Conformal Maps, Hyperbolic Geometry and Optimal Meshing

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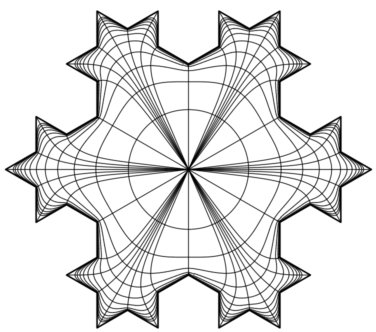
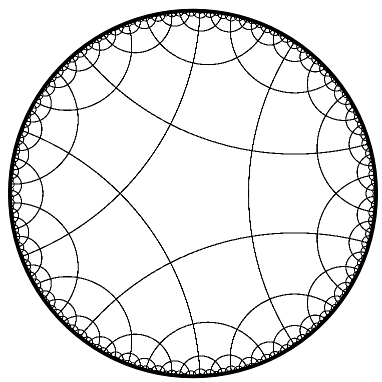
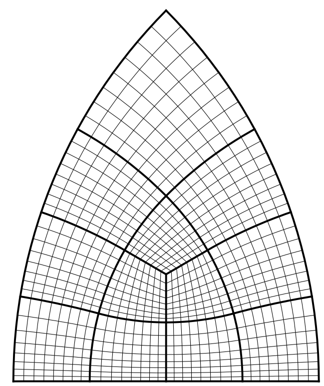
FWCG, Oct 30, 2010

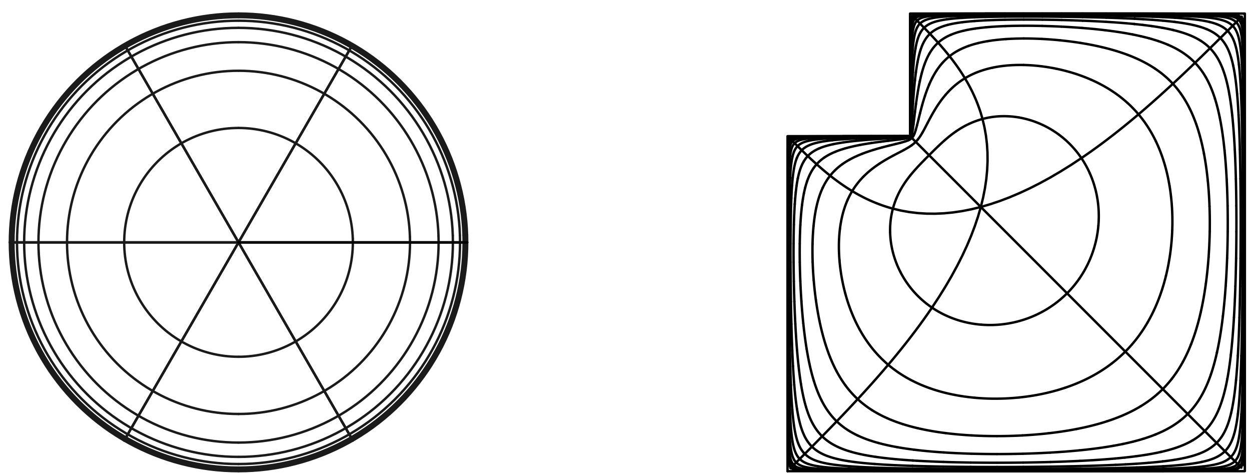




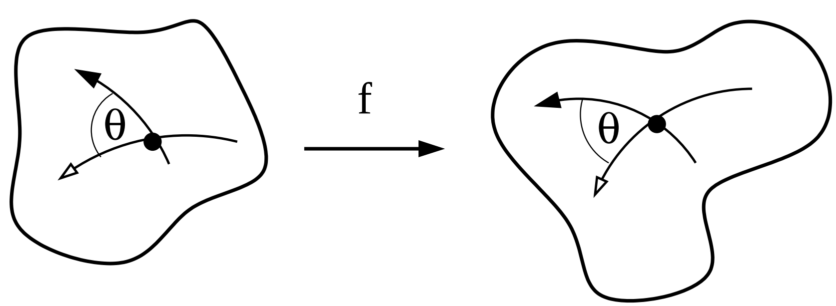


www.math.sunysb.edu/~bishop/lectures

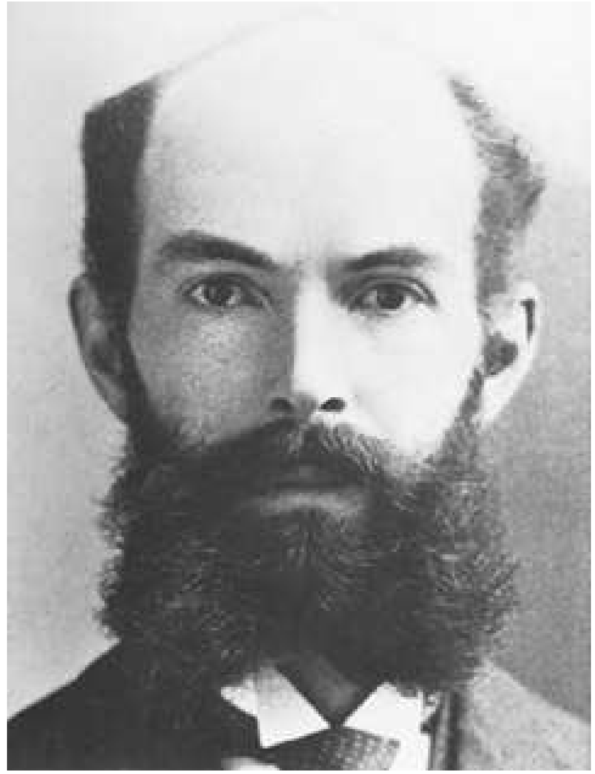
Riemann Mapping Theorem: If Ω is a simply connected, proper subdomain of the plane, then there is a conformal map $f : \mathbb{D} \to \Omega$.



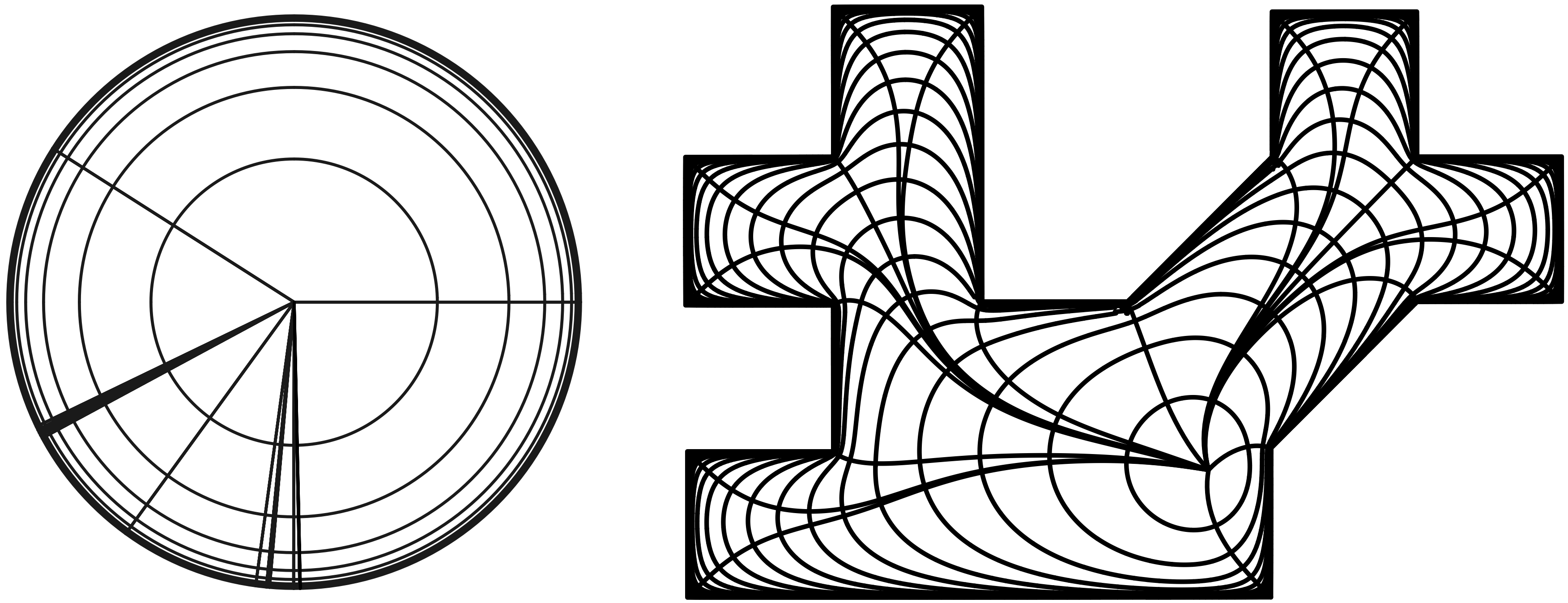
Conformal = angle preserving

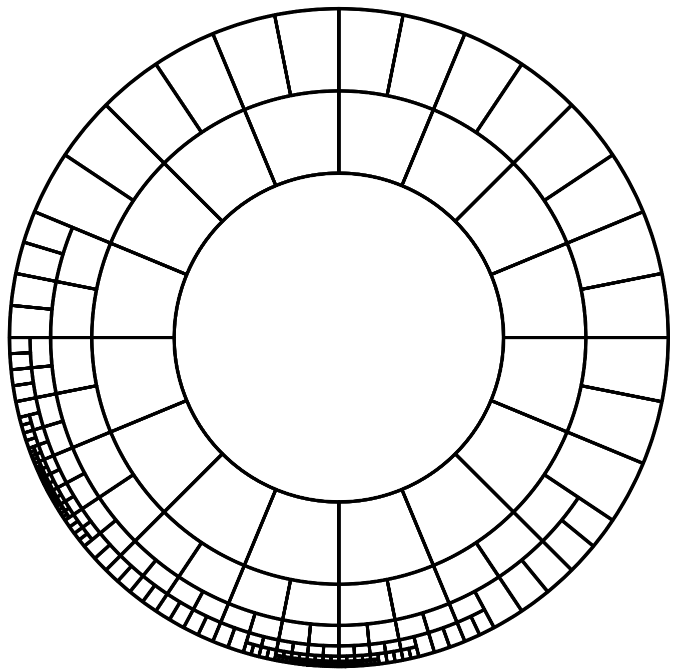
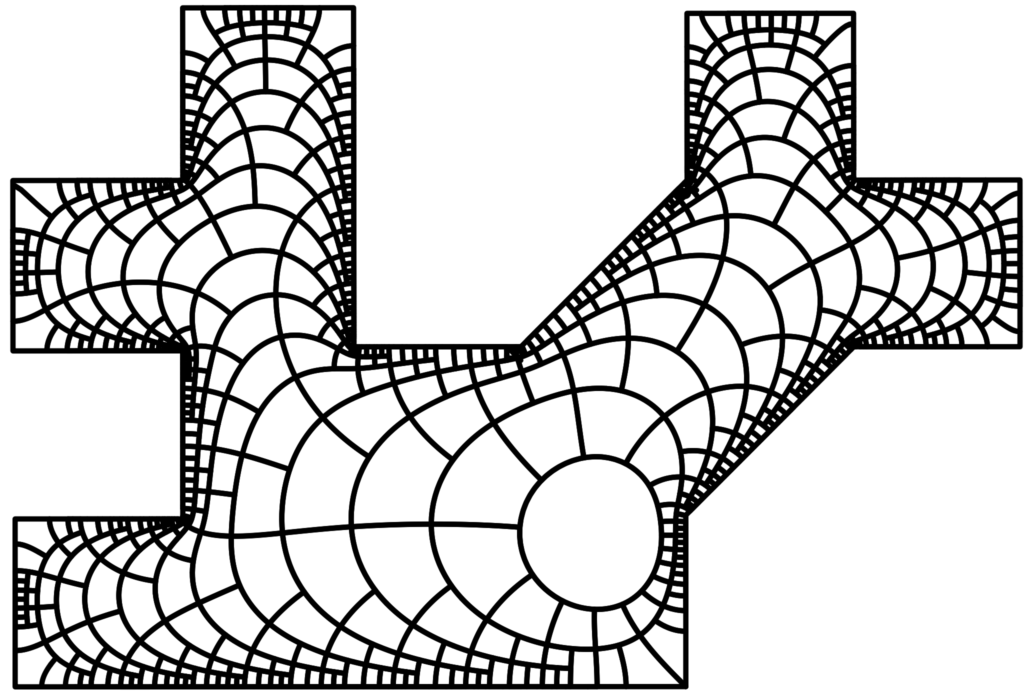


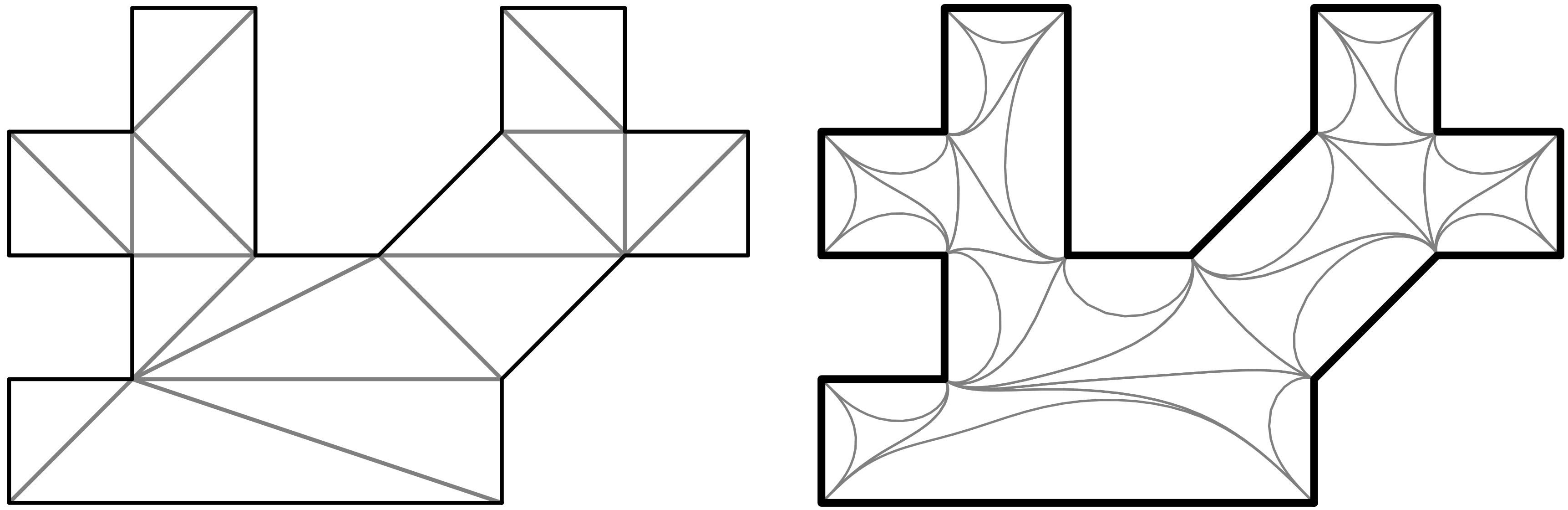
Our Founder



William Fogg Osgood
First proof of RMT, 1900



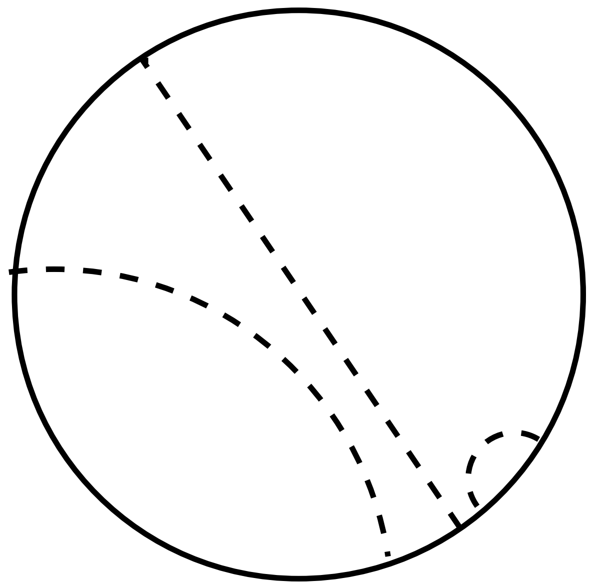



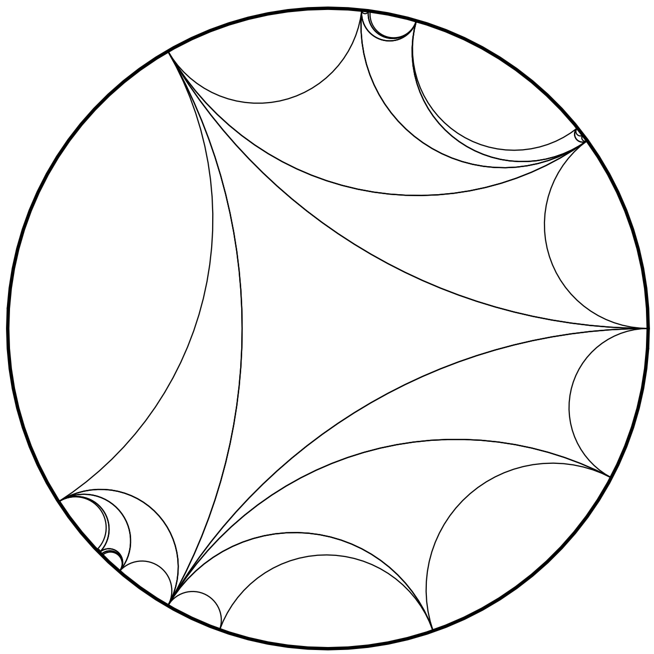


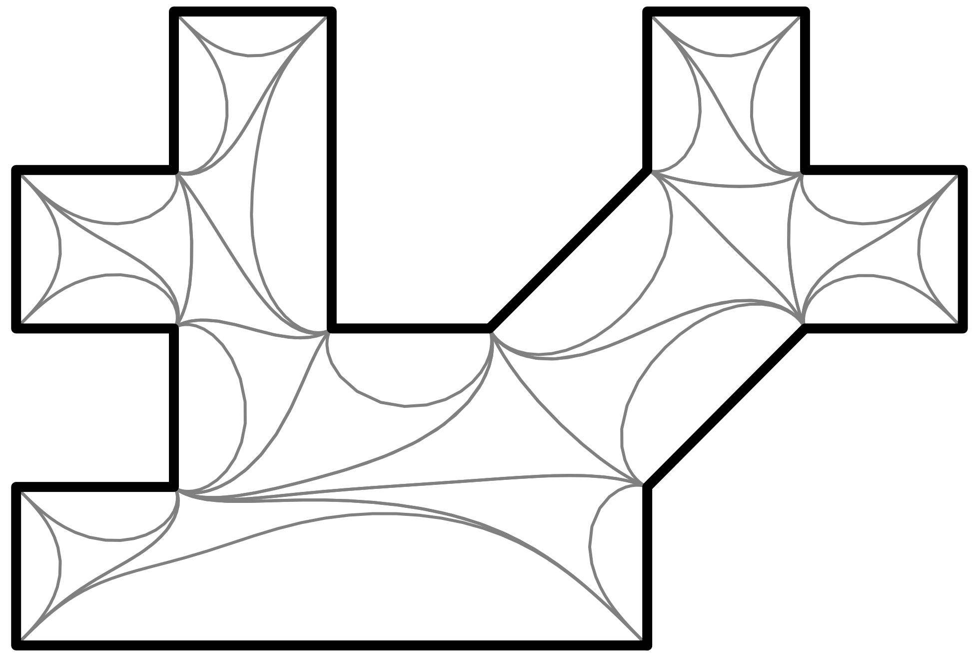
Hyperbolic metric on disk given by

$$d\rho = \frac{ds}{1-|z|^2} \simeq \frac{ds}{\mathrm{dist}(z, \partial D)}.$$

- Geodesics are circles perpendicular to boundary.
- Metric transfers via conformal maps to other domains.







How much time is needed to compute a conformal map?

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Theorem (DCG Sept 2010): We can compute a ϵ-conformal map onto an n-gon in $O(n \log \frac{1}{\epsilon} \log\log \frac{1}{\epsilon})$.

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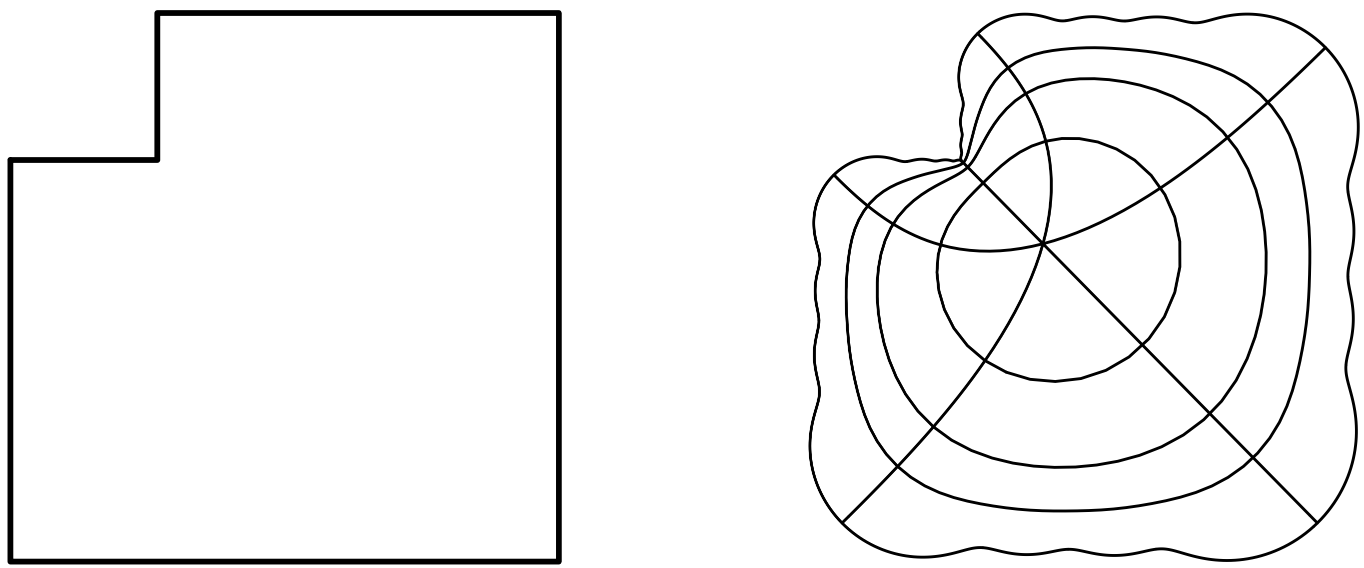
ϵ-conformal = ϵ-distortion of angles.

Goal: approximate conformal map onto n-gon at $O(n)$ points in $O(n)$ time (fixed ϵ).

Proof of the fast mapping theorem:

- Local representation of maps
- Newton's method for Beltrami's equation
- Use medial axis to find initial guess

Conformal maps have power series, but corners of polygon create singularities on circle. Convergence is slow.



