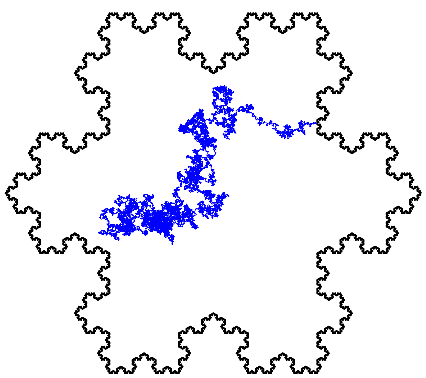
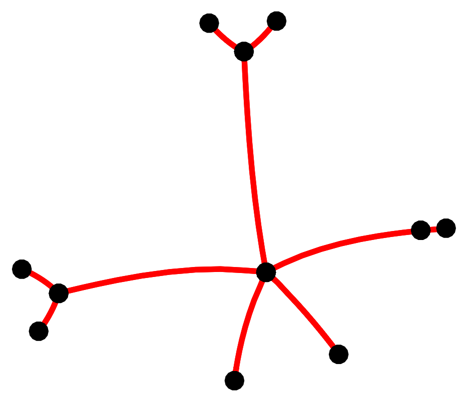
HARMONIC MEASURE, TRUE TREES AND EQUILATERAL TRIANGULATIONS

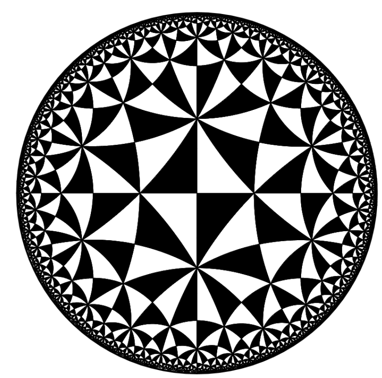
Christopher Bishop, Stony Brook

Boole Colloquim, University College Cork
Wednesday, March 4, 2026

www.math.sunysb.edu/~bishop/lectures

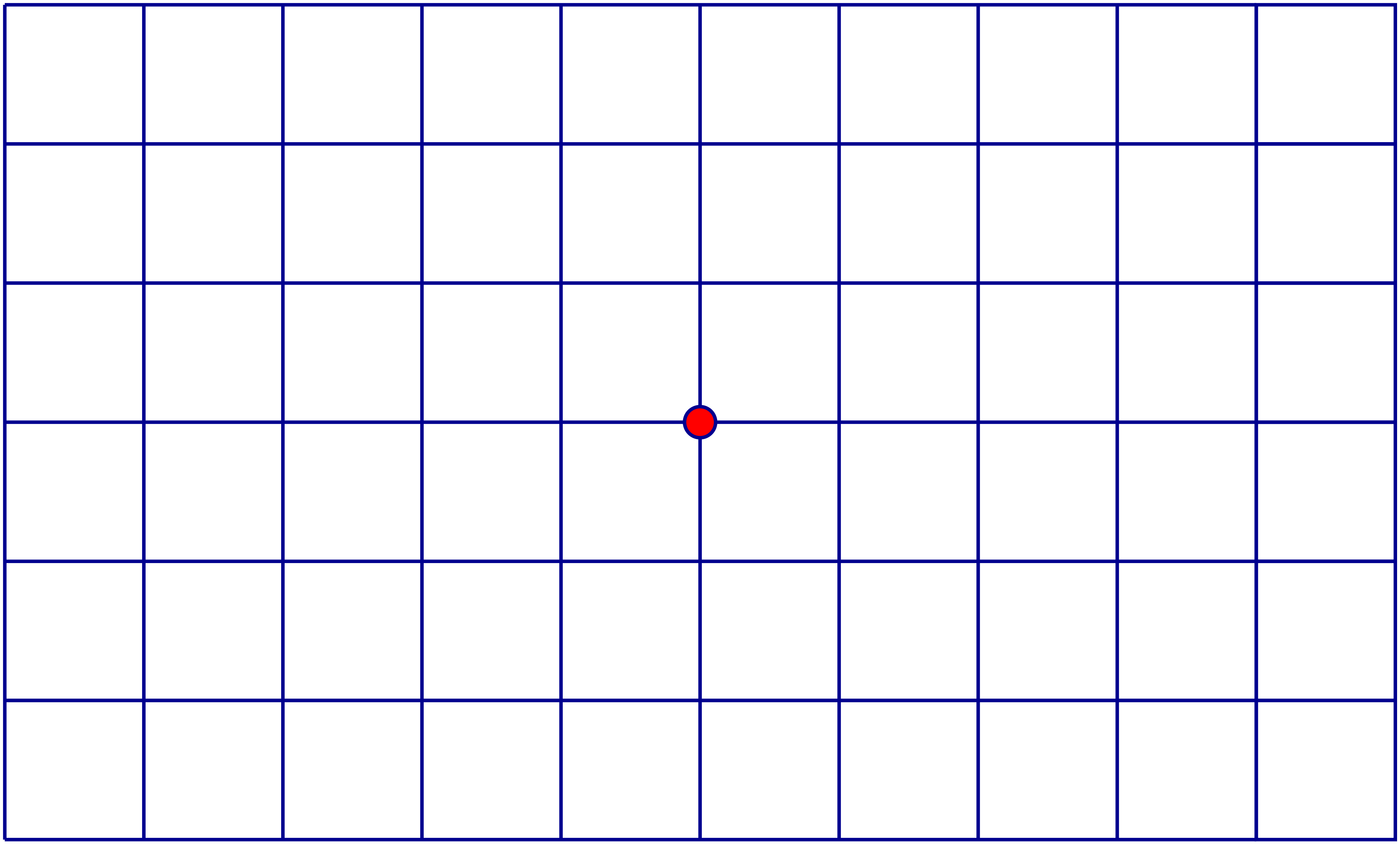


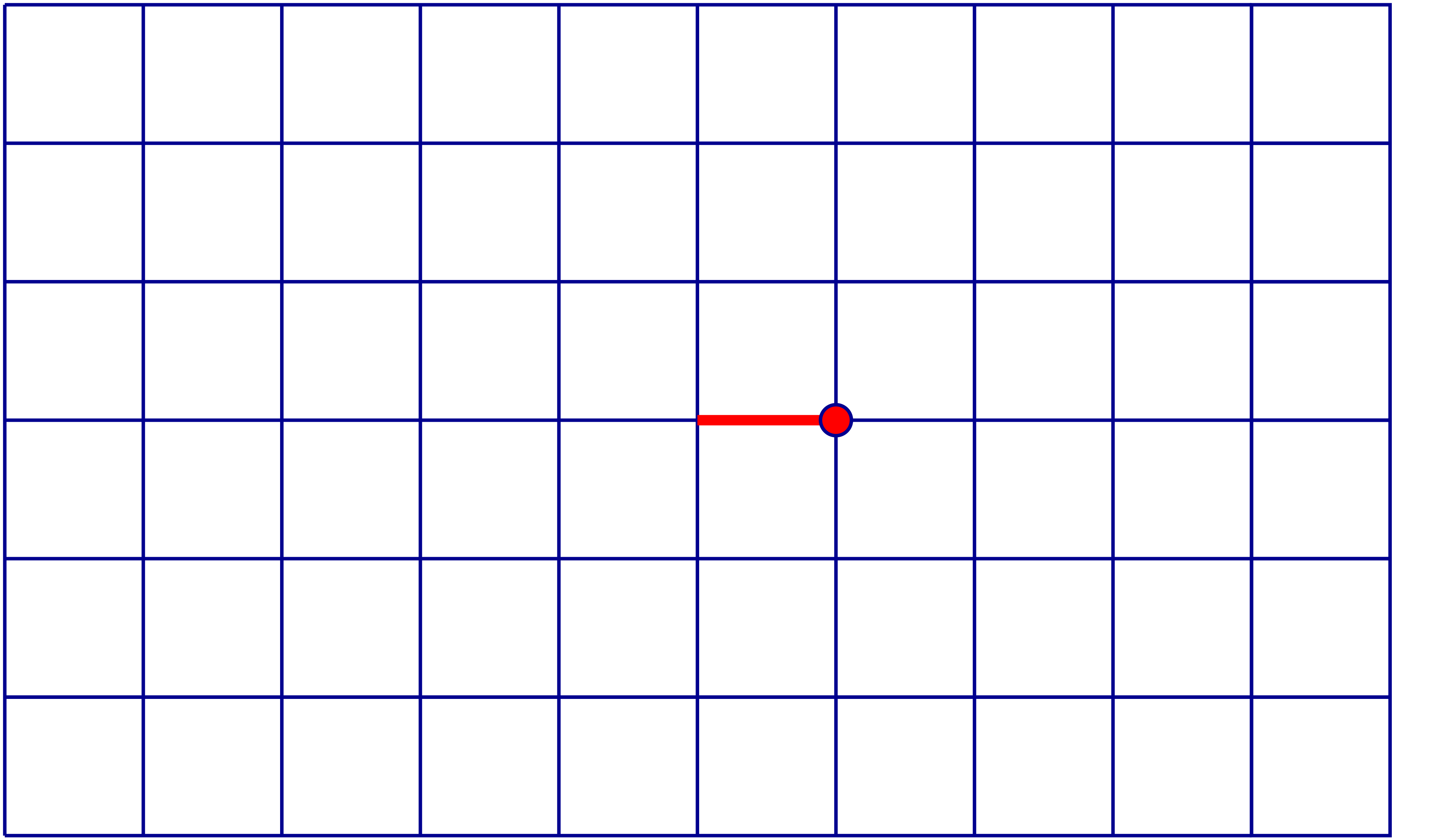


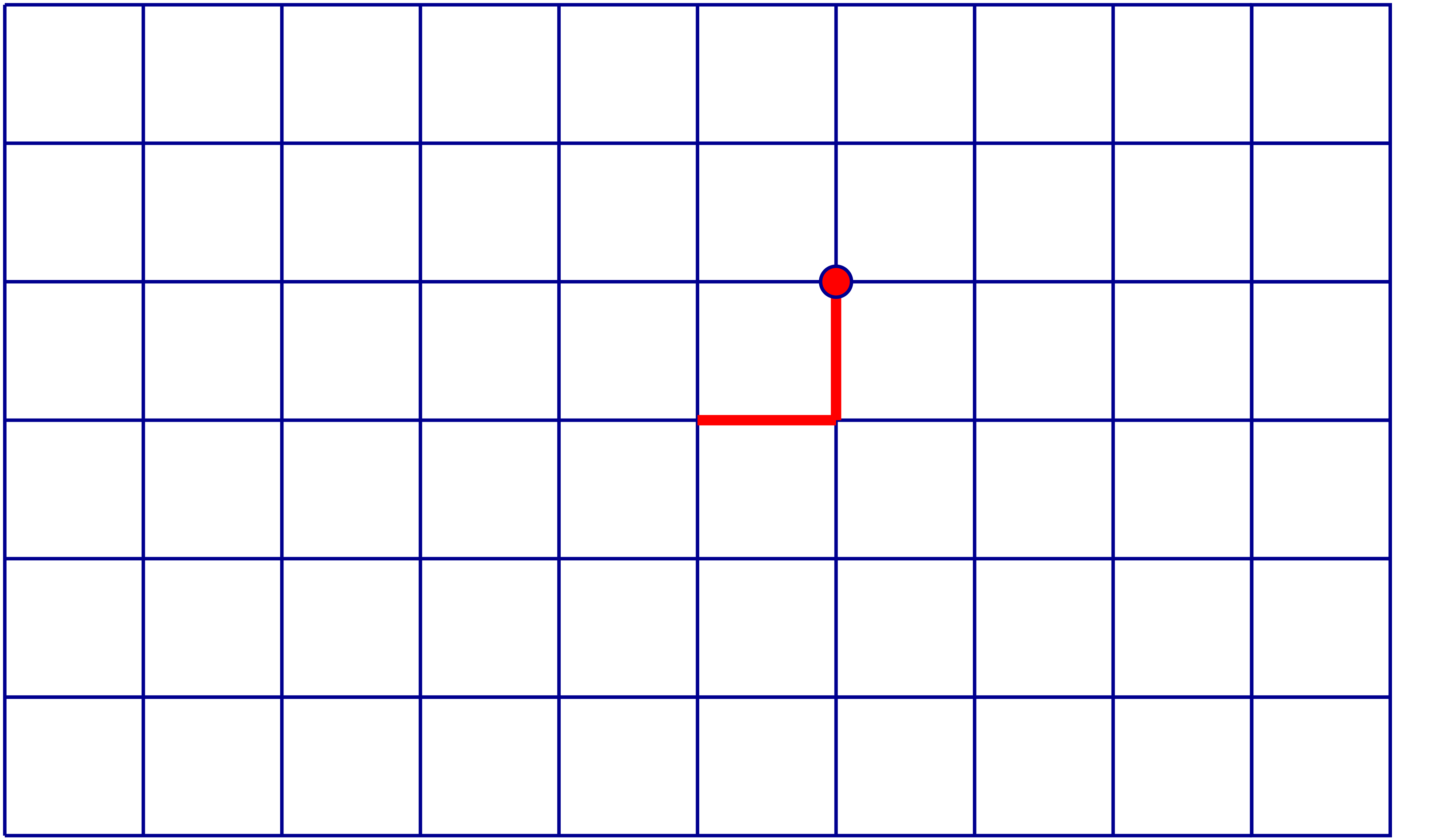


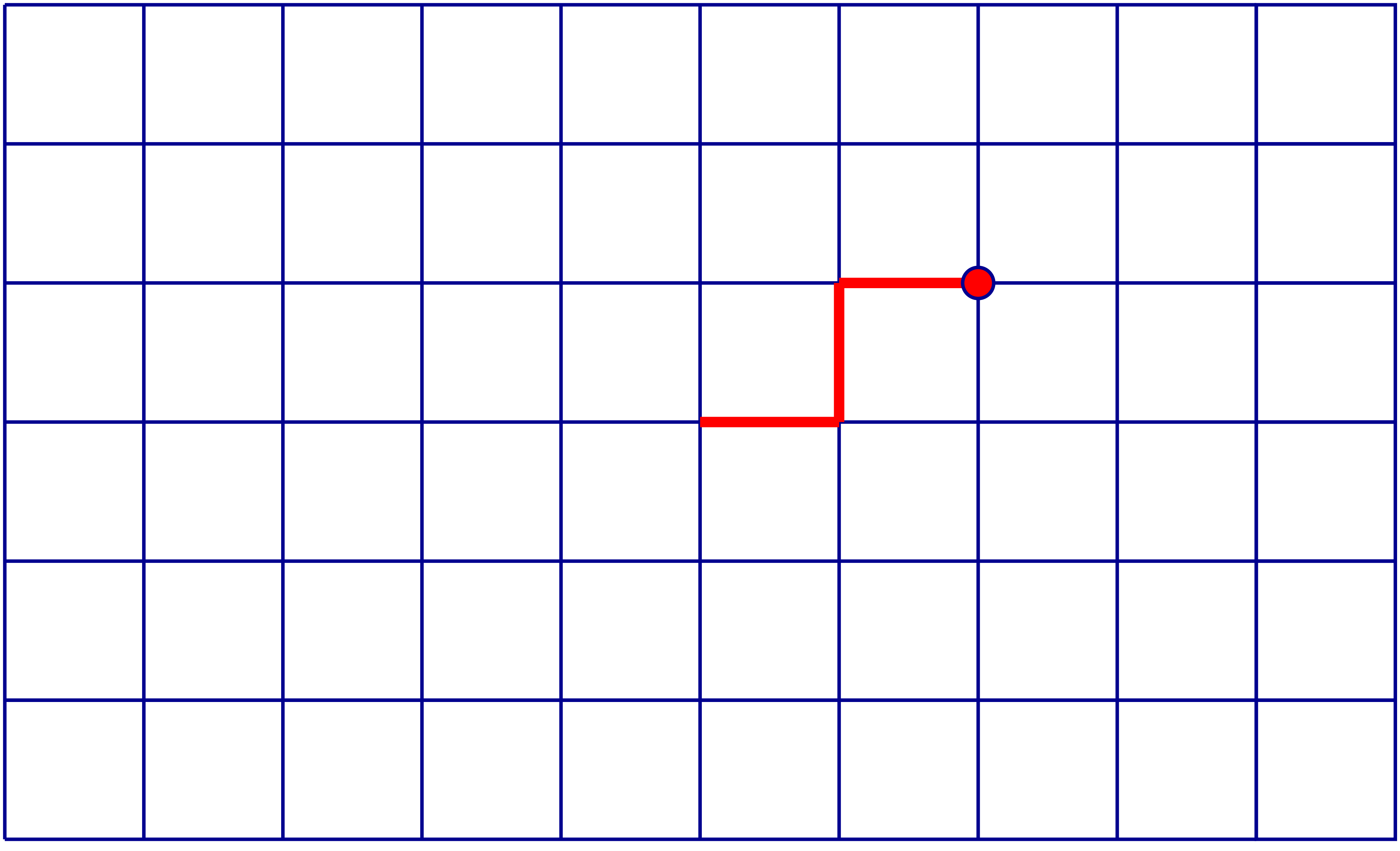
University College Cork, Ireland
Coláiste na hOllscoile Corcaigh

