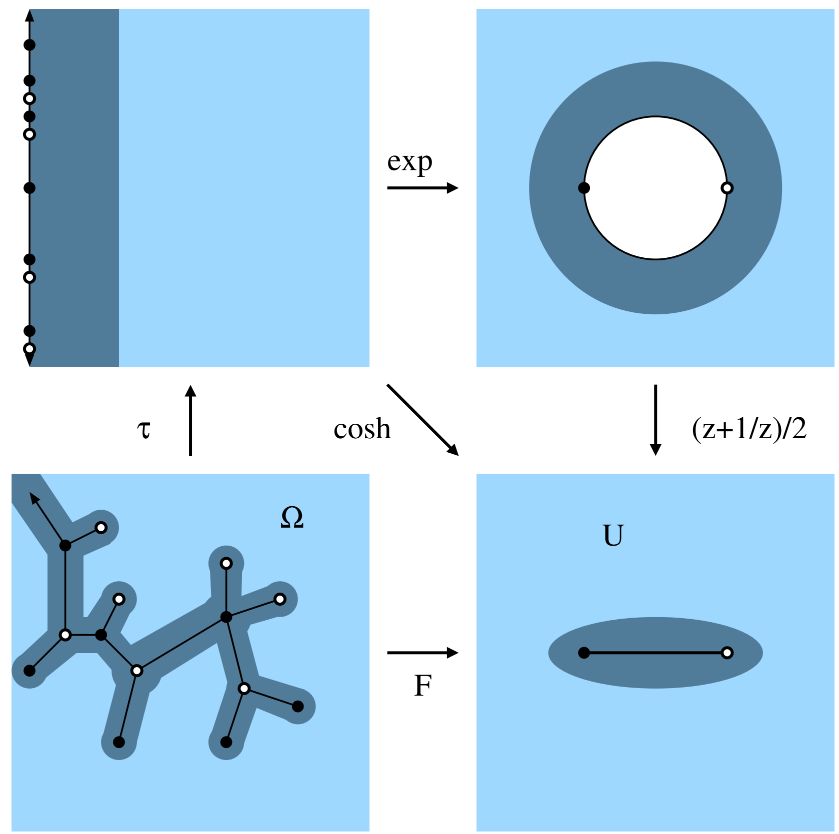
$$e(r) = \{z : \mathrm{dist}(z, e) \le r \cdot \mathrm{diam}(e)\}$$

Define neighborhood of T: $T(r) = \cup\{e(r) : e \in T\}$.



$T(r)$ for infinite tree replaces Hausdorff metric in finite case.

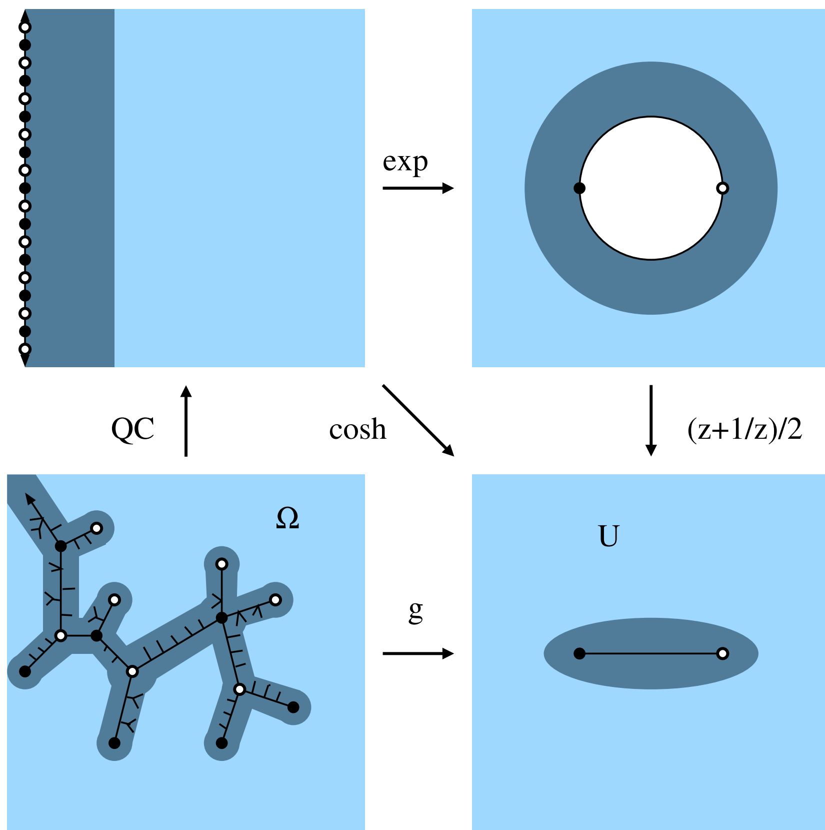
Thm: If T has bounded geometry and satisfies a τ-lower bound, then there is a QR g and $r > 0$ with $g = \cosh \circ \tau$ off $T(r)$ and $\mathrm{CV}(g) = \pm 1$.



This is the "QC-Folding Theorem".

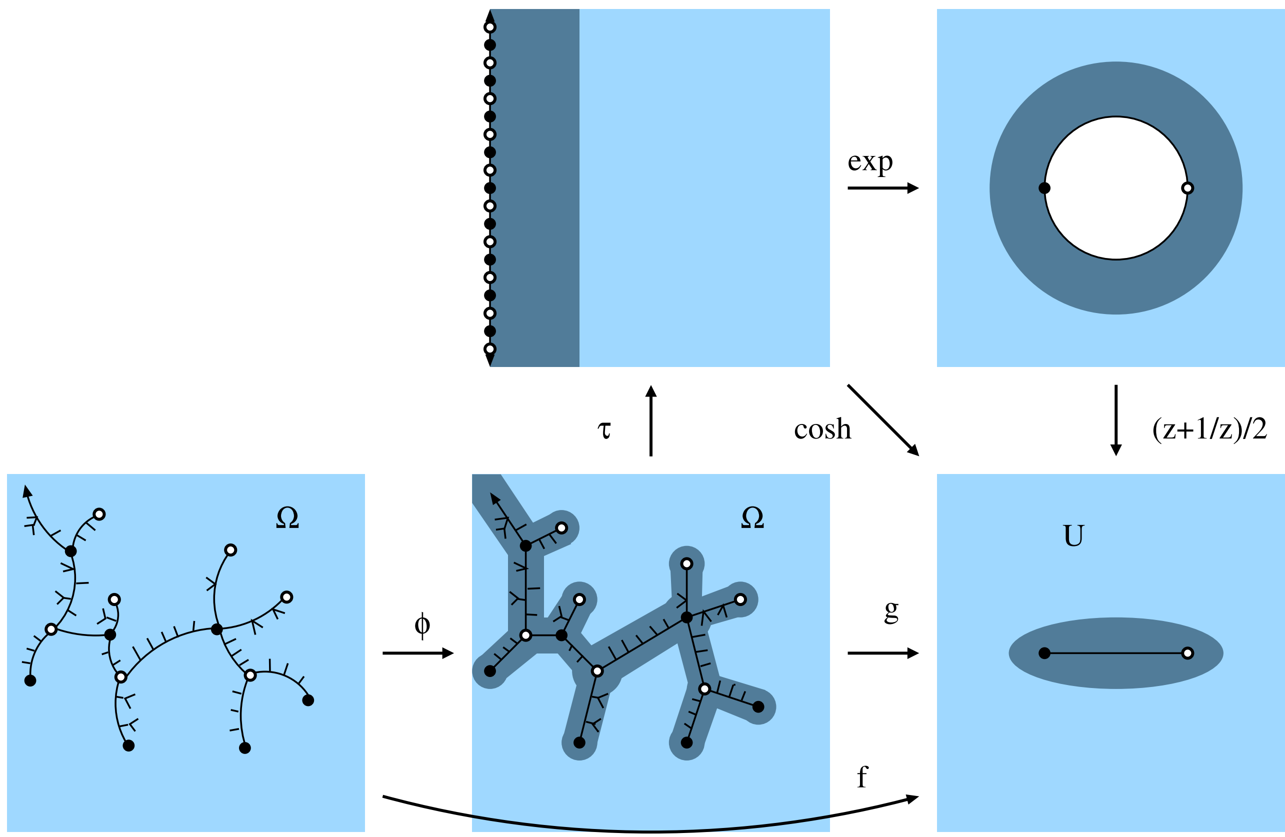
$F = \cosh \circ \tau$ is usually discontinuous across T.

Thm: If T has bounded geometry and satisfies a τ-lower bound, then there is a QR g and $r > 0$ with $g = \cosh \circ \tau$ off $T(r)$ and $\mathrm{CV}(g) = \pm 1$.



But g is continuous everywhere. Tree combinatorics are changed.

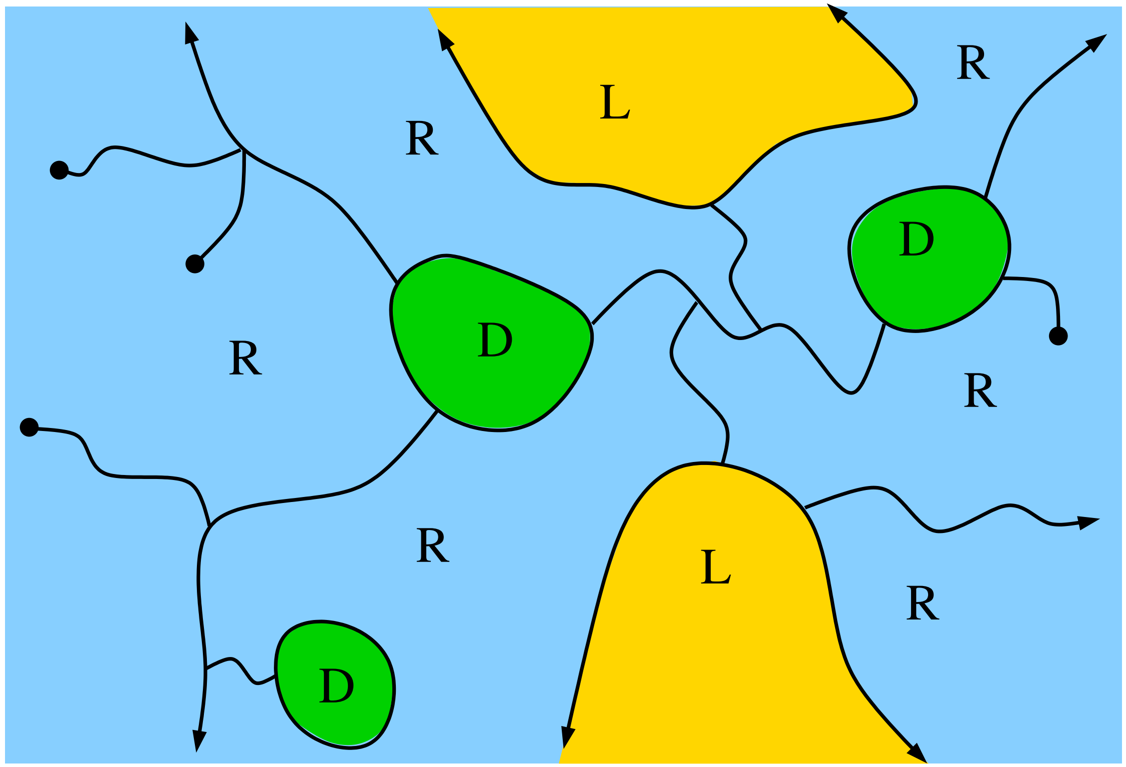
Thm: If T has bounded geometry and satisfies a τ-lower bound, then there is a QR g and $r > 0$ with $g = \cosh \circ \tau$ off $T(r)$ and $\mathrm{CV}(g) = \pm 1$.



By MRMT there is QC ϕ so $f = g \circ \phi$ is entire.

Often can prove $\phi(z) \approx z$, so $T \approx f^{-1}([-1, 1])$

More generally: replace tree by graph:



R = unbounded domains ($F = e^{\tau(z)}$, previous case)

D = bounded Jordan domains ($F = (z-a)^n$, high degree critical points)

L = unbounded Jordan domains ($F = e^{-\tau(z)}$, finite asymptotic values)

Can specify singular values in last two cases **exactly**.

Transcendental = entire, not polynomial

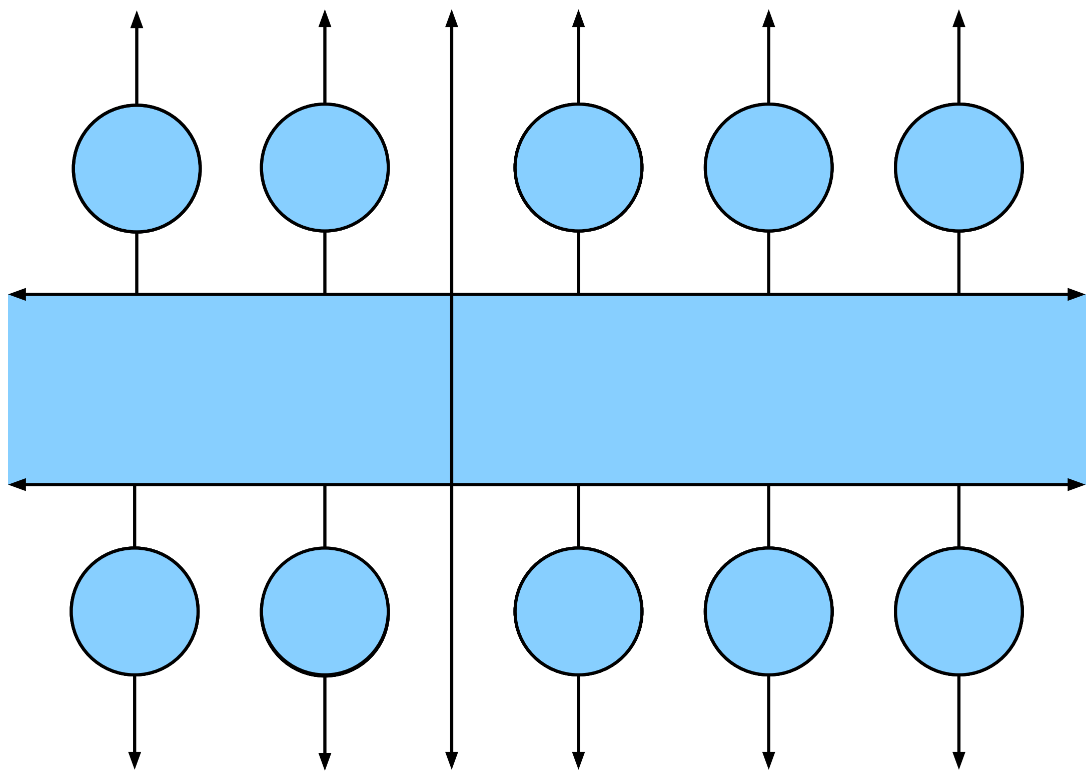
Singular set = closure of critical values and finite asymptotic values
= smallest set so that f is a covering map onto $\mathbb{C} \setminus S$

Eremenko-Lyubich class = bounded singular set = $\mathcal{B}$

Speiser class = finite singular set = $\mathcal{S} \subset \mathcal{B}$

Generalized folding produces "all" functions in both classes.

Entire functions with wandering domains.



Given an entire function f,

Fatou set $= \mathcal{F}(f) =$ open set where iterates are normal family.

Julia set $= \mathcal{J}(f) =$ complement of Fatou set.

f maps Fatou components into other Fatou components.