20 terms

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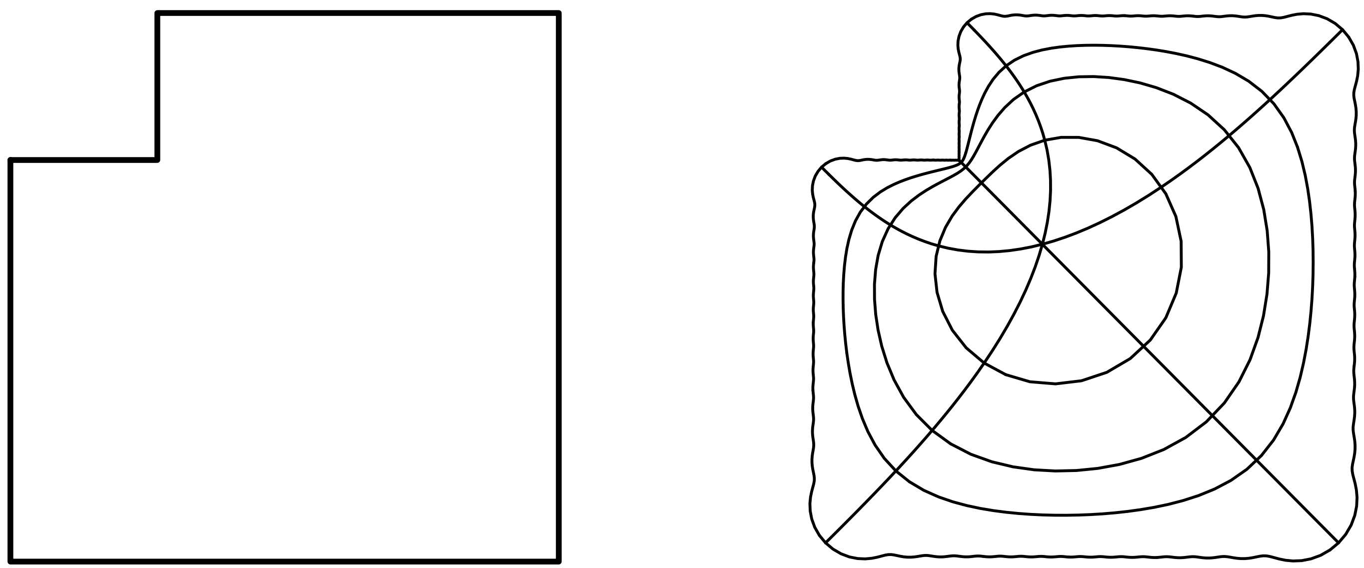

100 terms

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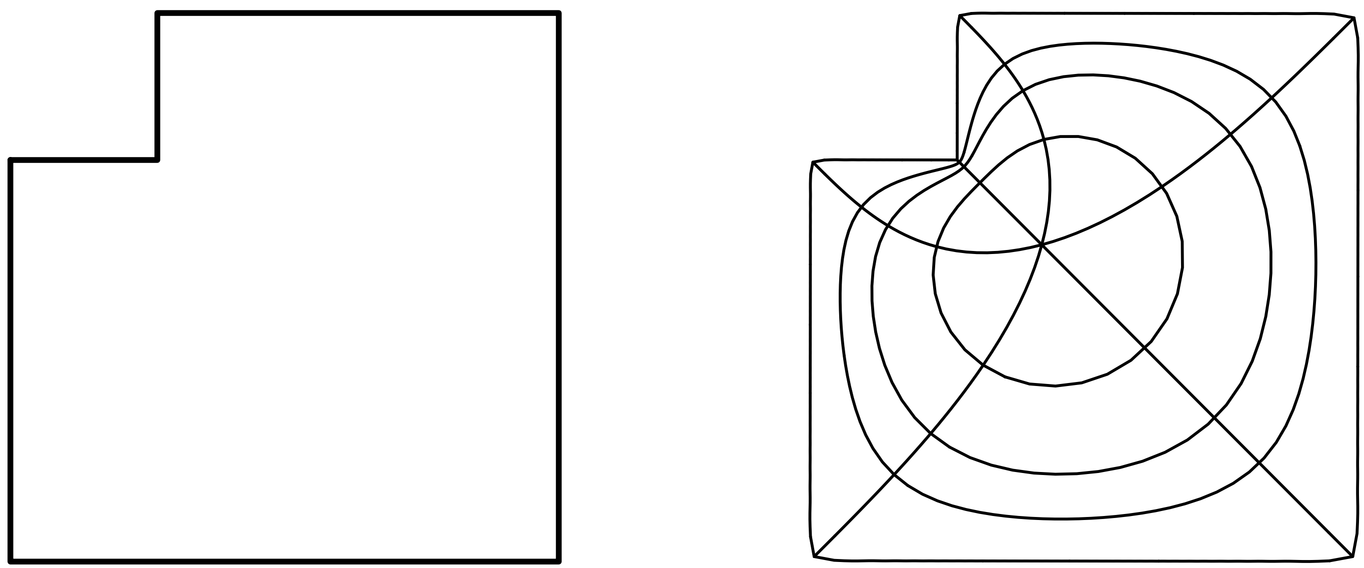

1000 terms

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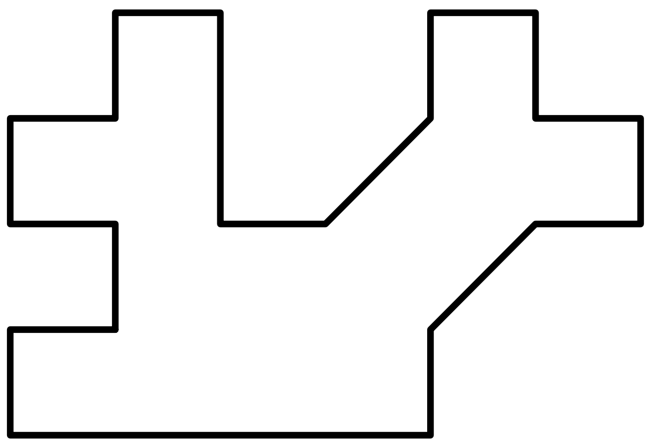

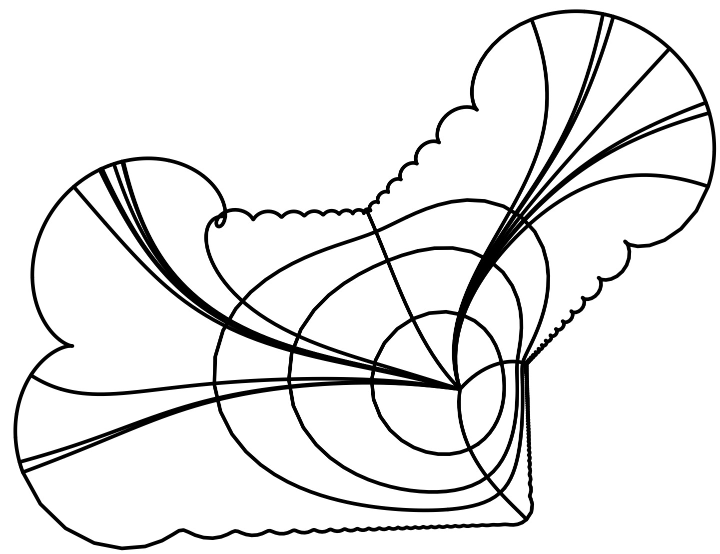

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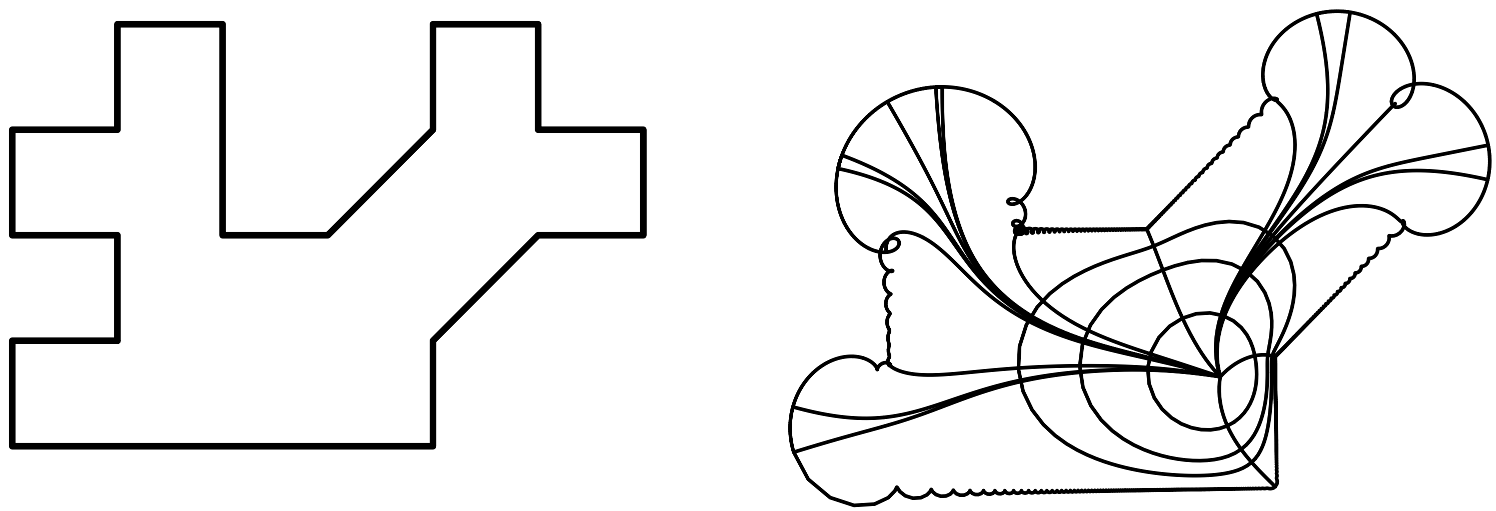

500 terms

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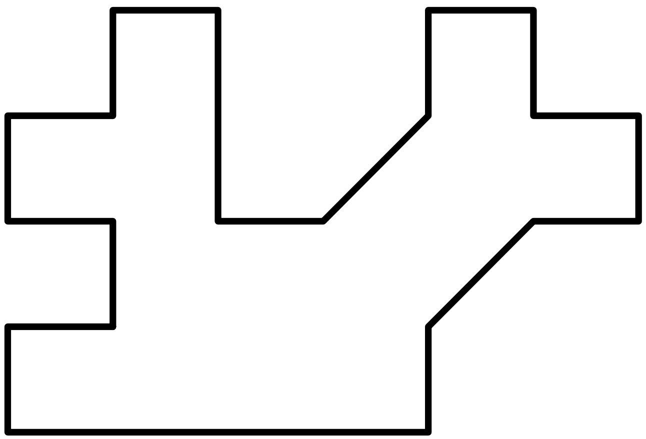

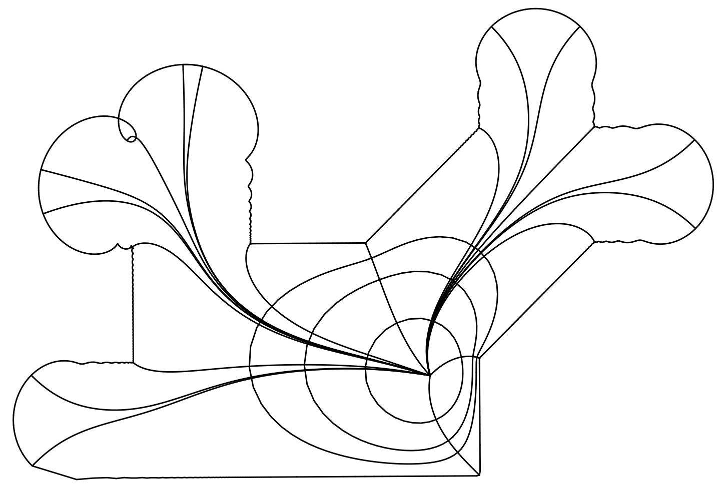

2500 terms

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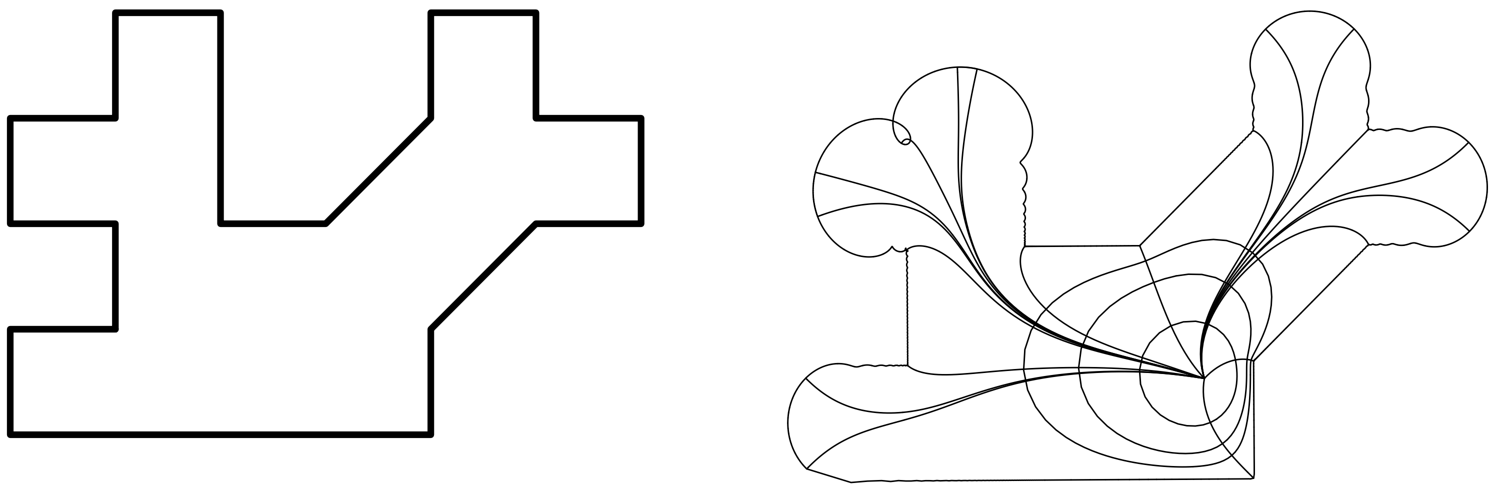

2500 terms

1×20 rectangle would require about 10^{15} terms.

Need more efficient representation.

Schwarz-Christoffel formula (1867):

$$f(z) = A + C \int^z \prod_{k=1}^{n} (1 - \frac{w}{z_k})^{\alpha_k - 1} dw,$$

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Christoffel

Schwarz

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$\{\alpha_1\pi, \ldots, \alpha_n\pi\}$, are interior angles of polygon.
$\{z_1, \ldots, z_n\}$ are points on circle mapping to vertices.

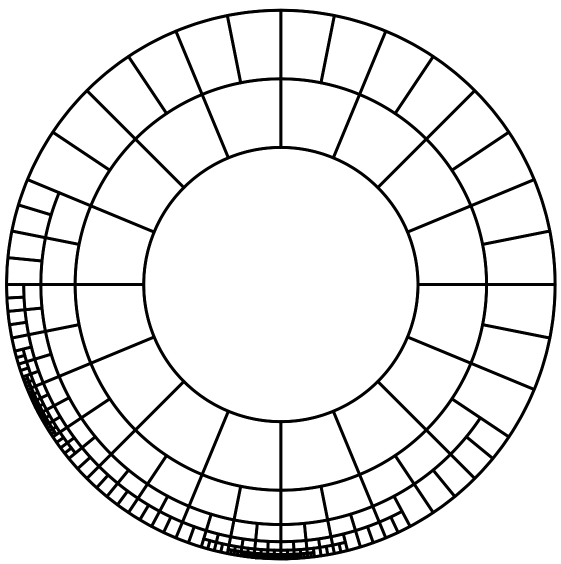
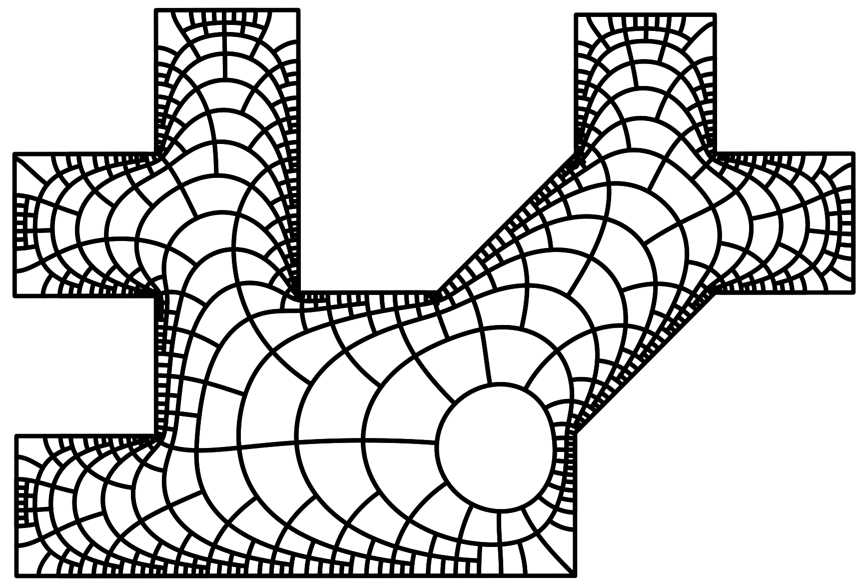
α's are known.

z's must be solved for.

Each evaluation of integrand is n-fold product.

We want n evaluations of f in time $O(n)$.

Local series representation: We cut disk into $O(n)$ regions and use a p-term series on each piece to approximate map with accuracy $\epsilon \approx 2^{-p}$ in hyperbolic metric.

Use partition of unity to get global map.

Easy to evaluate; just plug in.

Also more subtle advantage.

Schwarz-Christoffel always gives conformal map, but onto wrong polygon if z-parameters are only approximate.

Hard to understand relationship between parameters and image domain, so hard to update parameters in provably correct way (many unproven heuristics which seem to work in practice, e.g., Davis, CRDT.)

Interpolation of local series maps onto correct domain, but is not conformal if series are only approximate.

Easy to make a map conformal and preserve image. (Method has been known for 50 years.)

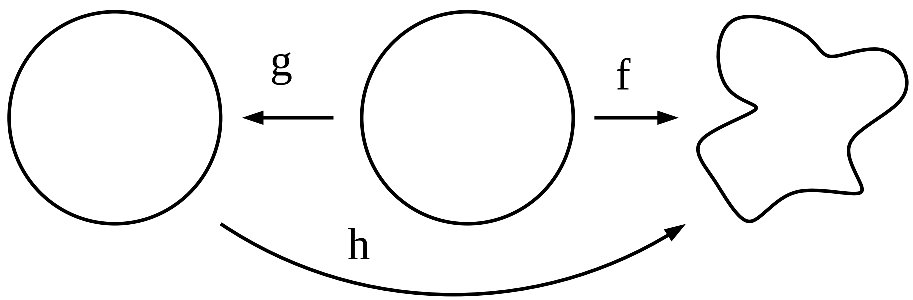
$$\partial f = \frac{1}{2}(f_x - i f_y), \quad \overline{\partial} f = \frac{1}{2i}(f_x + i f_y).$$

We want $f : \mathbb{D} \to \Omega$ with $\overline{\partial} f = 0$ (Cauchy-Riemann).

We measure distance to conformality by dilatation

$$\|f\| = \sup |\mu_f| = \sup |\overline{\partial} f / \partial f|.$$

Main point: If $f : \mathbb{D} \to \Omega$, $g : \mathbb{D} \to \mathbb{D}$ and $\mu_g = \mu_f$, then $h = f \circ g^{-1} : \mathbb{D} \to \Omega$ is conformal.



Beltrami equation: given μ find g with $\mu_g = \mu$,

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Set $g = P[\mu(h+1)] + z$, where

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T is the Beurling transform

$$T\varphi(w) = \lim_{r\to 0} \frac{1}{\pi} \iint_{|z-w|>r} \frac{\varphi(z)}{(z-w)^2} dxdy,$$

P is the Cauchy transform

$$P\varphi(w) = -\frac{1}{\pi} \iint \varphi(z)\left(\frac{1}{z-w} - \frac{1}{z}\right) dxdy.$$

If f has local representation by $O(n)$ p-term series, we can compute a g in time $O(np \log p)$ so that

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Iteration gives quadratic convergence to conformal map.

Uses fast multipole method and FFT

For $O(n)$ bound iteration needs a starting map $\mathbb{D} \to \Omega$ that is close to conformal (independent of Ω).

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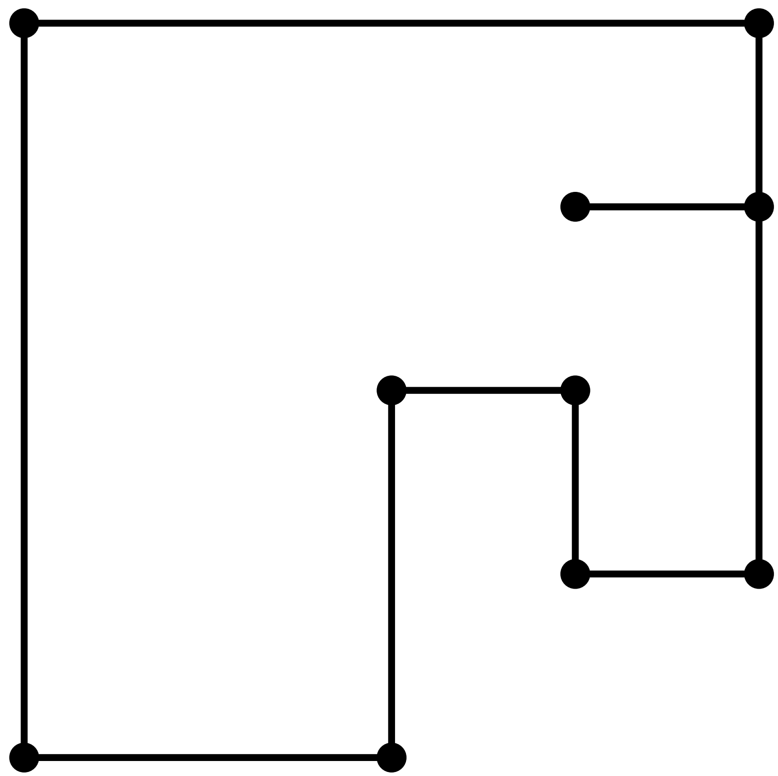
For $O(n)$ bound iteration needs a starting map $\mathbb{D} \to \Omega$ that is close to conformal (independent of Ω).

This is how I got involved with numerical mapping.

Initial guess comes from **medial axis flow**.



Start with a polygon.

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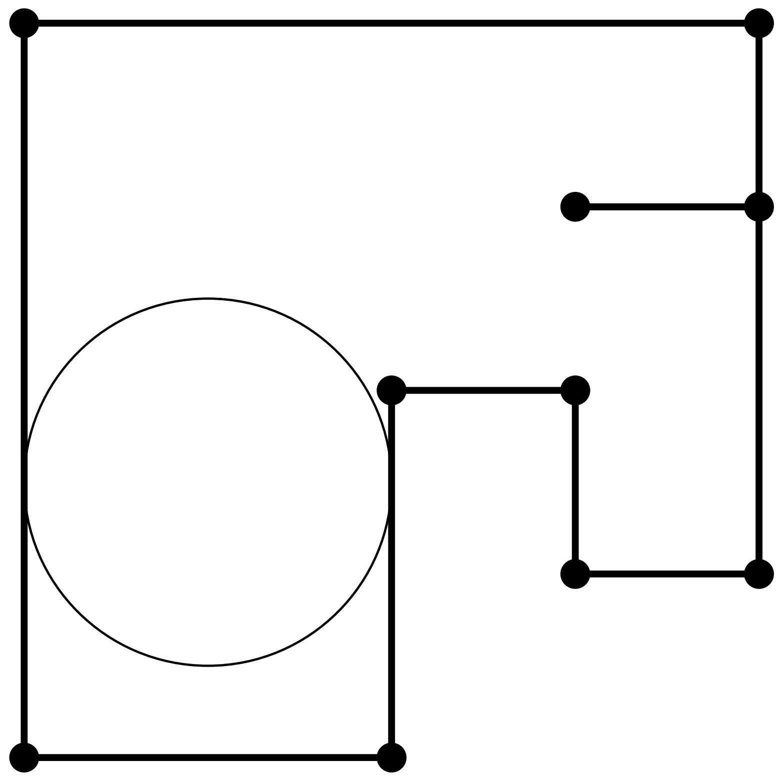

Consider interior disks with ≥ 2 contacts on boundary.

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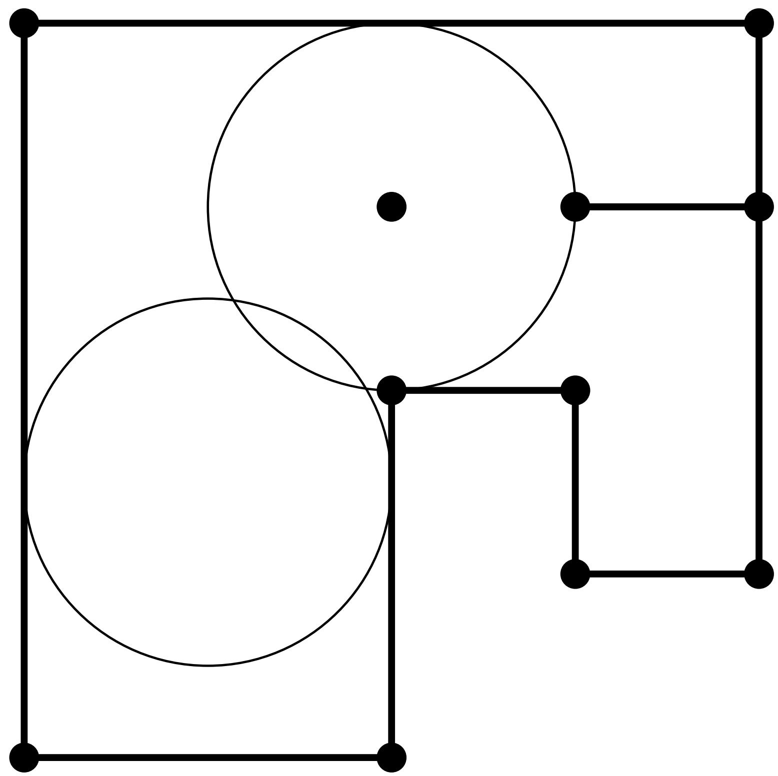

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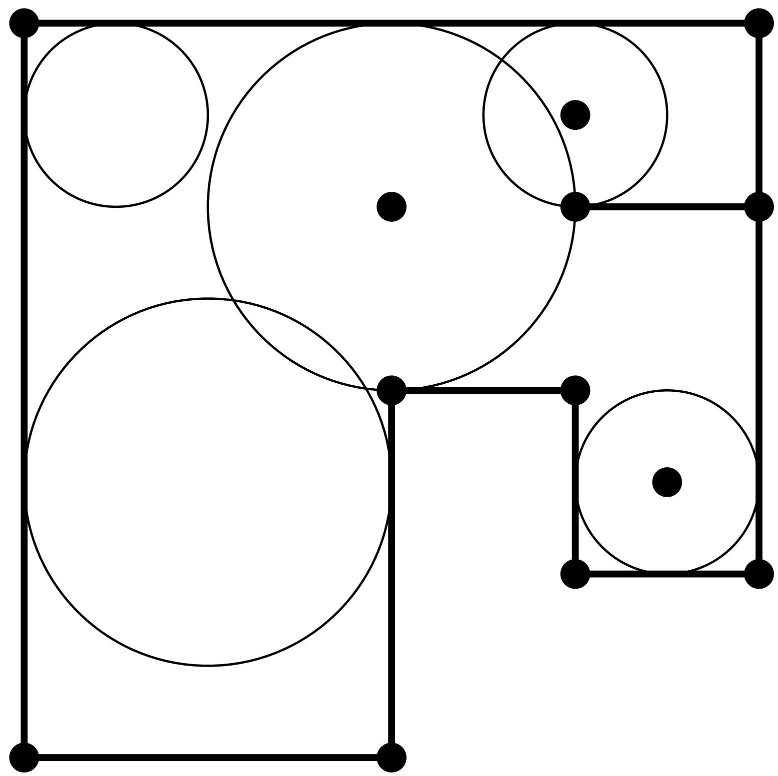

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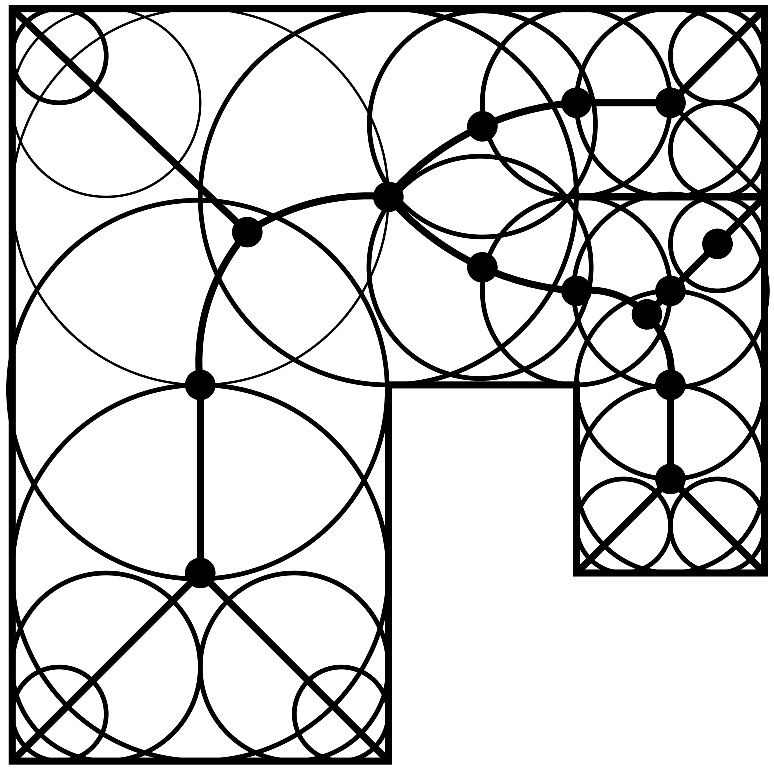

Centers of all such disks define **medial axis**.