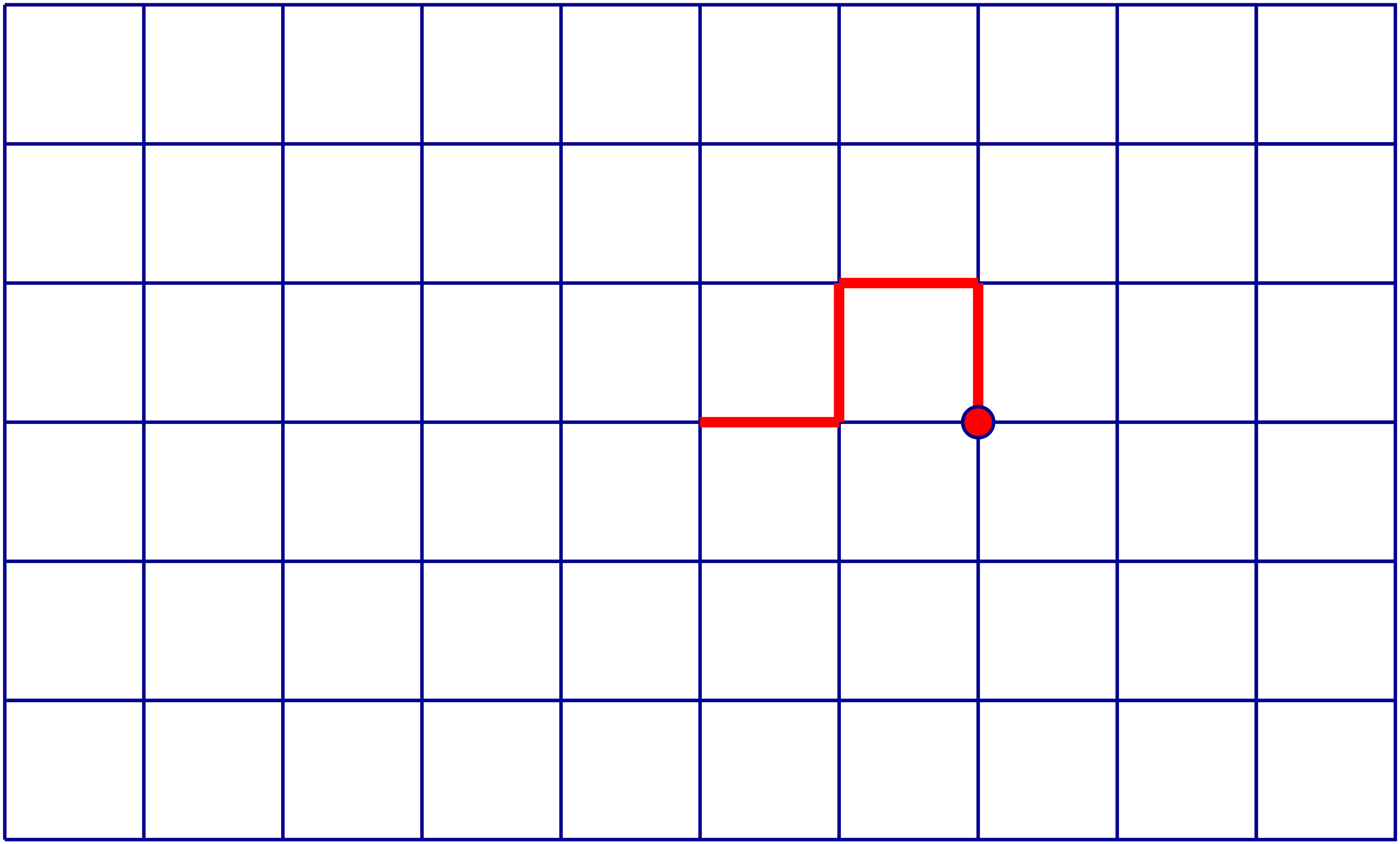
Harmonic Measure

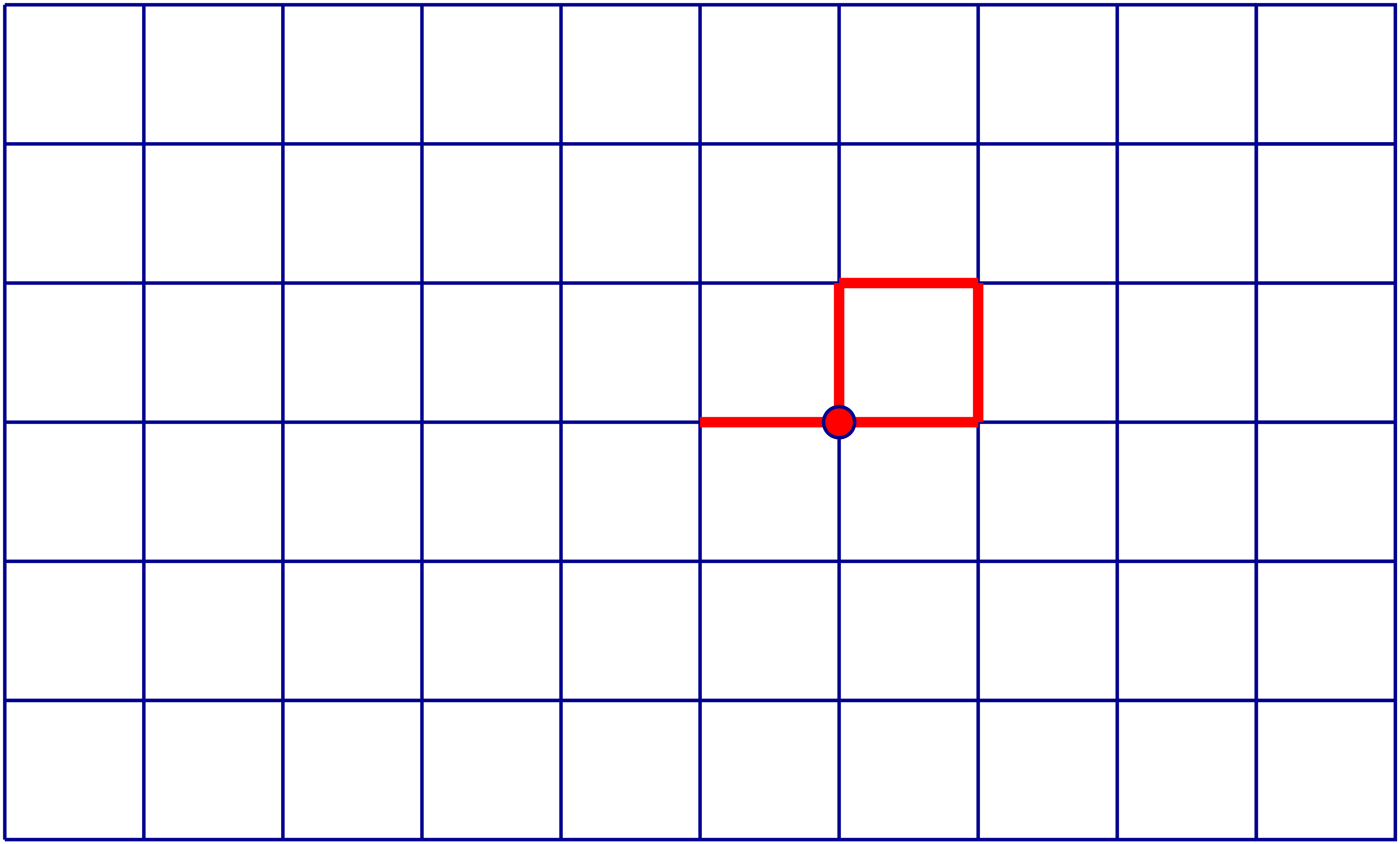


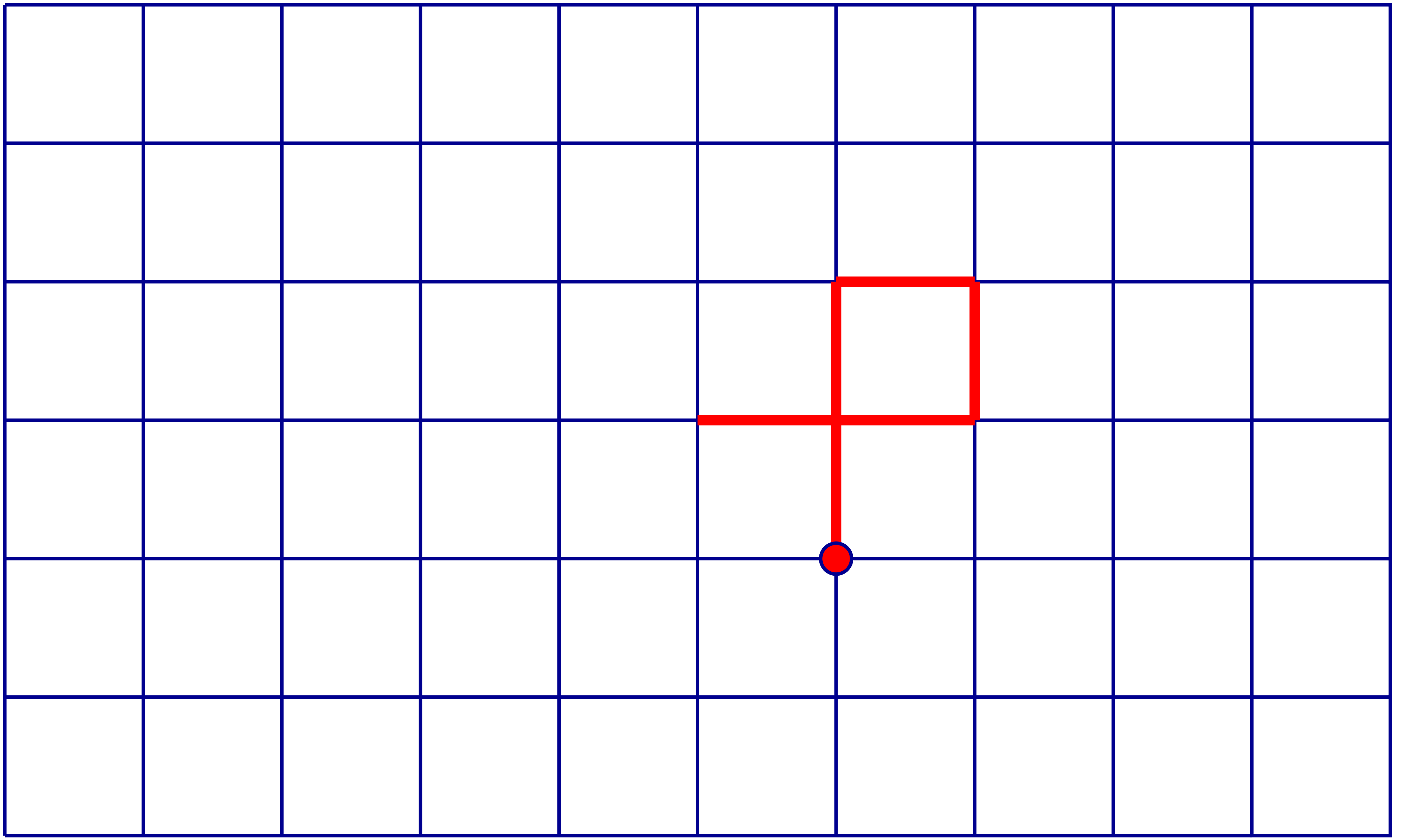


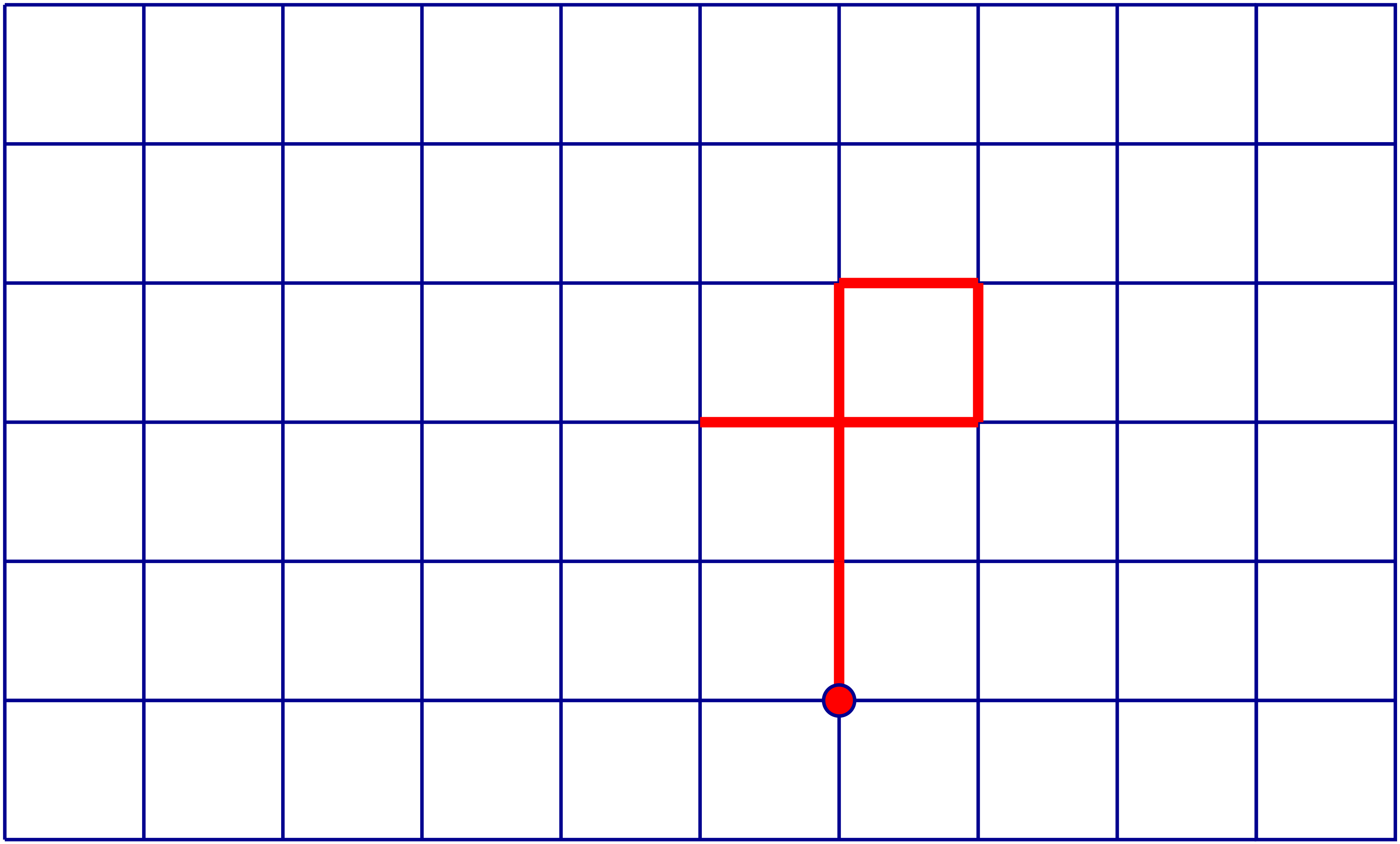


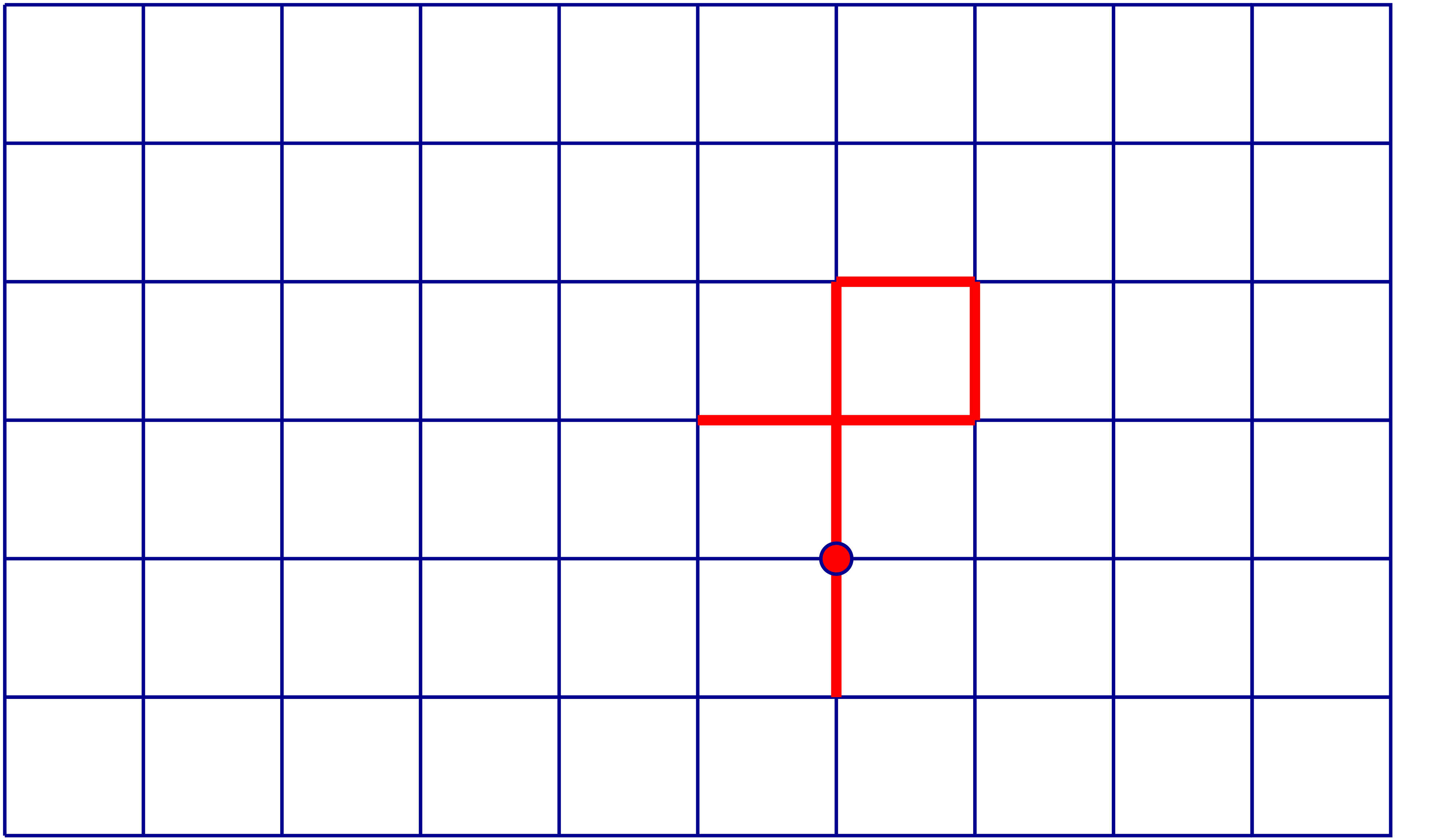


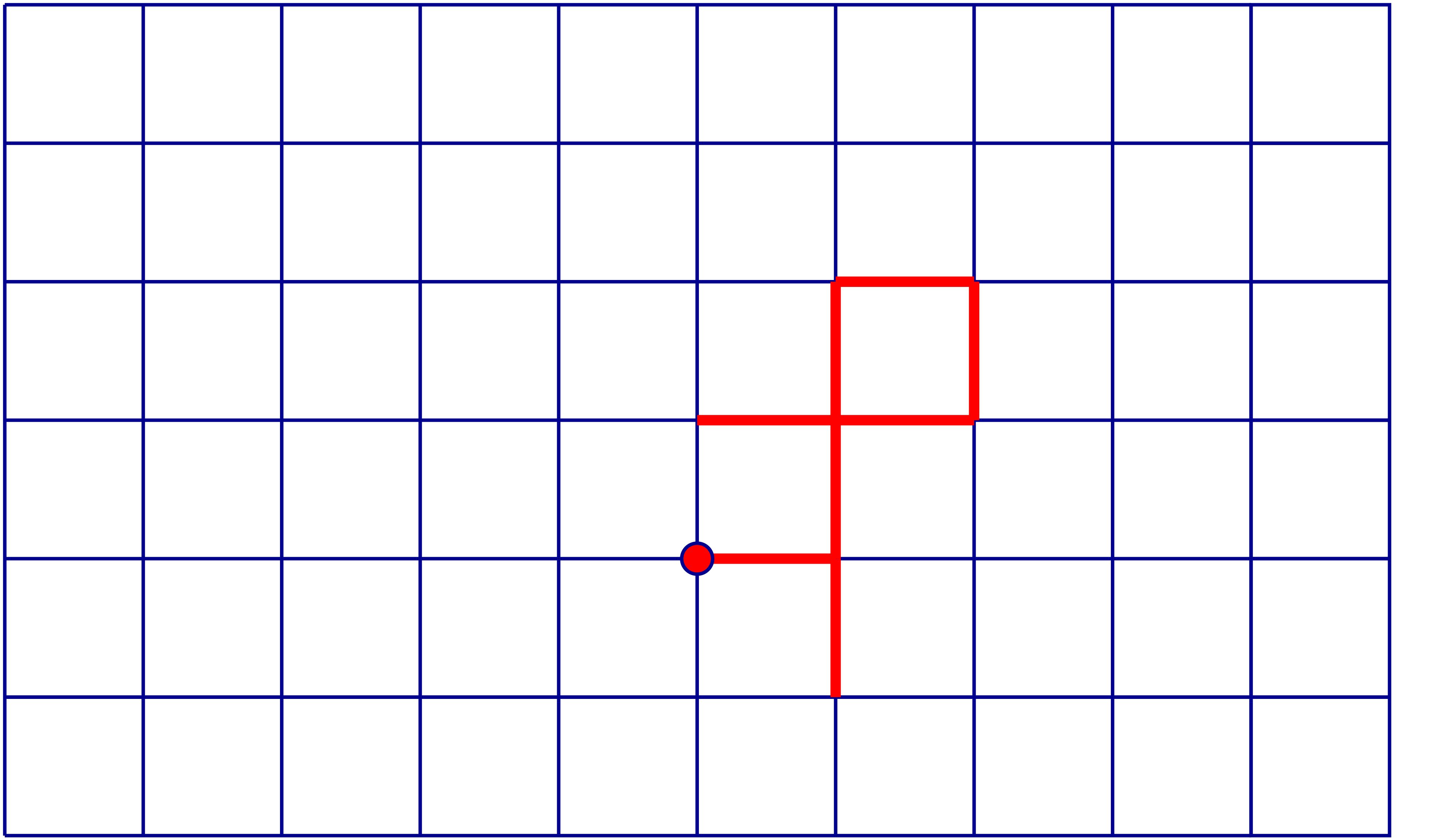


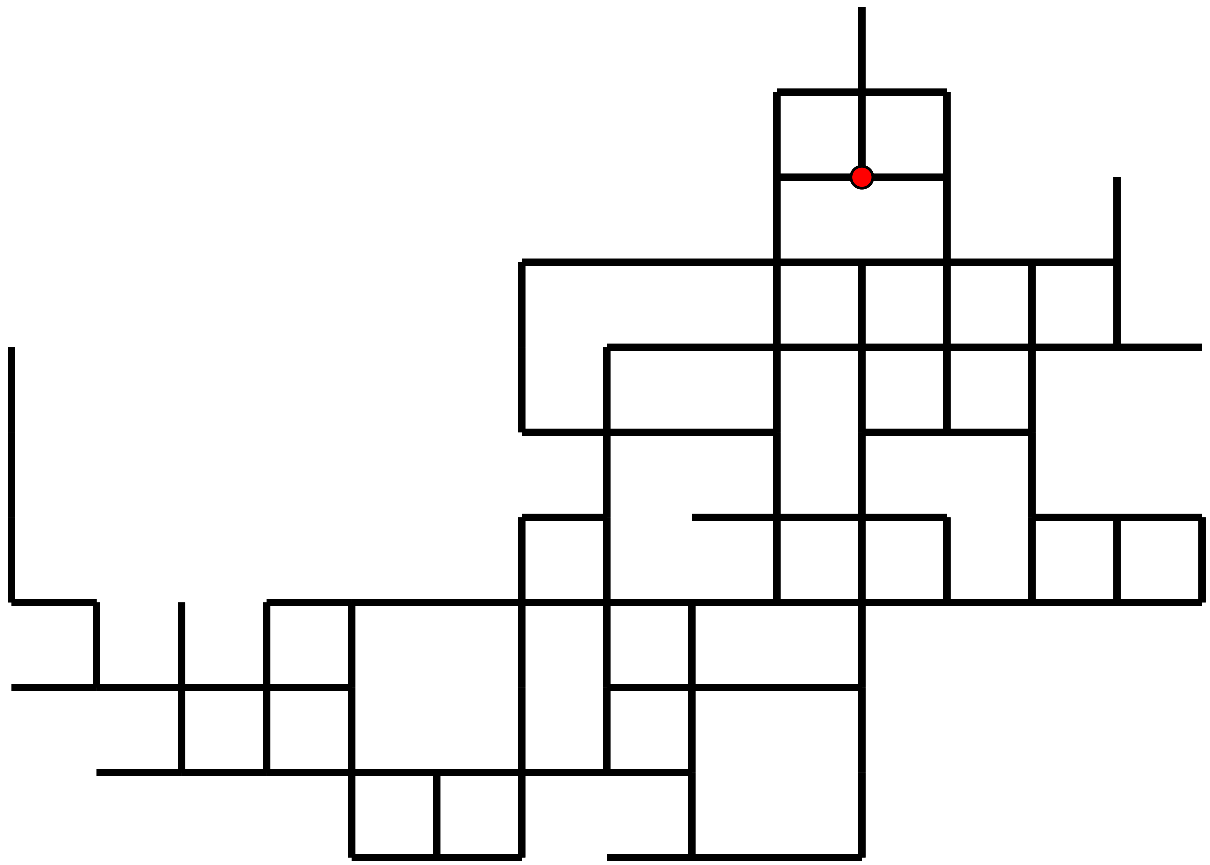




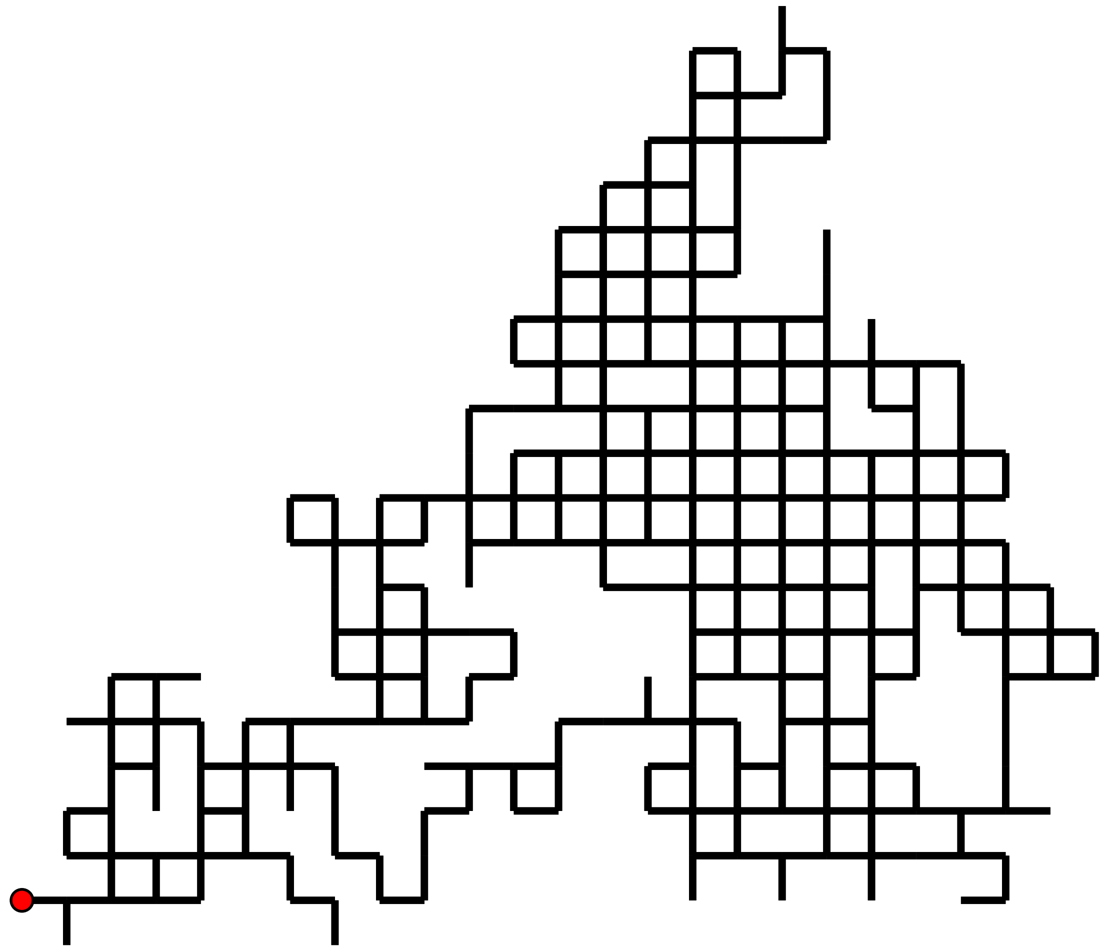




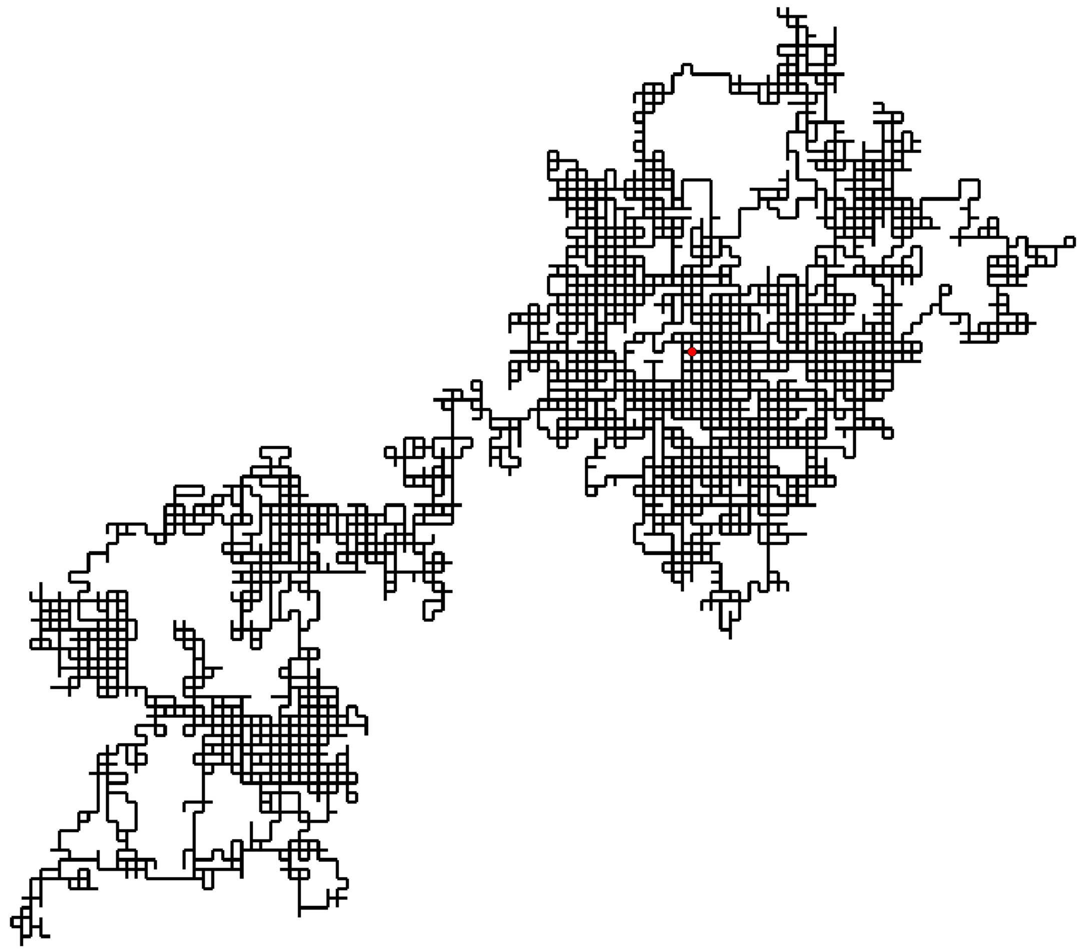




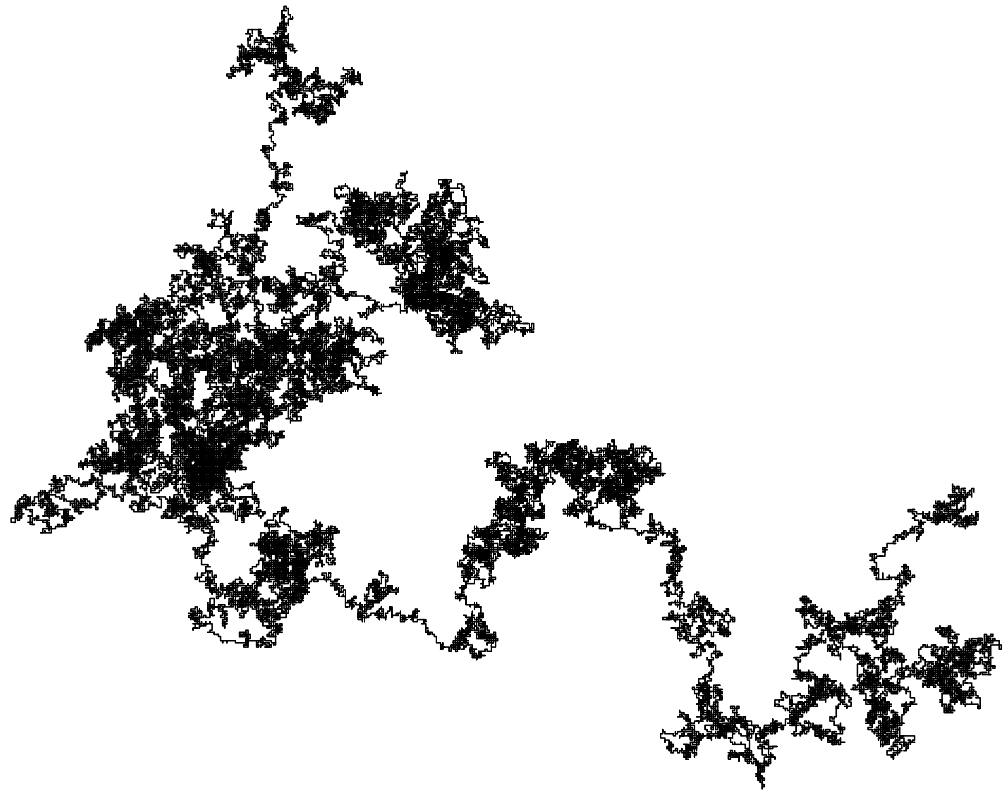
200 step random walk.



1000 step random walk.

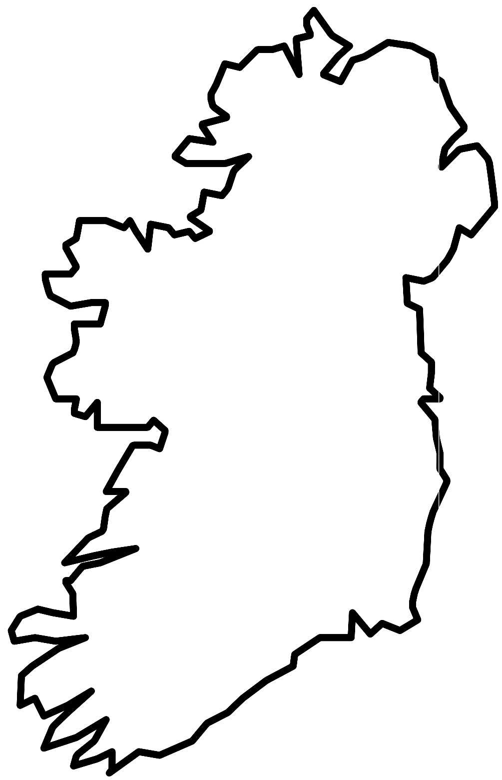


10,000 step random walk.



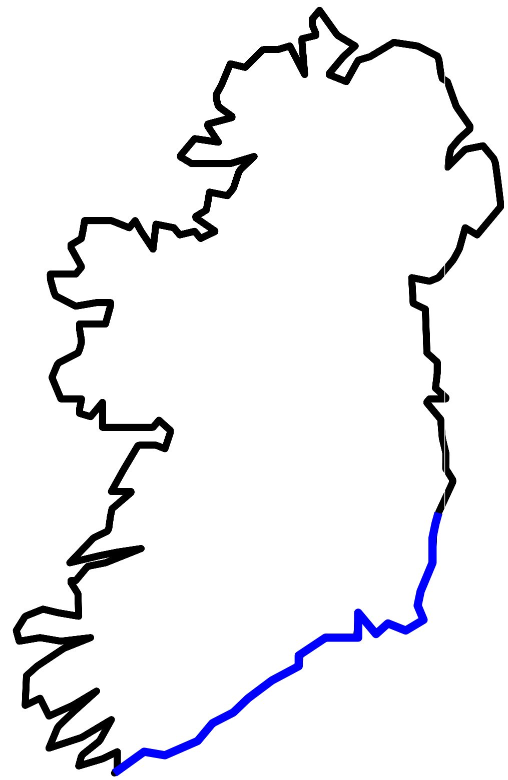
100,000 step random walk.

Harmonic measure = hitting distribution of Brownian motion



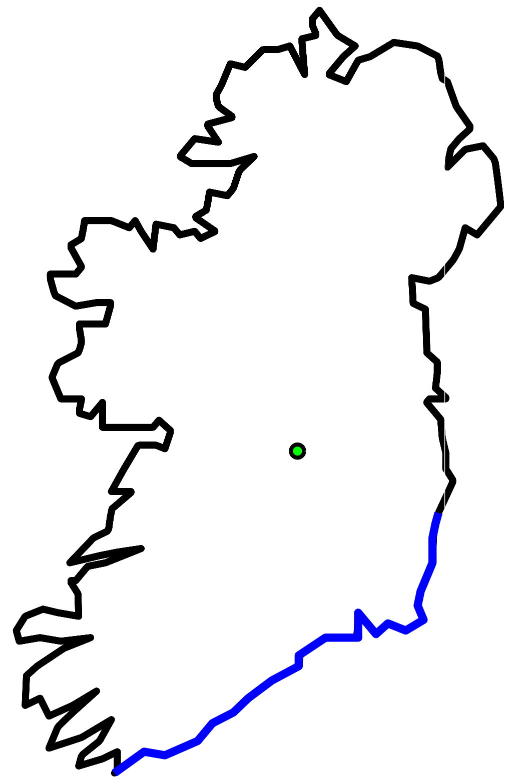
Suppose Ω is a planar Jordan domain.

Harmonic measure = hitting distribution of Brownian motion



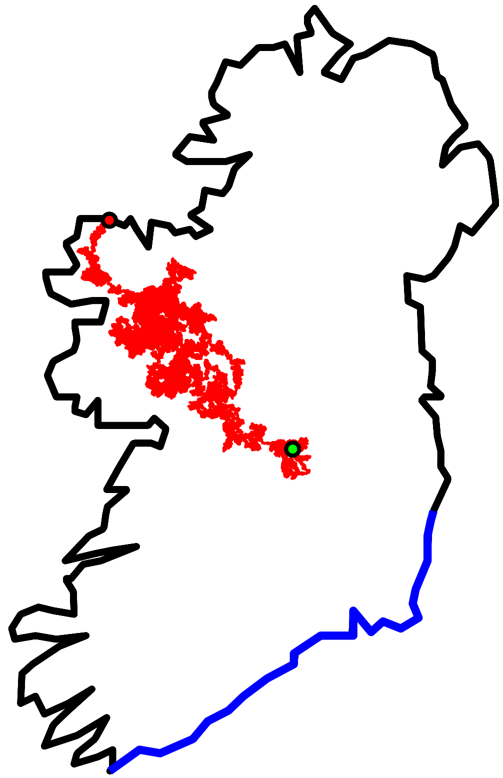
Let E be a subset of the boundary, $\partial\Omega$.

Harmonic measure = hitting distribution of Brownian motion



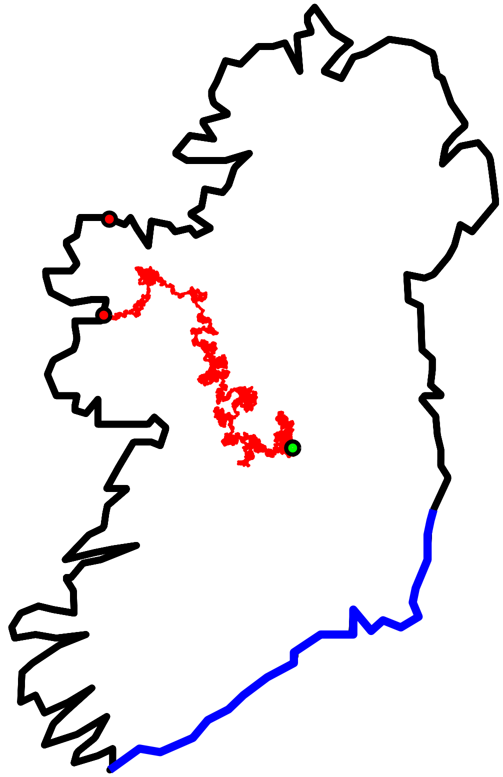
Choose a point z inside Ω.

Harmonic measure = hitting distribution of Brownian motion



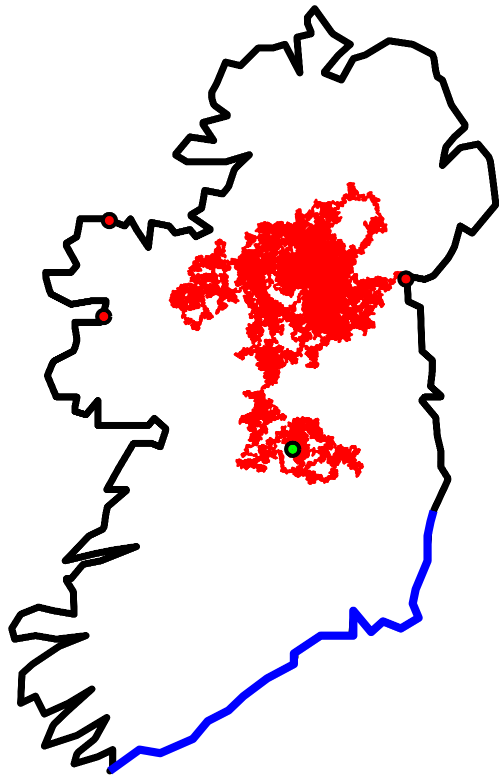
$\omega(z, E, \Omega)$ = probability a particle started at z first hits $\partial\Omega$ in E.

Harmonic measure = hitting distribution of Brownian motion



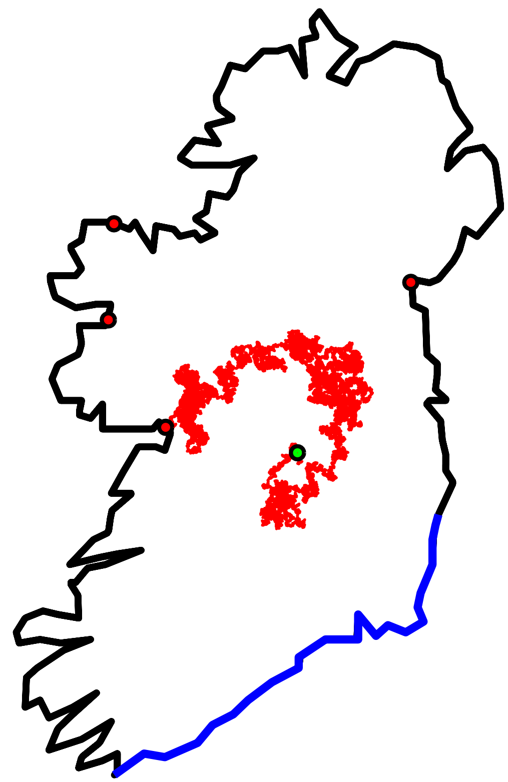
$\omega(z, E, \Omega)$ = probability a particle started at z first hits $\partial\Omega$ in E.

Harmonic measure = hitting distribution of Brownian motion



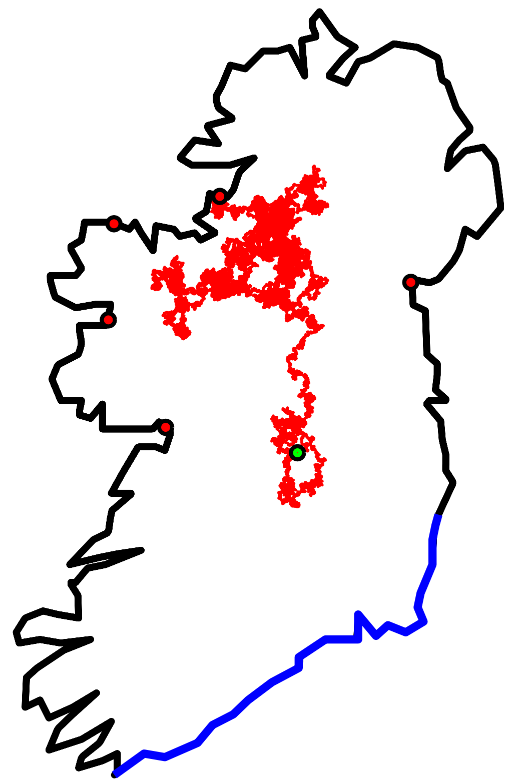
$\omega(z, E, \Omega)$ = probability a particle started at z first hits $\partial\Omega$ in E.

Harmonic measure = hitting distribution of Brownian motion



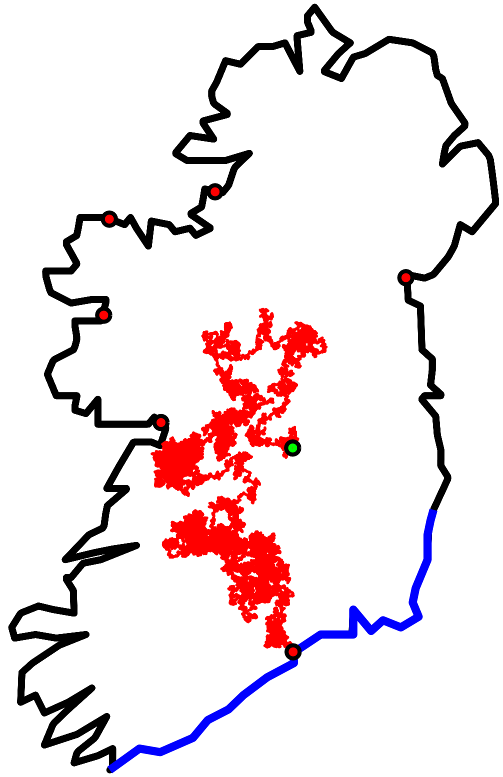
$\omega(z, E, \Omega)$ = probability a particle started at z first hits $\partial\Omega$ in E.