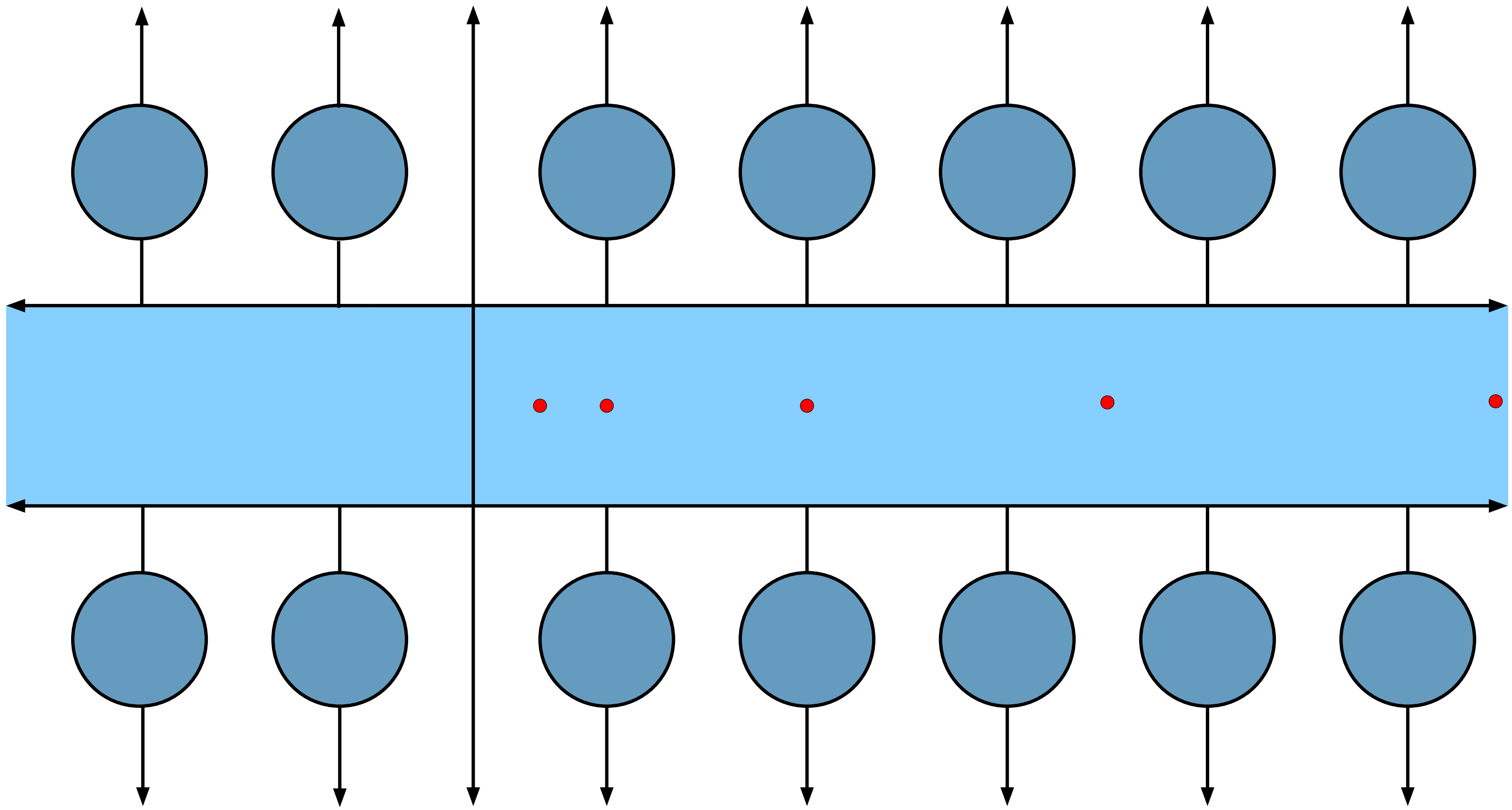
Wandering domain = Fatou component with infinite orbit.

- Entire functions can have wandering domains (Baker 1975).

- No wandering domains for rational functions (Sullivan 1985).

- Also none in Speiser class (Eremenko-Lyubich, Goldberg-Keen).

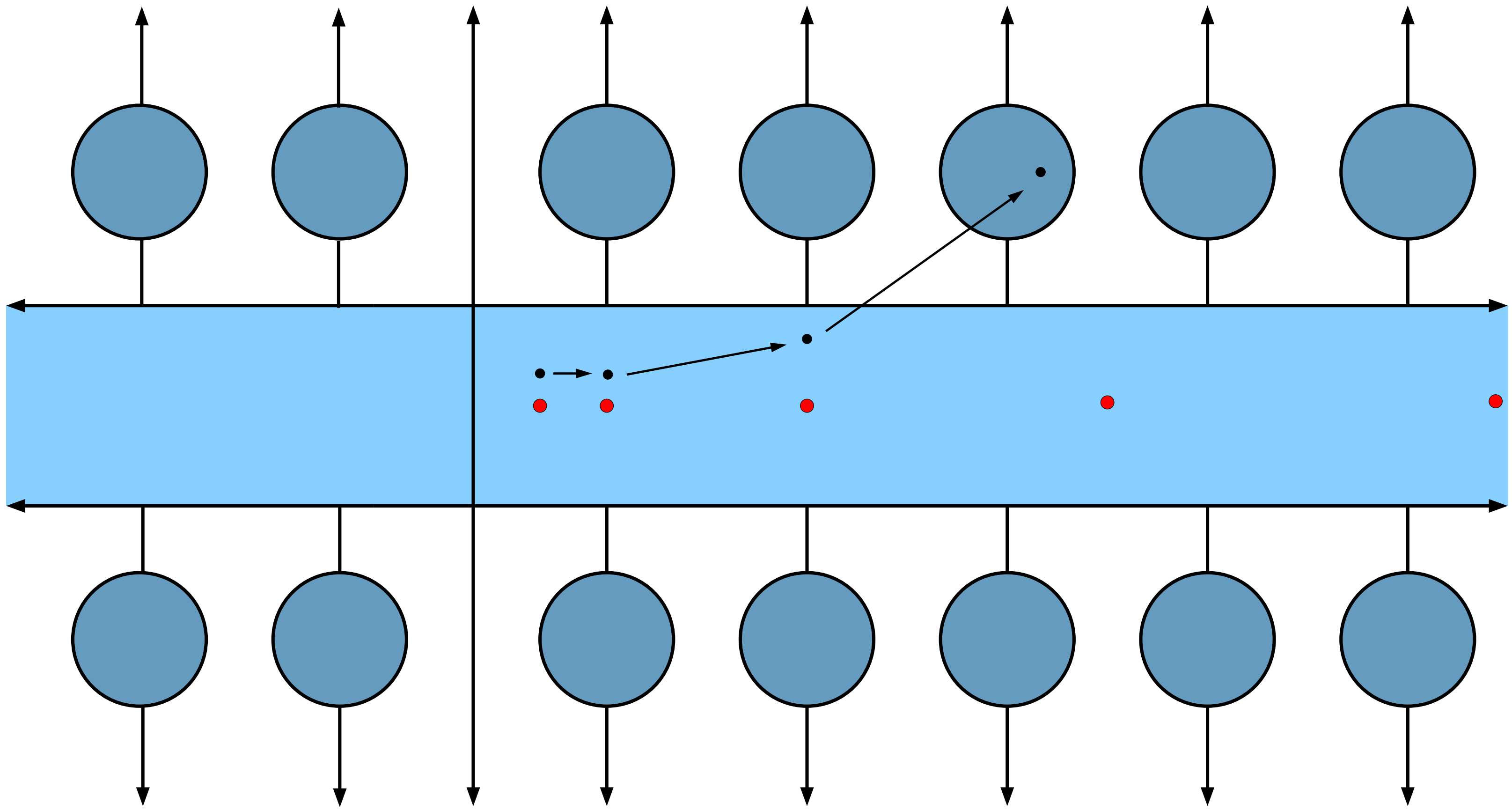
Wandering domains in Eremenko-Lyubich class? Open since 1985.



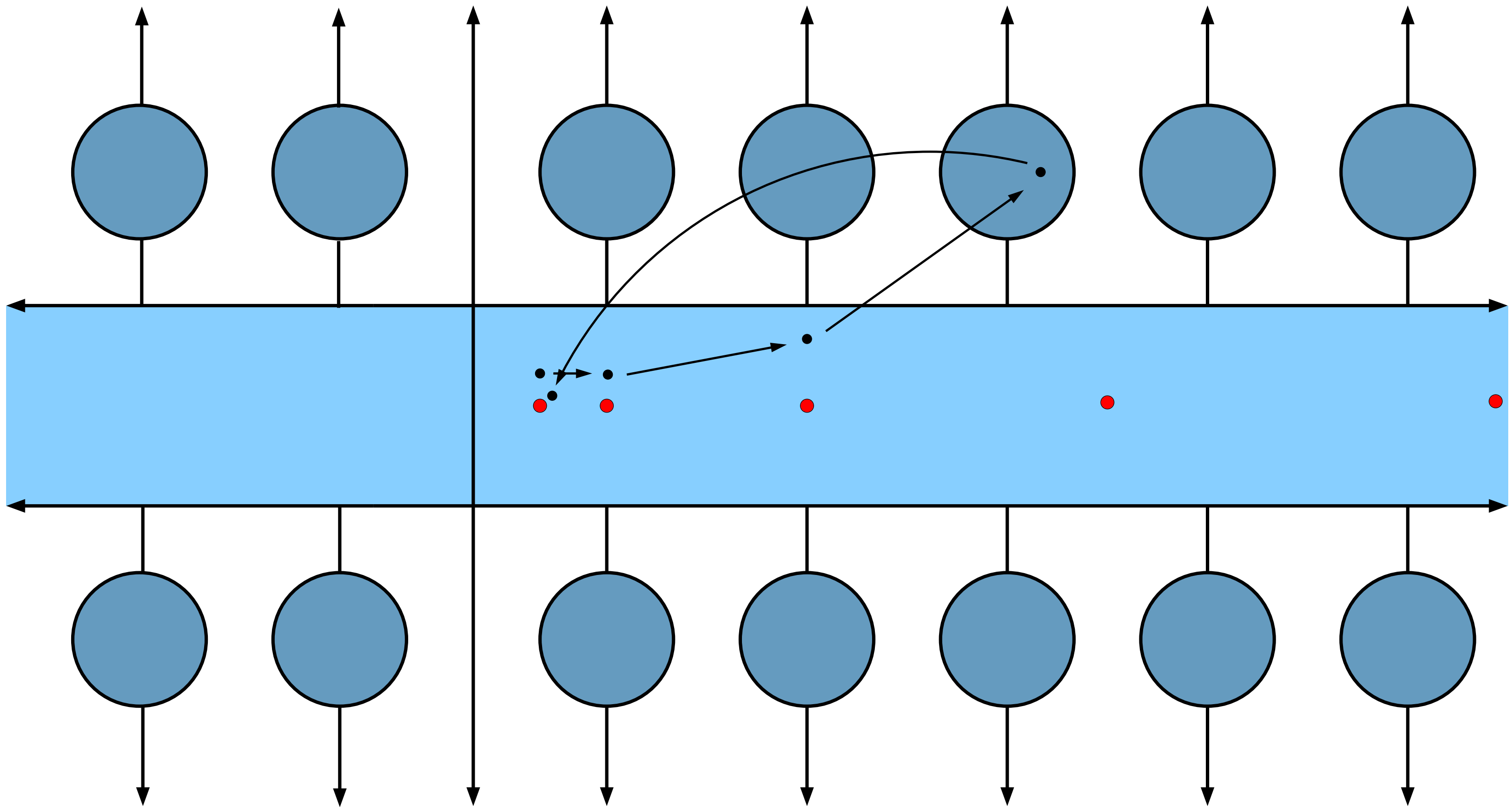
Graph giving EL wandering domain via folding (B 2015).

R and D components only.

Symmetry $\Rightarrow x = \frac{1}{2}$ iterates to $+\infty$ on $\mathbb{R}$.



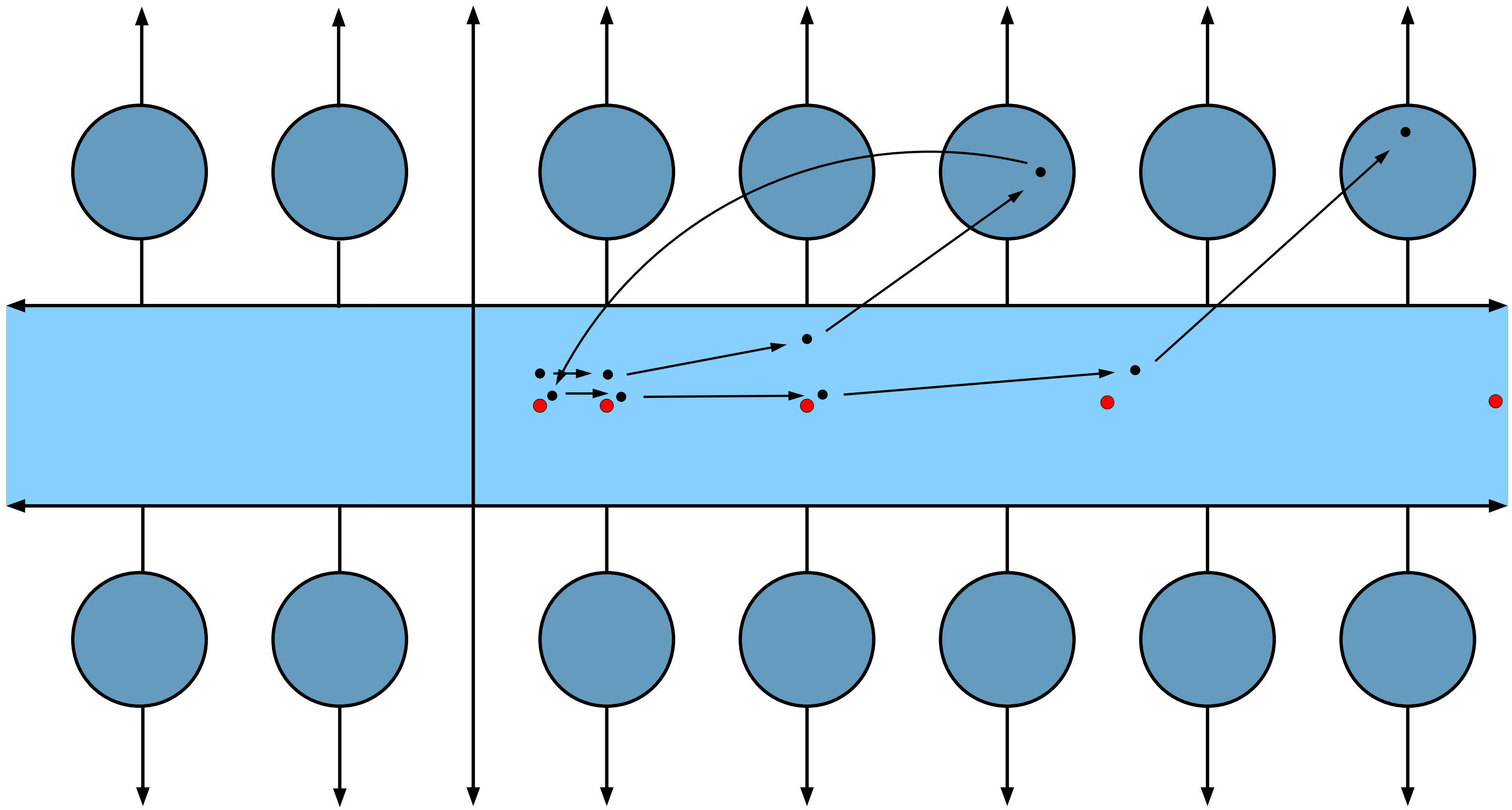
Iterates of $\frac{1}{2} + i\epsilon$ follow orbit of $\frac{1}{2}$ for a time, eventually diverge.