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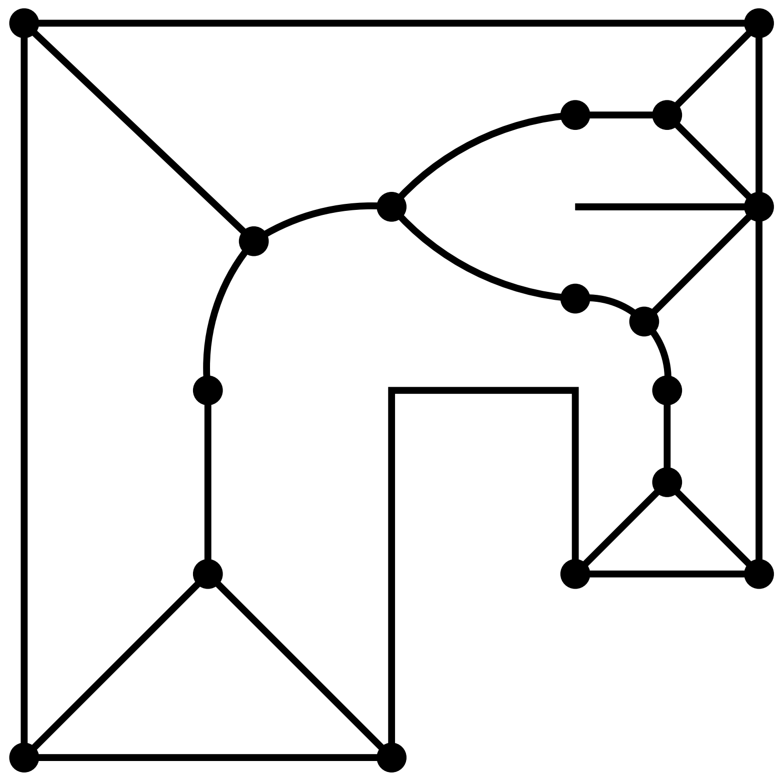

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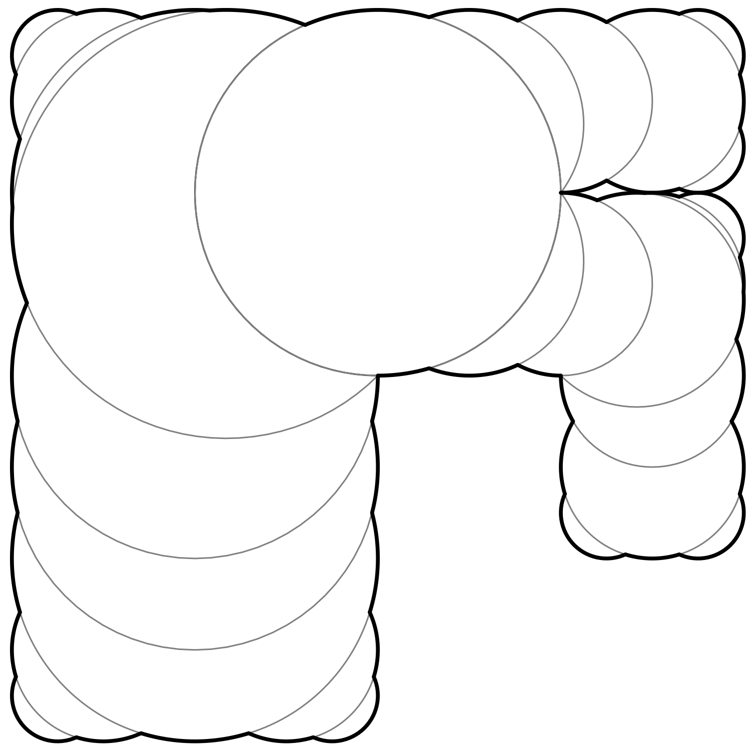
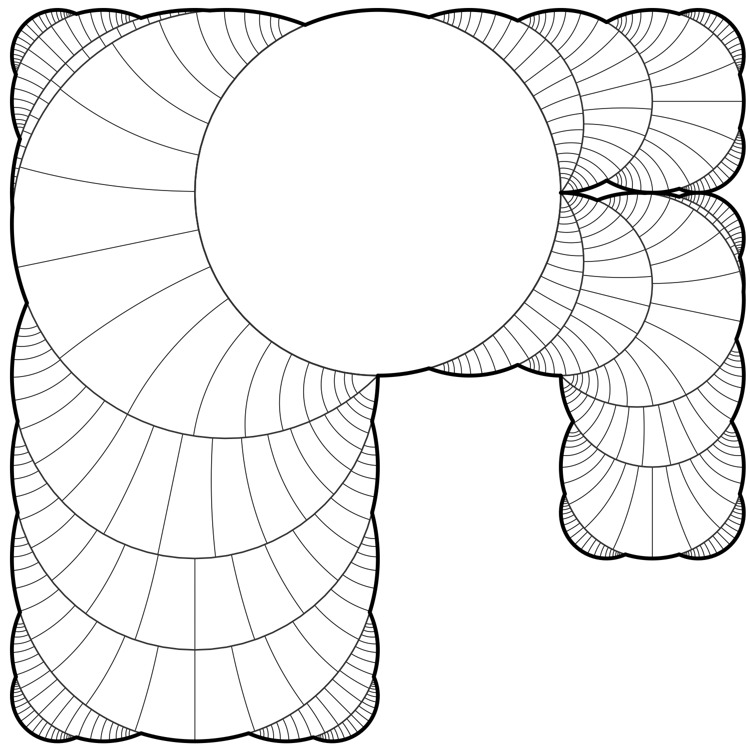


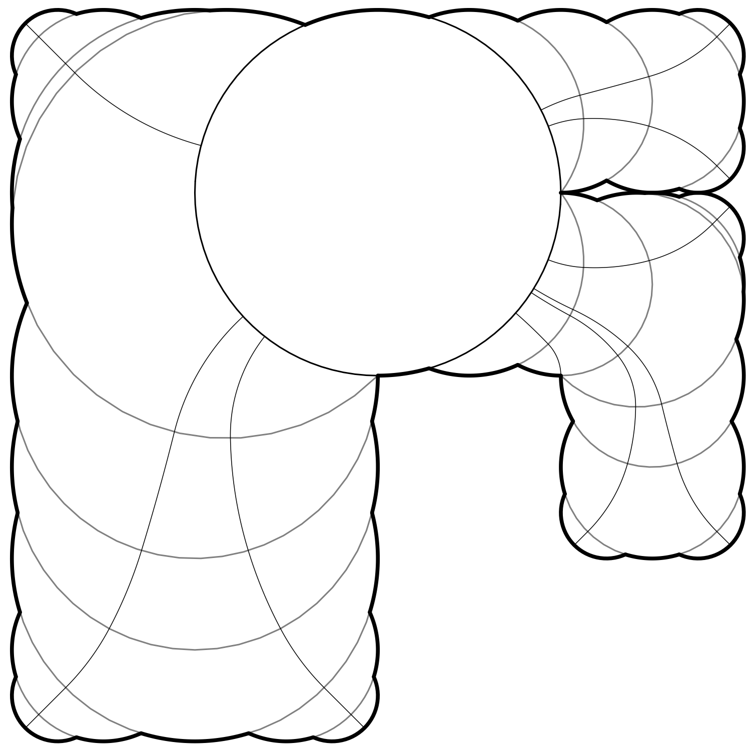
Take a finite set of medial axis disks. Choose a root.

Initial guess comes from **medial axis flow**.



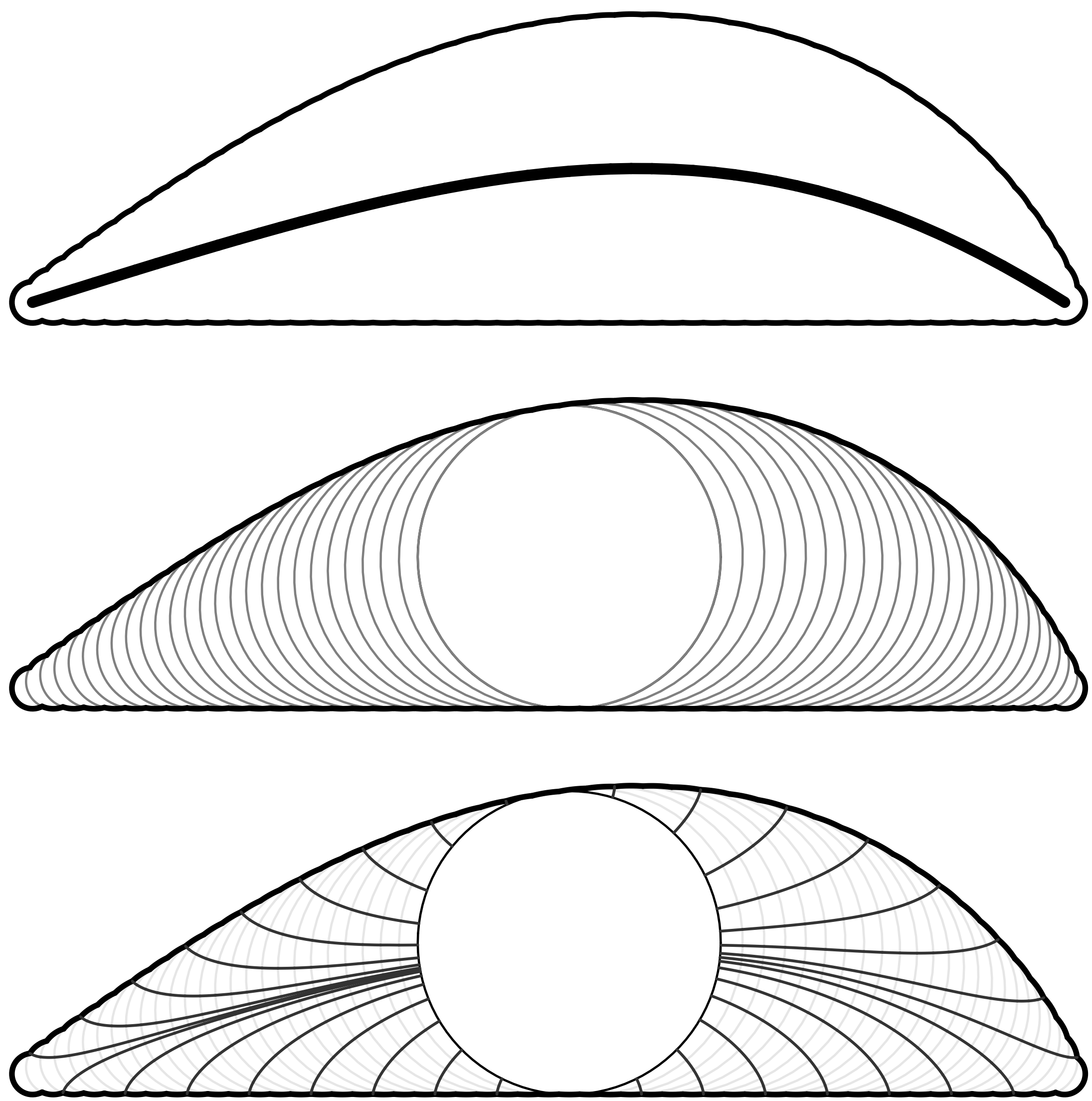
Foliate crescents by orthogonal arcs.

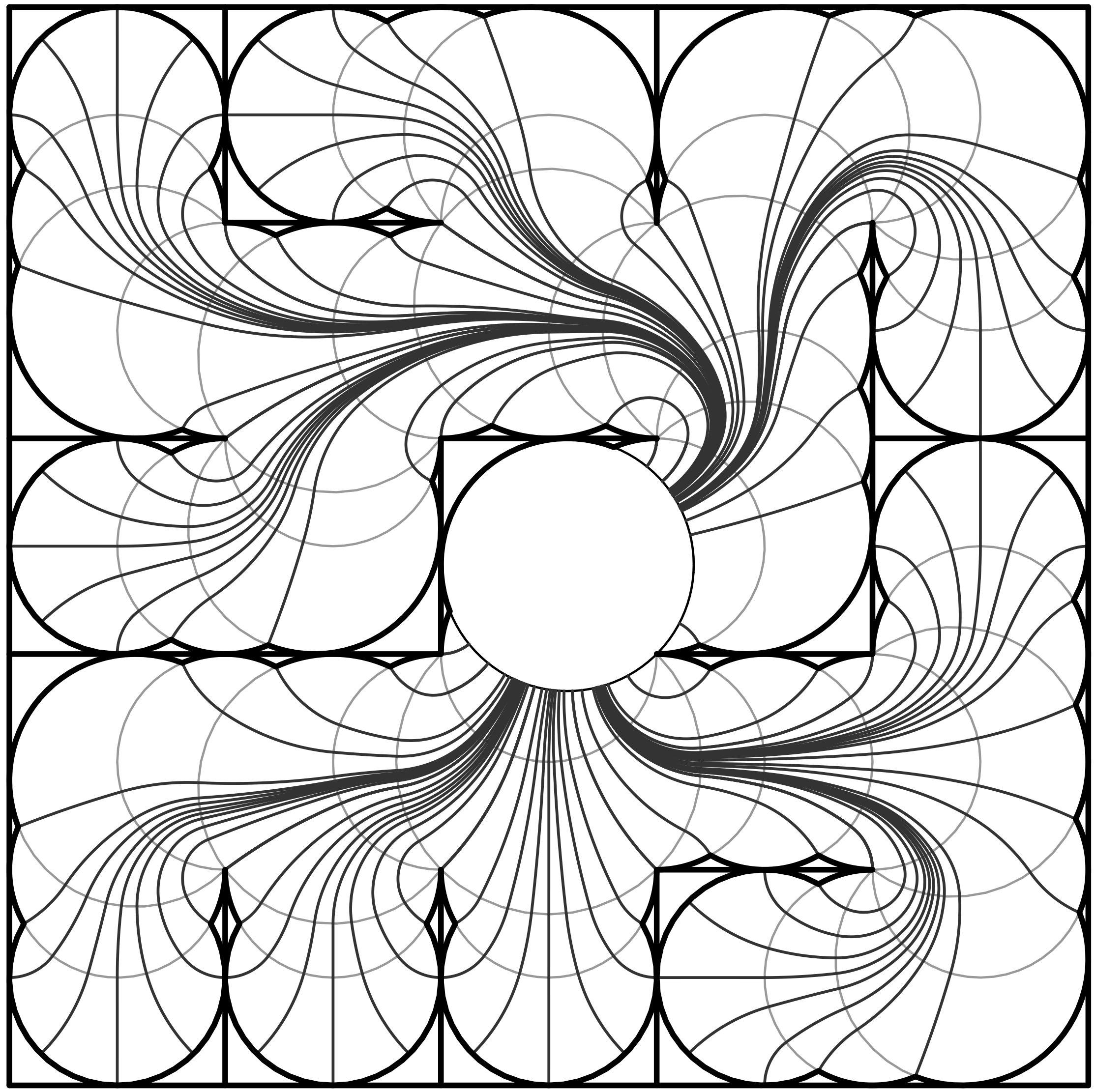
Initial guess comes from **medial axis flow**.

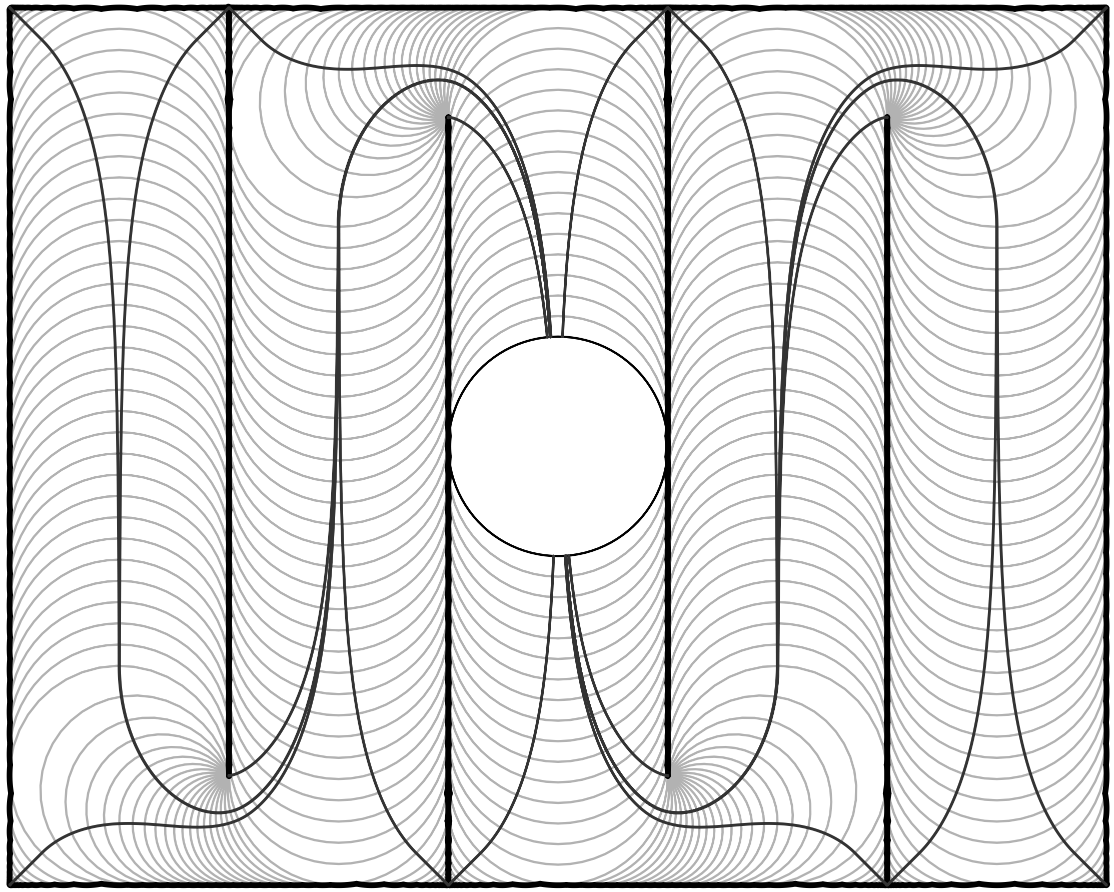


Follow arcs to define map of boundary to circle.

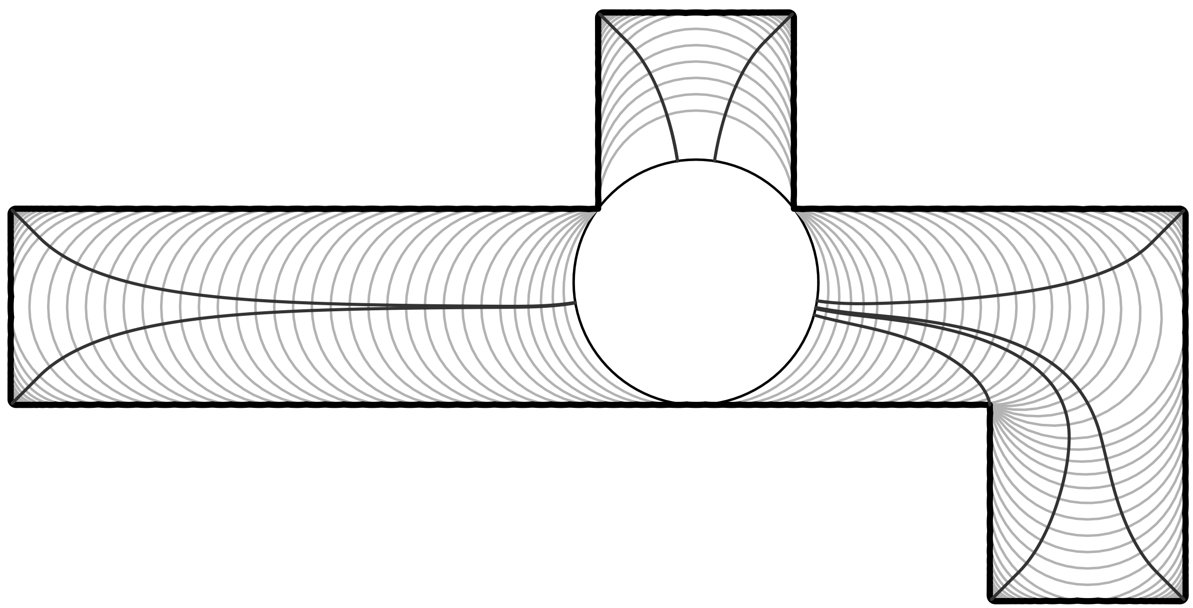
Similar flow for any simply connected domain.





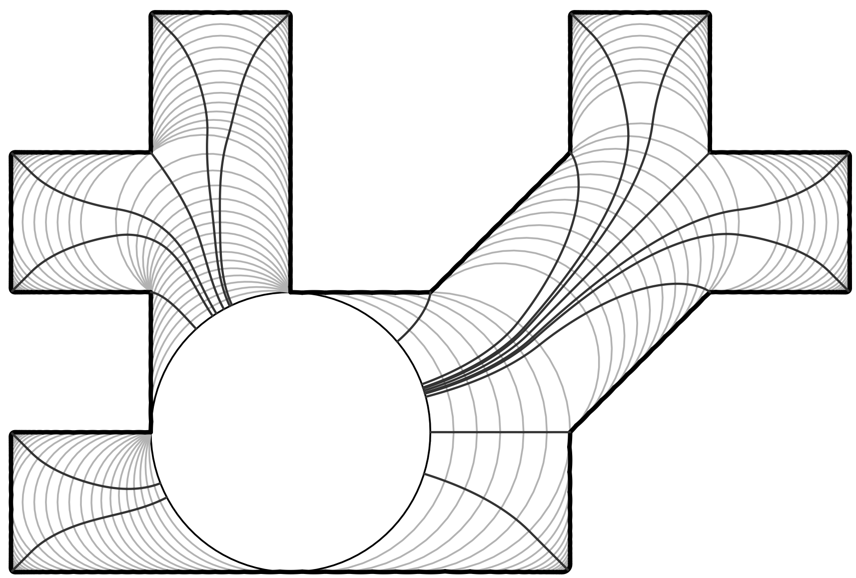


Take limit as disks become denser to define map for any polygon. Images of vertices computable in $O(n)$ using medial axis and cross ratios. Medial axis computable in $O(n)$ due to Chin, Snoeyink and Wang.



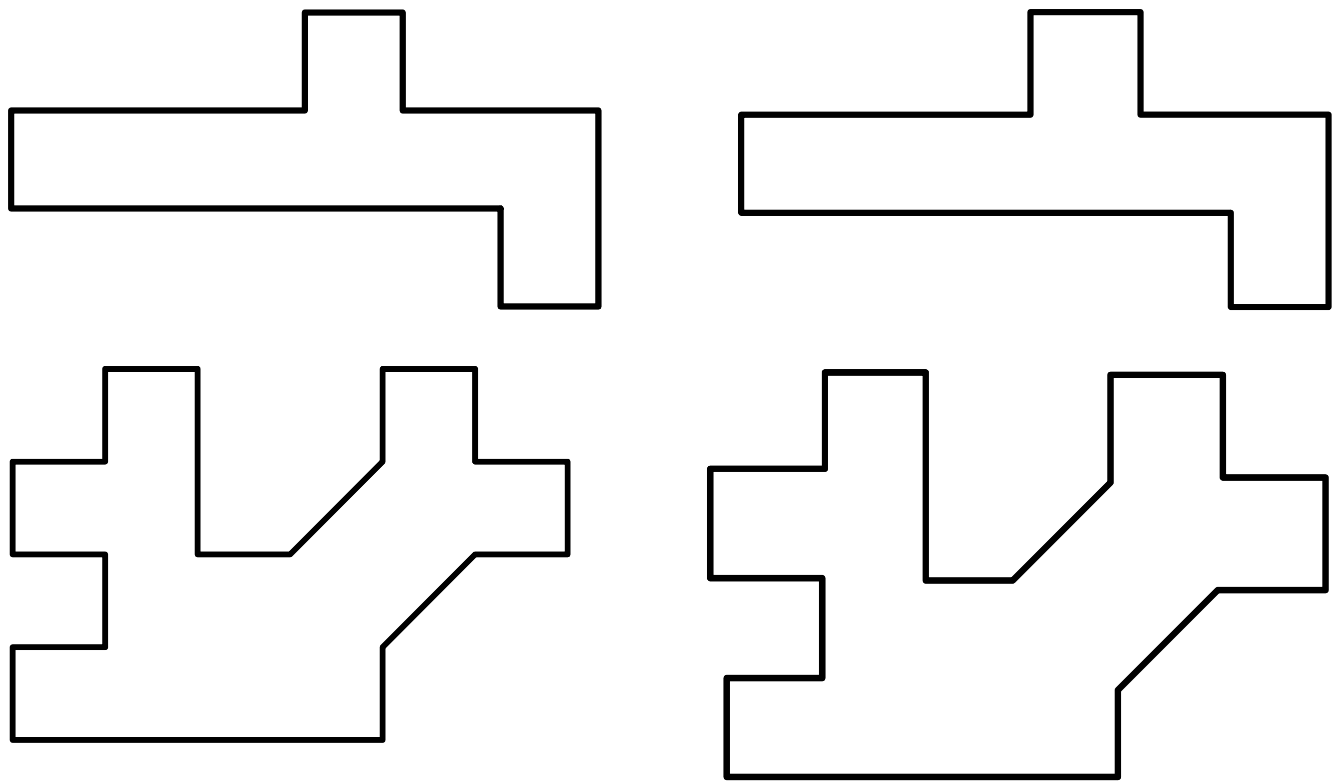
Thm: Medial axis flow is uniformly close to conformal.

More precisely, flow map on boundary extends to interior map f with $|\mu_f| < k < 1$, universal k.



How big is k?

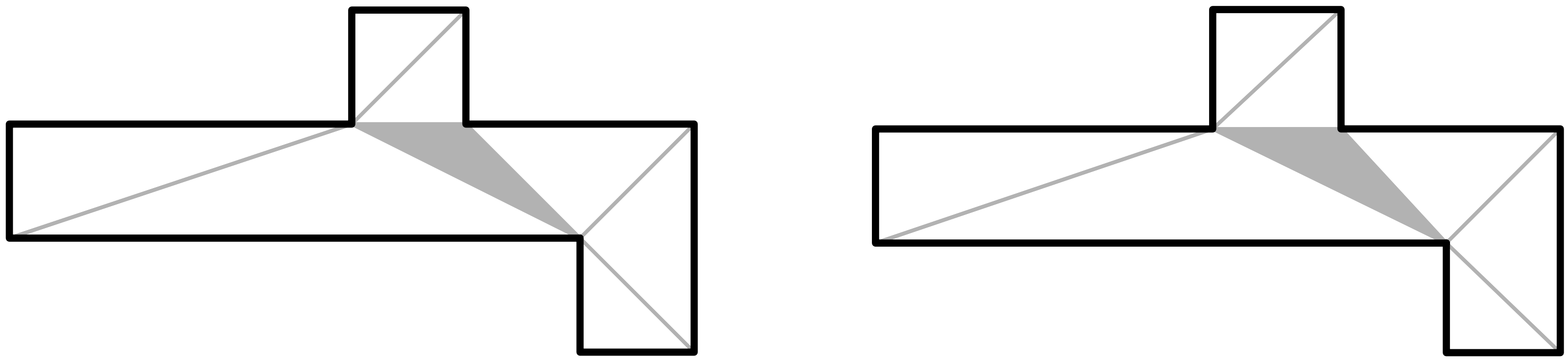
Use "iota parameters" in Schwarz-Christoffel formula.