Harmonic measure = hitting distribution of Brownian motion



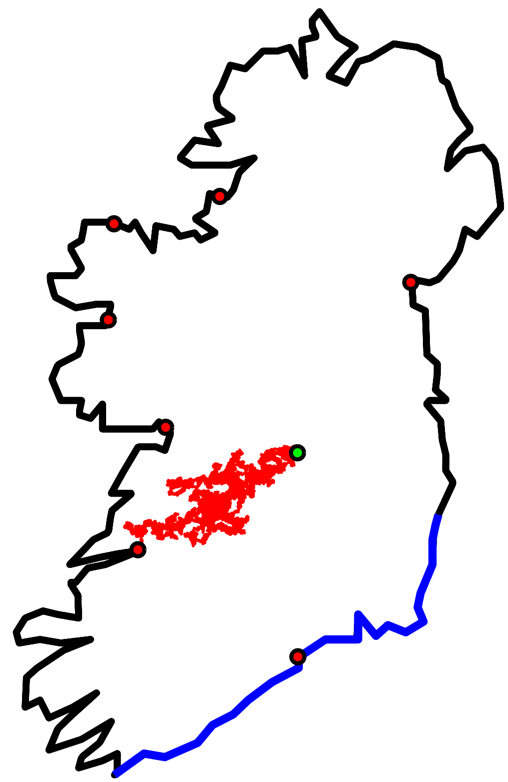
$\omega(z, E, \Omega)$ = probability a particle started at z first hits $\partial\Omega$ in E.

Harmonic measure = hitting distribution of Brownian motion



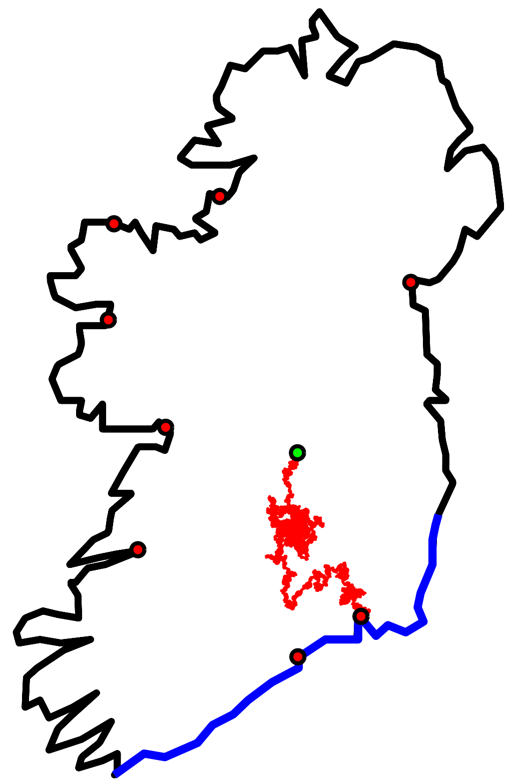
$\omega(z, E, \Omega)$ = probability a particle started at z first hits $\partial\Omega$ in E.

Harmonic measure = hitting distribution of Brownian motion



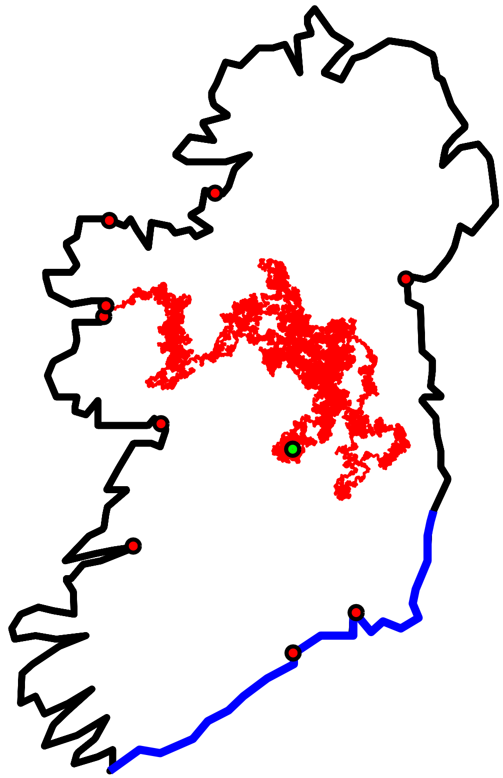
$\omega(z, E, \Omega)$ = probability a particle started at z first hits $\partial\Omega$ in E.

Harmonic measure = hitting distribution of Brownian motion



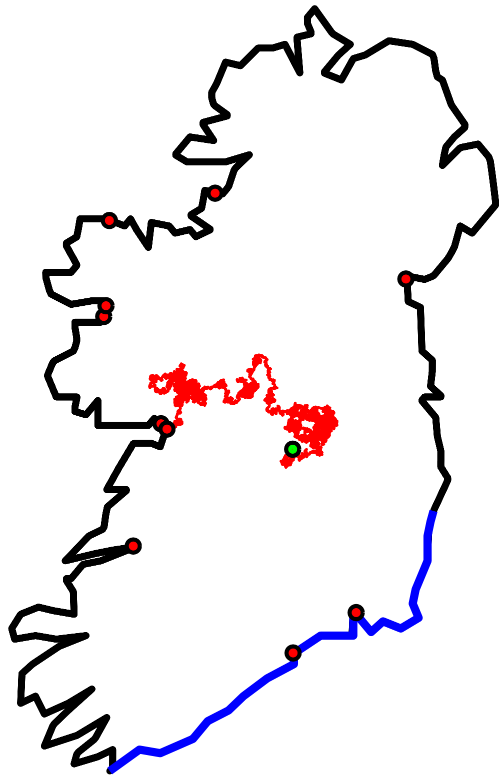
$\omega(z, E, \Omega)$ = probability a particle started at z first hits $\partial\Omega$ in E.

Harmonic measure = hitting distribution of Brownian motion



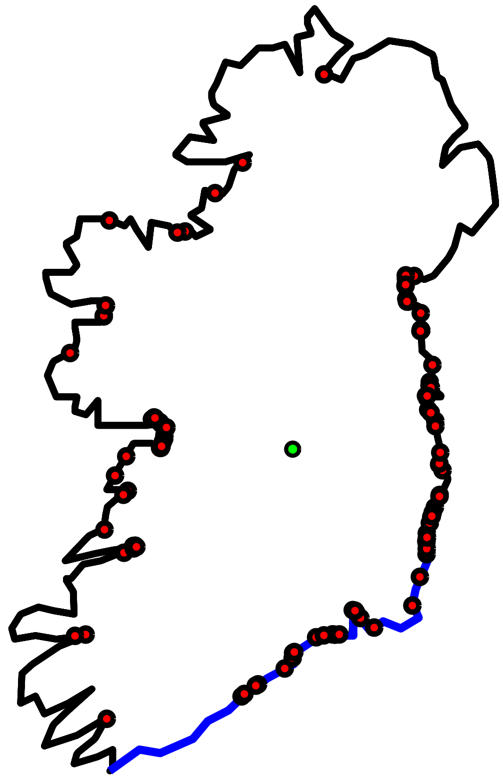
$\omega(z, E, \Omega)$ = probability a particle started at z first hits $\partial\Omega$ in E.

Harmonic measure = hitting distribution of Brownian motion



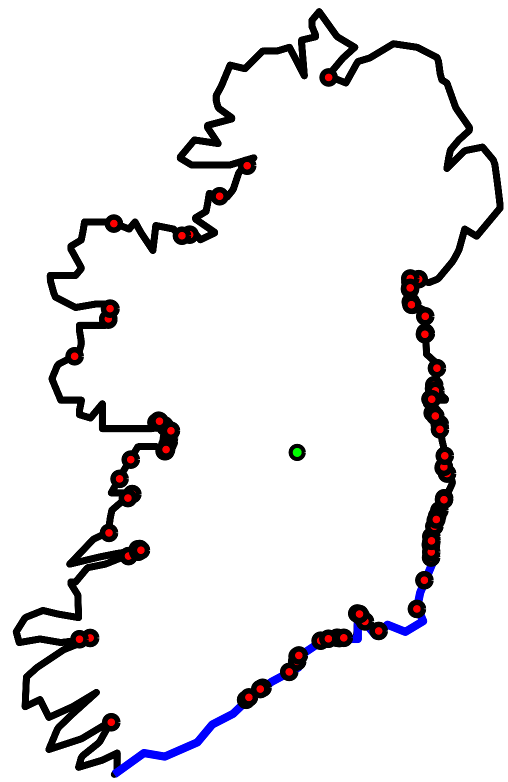
$$\omega(z, E, \Omega) \approx 3/10$$

Harmonic measure = hitting distribution of Brownian motion



$$\omega(z, E, \Omega) \approx 33/100.$$

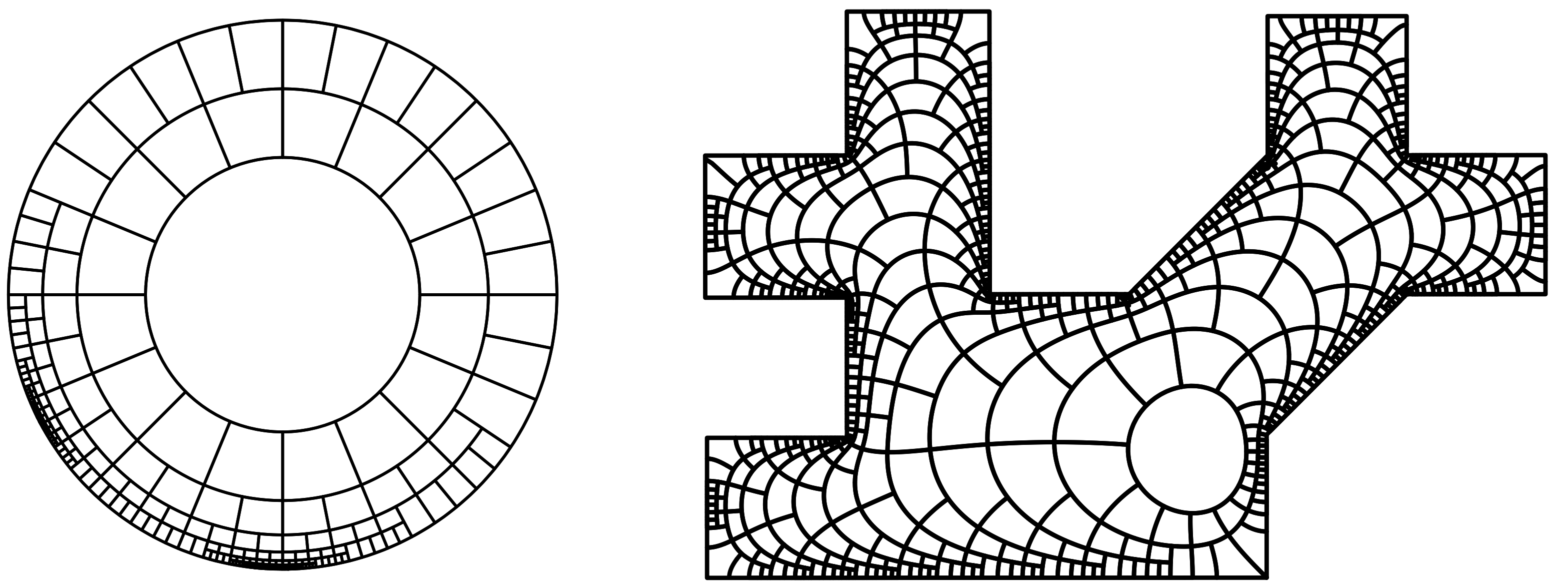
Harmonic measure = hitting distribution of Brownian motion



$\omega(z, E, \Omega)$ is a harmonic function of z on Ω.
Boundary values are 1 on E and 0 elsewhere.
Easy to solve this PDE on the disk: Poisson kernel.

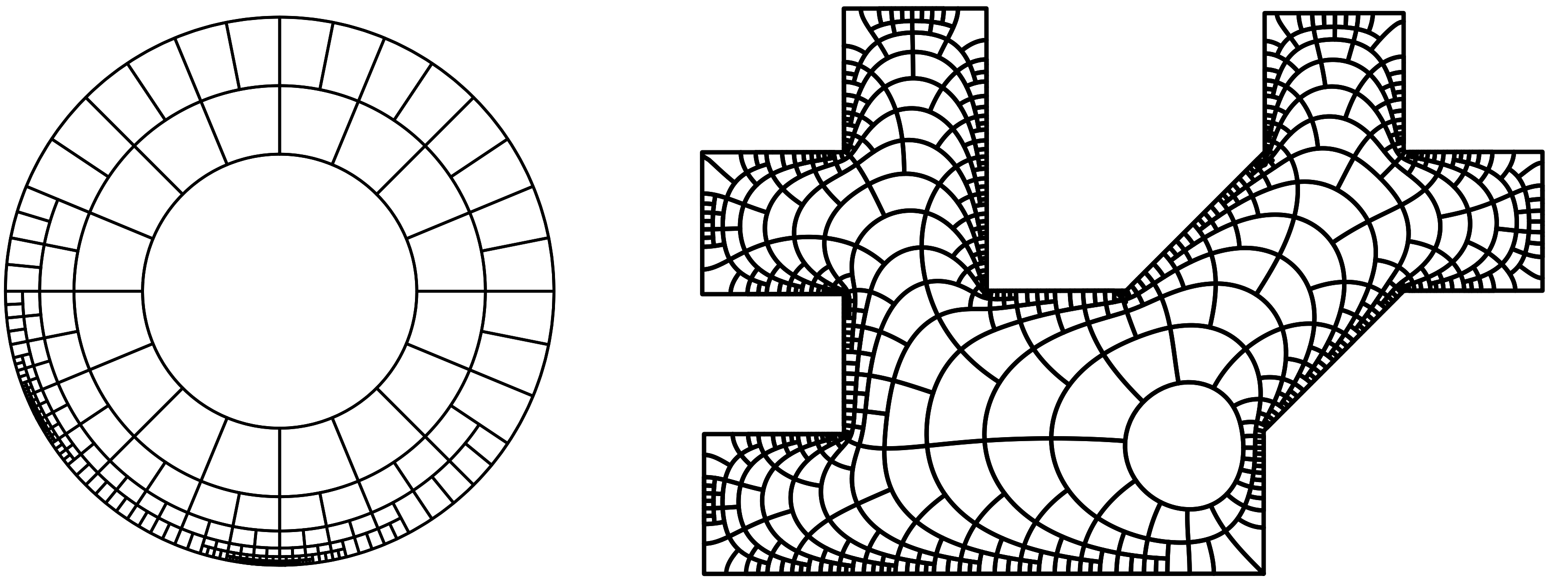
Riemann Mapping Theorem: If $\Omega \subsetneq \mathbb{R}^2$ is simply connected and $z \in \Omega$, then there is a conformal map $f : \mathbb{D} \to \Omega$ with $f(0) = z$.

conformal = 1-1, holomorphic= angle and orientation preserving



Riemann Mapping Theorem: If $\Omega \subsetneq \mathbb{R}^2$ is simply connected and $z \in \Omega$, then there is a conformal map $f : \mathbb{D} \to \Omega$ with $f(0) = z$.

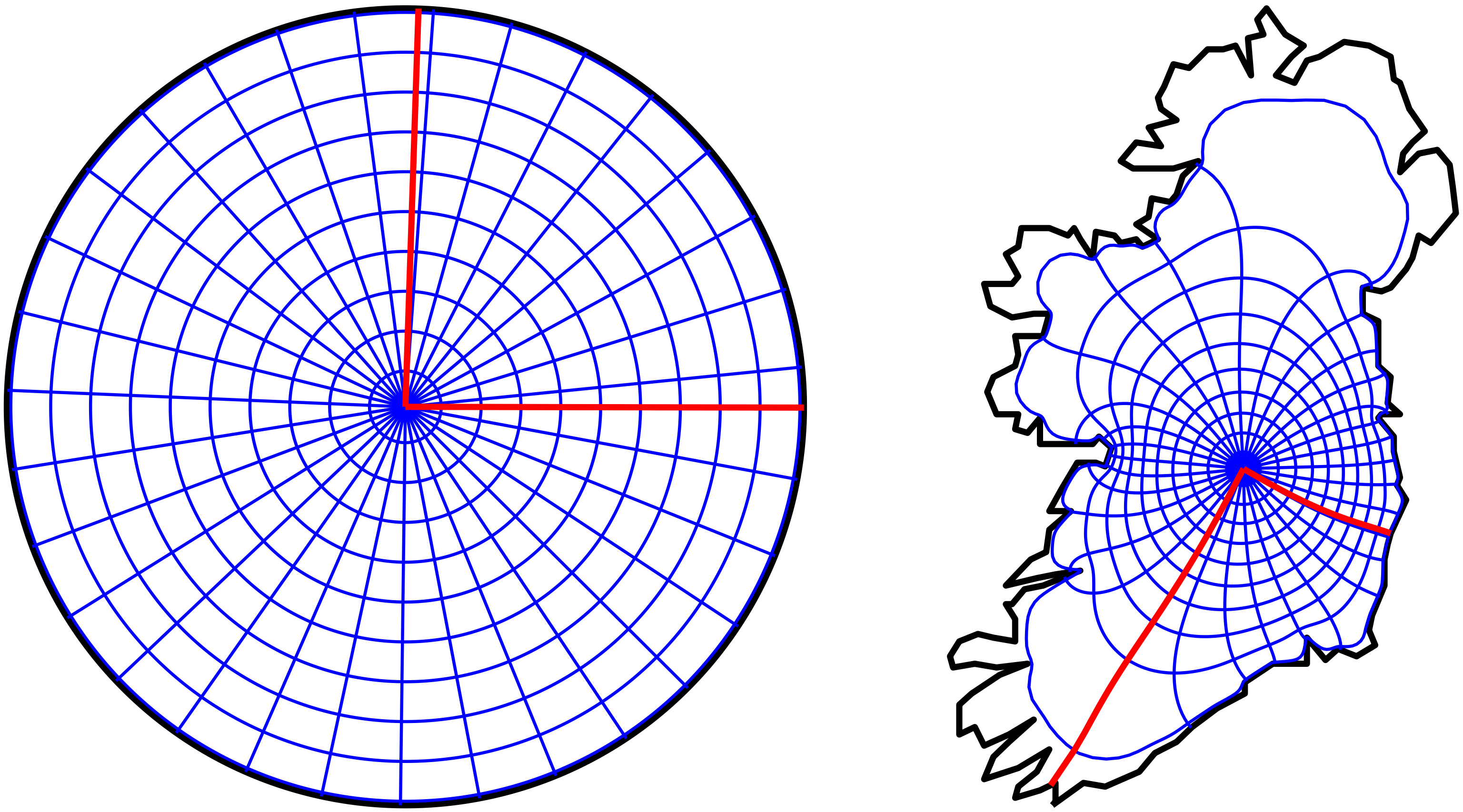
conformal = 1-1, holomorphic= angle and orientation preserving



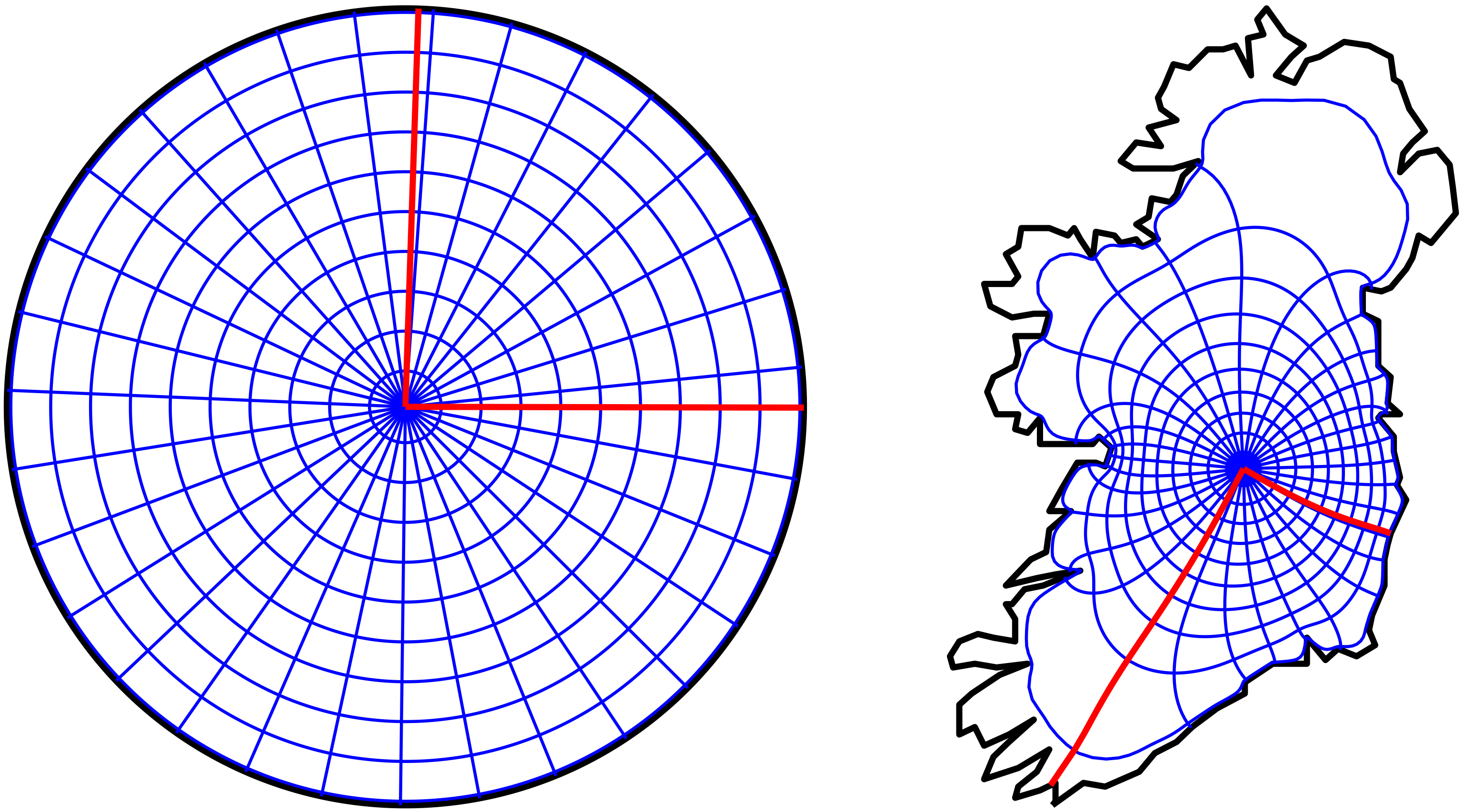
If ω is harmonic on Ω, then $\omega \circ f$ is harmonic on $\mathbb{D}$.

$\Rightarrow \omega(z, E, \Omega) = \omega(0, f^{-1}(E), \mathbb{D}) = |E|/2\pi.$

$\Rightarrow \omega$ on $\partial\Omega$ = conformal image of (normalized) length on $\mathbb{T}$



Harmonic measure ≈ .24507

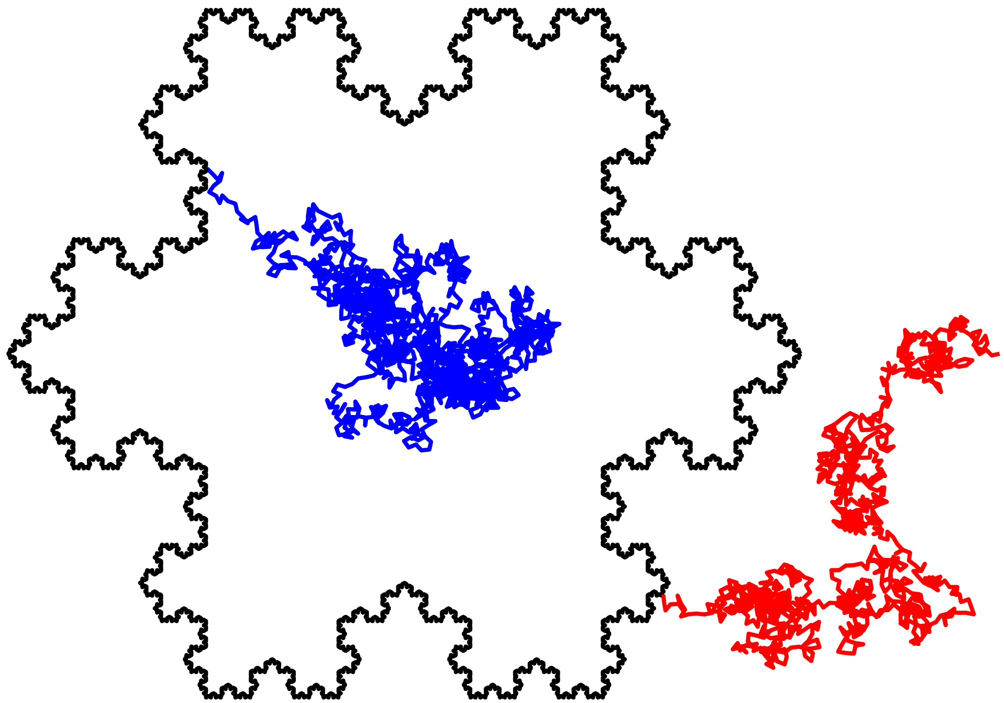


• different base points $z \in \Omega$ give different distributions on $\partial\Omega$,

• but all points in Ω give mutually continuous distributions.

What about points on different sides of $\partial\Omega$?

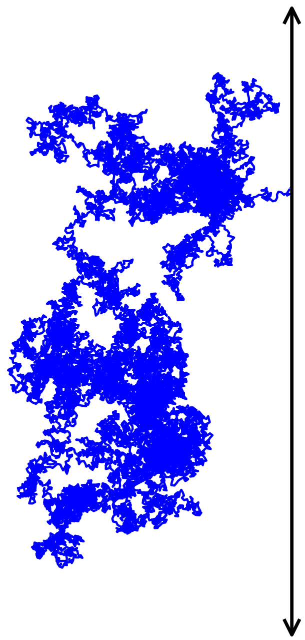
For a fractal curve, inside and outside harmonic measures are singular.



$\omega_1 \perp \omega_2$ iff tangents points have zero length.

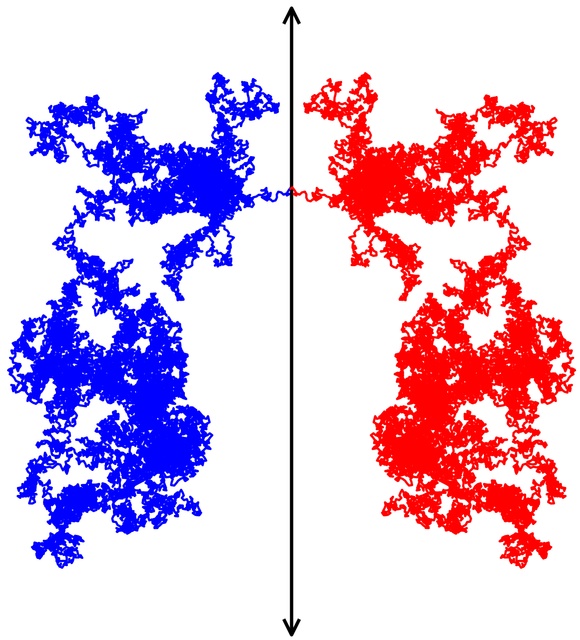
(B. 1987, based on results of N.G. Makarov)

For which curves is $\omega_1 = \omega_2$? (inside measure = outside measure)



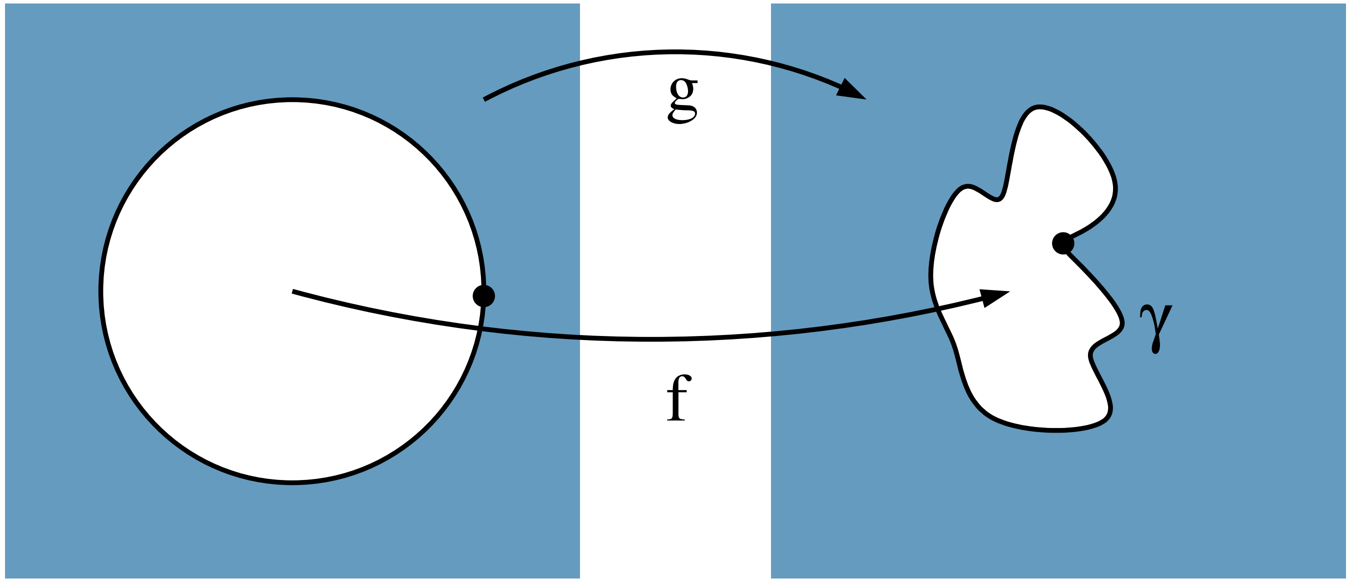
True for lines and circles.

For which curves is $\omega_1 = \omega_2$? (inside measure = outside measure)



True for lines and circles.

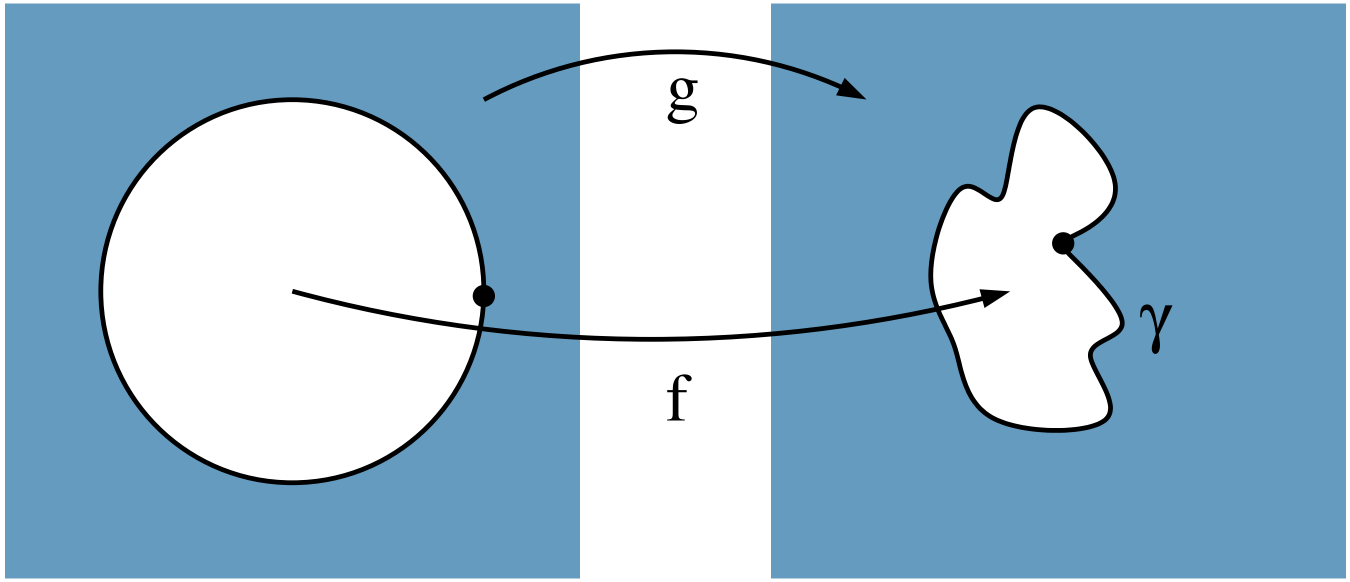
Is converse true? Equal harmonic measures $\Rightarrow$ circle?