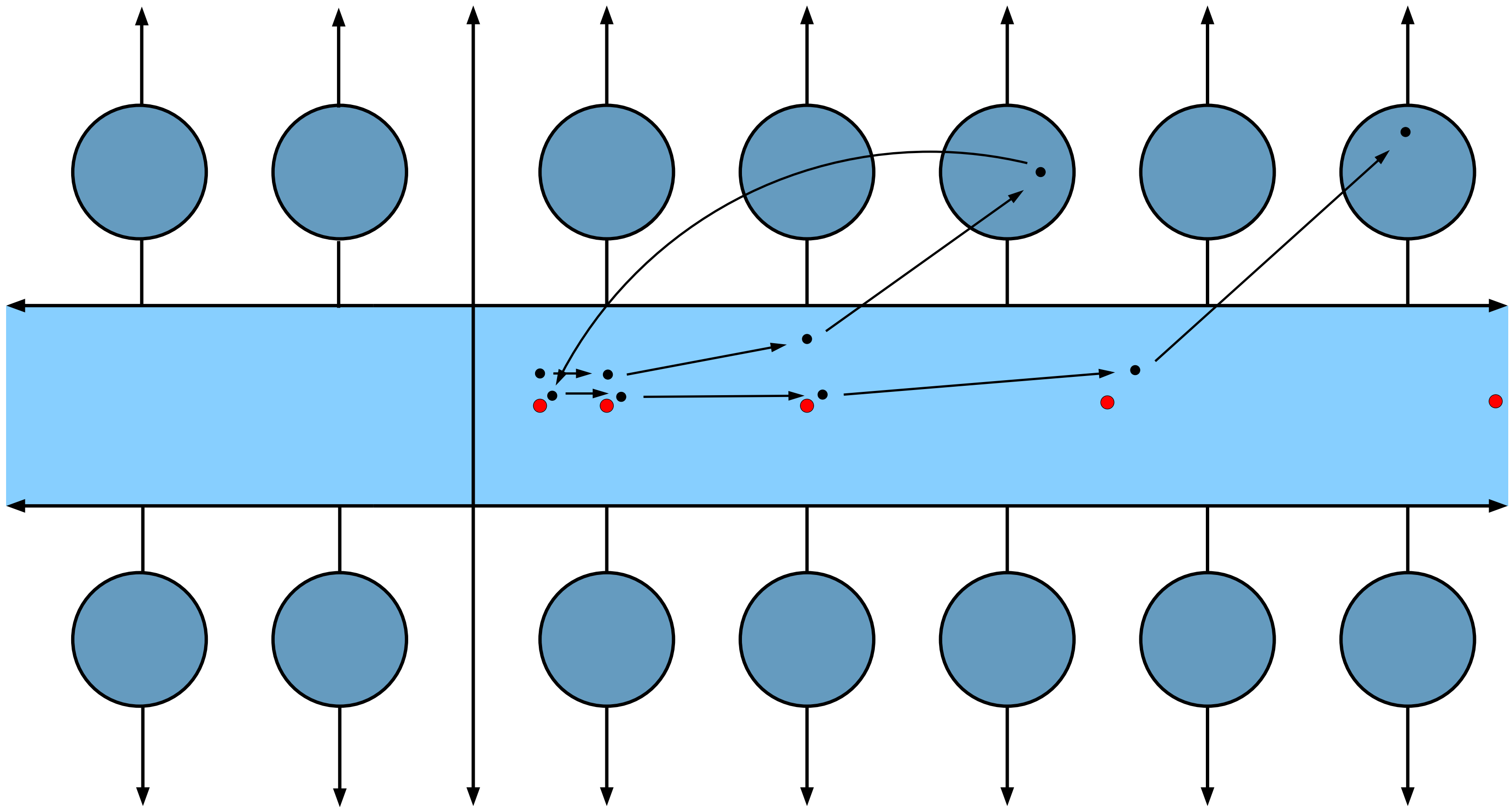
Orbit lands in D-component. Critical point compresses.

Can specify that orbit lands very near $\frac{1}{2}$.

Closer than before.



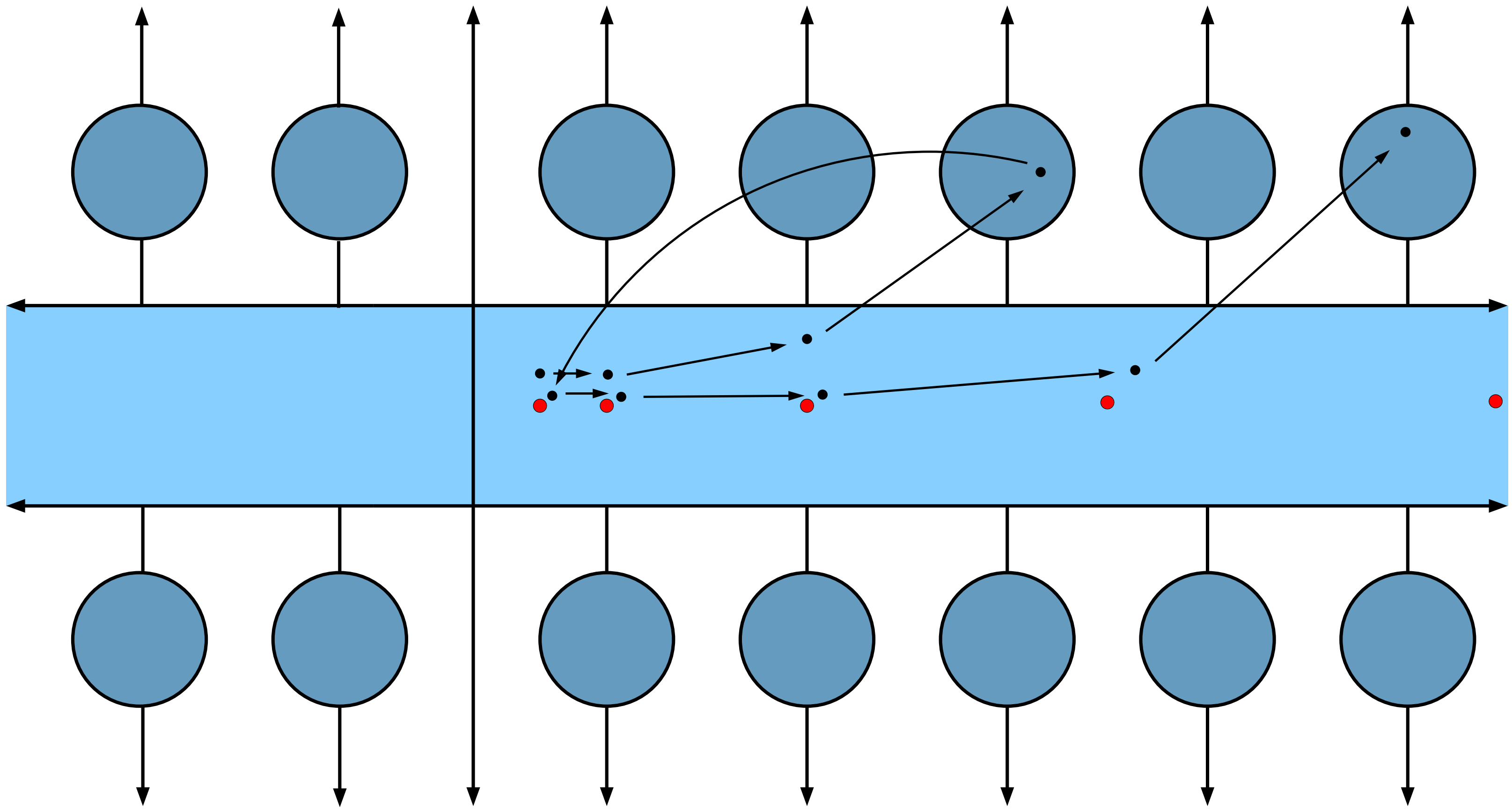
Follows $\frac{1}{2}$-orbit longer, then diverges and returns near $\frac{1}{2}$.



Compression $\Rightarrow$ orbit in Fatou component.

Oscillation between $\frac{1}{2}$ and ∞ $\Rightarrow$ wandering domain.

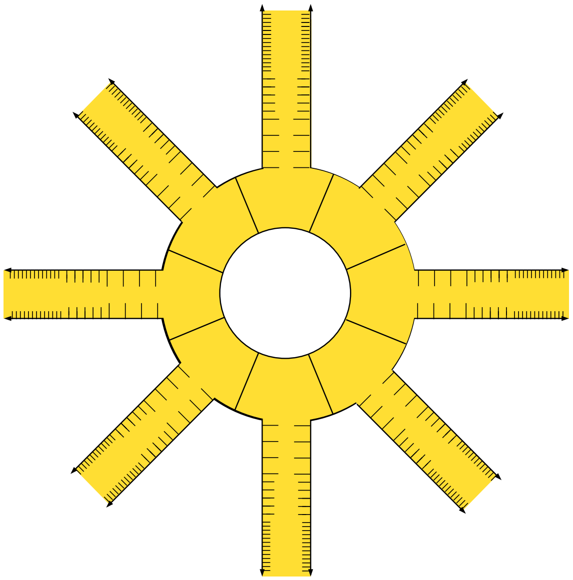
(Schwarz lemma: iteration decreases hyperbolic metric.)



Alternate proof by Marti-Pete and Shishikura.

Variations by Lazebnik, Fagella-Godillon-Jarque, Osborne-Sixsmith.

Dimensions of transcendental Julia sets:



Given an entire function f,

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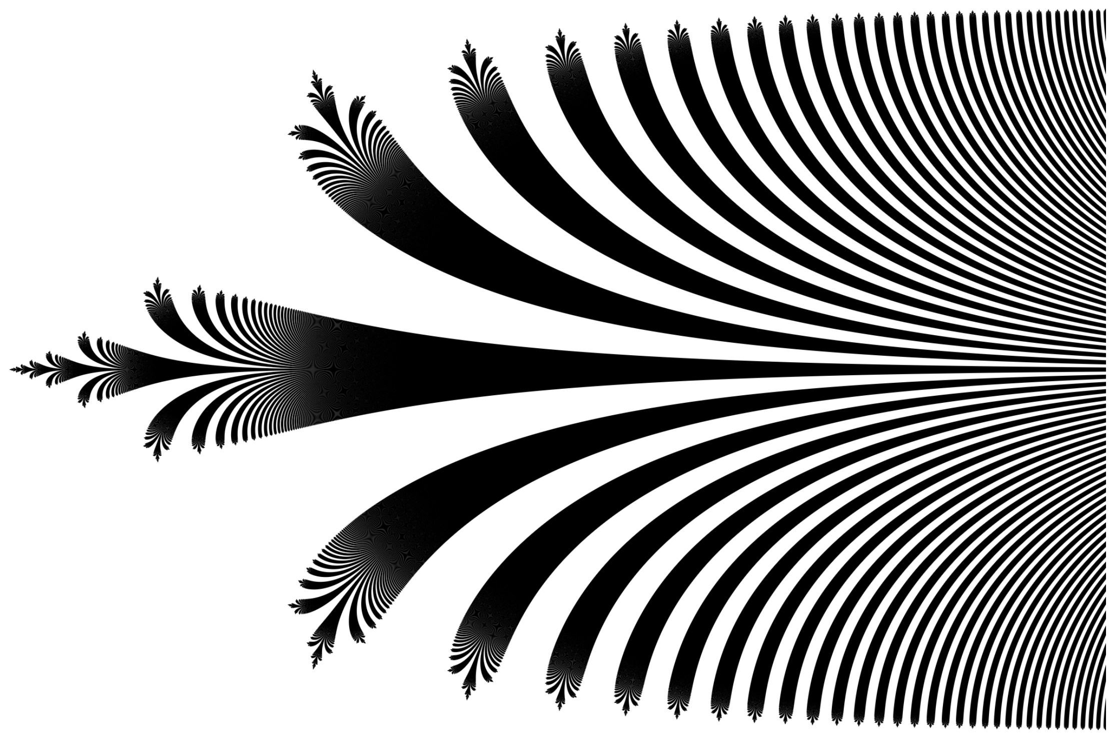
Julia set $= \mathcal{J}(f) =$ complement of Fatou set.

Julia set is usually fractal. What is its (Hausdorff) dimension?



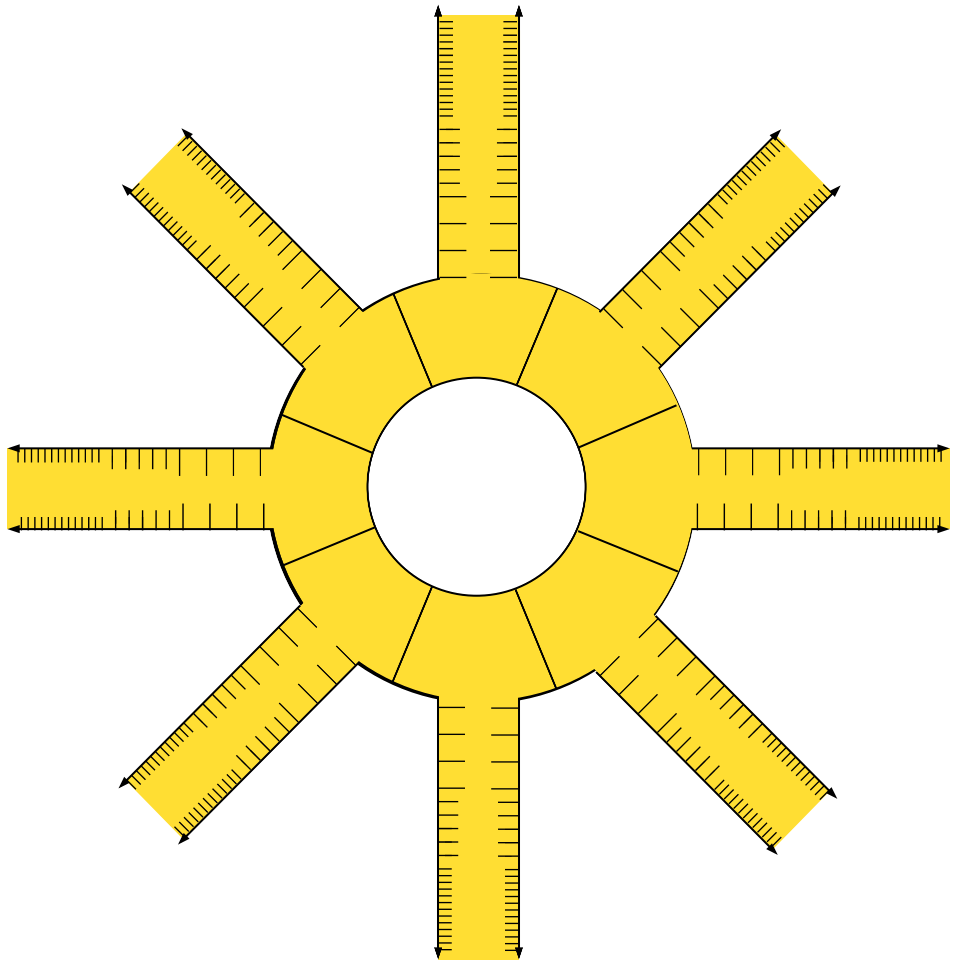
$\mathcal{J}((e^z - 1)/2)$, courtesy of Arnaud Chéritat

A short (and incomplete) history:

- Baker (1975): f transcendental $\Rightarrow \mathcal{J}$ contains a continuum $\Rightarrow$ dim ≥ 1.
- Misiurewicz (1981): $\mathcal{J}(e^x) = \mathbb{C}$.
- McMullen (1987): H-dim = 2, zero area.
- Stallard (1997, 2000): $\{\dim(\mathcal{J}(f)) : f \in \mathcal{B}\} = (1, 2]$.
- Albrecht-B (2020): 1 < H-dim < 2 in Speiser class (folding thm).
- B (2018): H-dim = 1 example exists (folding thm).

Transcendentals: easy to get dim = 2, hard to get dim ≈ 1.

