Nonobtuse Triangulation of PSLGs

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What is a triangulation of a point set?

What is a triangulation of a PSLG?

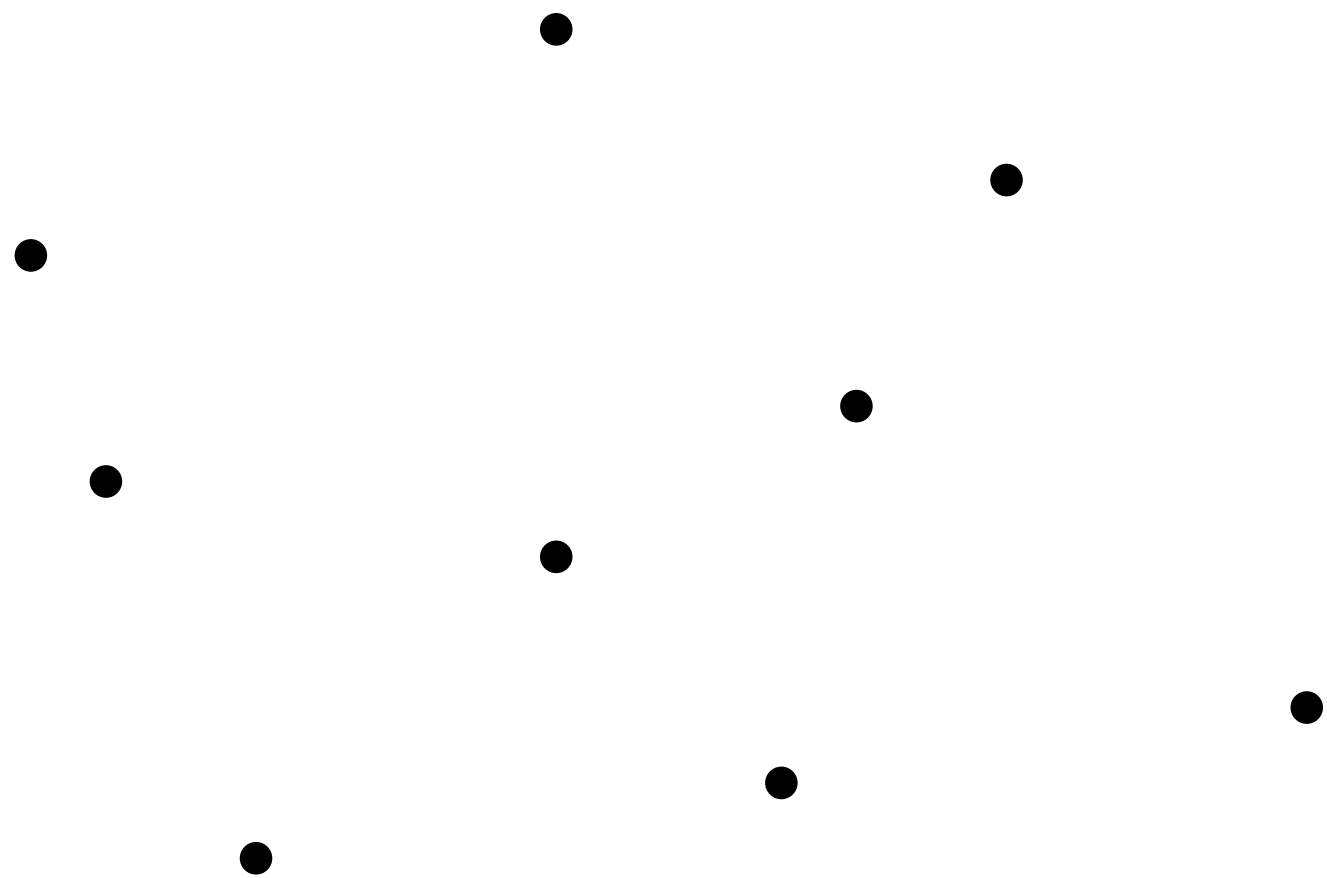
What is a non-obtuse triangulation?

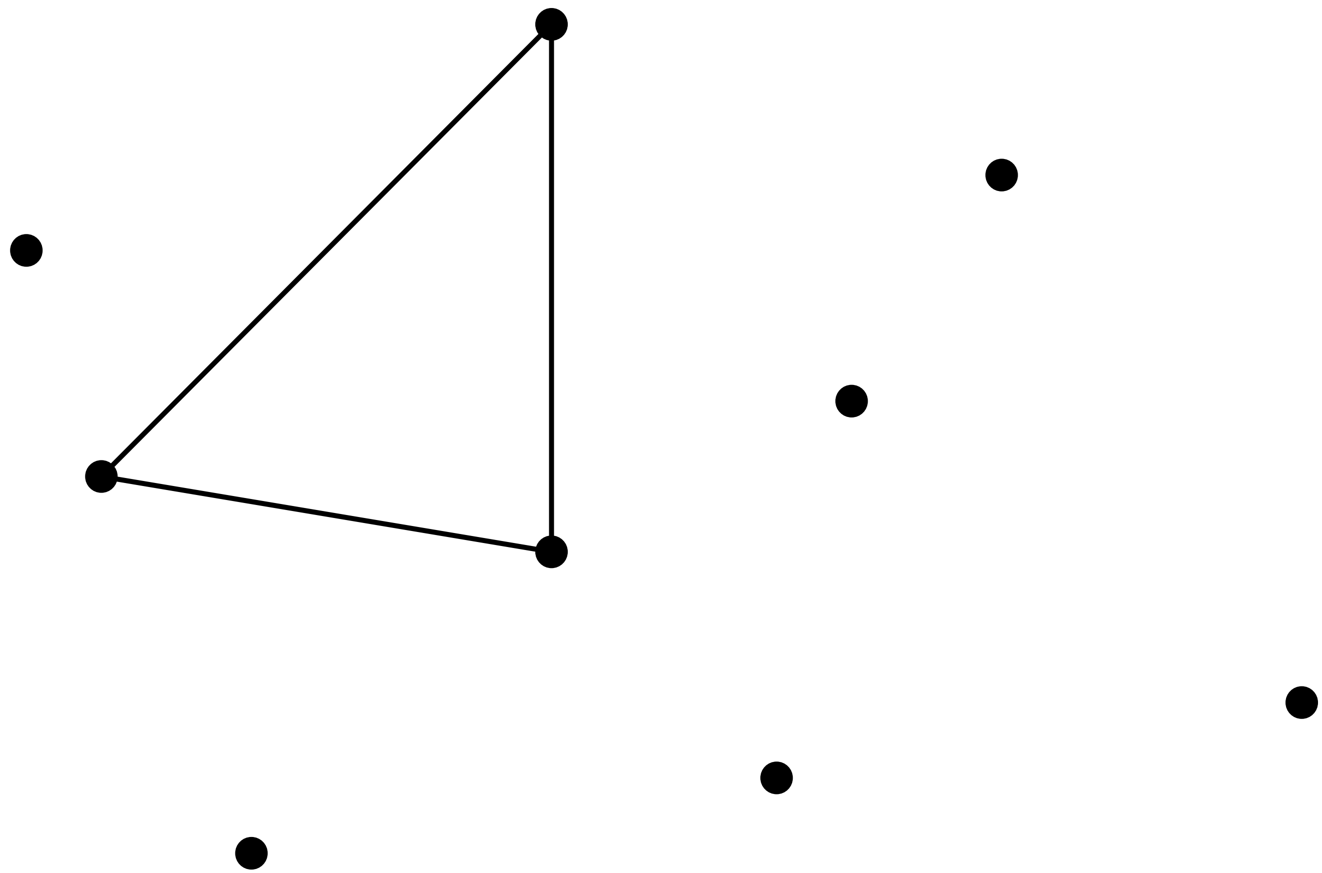
What are they good for?

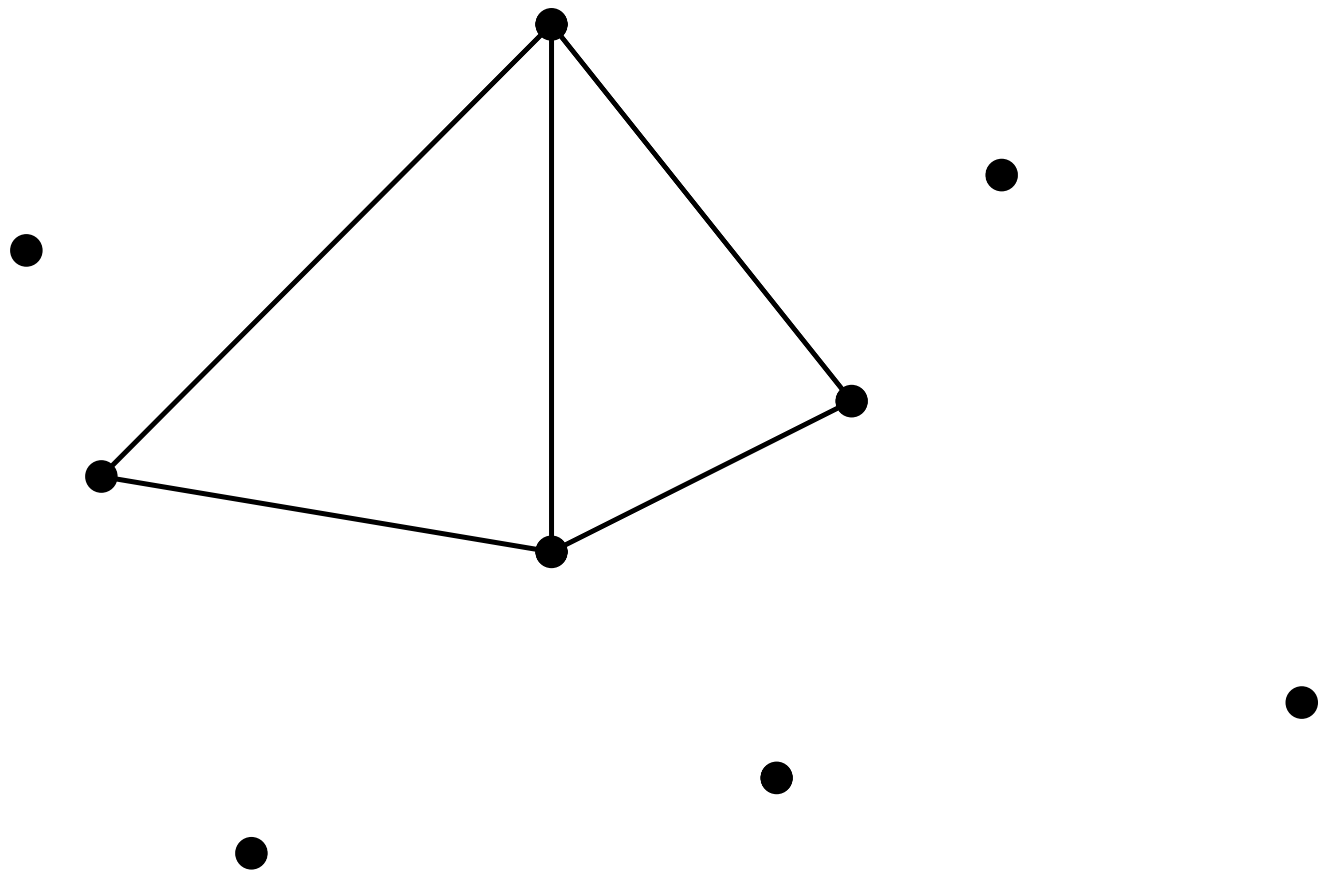
Do they exist?

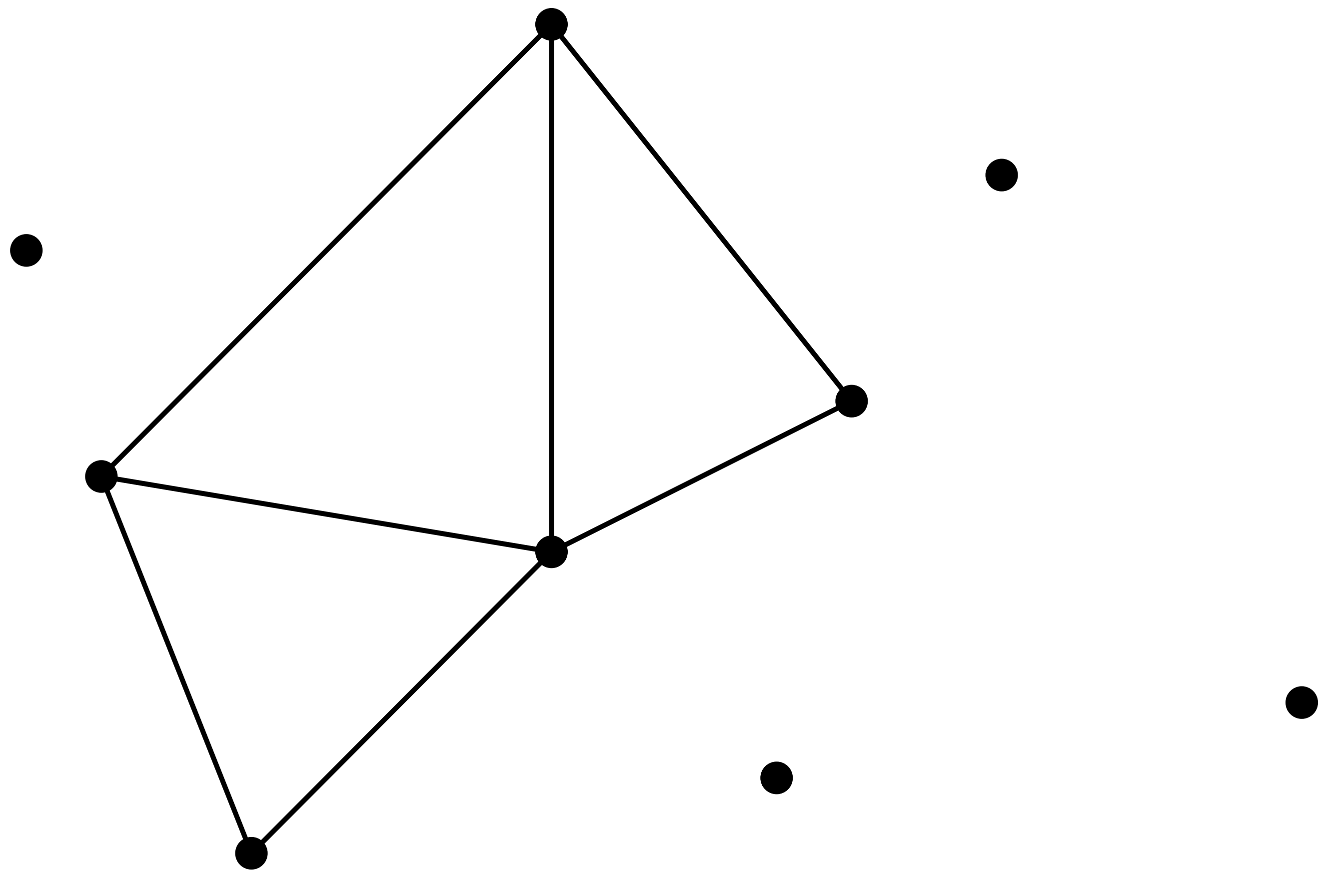
Are they efficient to compute?

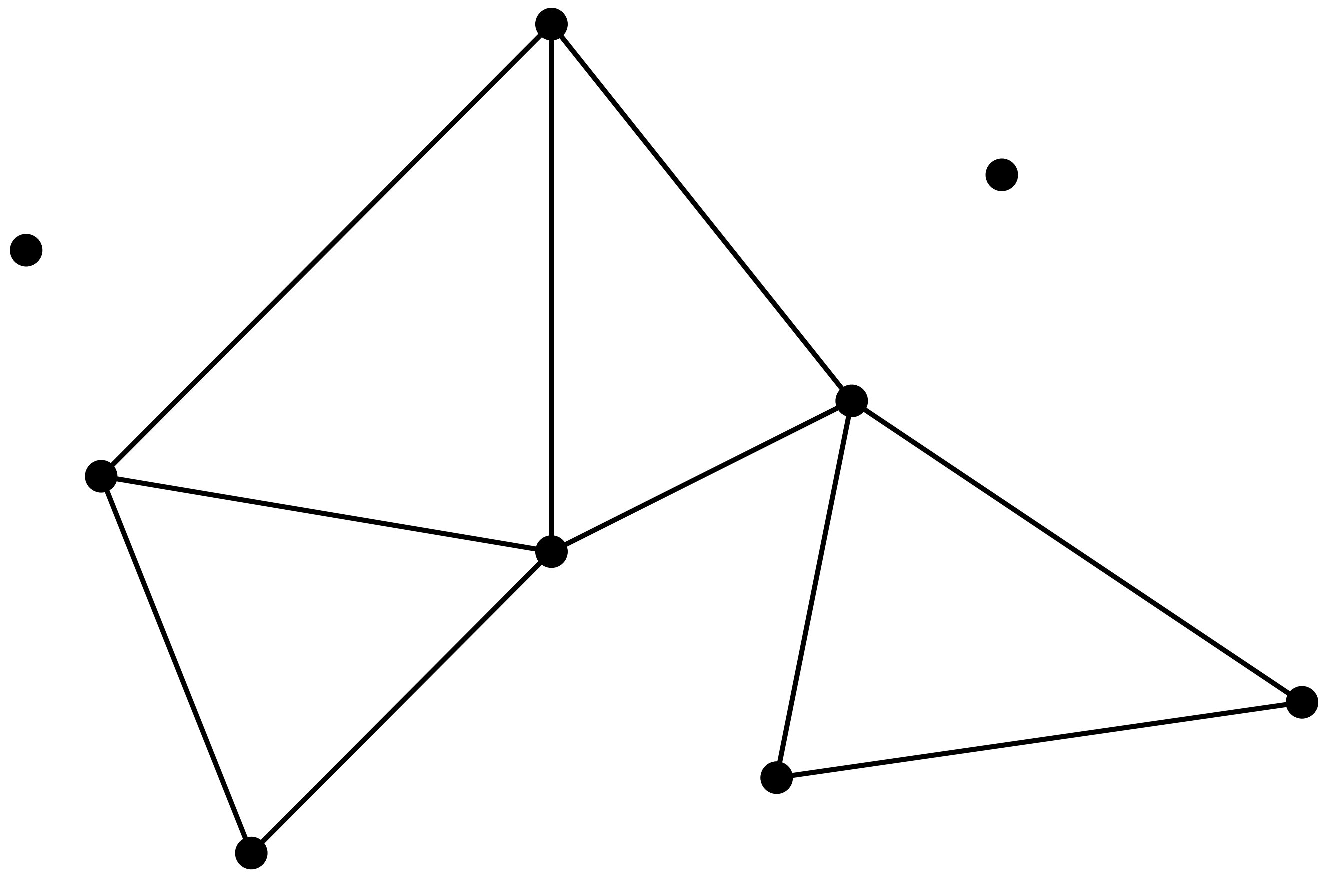
Quadrilateral meshes?