Suppose $\omega_1 = \omega_2$ for a curve γ.

Conformally map two sides of circle to two sides of γ so $f(1) = g(1)$.

$\omega_1 = \omega_2$ implies maps agree on whole boundary.



Suppose $\omega_1 = \omega_2$ for a curve γ.

Conformally map two sides of circle to two sides of γ so $f(1) = g(1)$.

$\omega_1 = \omega_2$ implies maps agree on whole boundary.

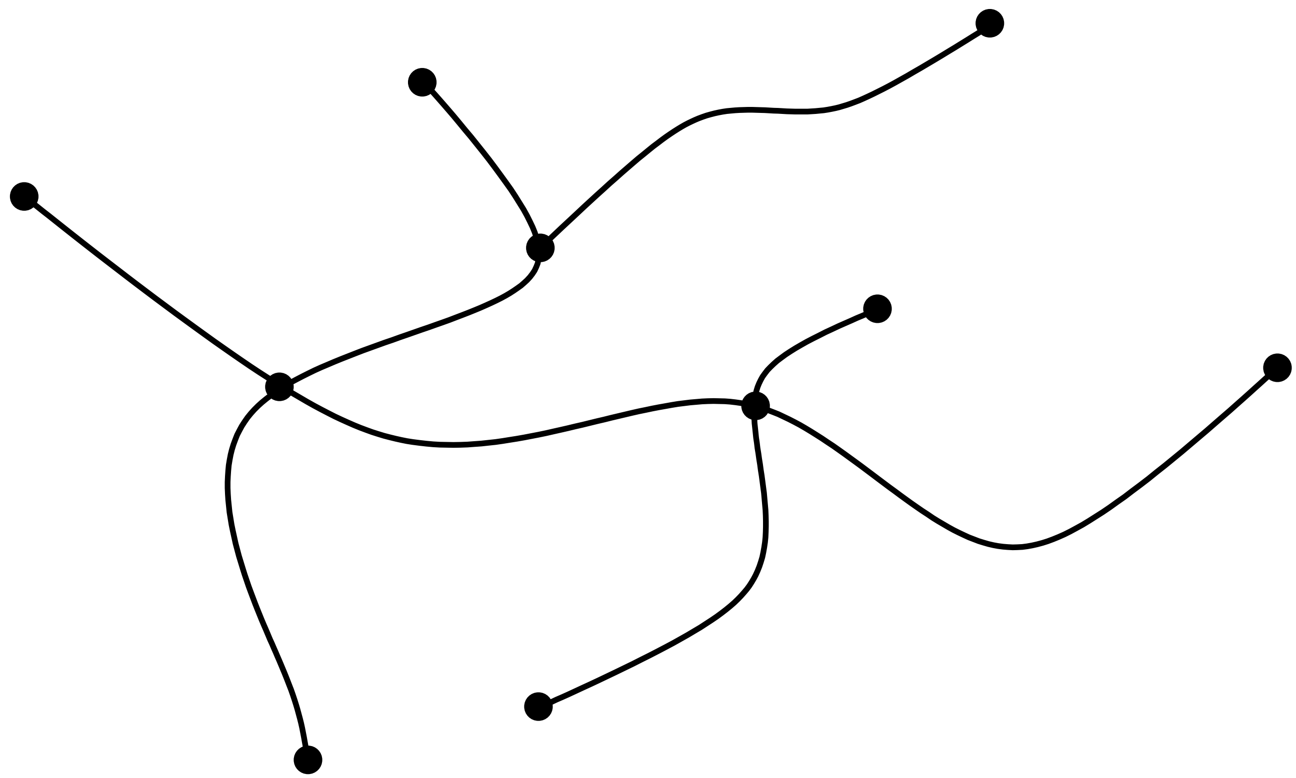
So f, g define homeomorphism h of plane holomorphic off circle.

Then h is entire by Morera's theorem.

Entire and 1-1 implies h is linear (Liouville's thm), so γ is a circle.

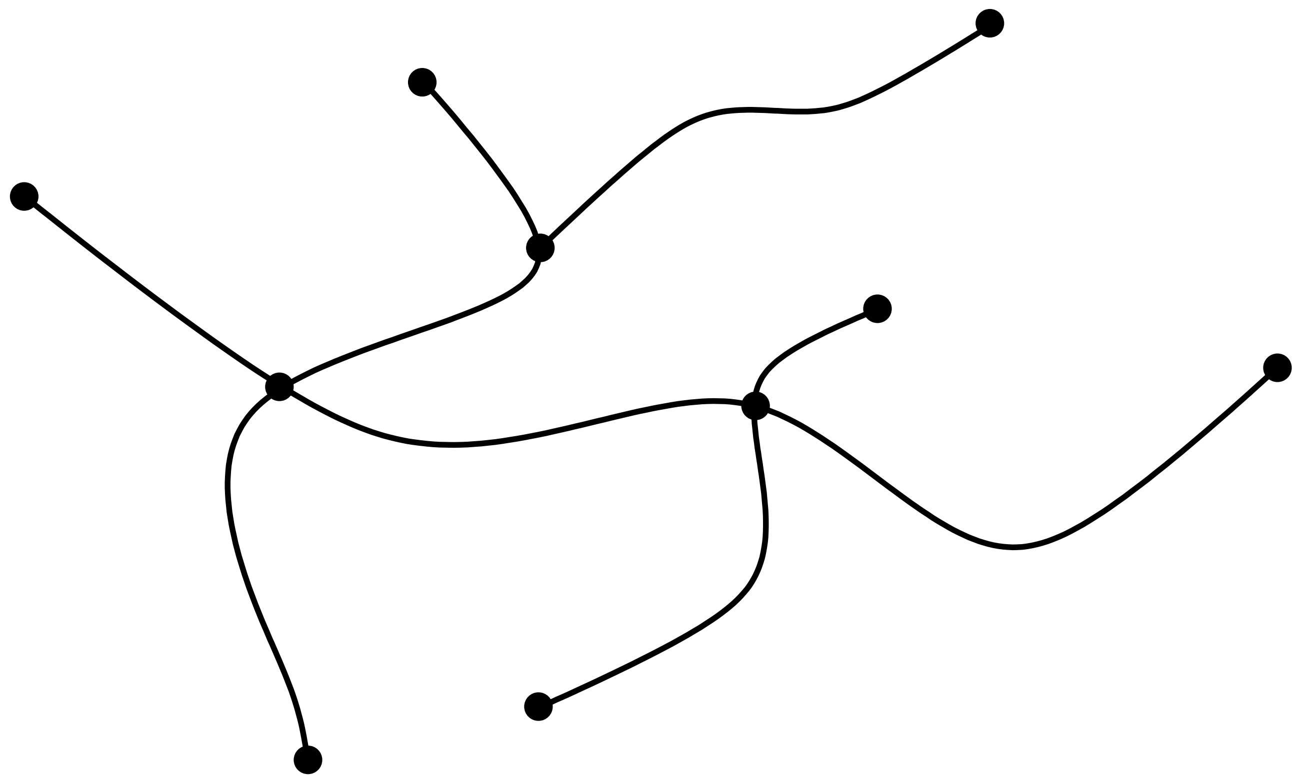
Circles are only closed curves with equal harmonic measures on both sides.

What about other kinds of 2-sided objects?



A planar graph is a finite set of points connected by non-crossing edges.

It is a tree if there are no closed loops.

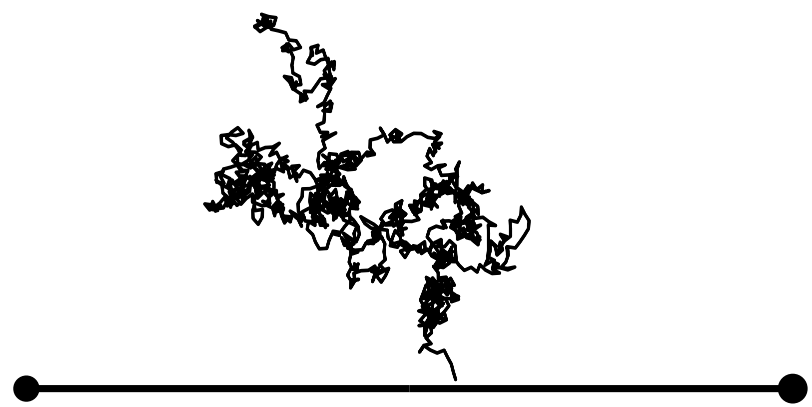


A planar tree is **conformally balanced** if

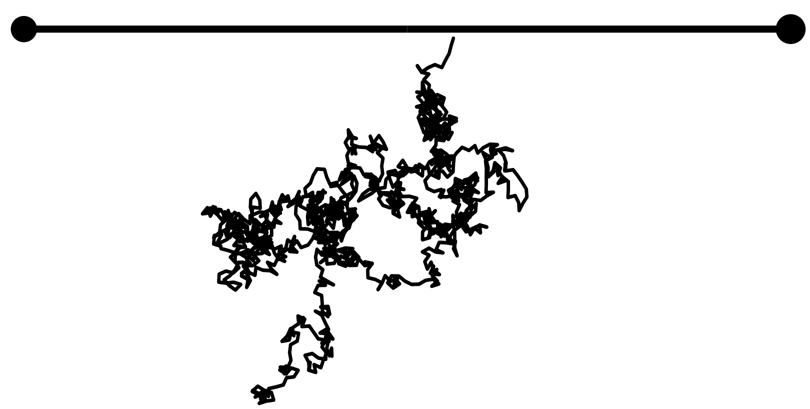
- every edge has equal harmonic measure from ∞
- edge subsets have same measure from both sides

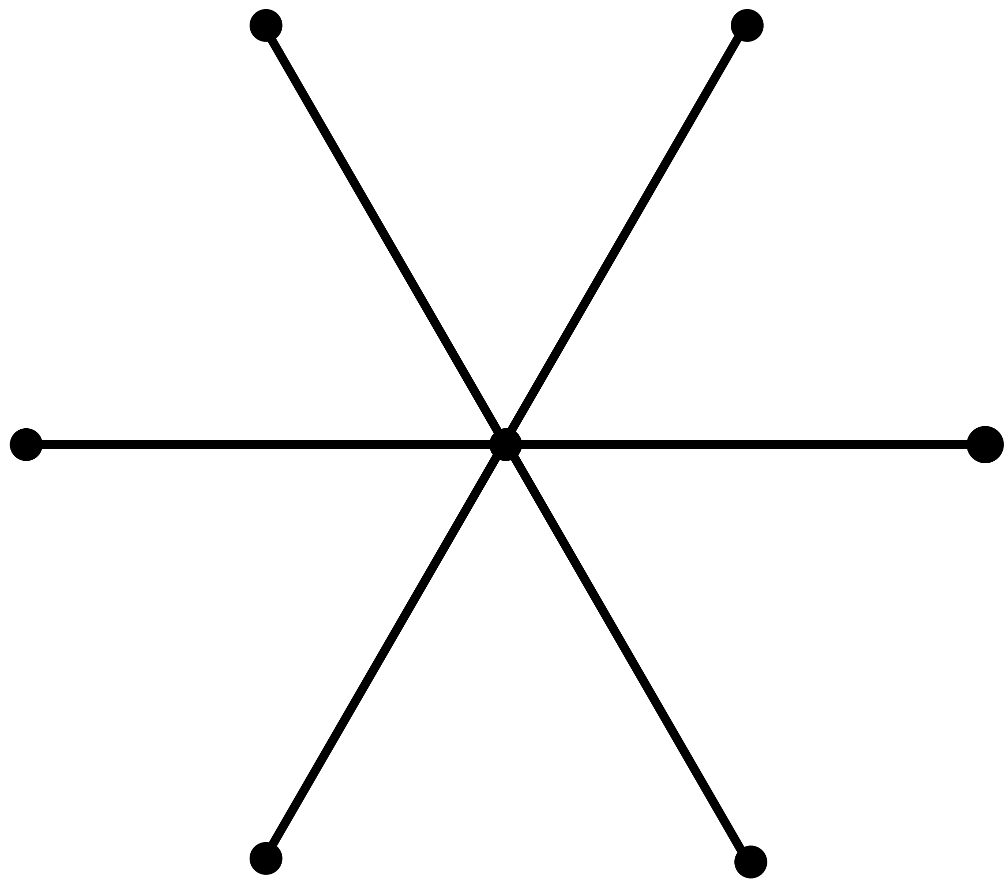
This is also called a "**true tree**". A line segment is an example.

This is also called a "**true tree**". A line segment is an example.

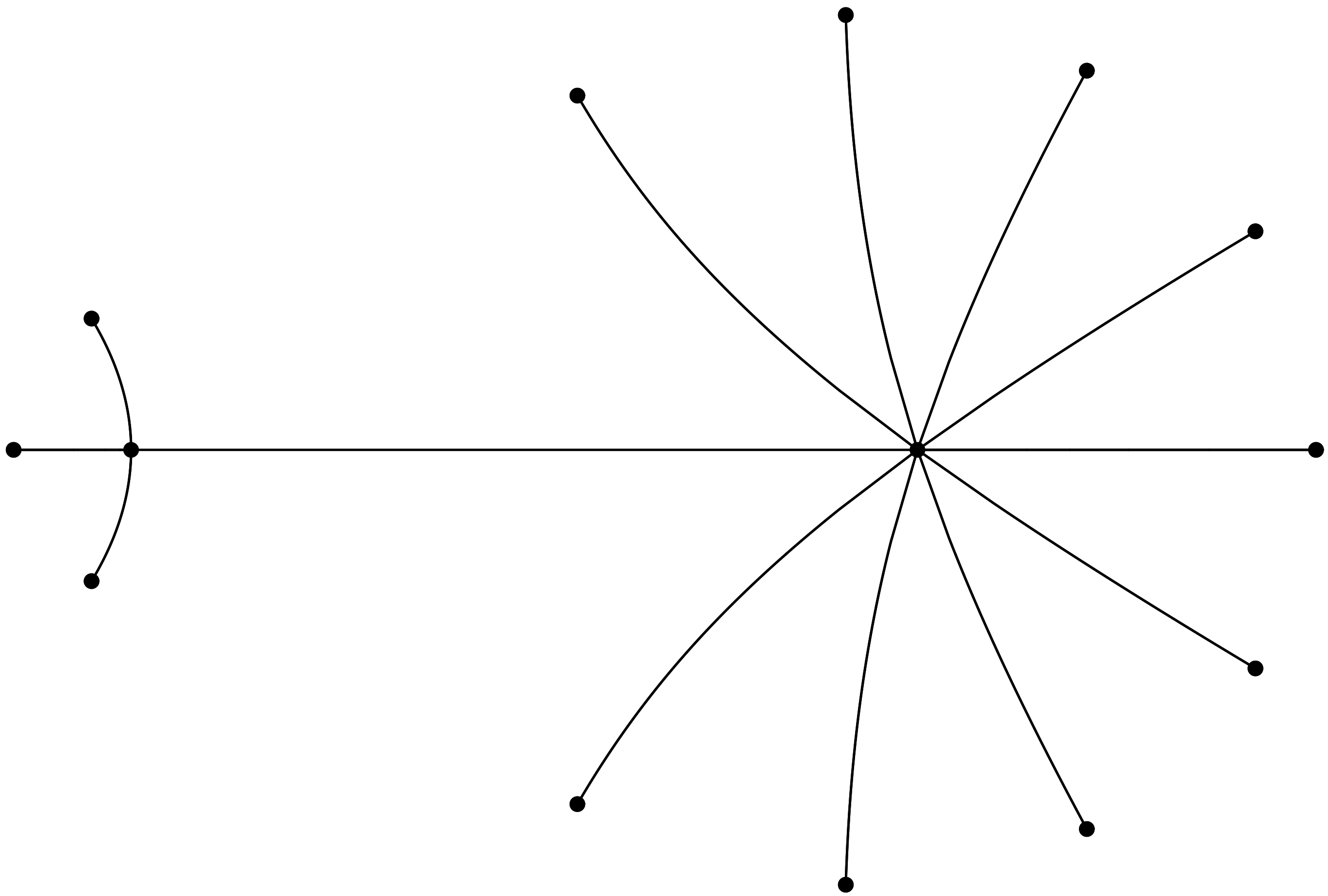


This is also called a "**true tree**". A line segment is an example.





Trivially true by symmetry



Non-obvious true tree

Definition of critical value: if p = polynomial, then

$$\mathrm{CV}(p) = \{p(z) : p'(z) = 0\} = \text{critical values}$$

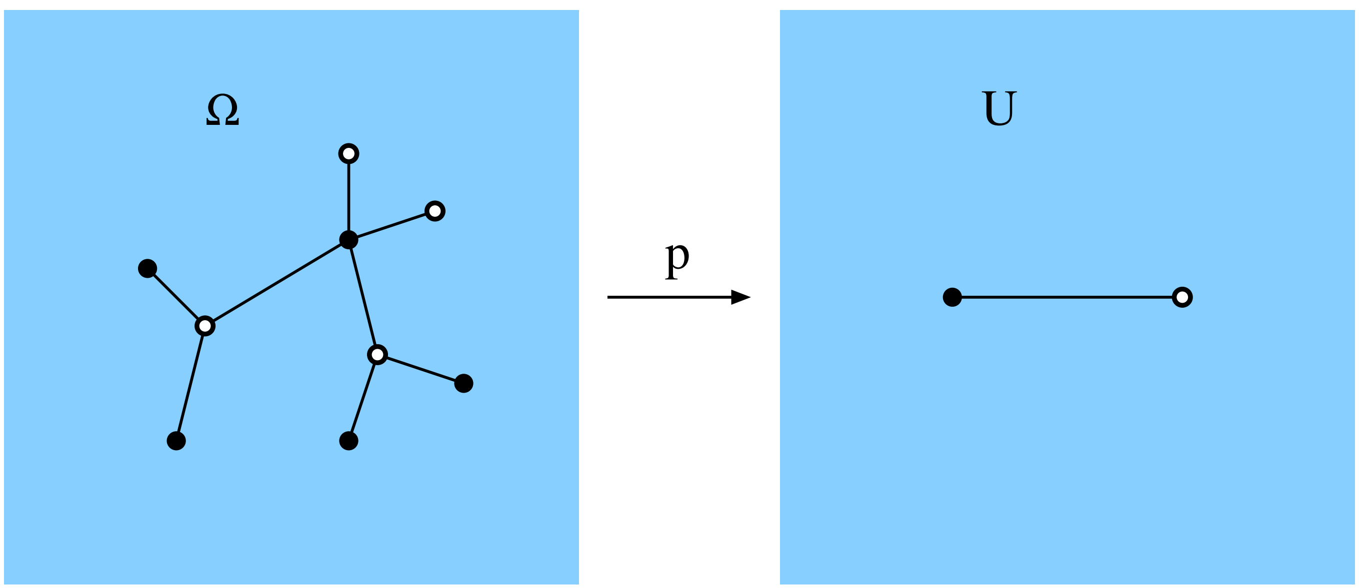
If $\mathrm{CV}(p) = \pm 1$, p is called **generalized Chebyshev** or **Shabat**.

Definition of critical value: if p = polynomial, then

$$\mathrm{CV}(p) = \{p(z) : p'(z) = 0\} = \text{critical values}$$

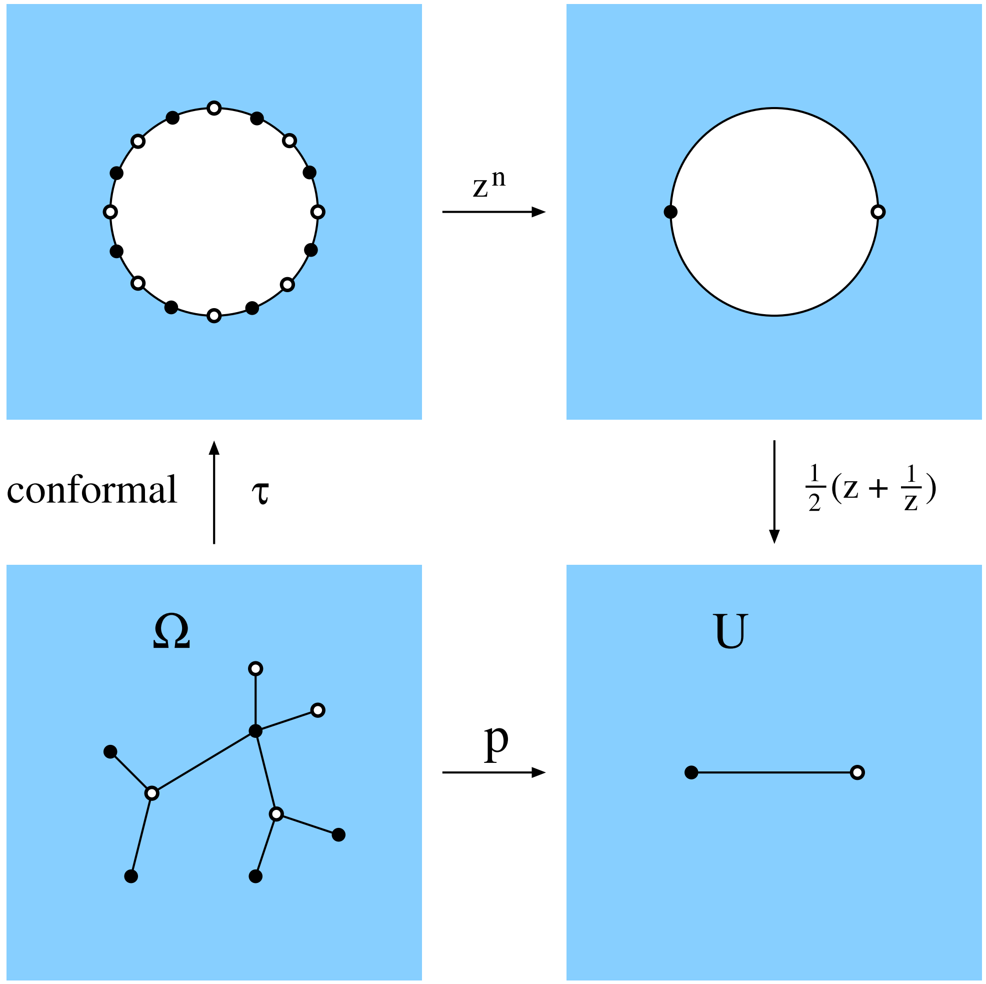
If $\mathrm{CV}(p) = \pm 1$, p is called **generalized Chebyshev** or **Shabat**.

Thm: T is balanced iff $T = p^{-1}([-1,1])$, p = Shabat.



$$\Omega = \mathbb{C} \setminus T \qquad\qquad U = \mathbb{C} \setminus [-1,1]$$

T conformally balanced $\Leftrightarrow$ p Shabat.



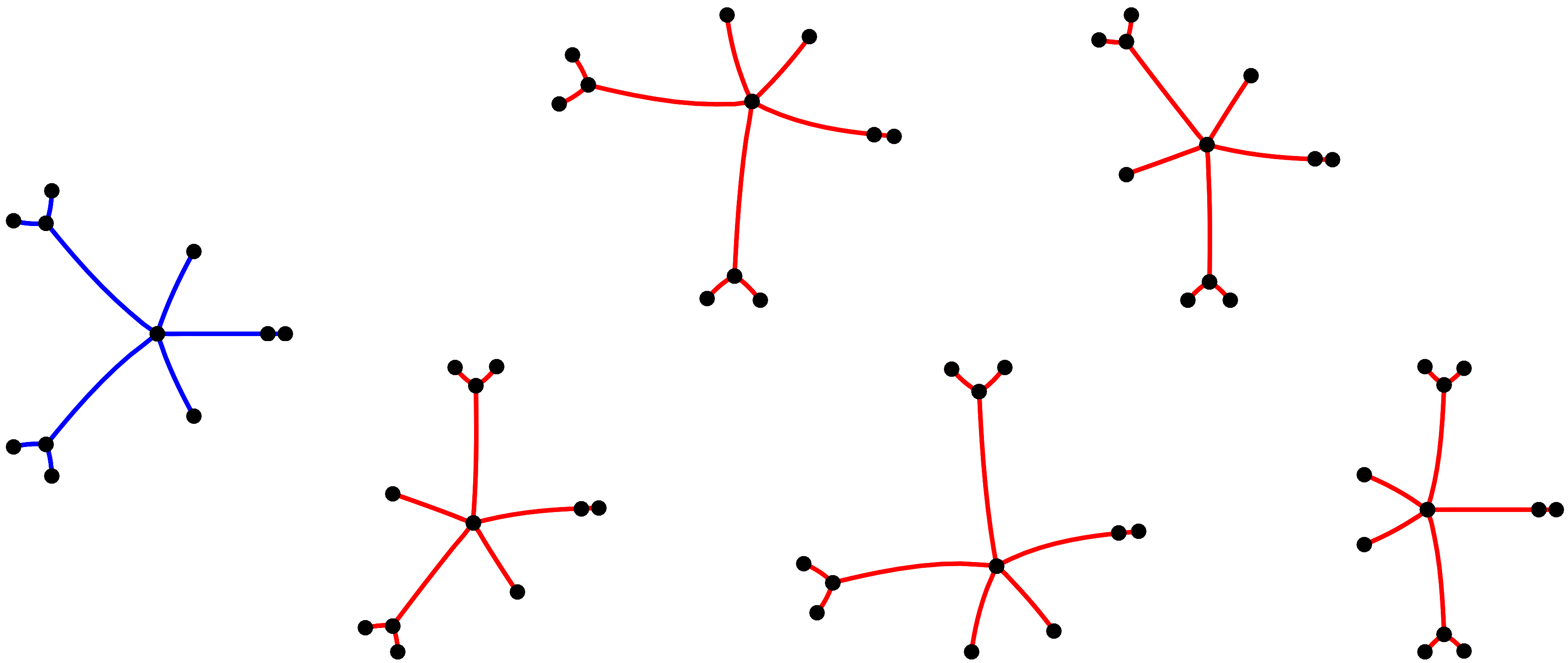
p is entire and n-to-1 $\Leftrightarrow$ p = polynomial.

$\mathrm{CV}(p) \notin U \leftrightarrow p : \Omega \to U$ is covering map.

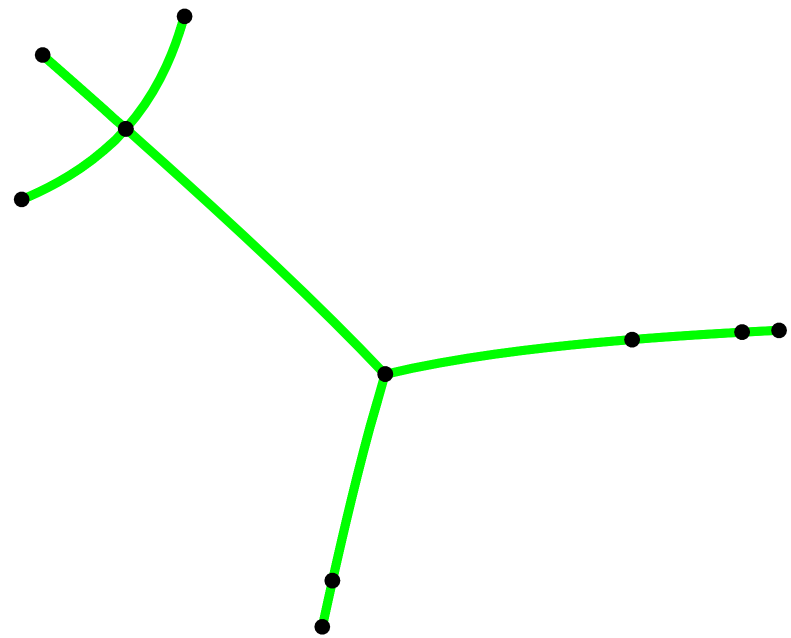
Algebraic aside:

True trees are examples of Grothendieck's *dessins d'enfants* on sphere.

Normalized polynomials are algebraic, so trees correspond to number fields. Computing number field from tree is difficult.



Kochetkov (2009, 2014): cataloged all trees with 9 and 10 edges.



For example, the polynomial for this 9-edge tree is

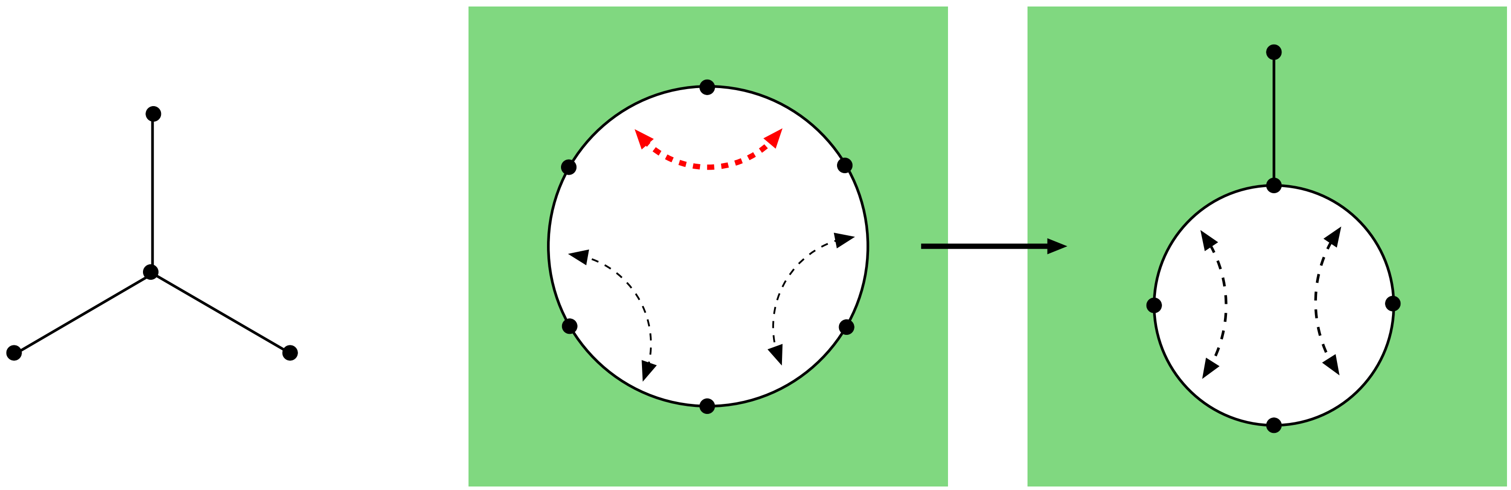
$$p(z) = z^4(z^2 + az + b)^2(z - 1),$$

where a is a root of ...