Polynomials: dim = 2 is hard (Shishikura), area > 0 (Buff-Cheritat).

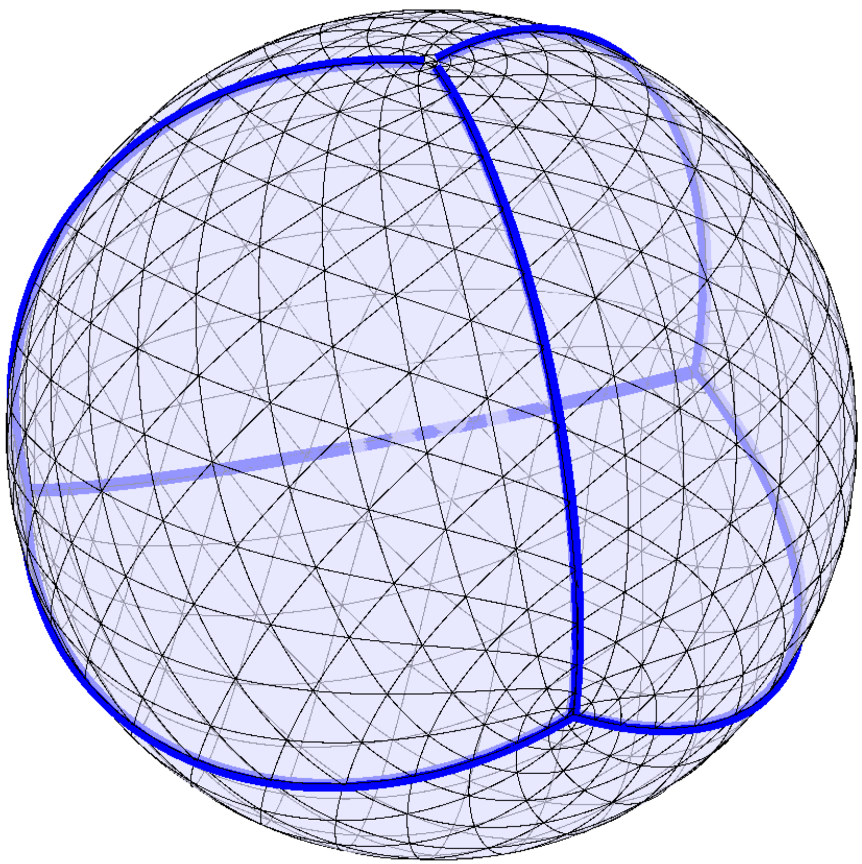


Speiser class Julia sets with dimension < 2.

Build explicit QR map with small Julia set.

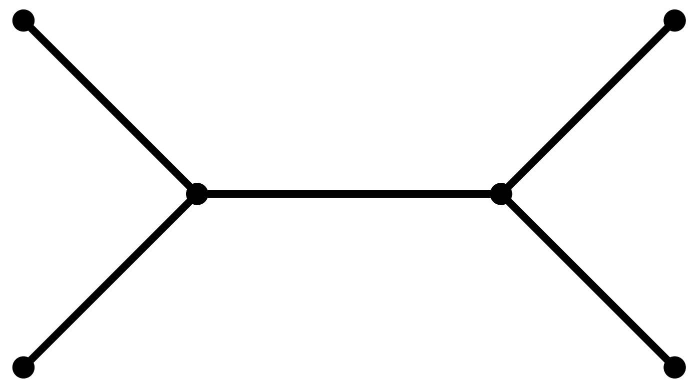
Prove QC correction map is bi-Lipschitz near Julia set.

Dessins and equilateral triangulations

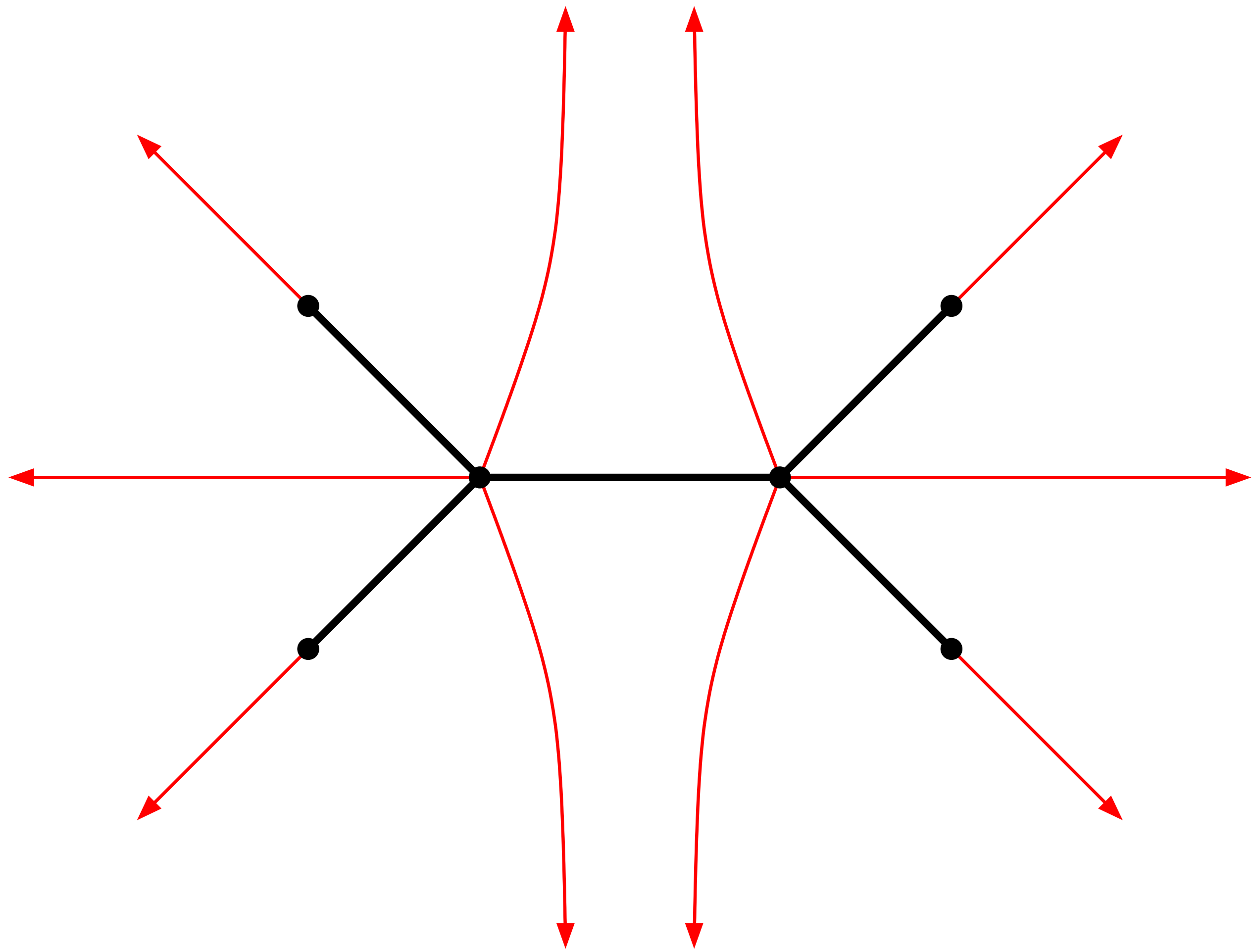


Grothendieck observed a graph on a surface induces a conformal structure.

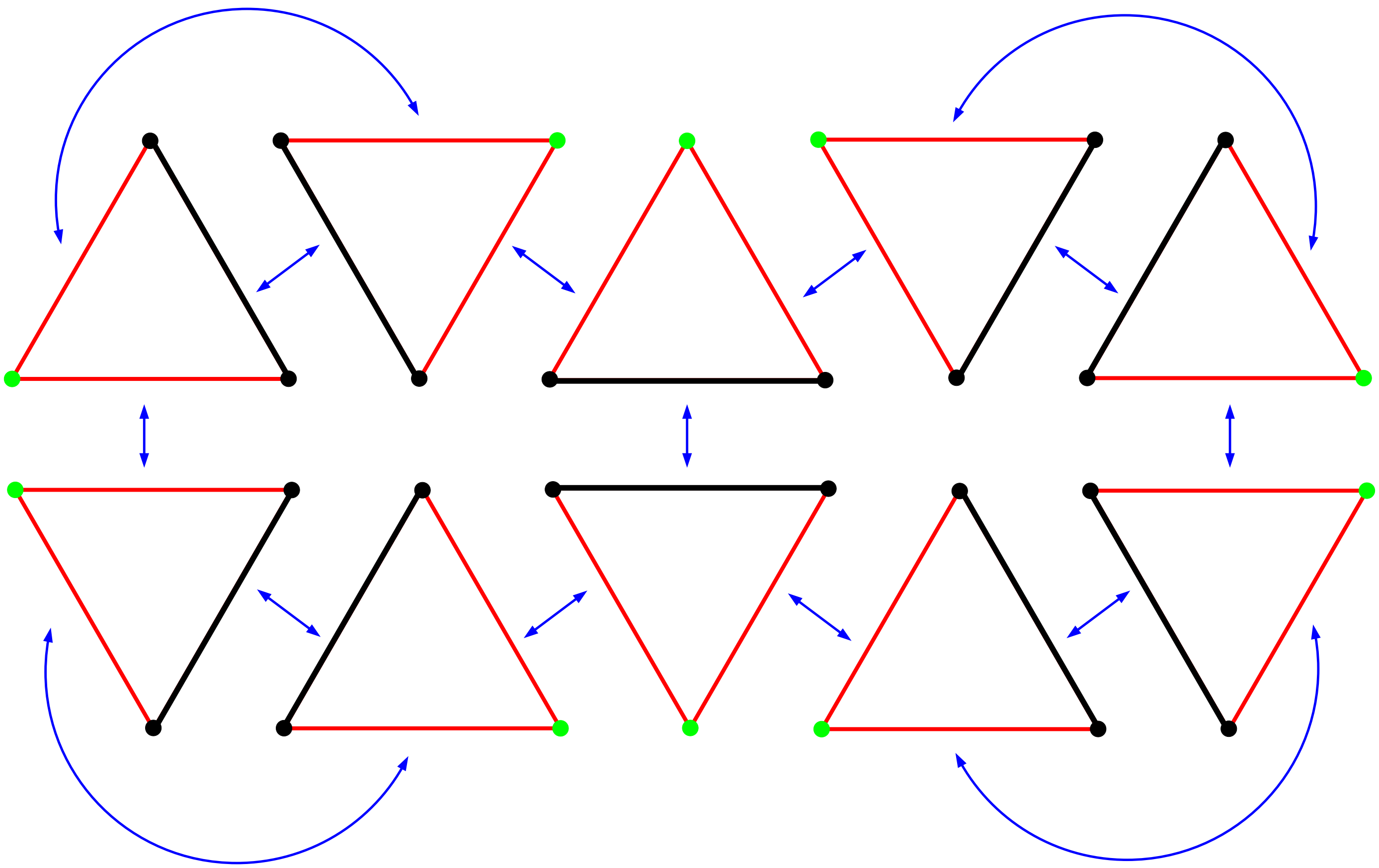
Consider trees on the sphere.



- start with a finite tree.



- connect vertices of T to infinity; gives finite triangulation of sphere.
- Defines adjacencies between triangles.



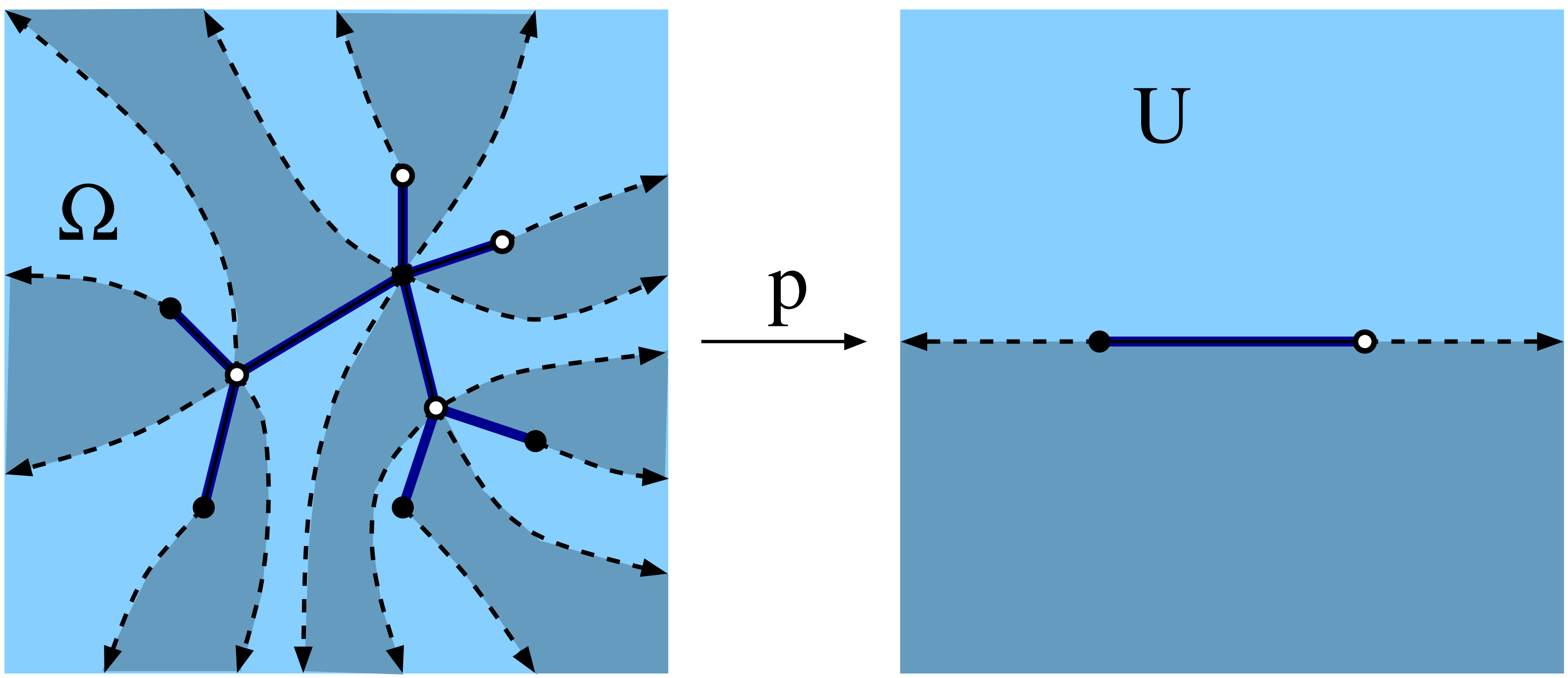
- Glue equilateral triangles using adjacencies: get a conformal 2-sphere.
- works for graphs on compact surfaces
- graph on surface ⇒ triangulation of surface ⇒ conformal structure

Only countably many ways to glue together finitely many triangles.

Which compact Riemann surfaces occur in this way?

Belyi function = holomorphic map from Riemann surface with 3 critical values. Shabat polys are examples on sphere.