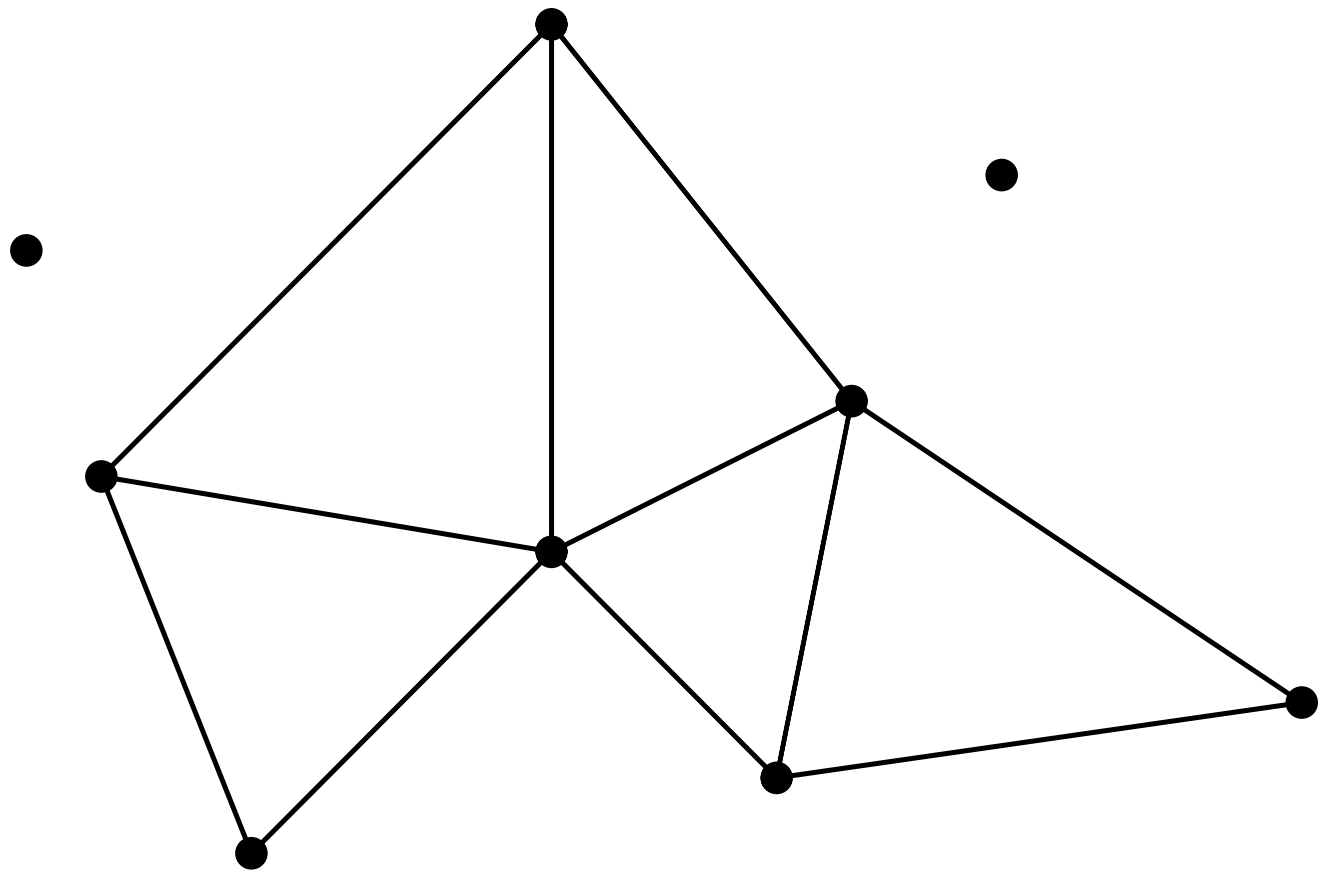


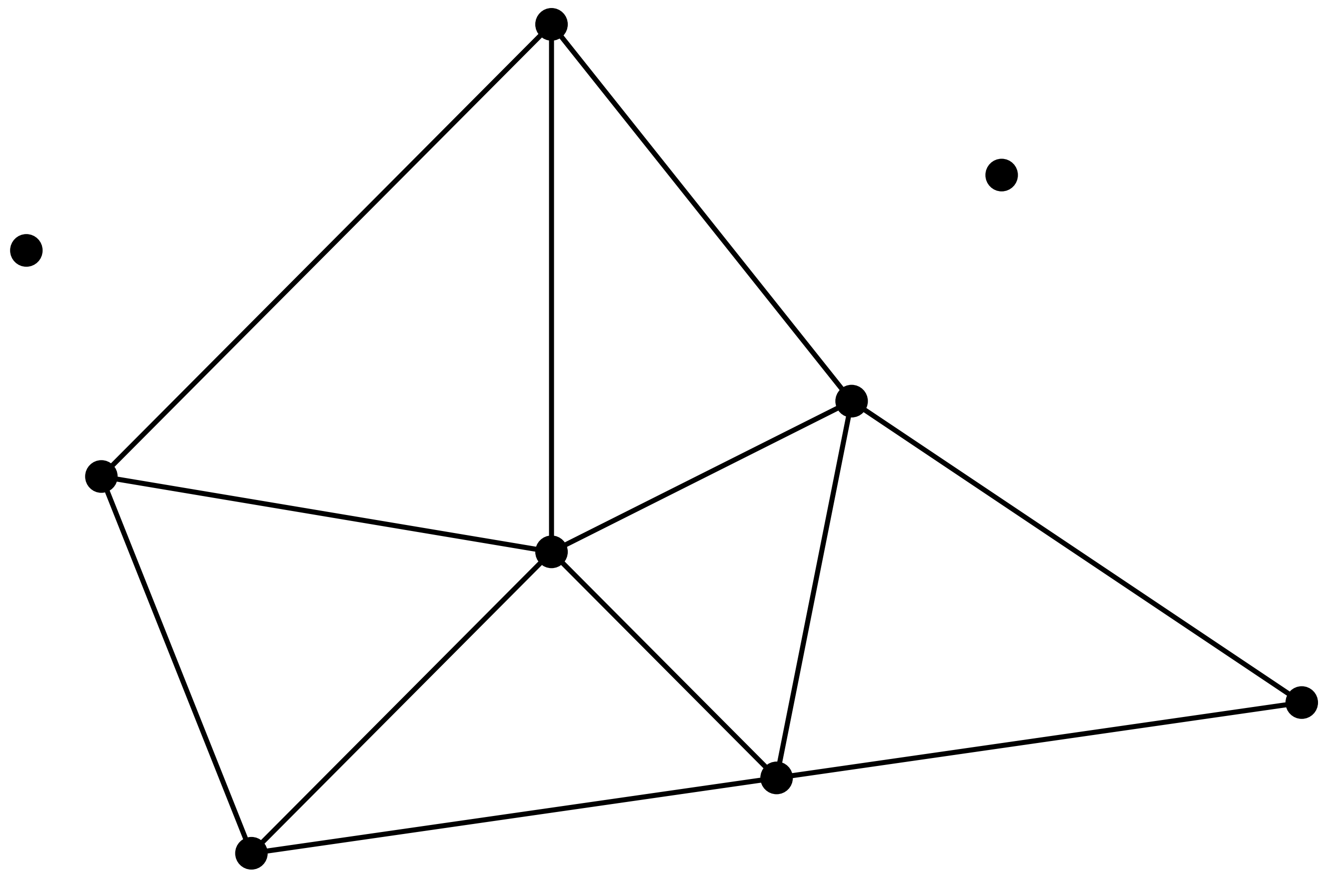


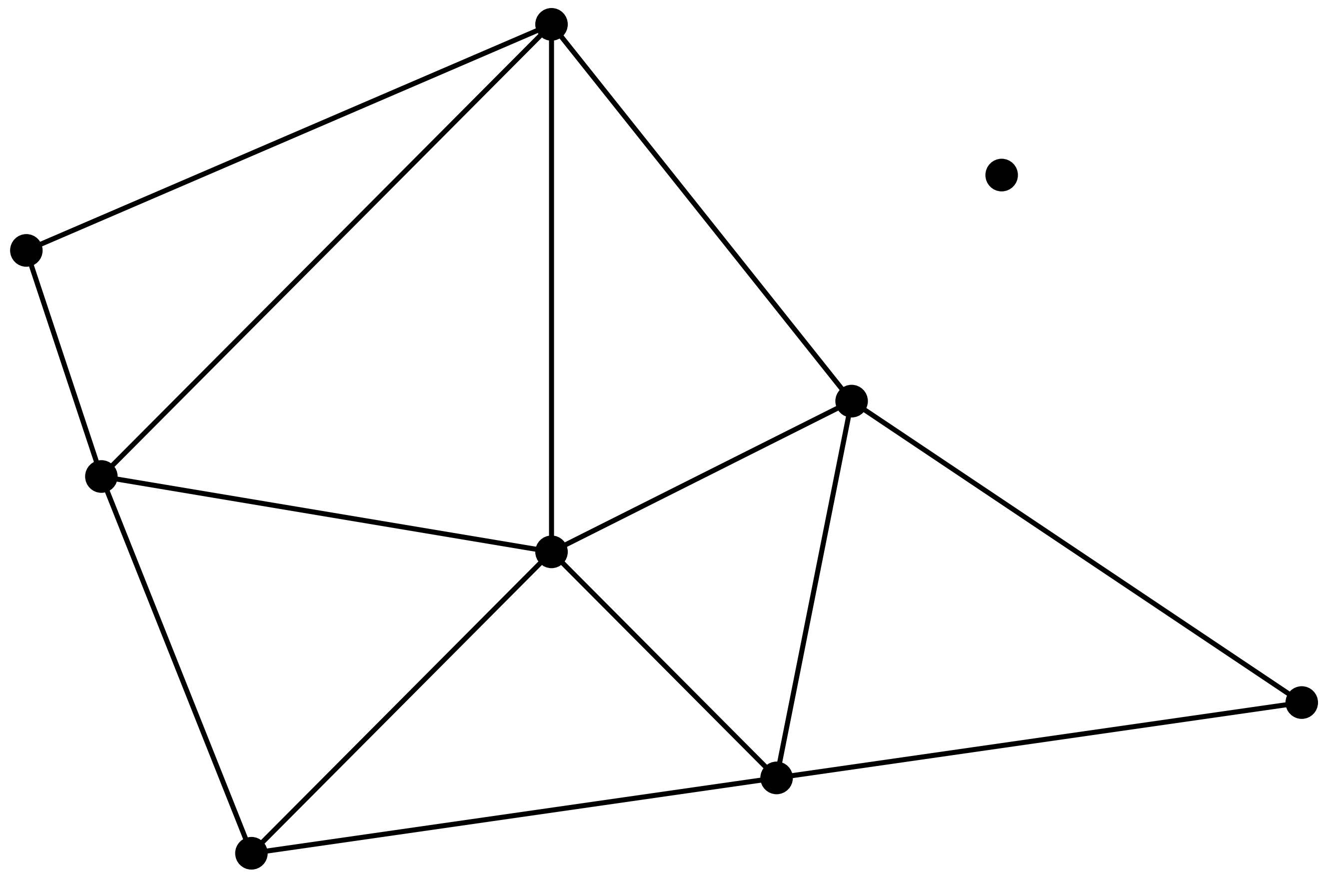


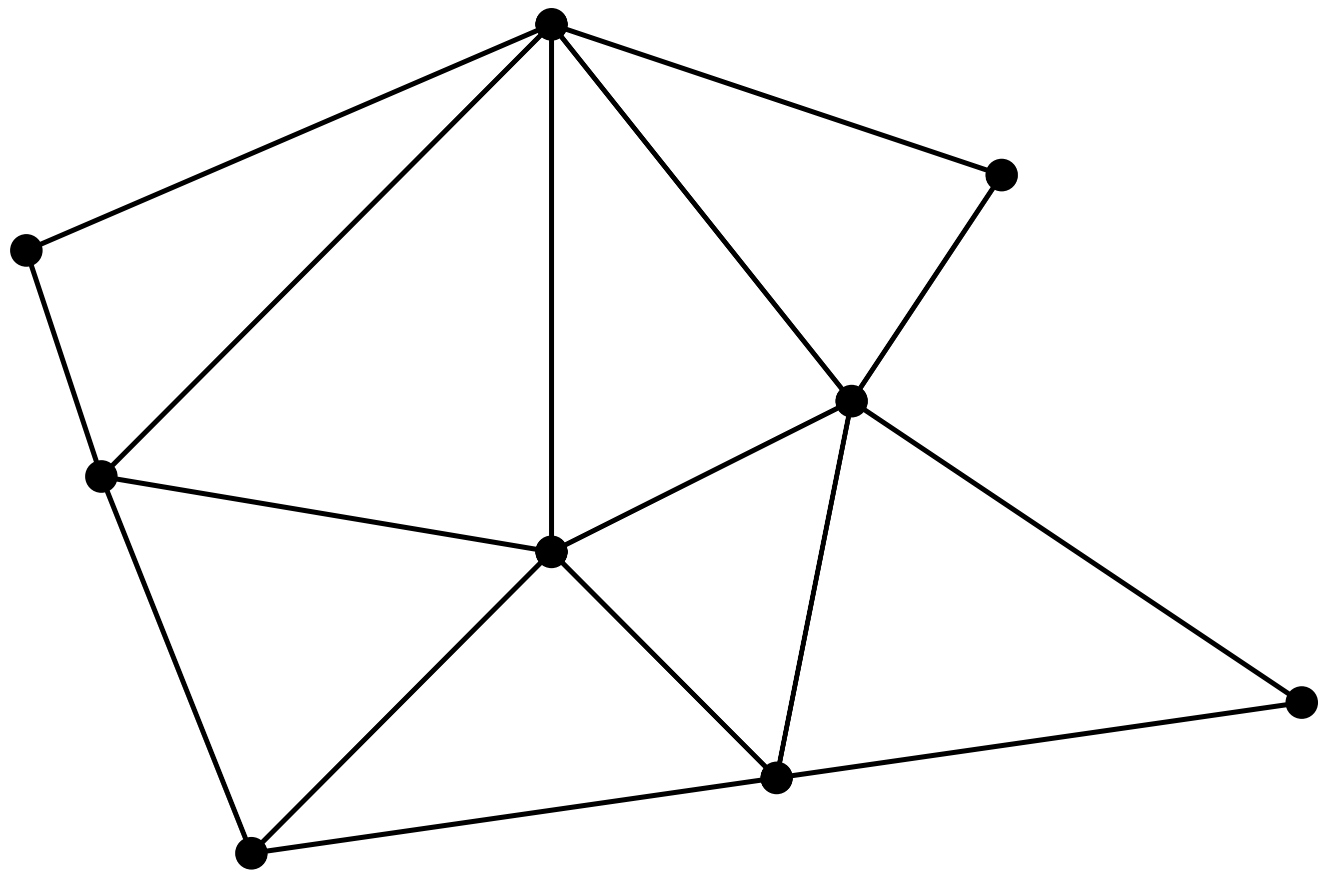


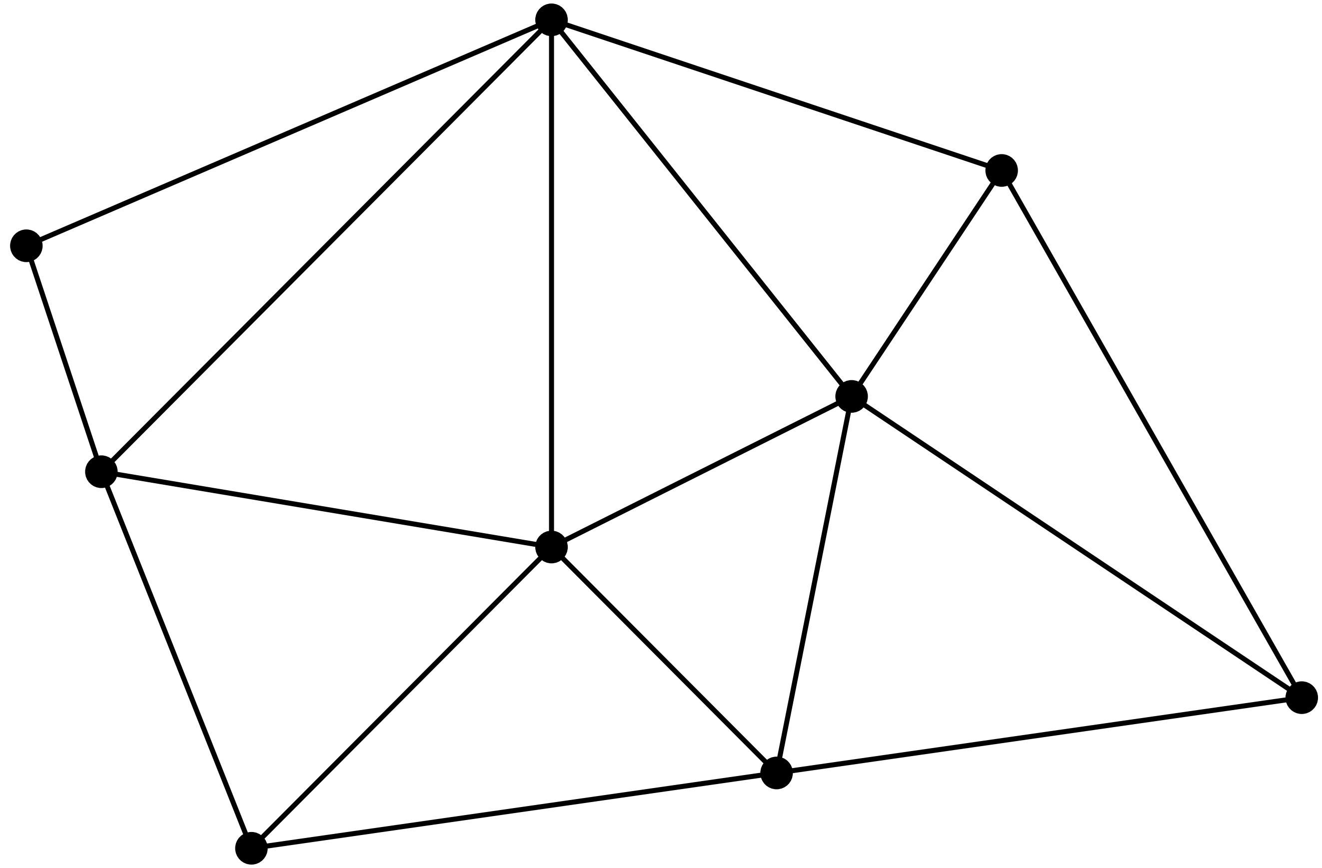




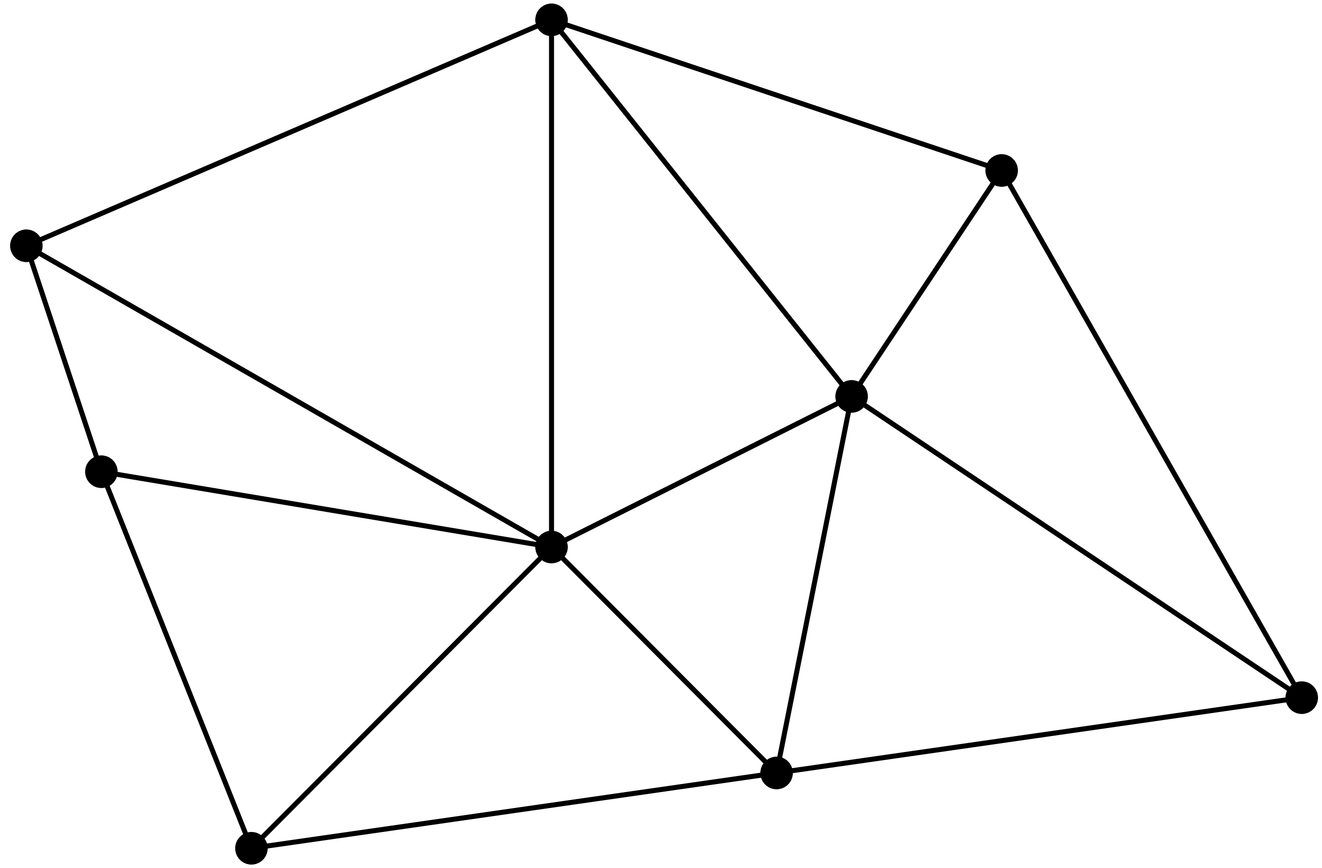




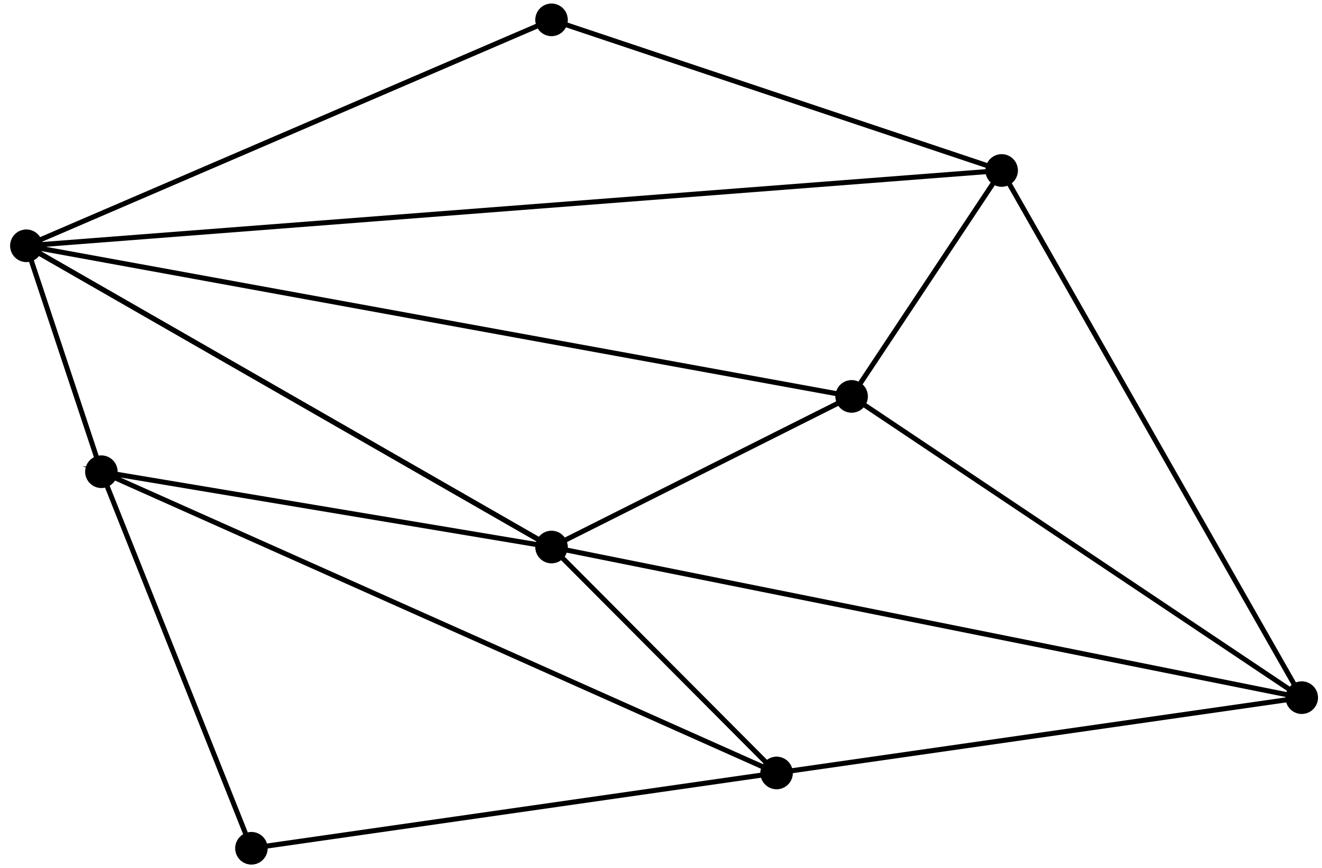




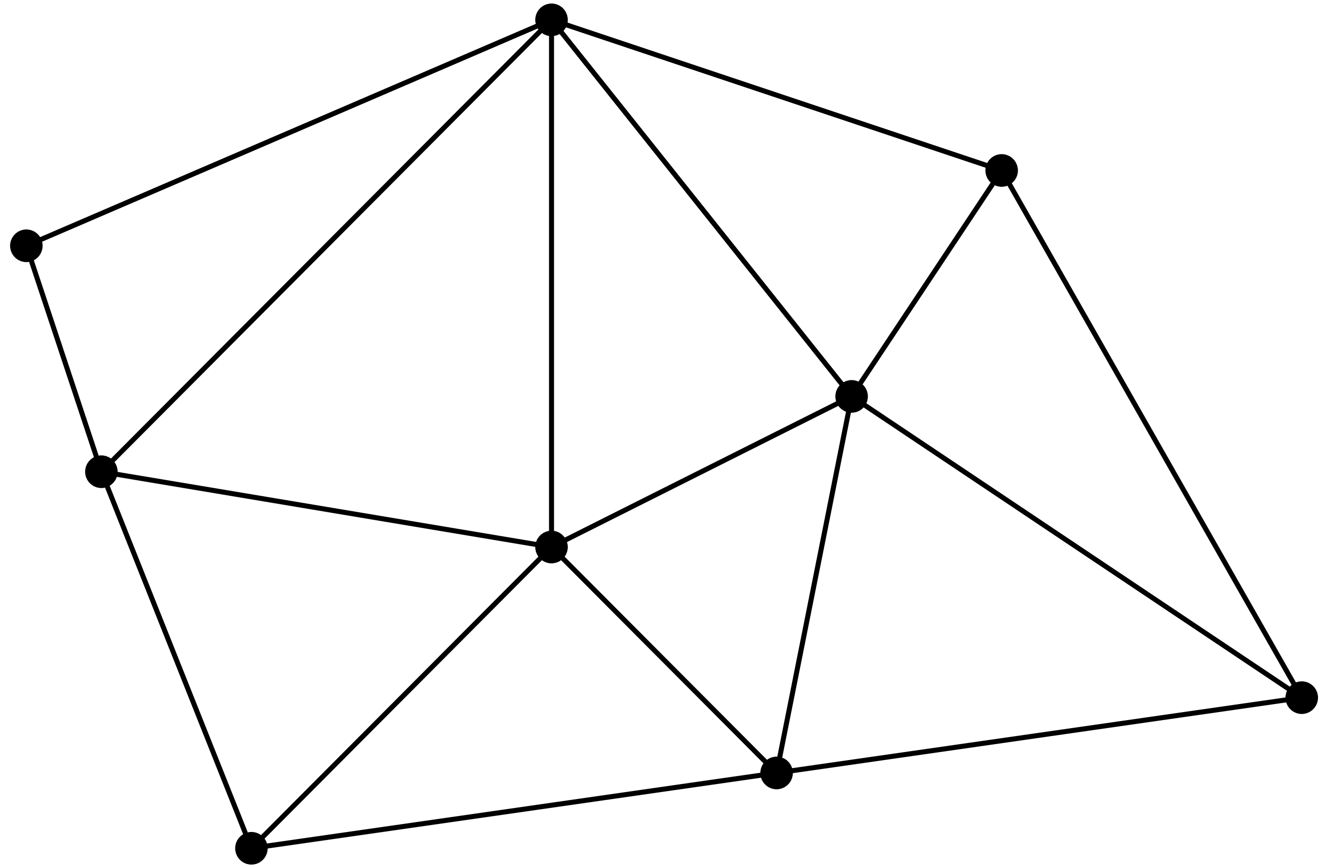
Triangulation = maximal collection of disjoint edges



Another triangulation (flipped a diagonal)

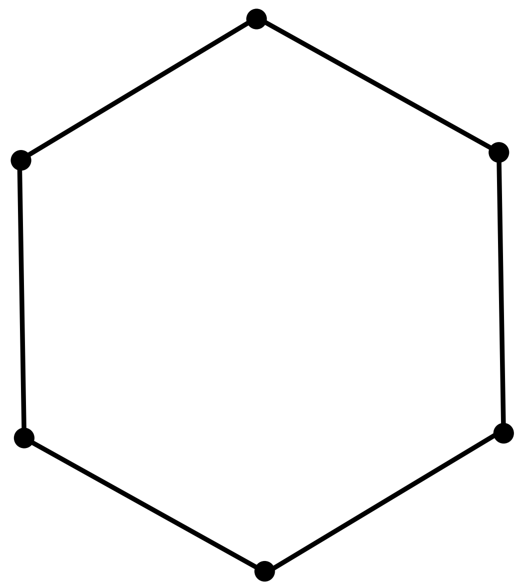


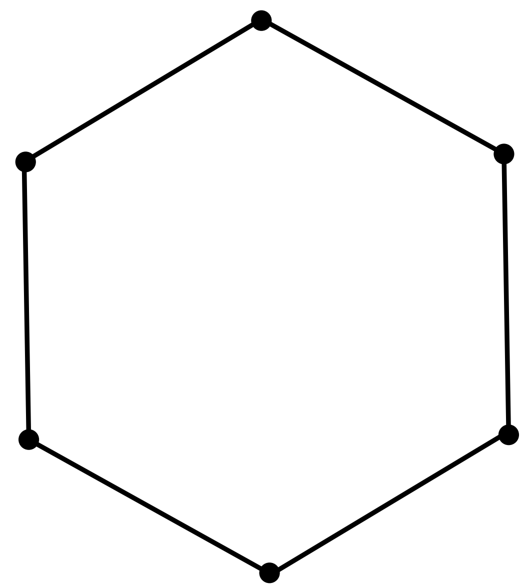
Yet another



smallest maximum angle = Delaunay triangulation

We can add extra points (**Steiner points**) to get better shaped triangles.





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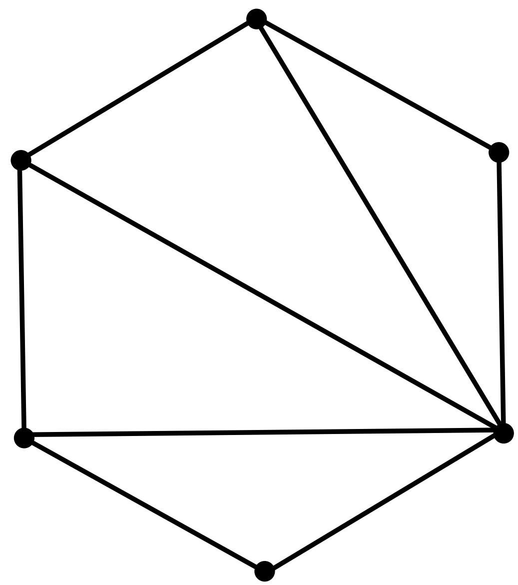

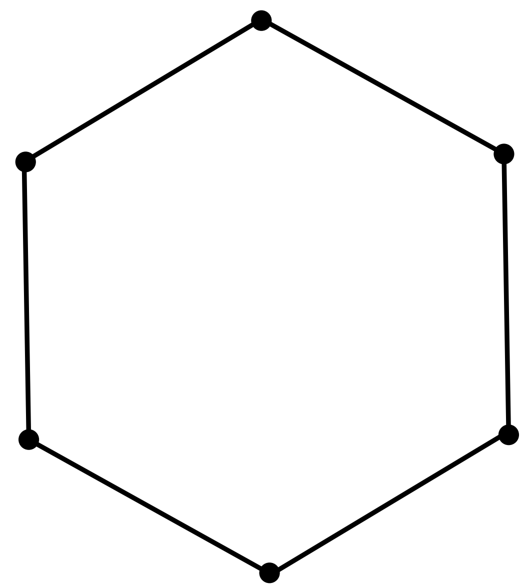

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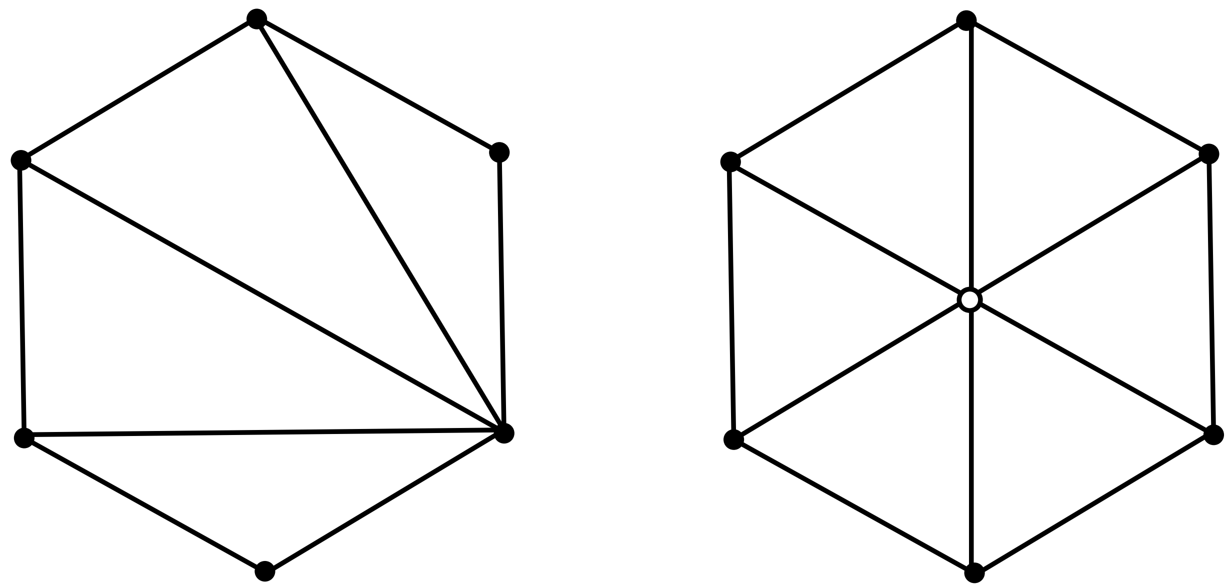

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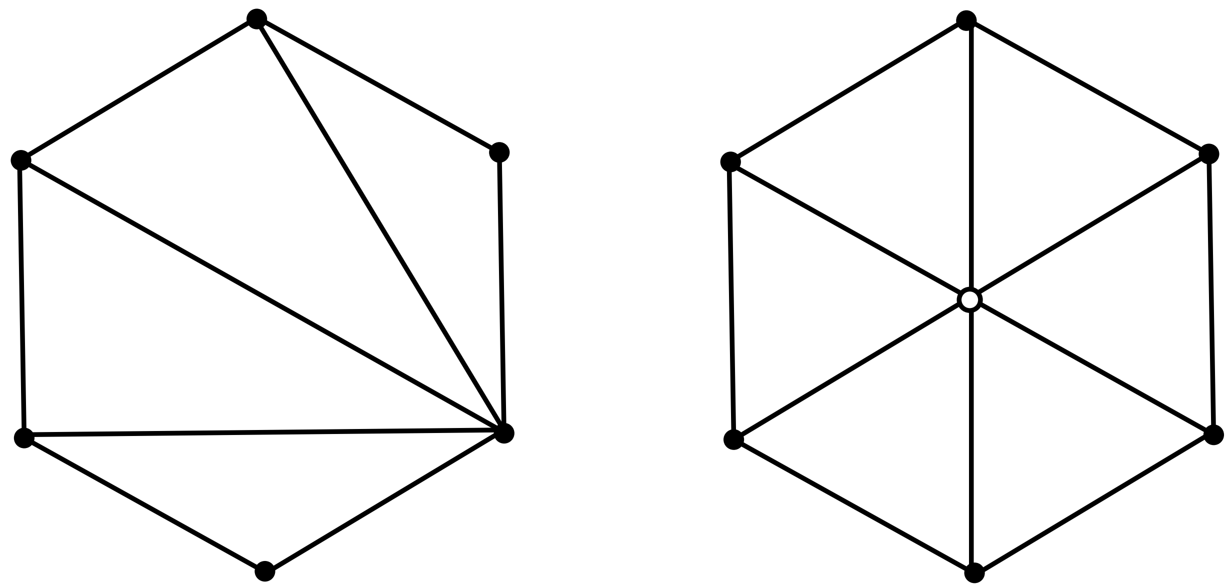
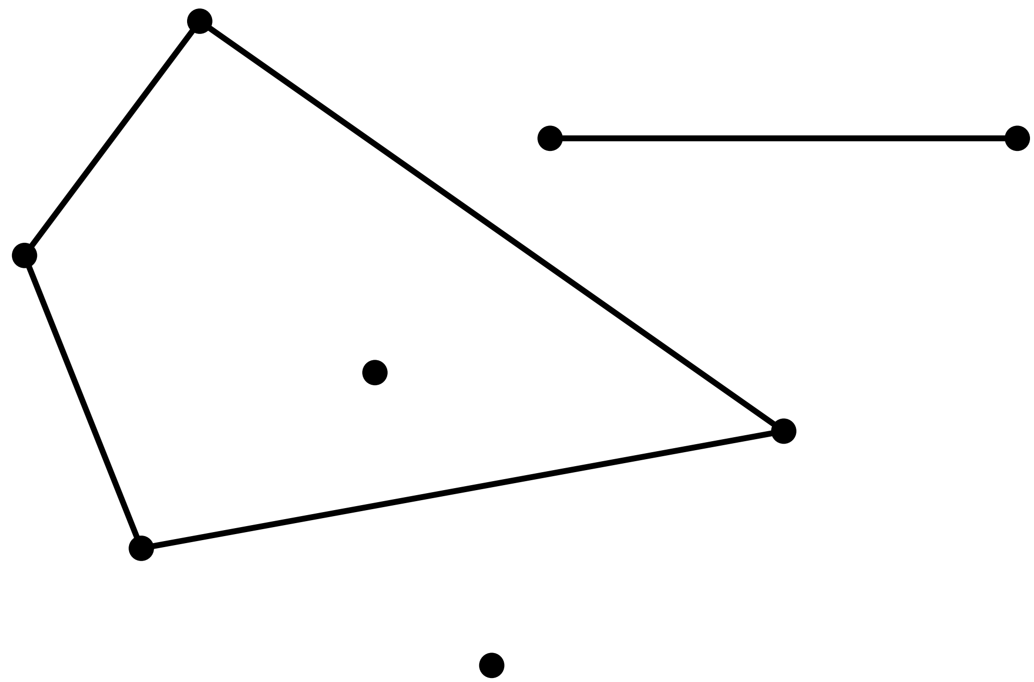


Good geometry = no small angles, no big angles

Non-obtuse = all angles $\leq 90^\circ$

Sometimes we force certain edges in the triangulation.

A **Planar Straight Line Graph** (PSLG) is a finite point set plus a set of disjoint edges between them.



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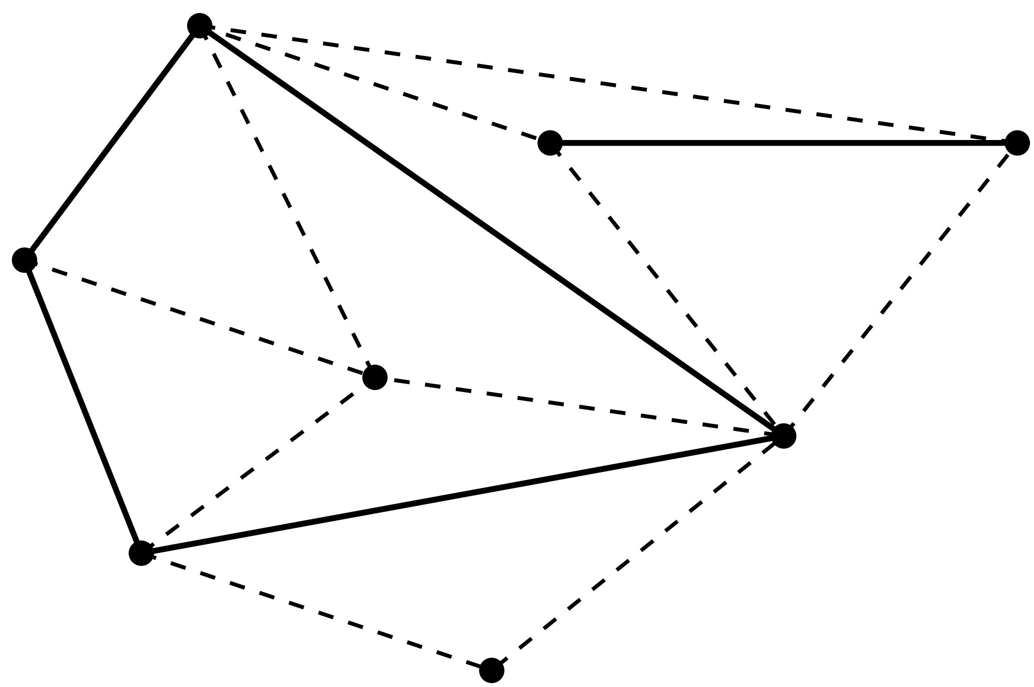

A triangulation of a PSLG is a triangulation of the point set which covers the edges of the PSLG.