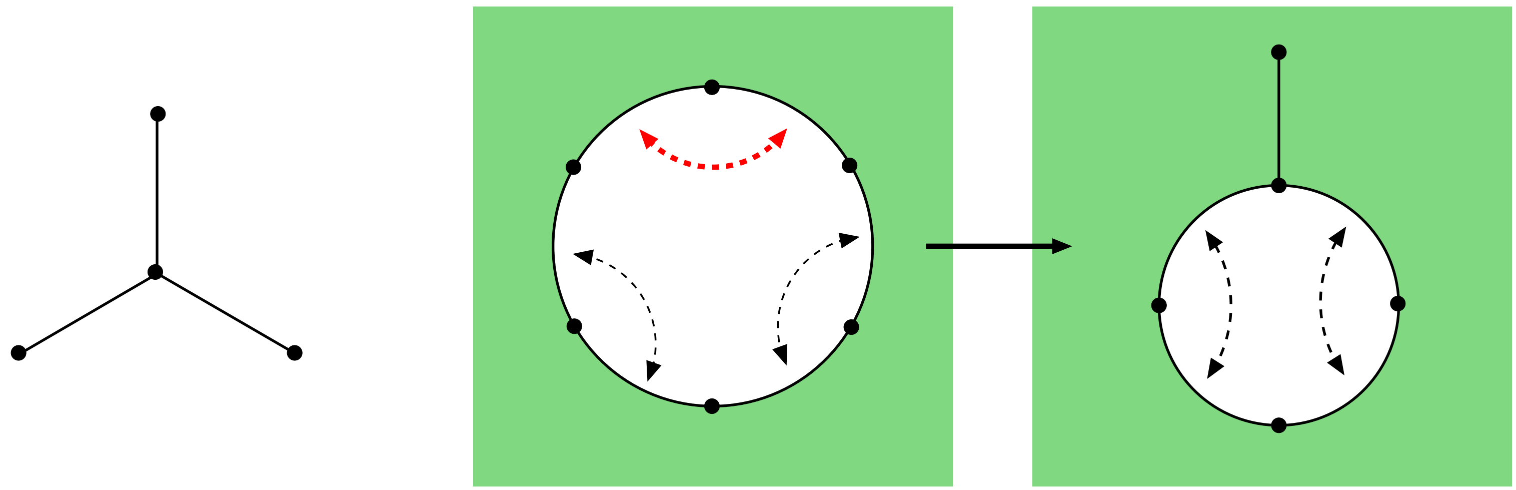
$$
\begin{aligned}
0 = \;& 126105021875\, a^{15} + 873367351500\, a^{14} \\
& +2340460381665\, a^{13} + 2877817869766\, a^{12} \\
& +3181427453757\, a^{11} - 68622755391456\, a^{10} \\
& -680918281137097\, a^{9} - 2851406436711330\, a^{8} \\
& -7139130404618520\, a^{7} - 12051656256571792\, a^{6} \\
& -14350515598839120\, a^{5} - 12058311779508768\, a^{4} \\
& -6916678783373312\, a^{3} - 2556853615656960\, a^{2} \\
& -561846360735744\, a - 65703906377728
\end{aligned}
$$

I did **not** use this to draw tree on previous slide.

Don Marshall's `ZIPPER` uses conformal mapping to draw true trees.



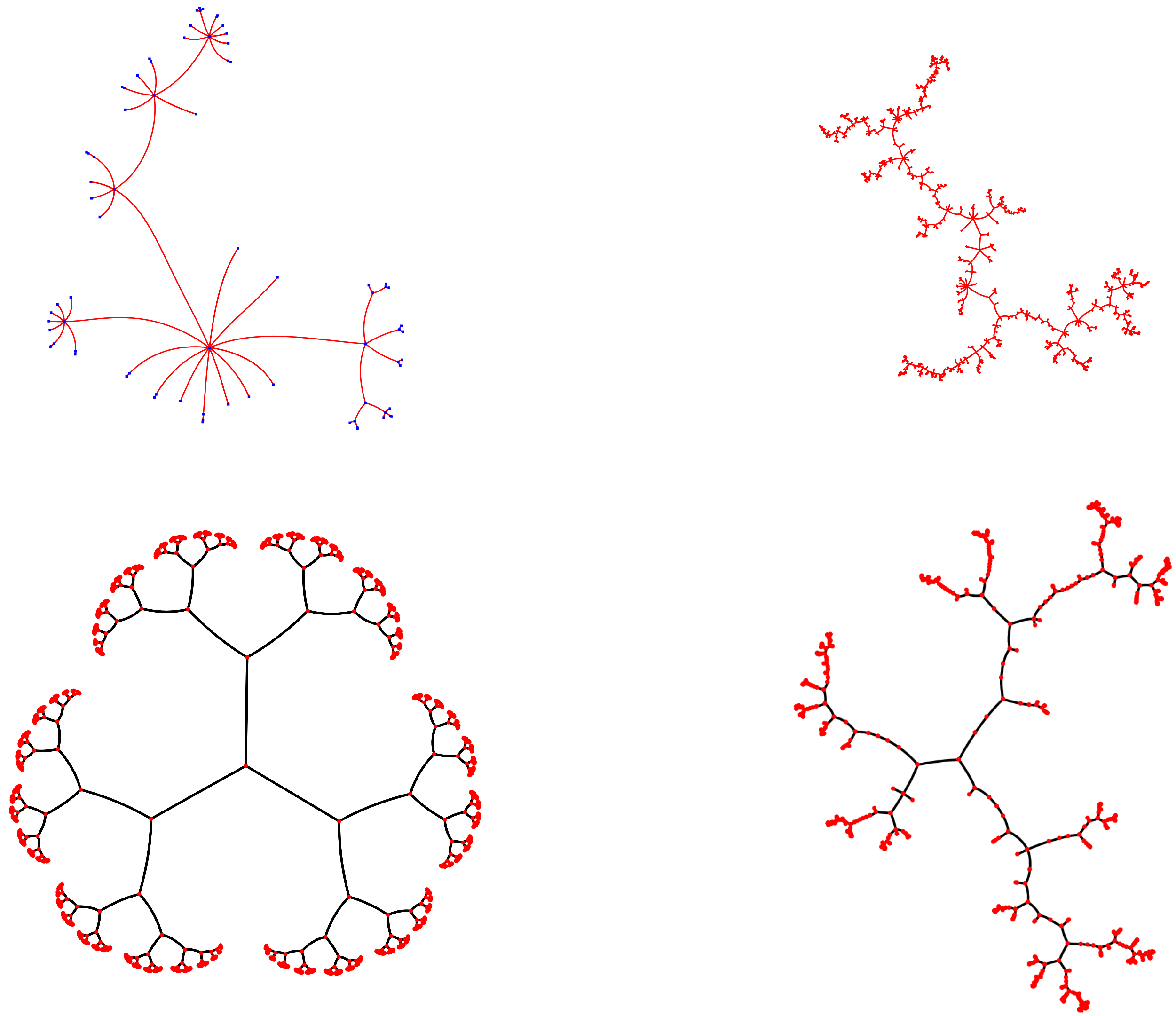
Don Marshall's `ZIPPER` uses conformal mapping to draw true trees.



Conformal mapping can handle trees with thousands of edges.

Can obtain polynomial roots α with thousands of digits of accuracy.

Marshall and Rohde compute number field by finding integer relation between $1, \alpha, \alpha^2, \ldots$ using Ferguson's PSLQ algorithm.



Some true trees, courtesy of Marshall and Rohde

Which planar trees have a true form?

What can that true form look like?

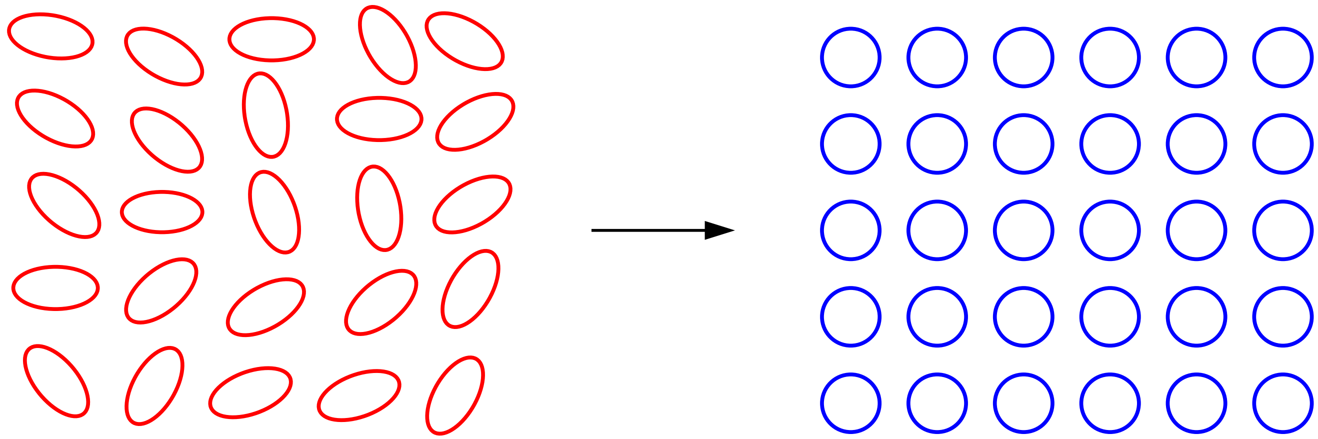
Theorem: Every finite planar tree has a true form.

Unique up to similarities (Morera + Liouville).

Standard proof uses the uniformization theorem.

I will describe an alternate proof using quasiconformal maps.

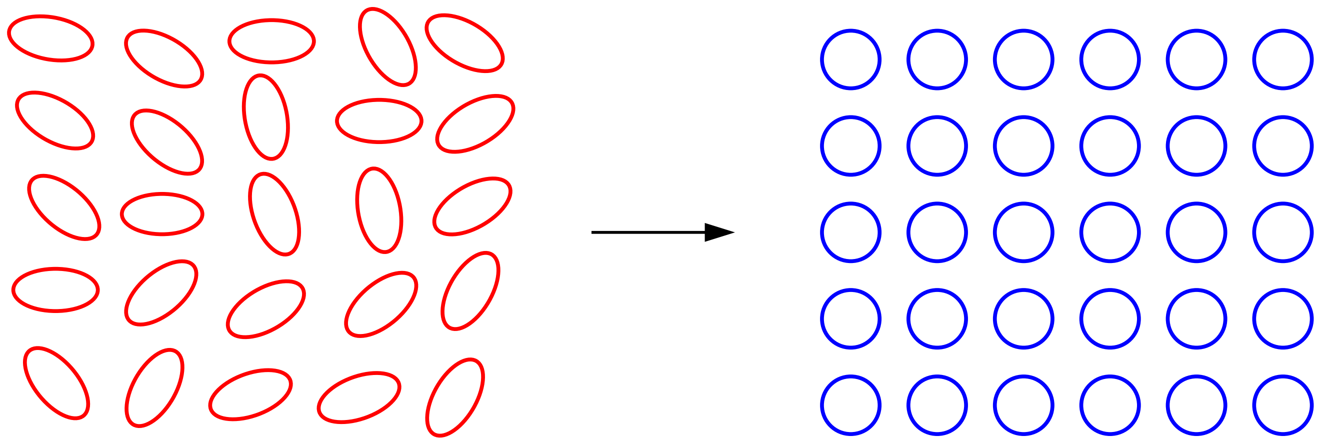
Diffeomorphisms send infinitesimal ellipses to circles.



Eccentricity = ratio of major to minor axis of ellipse.

K-quasiconformal = ellipses have eccentricity $\leq K$ almost everywhere

Diffeomorphisms send infinitesimal ellipses to circles.



Eccentricity = ratio of major to minor axis of ellipse.

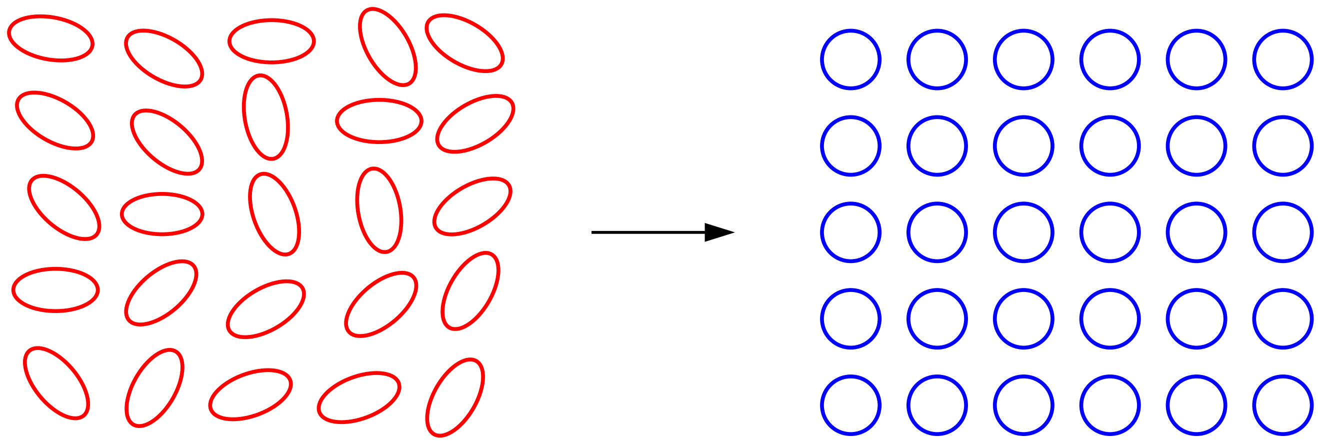
K-quasiconformal = ellipses have eccentricity $\leq K$ almost everywhere

Ellipses determined a.e. by measurable dilatation $\mu = f_{\overline{z}}/f_z$ with

$$|\mu| \leq \frac{K-1}{K+1} < 1.$$

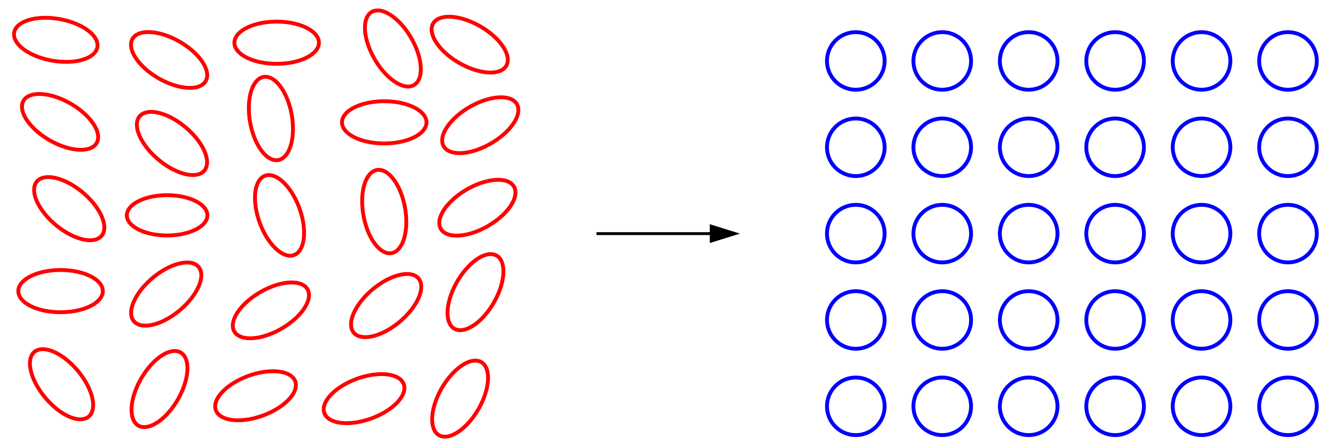
Conversely, ...

Diffeomorphisms send infinitesimal ellipses to circles.



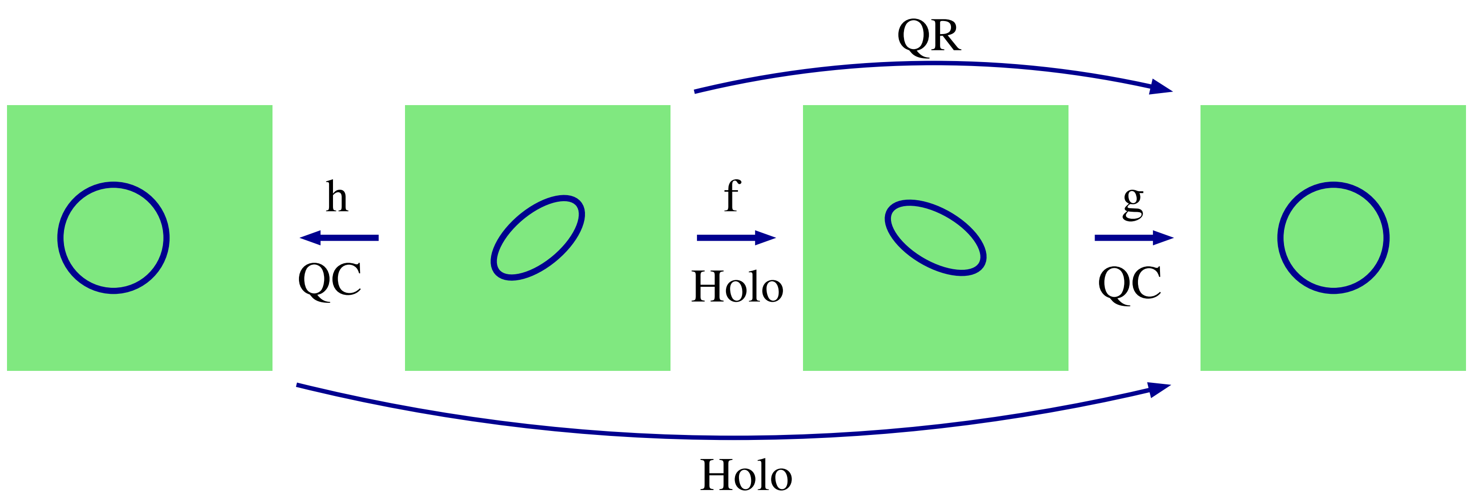
Mapping theorem: any such μ comes from some QC map f.

Diffeomorphisms send infinitesimal ellipses to circles.



Mapping theorem: any such μ comes from some QC map f.

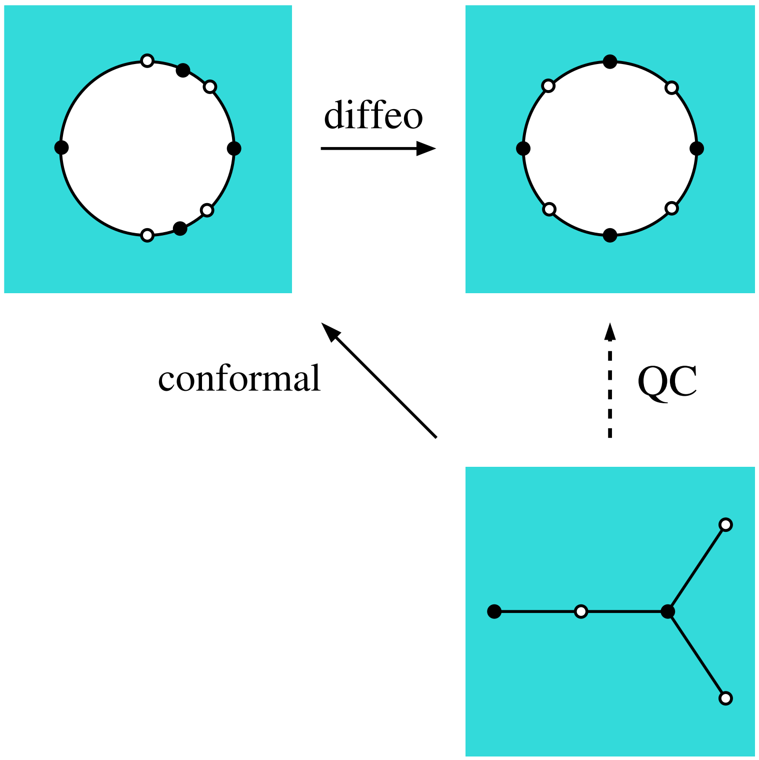
Cor: If f is holomorphic and g is QC, then there is a QC map h so that $F = g \circ f \circ h^{-1}$ is also holomorphic. ($g \circ f$ is called QR = quasiregular)



QC proof that every finite tree has a true form:

Map $\Omega = \mathbb{C} \setminus T$ to $\{|z| > 1\}$ conformally.

"Equalize intervals" by diffeomorphism. Composition is quasiconformal.

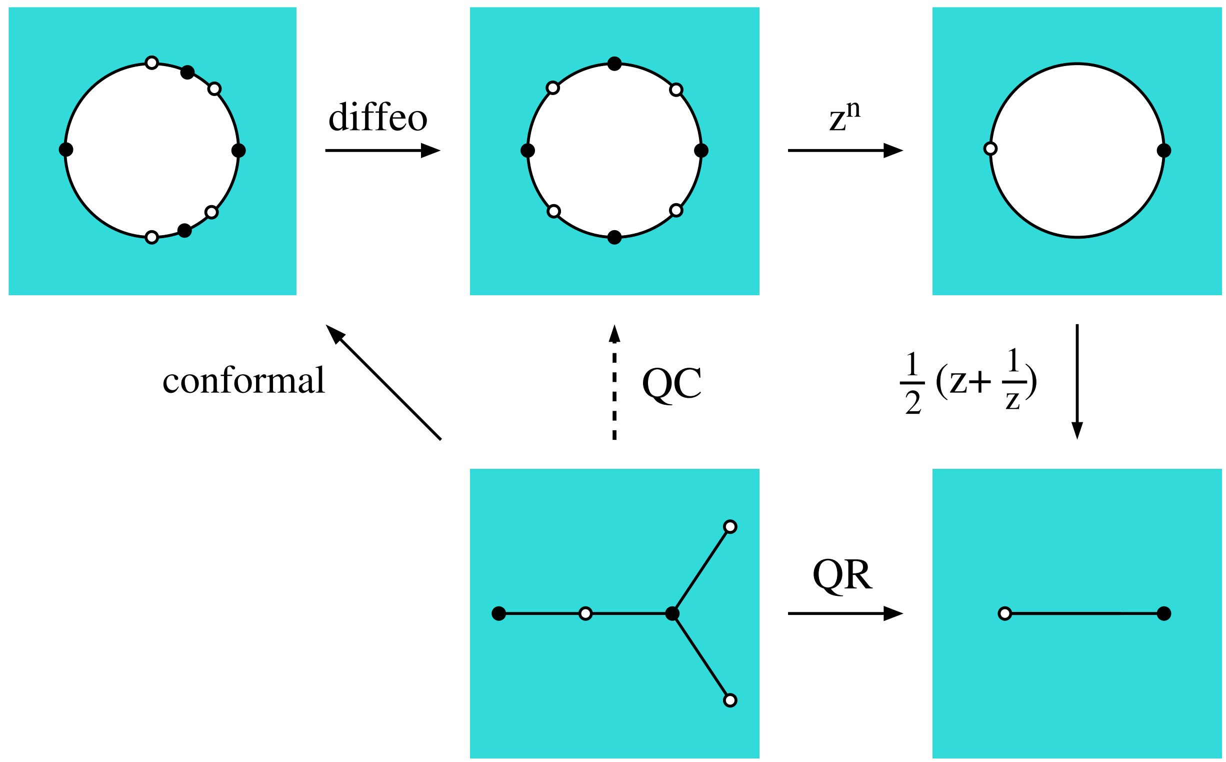


Dilatation of QC map depends degree of "imbalance" of harmonic measure.

QC proof that every finite tree has a true form:

Map $\Omega = \mathbb{C} \setminus T$ to $\{|z| > 1\}$ conformally.

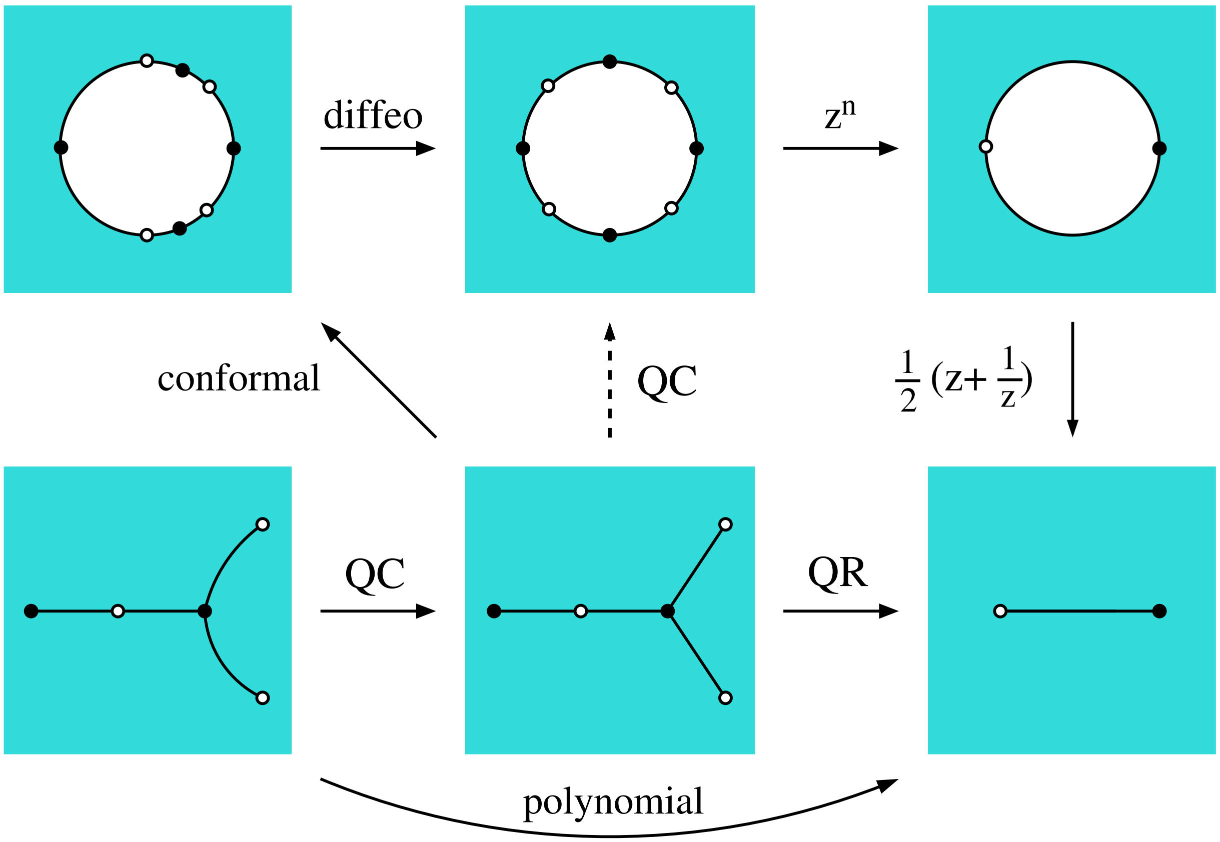
"Equalize intervals" by diffeomorphism. Composition is quasiconformal.



QC proof that every finite tree has a true form:

Map $\Omega = \mathbb{C} \setminus T$ to $\{|z| > 1\}$ conformally.

"Equalize intervals" by diffeomorphism. Composition is quasiconformal.



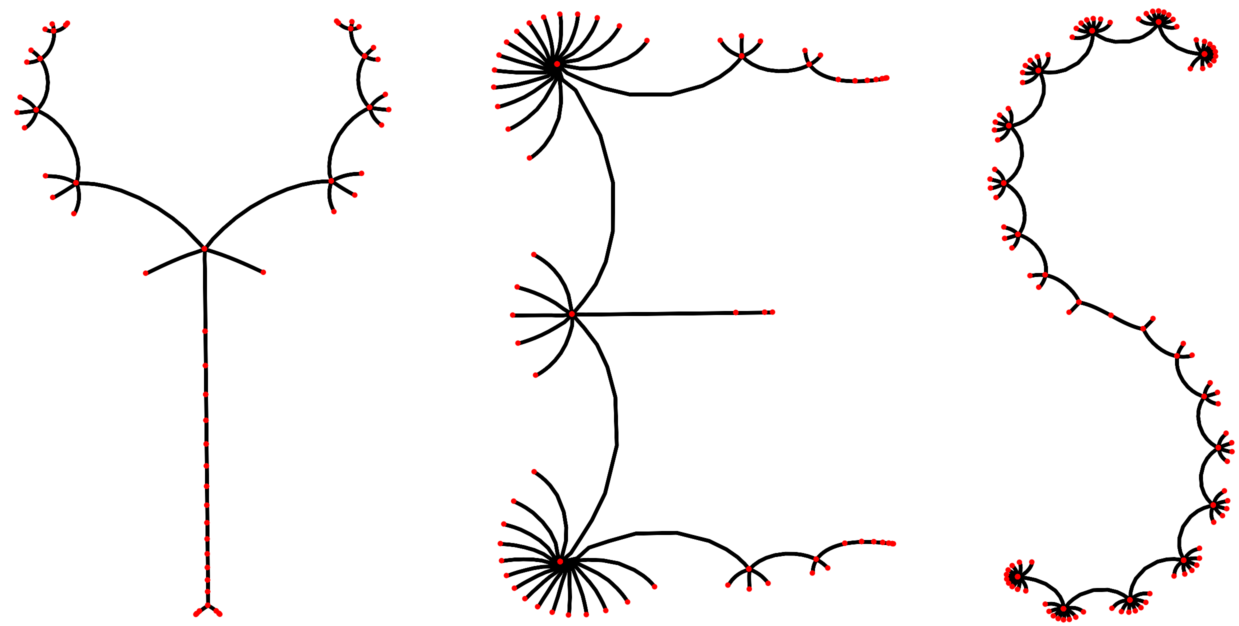
Mapping theorem implies there is a QC φ so $p = q \circ \varphi$ is a polynomial.

Thus true trees can have any combinatorics.

Alex Eremenko: can they have any shape?

Thus true trees can have any combinatorics.

Alex Eremenko: can they have any shape?



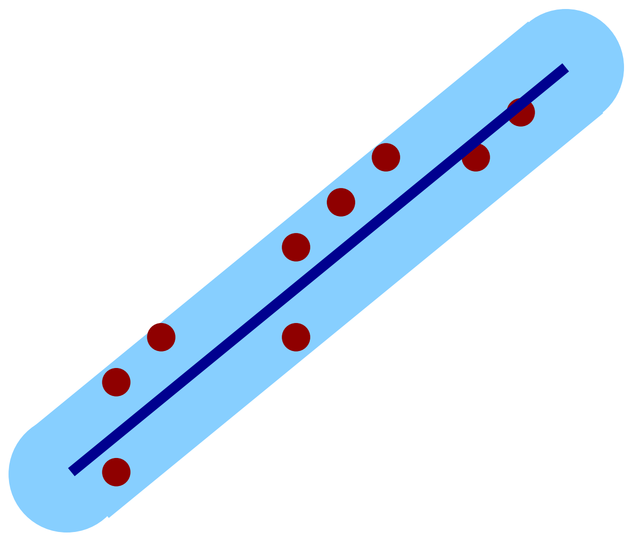
Thm: Every planar continuum is Hausdorff limit of true trees.