Target Polygon

Iota Parameters

Triangulate polygons and form piecewise linear maps. Max $|\mu|$ gives upper bound for distance to conformal.



The most distorted triangle is shaded. Here $|\mu| = .108$.

We can bound conformal distance to true SC parameters even though we don't know what the are.

Why is medial axis flow close to conformal on Ω?

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Because it is conformal on the dome of Ω.

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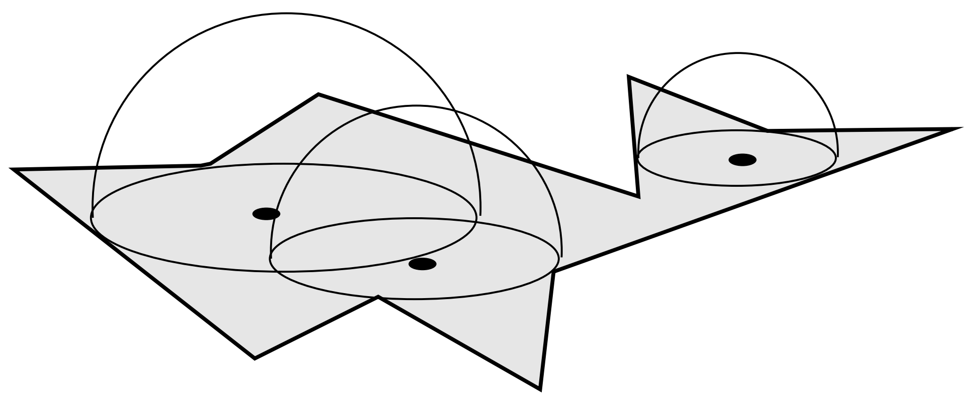
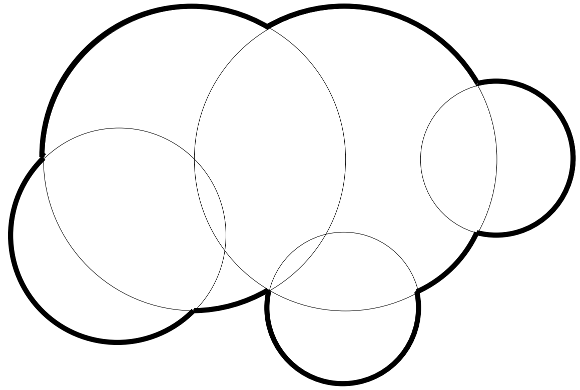
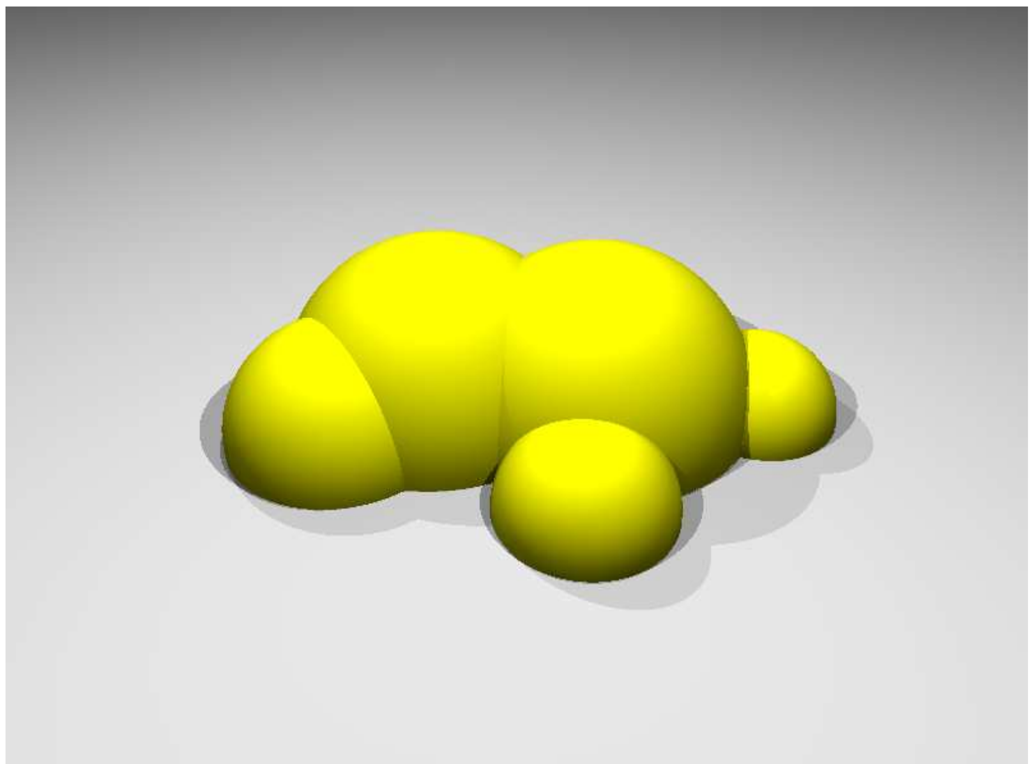
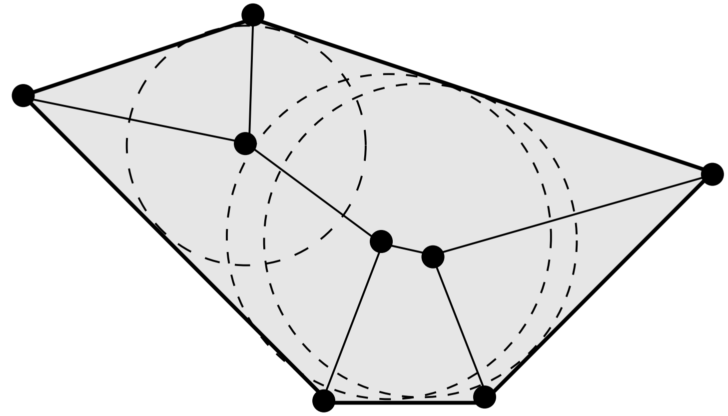
Take union of all hemispheres whose bases lie inside Ω.
Upper envelope is the "dome" of Ω.

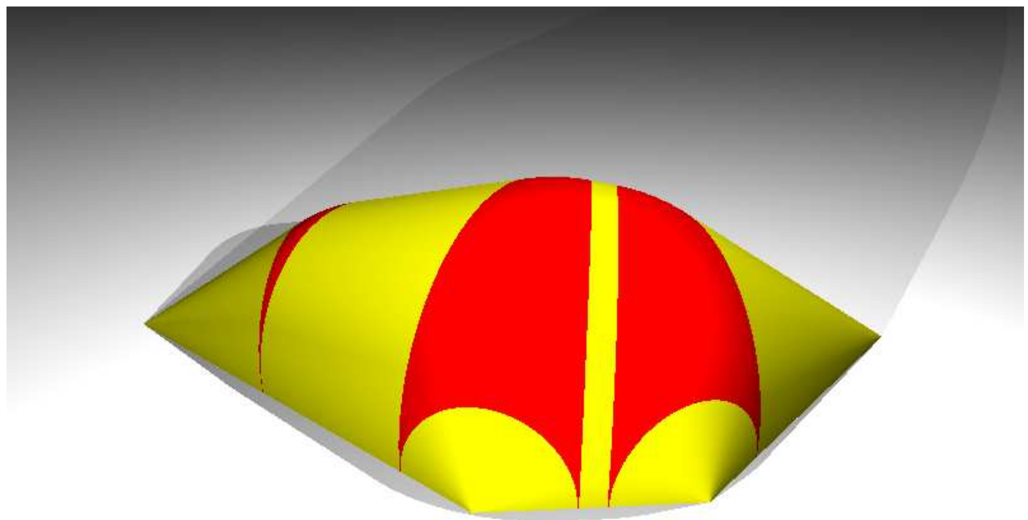


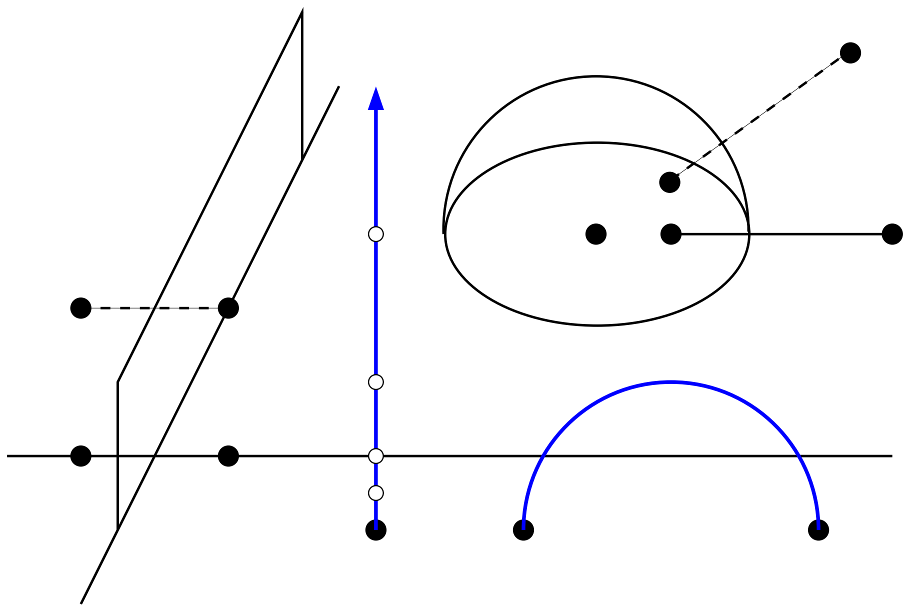
Only need medial axis disks.





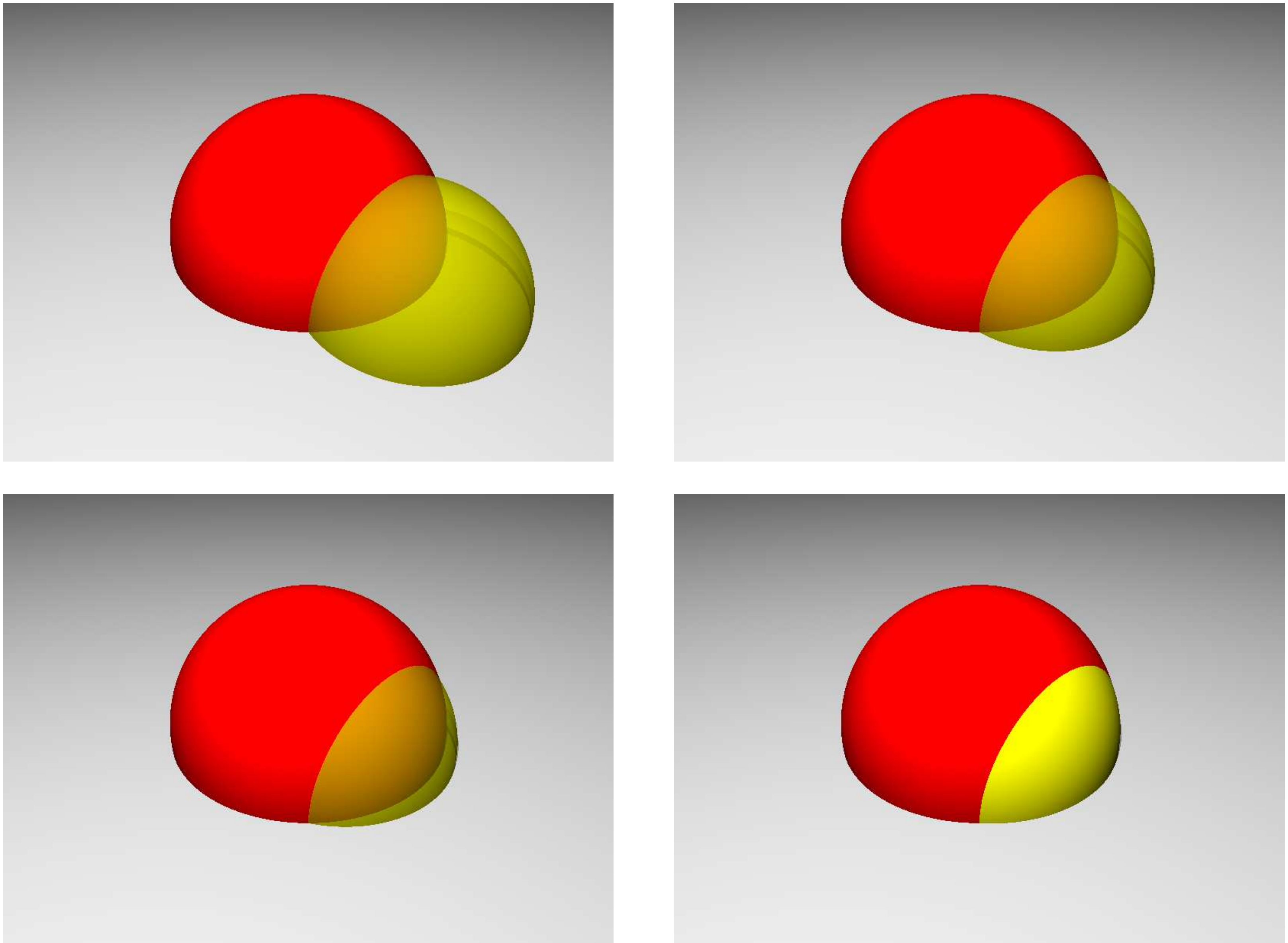




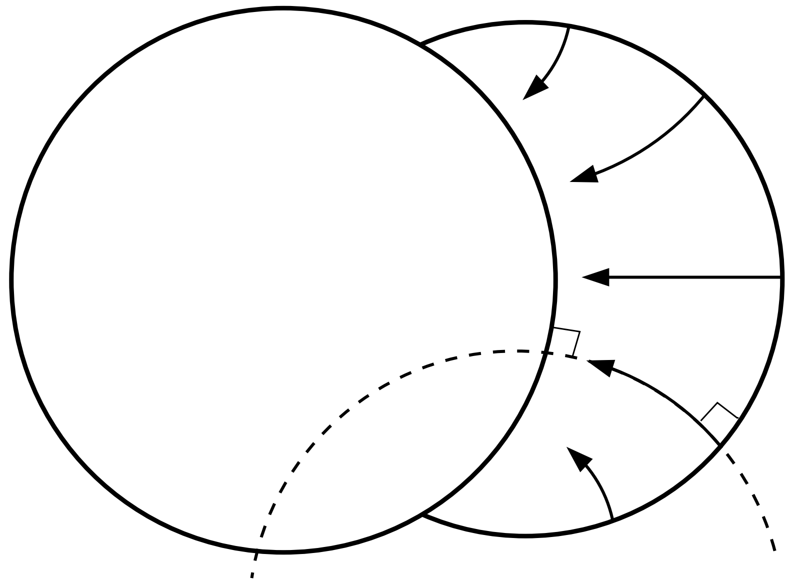


- Hyperbolic metric $d\rho = \frac{ds}{\mathrm{dist}(z,\mathbb{R}^2)}$.
- Geodesics are circles perpendicular to boundary.
- Hemispheres = flat planes = hyperbolic disk.
- We can define hyperbolic length on dome.
- Every dome is isometric to a hemisphere.

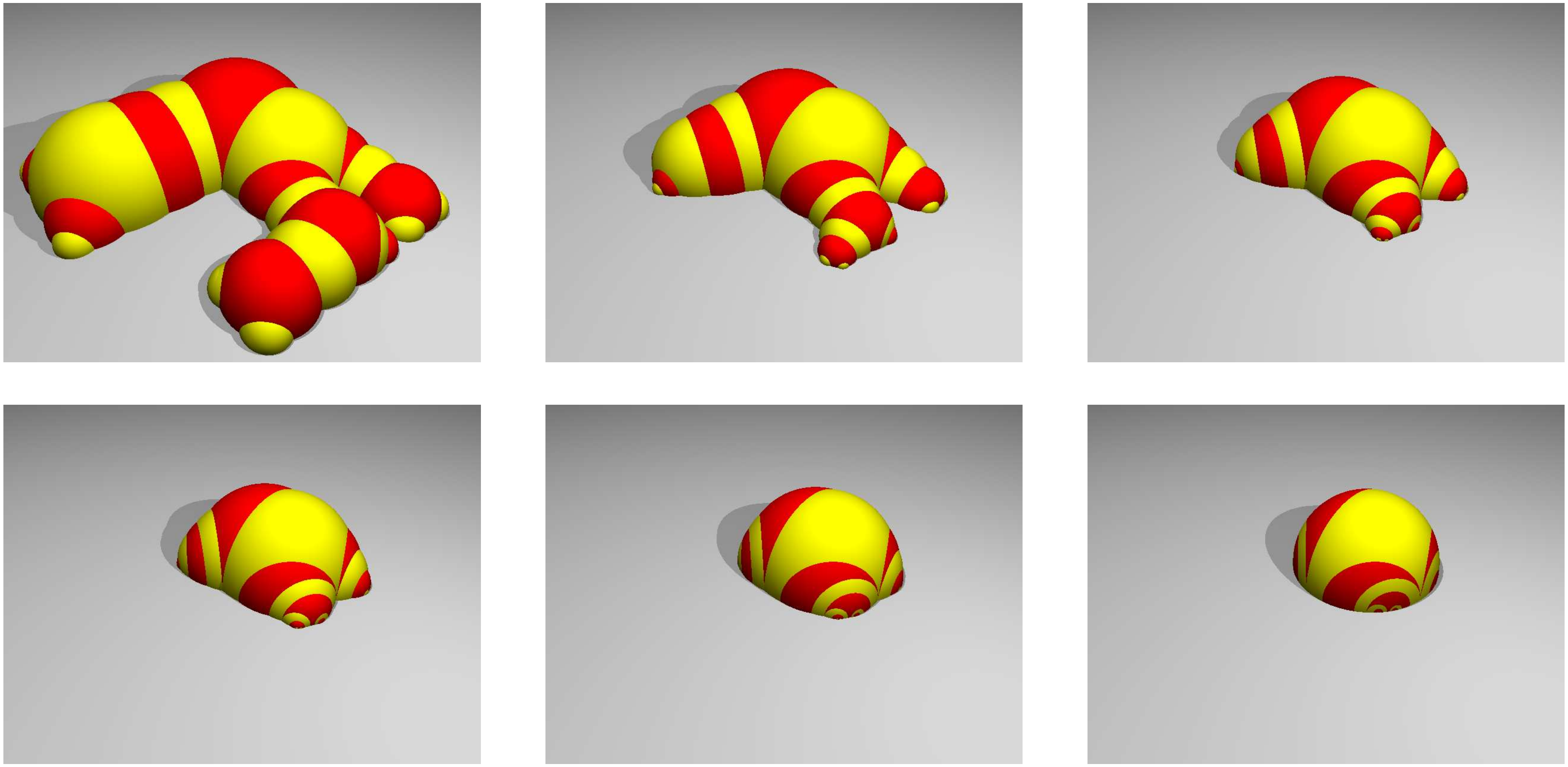
It is easy to map any dome conformally to a disk.



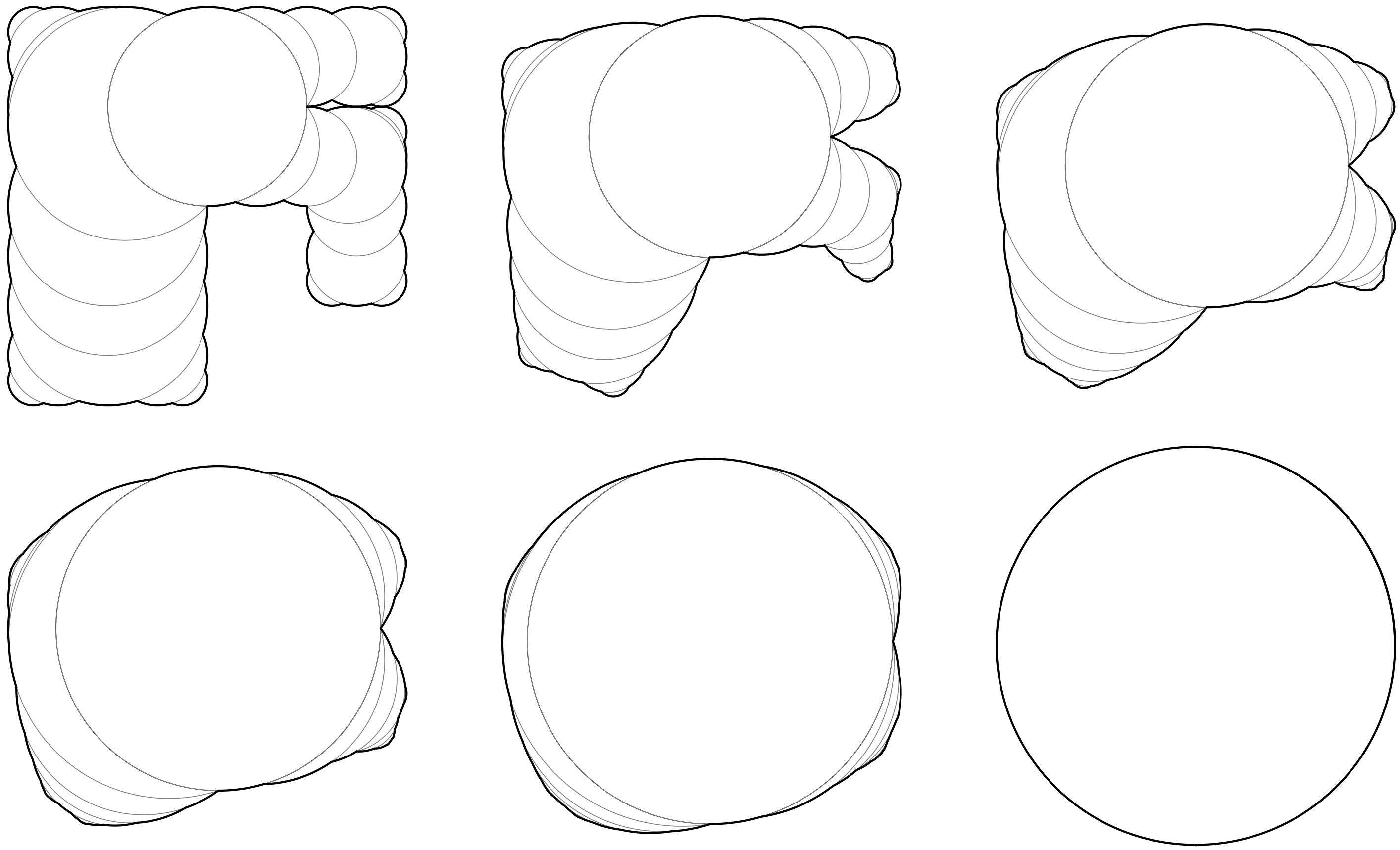
Flattening dome collapses crescent in base by collapsing orthogonal arcs.



Medial axis flow = boundary of flattening map



A dome is a hinged surface. We map it to a hemisphere by making all faces flush with each other. More interesting in hyperbolic space than Euclidean space because parallel postulate fails (more non-intersecting lines).

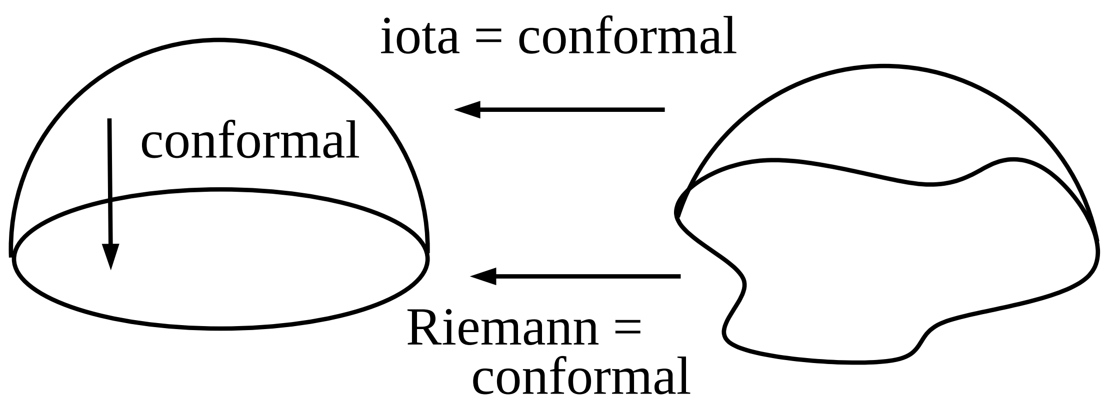


Boundary of base follows medial axis flow lines.

Iota = conformal from dome to disk.

Medial axis flow = boundary values of iota

Riemann map = conformal from base to disk



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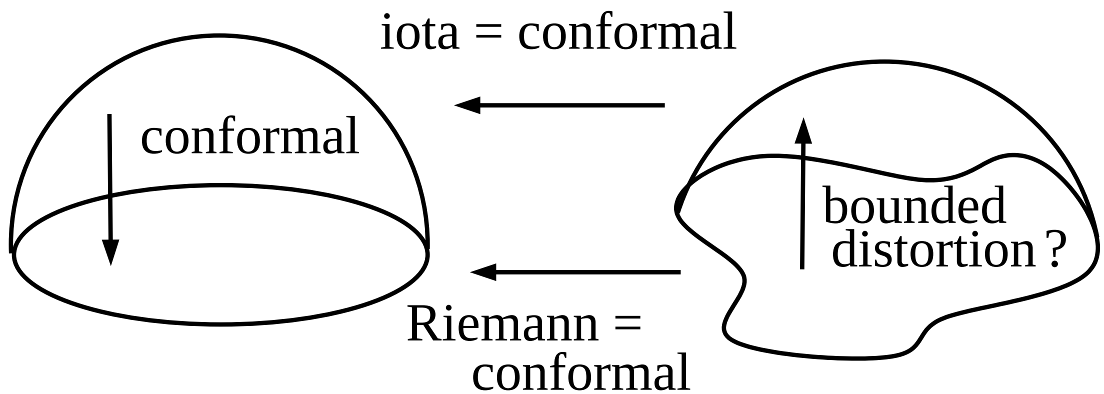


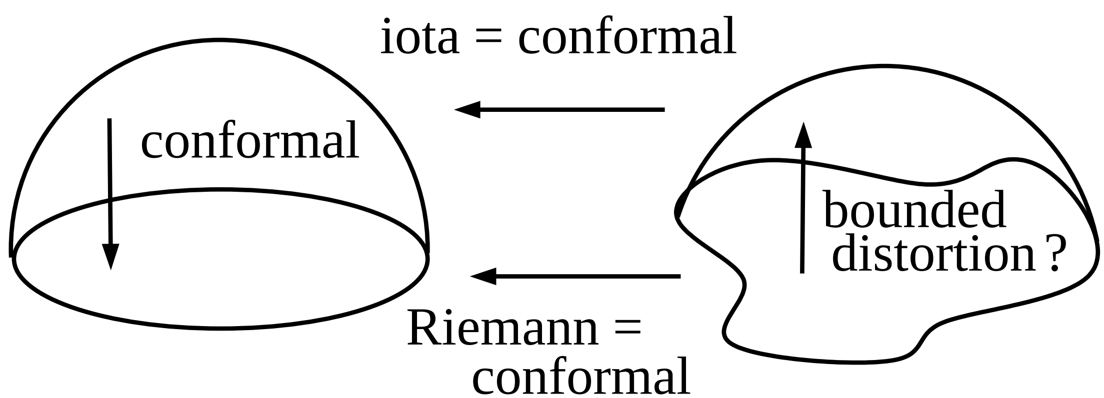
Medial Axis Flow ≈ Riemann Mapping

if and only if there is bounded distortion map from base to dome which is the identity on the boundary.