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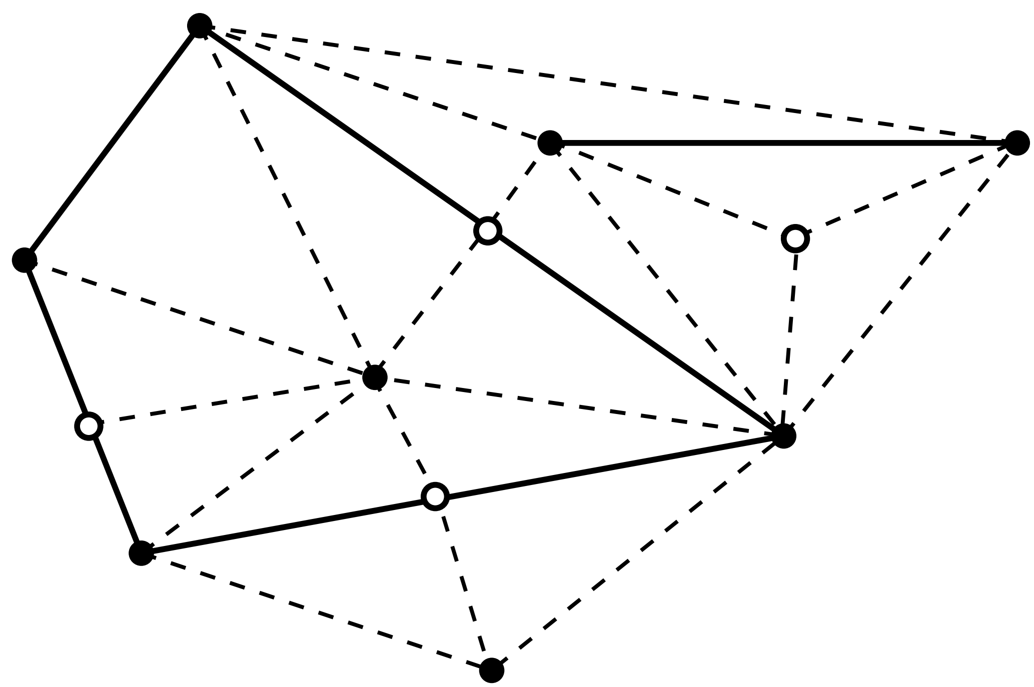
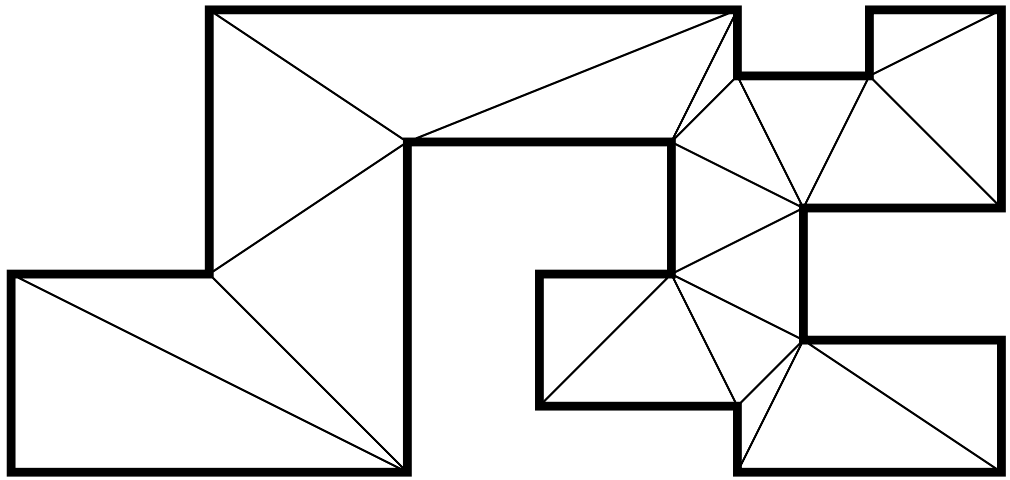


Steiner points will be allowed.

Special case of PSLG is a polygon.



A triangulation of a polygon only covers the interior.

A triangulation of a polygon has a tree structure.

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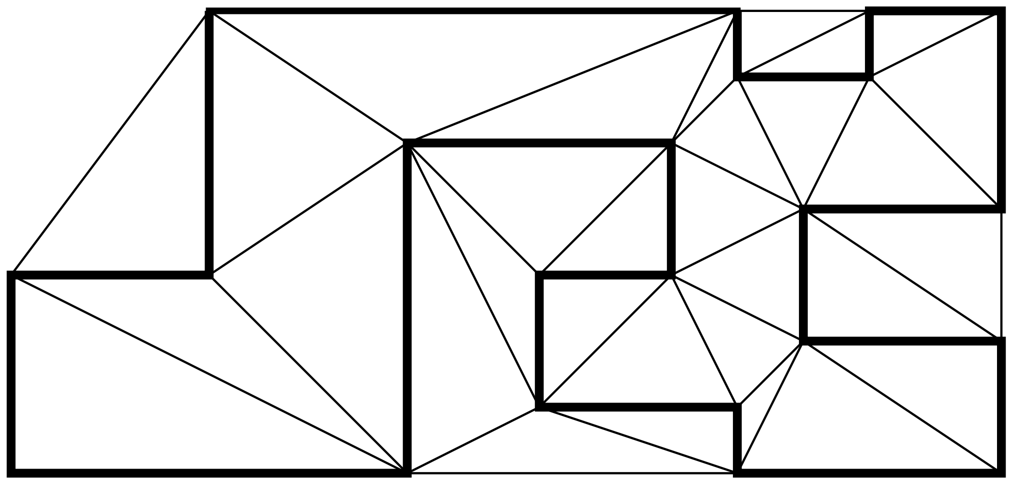

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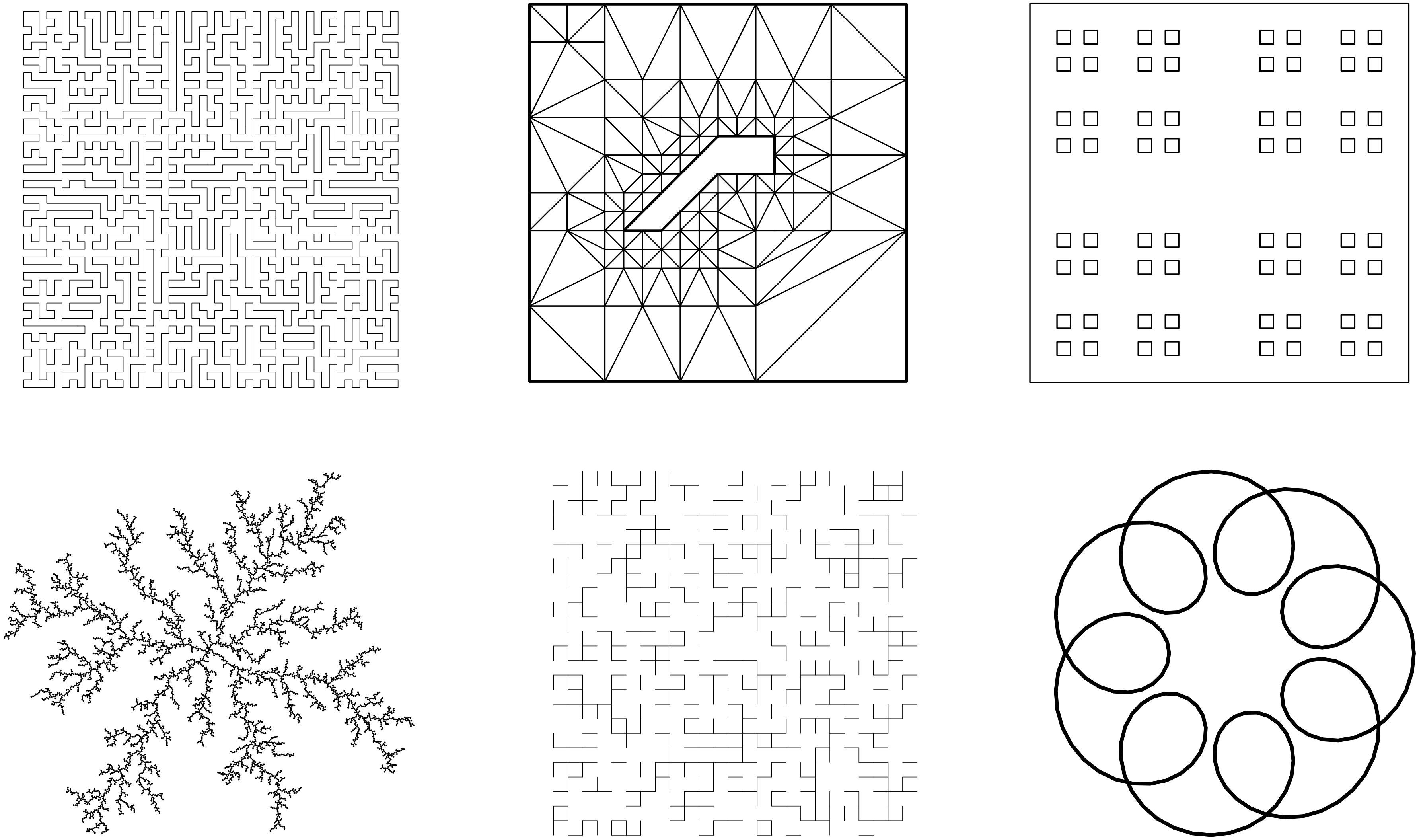
A triangulation of a PSLG need not.



More examples of PSLGs

Two conflicting goals: add Steiner points so we

- Triangulate with best geometry (angles bounded)
- Triangulate with least complexity (fewest elements).

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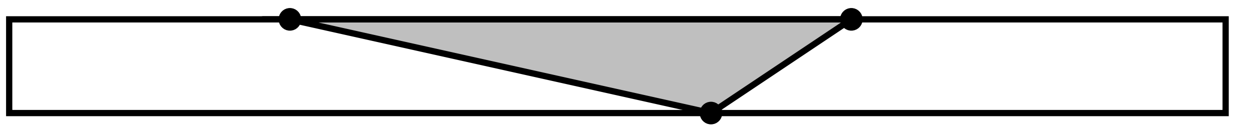
Nonobtuse triangulation ($\leq 90^\circ$) is best we can do.

Why?

For $1 \times R$ rectangle

$$\text{number of triangles} \gtrsim R \times (\text{smallest angle})$$



So uniform complexity $\Rightarrow$ no lower angle bound.

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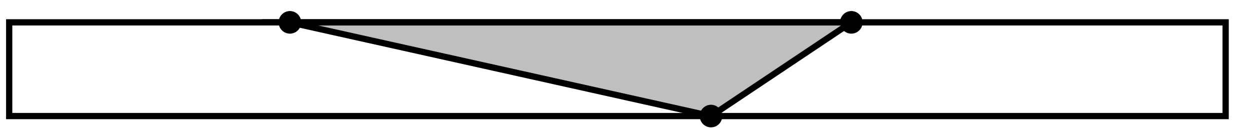
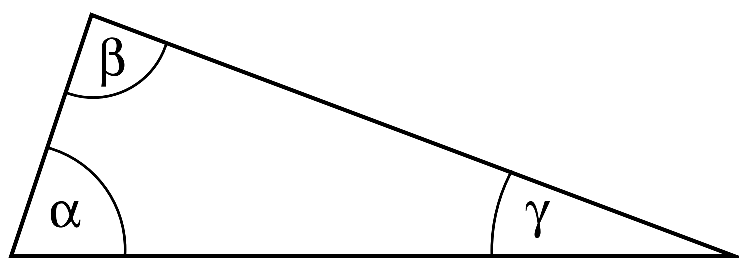


So uniform complexity $\Rightarrow$ no lower angle bound.

If all angles are $\leq 90° - \epsilon$ then all angles are $\geq 2\epsilon$.



$$\alpha, \beta < 90 - \epsilon \quad \Rightarrow \quad \gamma = 180 - \alpha - \beta > 2\epsilon.$$

Some history:

- Nonobtuse triangulation is always possible: Burago, Zalgaller 1960 and Baker, Grosse, Rafferty, 1988

- $O(n)$ for points sets: Bern, Eppstein, Gilbert 1990

- $O(n^2)$ for polygons: Bern, Eppstein 1991

- $O(n)$ for polygons: Bern, Mitchell, Ruppert 1994

- If there is a nonobtuse triangulation, there is an acute triangulation: Maehara 2002, Yuan 2005

- Many heuristics for nonobtuse triangulation.

No known polynomial bounds for PSLGs.

Applications of non-obtuse triangulations:

- Discrete maximum principle (Ciarlet, Raviart, 1973)
- Convergence of finite element methods (Vavasis, 1996)
- Fast marching method (Sethian, 1999)
- Meshing space-time (Ungör, Sheffer, 2002)
- Machine learning

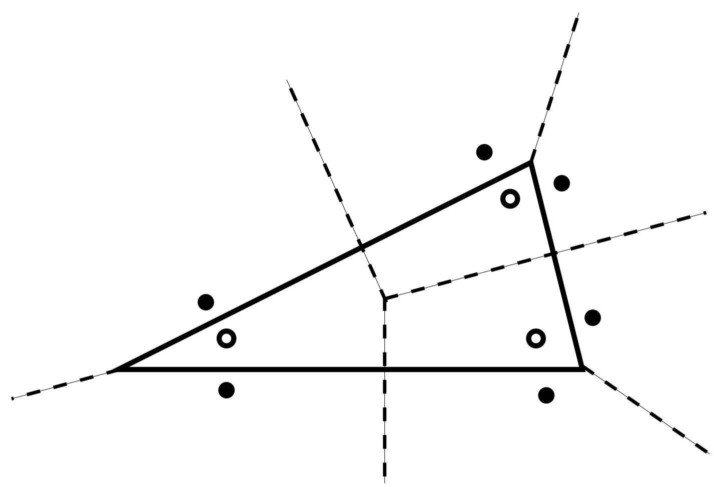
Salzberg, Delcher, Heath, Kasif, 1995, *Best-case results for nearest-neighbor learning*.

Given polygon Γ find point sets I, O so that

$$\mathrm{int}(\Gamma) = \{z : \mathrm{dist}(z, I) < \mathrm{dist}(z, O)\},$$

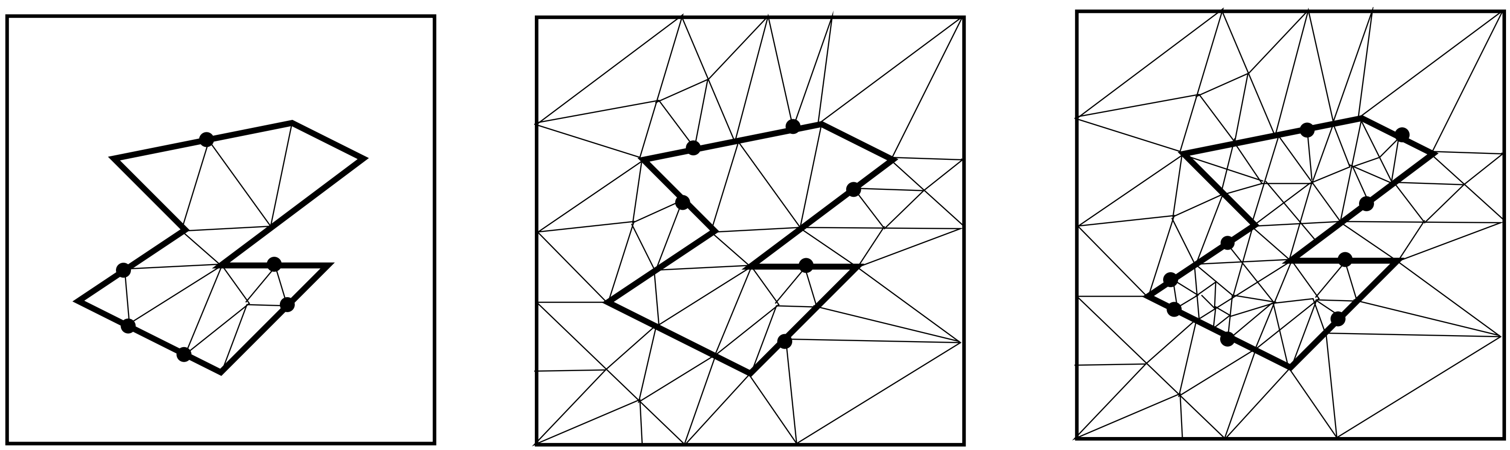
i.e., Voronoi diagram of $I \cup O$ covers Γ.

Easy for nonobtuse triangles.



Closed curve may represent boundary and we must mesh both sides using same points on boundary.

Triangulate one side, then the other, creating new boundary vertices. Then redo first side. Does process stop?



Can we non-obtusely triangulate both sides at once?

What if we weaken angle bound?

Replace 90° with some $\theta < 180^\circ$?

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- S. Mitchell (1993): every PSLG has $O(n^2)$ triangulation with all angles $\leq 157.5^\circ = \frac{7}{8}\pi$
- Tan (1996): same for angles $\leq 132^\circ = \frac{11}{15}\pi$.

The $O(n^2)$ is sharp because any mesh with maximum angle $\leq \theta < 180°$ sometimes requires n^2 elements.

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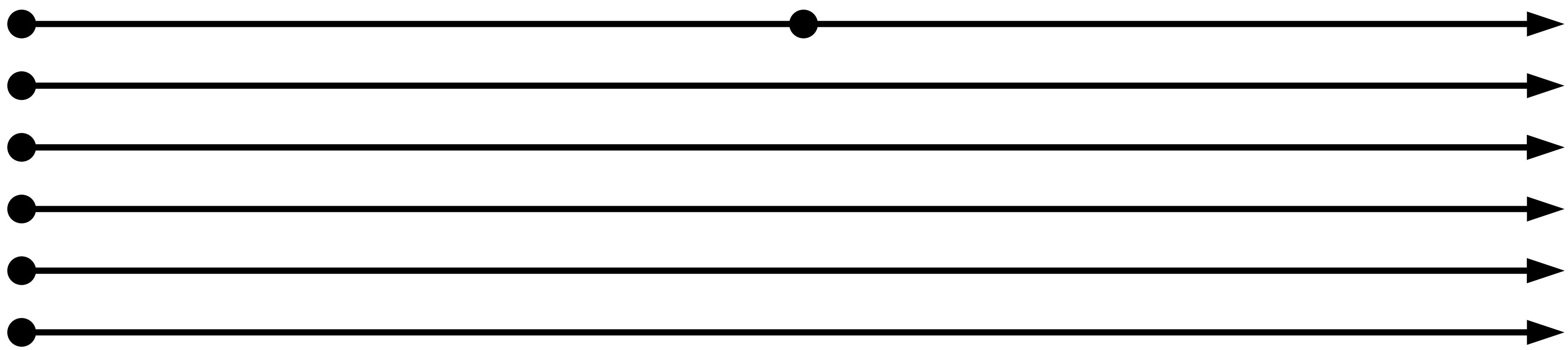

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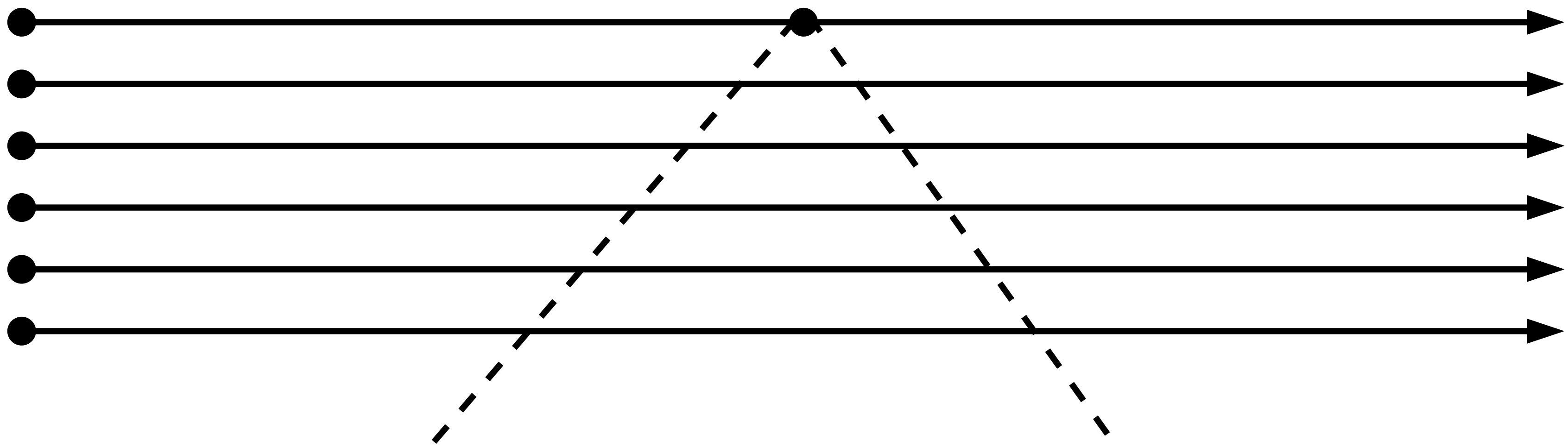

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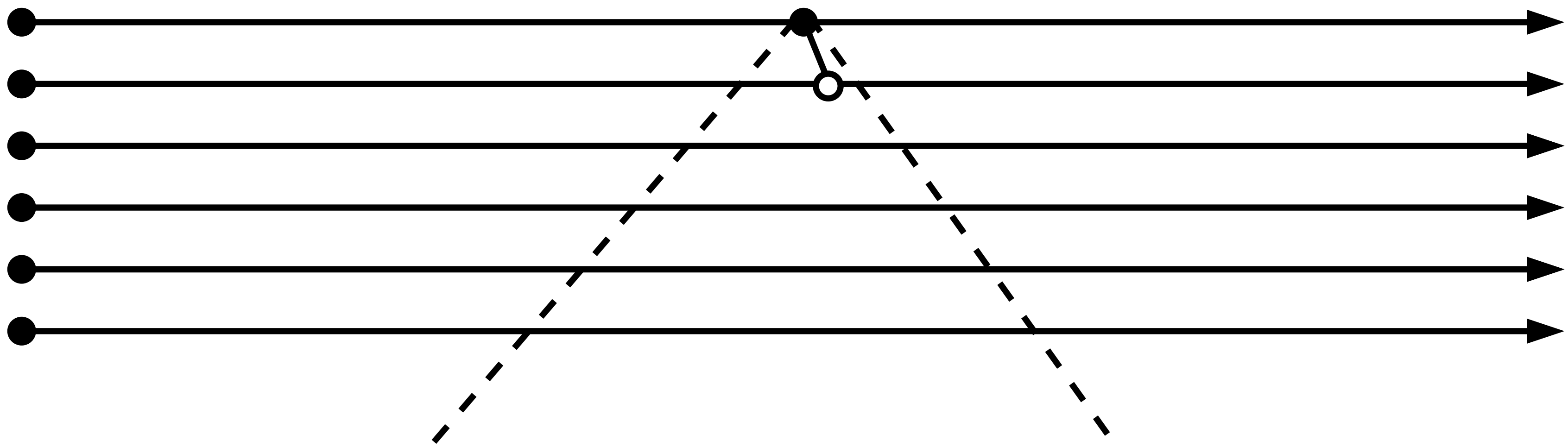

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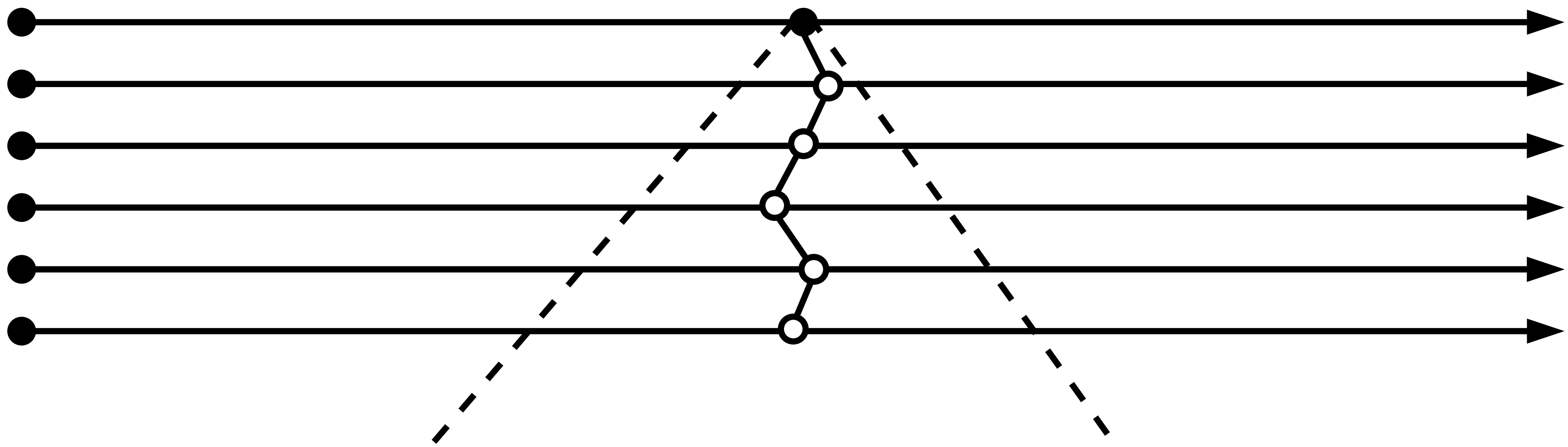

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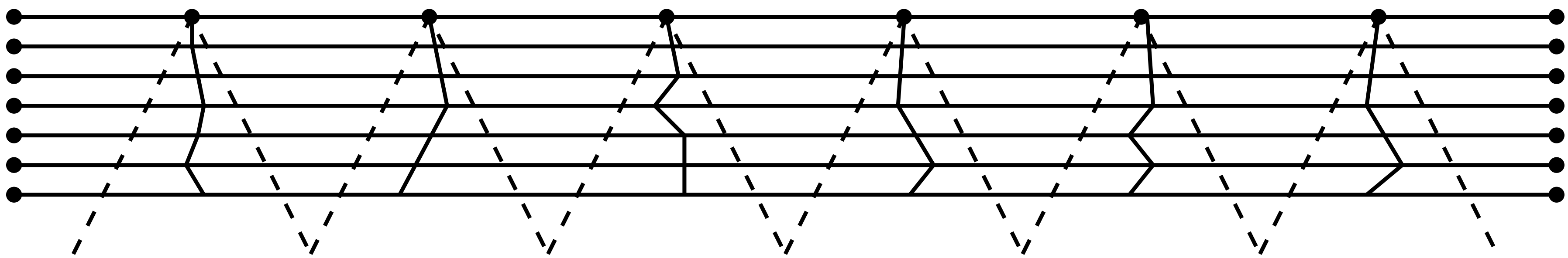
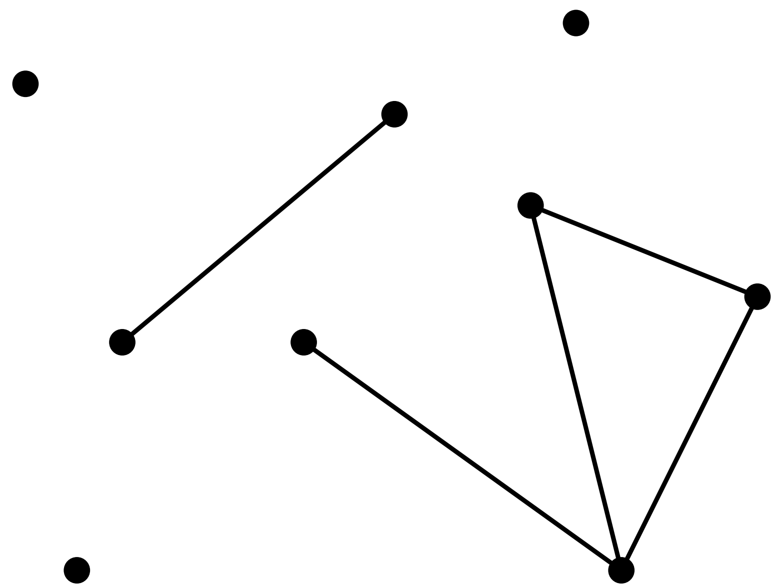


NOT-Theorem: Every PSLG has a non-obtuse triangulation with $O(n^{2.5})$ elements.

NOT = Non-Obtuse Triangulation

Theorem: Every PSLG has a triangulation with all angles $\leq 90^\circ + \epsilon$ and $O(n^2/\epsilon^2)$ elements.

If Γ is a PSLG, then any triangulation (no Stiener points) is also a PSLG with a comparable number of elements.



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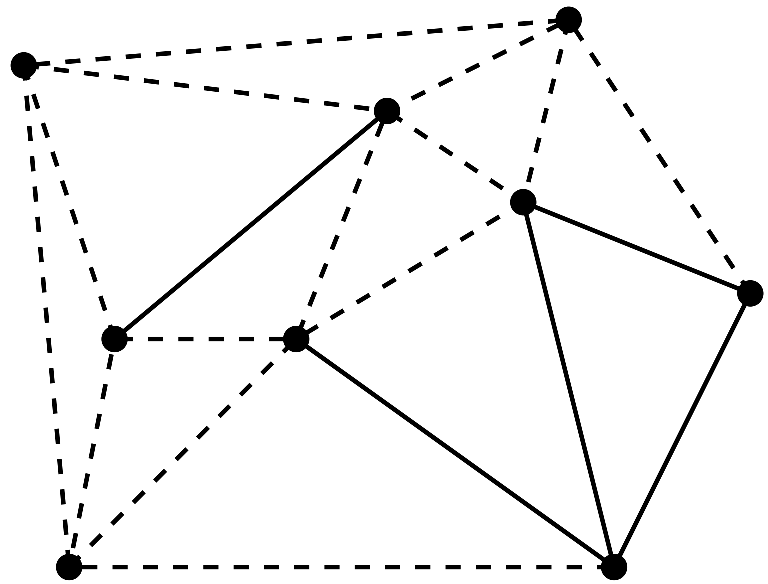

Trinagulation can be computed in $O(n \log n)$.

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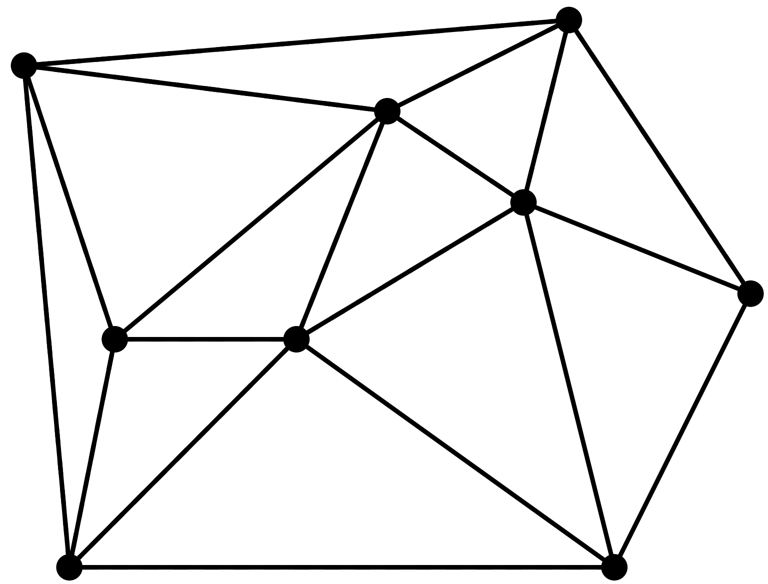

Suffices to assume PSLG is a triangulation.

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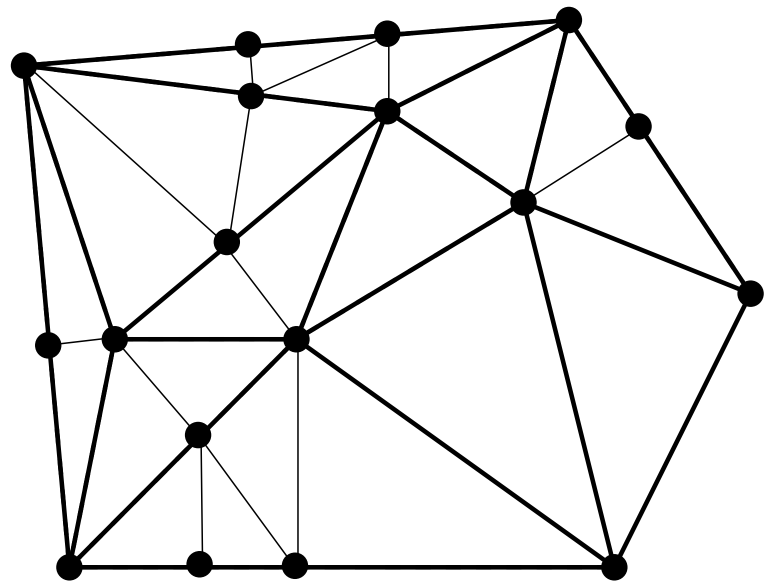
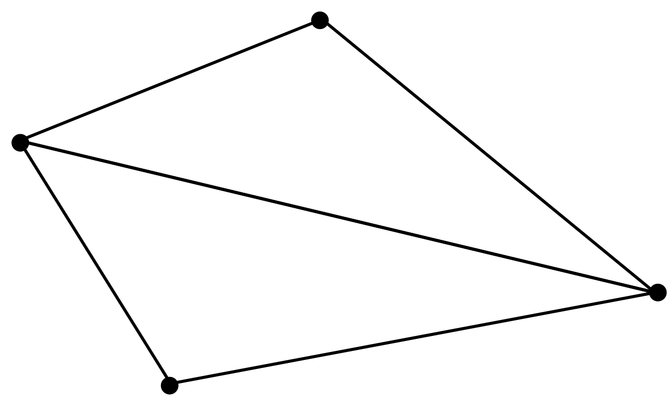


A non-obtuse refinement of this triangulation is also a non-obtuse triangulation of the original Γ.