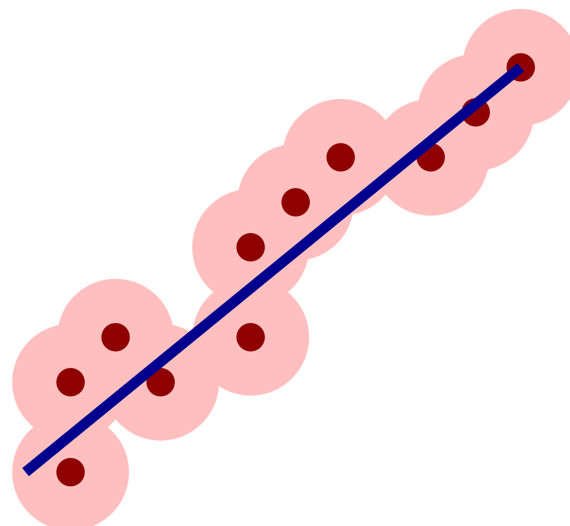
For a set E and $\epsilon > 0$ define ϵ-neighborhood of E:

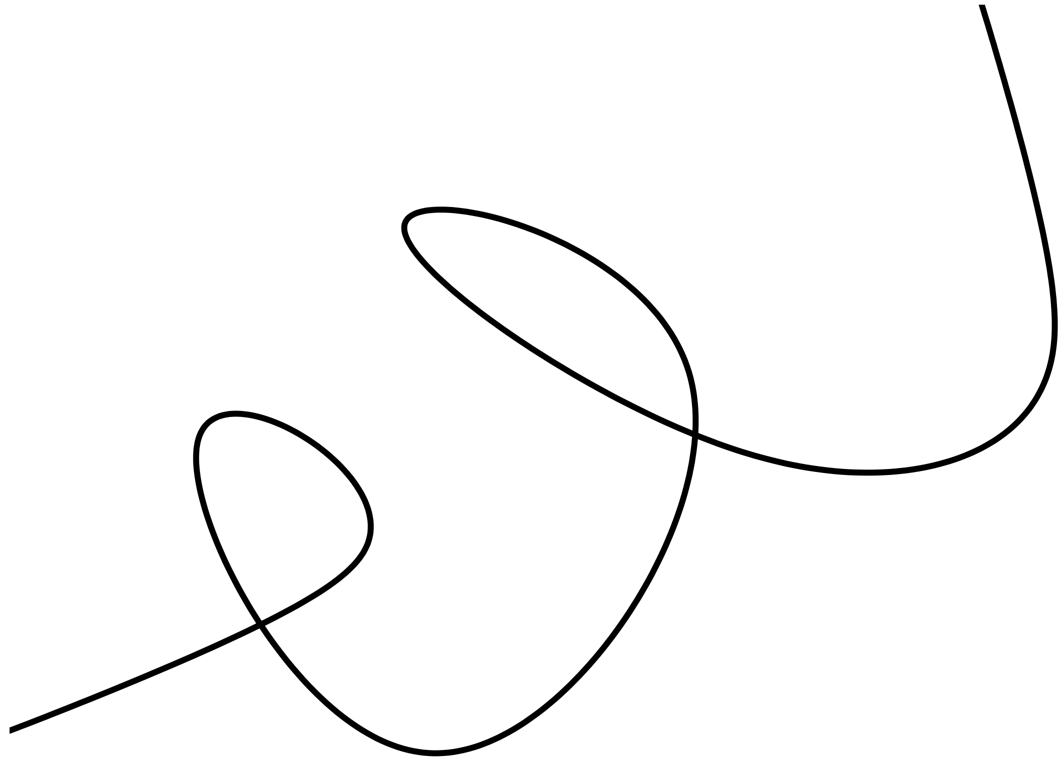
$$E(\epsilon) = \{z : \operatorname{dist}(z, E) < \epsilon\}.$$

Defn: Hausdorff distance between sets is

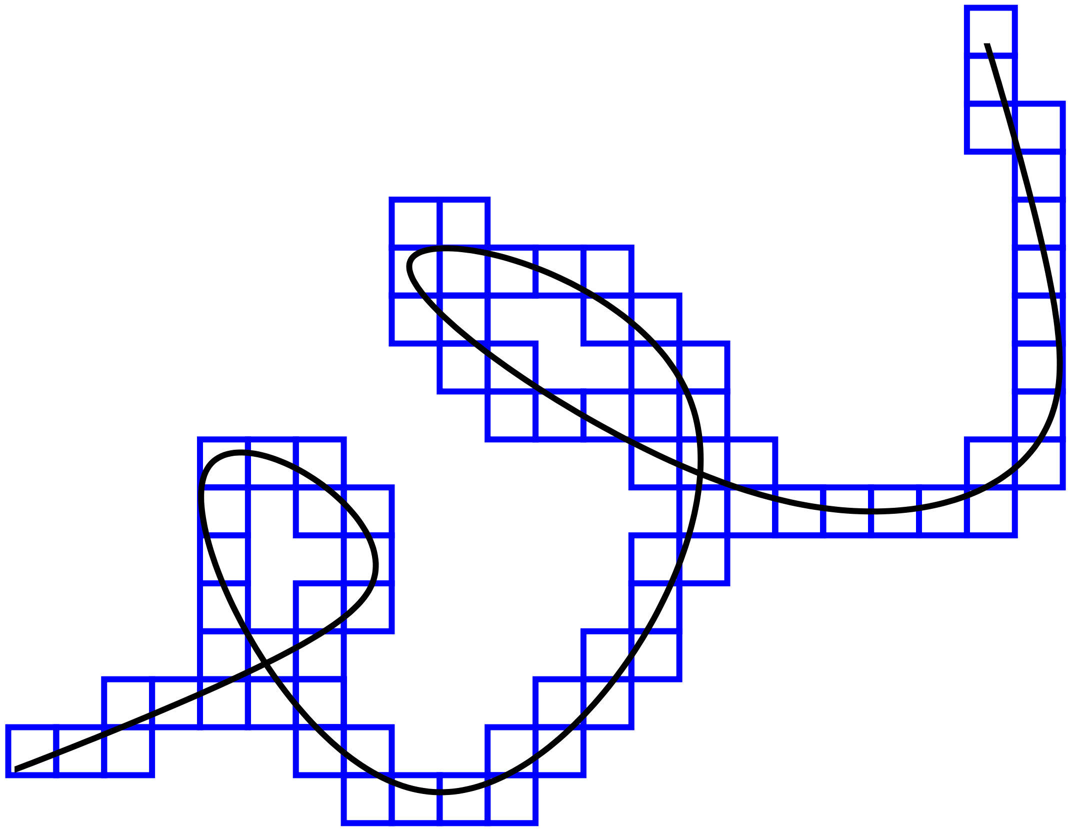
$$d(E, F) = \inf\{\epsilon > 0 : E \subset F(\epsilon), F \subset E(\epsilon)\}.$$



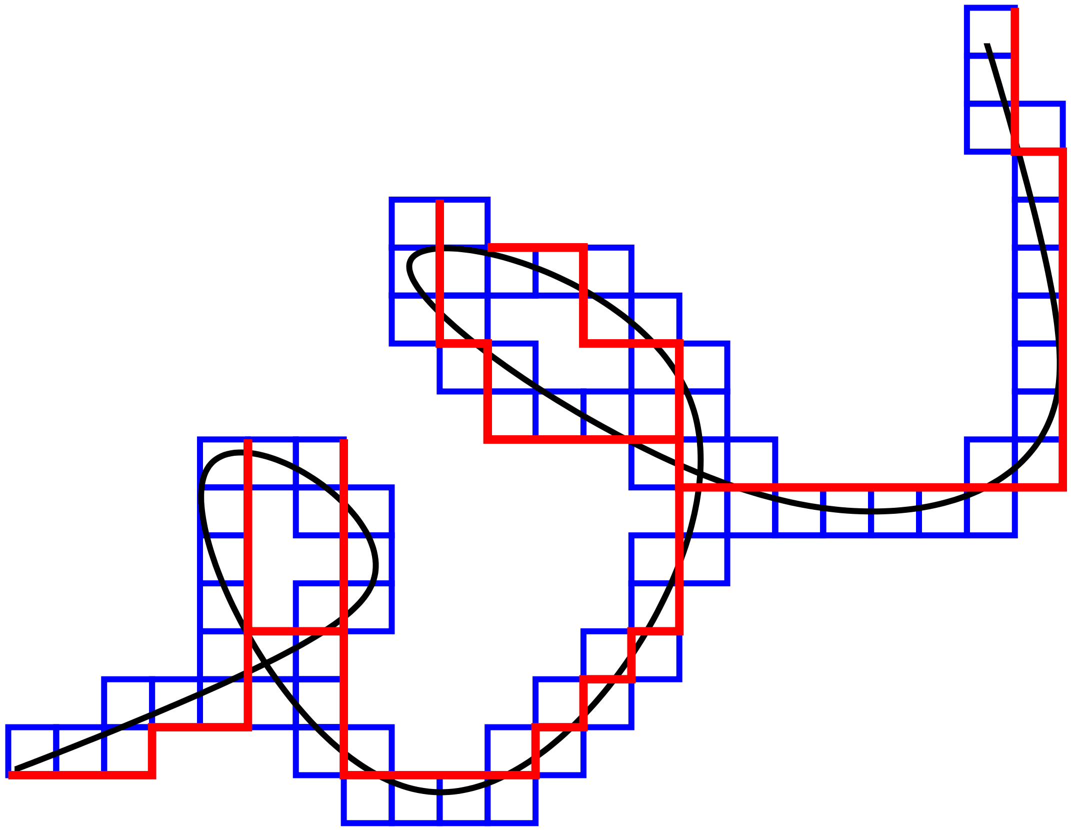




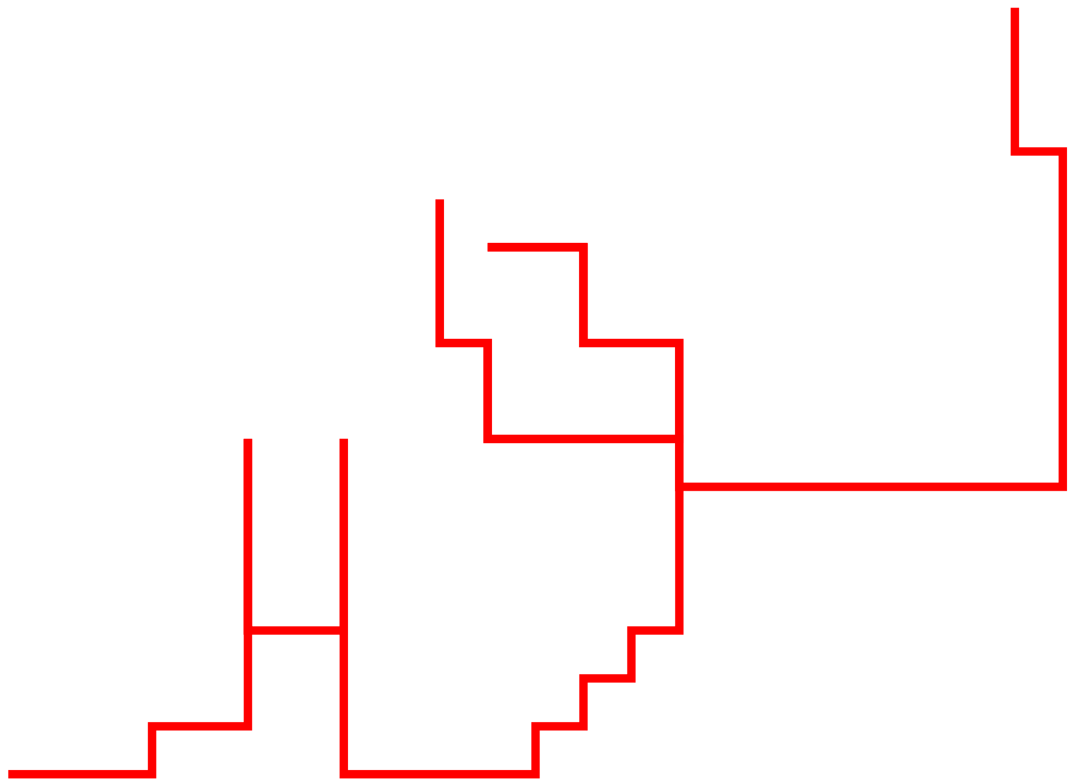
Suffices to approximate subtrees of a grid.



Suffices to approximate subtrees of a grid.



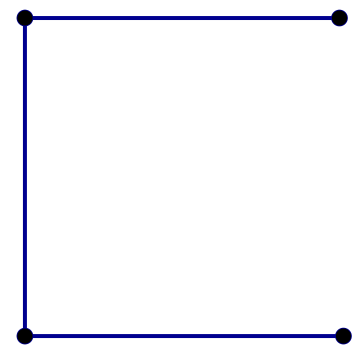
Suffices to approximate subtrees of a grid.



Suffices to approximate subtrees of a grid.

Theorem: Every planar continuum is a limit of true trees.

Idea of Proof: reduce harmonic measure ratio by adding edges.

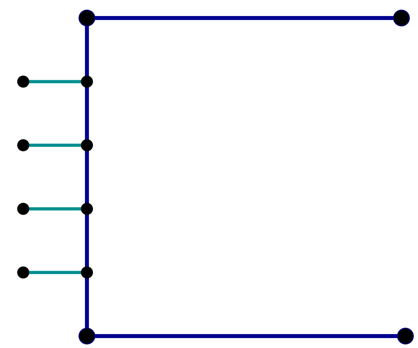


Vertical side has much larger harmonic measure from left.

Add edges ($\Rightarrow$ change combinatorics) to "balance" harmonic measure.

Theorem: Every planar continuum is a limit of true trees.

Idea of Proof: reduce harmonic measure ratio by adding edges.

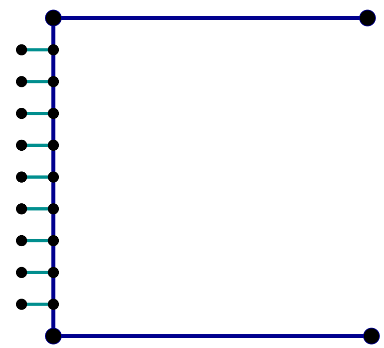


"Left" harmonic measure is reduced (roughly 3-to-1).

New edges are approximately balanced (universal constant).

Theorem: Every planar continuum is a limit of true trees.

Idea of Proof: reduce harmonic measure ratio by adding edges.



“Left” harmonic measure is reduced (roughly 3-to-1).

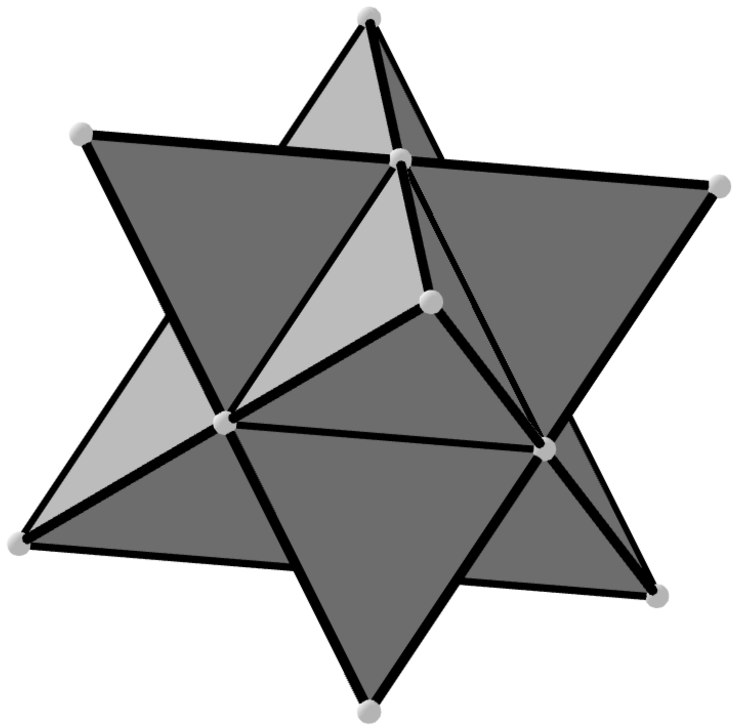
New edges are approximately balanced (universal constant).

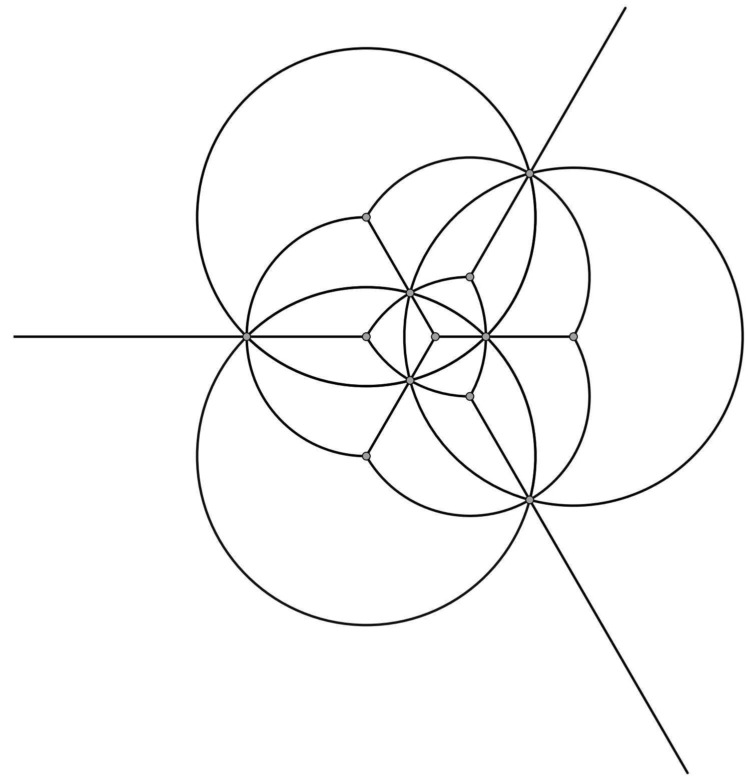
Mapping theorem gives exactly balanced.

QC correction map is near identity if “spikes” are short.

New tree approximates shape of old tree; different combinatorics.

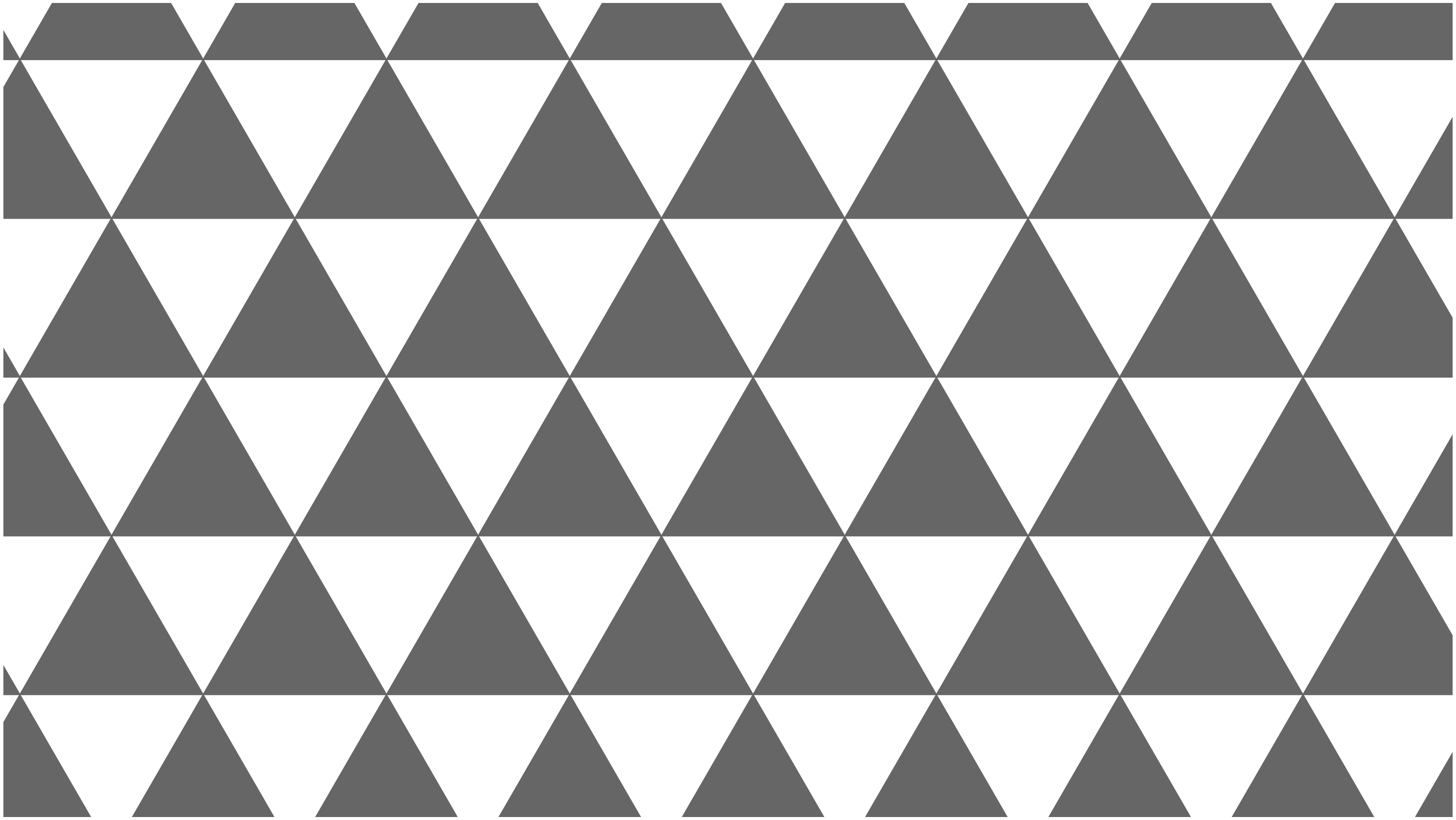
Equilateral triangulations



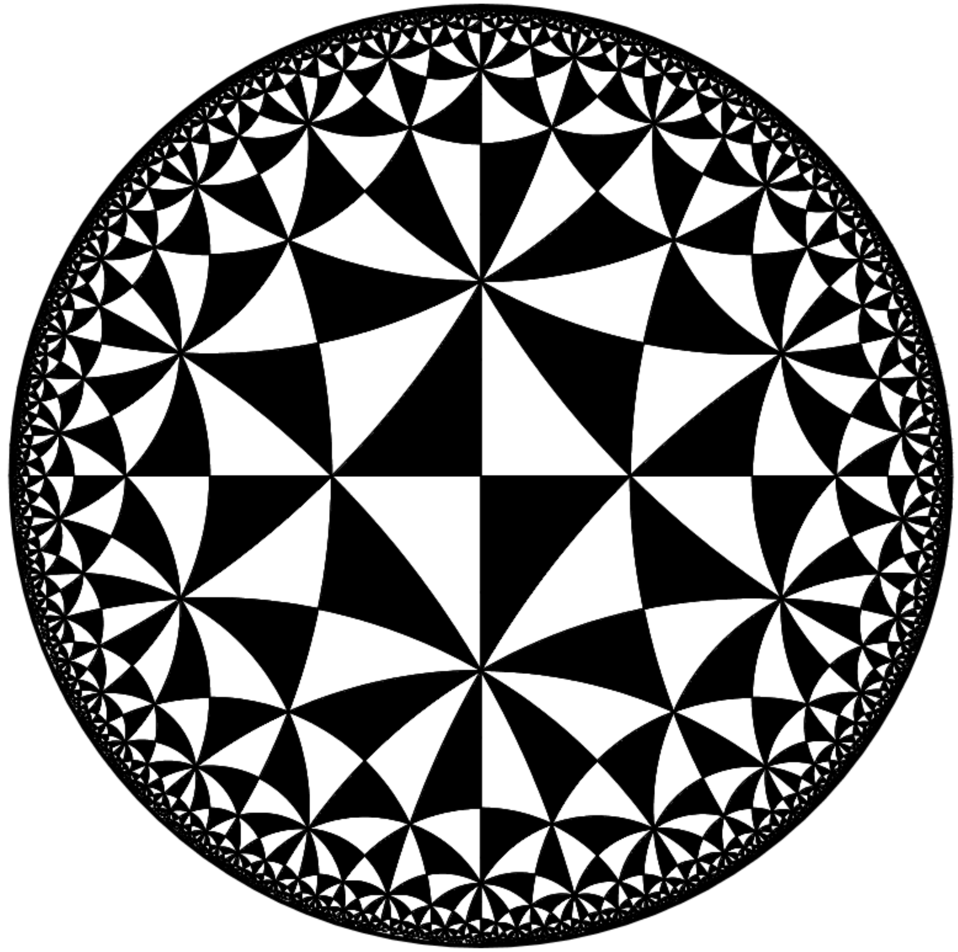


Given a finite or countable collection of equilateral triangles, identify edges pairwise so each vertex is identified with finitely many others.

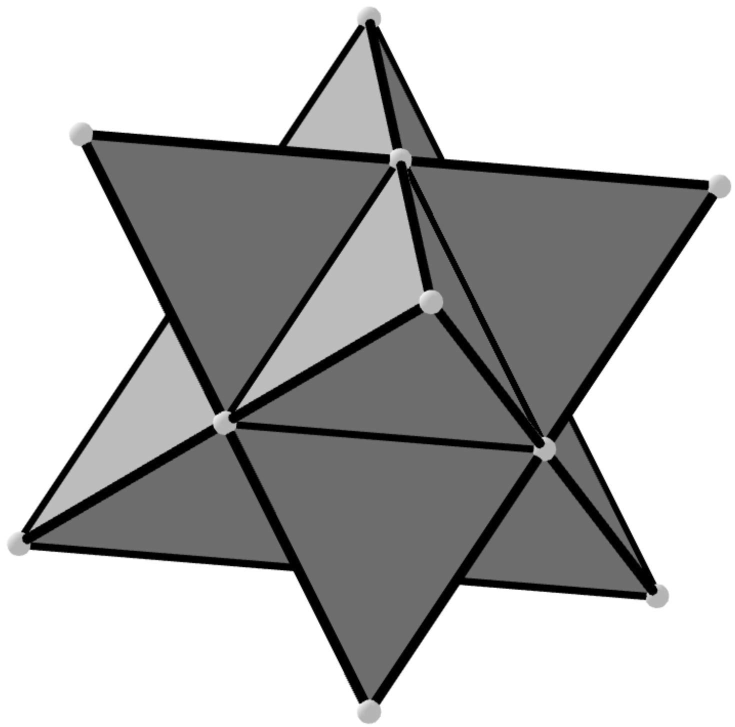
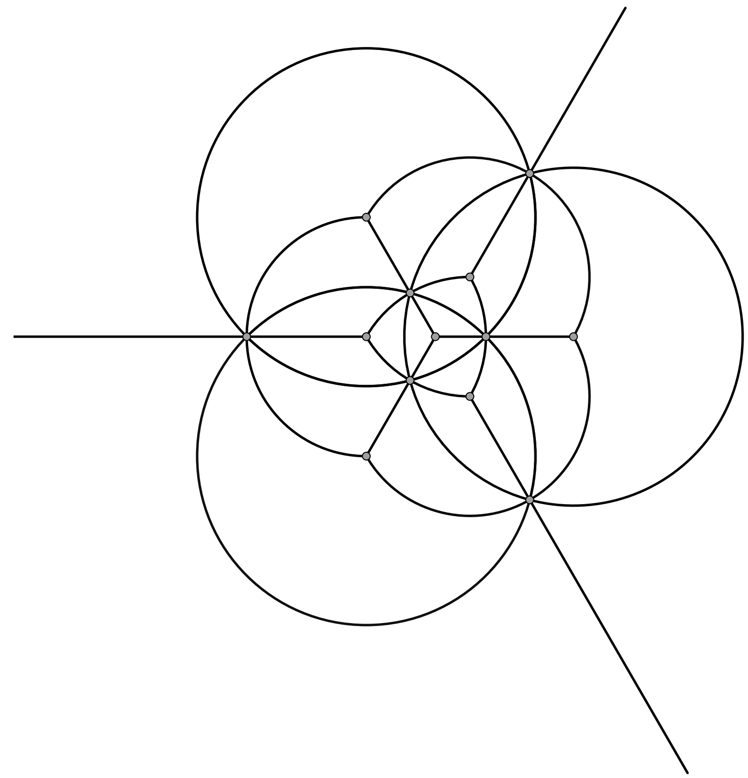
Resulting surface has a conformal structure = is a Riemann surface.



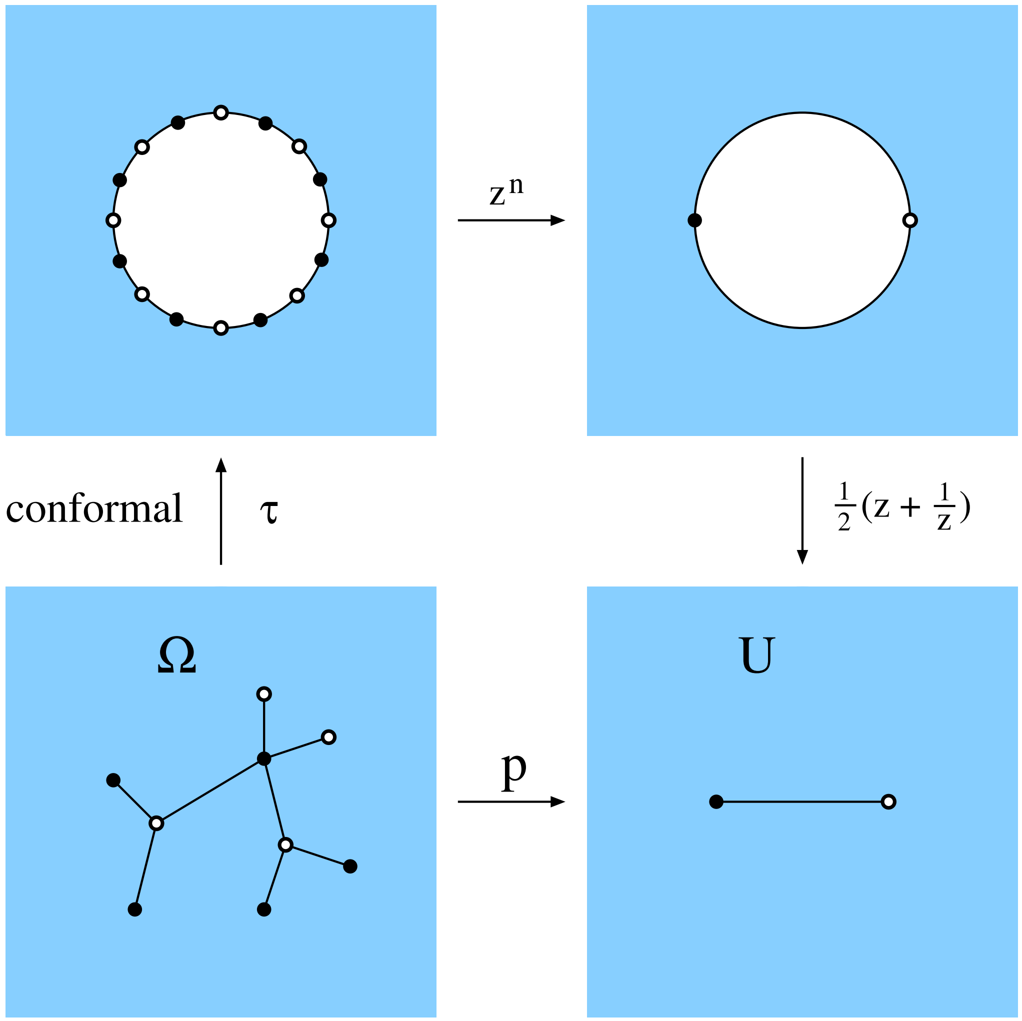
Equilateral triangulation of the plane



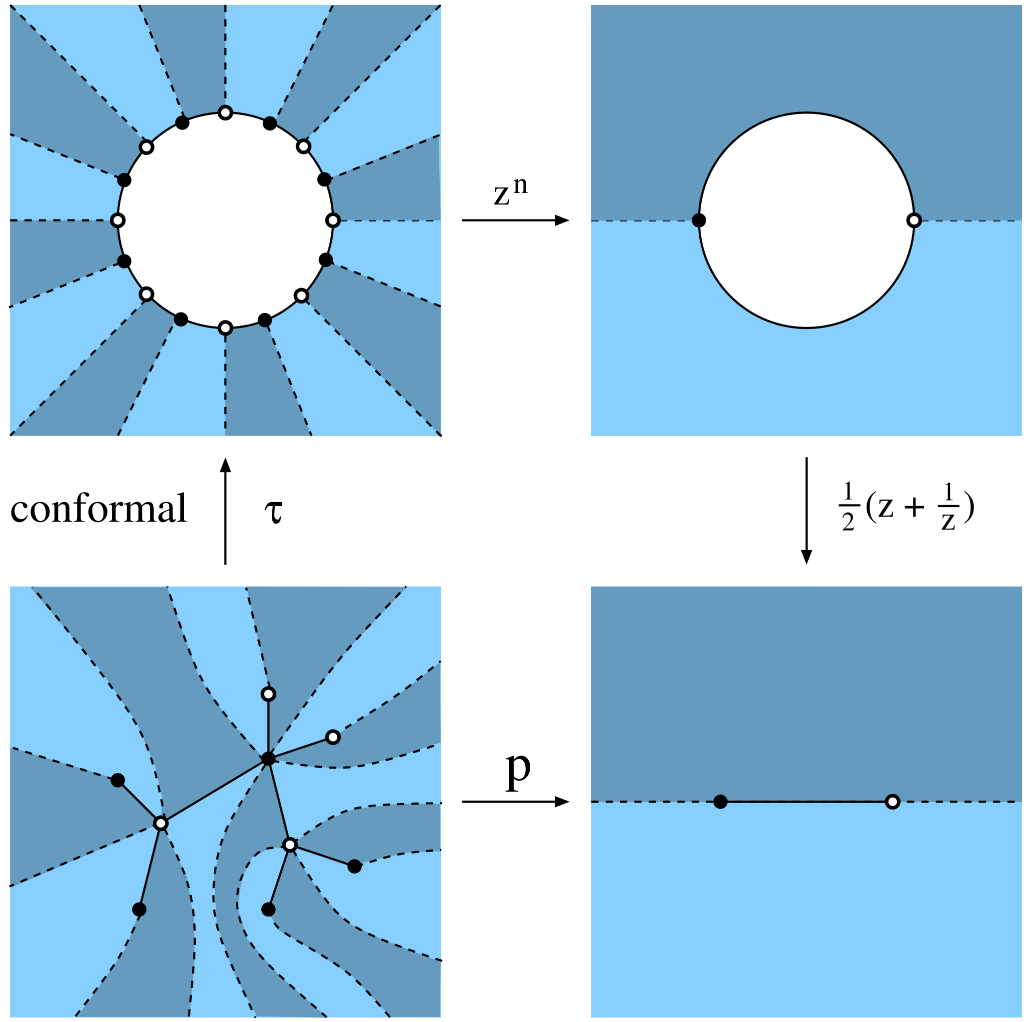
Equilateral triangulation of the disk

Fact: A triangulation of a surface is equilateral iff any two triangles sharing an edge have an anti-holomorphic reflection across that edge.