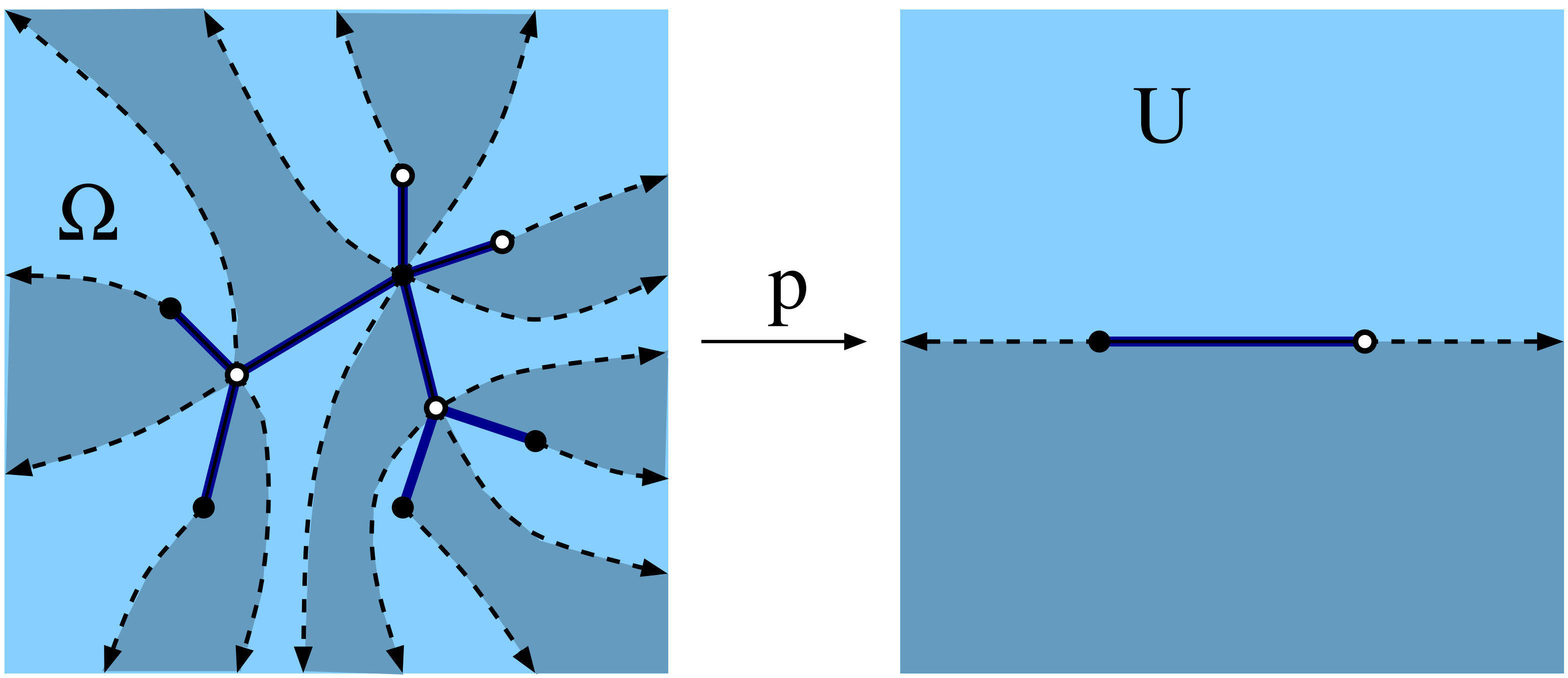
Theorem (Voevodsky-Shabat): A Riemann surface can constructed from equilateral triangles iff it has a Belyi function.



Triangles are inverse images of upper and lower half-planes.

Belyi function = holomorphic map from Riemann surface with 3 critical values. Shabat polys are examples on sphere.

Theorem (Voevodsky-Shabat): A Riemann surface can constructed from equilateral triangles iff it has a Belyi function.



Equilateral iff anti-holomorphic reflection between adjacent triangles.
⇒ whole triangulation is determined by a single triangle.

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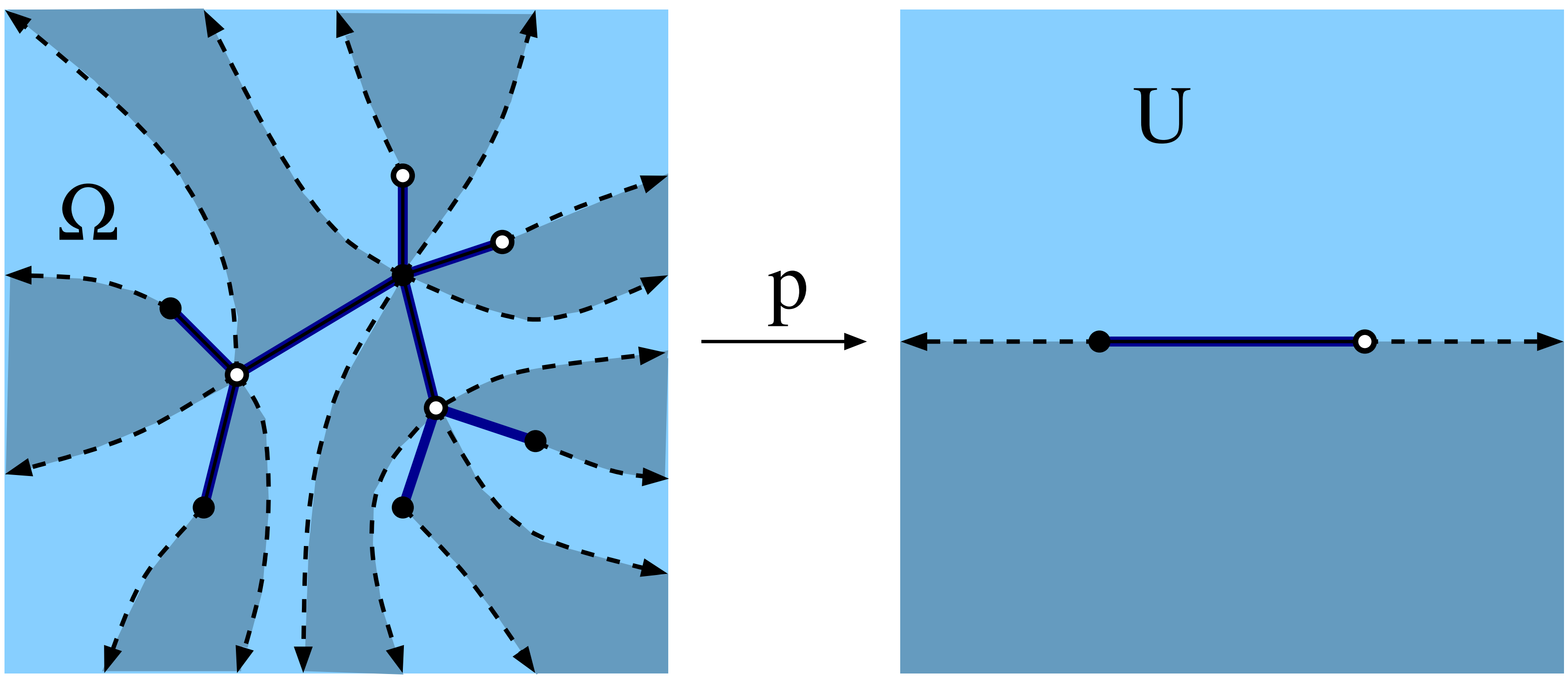


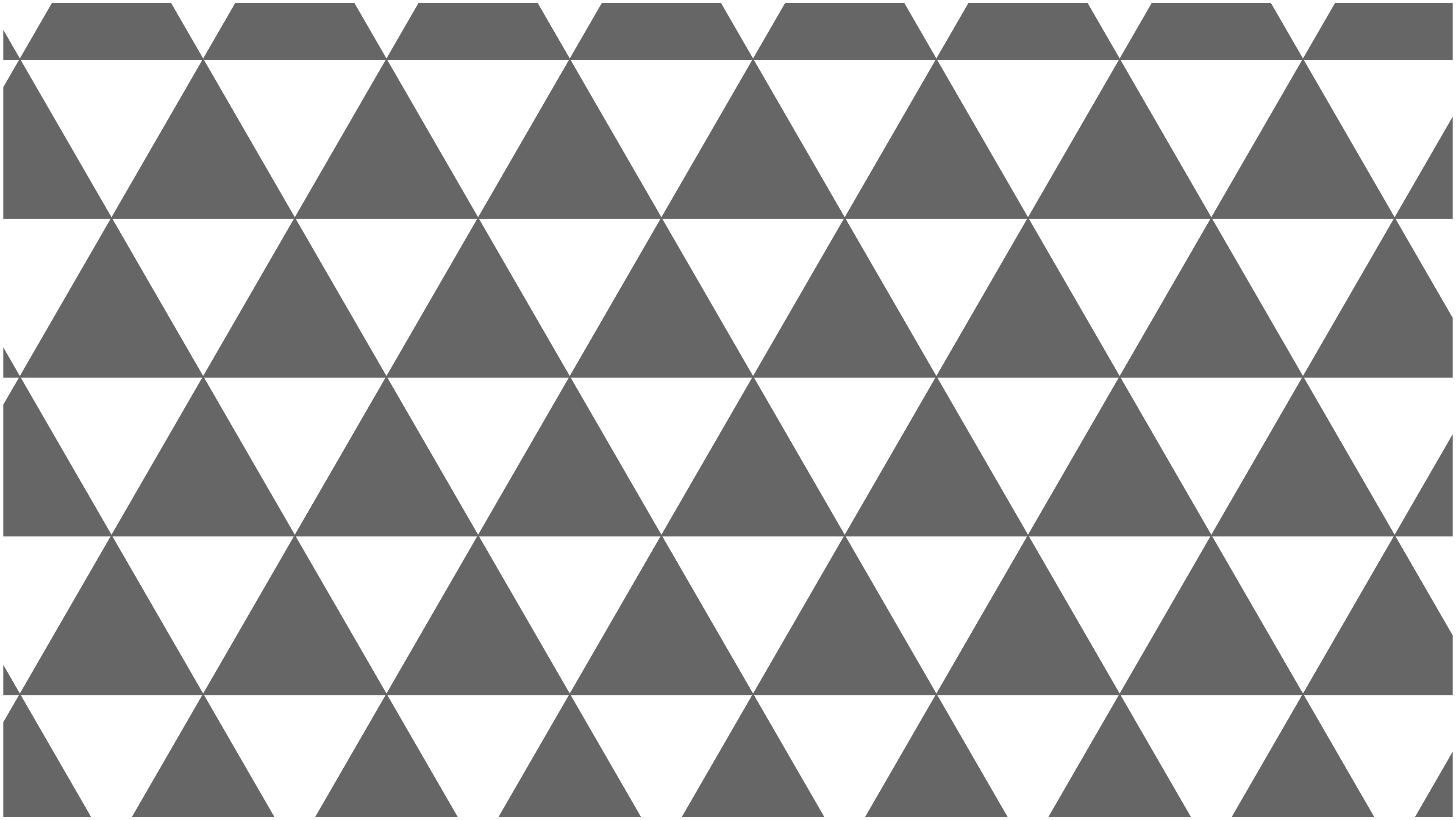
Belyi's Thm: A compact surface has a Belyi function iff it is algebraic.

(Zero set of $P(z, w)$ with algebraic integers as coefficients).

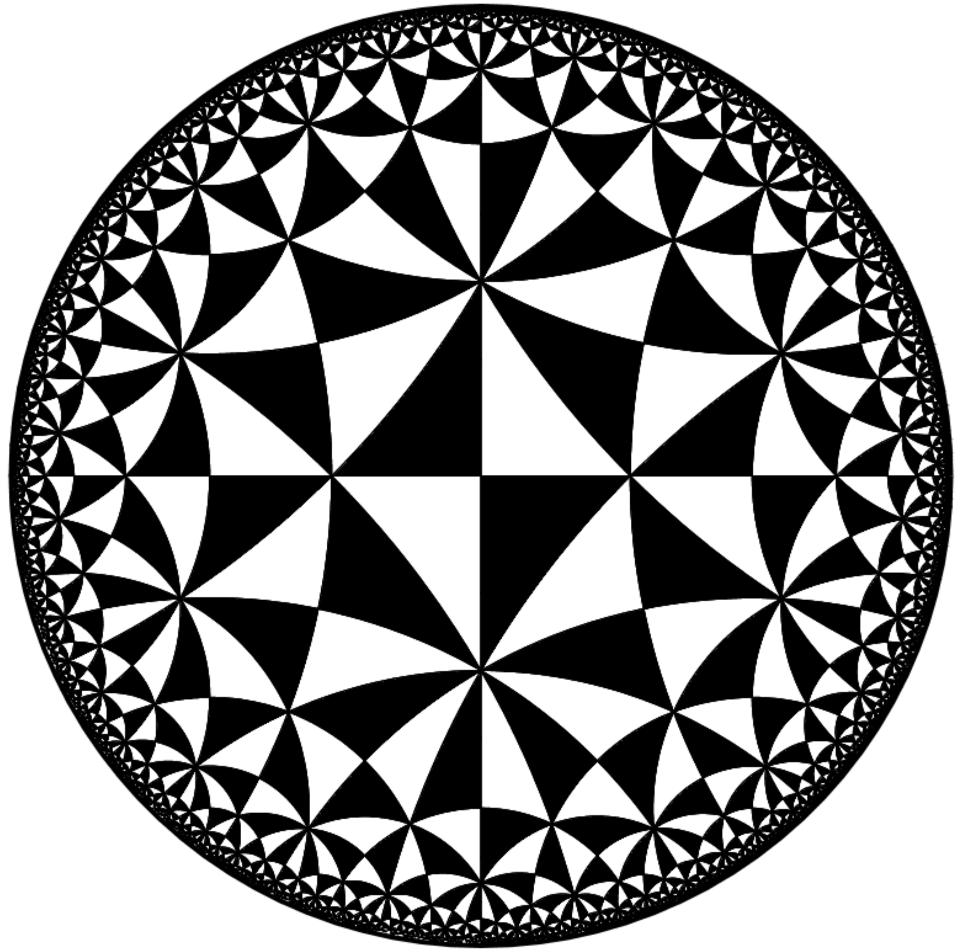
What about non-compact Riemann surfaces?

Now we can use countably many triangles.

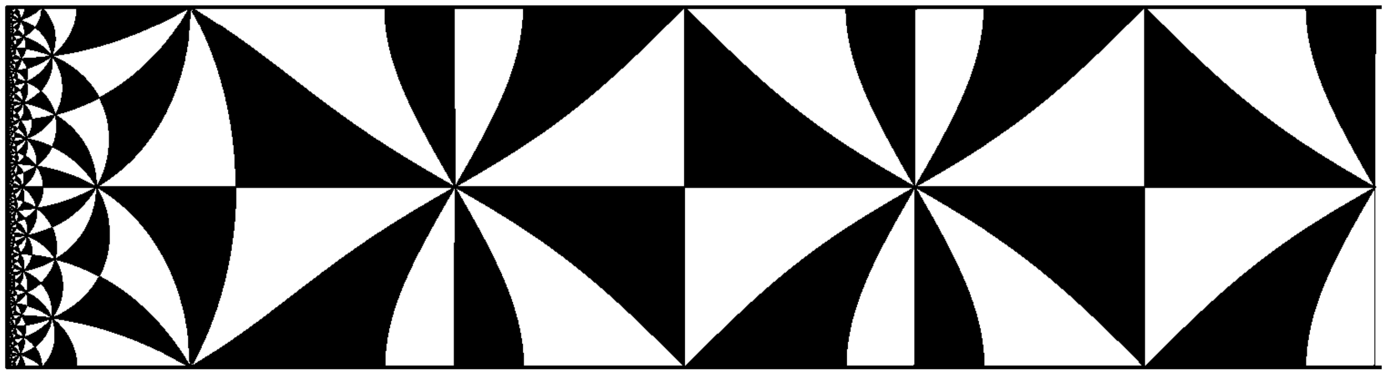
$\Rightarrow$ uncountably many way to glue them together.