So to prove the NOT-theorem, we can assume the PSLG is a triangulation of a point set.

Idea: replace each triangle by nonobtuse triangles that "match" along common boundaries.



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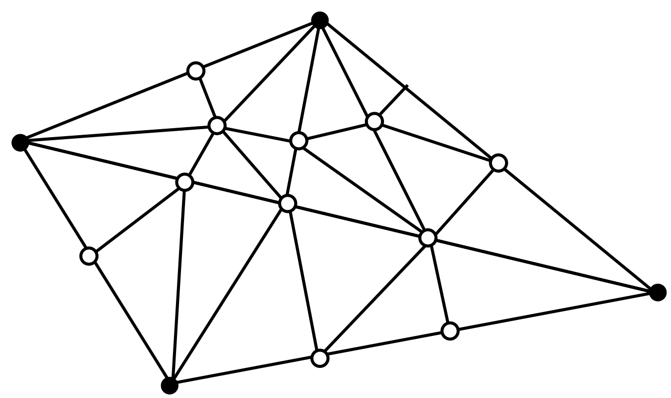
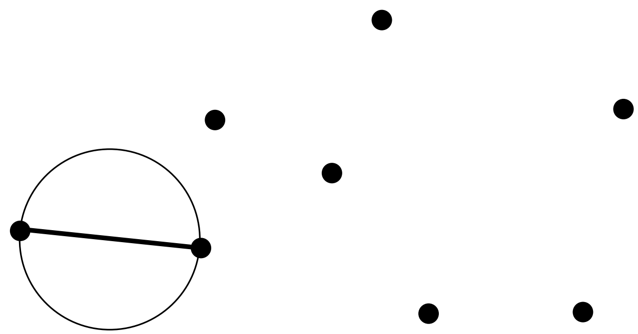


A method was given by Bern, Mitchell and Ruppert.

Uses **Gabriel** edges.

The segment $[v, w]$ is a **Gabriel** edge of a point set V if it is the diameter of an open disk missing V.



Gabriel edge.

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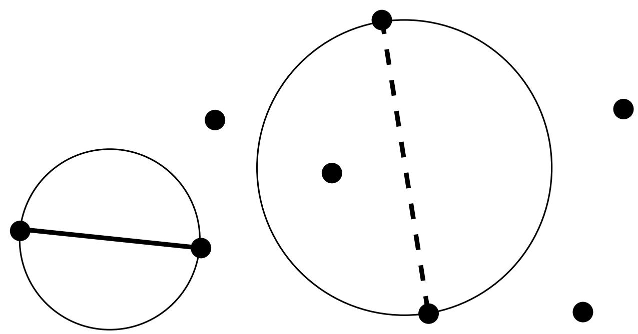

Not a Gabriel edge.

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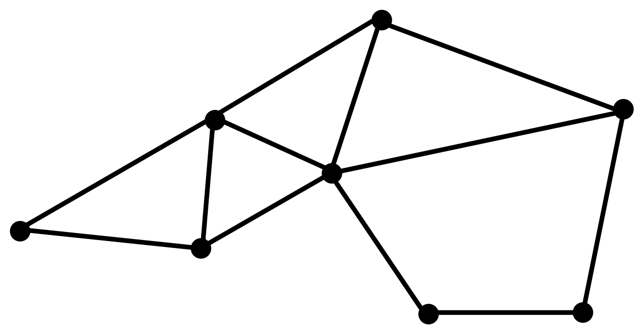
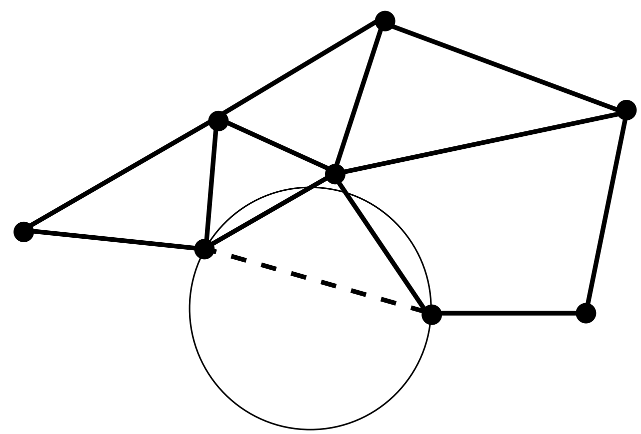


Gabriel graph contains the minimal spanning tree.

Gabriel and Sokol, *A new statistical approach to geographic variation analysis*, Systematic Zoology, 1969.

Gabriel edge is a special case of a **Delaunay** edge: $[v, w]$ is a **chord** of an open disk not hitting V.



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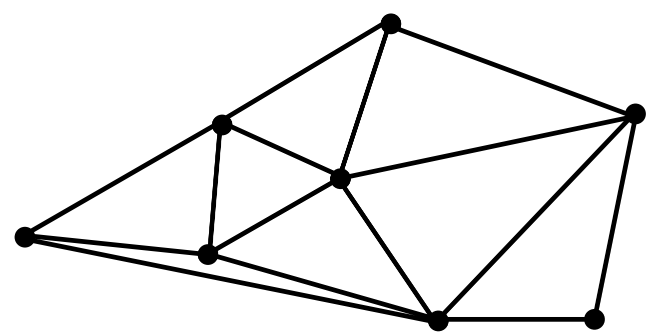
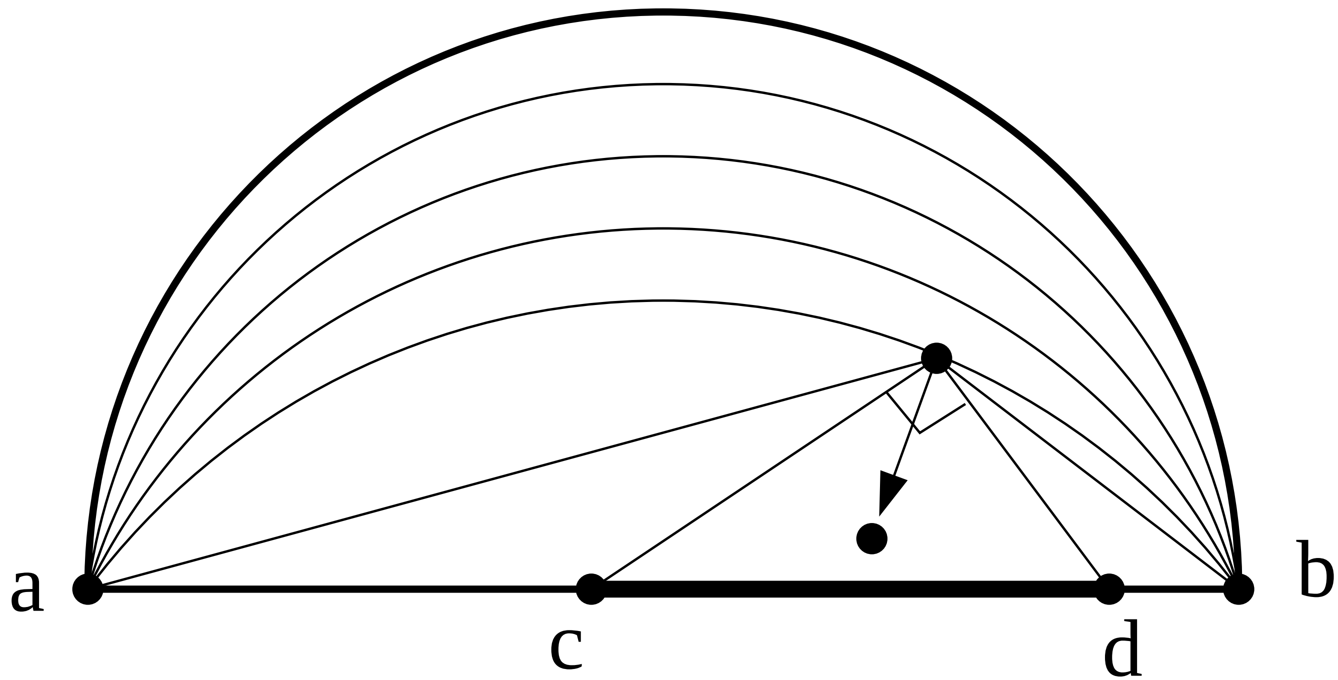


Adding all Delaunay edges triangulates with smallest maximum angle (best possible without Steiner points).

Every edge of a non-obtuse triangulation is Gabriel.



Proof: non-Gabriel $\Rightarrow$ some angle $> 90°$.

Every edge of a non-obtuse triangulation is Gabriel.

Thus the NOT-Theorem implies

Gabriel Edges Thm: For any PSLG Γ of size n there are $O(n^{2.5})$ points whose Gabriel graph covers Γ.

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Corollary: For any PSLG Γ of size n there is set of $O(n^{2.5})$ points whose Delaunay graph covers Γ.

Improves $O(n^3)$ by Edelsbrunner and Tan (1993).

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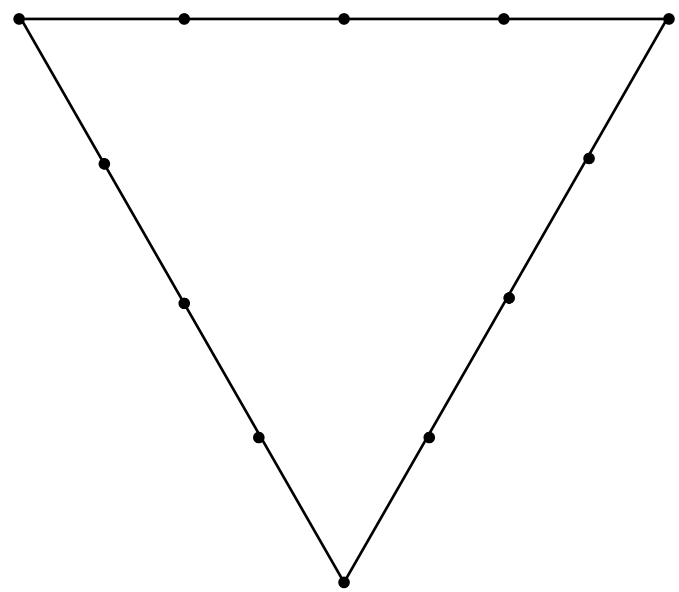
Gabriel Edges Thm: For any PSLG Γ of size n there are $O(n^{2.5})$ points whose Gabriel graph covers Γ.

In fact, GE-theorem $\Rightarrow$ NOT-Theorem

Bern-Mitchell-Ruppert (1994)

BMR Lemma: Add k vertices to sides of triangle (at least one per side) so all edges become Gabriel, then add all midpoints. Resulting polygon has a $O(k)$ NOT, with no additional vertices on boundary.



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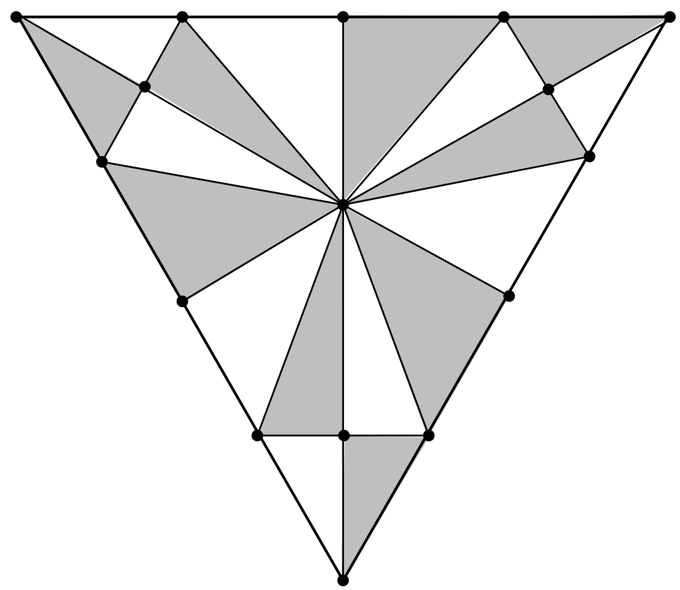

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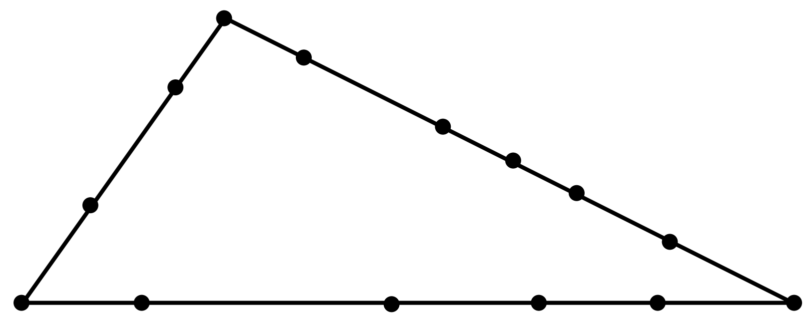
Building a NOT for a PSLG:

- Replace PSLG by triangulation of itself.
- Add vertices to make all edges Gabriel.
- Apply BMR lemma to each triangle.

Sketch of BMR lemma:



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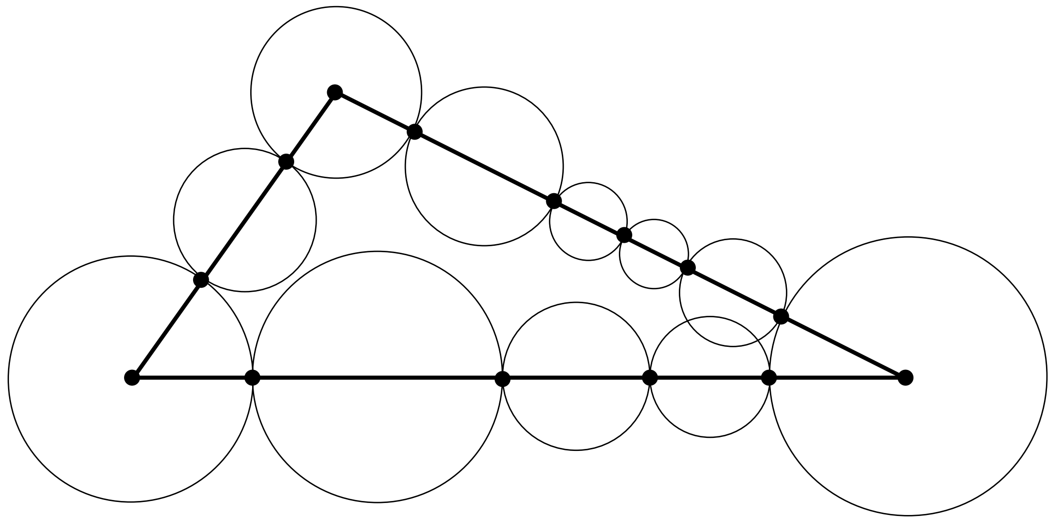

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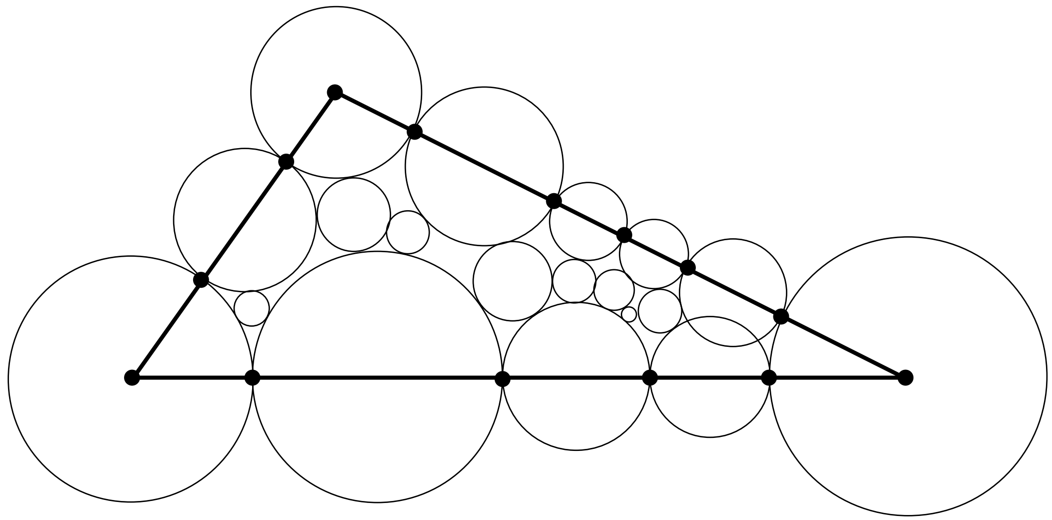

• Pack interior with disjoint disks so only 3-sided and 4-sided regions remain.

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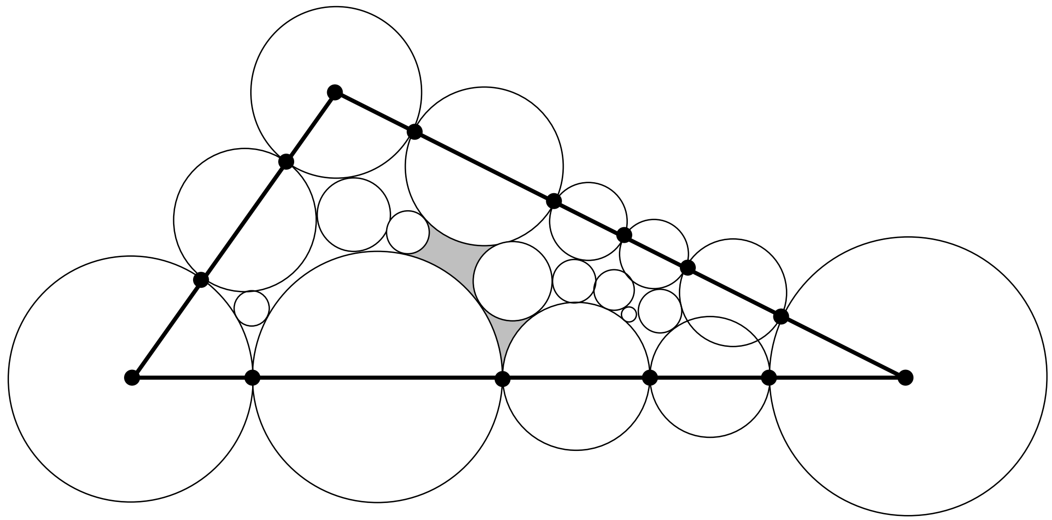

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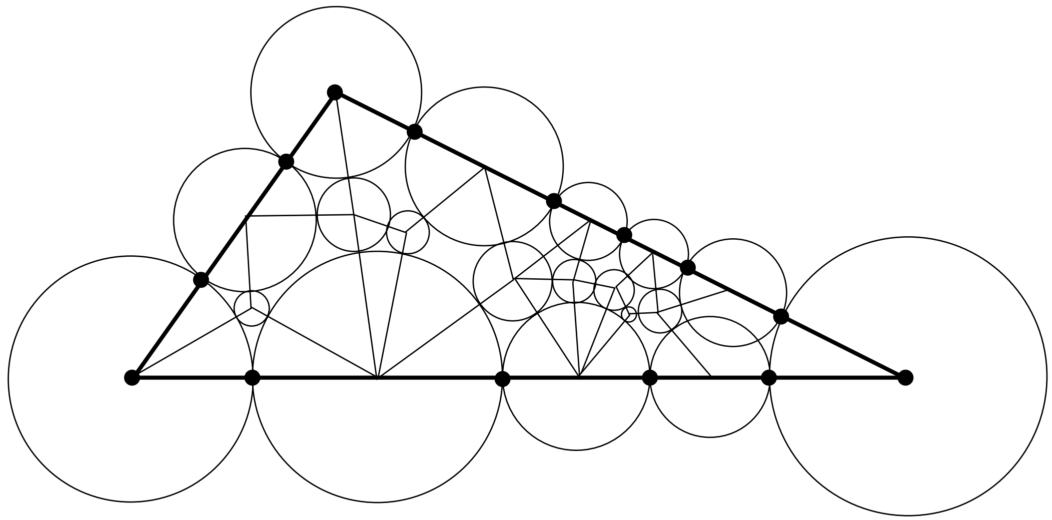
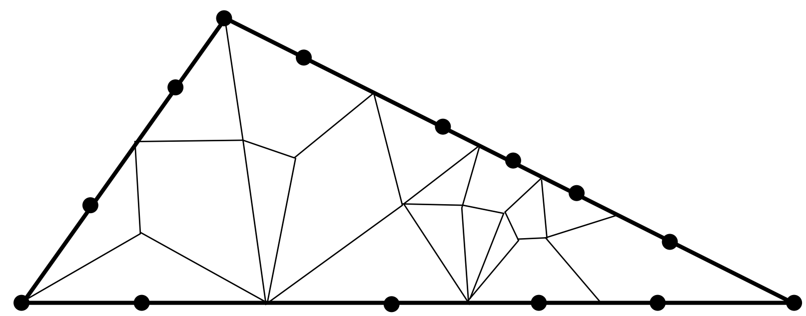


- Pack interior with disjoint disks so only 3-sided and 4-sided regions remain.
- Connect centers.

Sketch of BMR lemma:



• Pack interior with disjoint disks so only 3-sided and 4-sided regions remain.
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• Divides triangle into triangles and quadrilaterals.

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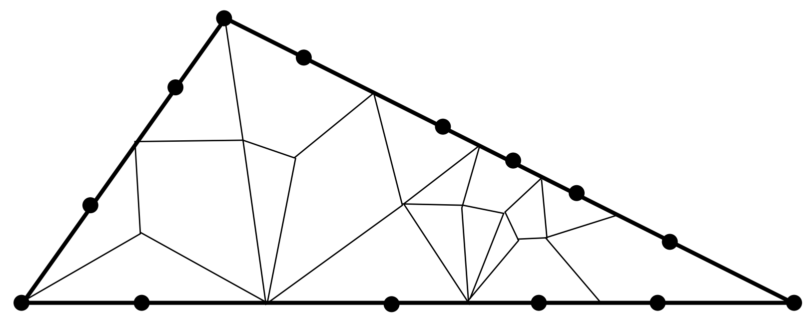
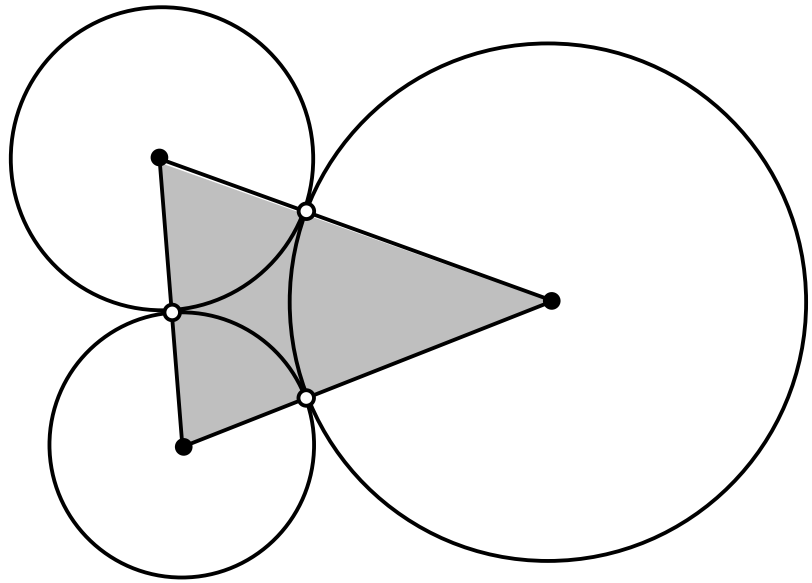


• Pack interior with disjoint disks so only 3-sided and 4-sided regions remain.
• Connect centers.
• Divides triangle into triangles and quadrilaterals.

We want to nonobtusely triangulate each region without adding new vertices along boundary. Several cases.

First case: decompose 3-region into right triangles



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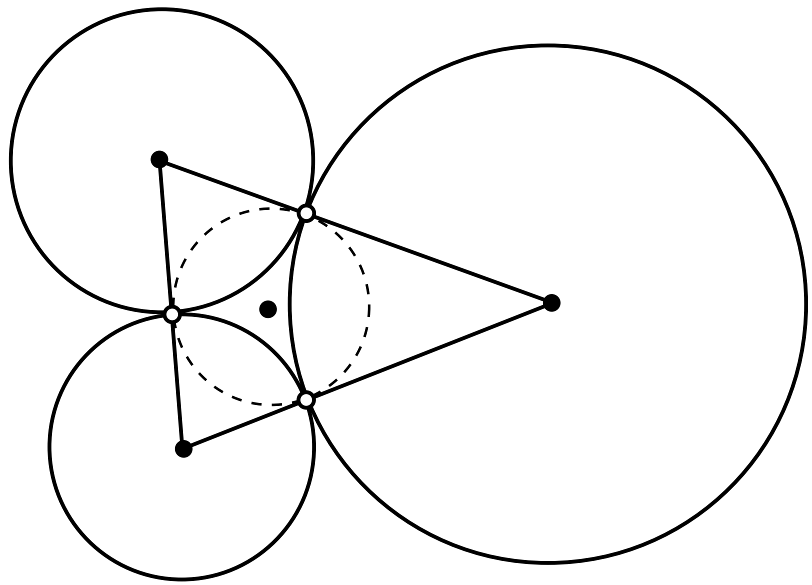

- Add center of circle through the three tangent points.

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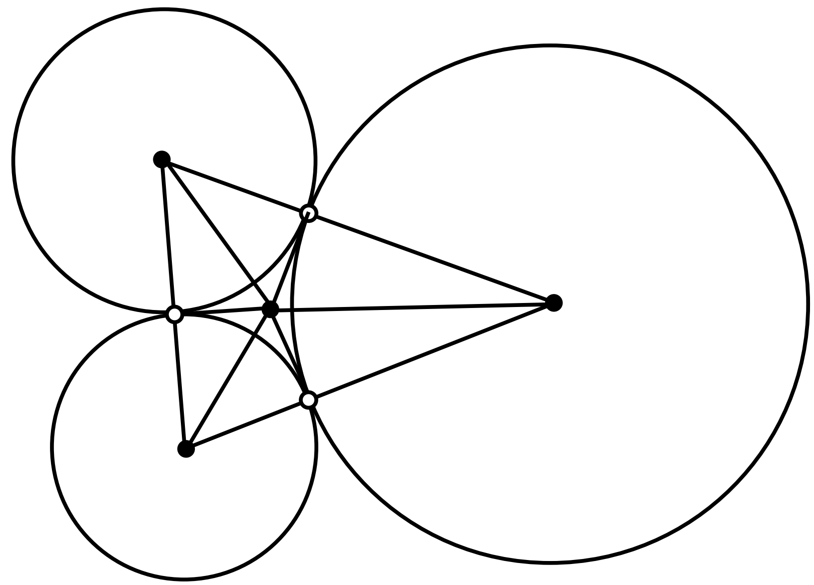

- Add center of circle through the three tangent points.
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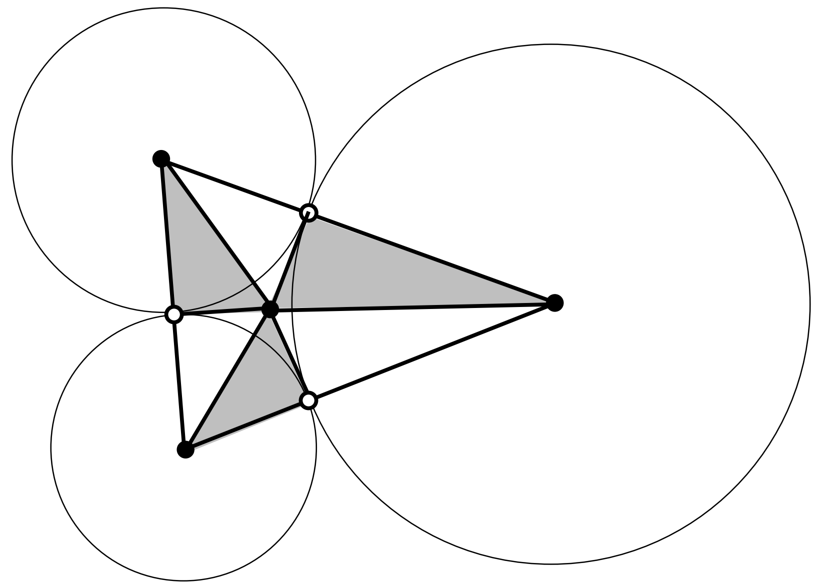
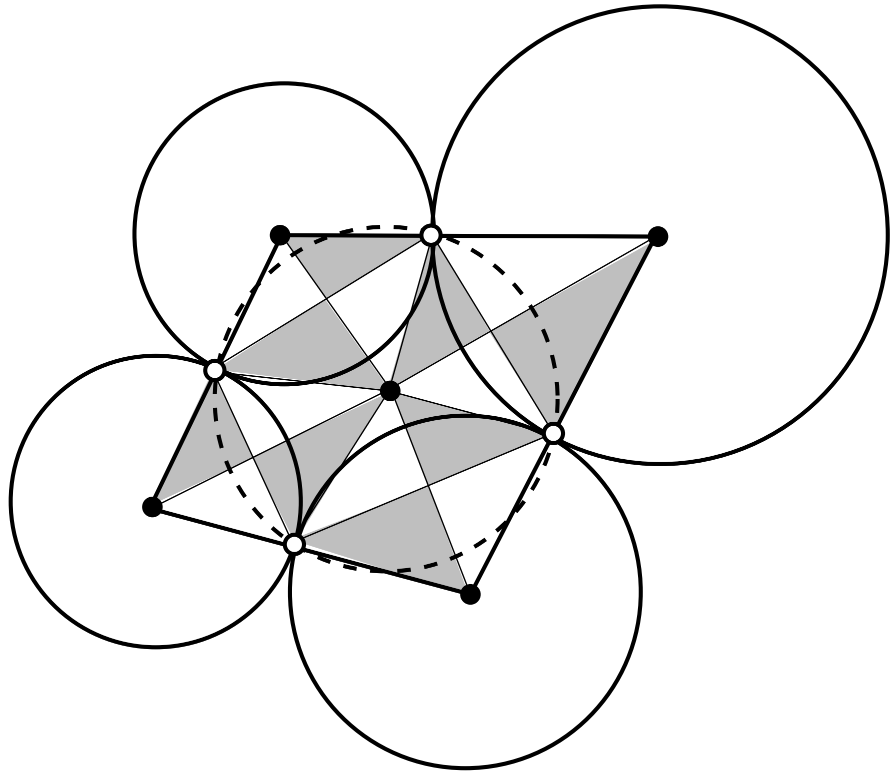


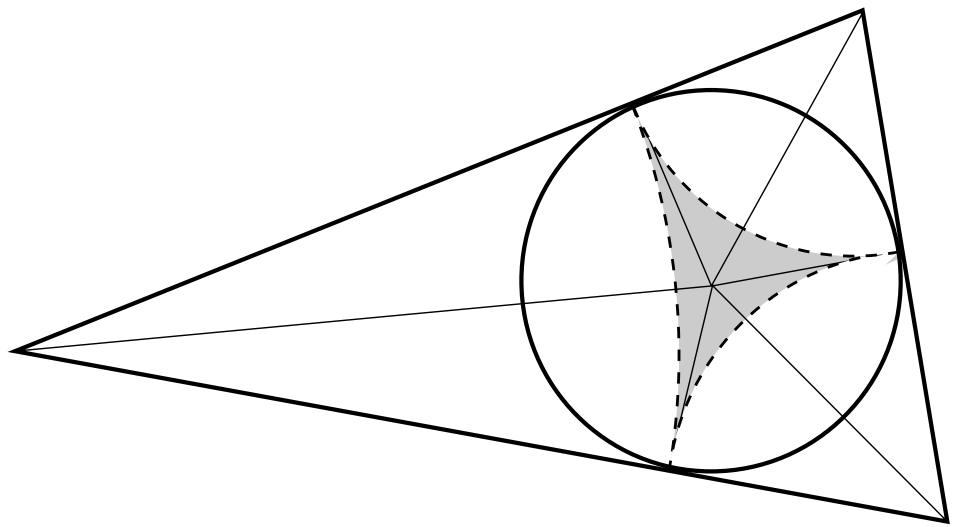
- Add center of circle through the three tangent points.
- Connect center to tangent points and centers of circles

The 4-regions are similar (but several cases arise).