Iota = conformal from dome to disk.

Medial axis flow = boundary values of iota

Riemann map = conformal from base to disk

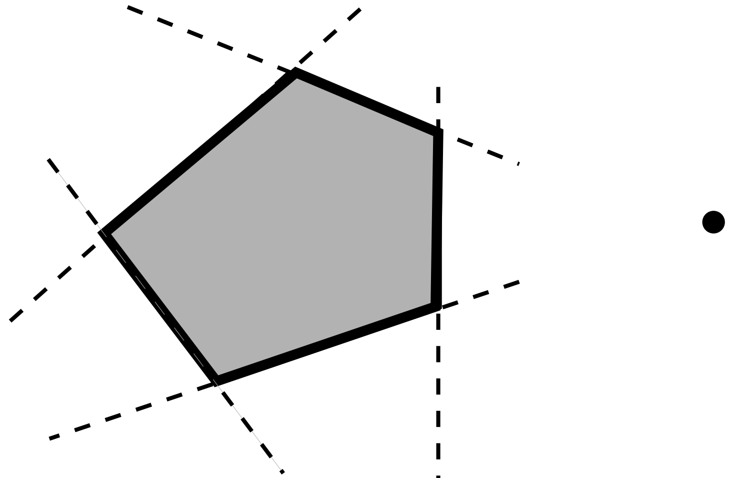


Medial Axis Flow ≈ Riemann Mapping

if and only if there is bounded distortion map from base to dome which is the identity on the boundary.

This is Dennis Sullivan's convex hull theorem.

Nearest point map in $\mathbb{R}^n$ is Lipschitz.



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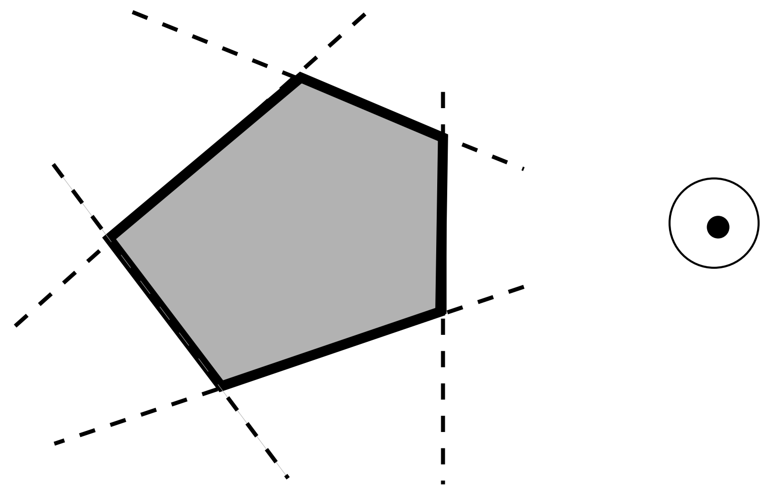

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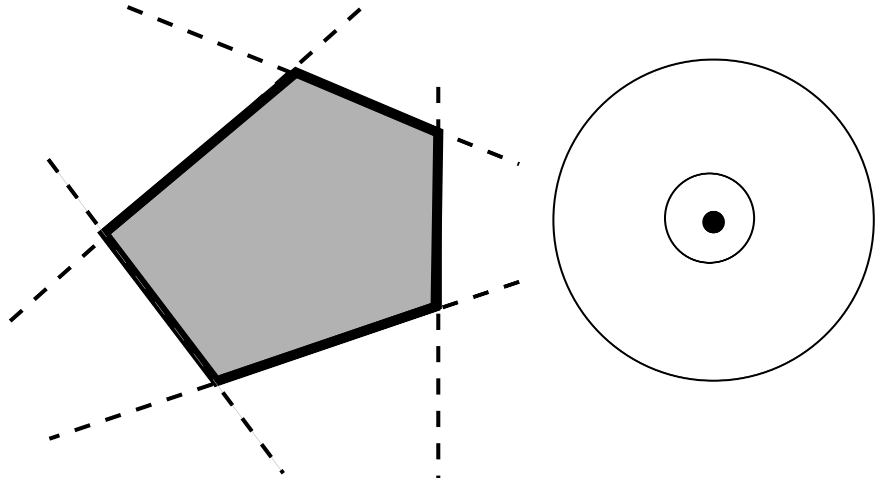

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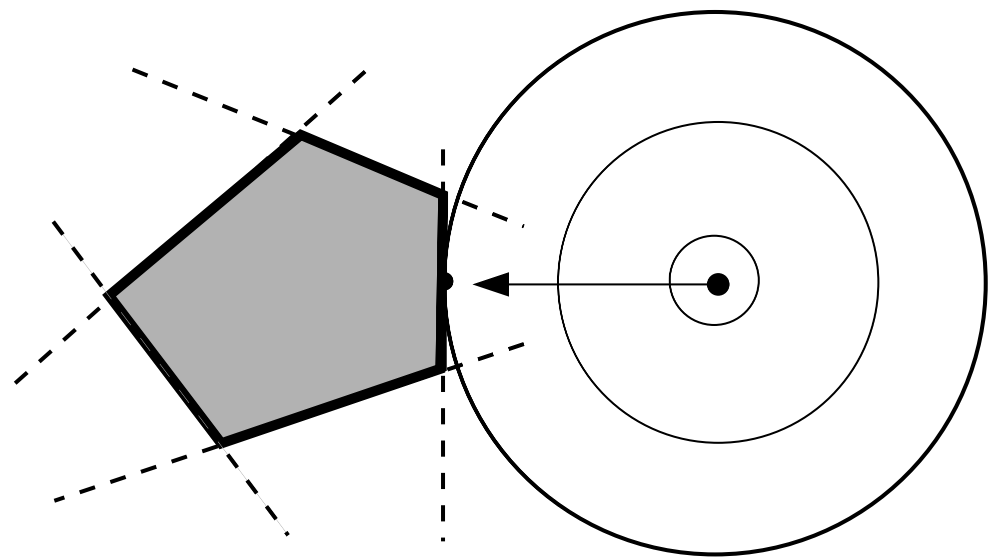

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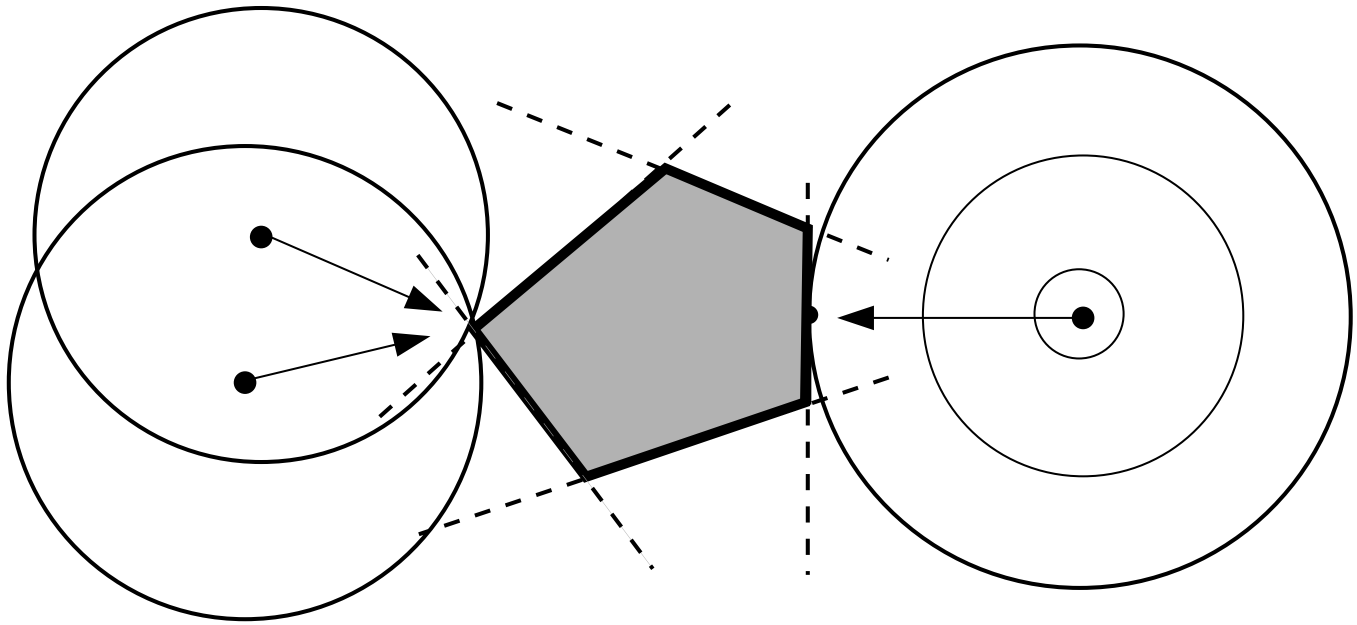
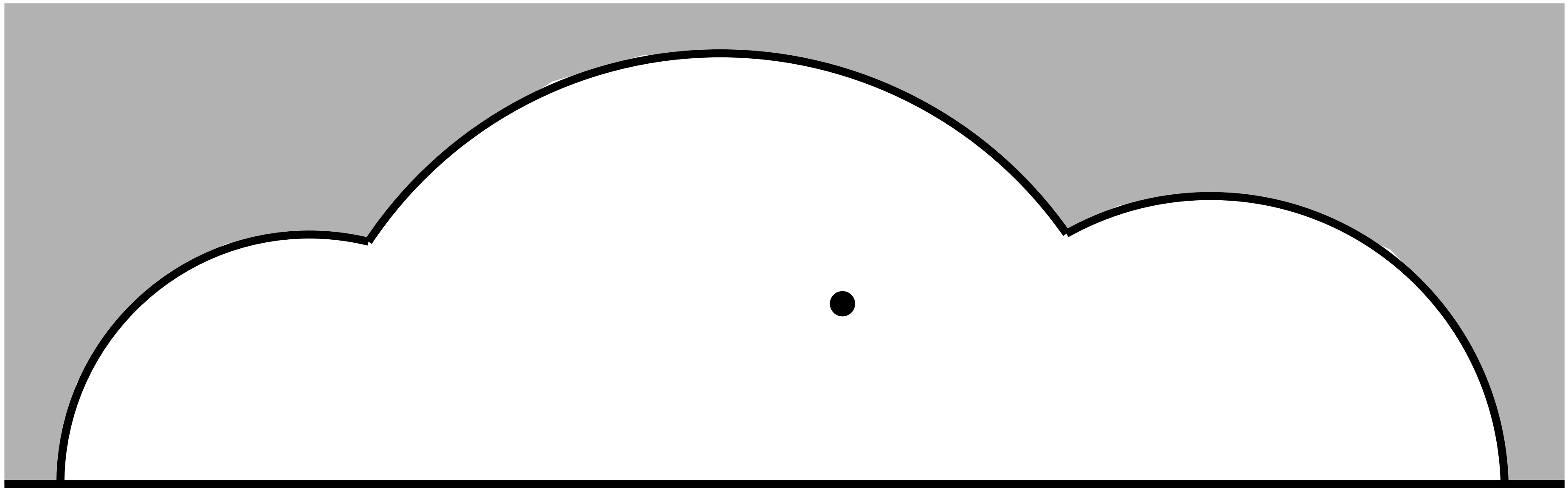
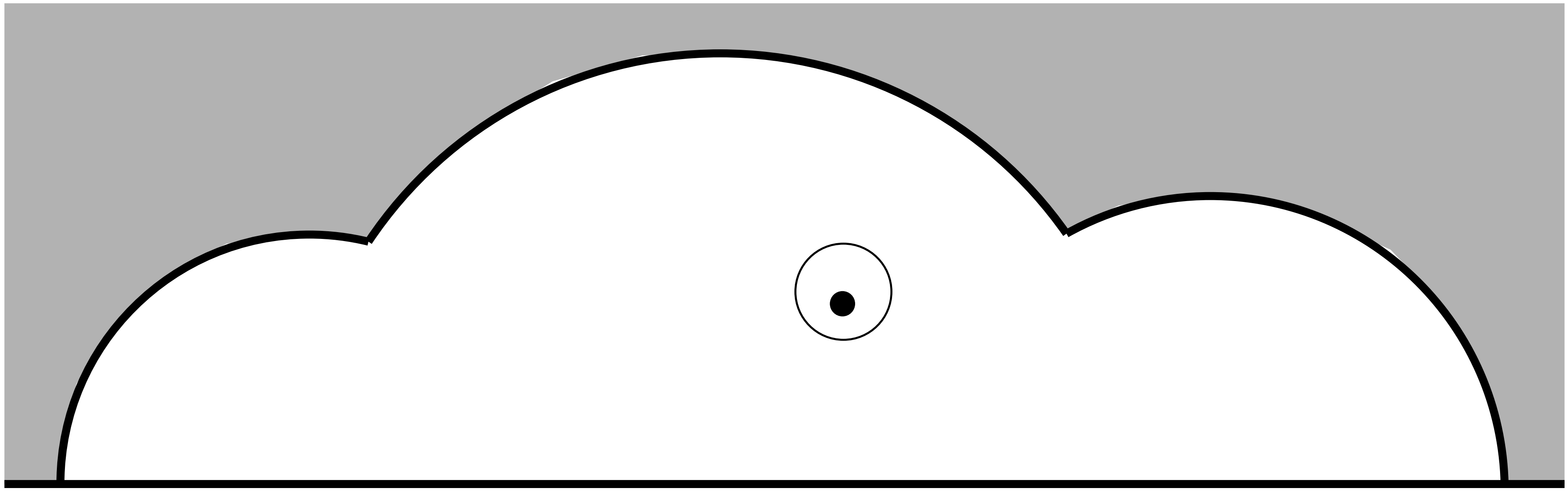
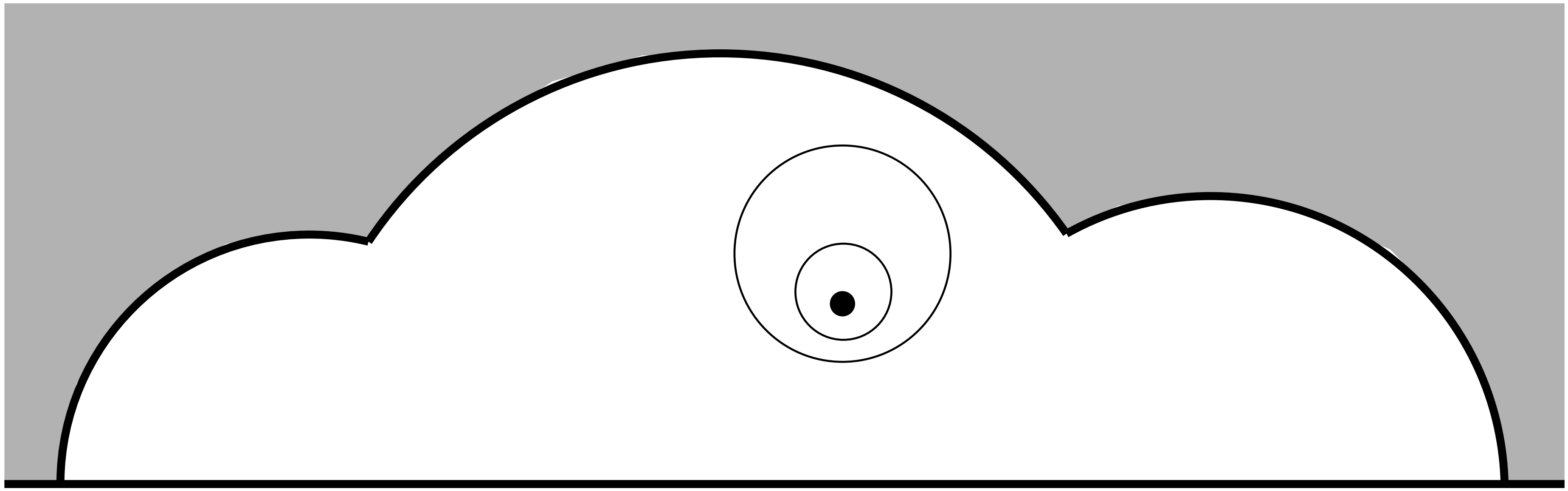
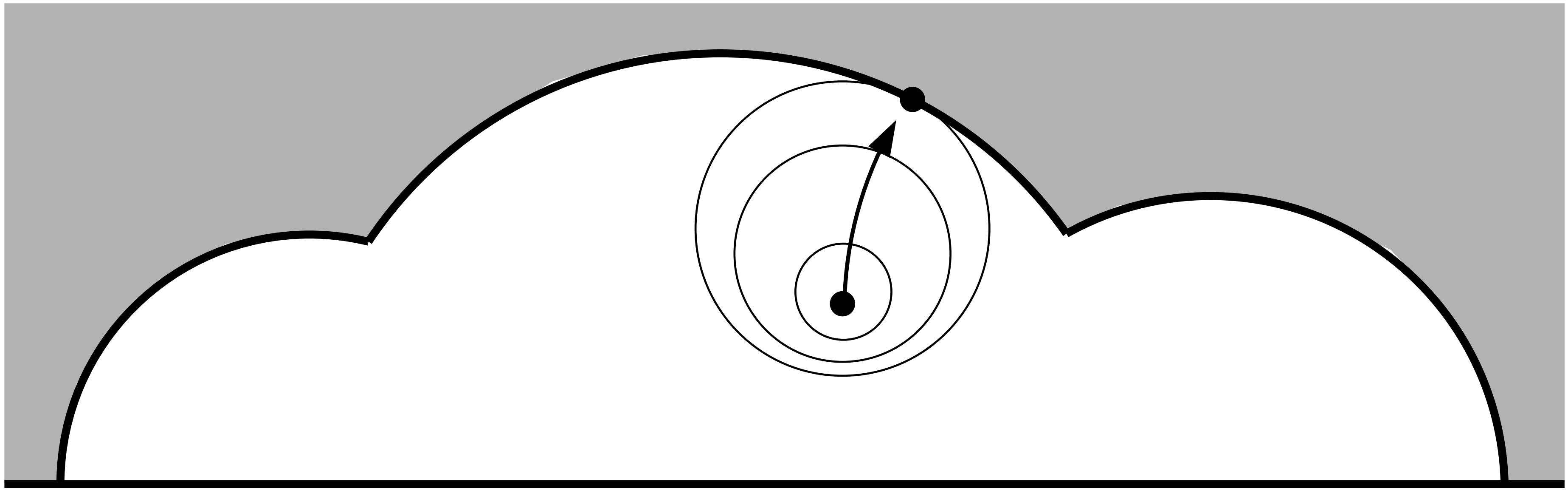
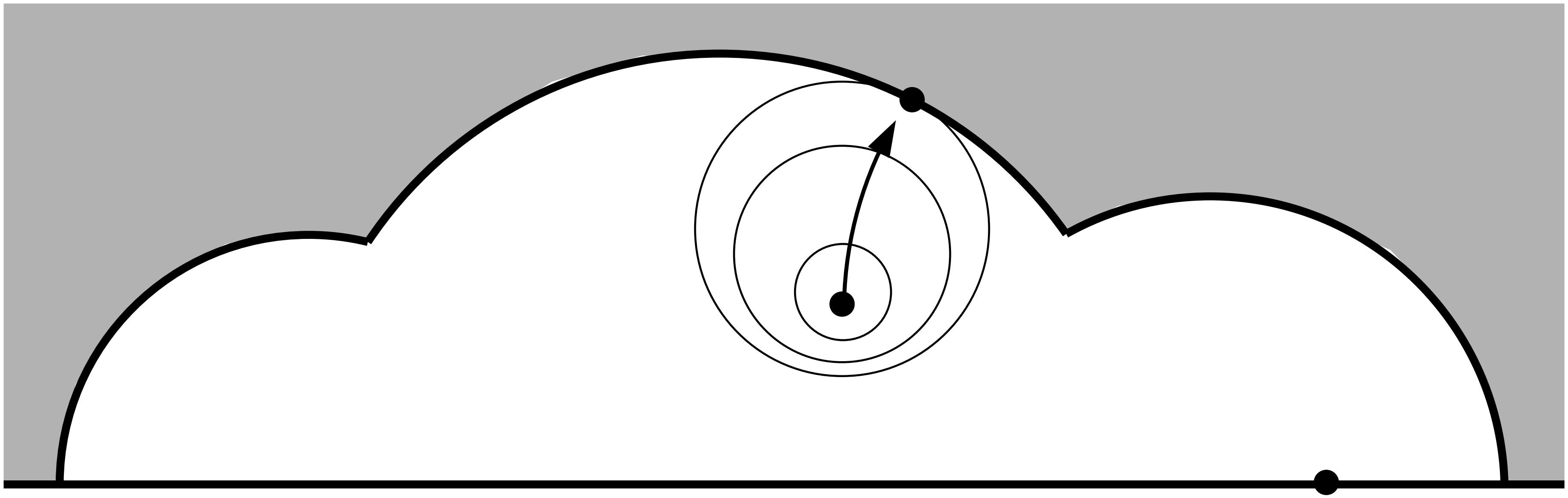