Shabat polynomials give equilateral triangulations.



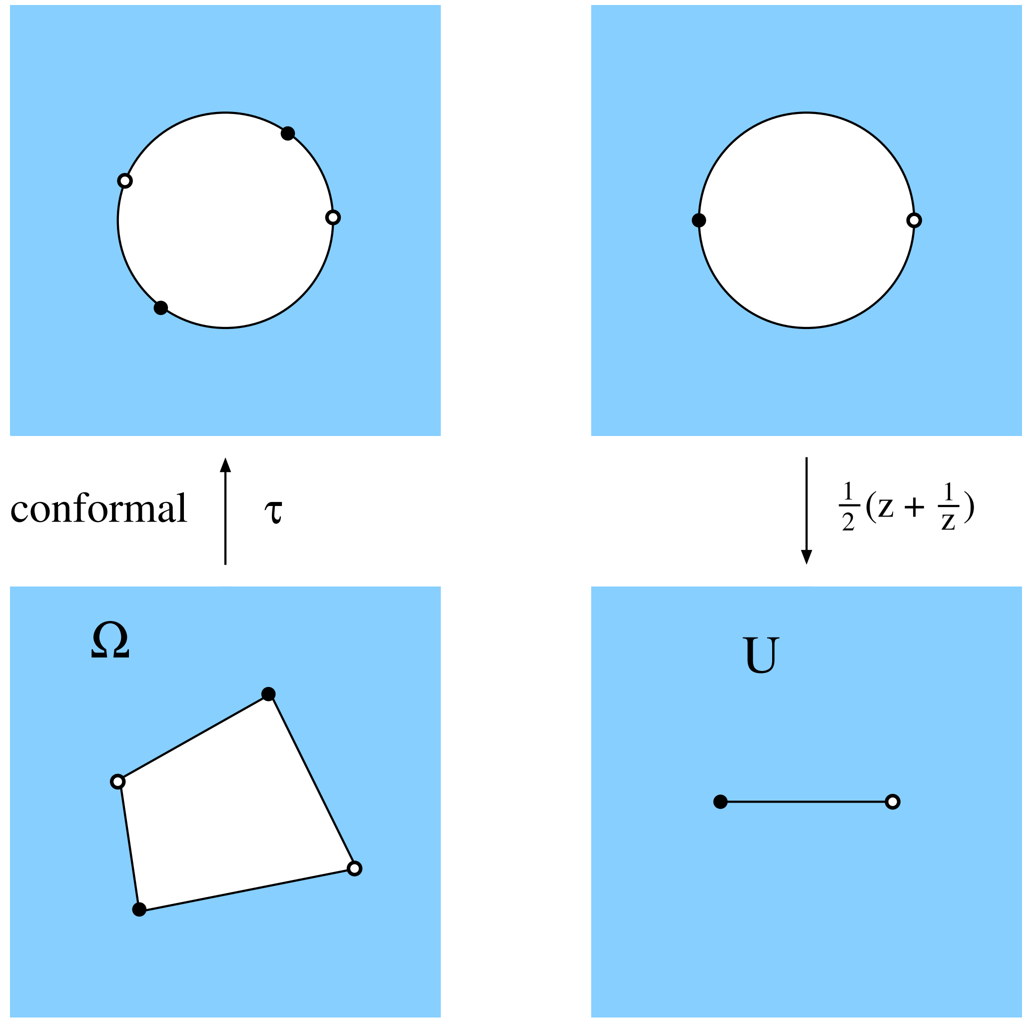
Upper/lower halfplanes give equilateral triangulation of plane.
Inverse image under covering map is also equilateral triangulation.

Belyi function on Riemann surface $R =$
holomorphic map from R to S^2 branched over $-1, 1, \infty$.

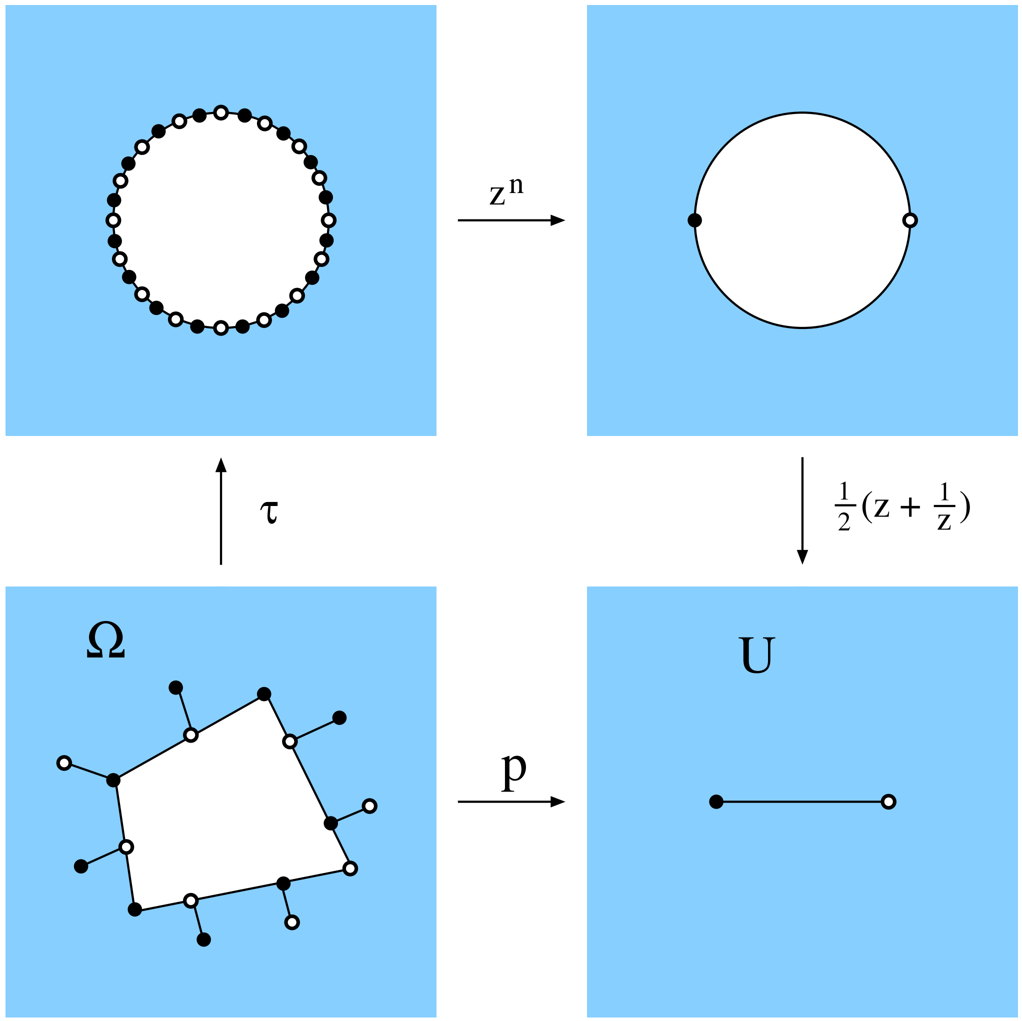
Shabat polynomial is a Belyi function on sphere.

Theorem (Voevodsky-Shabat): A surface has a Belyi function iff it has an equilateral triangulation.

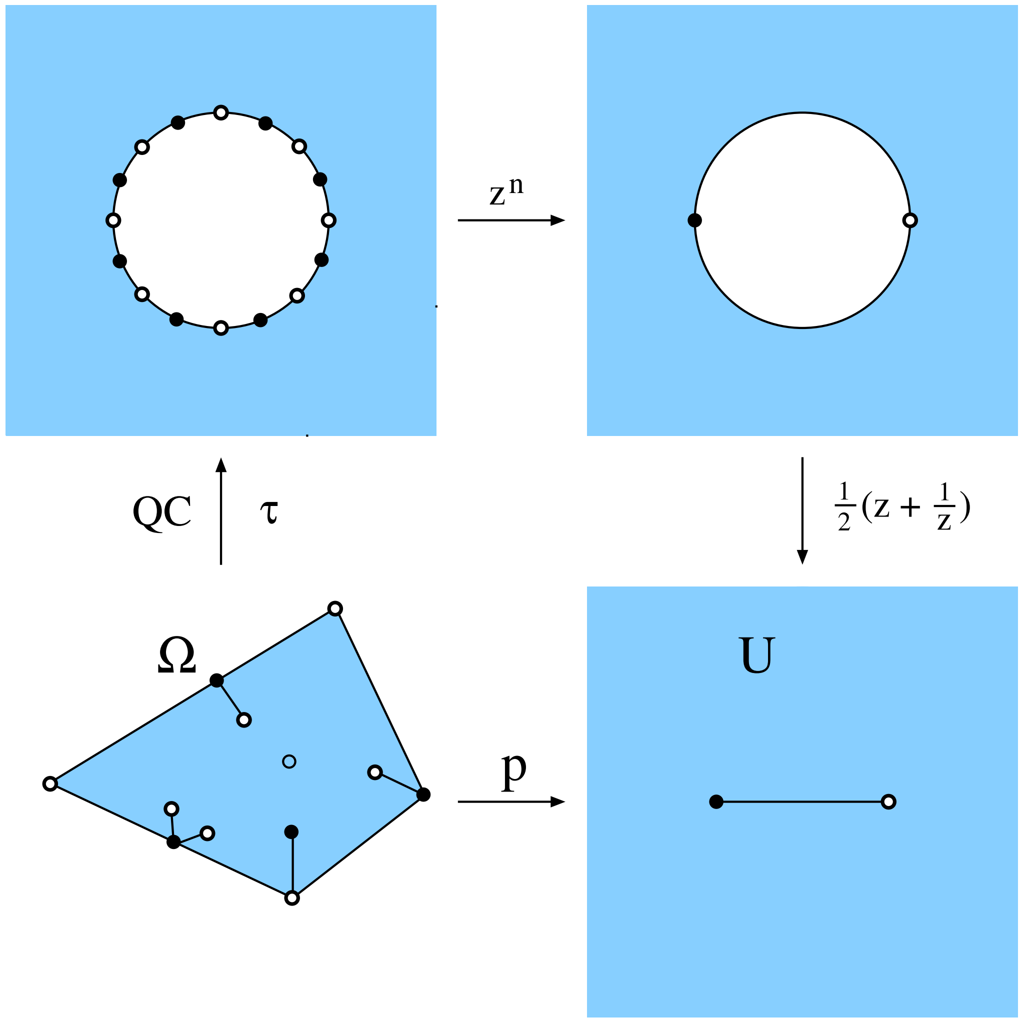
Which Riemann surfaces have a Belyi function?



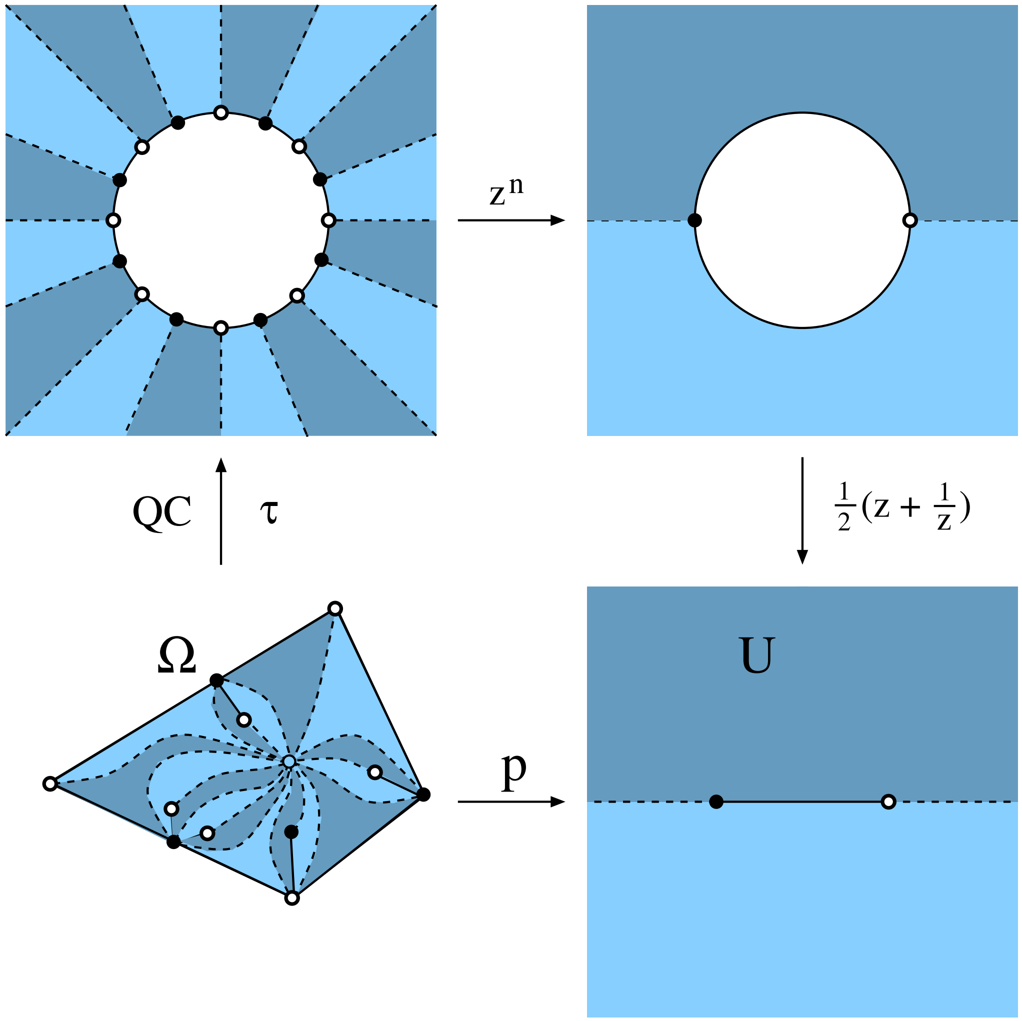
Apply folding idea to a polygon instead of tree.
For general polygon, edges have different harmonic measures.



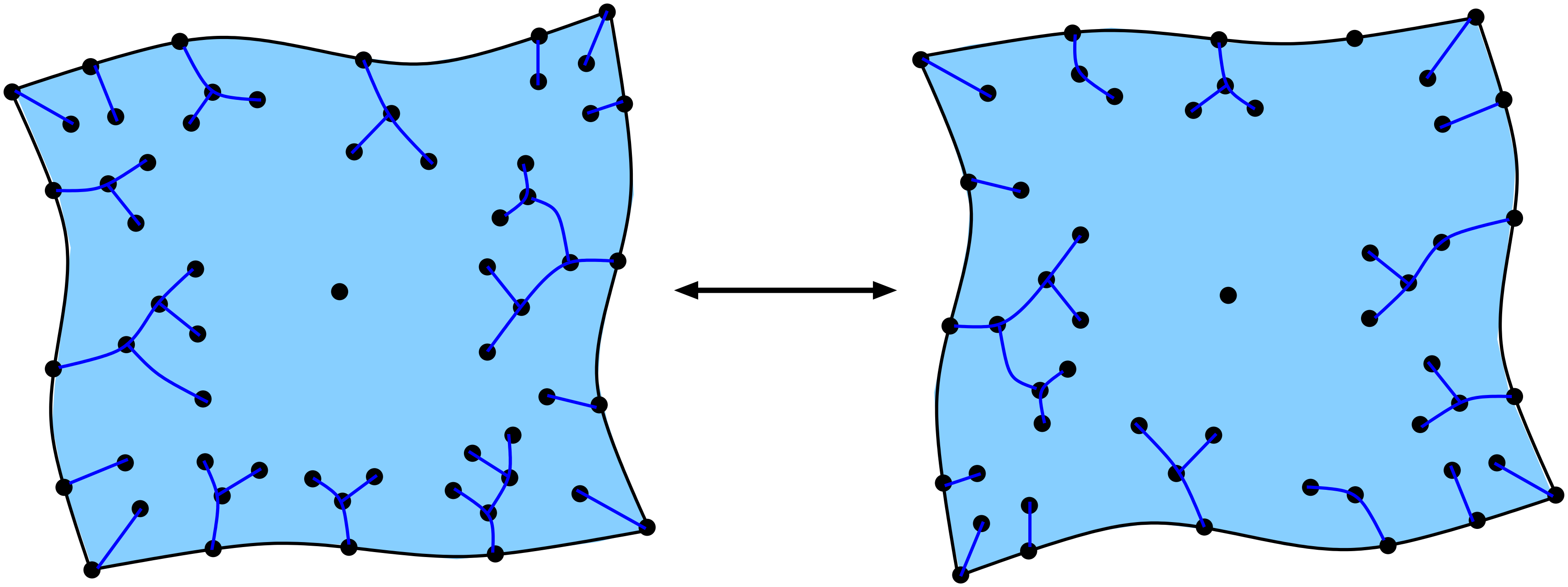
Add decorations to nearly equalize harmonic measures.
Length measure maps to multiple of length measure.



Inverting allows same construction for interior domains.
Now p has a degree n pole.



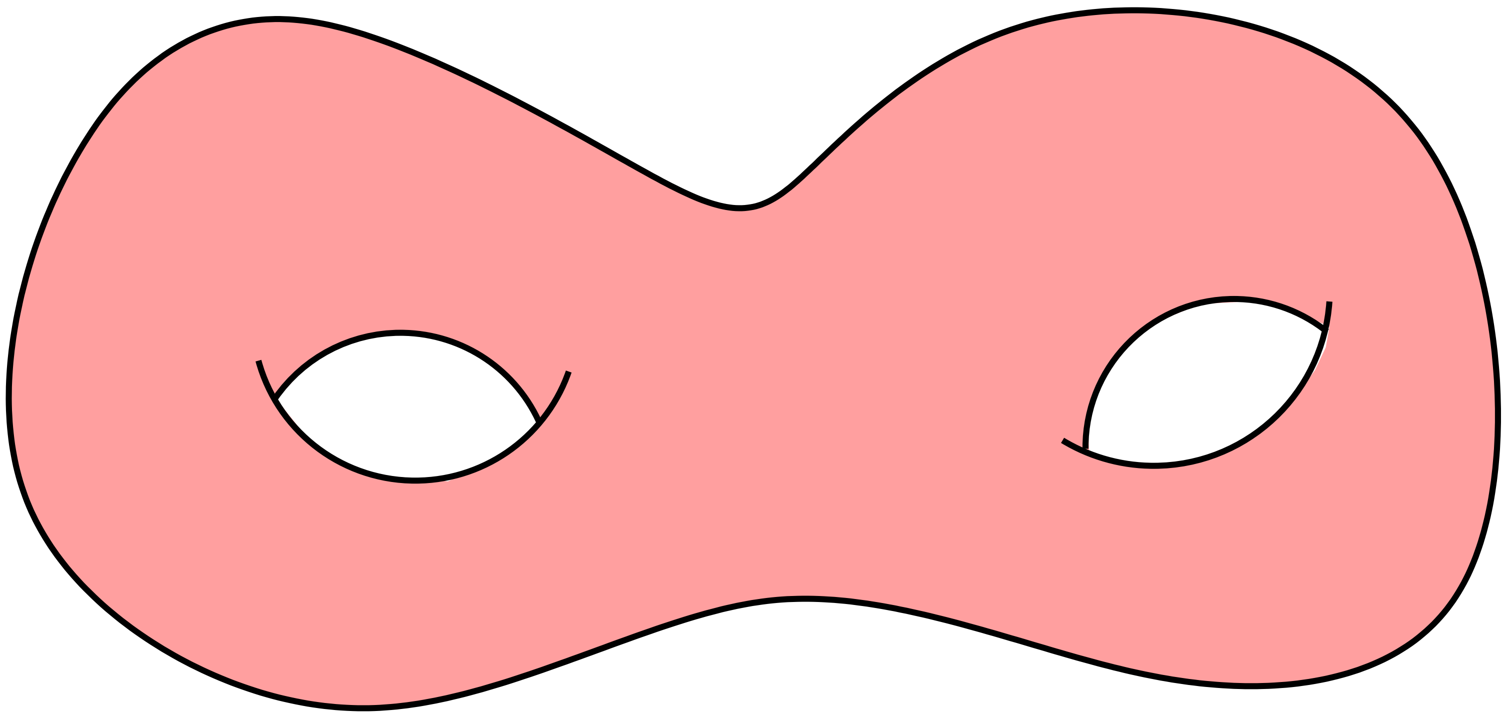
Preimages of half-planes gives equilateral triangulation.
Normalized arclength on boundary edges maps to arclength on $[-1, 1]$.



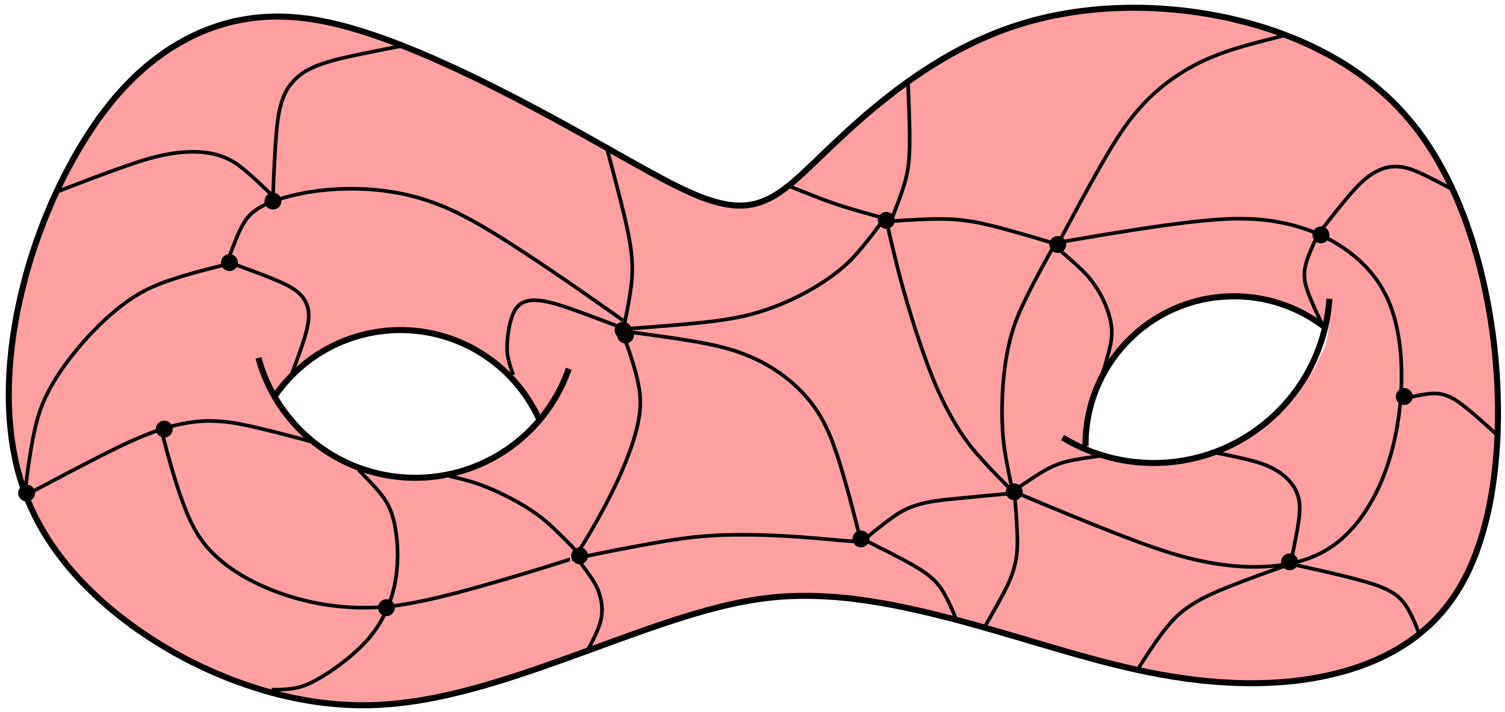
Boundary conditions allow triangulated pieces to be glued together.

We get a well defined, QR function on union.

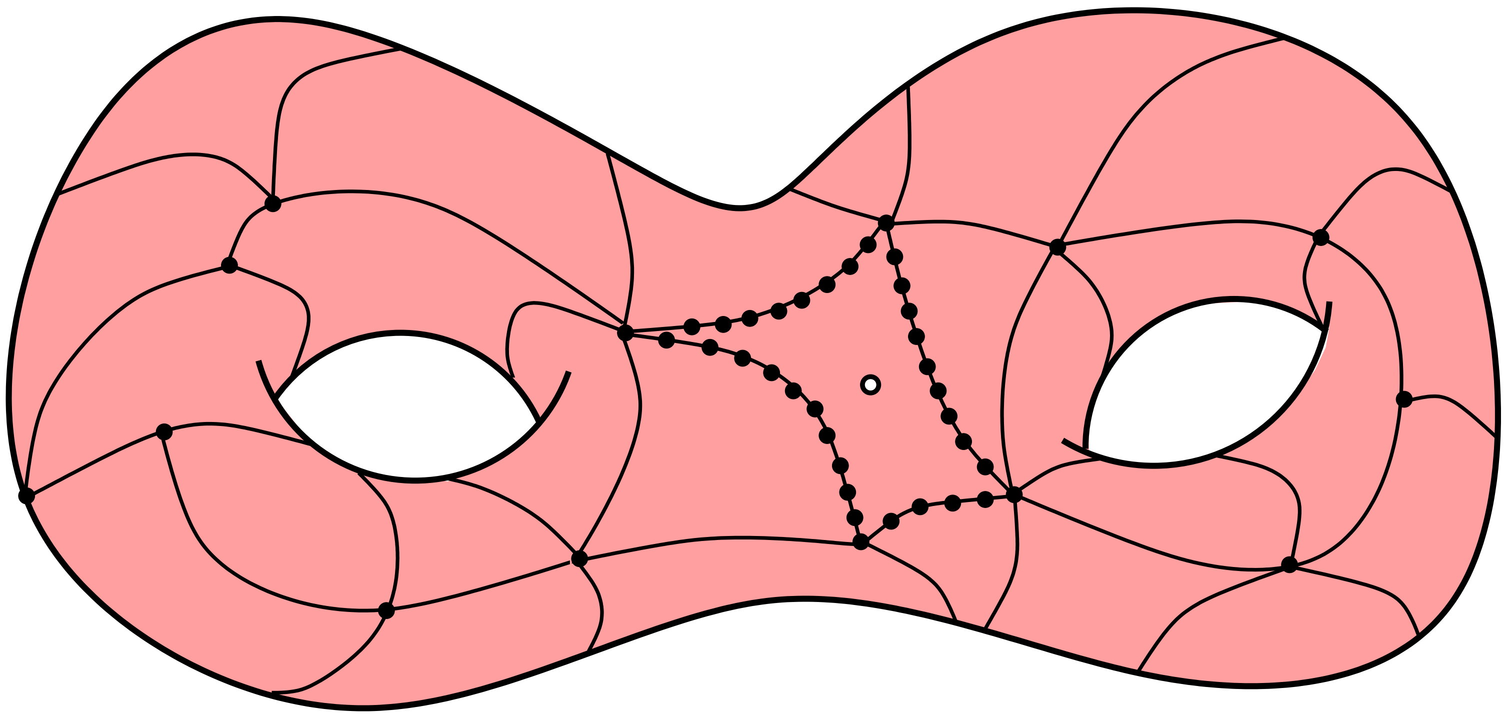
Holomorphic except on small area. QC correction close to identity.



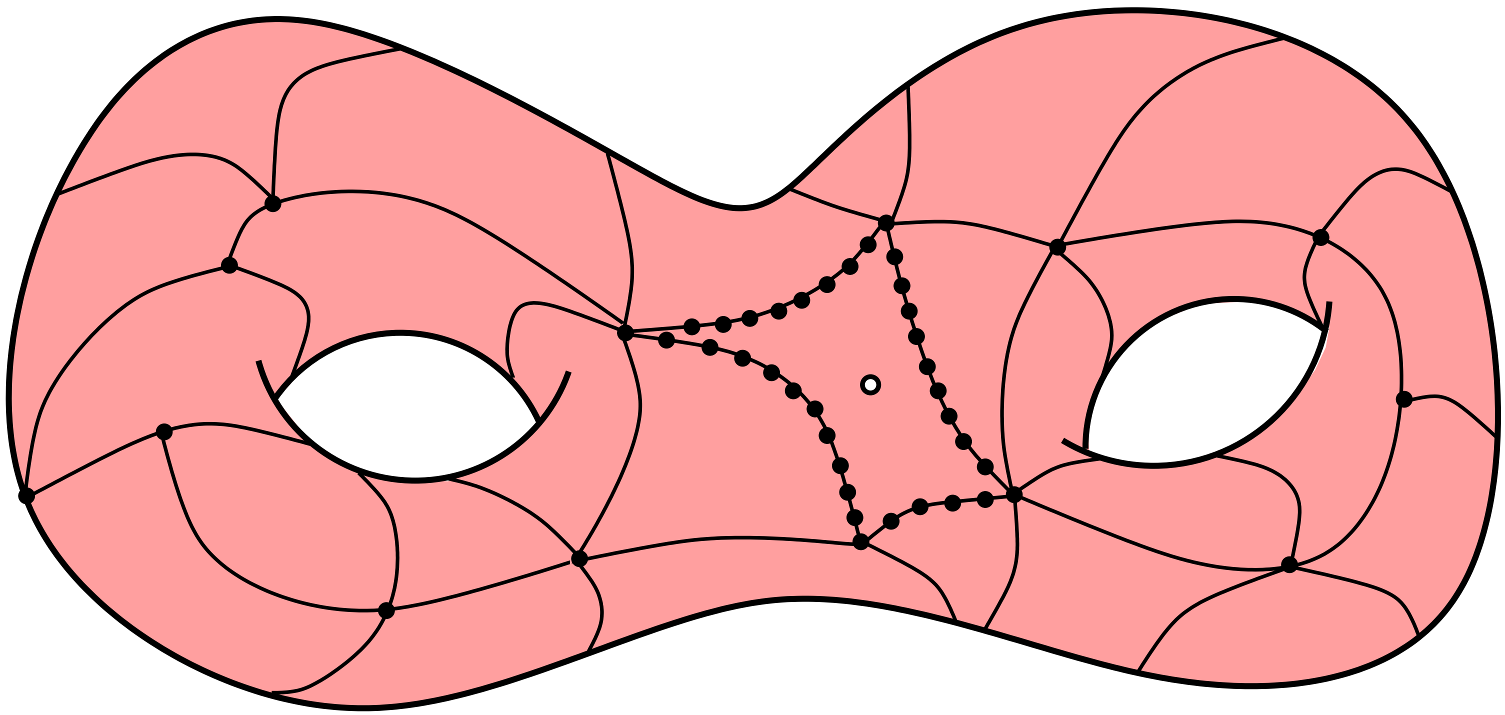
Now do this on a Riemann surface.