So Gabriel covering gives a nonobtuse triangulation.

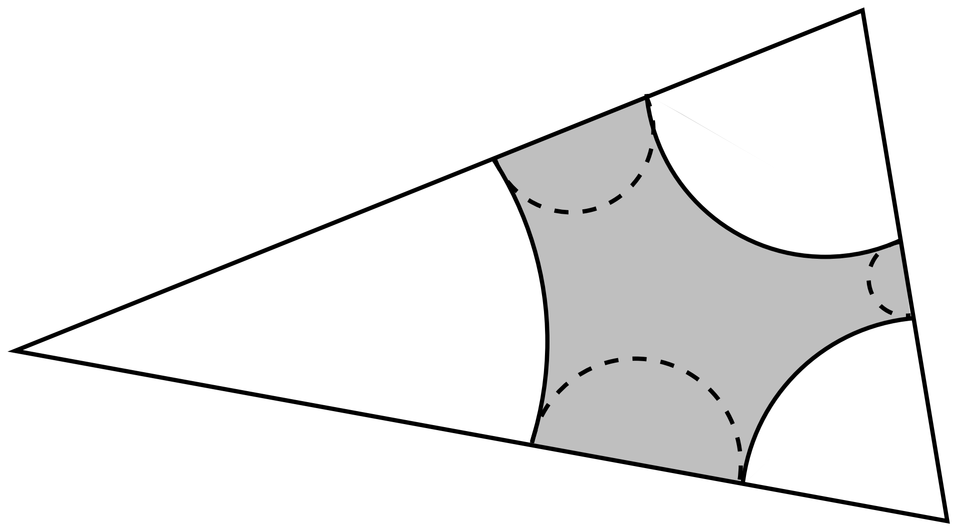
Construct Gabriel points:



Break every triangle into thick and thin parts.

Thin parts = corners, Thick part = central region

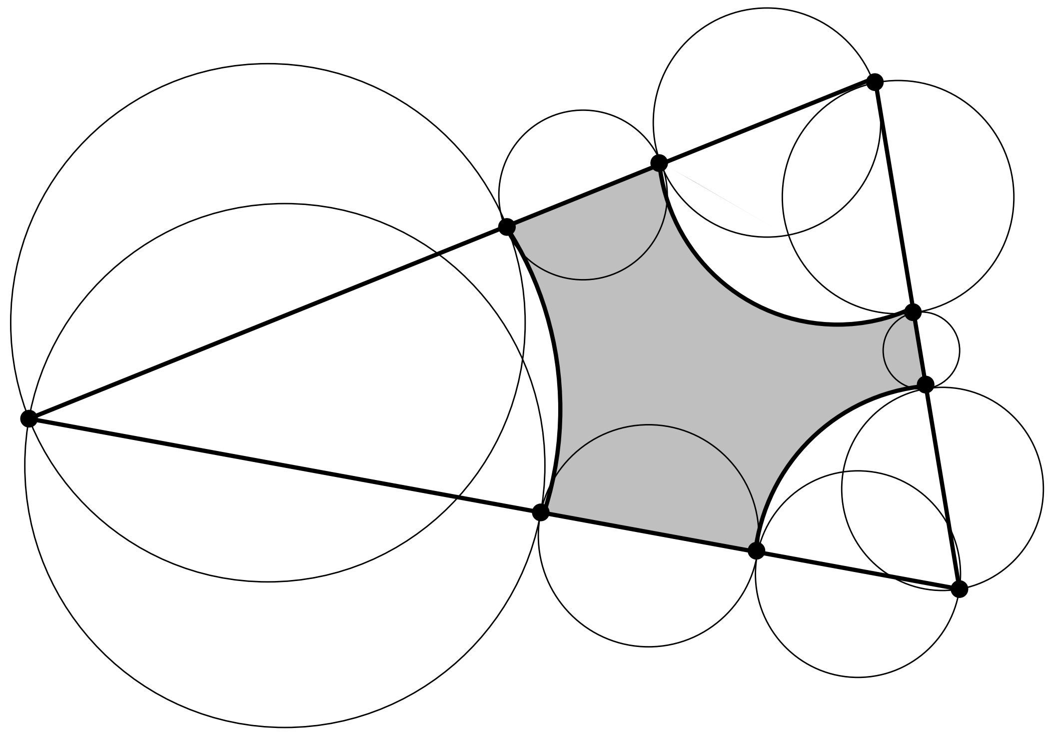
Construct Gabriel points:



Advantageous to increase thick part.

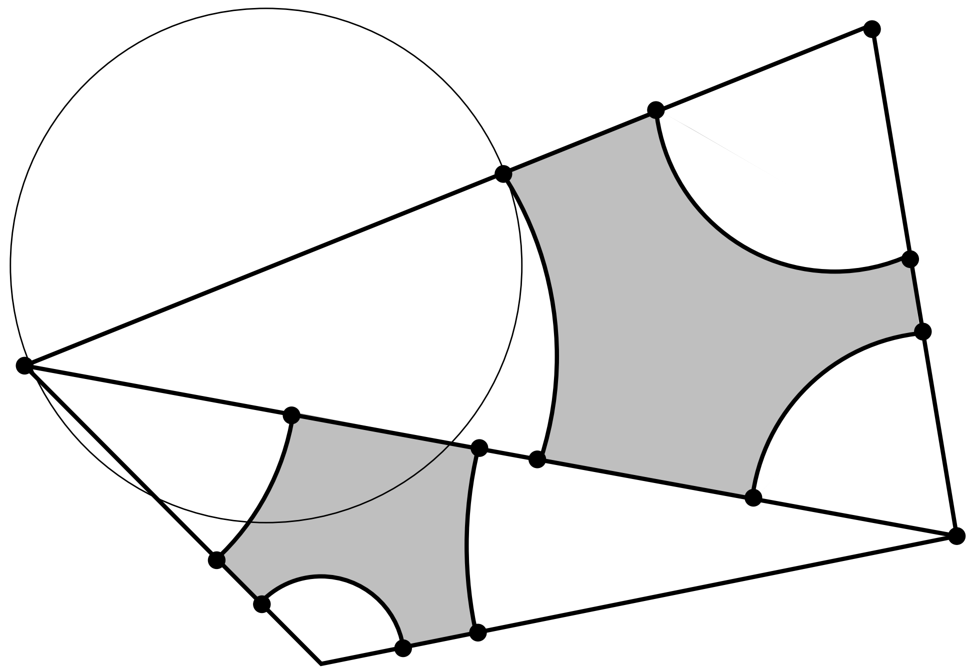
Thick sides are base of half-disk inside triangle.

Construct Gabriel points:



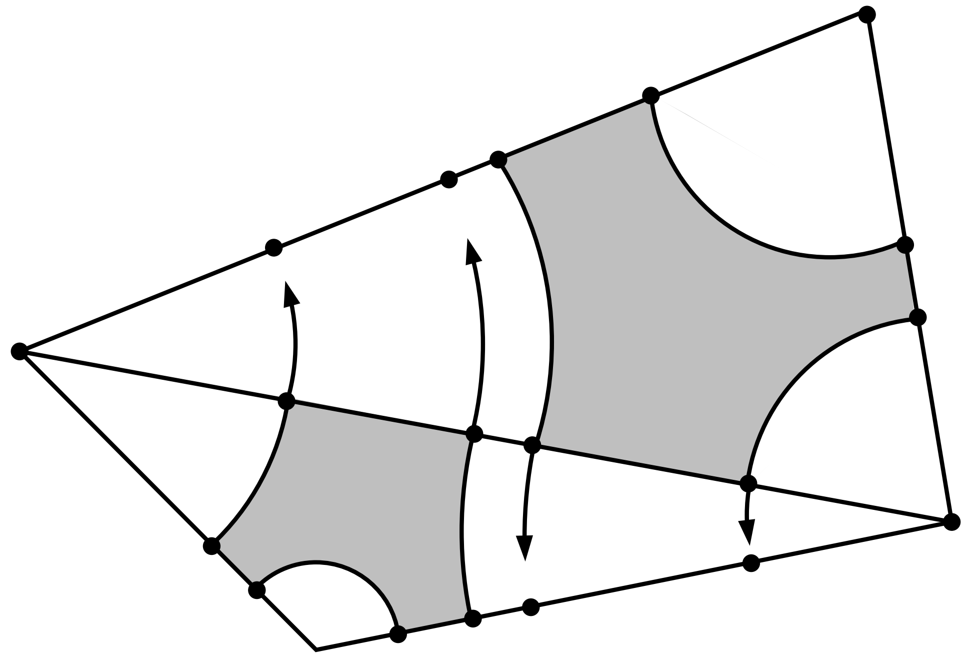
Then vertices of thick part give Gabriel edges.

Construct Gabriel points:



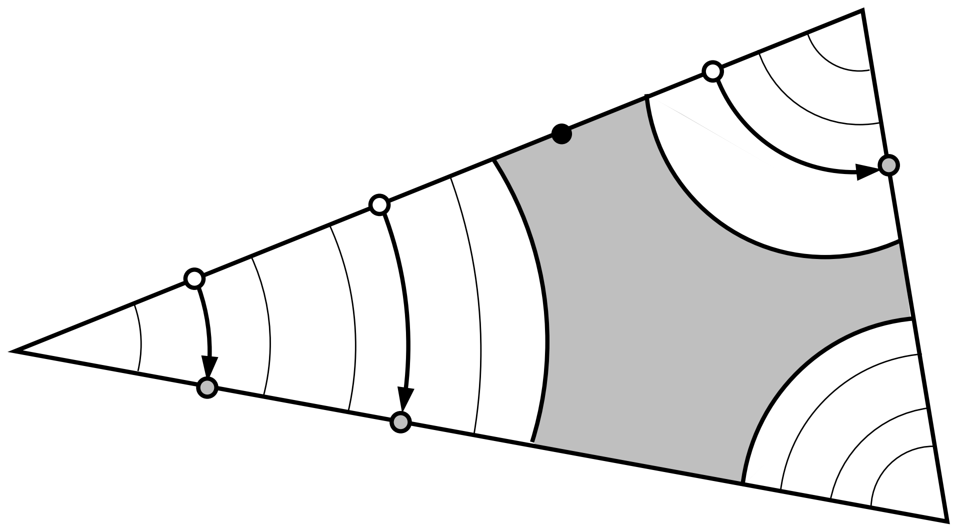
But, adjacent triangle can make Gabriel condition fail.

Construct Gabriel points:

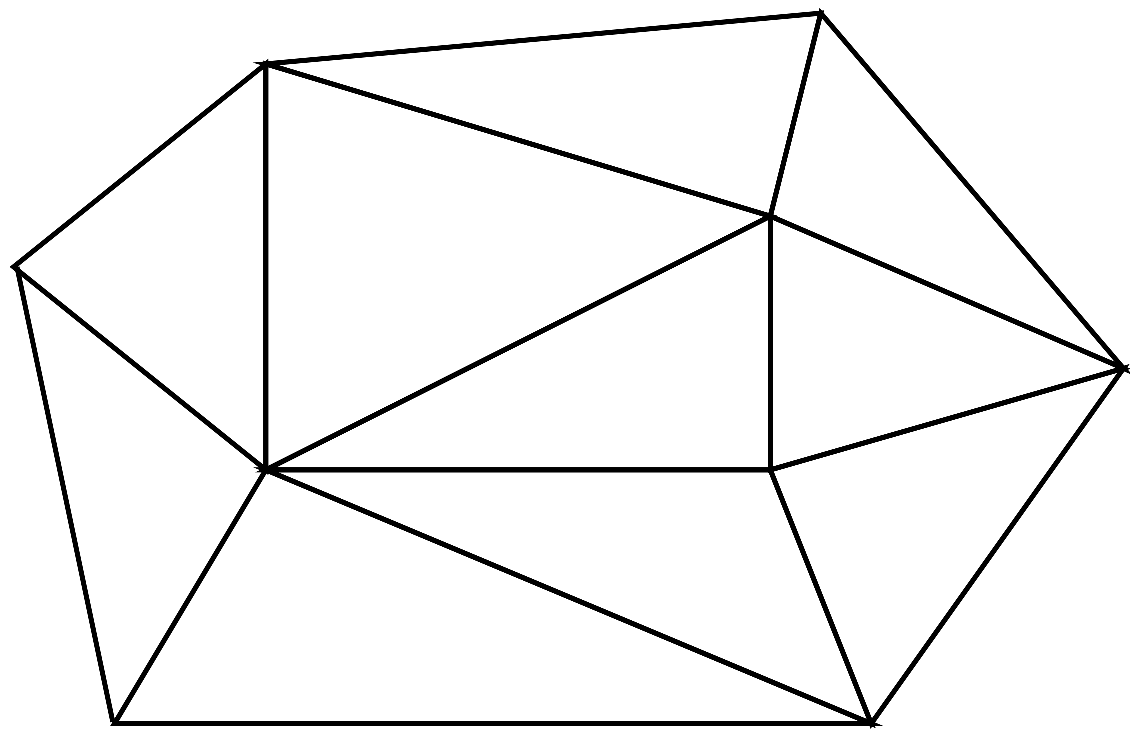


Idea: "Push" vertices across the thin parts.

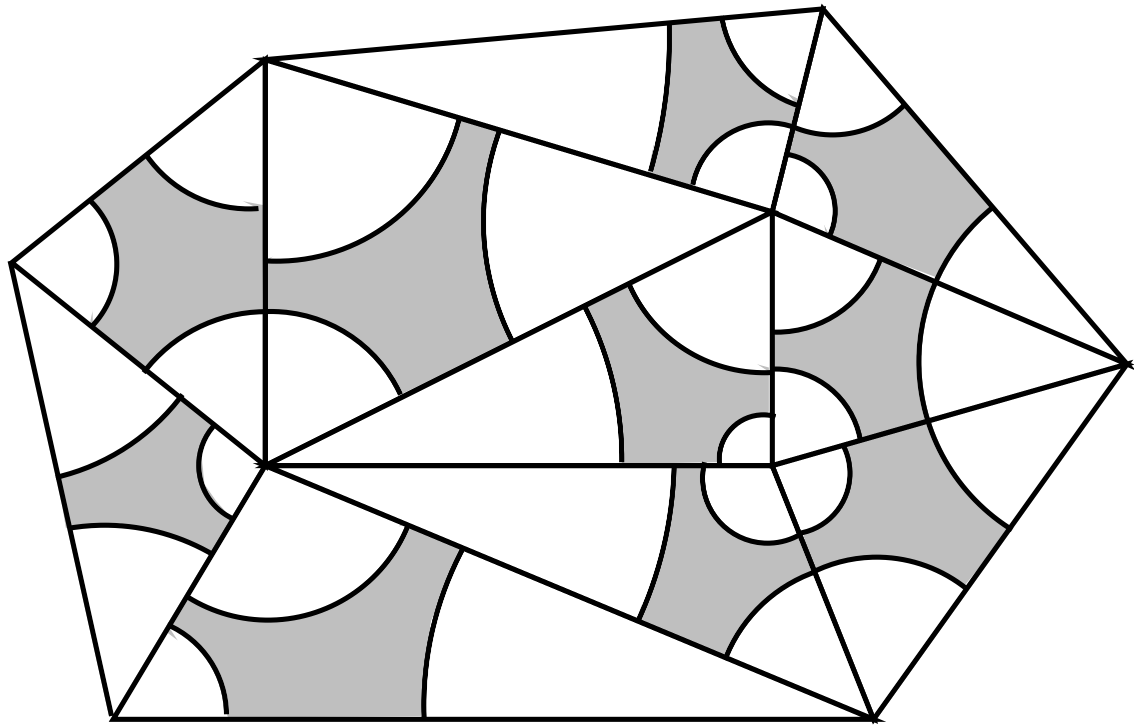
Construct Gabriel points:



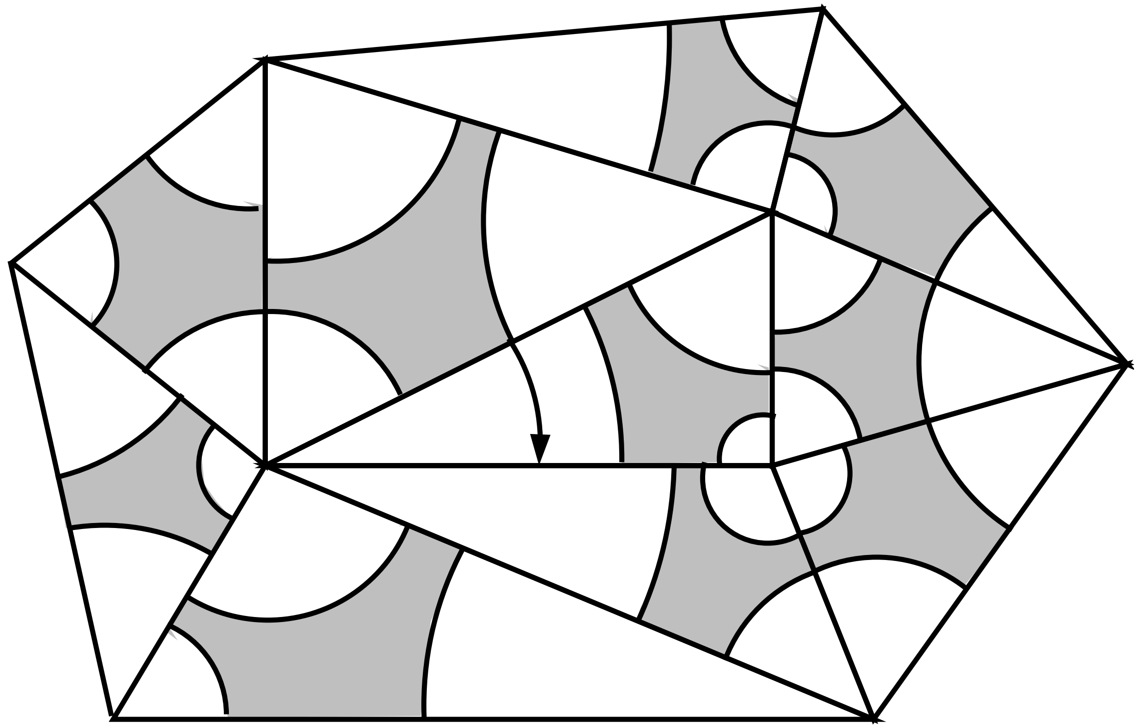
Thin parts foliated by circles centered at vertices.



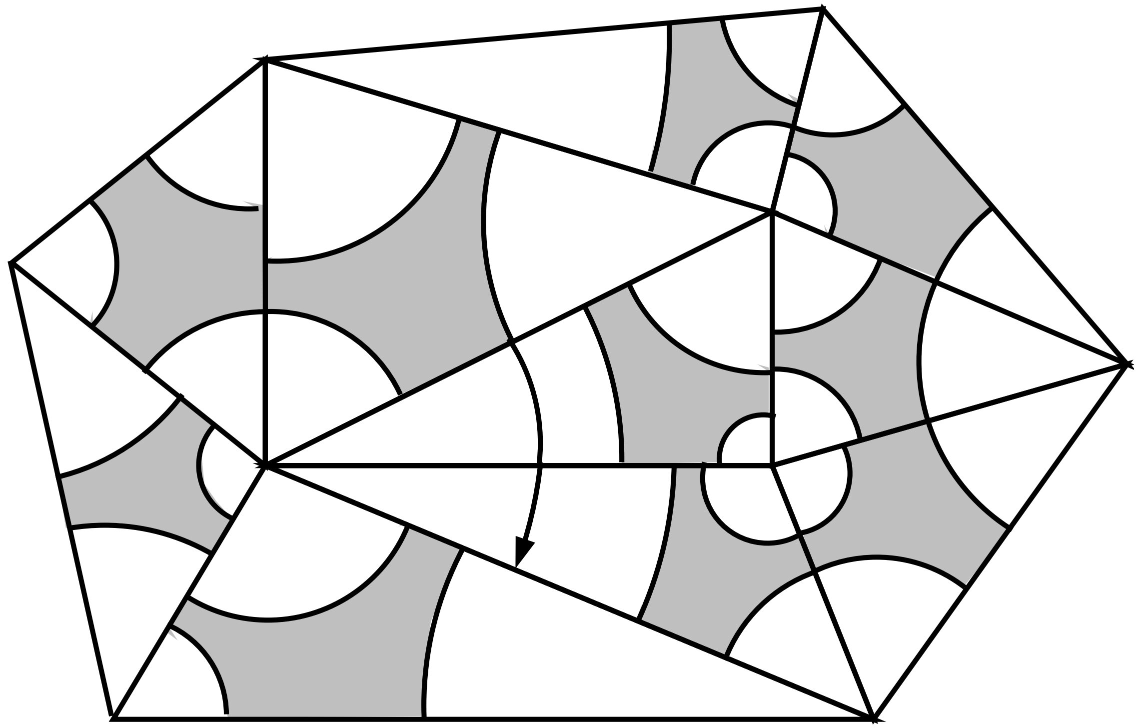
- Start with any triangulation.



- Start with any triangulation.
- Make thick/thin parts.



- Start with any triangulation.
- Make thick/thin parts.
- Propagate vertices until they leave thin parts.



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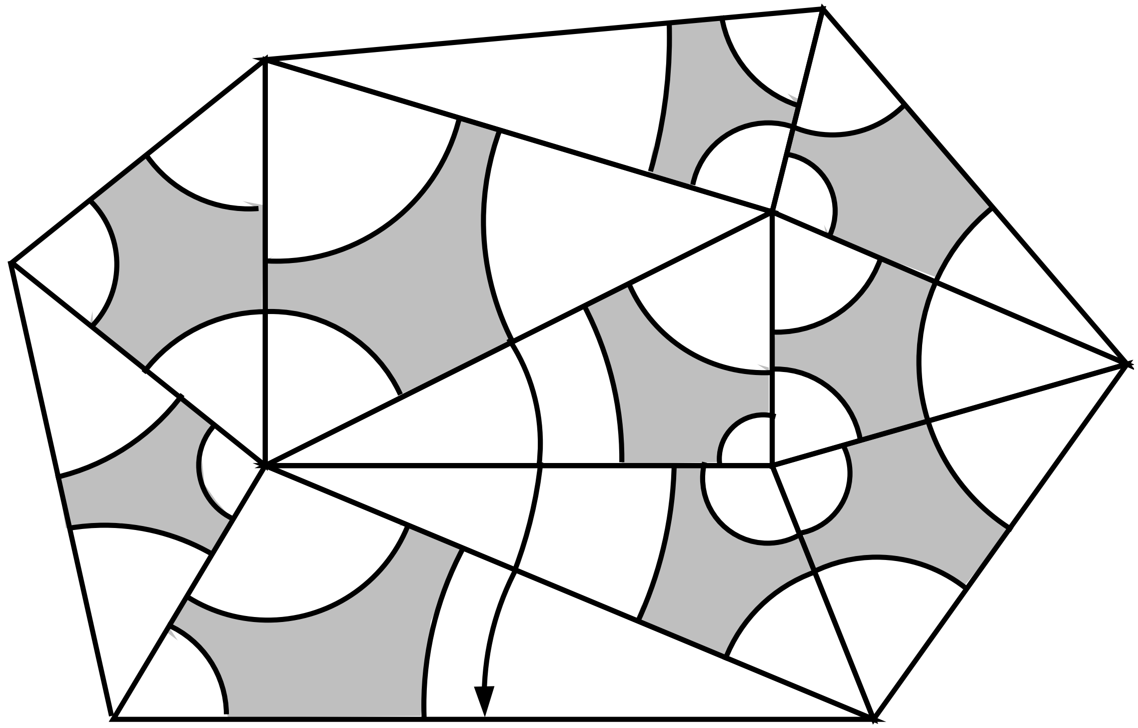

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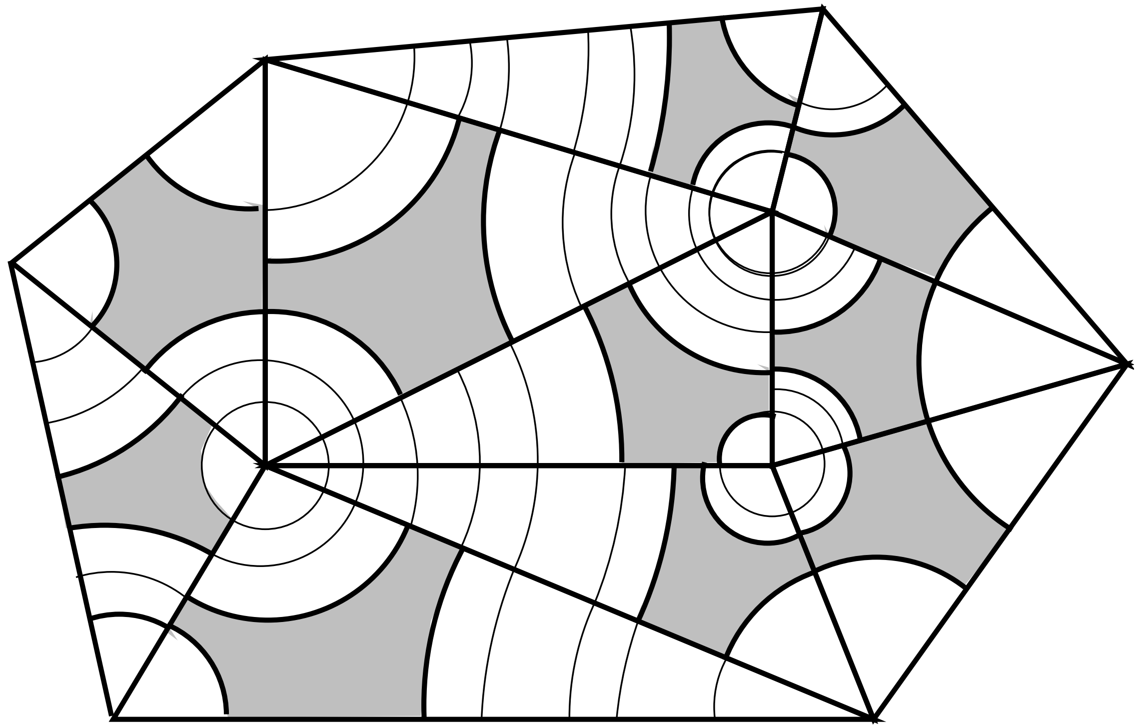

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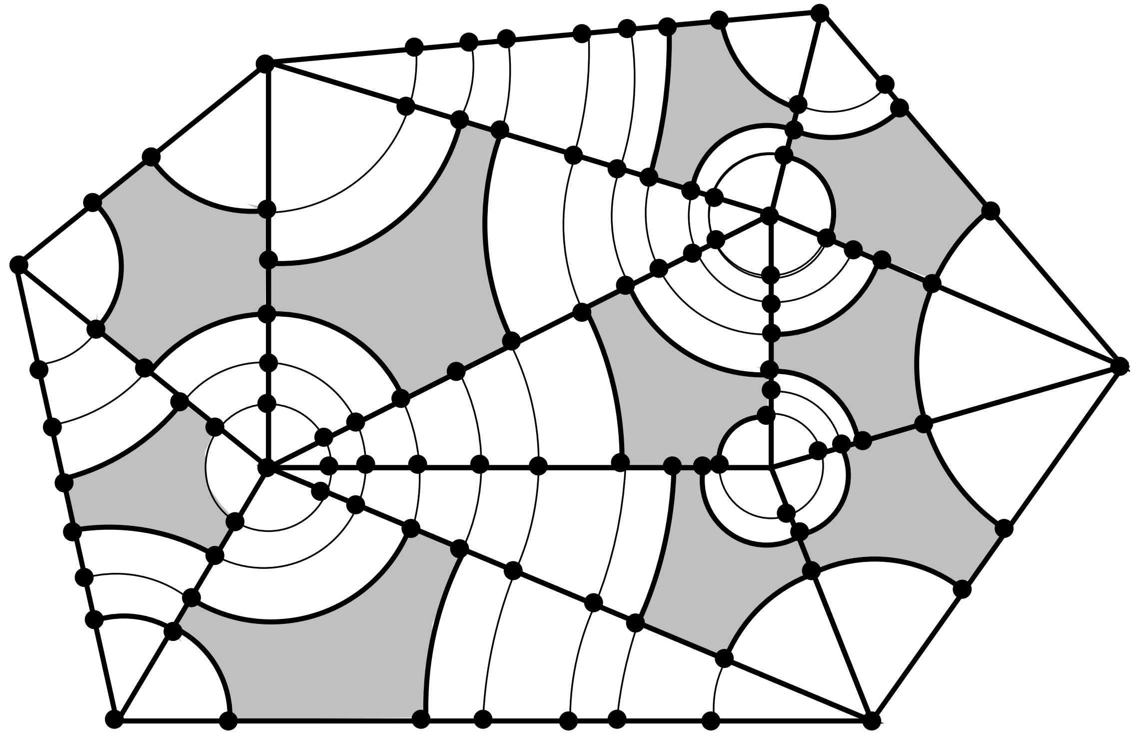
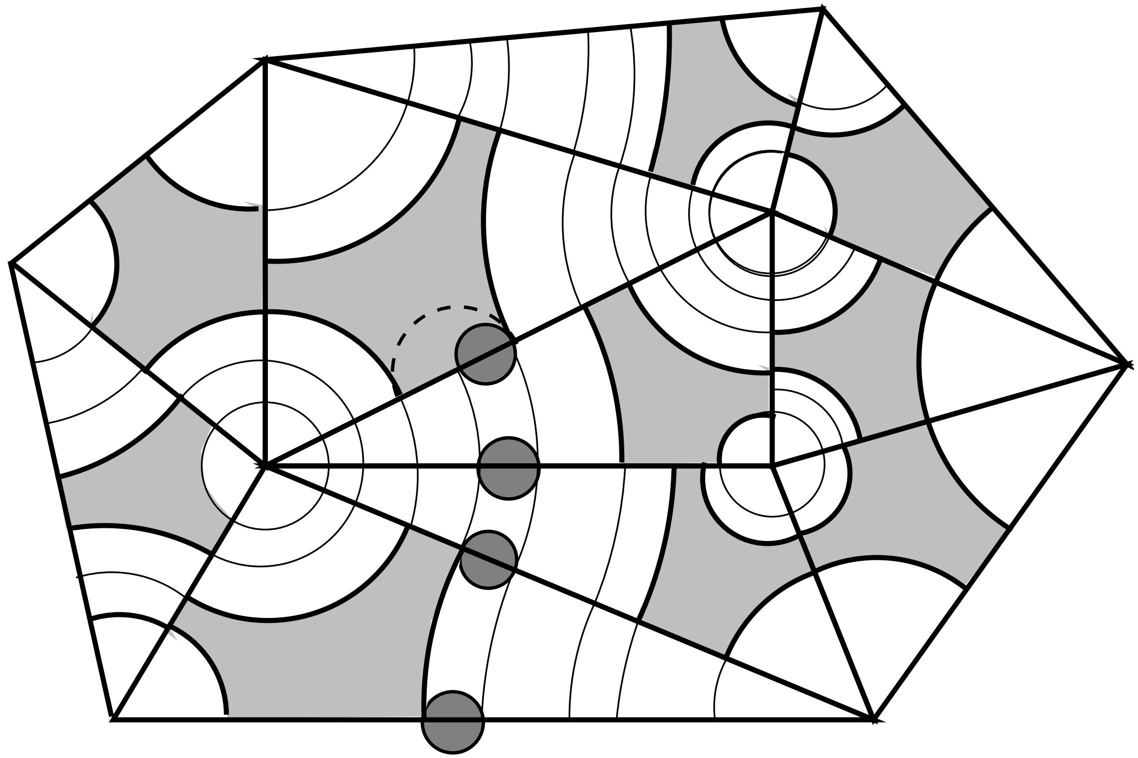


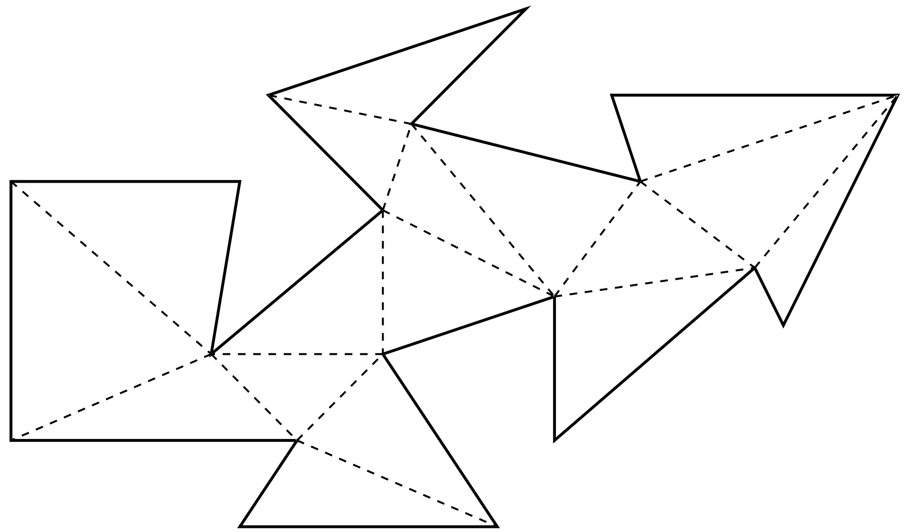
- Start with any triangulation.
- Make thick/thin parts.
- Propagate vertices until they leave thin parts.
- Intersections satisfy Gabriel condition. Why?