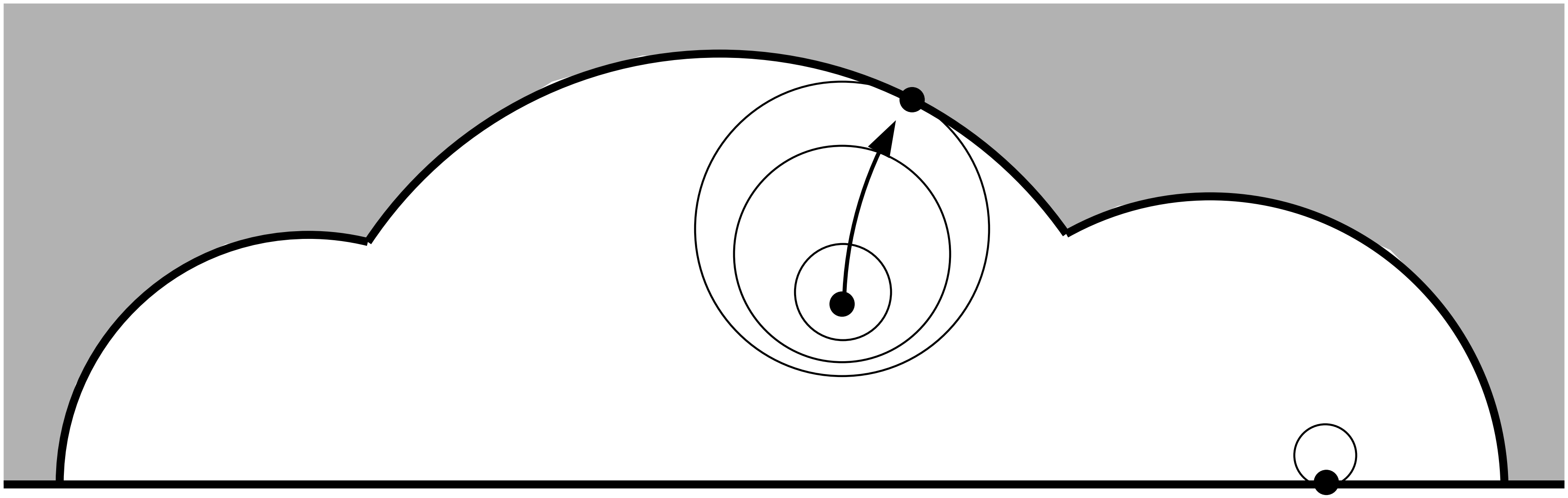


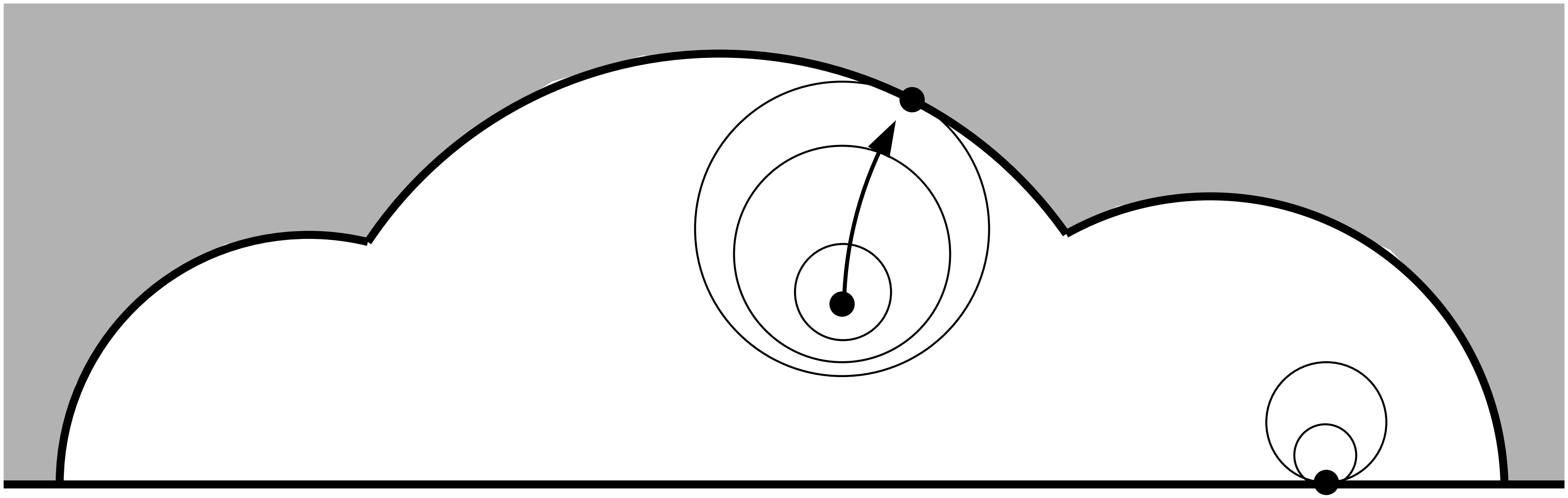


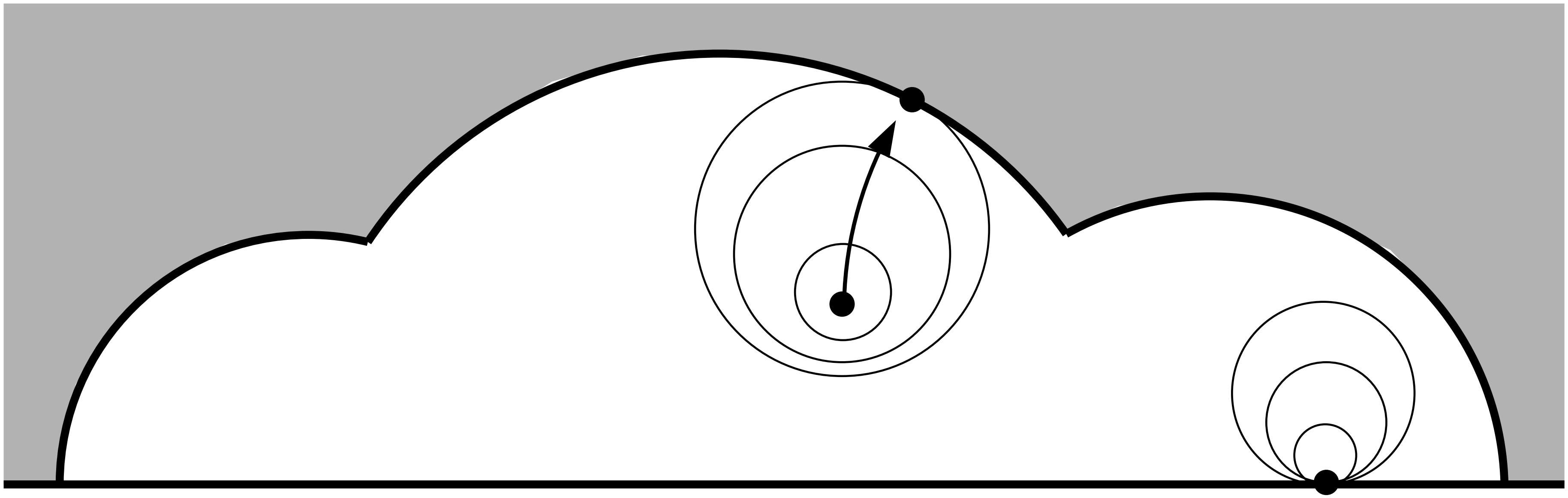


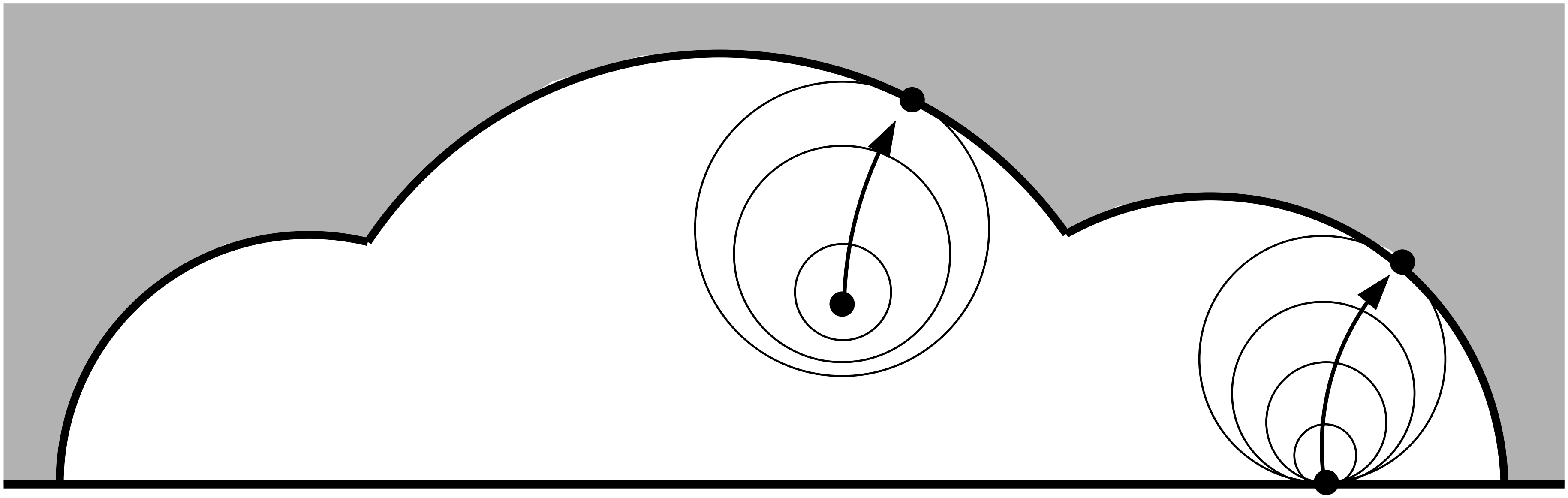


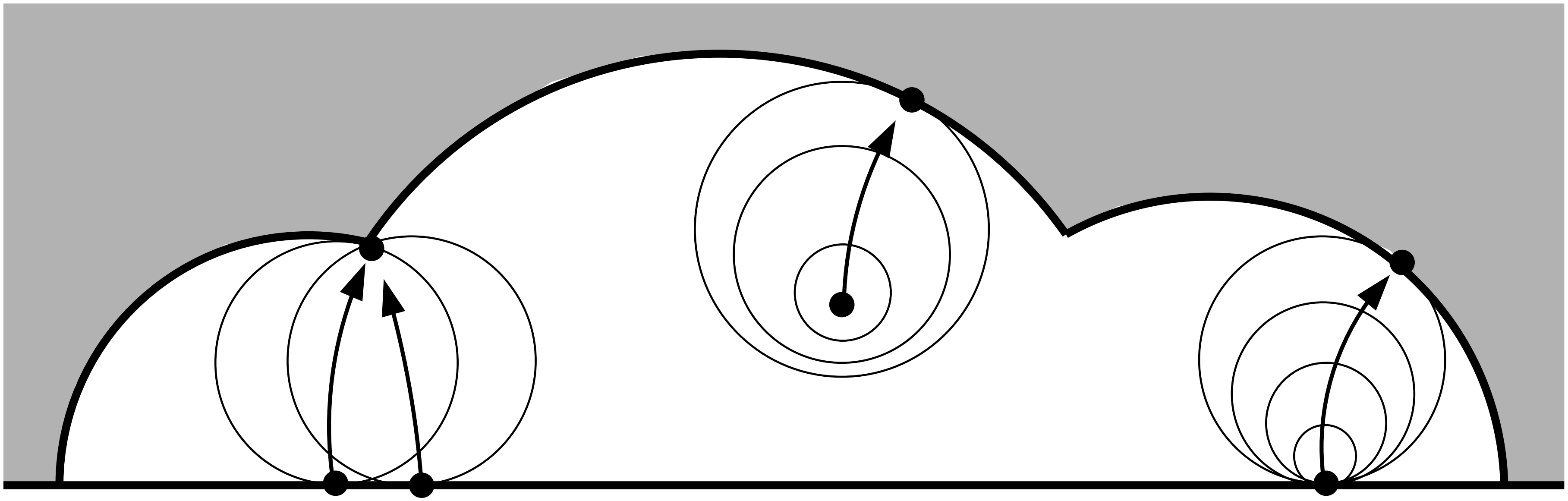


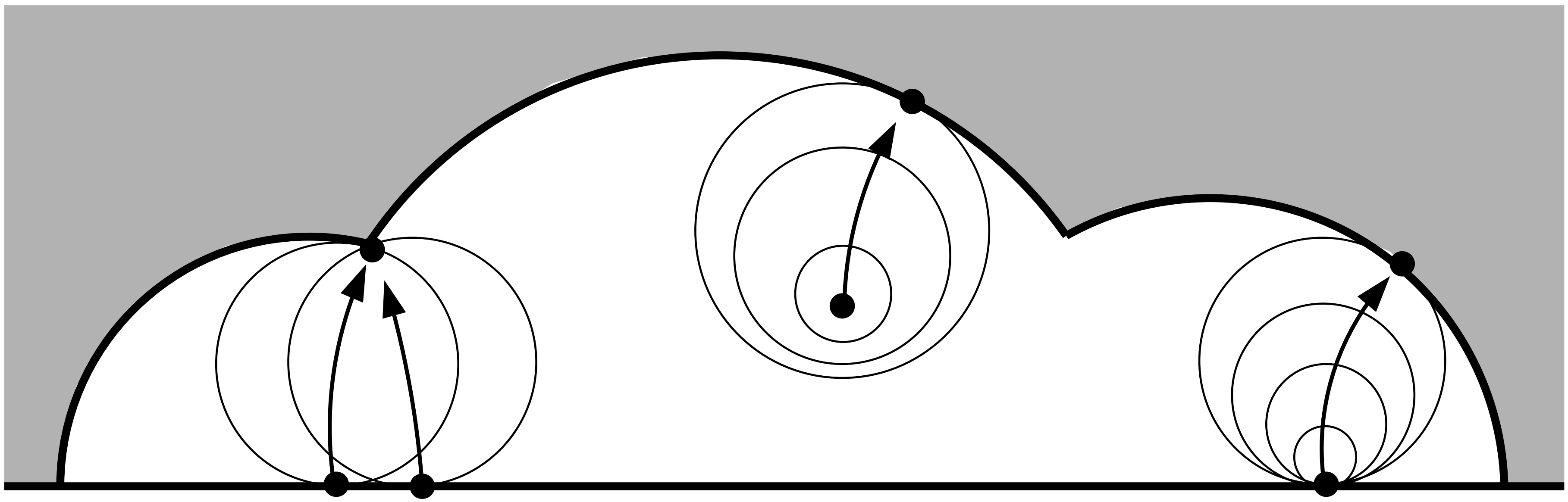












Nearest point retraction in hyperbolic space extends to map $R : \Omega \to S =$ Dome and is a quasi-isometry

$$\frac{1}{A}\rho_\Omega(x,y) - B \leq \rho_S(R(x),R(y)) \leq A\rho_\Omega(x,y).$$

(Sullivan, David Epstein and Al Marden, Bishop)

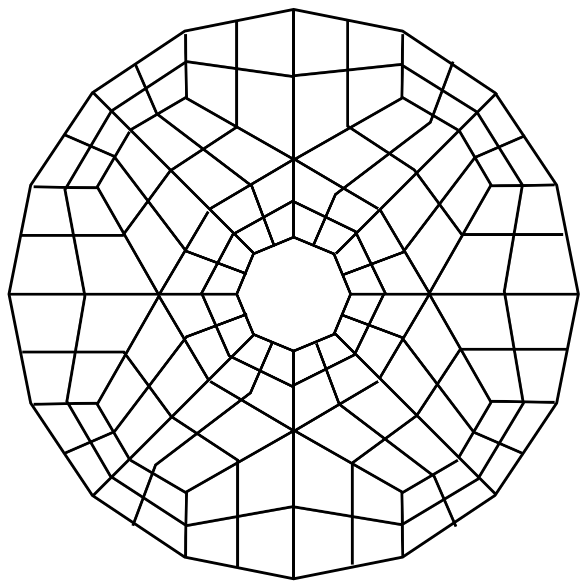
P

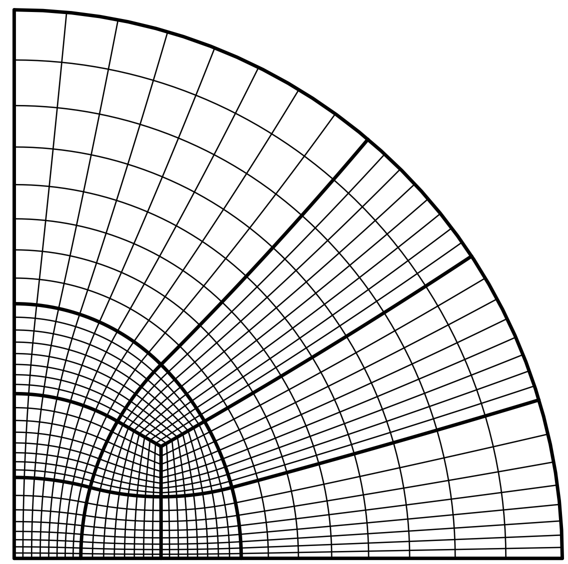
P^2

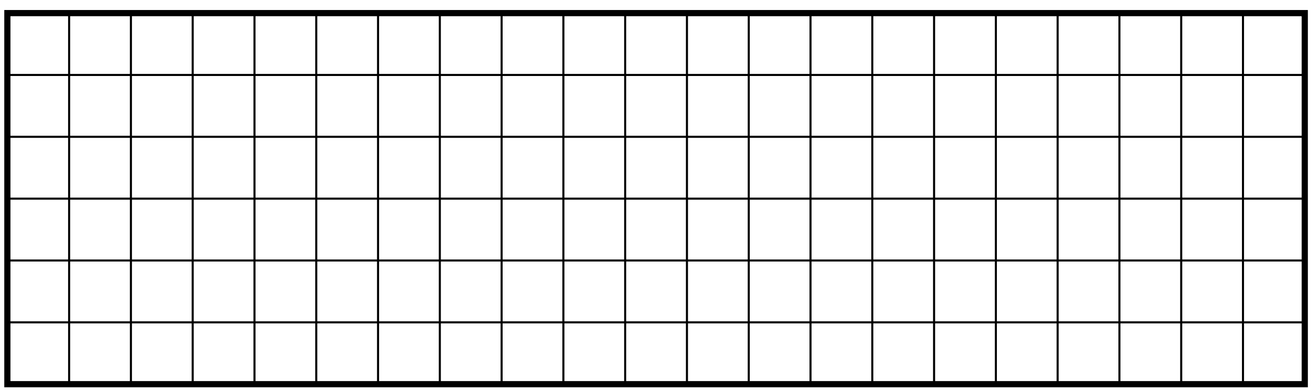
$P =$ hyperbolic geometry, University of Warwick

$P^2 =$ computational geometry, UC Irvine

Application: Quadrilateral meshes







- Every simple n-gon has $O(n)$ quad mesh with angles $\leq 120°$. Bern and Eppstein, 2000. $O(n \log n)$ work.
- Any quad mesh of hexagon has an angle $\geq 120°$.

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Theorem: Every n-gon has $O(n)$ quad mesh with all angles $\leq 120°$ and new angles $\geq 60°$. $O(n)$ work.

- Every simple n-gon has $O(n)$ quad mesh with angles $\leq 120^\circ$. Bern and Eppstein, 2000. $O(n \log n)$ work.

- Any quad mesh of hexagon has an angle $\geq 120^\circ$.

Theorem: Every n-gon has $O(n)$ quad mesh with all angles $\leq 120^\circ$ and new angles $\geq 60^\circ$. $O(n)$ work.

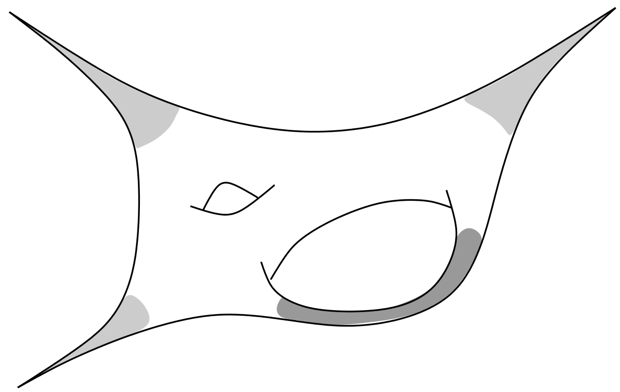
Angles bounds and complexity are sharp.

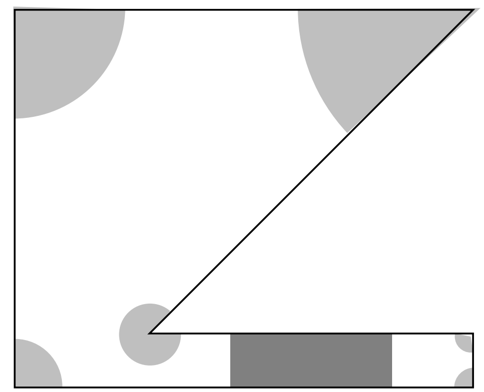
Idea of proof:

- Decompose polygon into thick and thin parts.
- Mesh thin parts by explicit construction
- Mesh thick parts using hyperbolic geometric in disk and conformal map to polygon.

Thick and Thin parts

An ϵ**-thin part** of a surface is a union of non-trivial loops of length $\leq \epsilon$ (parabolic/hyperbolic).

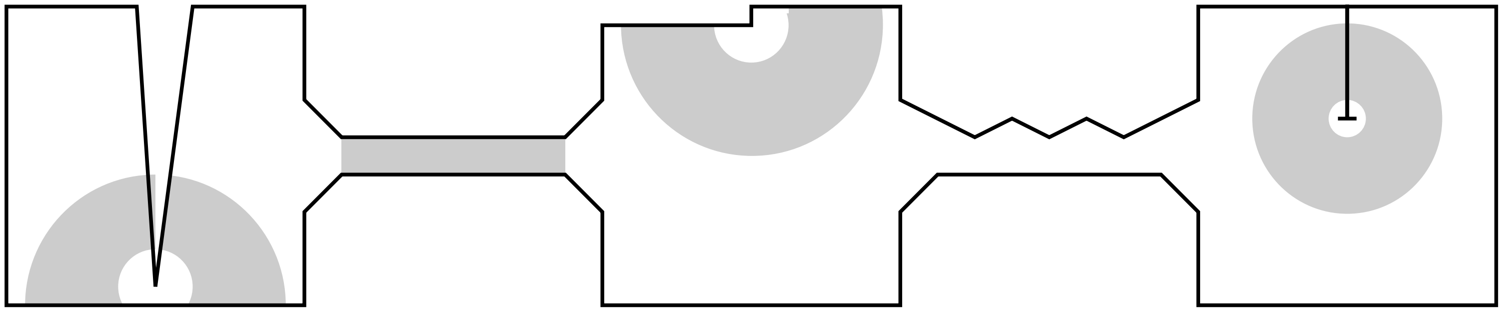




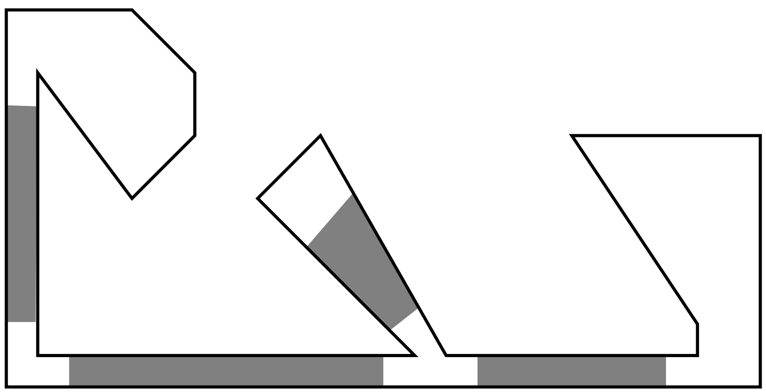
Thin piece is a sector whose two straight sides satisfy

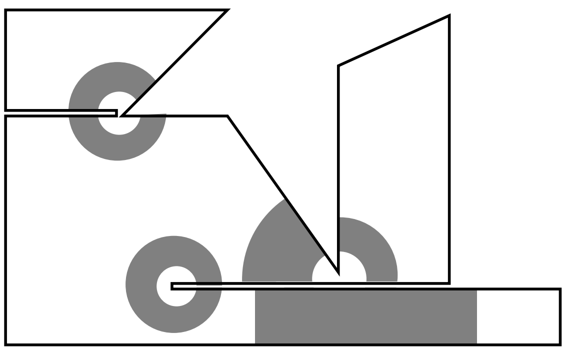
$$\mathrm{dist}(I, J) \ll \min(|I|, |J|).$$

Precise definition uses extremal distance between edges.



Right channel is not thin because edges on top have length comparable to width of channel.

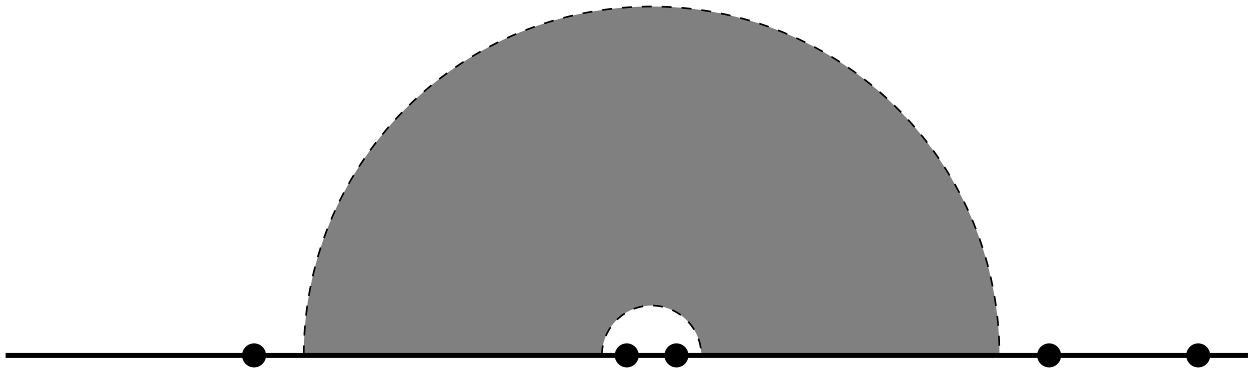




Thin parts computable in $O(n)$ using conformal map.

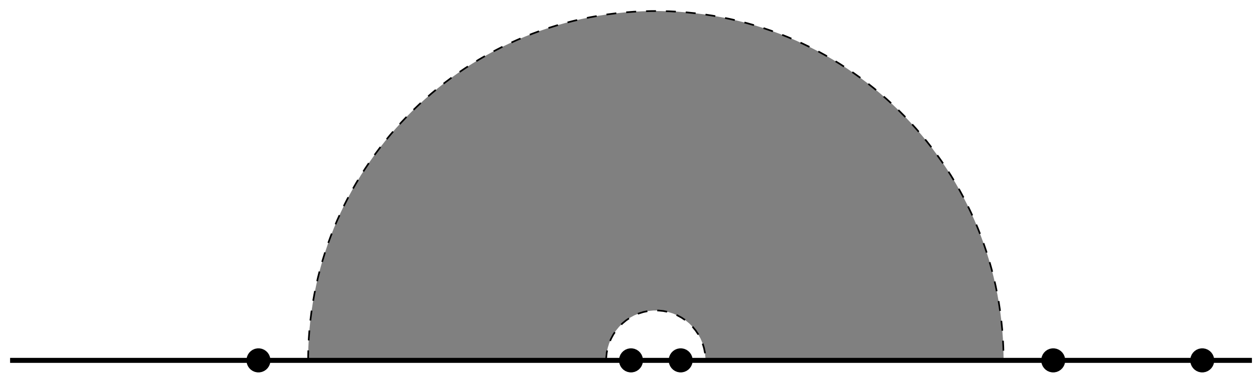
- Map polygon conformally to half-plane.
  Vertices map to points on line.

Thin parts computable in $O(n)$ using conformal map.



- Map polygon conformally to half-plane.
  Vertices map to points on line.
- Thin parts = wide annuli separating vertices.

Thin parts computable in $O(n)$ using conformal map.



- Map polygon conformally to half-plane.
  Vertices map to points on line.
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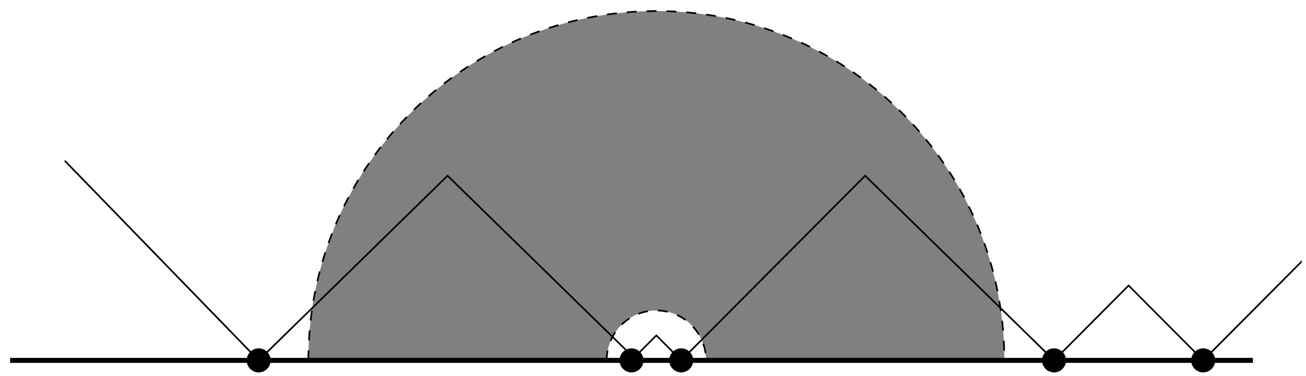
Find clusters so distances inside cluster are much smaller than connecting cluster to complement.

Thin parts computable in $O(n)$ using conformal map.