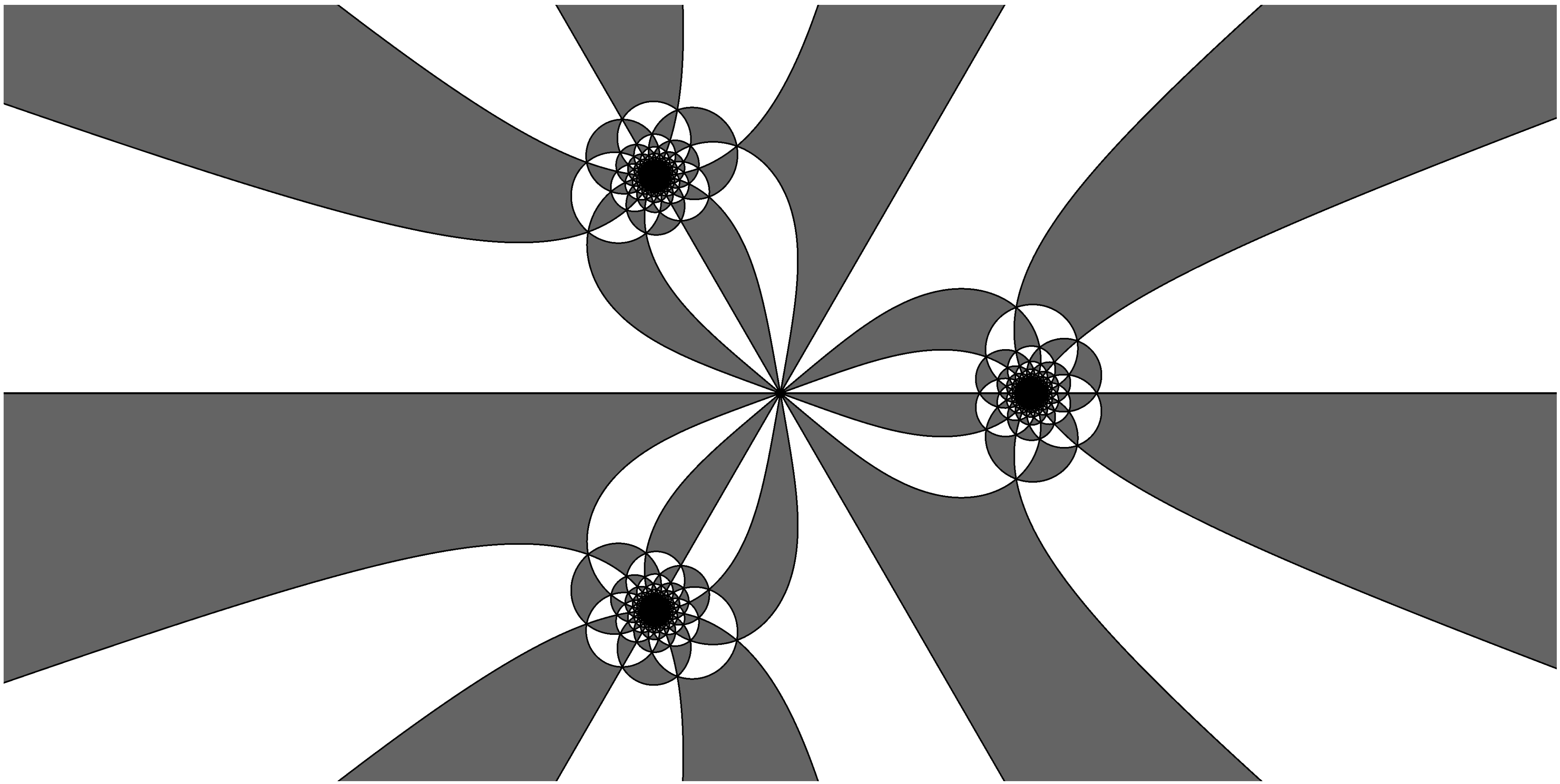
The plane



The disk



The punctured disk (identify top and bottom sides)



The trice punctured sphere

Thm (B-Rempe): Every non-compact surface has a Belyi function.

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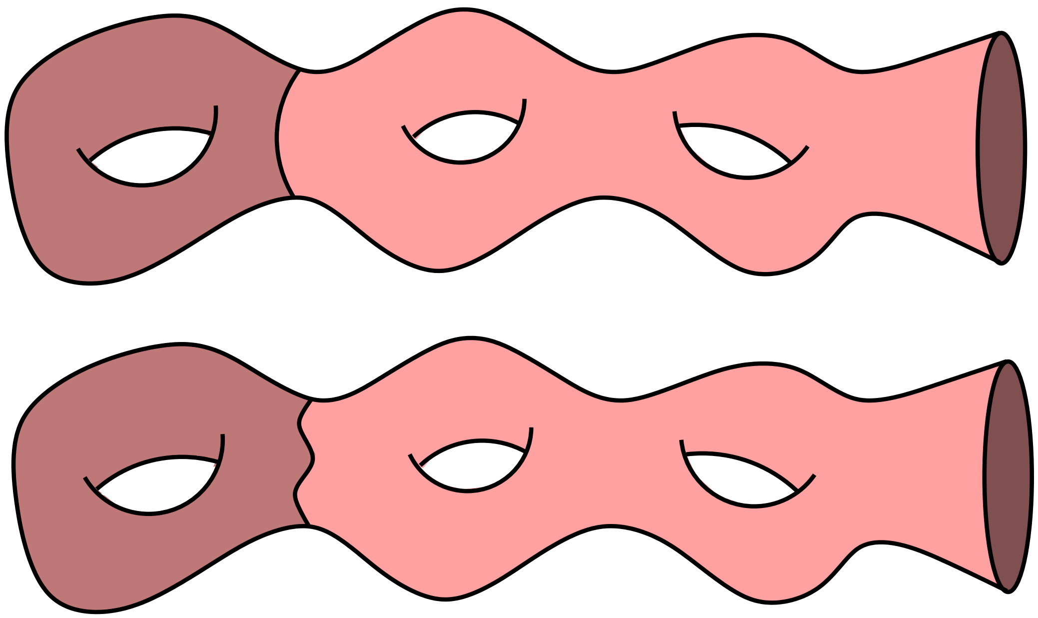
Idea of proof:

- Translate "true trees are dense" proof to Riemann surface S.
- Compact pieces can be approximated by triangulated surfaces.
- Conformal structure is changed, but as little as we wish.
- **Key fact:** small perturbation $\Rightarrow$ triangulated pieces re-embed in S.
- Triangulate a compact exhaustion of S. Take limit.



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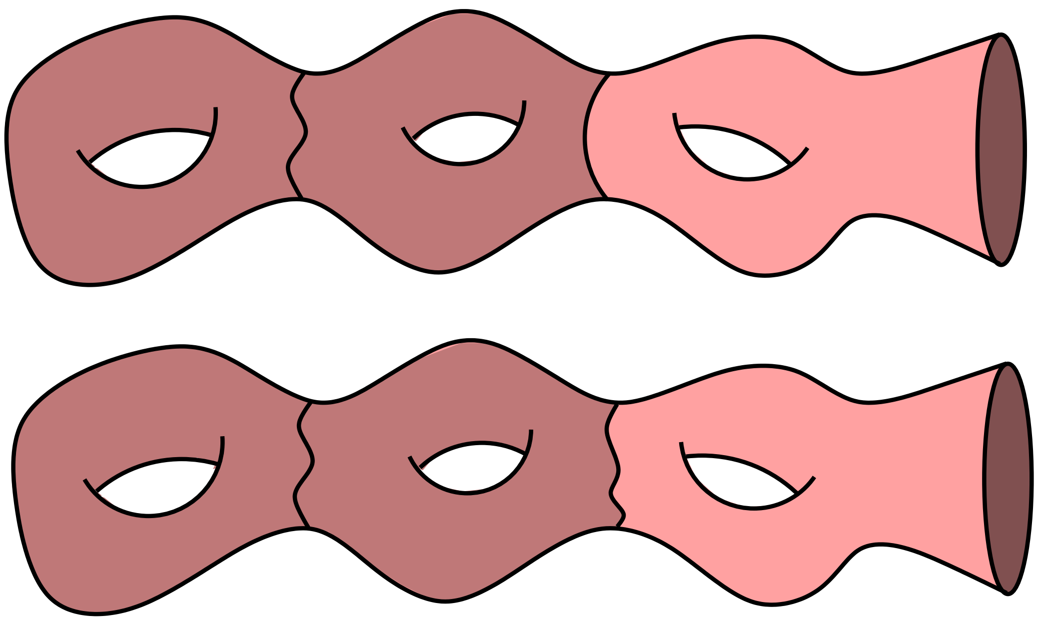
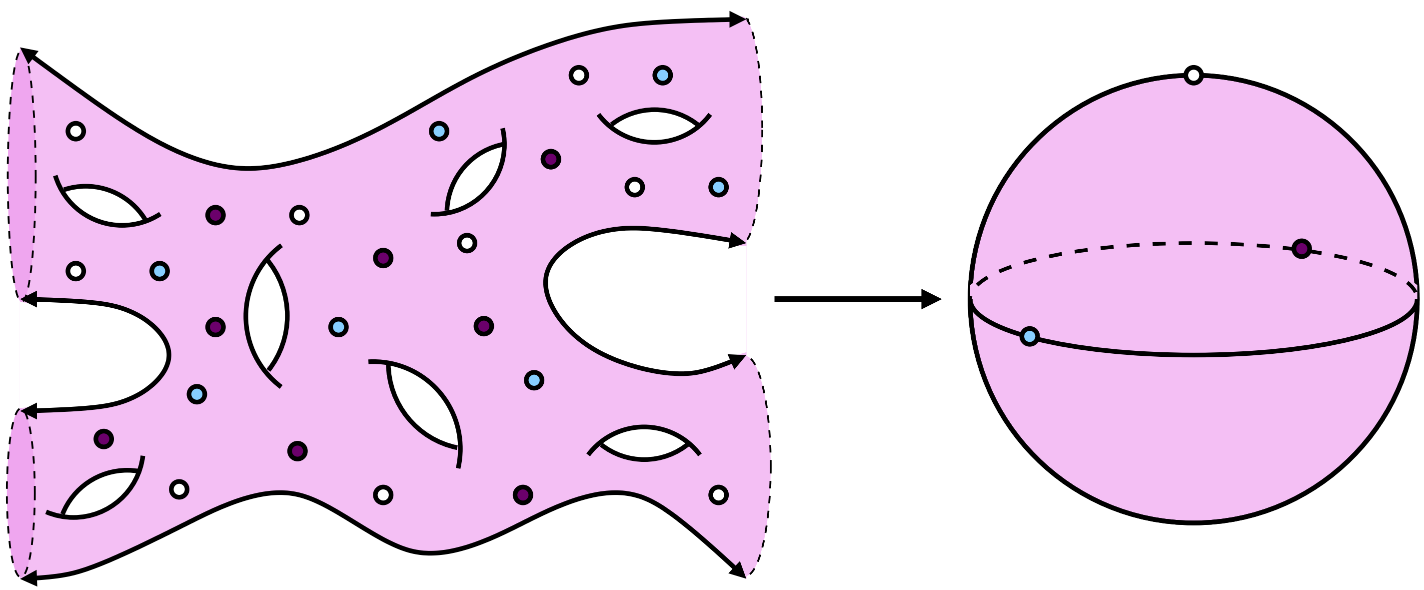


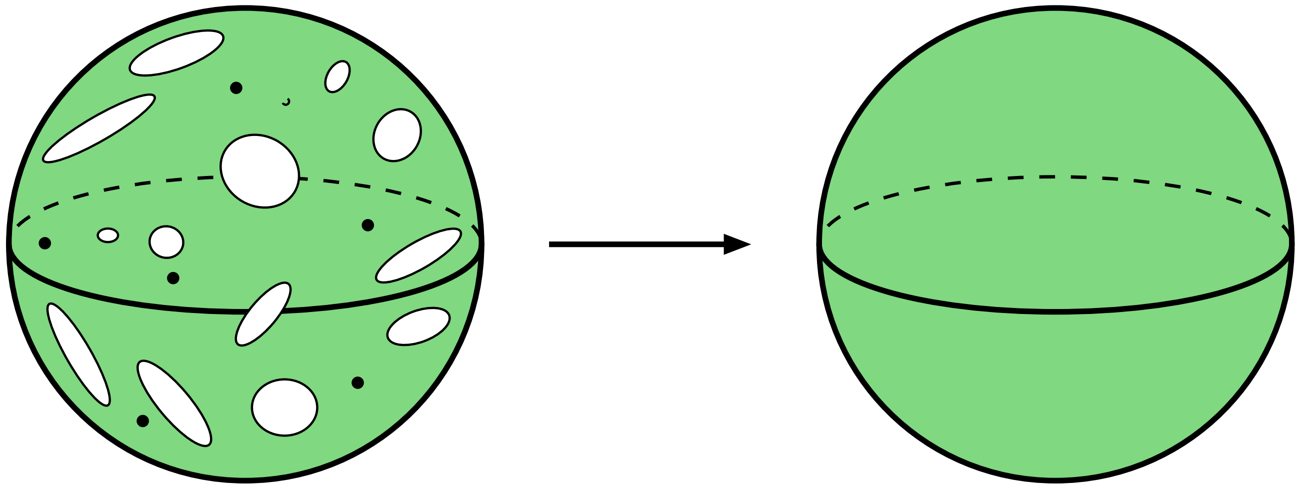
Thm (B-Rempe): Every non-compact surface has a Belyi function.

Corollary: Every Riemann surface is a branched cover of the sphere, branched over finitely many points.

- For compact surfaces, this is Riemann-Roch.
- Compact, genus g sometimes needs $3g$ branch points.
- 3 branch points suffice for all non-compact surfaces.



Corollary: Any open $U \subset \mathbb{C}$ is 3-branched cover of the sphere.



Adam Epstein: a holomorphic map $X \to Y$ is finite type if

- Y is compact,
- f is open with no isolated removable singularities,
- the set of singular values is finite.