Tube is “swept out” by fixed diameter disk.

Disk lies inside tube or thick part or outside convex hull.

In triangulation of a n-gon, adjacent triangles form tree.



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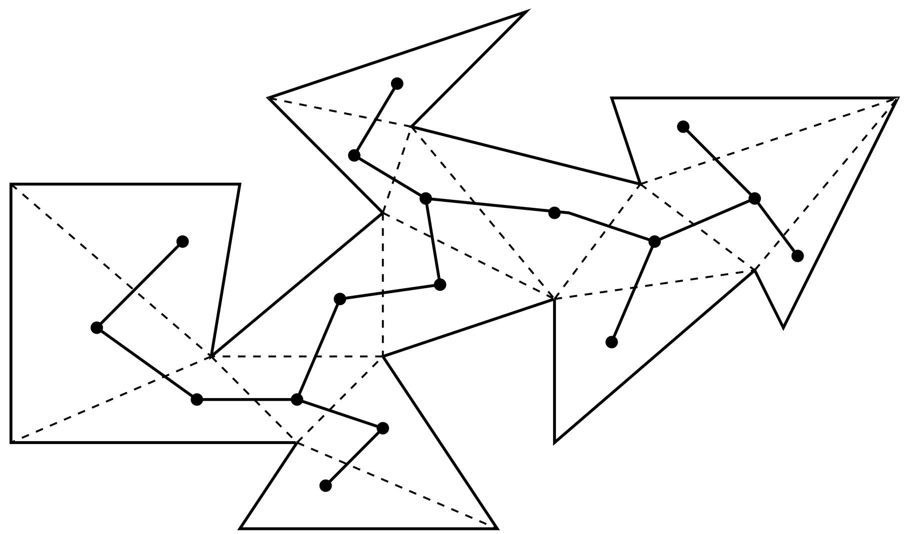

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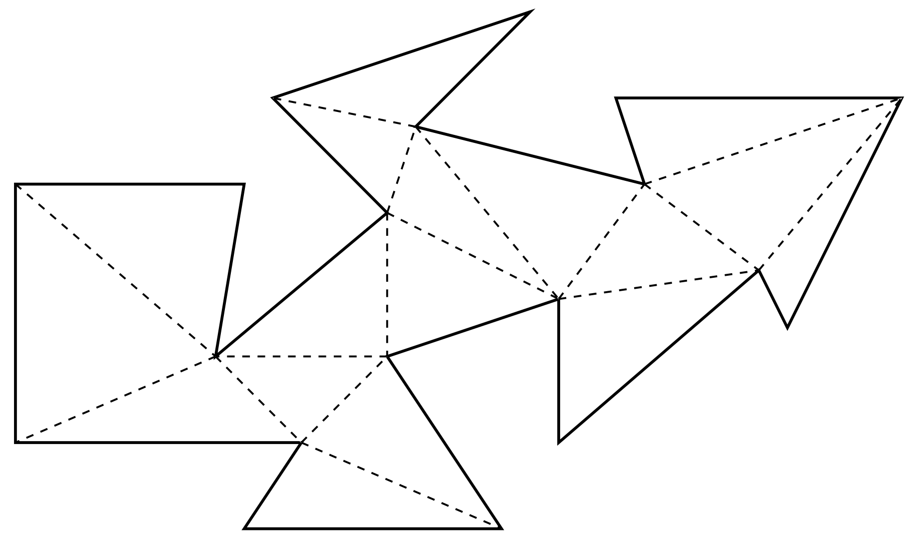

Hence paths never revisit a triangle.
$6n$ starting points, so $O(n^2)$ points are created.

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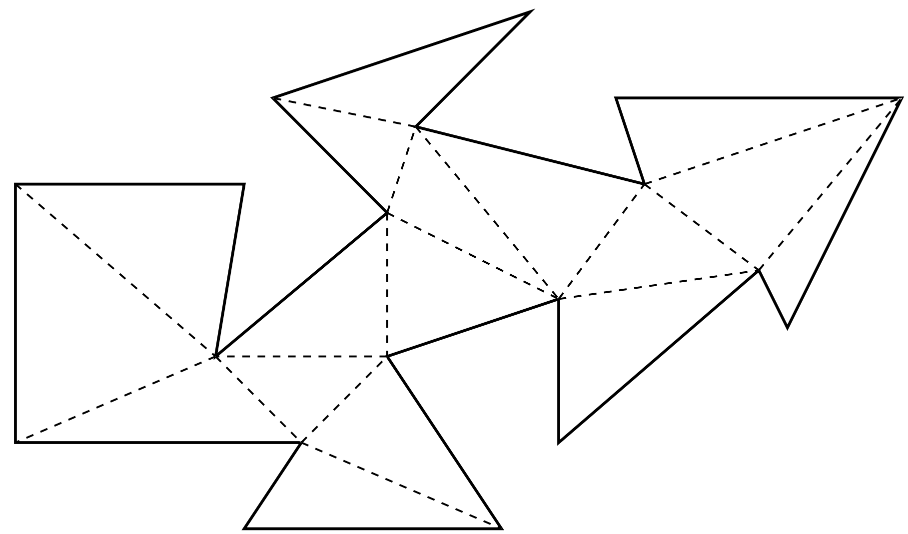

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Theorem: Any triangulation of a n-gon has a refinement into $O(n^2)$ non-obtuse triangles.

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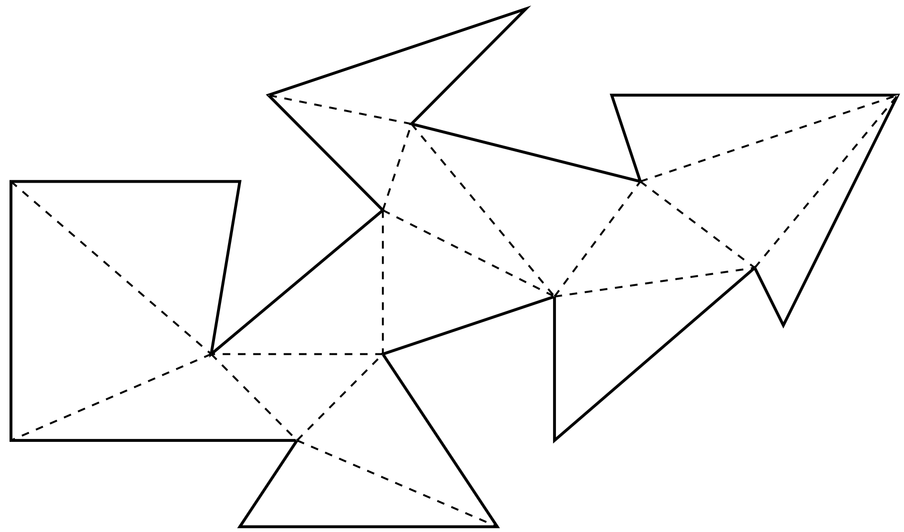

Hence paths never revisit a triangle.
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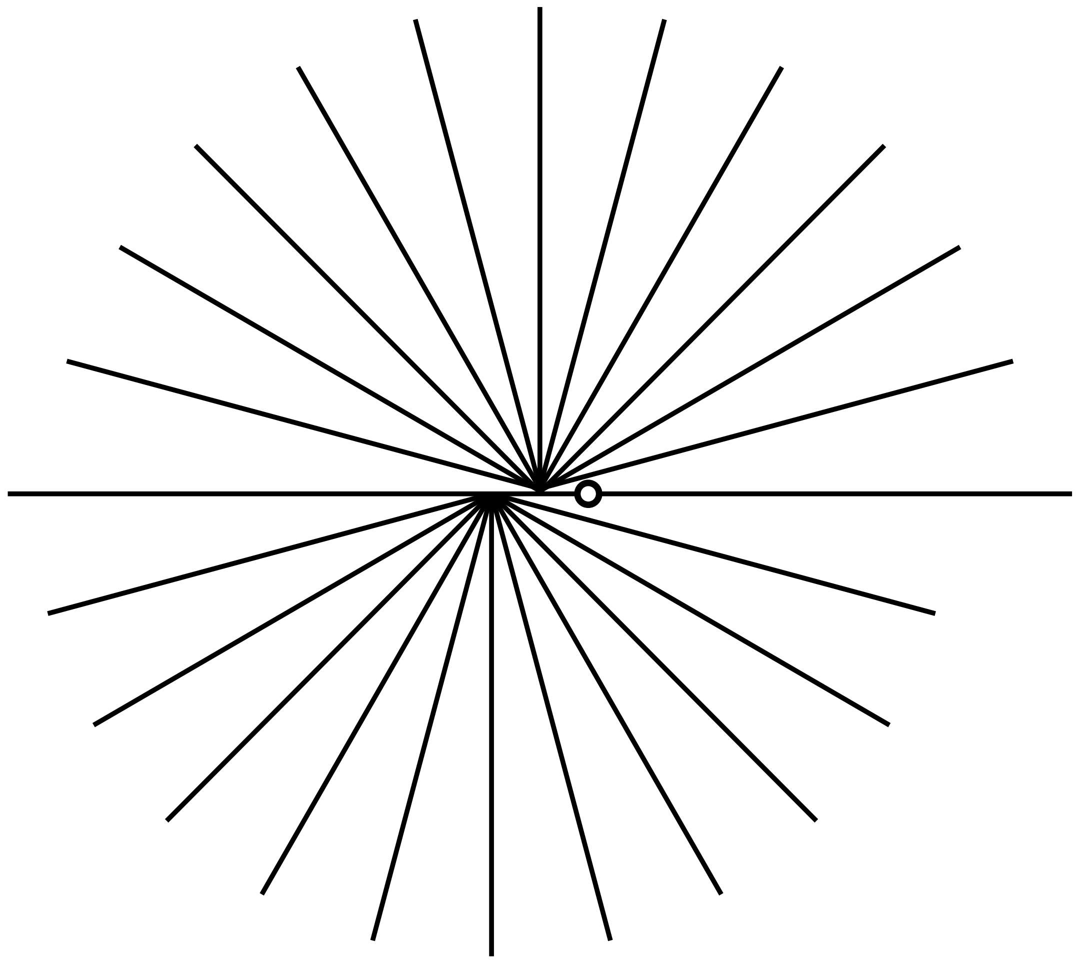
Theorem: Any triangulation of a n-gon has a refinement into $O(n^2)$ non-obtuse triangles.

Improves $O(n^4)$ bound by Bern and Eppstein (1992).

How do we get 2.5 in the NOT-Theorem?

In general, path can hit same thin part many times.



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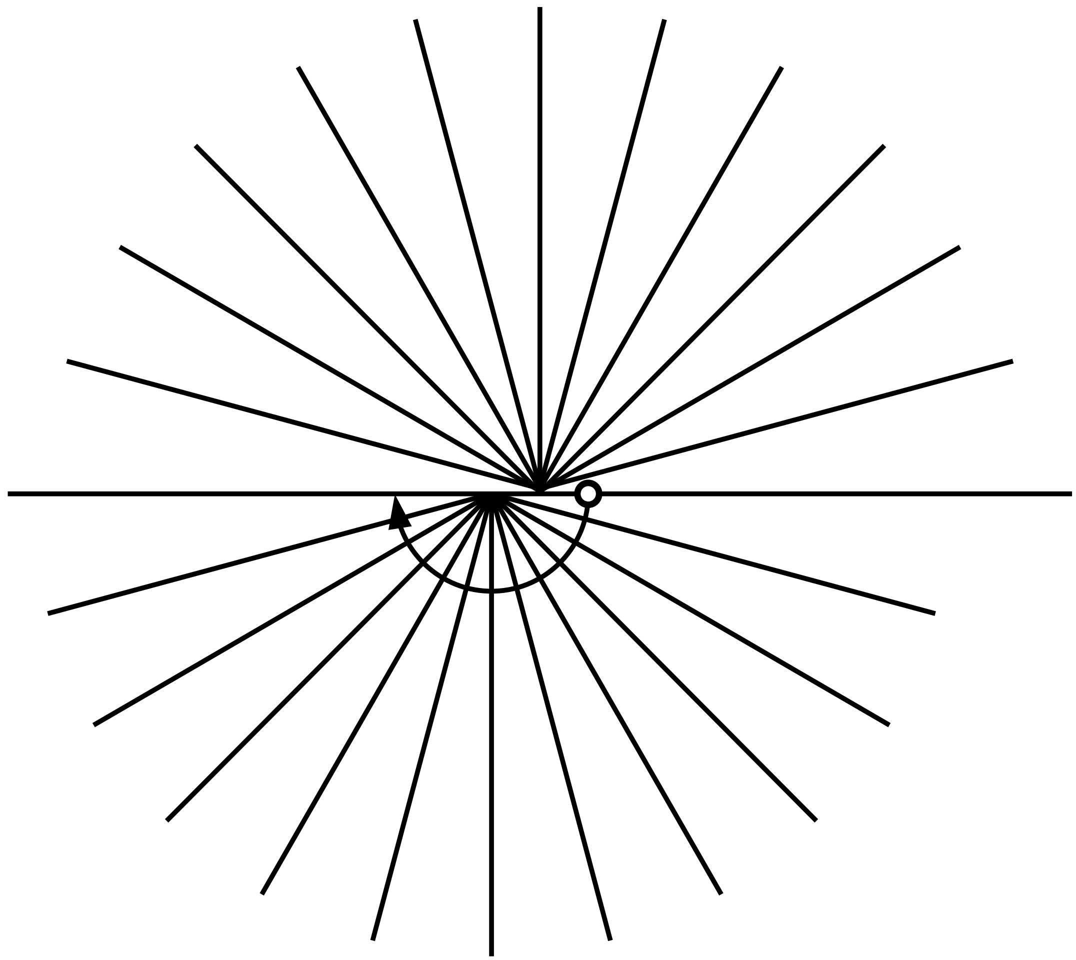

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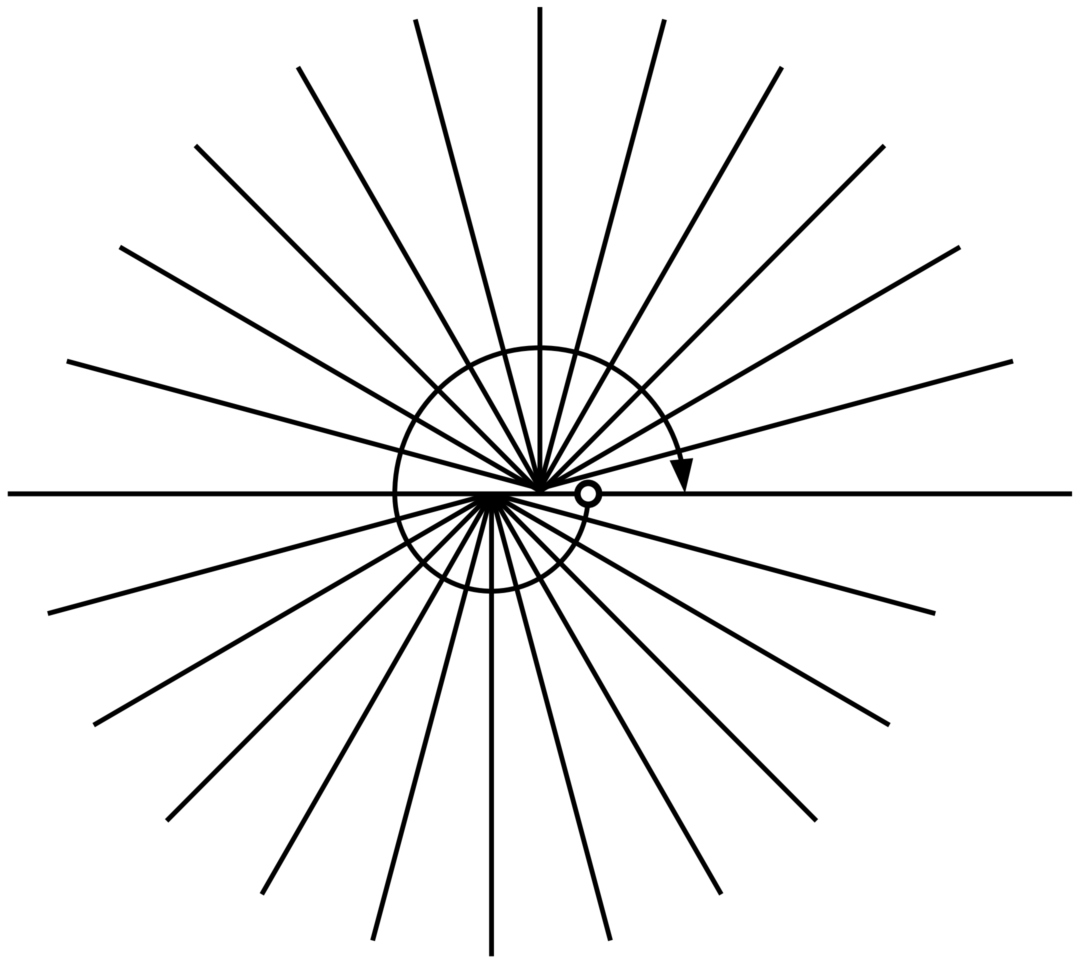

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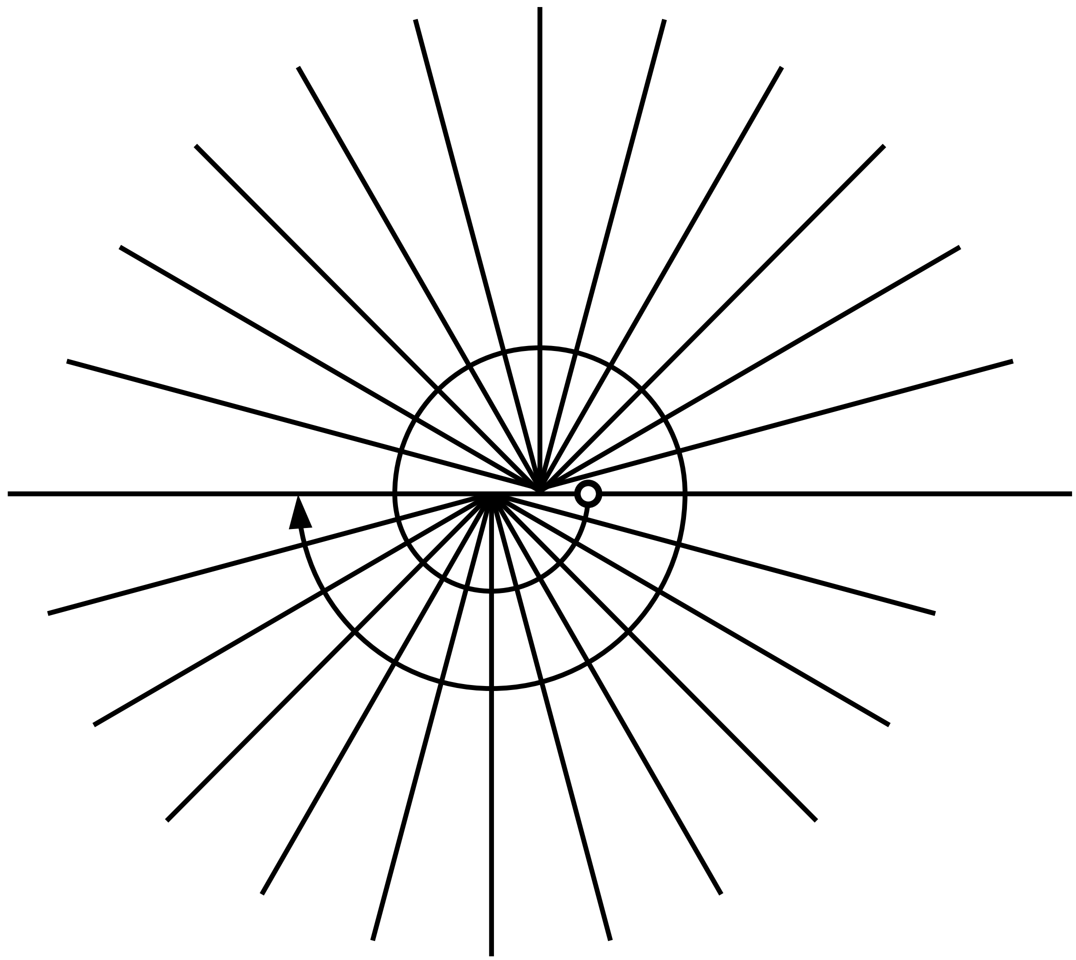

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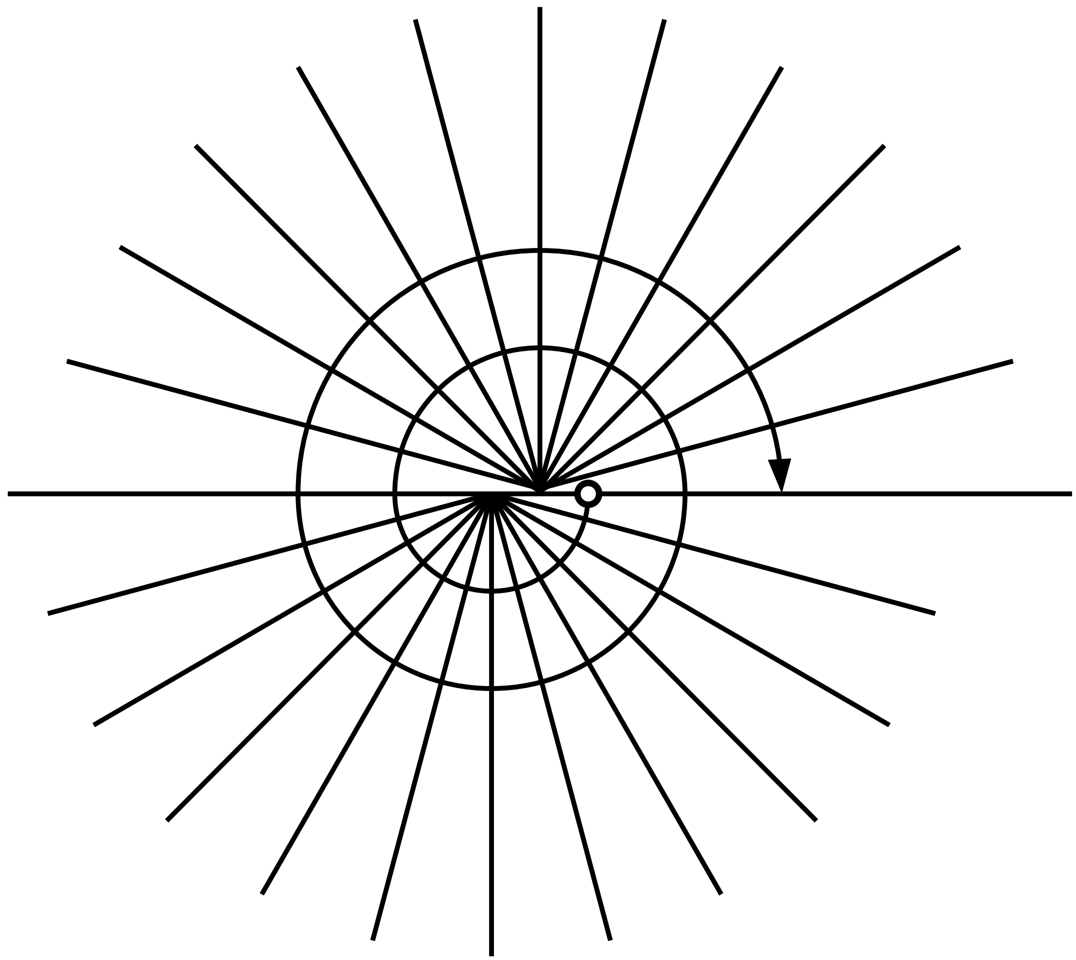

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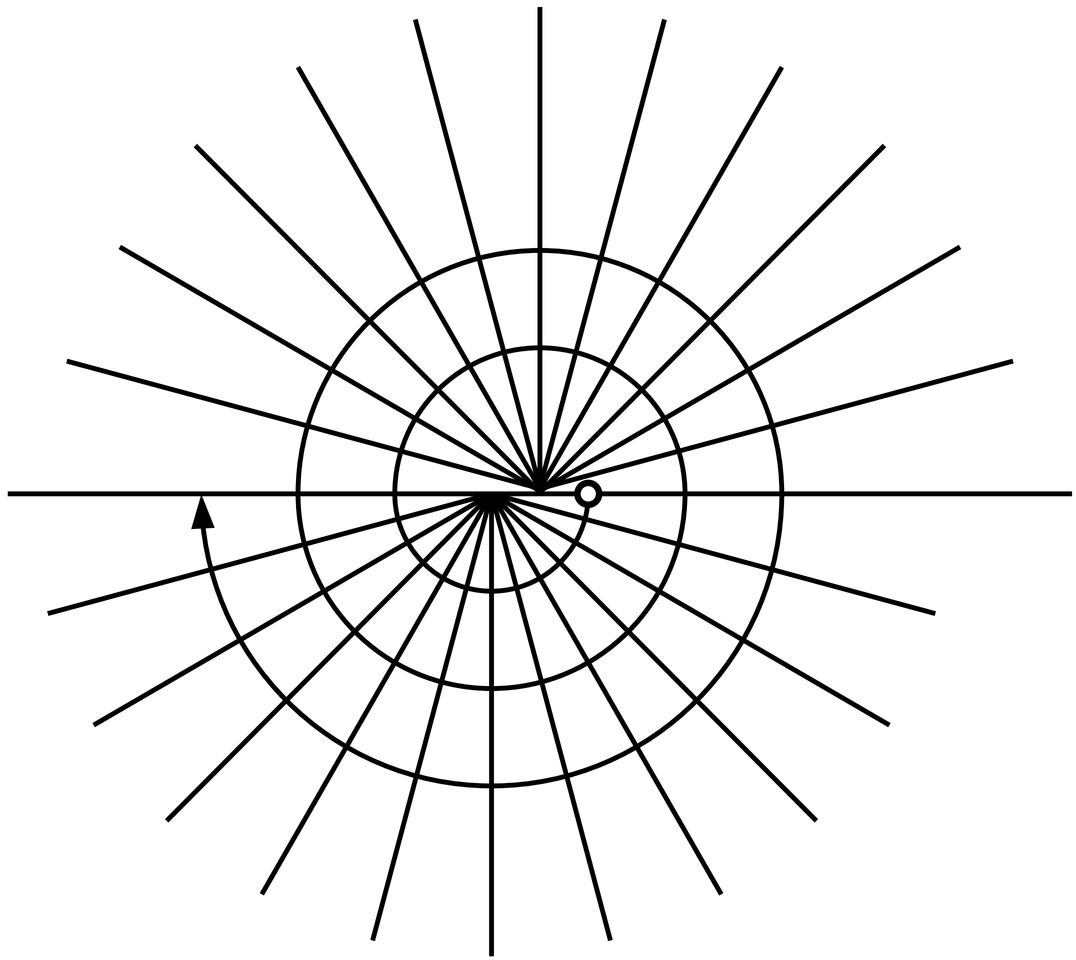

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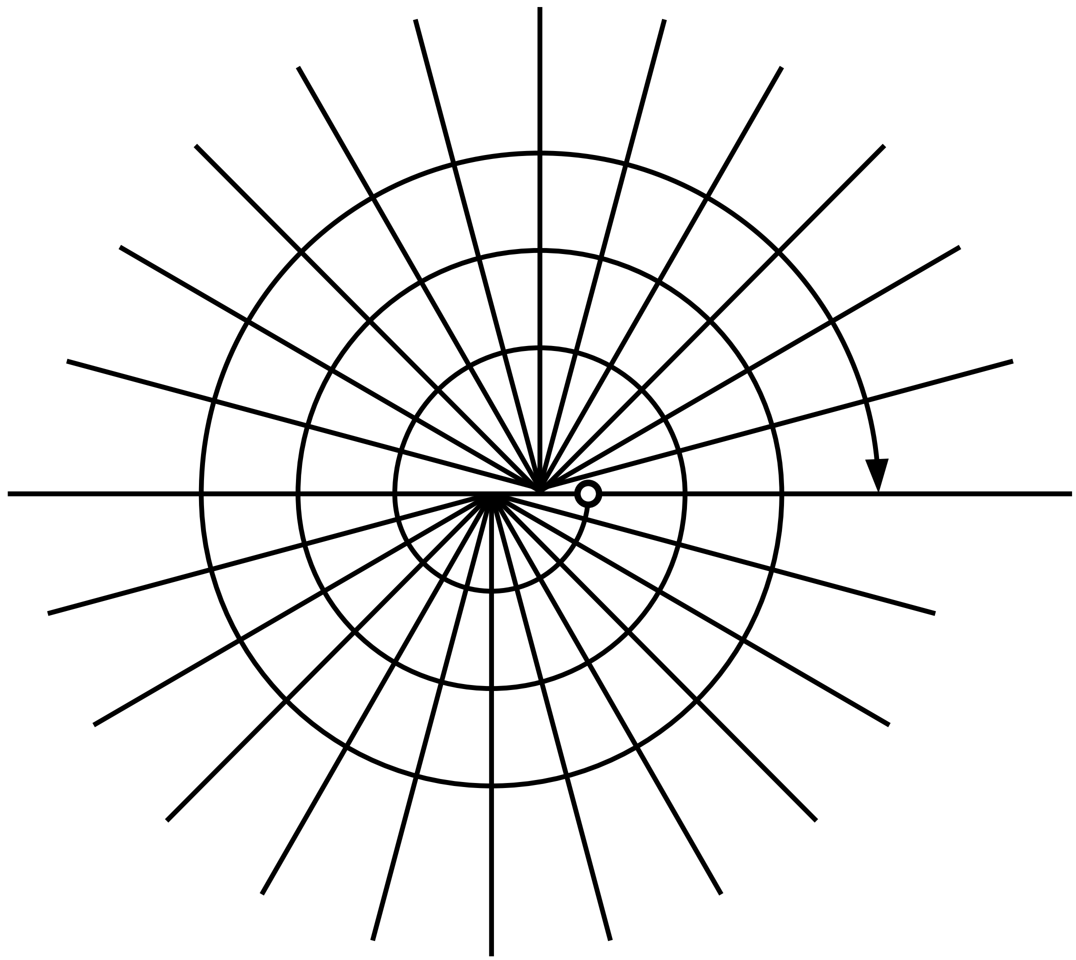
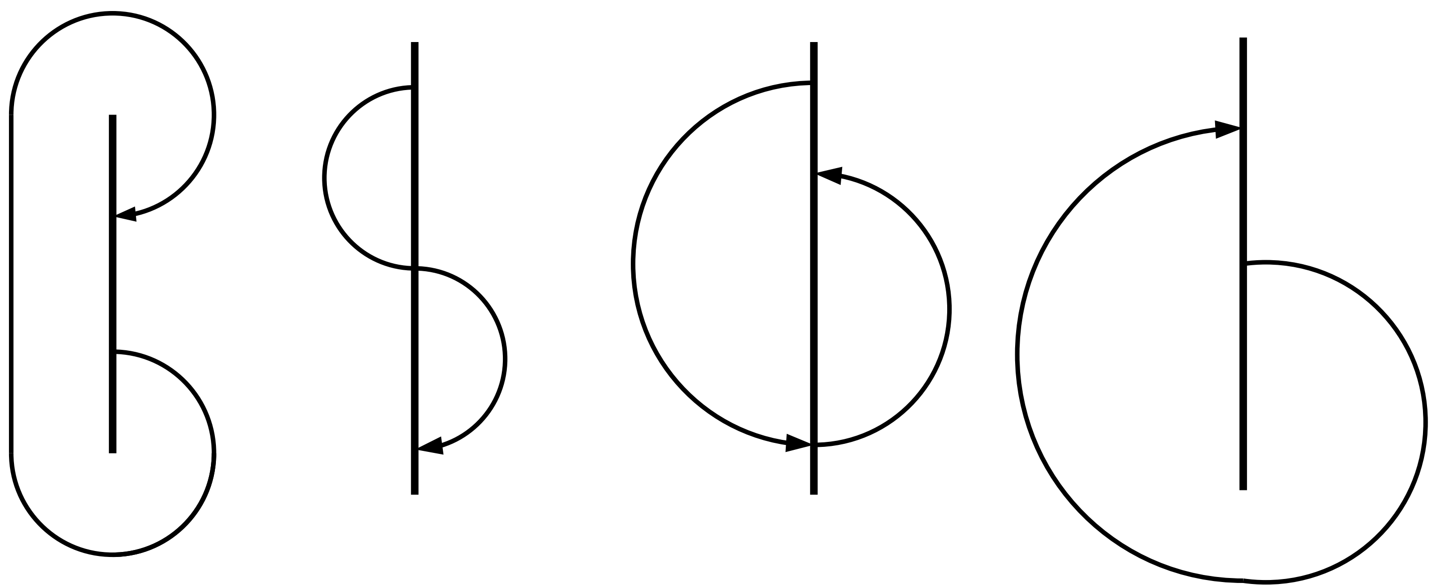
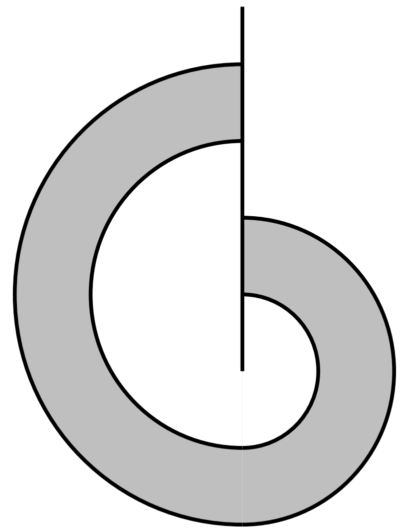


If a path returns to same thin edge at least 3 times it has a sub-path that looks like one of these:



C-curve, S-curve, G-curves

Return region consists of paths “parallel” to one of these.