- Map polygon conformally to half-plane.
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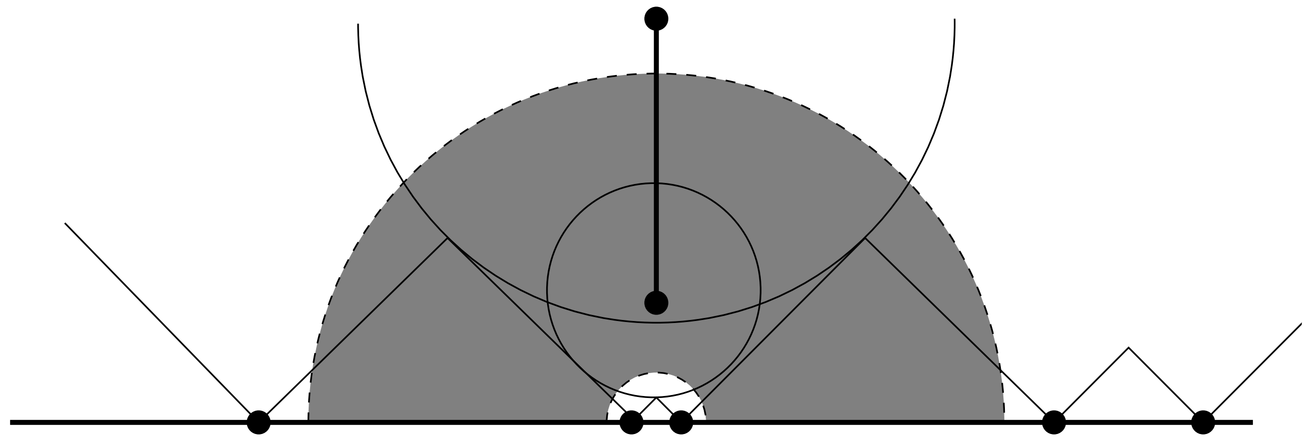
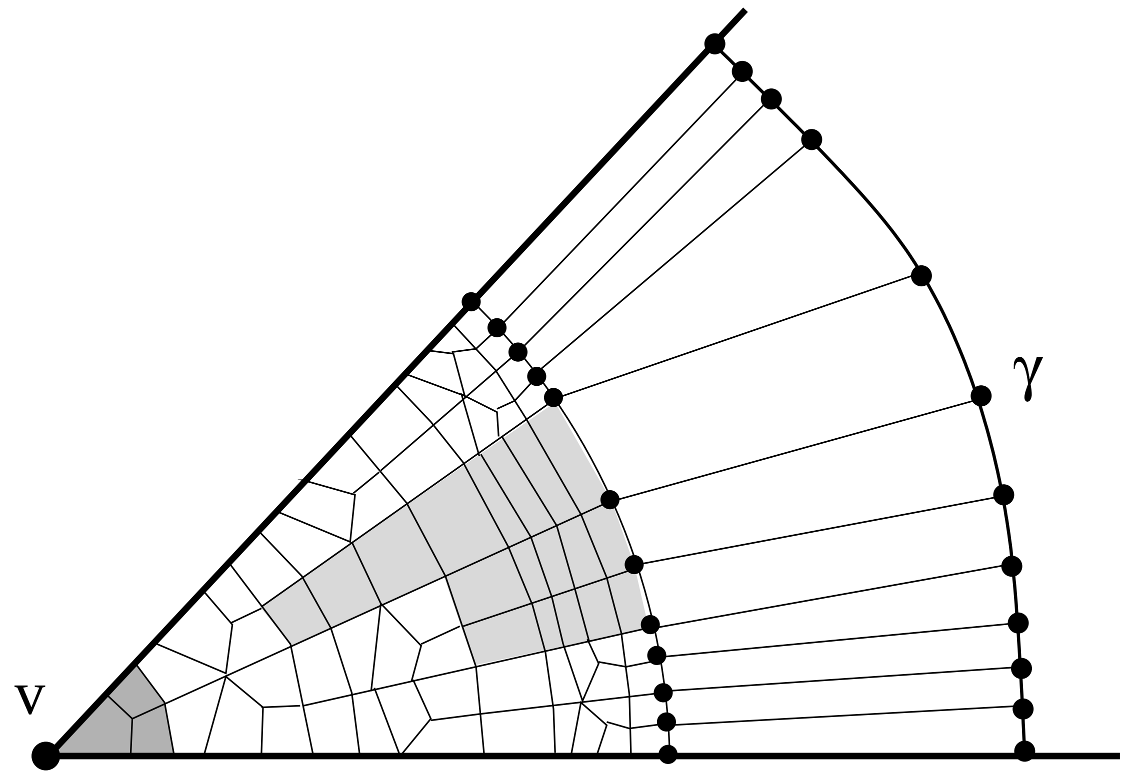
- Thin parts = wide annuli separating vertices.
- Draw sawtooth domain.

Thin parts computable in $O(n)$ using conformal map.

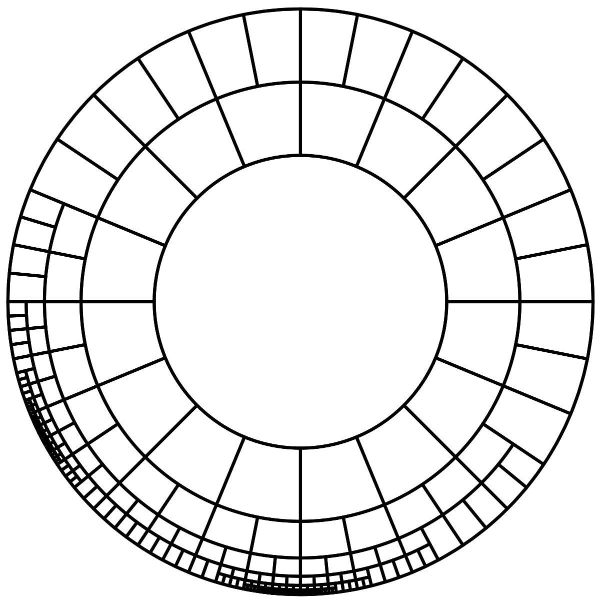


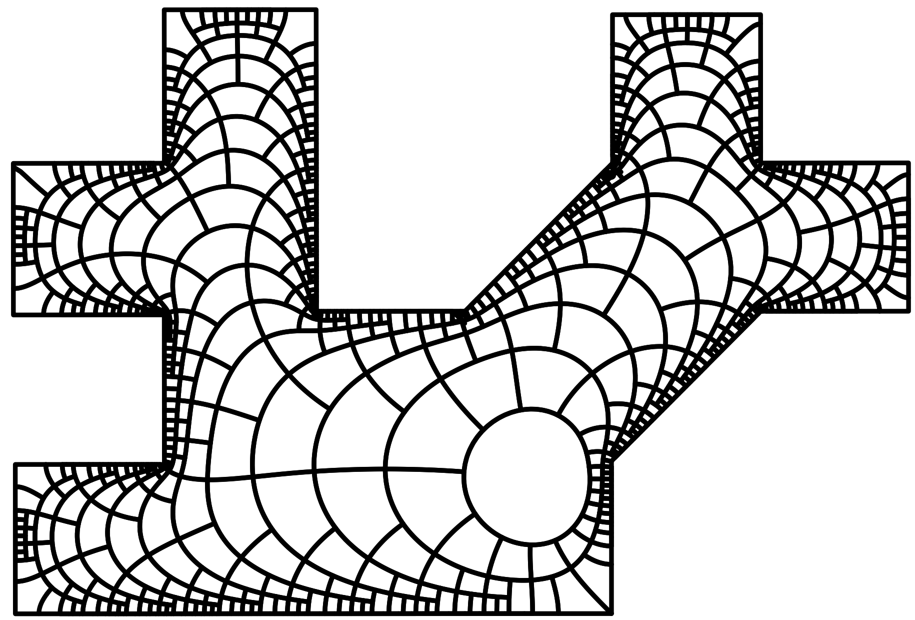
- Map polygon conformally to half-plane.
  Vertices map to points on line.
- Thin parts = wide annuli separating vertices.
- Draw sawtooth domain.
- Compute medial axis, find long vertical segments

Thin parts are meshed by explicit construction (easy).



Basic idea for meshing thick parts: Conformal map from disk preserves angles except near vertices.

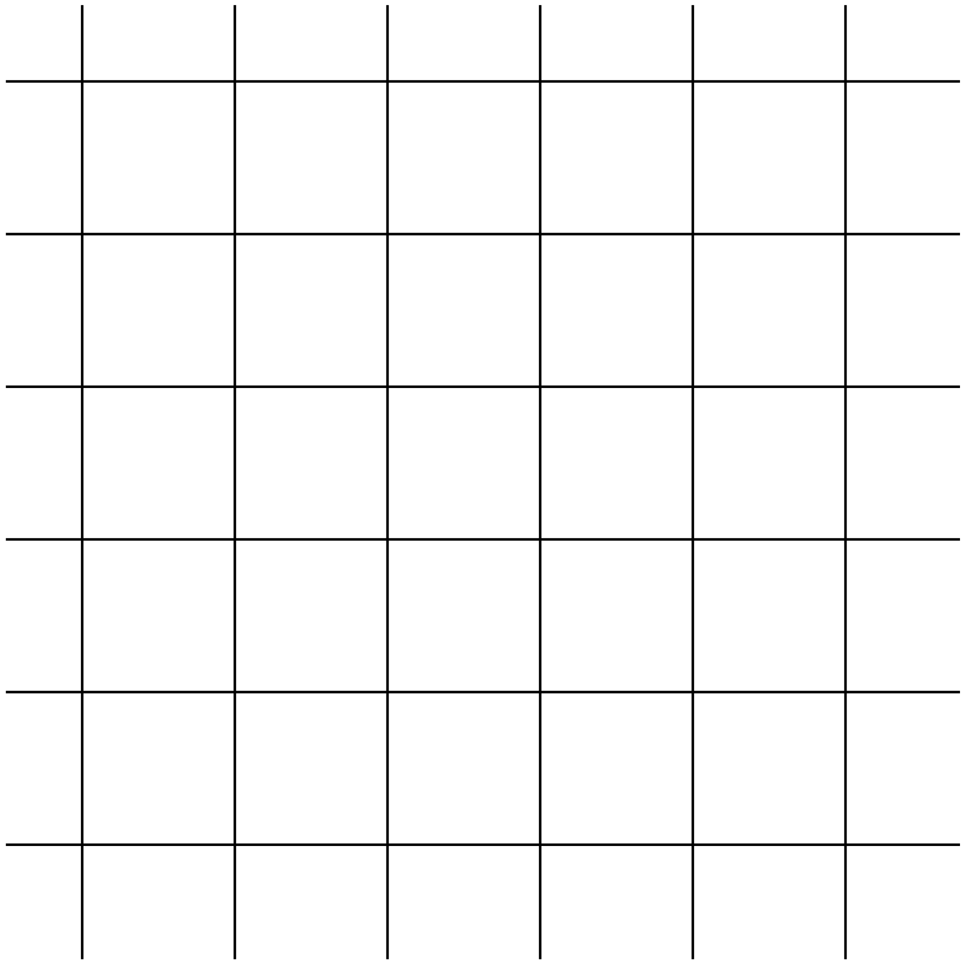




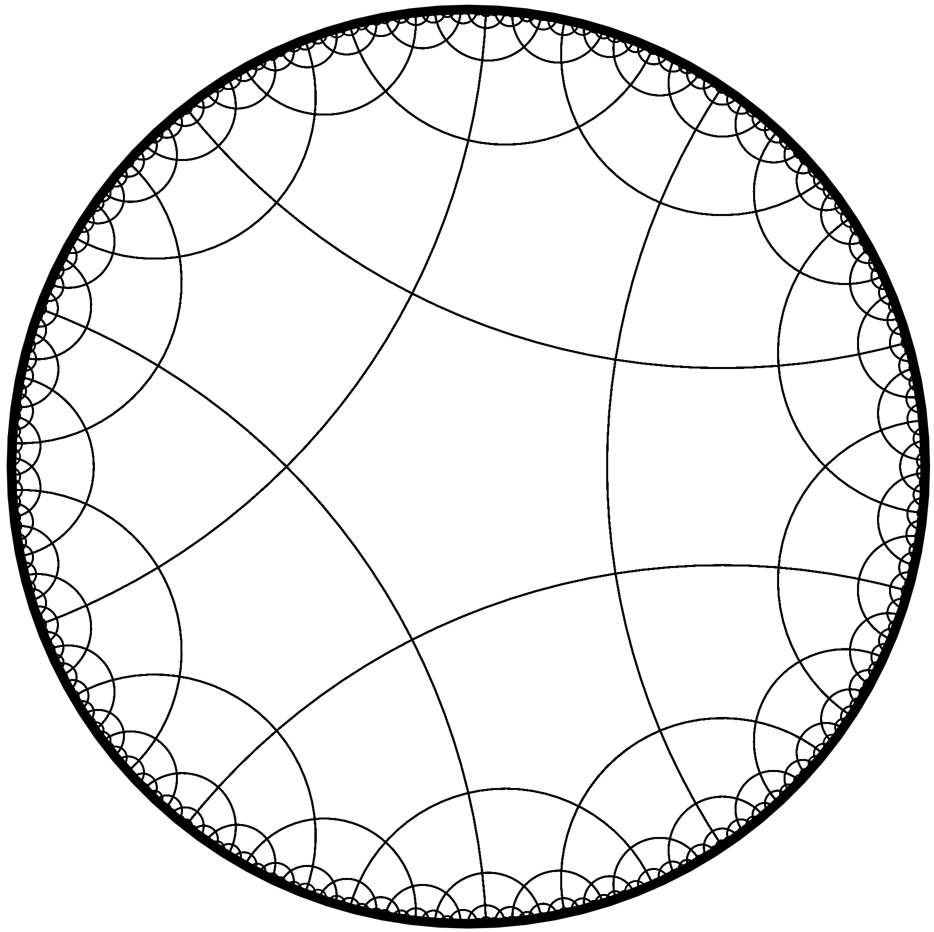
Transfer mesh on disk to mesh of polygon.

Need to be careful with tiles and timing.

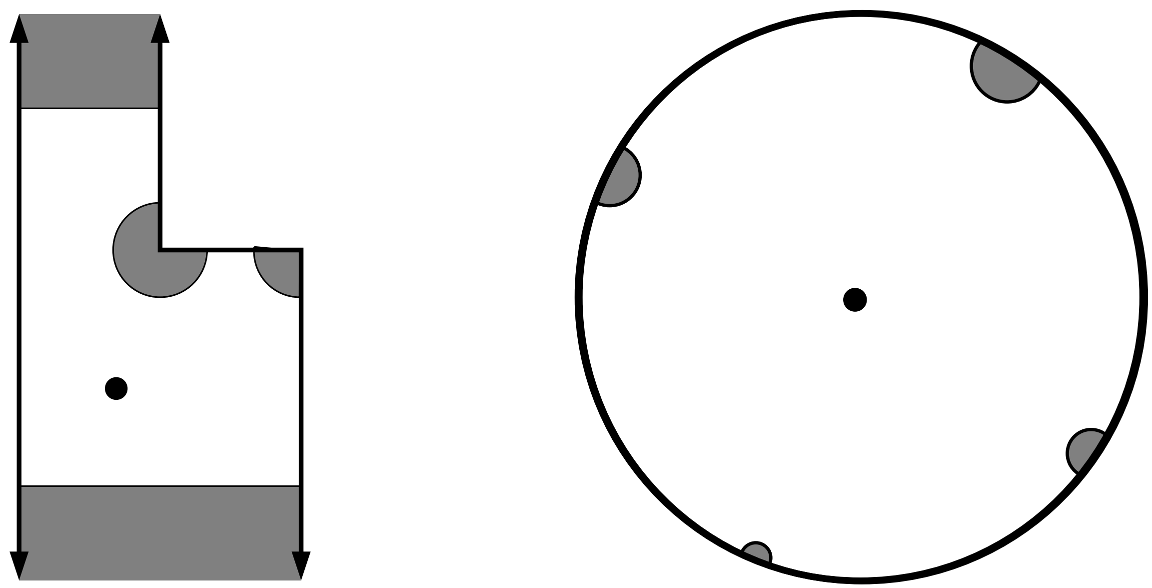
Euclidean plane can be tesselated by squares



Hyperbolic disk can be tesselated by right pentagons.



Conformal map from polygon to disk takes thick and thin parts to disk as shown.



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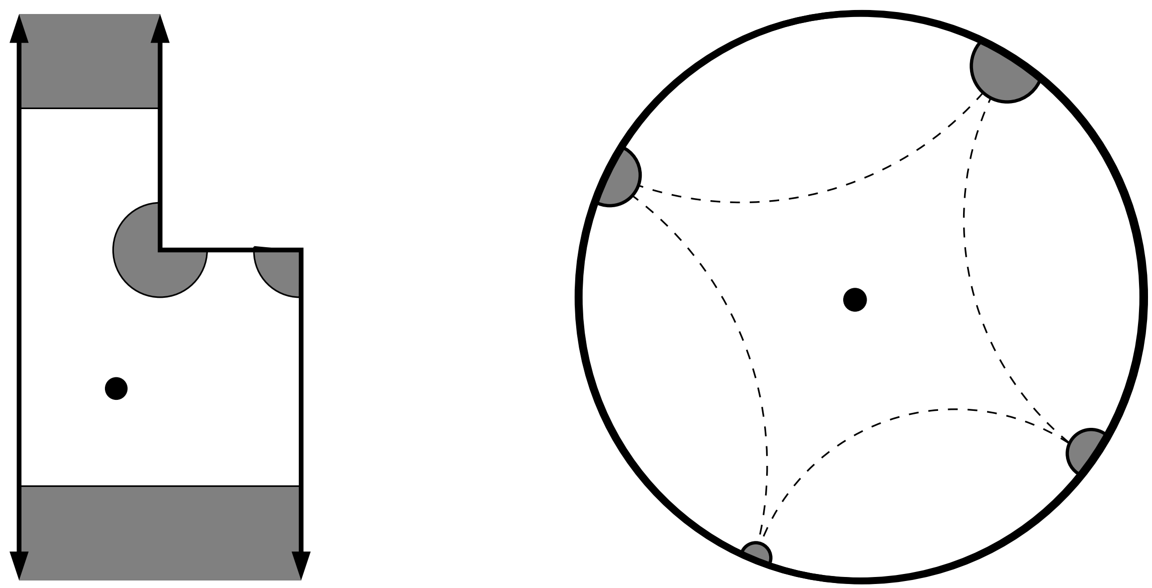
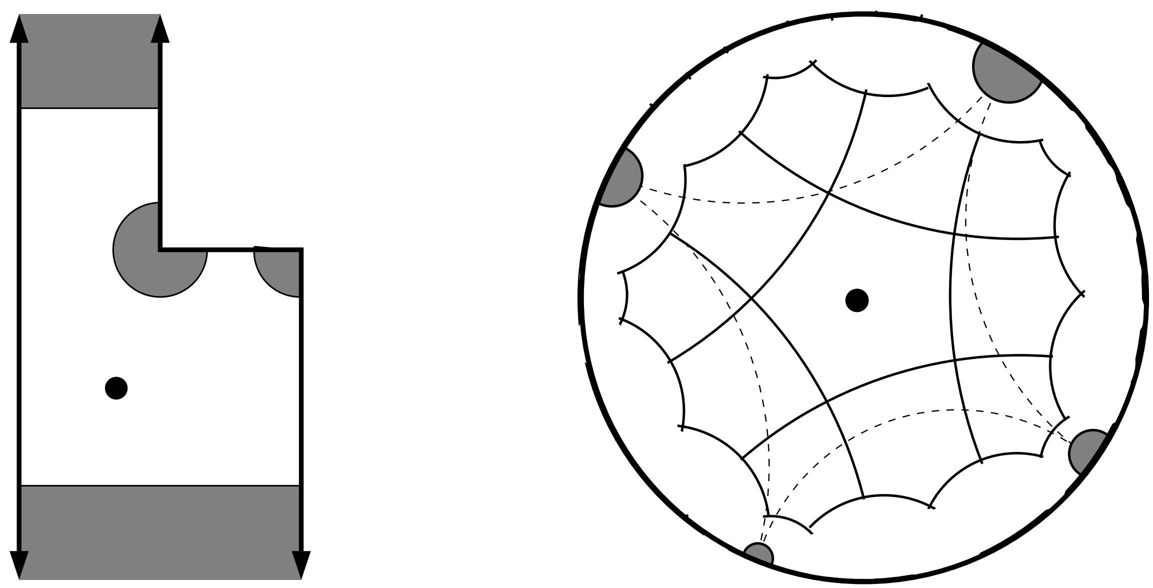


Draw (hyperbolic) convex hull of thin regions.

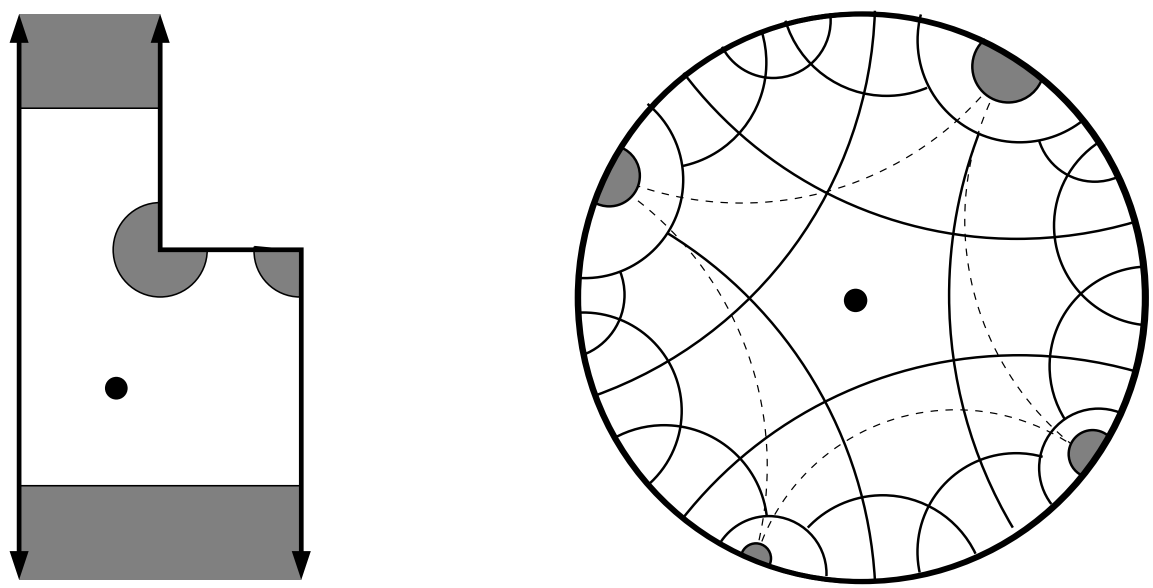
Conformal map from polygon to disk takes thick and thin parts to disk as shown.



Draw (hyperbolic) convex hull of thin regions.

Take pentagons from tesselation hitting convex hull but missing thin parts.

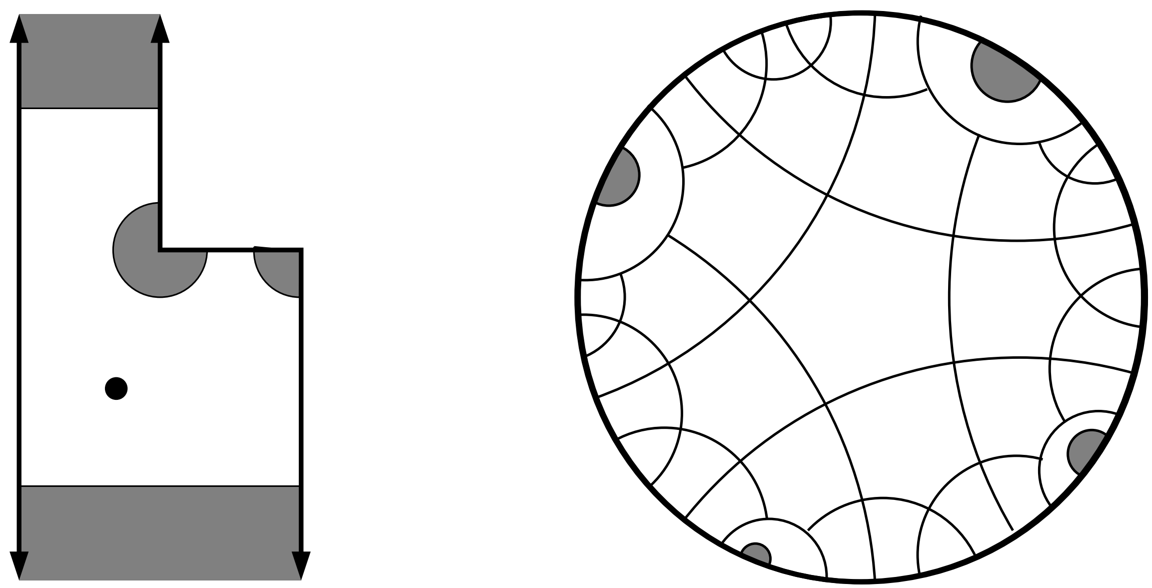
Conformal map from polygon to disk takes thick and thin parts to disk as shown.



Draw (hyperbolic) convex hull of thin regions.

Take pentagons from tesselation hitting convex hull but missing thin parts. Extend pentagon edges to boundary.

Conformal map from polygon to disk takes thick and thin parts to disk as shown.

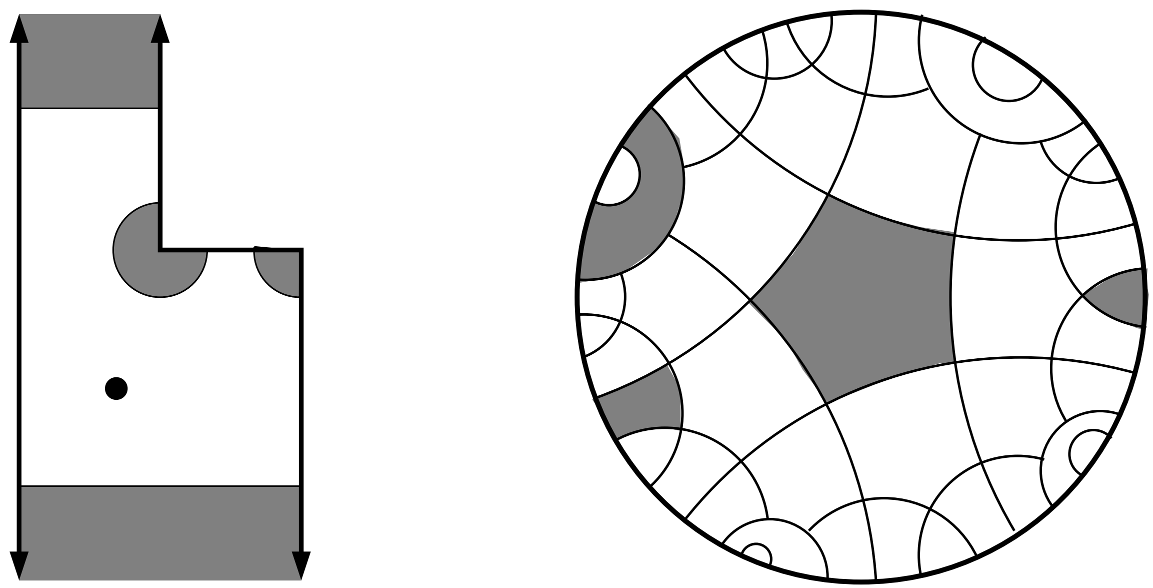


Draw (hyperbolic) convex hull of thin regions.

Take pentagons from tesselation hitting convex hull but missing thin parts. Extend pentagon edges to boundary.

Analog of Whitney or quadtreee construction.