If $X \subset Y$, map can be iterated while orbits stay in X, e.g.,

- ▶ Rational maps on $X = Y =$ 2-sphere
- ▶ Speiser class on $X = \mathbb{C}$, $Y =$ 2-sphere

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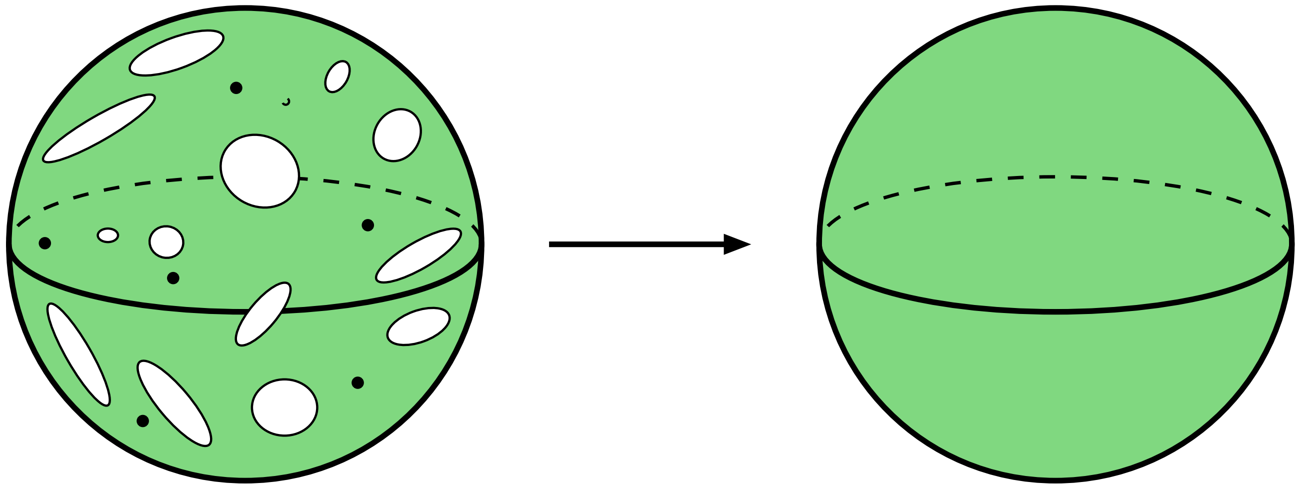
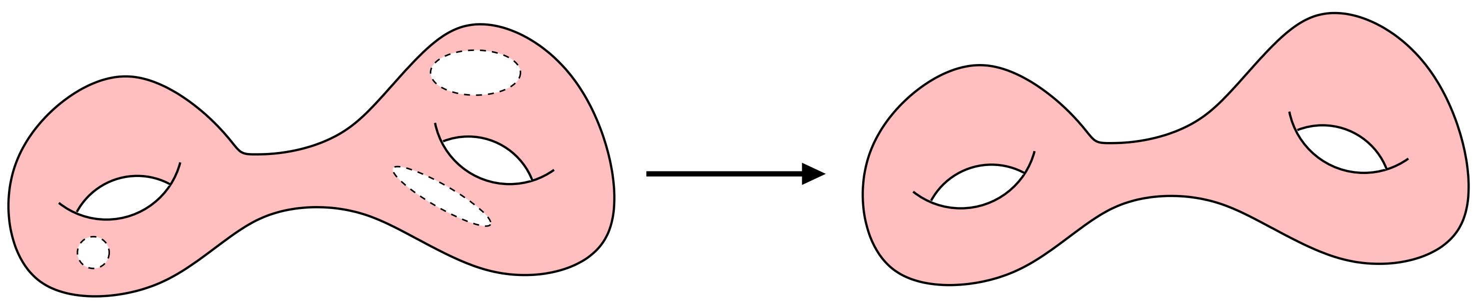


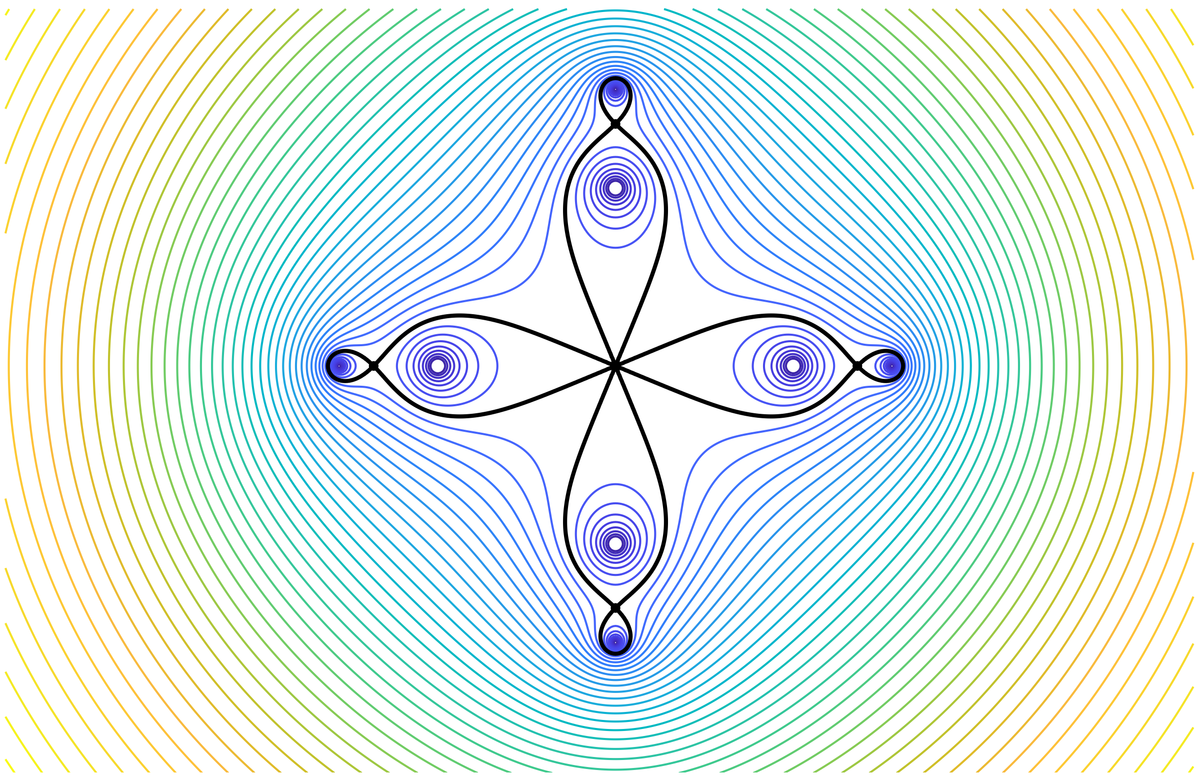
Gives many new dynamical systems of finite type.



Can have $f : X \to Y$ where $X \subset Y$ open, Y compact, genus > 0.

Unknown which surfaces Y work.

QC folding and rational approximation



Weierstrass's Theorem: Any continuous function on an interval $K = [a, b]$ can uniformly approximated by polynomials.

Runge's Theorem: If f is holomorphic on a neighborhood of a compact set K, then f can be uniformly approximated on K by rational maps.

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Runge's Theorem: If f is holomorphic on a neighborhood of a compact set K, then f can be uniformly approximated on K by rational maps.

Poles may be prescribed; one per complementary component of K.

Neither theorem says what the approximants look like off K.

Are there critical points outside K?

Theorem (B.-Lazebnik): if f is holomorphic on a neighborhood a compact set K, and if U is a connected neighborhood of K, then f can be uniformly approximated on K by rational functions with critical points only in U.

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Poles may be in any set P hitting each component of $S^2 \setminus K$.

Proof uses QC folding ideas.

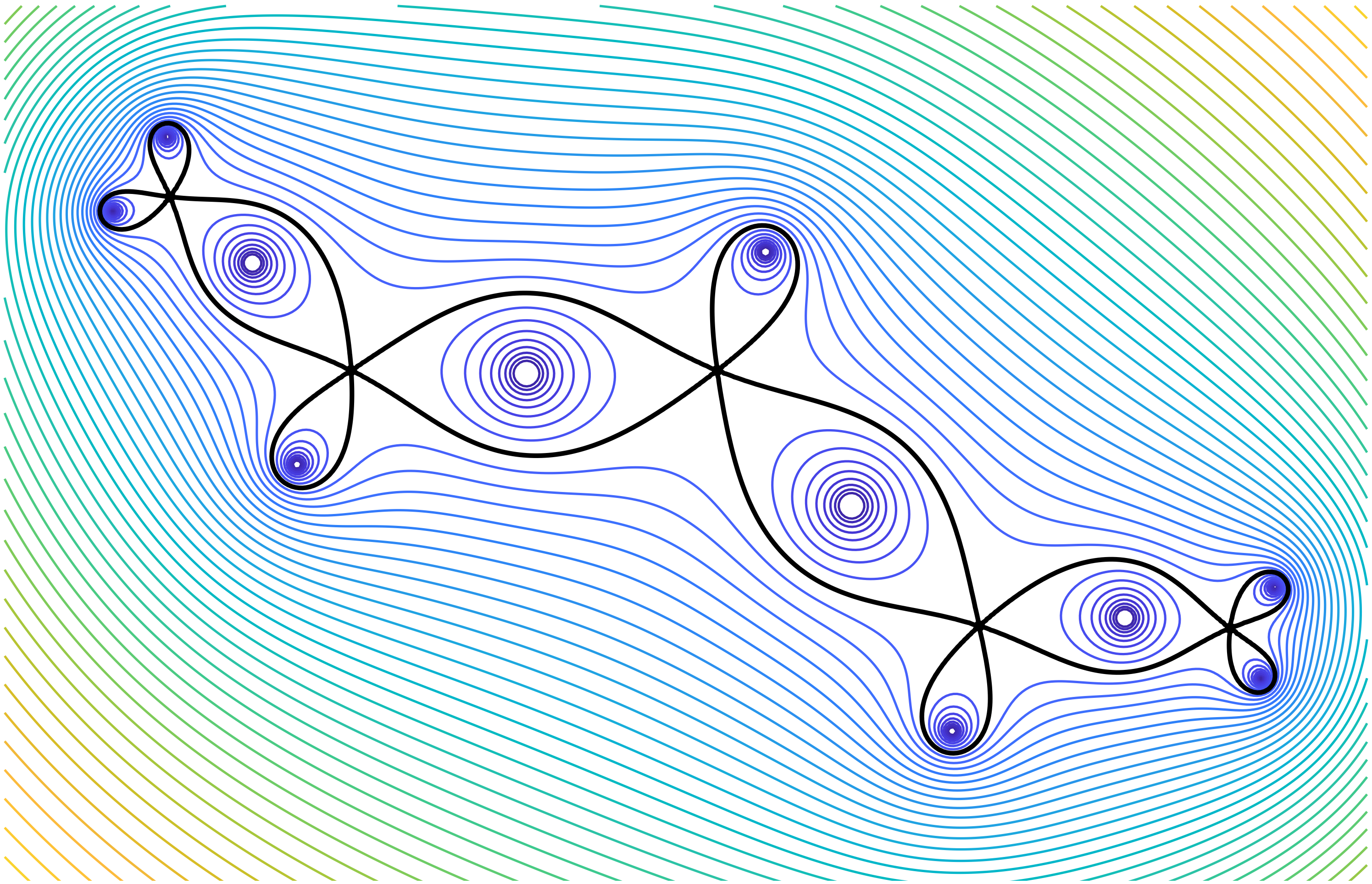
Theorem (D. Bishop, 2024) if f is continuous and real-valued on $K = [a, b] \subset \mathbb{R}$, then f can be uniformly approximated by polynomials with all critical points inside K.

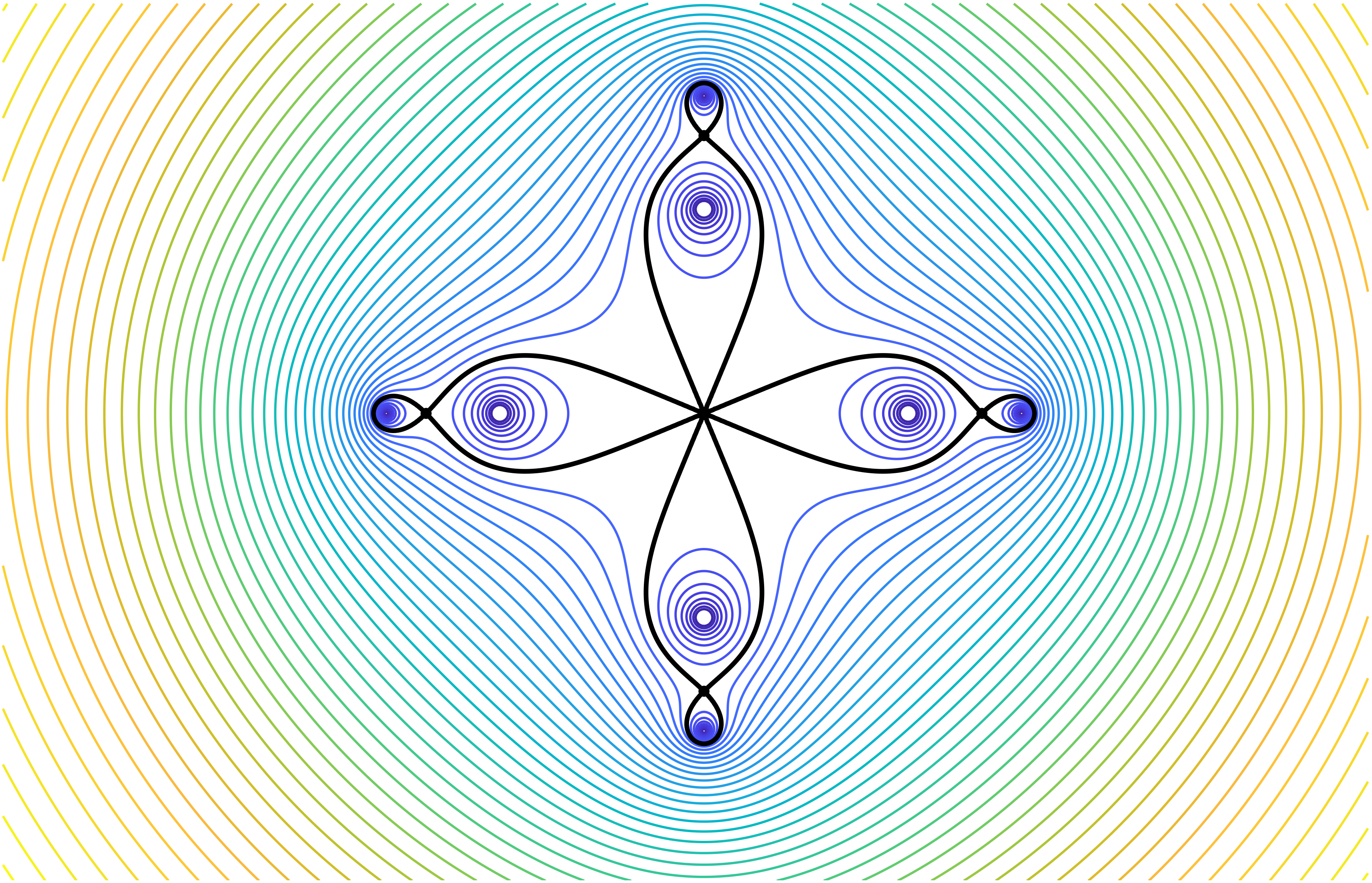
May fail if K is not an interval.

Another type of rational approximation.

A rational lemniscate is a level set $L = \{|r| = 1\}$ where r is rational.

Similarly for polynomial lemniscates.

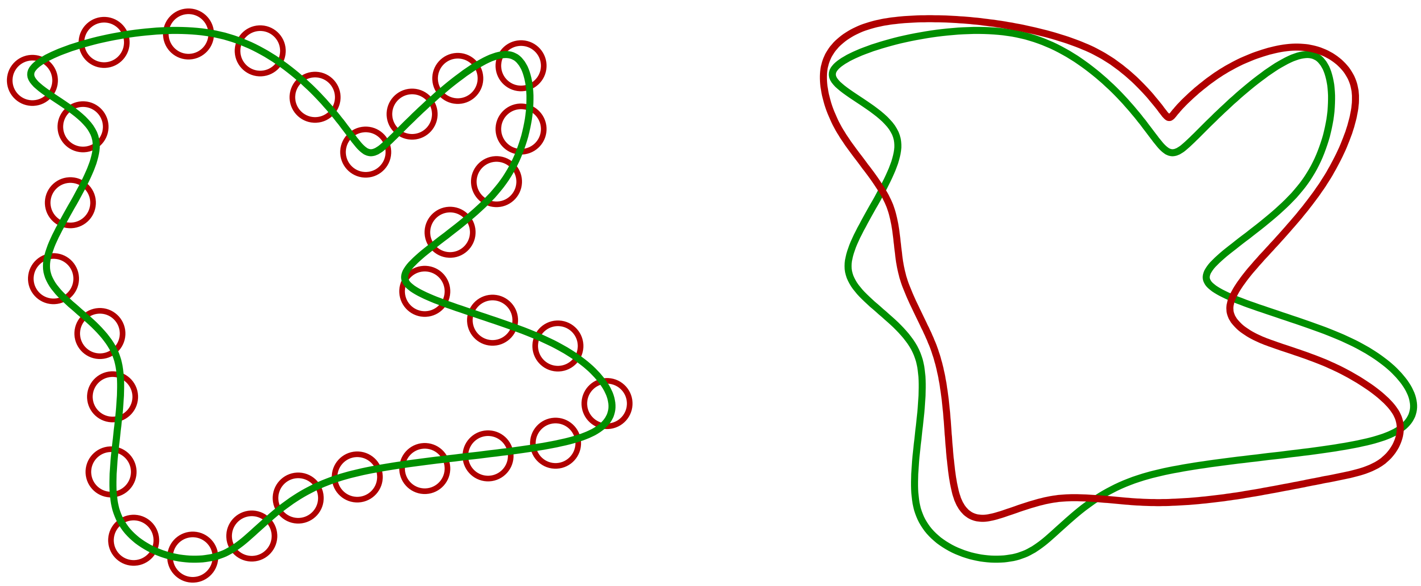




Rational lemniscates are Euler graphs: only even degree vertices.