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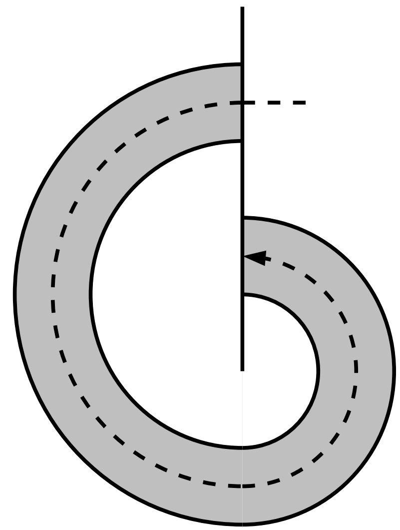

There are $O(n)$ return regions and every propagation path enters one after crossing at most $O(n)$ thin parts.

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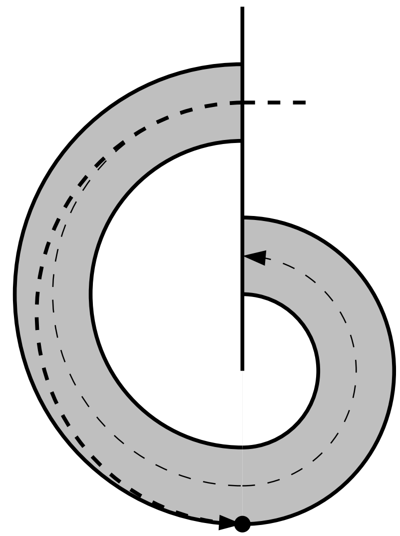
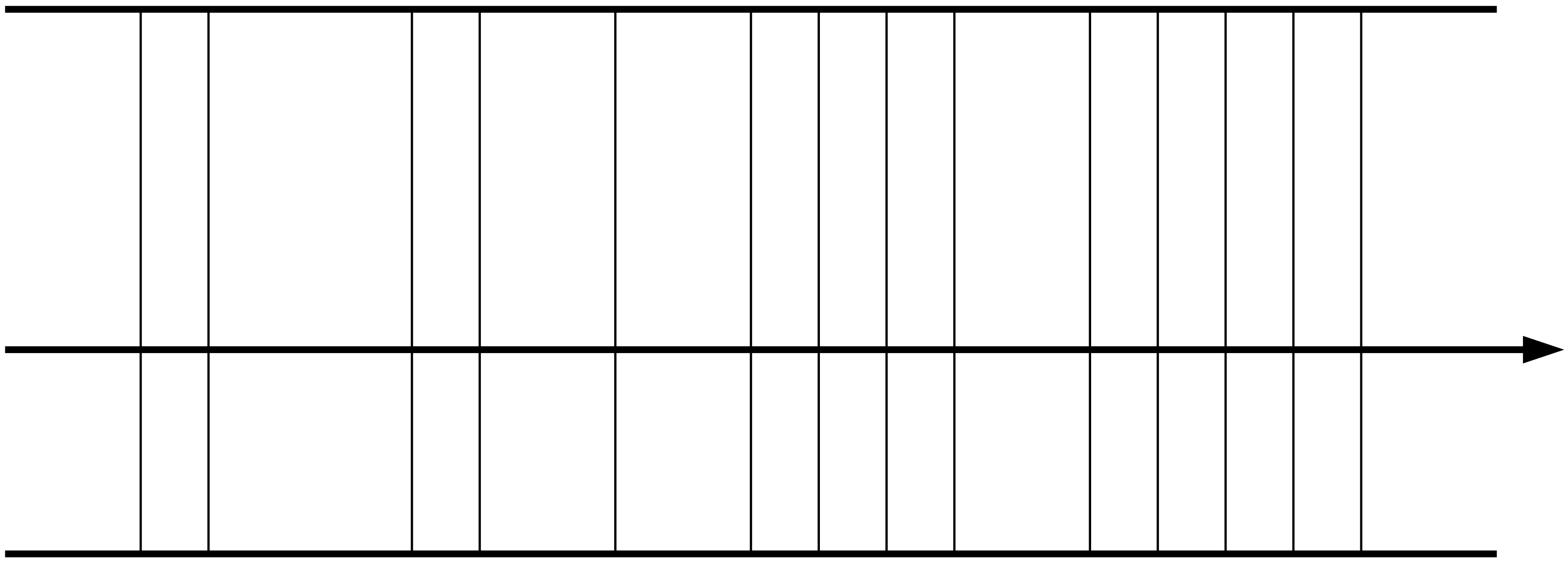


There are $O(n)$ return regions and every propagation path enters one after crossing at most $O(n)$ thin parts.

IDEA: bend paths to hit side before they exit.

Still need "disks inside tubes". Gives $O(n^2)$ if it works.

For simplicity, “straighten” region to rectangle.



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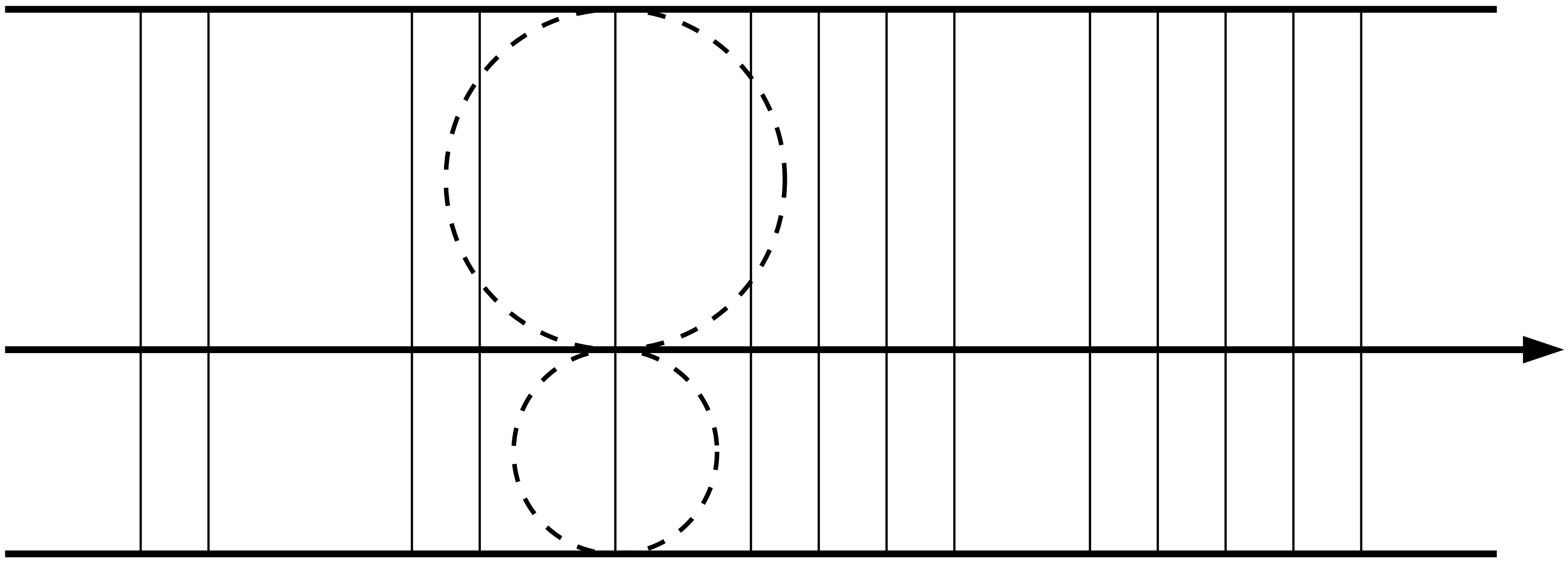
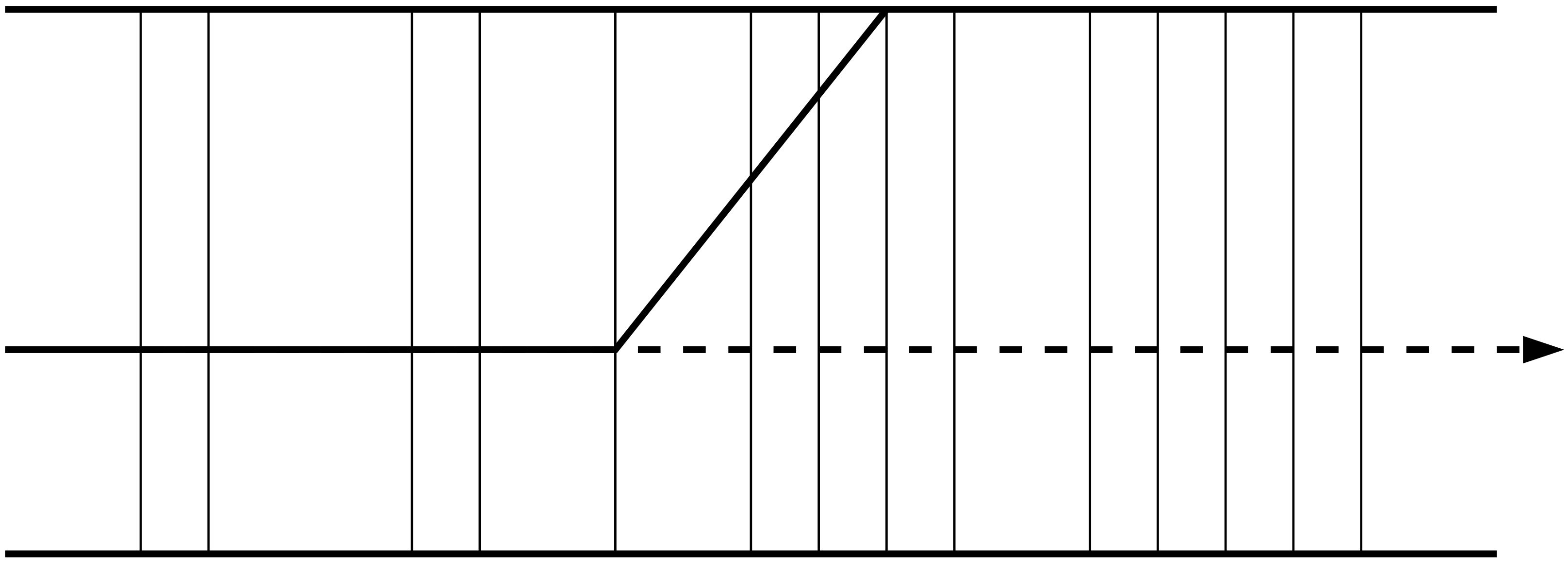


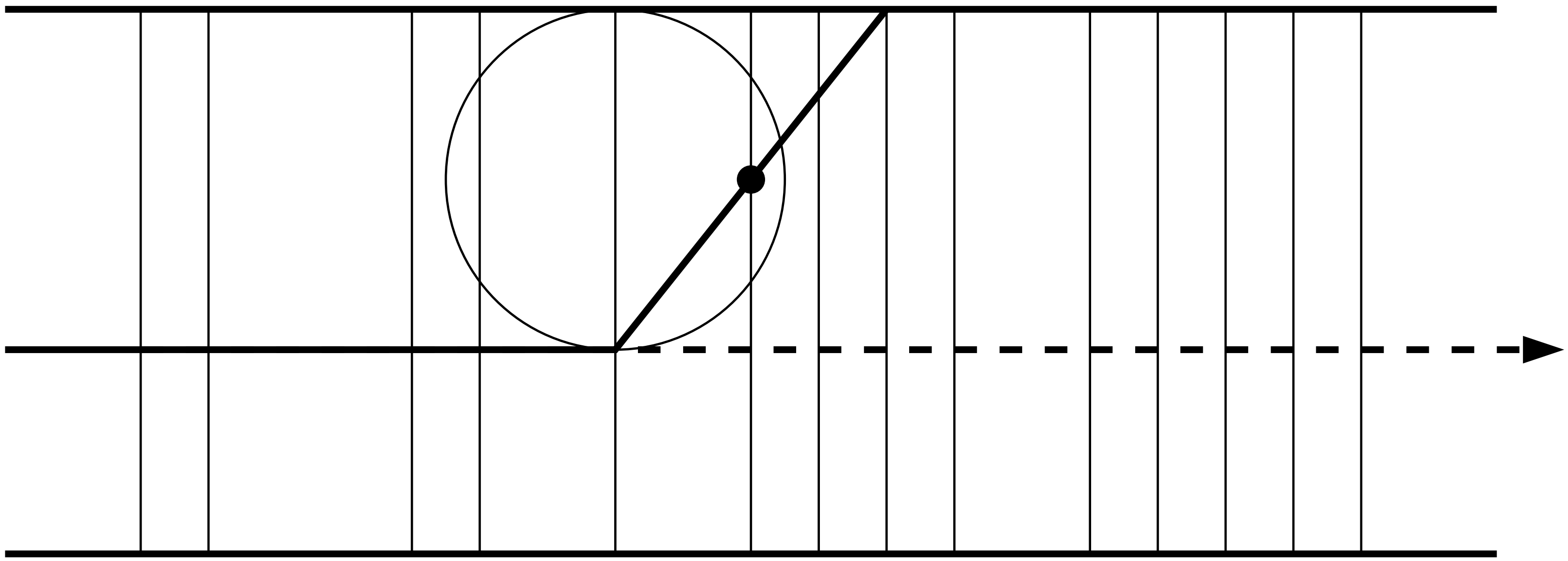
Gabriel condition is satisfied if path follows foliation.

For simplicity, "straighten" region to rectangle.



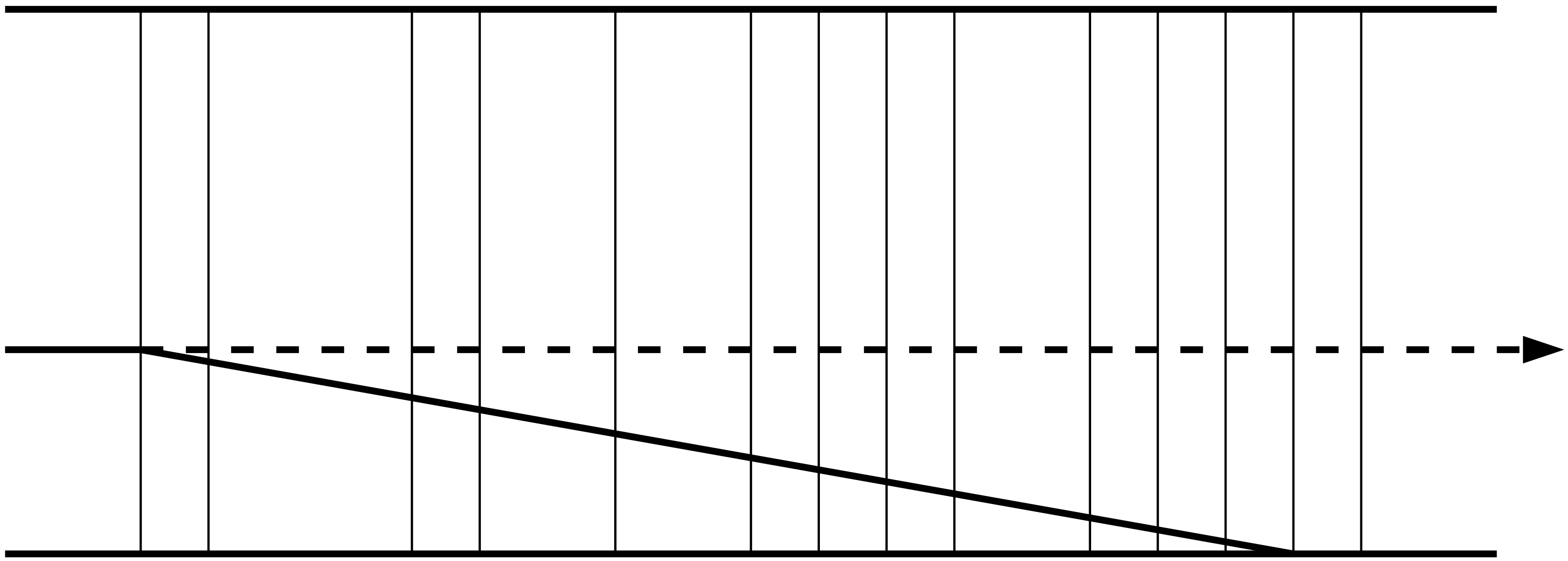
We want to bend path to hit side of tube. If it hits existing vertex, then path ends.

For simplicity, “straighten” region to rectangle.



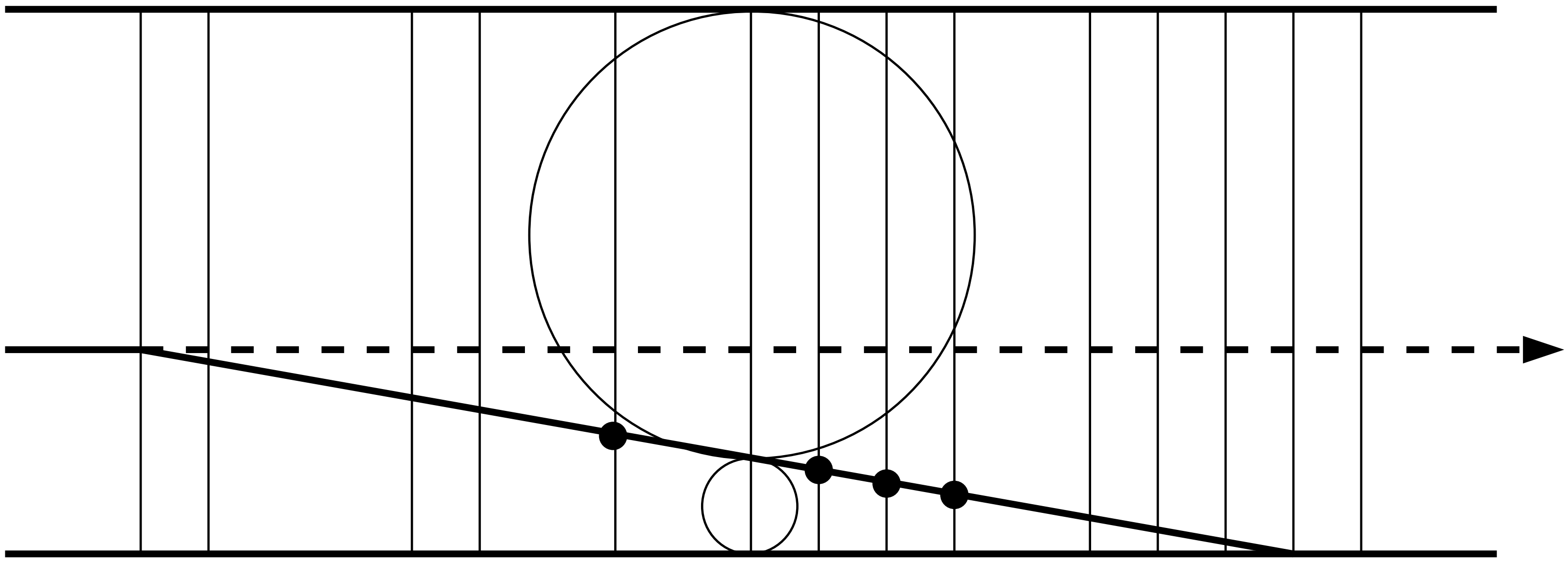
If path bends too fast, Gabriel condition can fail.

For simplicity, “straighten” region to rectangle.



Bend slowly enough to satisfy Gabriel condition.

For simplicity, "straighten" region to rectangle.



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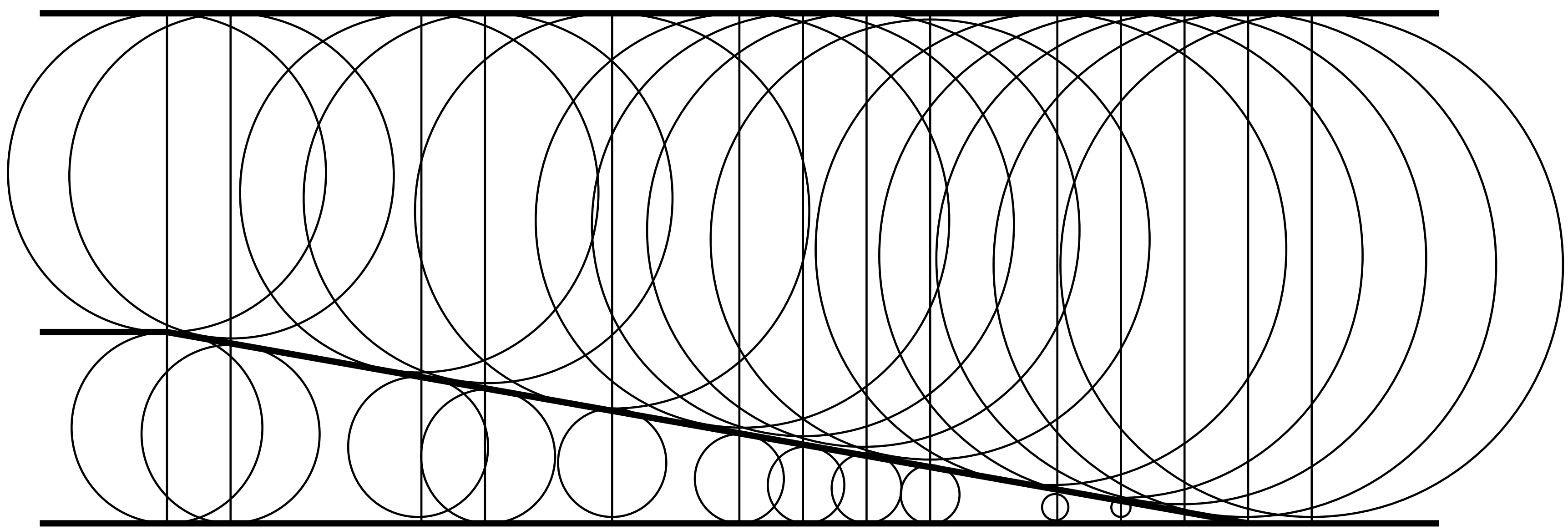
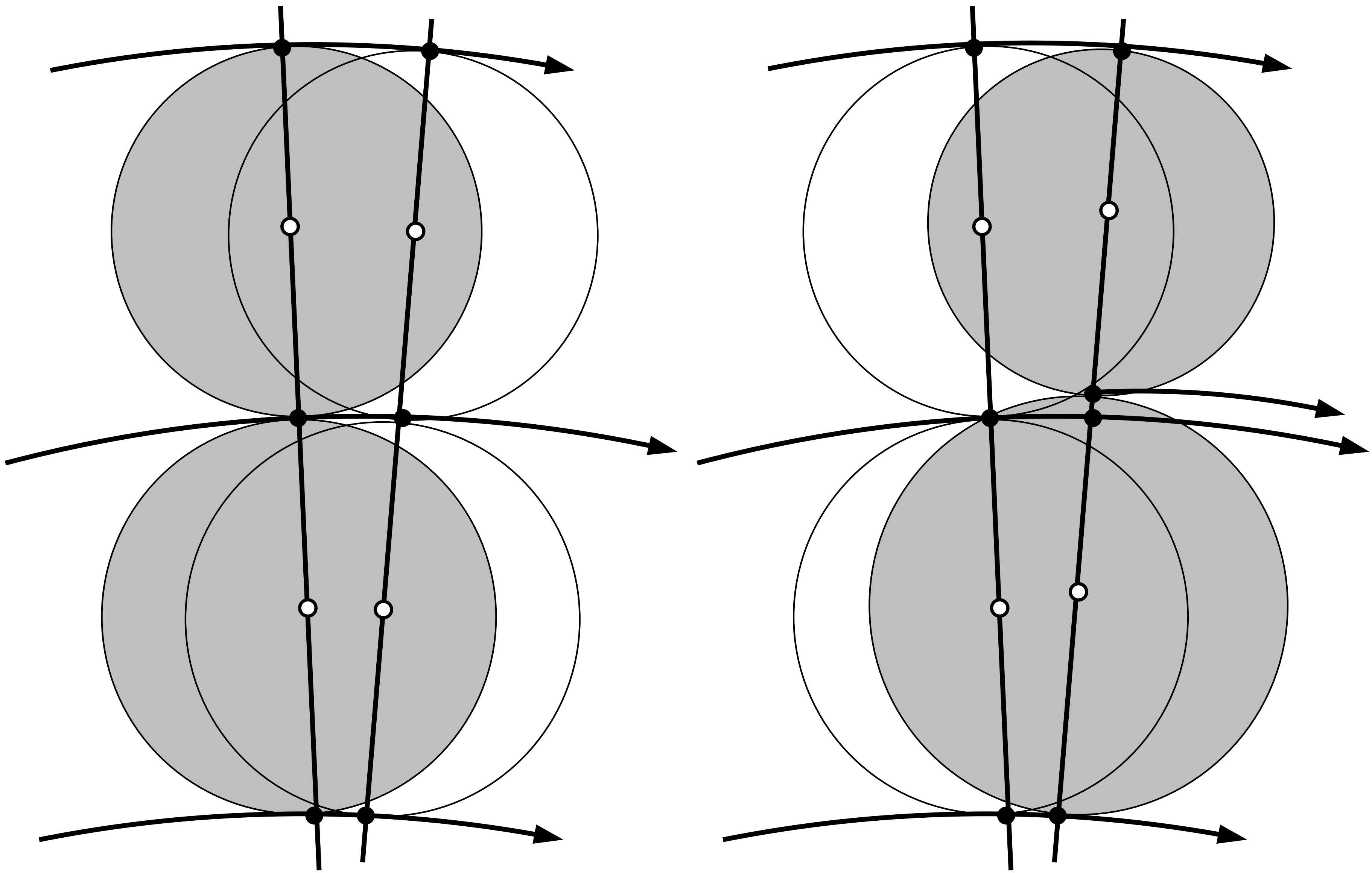
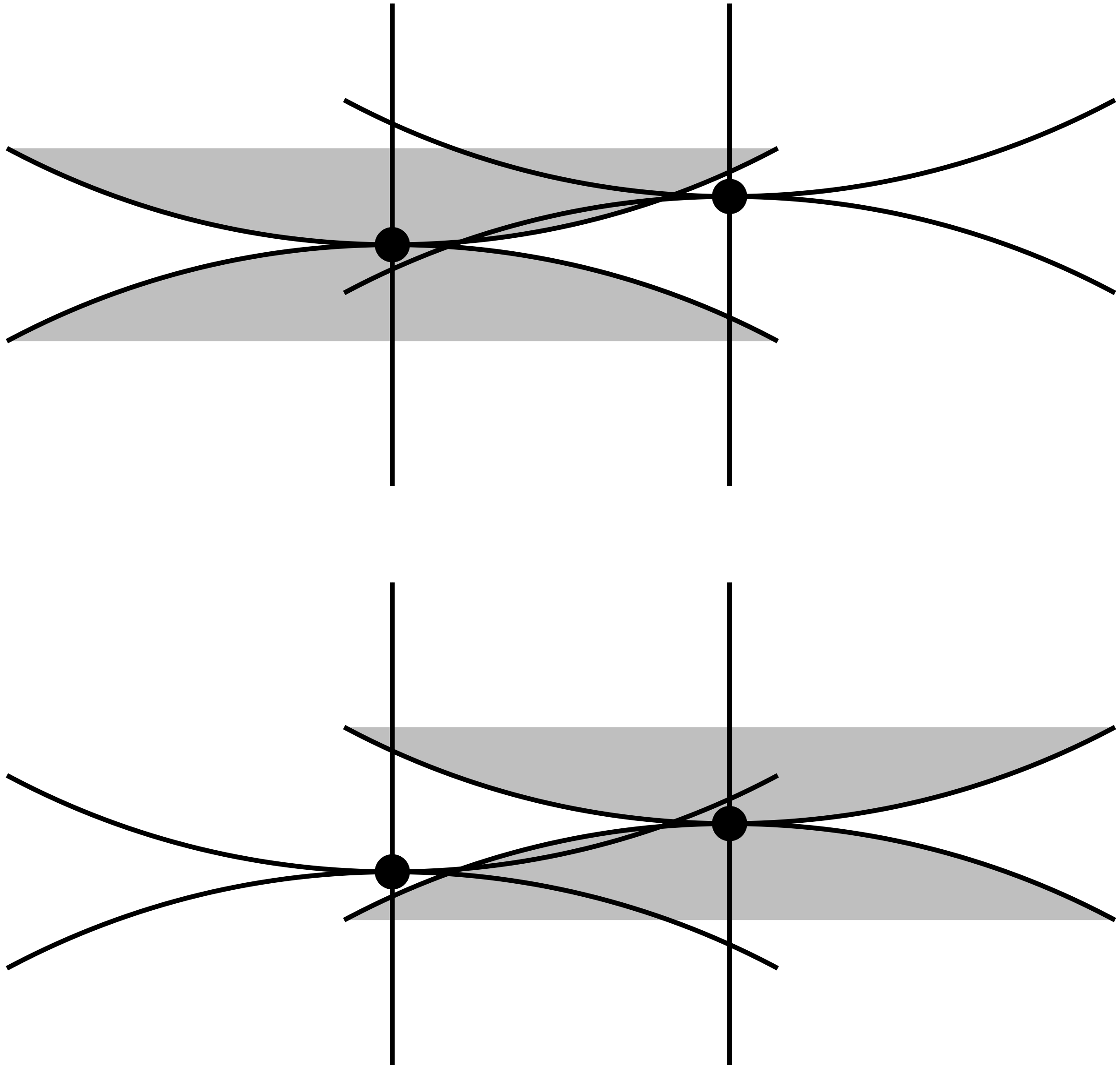
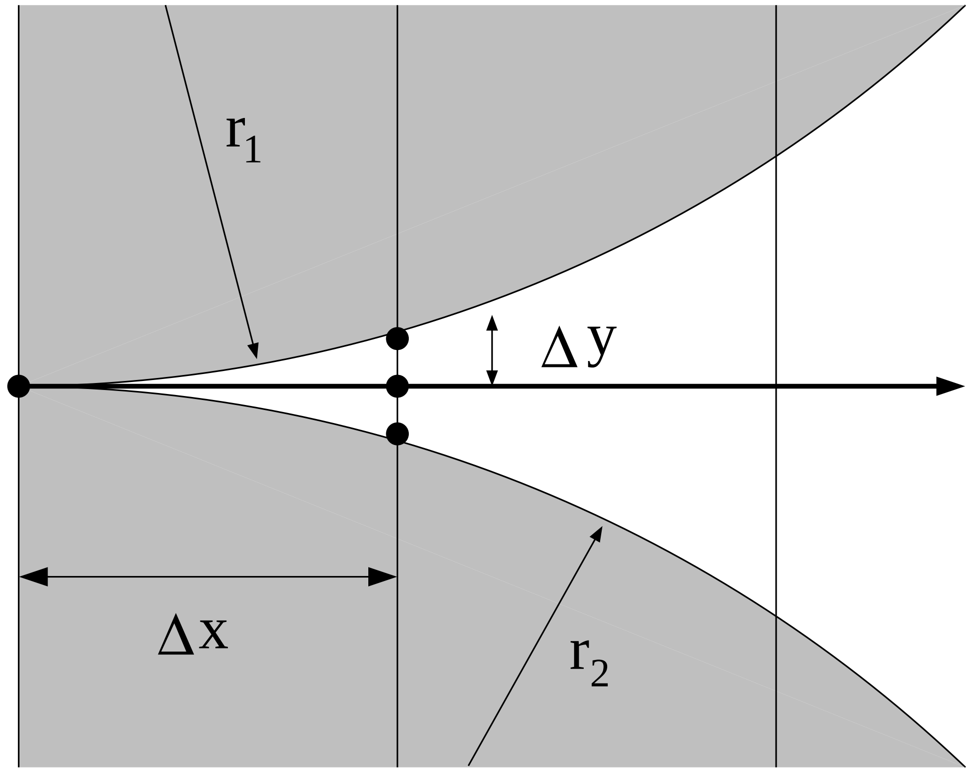


Bend slowly enough to satisfy Gabriel condition.