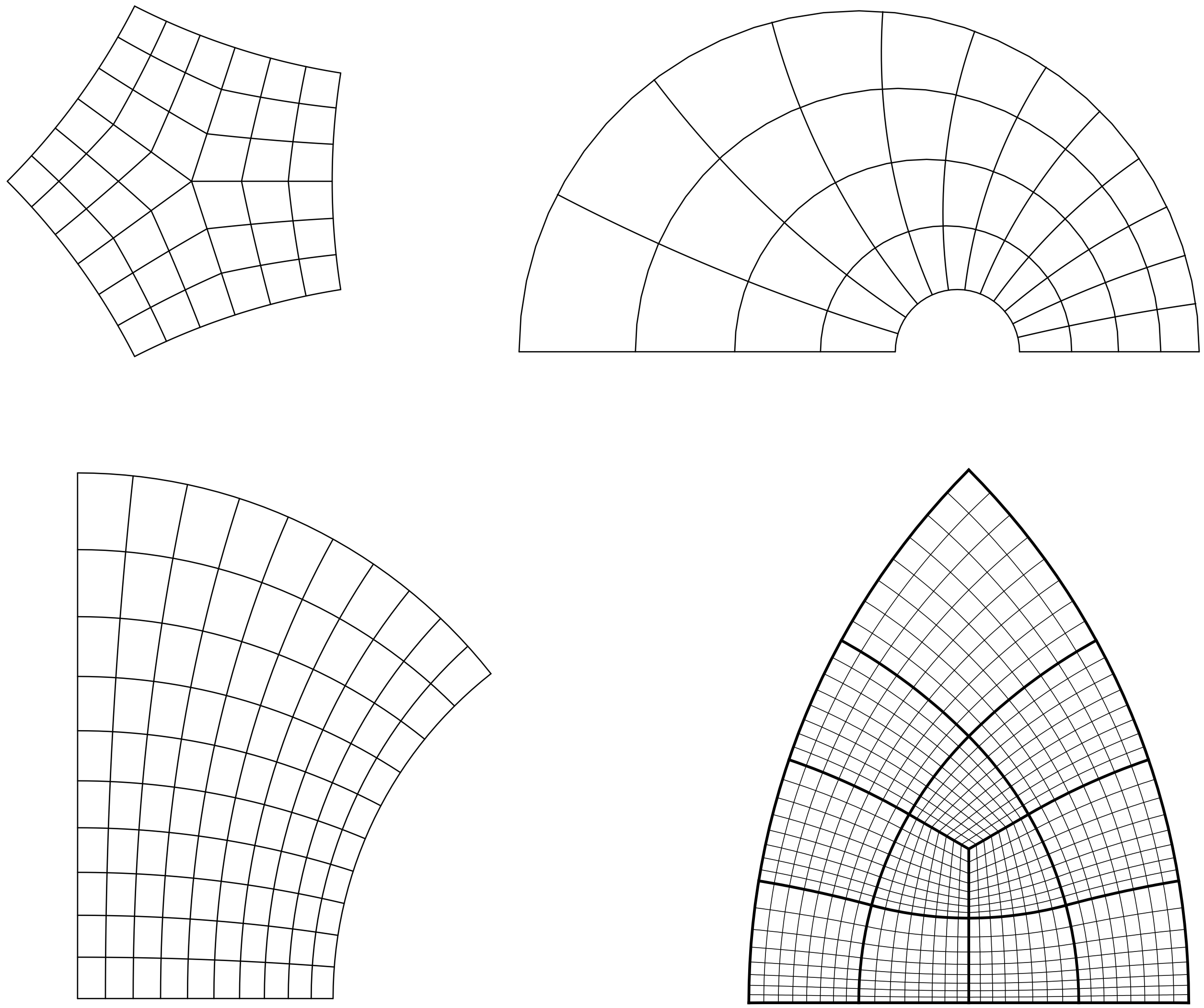
Conformal map from polygon to disk takes thick and thin parts to disk as shown.



Draw (hyperbolic) convex hull of thin regions.

Take pentagons from tesselation hitting convex hull but missing thin parts. Extend pentagon edges to boundary.

Pentagons, quadrilaterals, triangles and half-annuli.

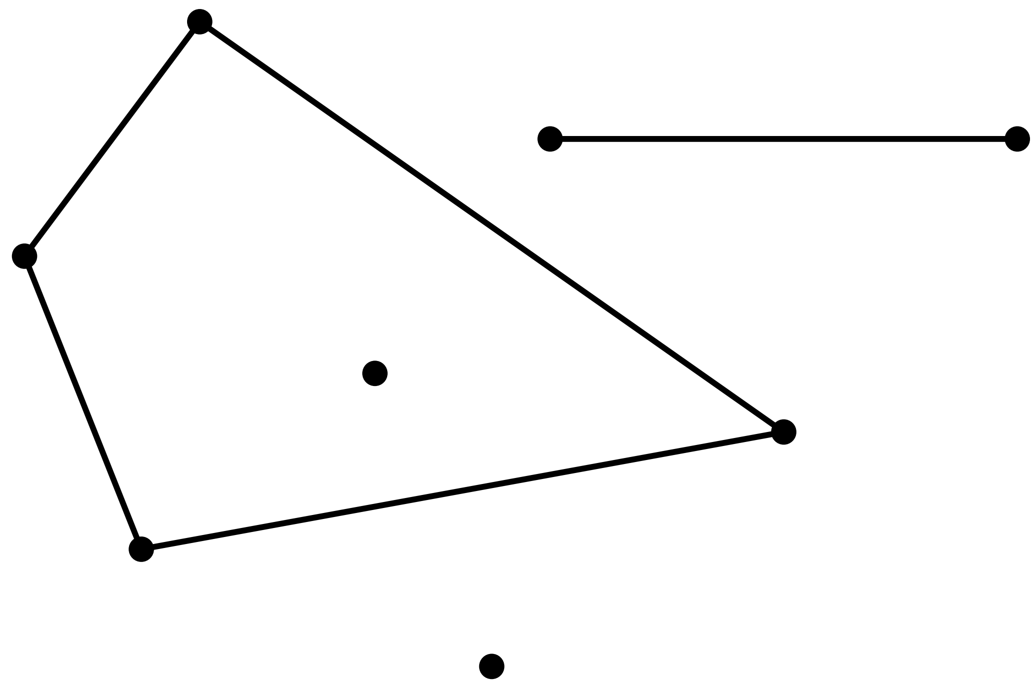


Shapes can be meshed to match along common edges.

This completes sketch of quad meshing of polygons.

Theorem can be extended from polygons to PSLGs.

A **Planar Straight Line Graph** (PSLG) is a finite point set plus a set of disjoint edges between them.



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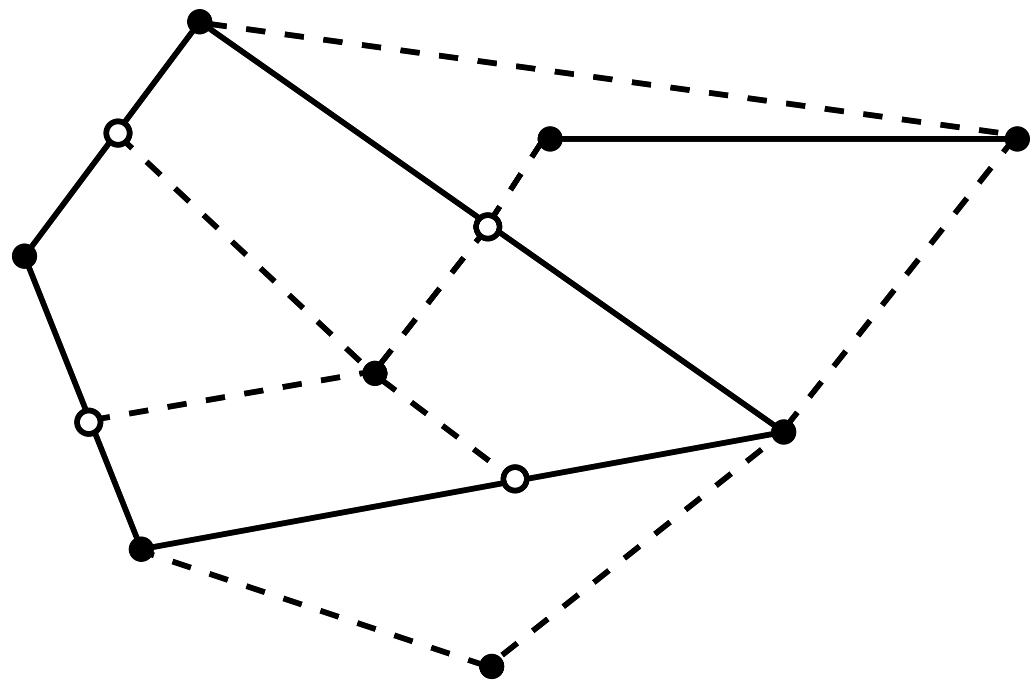
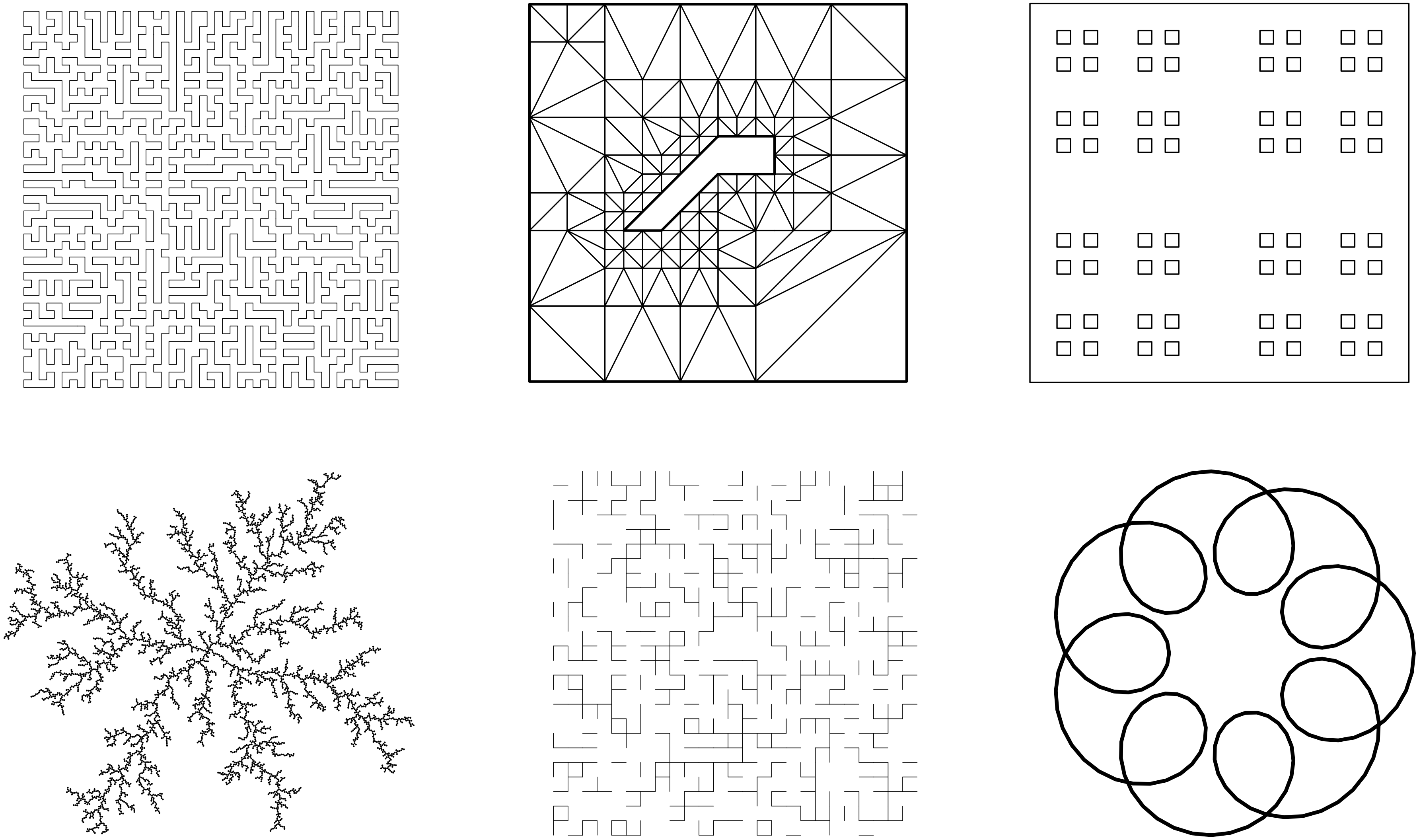


Mesh must **cover** the edges of the PSLG.

May be necessary to add Steiner points.

Fills convex hull.



More PSLGs

Theorem (B, 2010): Every PSLG has a quadrilateral mesh with $O(n^2)$ elements, all angles less than 120° and all new angles greater than 60°.

Theorem (B, 2010): Every PSLG has a quadrilateral mesh with $O(n^2)$ elements, all angles less than $120°$ and all new angles greater than $60°$.

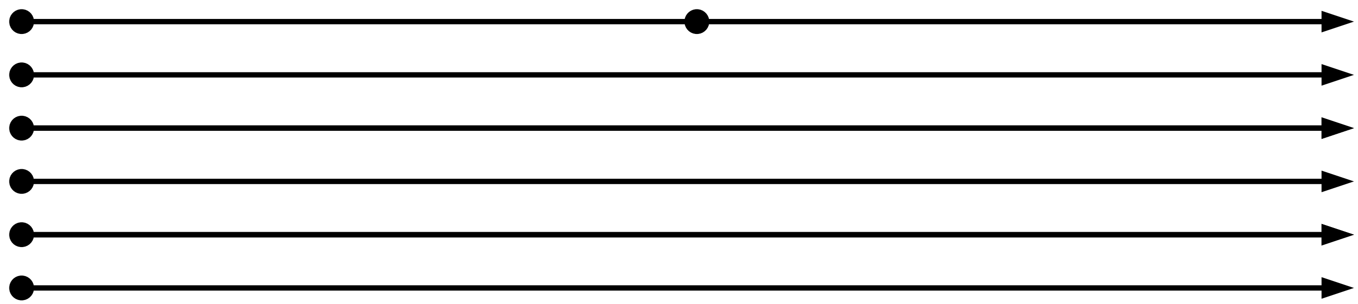
Angles and complexity sharp.

All but $O(n)$ vertices have angles in $[89°, 91°]$.

All but $O(n)$ vertices are degree 4.

Mesh can be decomposed into $O(n)$ sub-meshes, each equivalent to a rectangular grid. (Motorcycle graphs, Eppstein, Goodrich, Kim, Tamstorf)

Any bound $\theta < 180^\circ$ sometimes requires n^2 vertices.



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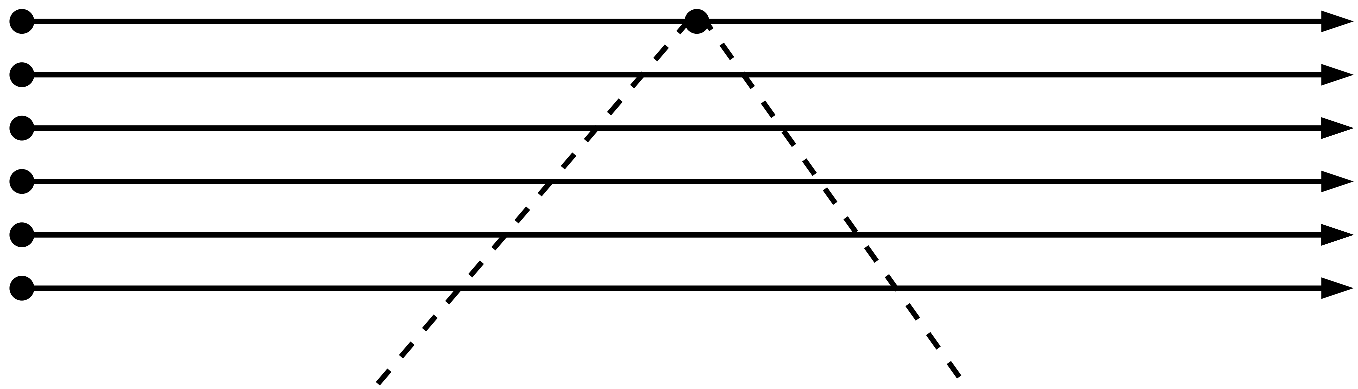

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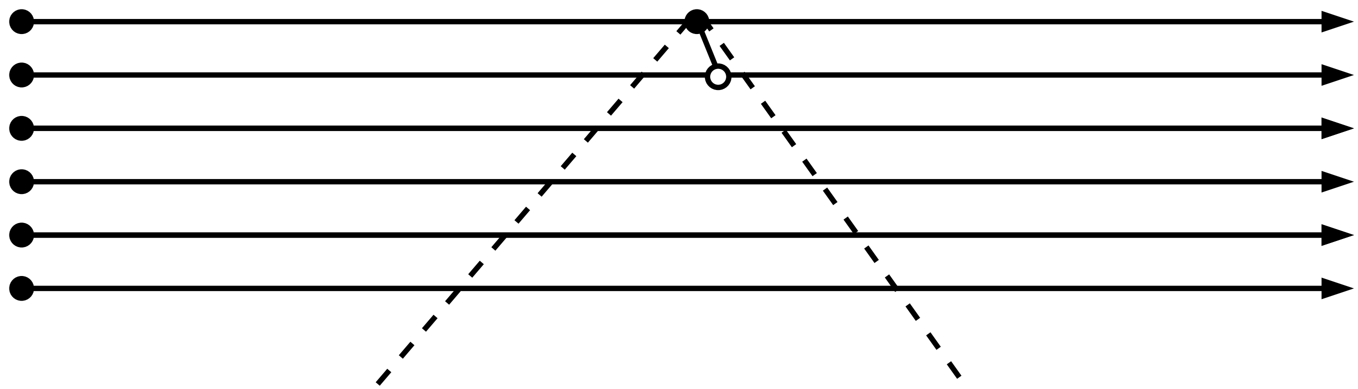

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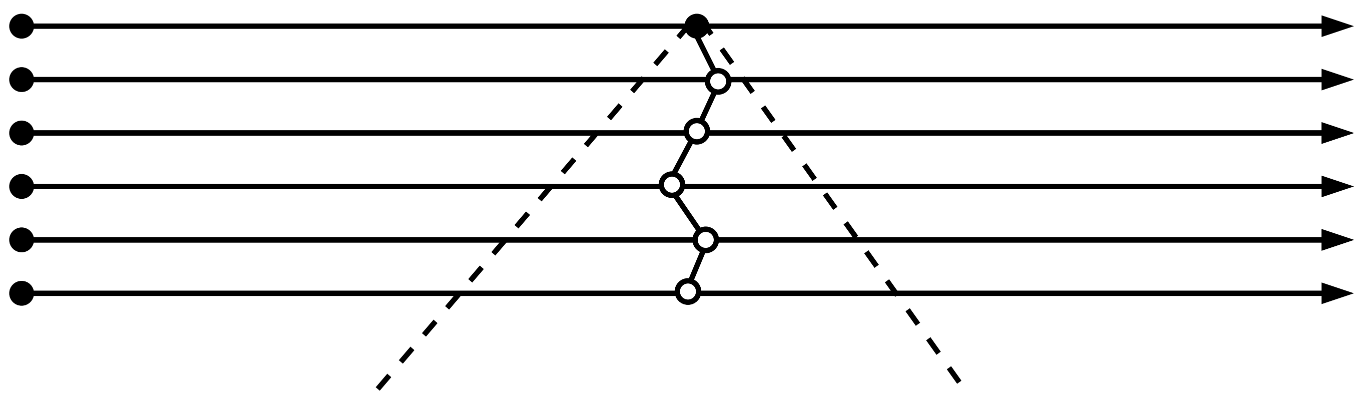

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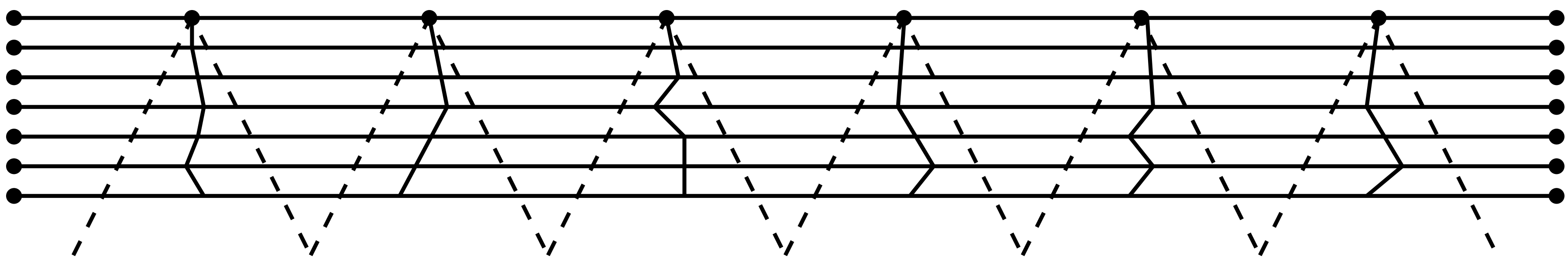

Proof quad meshing for PSLGs requires new ideas:

- connect PSLG without small angles
- foliation of thin parts
- propagation paths
- bending paths
- traps
- sinks

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Convert quadrilaterals to triangles by adding diagonals.

Corollary: Every PSLG has a $O(n^2)$ triangulation with maximum angle $\leq 120°$.

Compare S. Mitchell 1993 ($157.5°$) and Tan 1996 ($132°$).

Can the 120° bound be improved for triangulations?

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Can we get a lower angle bound?

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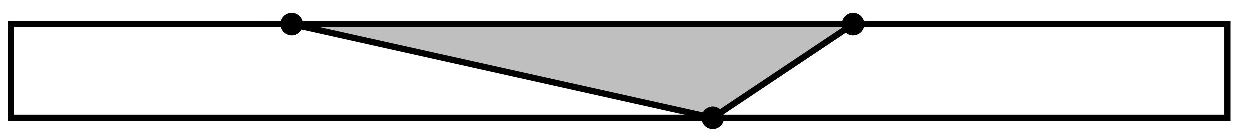
Can we get a lower angle bound?

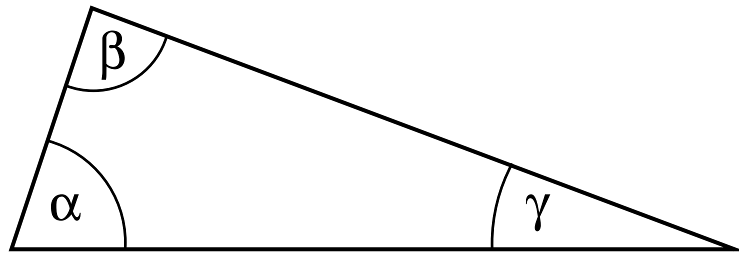
No lower angle bound. For $1 \times R$ rectangle

$$\text{number of triangles} \gtrsim R \times (\text{smallest angle})$$



So uniform complexity $\Rightarrow$ no lower angle bound.

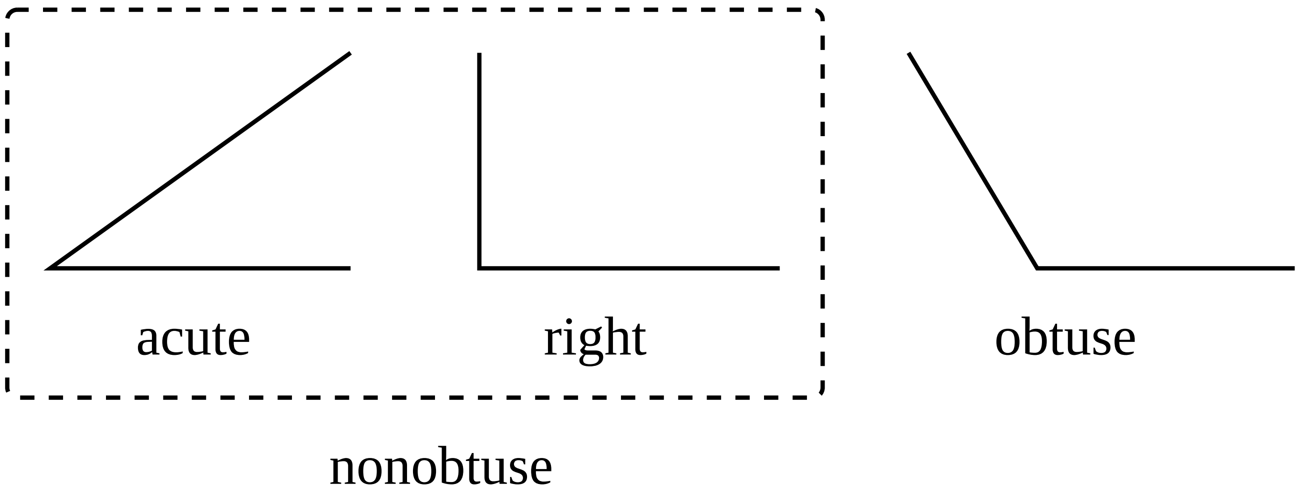
No upper bound $< 90°$**:**



If all angles are $\leq 90° - \epsilon$ then all angles are $\geq 2\epsilon$.

$$\gamma = 180 - \alpha - \beta \geq 180 - (90 - \epsilon) - (90 - \epsilon) \geq 2\epsilon.$$

So **nonobtuse** triangulation is best we can hope for.



Brief history of nonobtuse triangulation:

- Always possible: Burago, Zalgaller 1960.
- Rediscovered: Baker, Grosse, Rafferty, 1988.
- $O(n)$ for points sets: Bern, Eppstein, Gilbert 1990
- $O(n^2)$ for polygons: Bern, Eppstein 1991
- $O(n)$ for polygons: Bern, Mitchell, Ruppert 1994

Numerous applications, heuristics.

Is there a polynomial bound for PSLGs?

Theorem (B, 2010): Every PSLG has a nonobtuse triangulation with $O(n^{2.5})$ elements.

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Cor (of proof): Every PSLG has a triangulation with all angles $\leq 90° + \epsilon$ and $O(n^2/\epsilon^2)$ elements.

Theorem (B, 2010): Every PSLG has a nonobtuse triangulation with $O(n^{2.5})$ elements.

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Cor: Any PSLG has a conforming Delaunay triangulation of size $O(n^{2.5})$. (see Edelsbrunner, Tan 1993)

Theorem (B, 2010): Every PSLG has a nonobtuse triangulation with $O(n^{2.5})$ elements.

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Cor: Any PSLG has a conforming Delaunay triangulation of size $O(n^{2.5})$. (see Edelsbrunner, Tan 1993)

Cor: Any triangulation of a simple n-gon has an acute refinement of size $O(n^2)$. (see Bern, Eppstein 1992)

Theorem (B, 2010): Every PSLG has a nonobtuse triangulation with $O(n^{2.5})$ elements.

Cor (of proof): Every PSLG has a triangulation with all angles $\leq 90^\circ + \epsilon$ and $O(n^2/\epsilon^2)$ elements.

Cor: Any PSLG has a conforming Delaunay triangulation of size $O(n^{2.5})$. (see Edelsbrunner, Tan 1993)

Cor: Any triangulation of a simple n-gon has an acute refinement of size $O(n^2)$. (see Bern, Eppstein 1992)

Proofs have no conformal maps or hyperbolic geometry.

Summary:

• Advantagous to combine Euclidean and hyperbolic viewpoints.

• An ϵ-conformal map onto an n-gon can be computed in time $O(n \log \epsilon \log \log \epsilon)$.

• Any polygon has a quad mesh with $O(n)$ elements and all angles $\leq 120°$ and all new angles $\geq 60°$.

• Any PSLG has a quad mesh with $O(n^2)$ elements and all angles $\leq 120°$ and all new angles $\geq 60°$.

• Any PSLG can be triangulated with $O(n^{2.5})$ elements and all angles $\leq 90°$.

Questions:

- Implementable?
- Average versus worst case bounds?
- Can we replace 2.5 by 2?
- 3-D meshes? The eightfold way? Ricci flow?
- Other applications for thick/thin pieces?
- Applications of Mumford-Bers compactness?
- Finite precision version of mapping theorem?
- Best k for medial axis flow? $(\frac{1}{3} < k < \frac{7}{8})$
- Can we do better than medial axis flow?
- $\frac{1}{3}$ + contracting $\Rightarrow$ Brennan's Conj. $\Rightarrow$ Fields medal.