We say K_1 ϵ-approximates K_2 if there is a homeomorphism $h : \mathbb{C} \to \mathbb{C}$ so that $h(K_1) = K_2$ and $\sup_z |h(z) - z| \leq \epsilon$.

Stronger than Hausdorff topology. Both sets must have same topology.



Hilbert Lemniscate Theorem (1897): any closed Jordan curve can be ϵ-approximated by a polynomial lemniscate.

Theorem (B., Eremenko, Lazebnik): Any finite Euler graph can be ϵ-approximated by rational lemniscates.

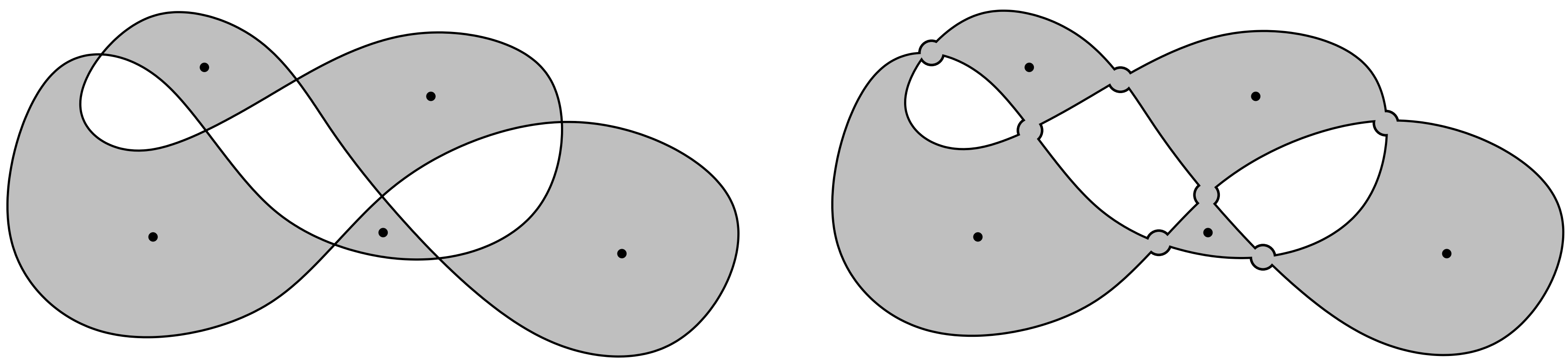
Faces of an Euler graph can be 2-colored, say white and gray.

We can specify one pole in each gray face.

Graph need not be connected.

Sketch of proof:

1. Approximate harmonic measures on gray faces by point masses.
2. Get rational r so $u = \log|r| \approx$ sum of Green's functions.
3. Join grey faces into a single face (add disks at each graph vertex).



Sketch of proof:

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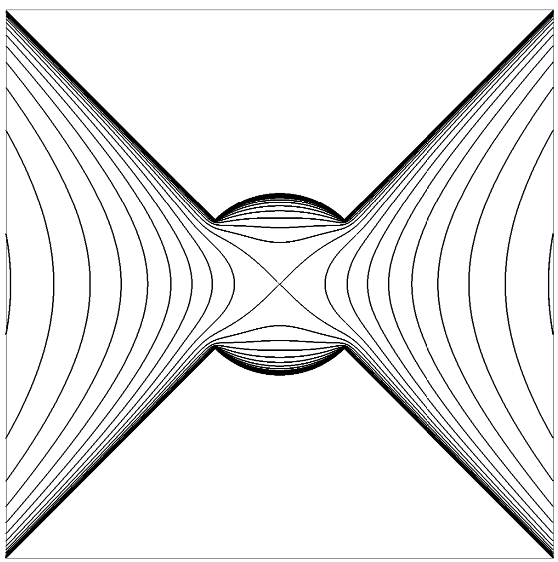
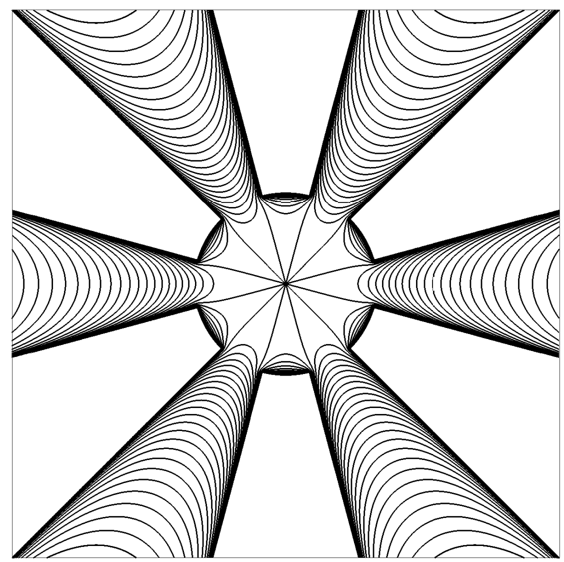
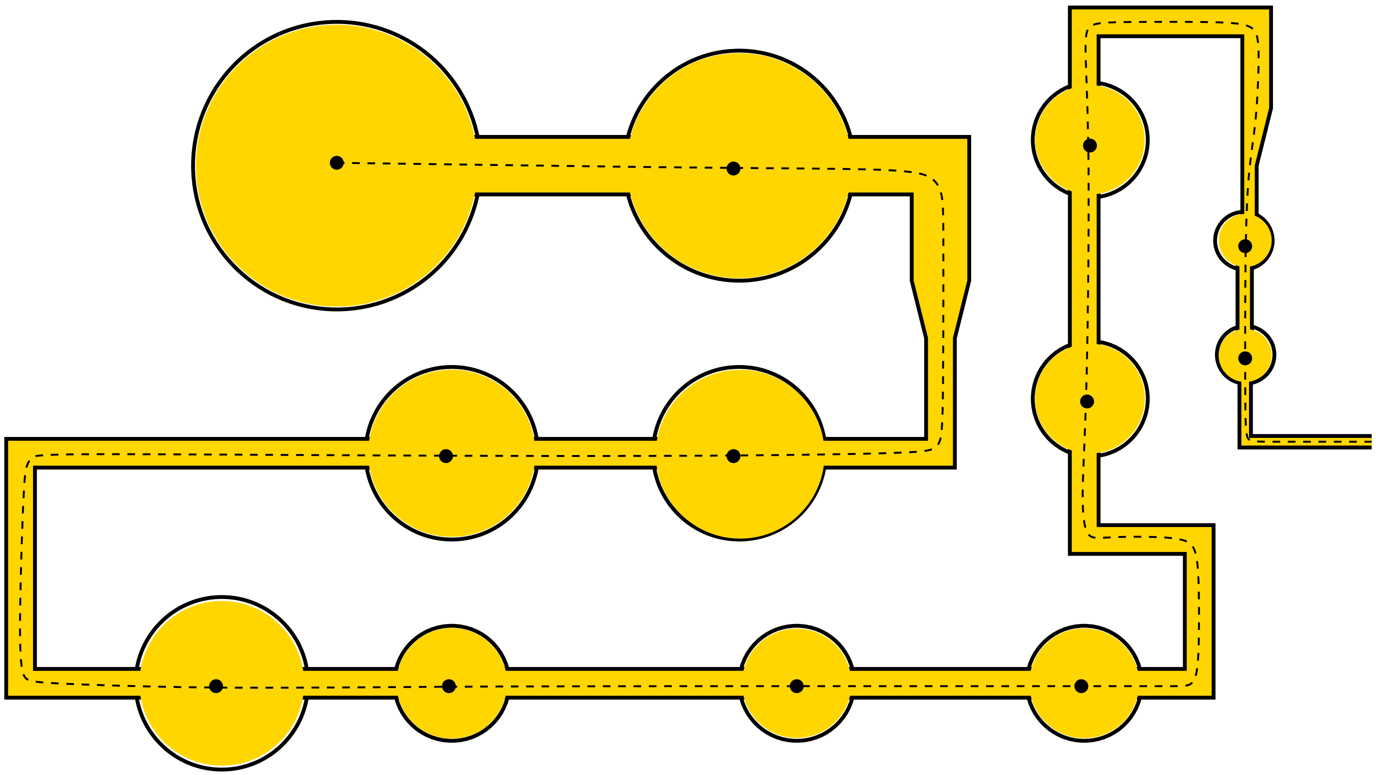
2. Get rational r so $u = \log|r| \approx$ sum of Green's functions.
3. Join grey faces into a single face (add disks at each graph vertex).
4. Near vertices construct explicit function is v with single critical point.
5. Merge to get QR function with one critical value (uses fixed pt thm).
6. Solve Beltrami to get rational map with correct level line. $\square$

Prescribing singular orbits



Question: What can the orbit of the singular points look like?

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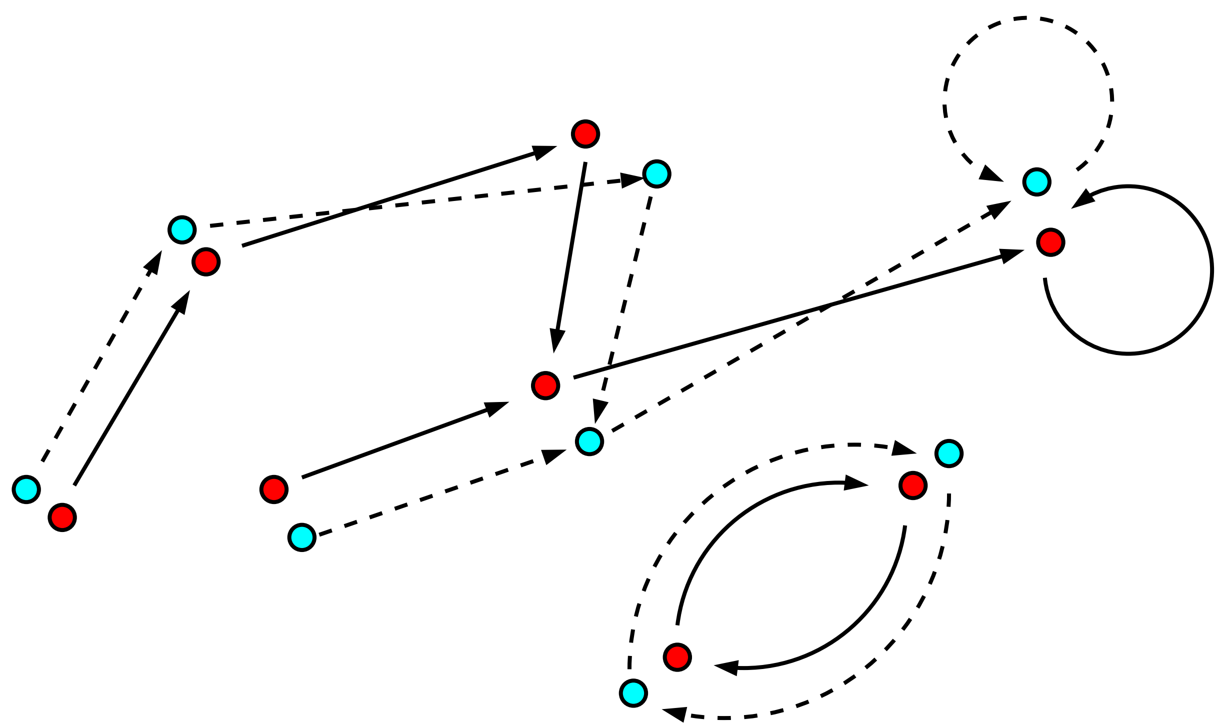
Answer: Anything (to within ϵ).

Thm (DeMarco-Koch-McMullen): For any finite $S \subset \mathbb{C}$ and $h : S \to S$, there is a rational f and a bijection $\psi : P(f) \to S$ so that

$$f = \psi^{-1} \circ h \circ \psi \qquad \text{and} \qquad |\psi(z) - z| \leq \epsilon.$$

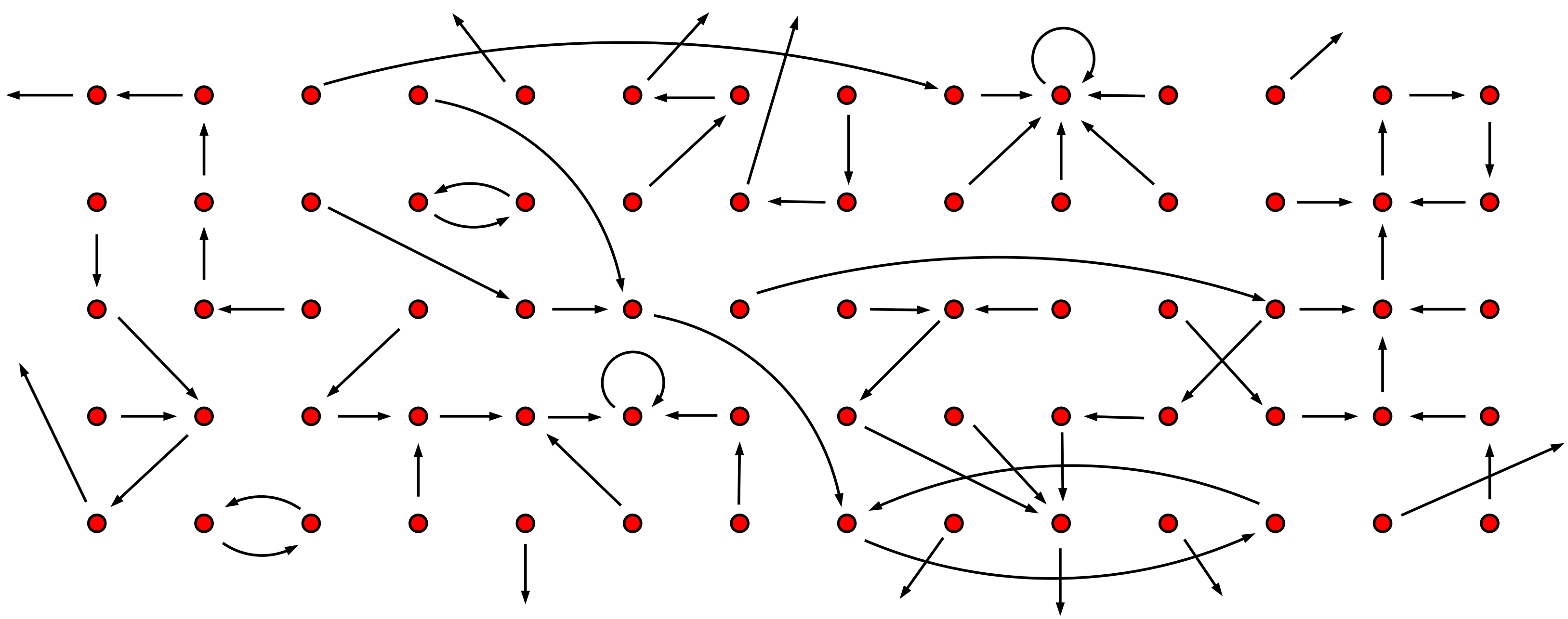
"Any finite post-critical dynamics can occur for rational maps."

Their proof uses iteration on Teichmüller space.



Theorem (B-Lazebnik):
Given any discrete set $S \subset \mathbb{C}$ with at least 3 points, and any $h : S \to S$, there is a meromorphic f and a bijection $\psi : P(f) \to S$ so that
(1) $f = \psi^{-1} \circ h \circ \psi$ on P,
(2) $|\psi(z) - z| \le \eta(z)$ for all $z \in P$.

Extended to general planar domains with M. Urbanski.



THANKS