How far c   an we bend?

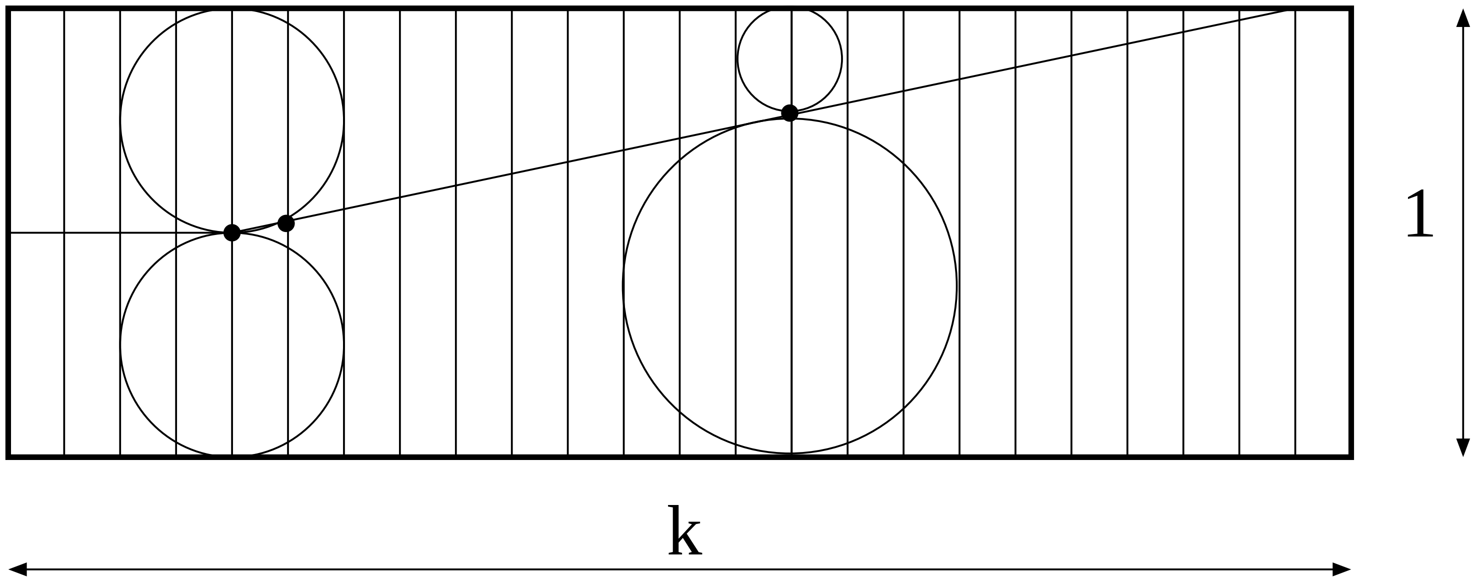






Answer: $\Delta y \approx (\Delta x/r)^2 r = (\Delta x)^2/r.$

$r = \max(r_1, r_2).$



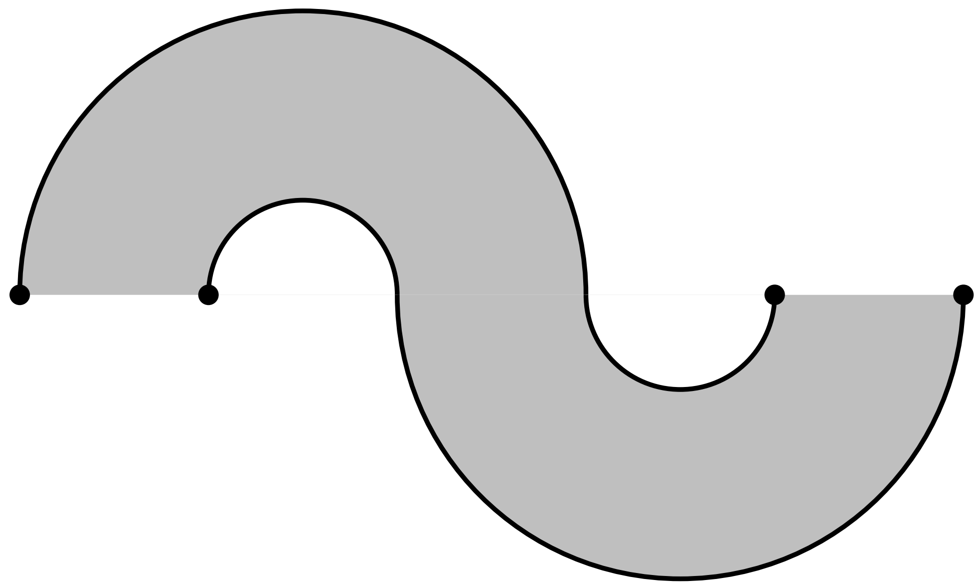
$k \times 1$ region crossing n (equally spaced) thin parts,

$$r \approx 1, \quad \Delta x \approx k/n, \quad \Rightarrow \quad \Delta y \approx k^2/n^2$$

Need $1 \leq \sum \Delta y = n\Delta y = k^2/n$.

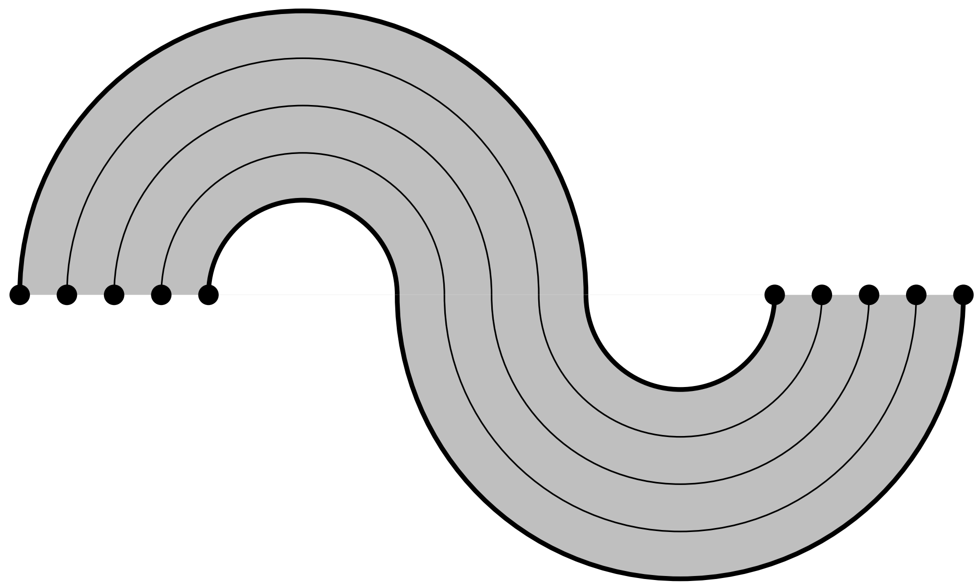
Bent path hits side of region if $k \gg \sqrt{n}$.

Easy case: return region length > width.



- Show there are $O(n)$ return regions.

Easy case: return region length > width.



- Show there are $O(n)$ return regions.
- Divide each region into $O(\sqrt{n})$ long parallel tubes.

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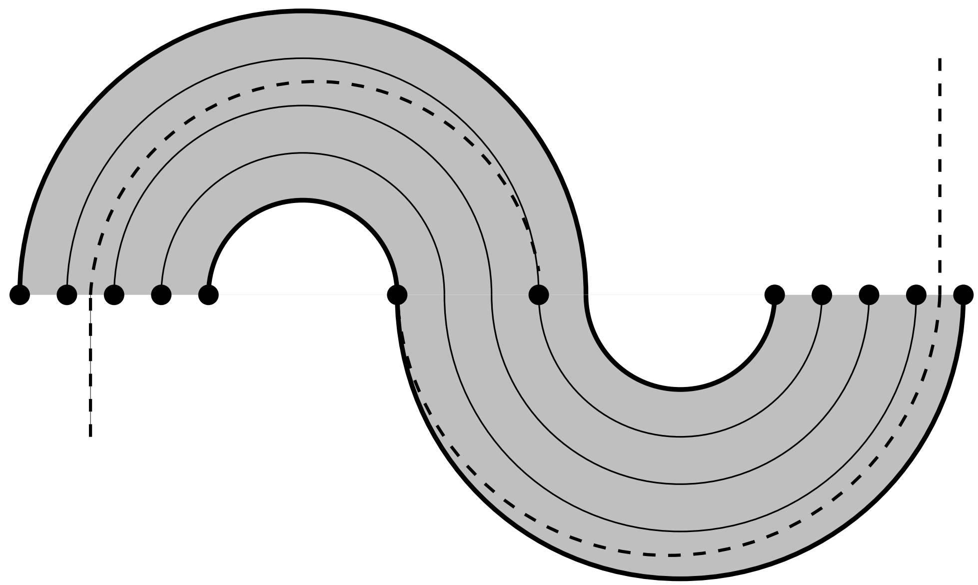

- Show there are $O(n)$ return regions.
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- Entering paths can be bent and terminated.
  Total vertices created $= O(n^2)$, but ...

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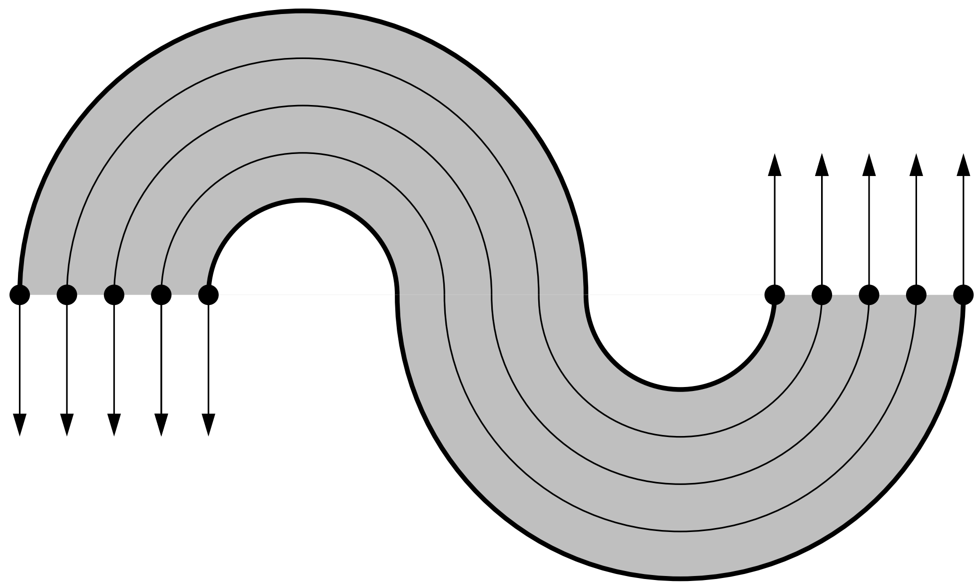
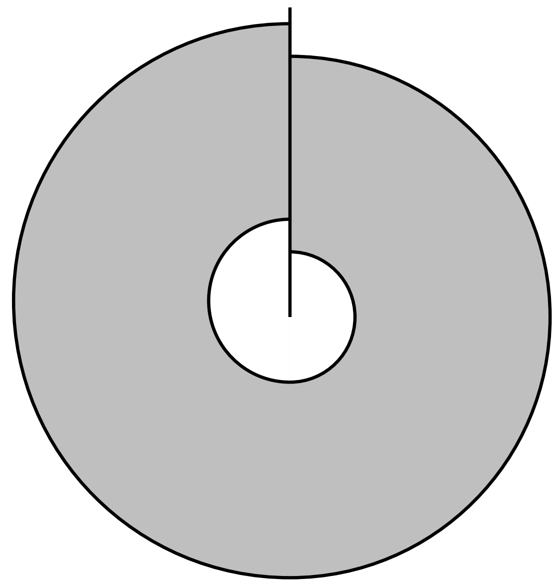


- Show there are $O(n)$ return regions.
- Divide each region into $O(\sqrt{n})$ long parallel tubes.
- Entering paths can be bent and terminated.
  Total vertices created $= O(n^2)$, but ...
- Each region has $O(\sqrt{n})$ new vertices to propagate.
  Vertices created is $O(\sqrt{n} \cdot n \cdot n) = O(n^{2.5})$.

Hard case is spirals:



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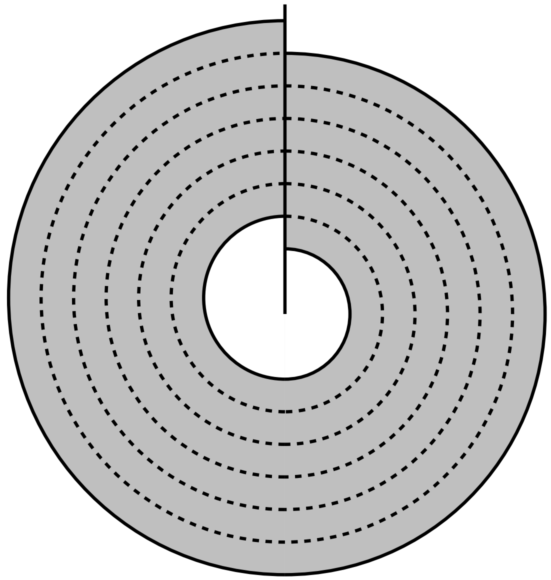
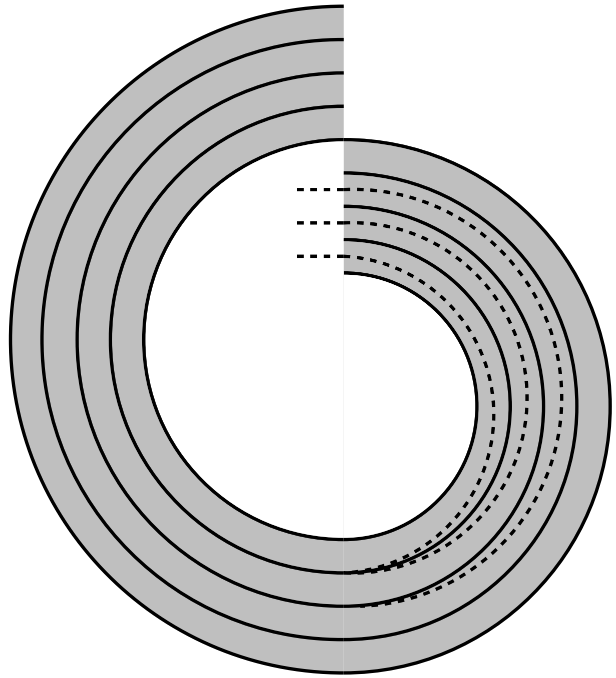


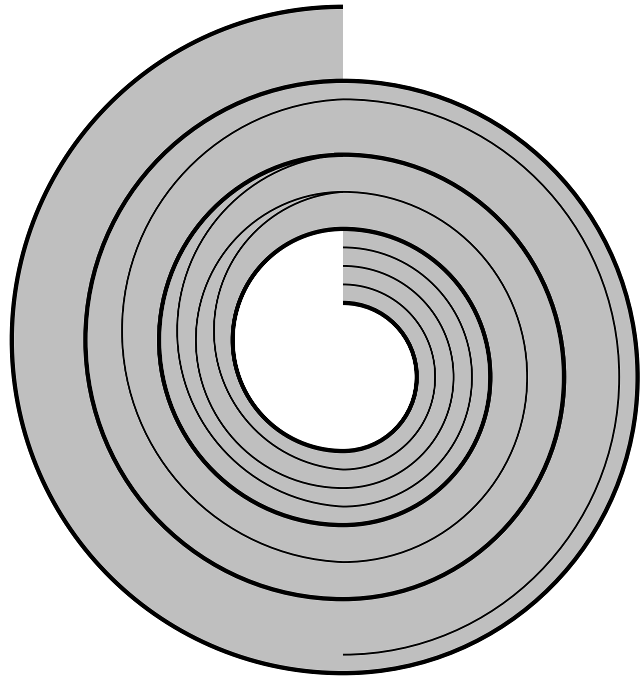
Curves may spiral arbitrarily often.

No curve can be allowed to pass all the way through the spiral. We stop them in a multi-stage construction.

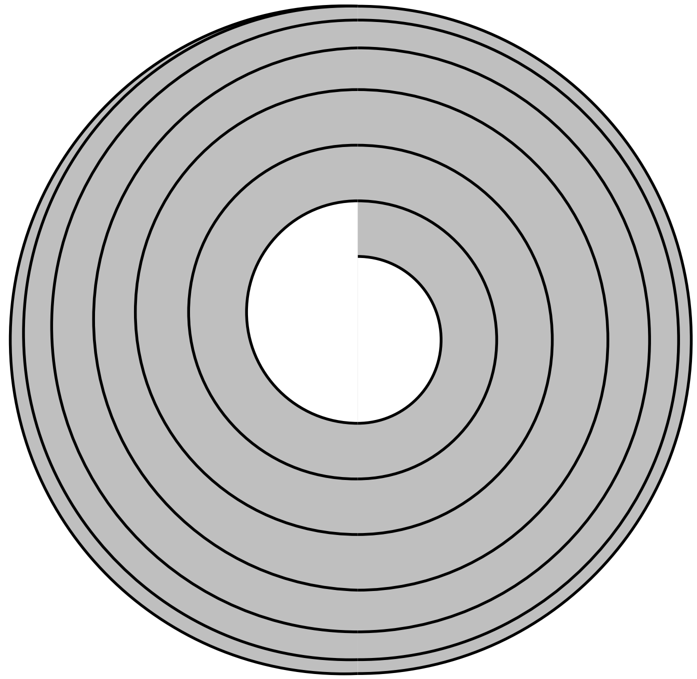
Normalize so "entrance" is unit width.



- Start with $\sqrt{n}$ parallel tubes at entrance of sprial. Terminate entering paths (1 spiral).



- Start with $\sqrt{n}$ parallel tubes at entrance of spiral. Terminate entering paths (1 spiral).
- Merge $\sqrt{n}$ tubes to single tube ($n^{1/3}$ spirals). (spirals get longer as we move out.)



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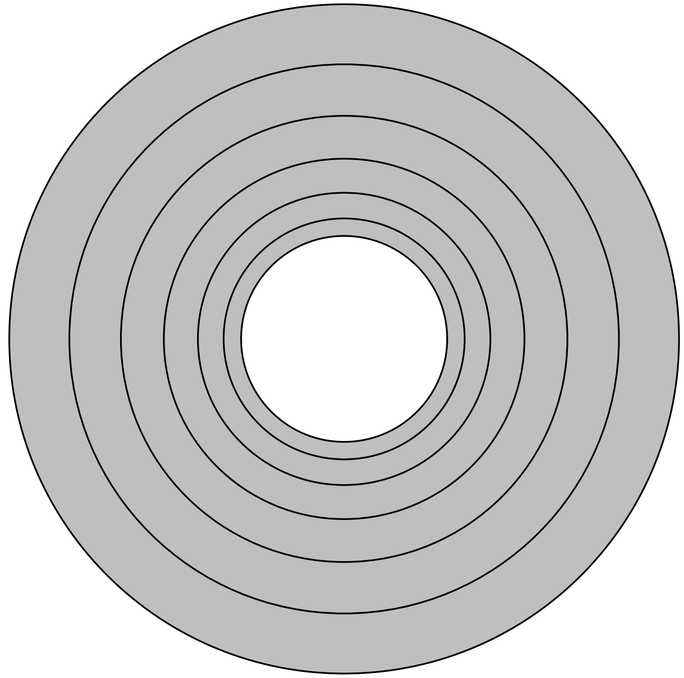

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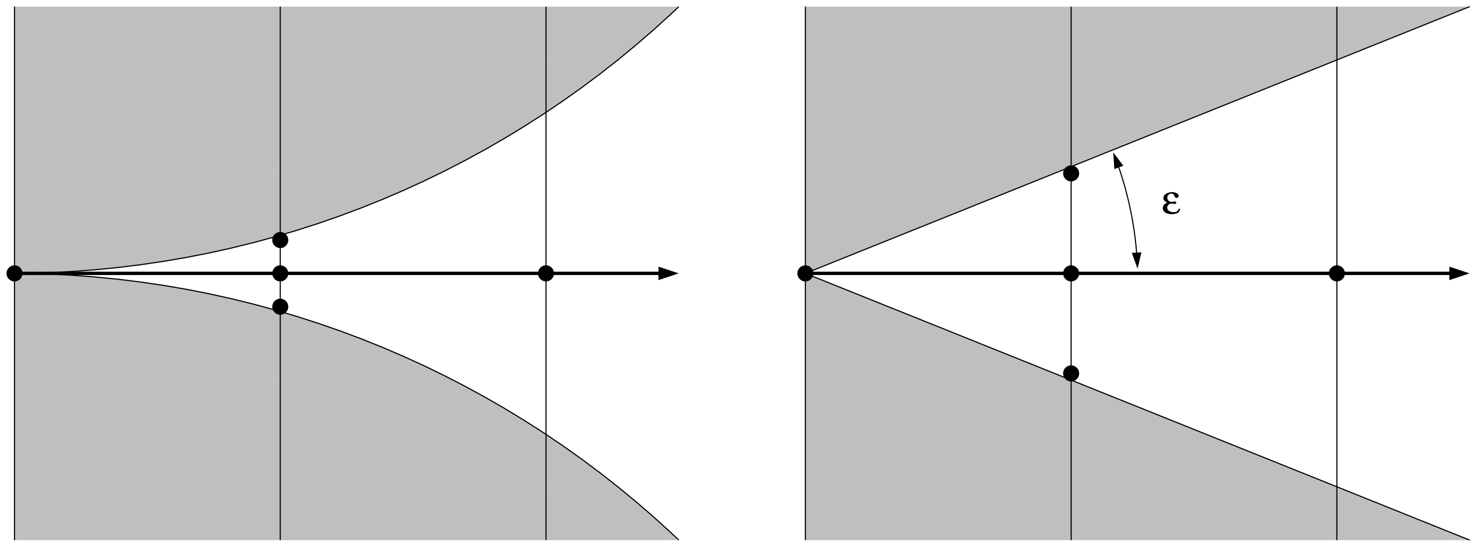
- Loops with increasing gaps ($n^{1/2}$ loops, to radius n)
- Beyond radius n spiral is empty. Gives $O(n^{2.5})$.

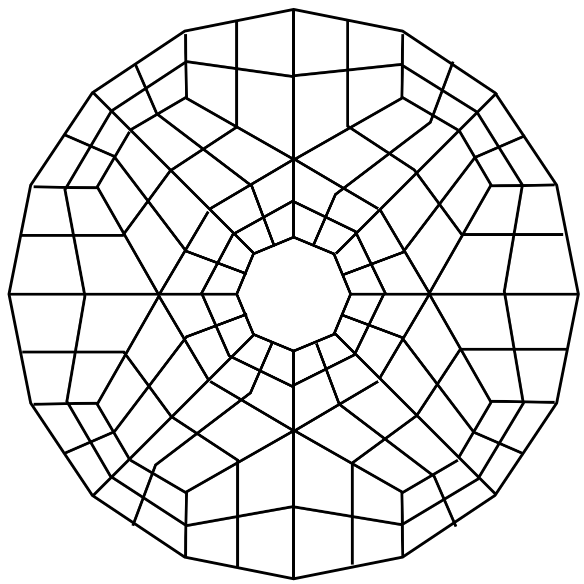
Almost Nonobtuse Triangulation: Replace cusps by cones of angle ϵ. Same construction in thick parts.

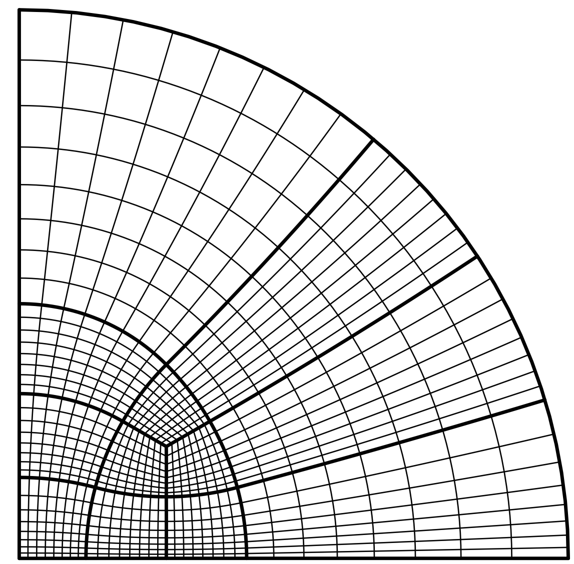


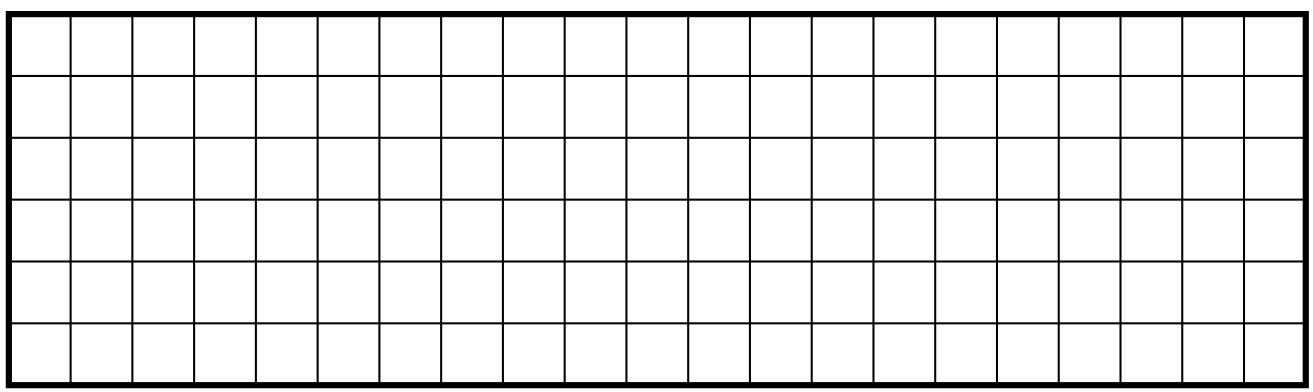
Paths can be terminated inside a $1 \times \frac{1}{\epsilon}$ tube.

Thm: Uses angles $\leq 90^\circ + \epsilon$ and $O(\frac{n^2}{\epsilon^2})$ triangles.

Quadrilateral meshes:







Some results

- Every n-gon has $O(n)$ quad mesh with angles $\leq 120^\circ$. Bern and Eppstein, 2000. $O(n \log n)$ work.

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Theorem, B, 2008: Every n-gon has $O(n)$ quad mesh with angles $\leq 120°$ and every new angle $\geq 60°$. Takes $O(n)$ work.

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Angles bounds and complexity are sharp.

At most $O(\frac{n}{\epsilon})$ angles outside $[90^\circ - \epsilon, 90^\circ + \epsilon]$.

Idea of proof:

- Connect Γ. Components now polygons, not triangles.
- Define thick/thin pieces.
- Mesh thin parts using propogation paths as before.
- Mesh thick parts using hyperbolic geometric in disk and conformal map to polygon.
- Interpolate between thick and thin parts using special meshes called “sinks”.