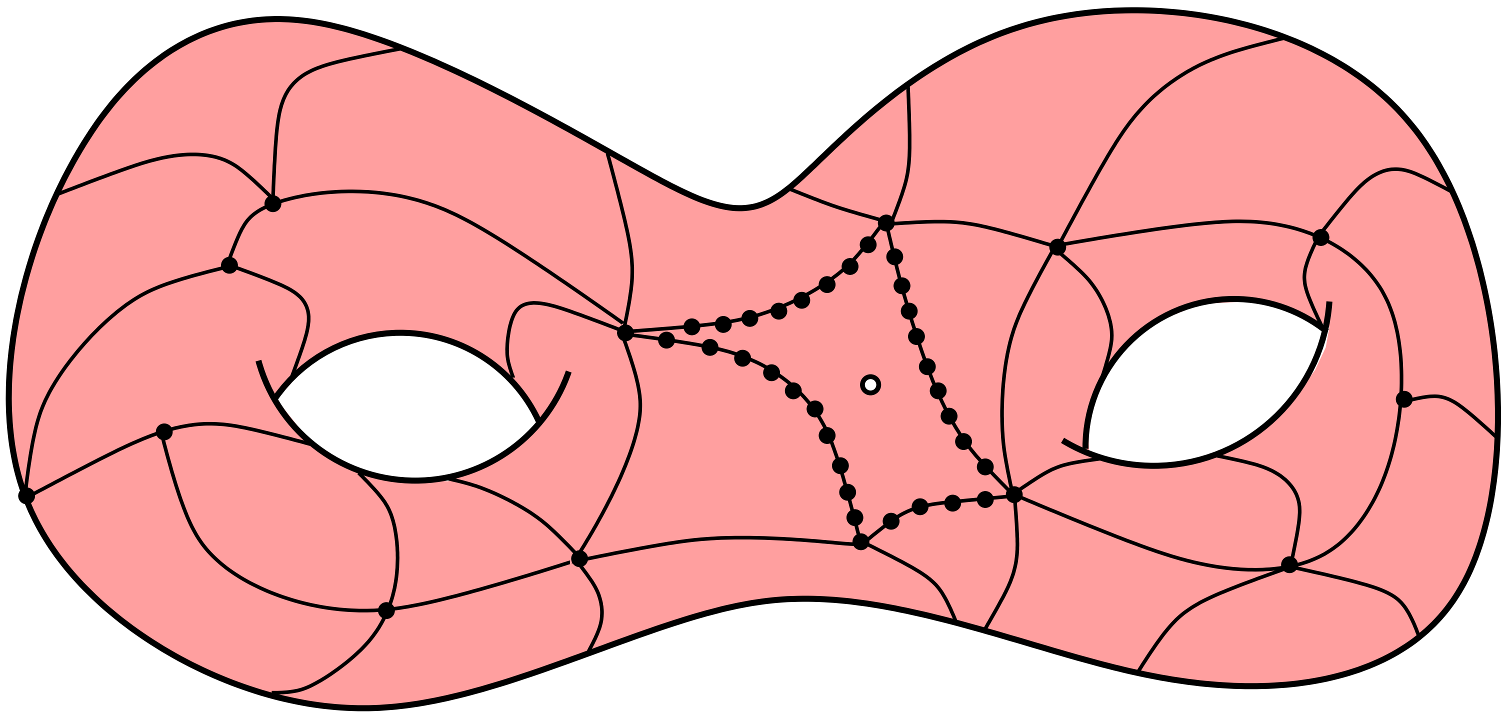
Cut the surface into nice pieces.



QC folding gives QR Belyi maps on each piece.
Boundary edges map to $[-1, 1]$ and match up between pieces.
We get a QR Belyi map on any compact surface surface.



Solving Beltrami equation gives holomorphic Belyi function,
but defined on slightly different Riemann surface.
(Only countably many Riemann surfaces are Belyi.)



Belyi's theorem: A compact surface is Belyi iff it is algebraic.

Algebraic = zero variety of algebraic polynomial.

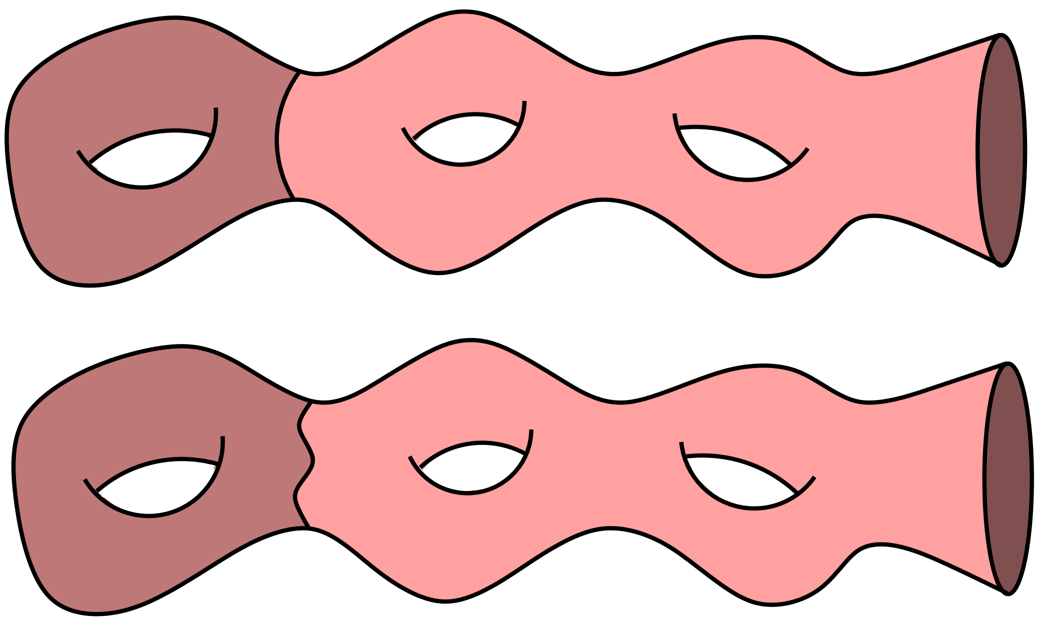
Which non-compact surfaces have equilateral triangulations?

Thm (B-Rempe): Every non-compact surface has a Belyi function.

Thm (B-Rempe): Every non-compact surface has a Belyi function.

Idea of proof:

- Compact pieces can be approximated by triangulated surfaces.
- Conformal structure is changed, but as little as we wish.
- **Key fact:** small perturbation $\Rightarrow$ triangulated pieces re-embed in S.
- Triangulate a compact exhaustion of S. Take limit.

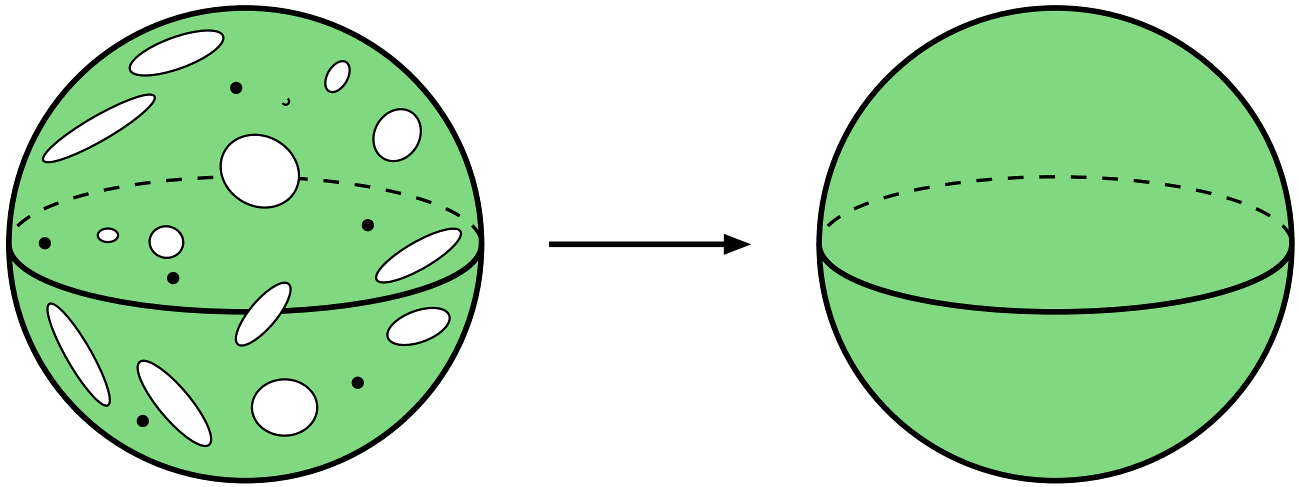


Thm (B-Rempe): Every non-compact surface has a Belyi function.

Corollary: Every Riemann surface is a branched cover of the sphere, branched over finitely many points.

- For compact surfaces, this is Riemann-Roch.
- Compact, genus g sometimes needs $3g$ branch points.
- 3 branch points suffice for all non-compact surfaces.

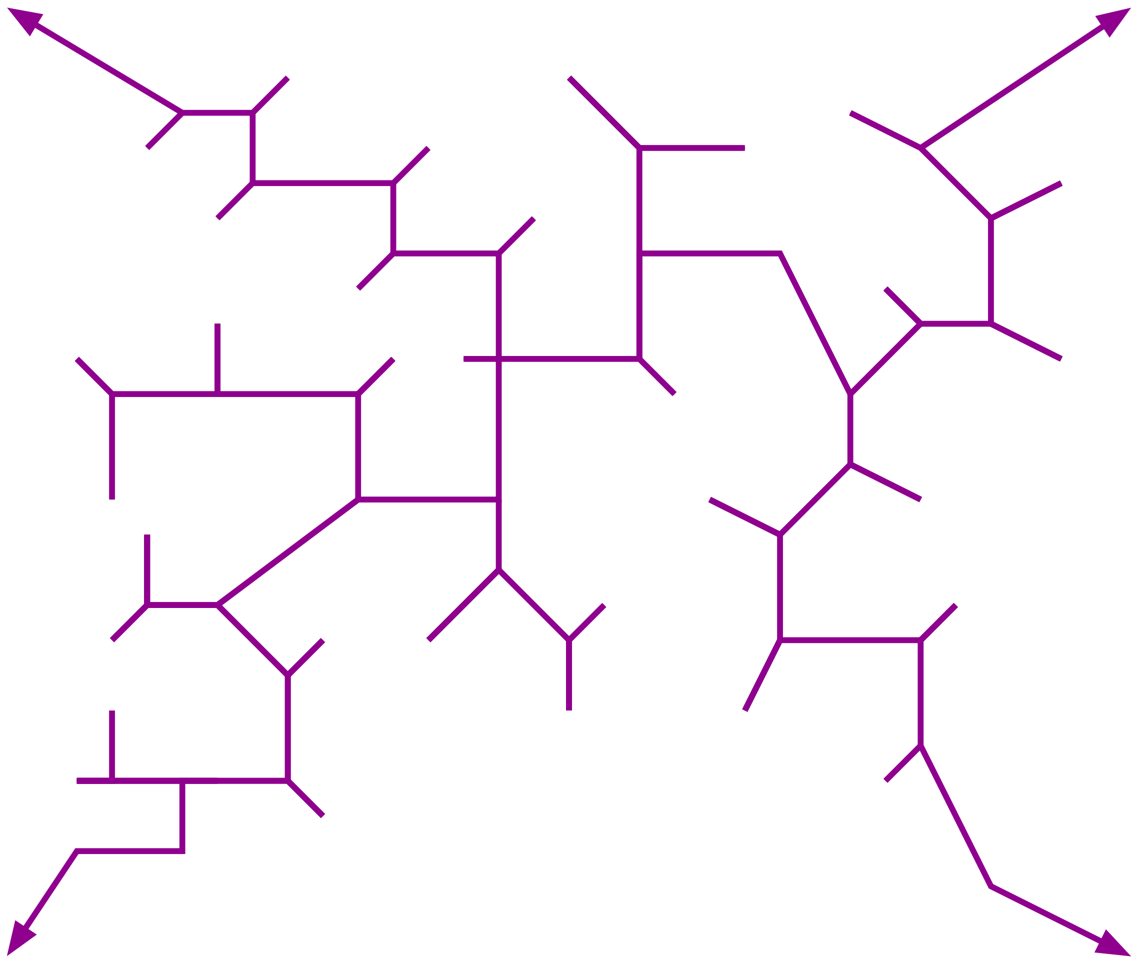
Corollary: Any open $U \subset \mathbb{C}$ is 3-branched cover of the sphere.



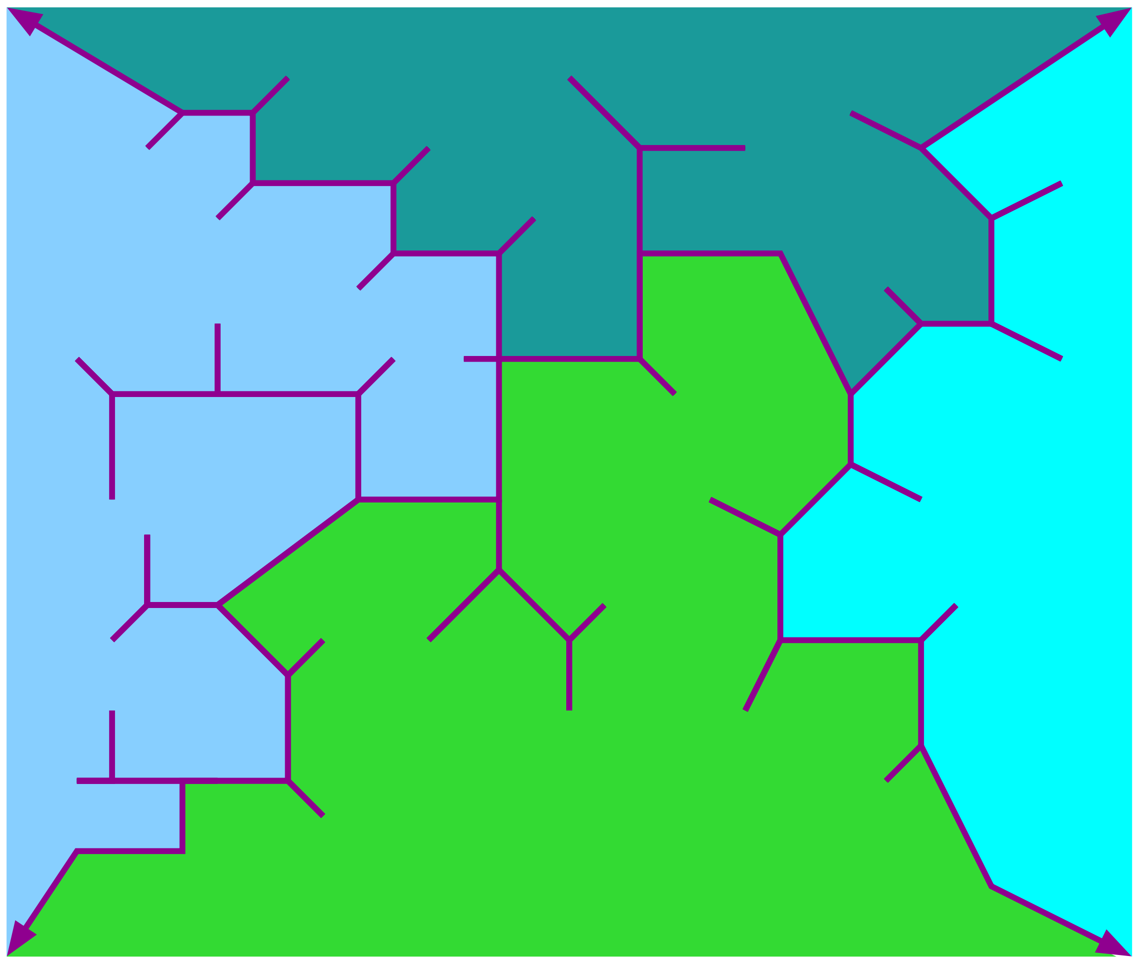
Gives many new dynamical systems of finite type (after A. Epstein).

Previous: rational maps on sphere, Speiser class on plane.

Quasiconformal Folding (infinite true trees)



Do infinite trees correspond to entire functions with 2 critical values?

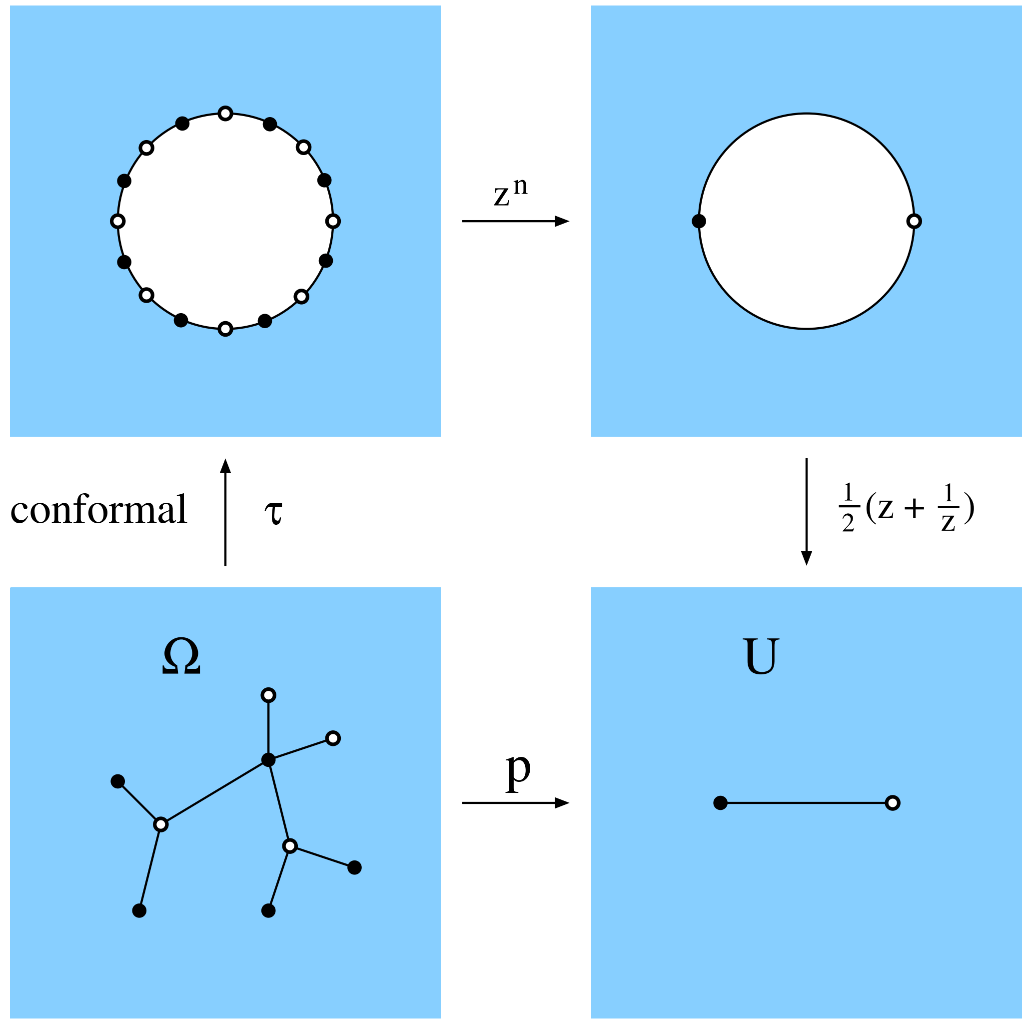


Main difference:

$\mathbb{C}\setminus$ finite tree = one topological annulus

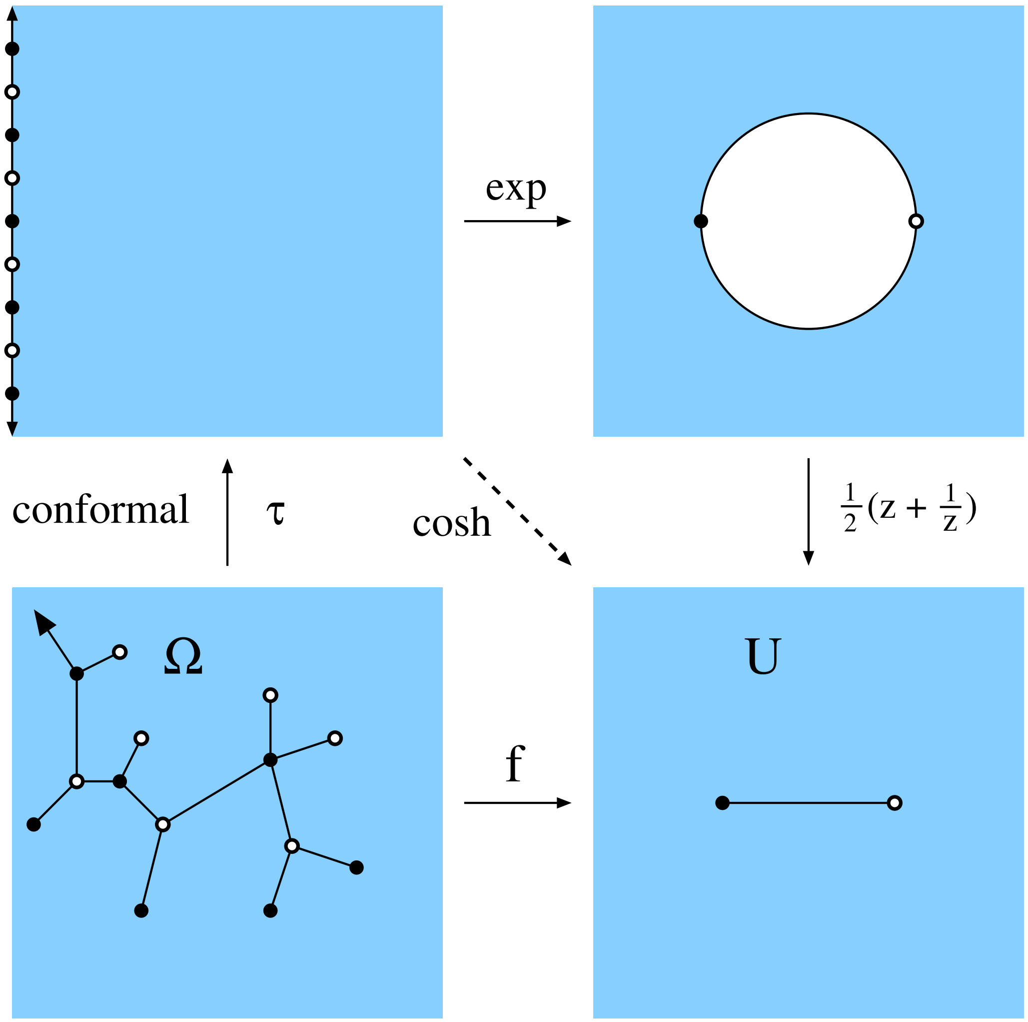
$\mathbb{C}\setminus$ infinite tree = many simply connected components

Recall finite case



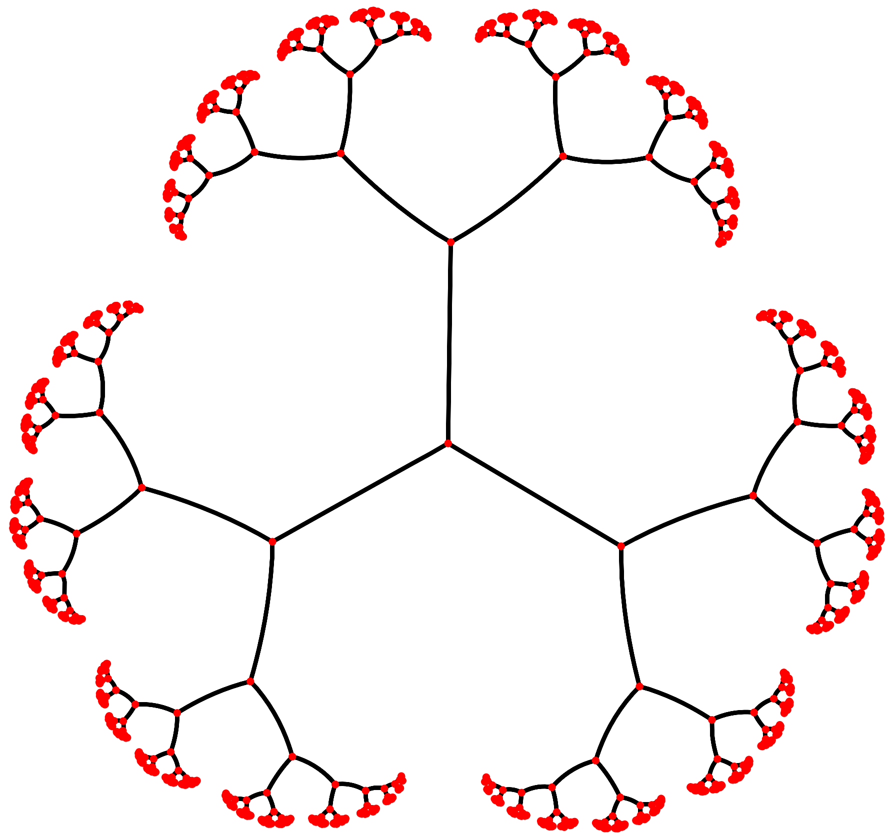
T is true tree $\Leftrightarrow p = \frac{1}{2}(\tau^n + 1/\tau^n)$ is continuous across T.

Infinite case



Infinite balanced tree $\Leftrightarrow f = \cosh \circ \tau$ is continuous across T.

Infinite 3-regular tree does not have a true form in the plane.



Instead it has a “true embedding” into the hyperbolic disk.

Studied by Oleg Ivrii, Peter Lin, Steffen Rohde, Emanuel Sygal.

Links to rational and group dynamics.

The Quasiconformal Folding Theorem (very roughly):

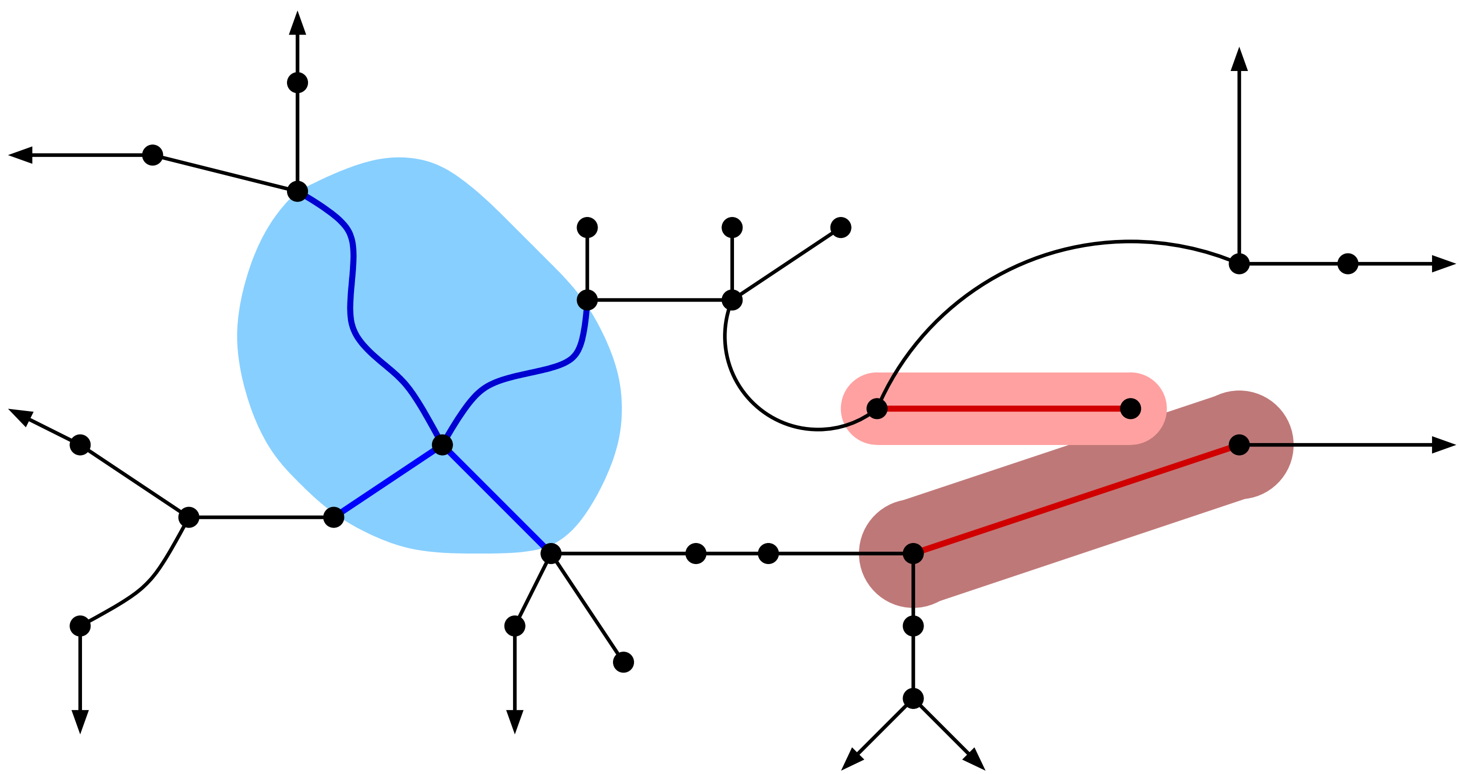
Thm: every "nice" infinite planar tree T is approximated by $f^{-1}([-1,1])$ for some entire function f with CV $= \{\pm 1\}$.

"nice" = two conditions that are automatic for finite trees.

(1) Bounded Geometry (local condition; easy to verify):

- edges are uniformly smooth.
- adjacent edges form bi-Lipschitz image of a star $= \{z^n \in [0, r]\}$
- non-adjacent edges are well separated,

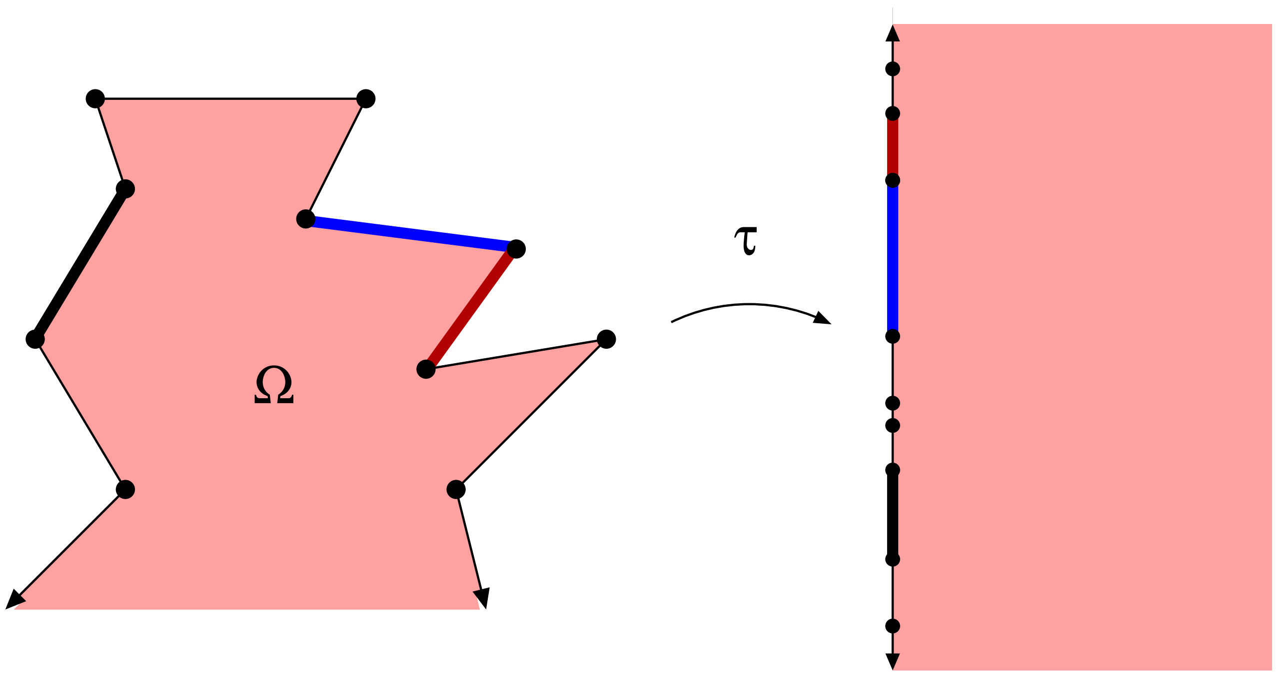
$$\text{dist}(e, f) \geq \epsilon \cdot \min(\text{diam}(e), \text{diam}(f)).$$



(2) τ-Lower Bound (global condition; harder to check):

Complementary components of tree are simply connected.

Each can be conformally mapped to right half-plane. Call map τ.

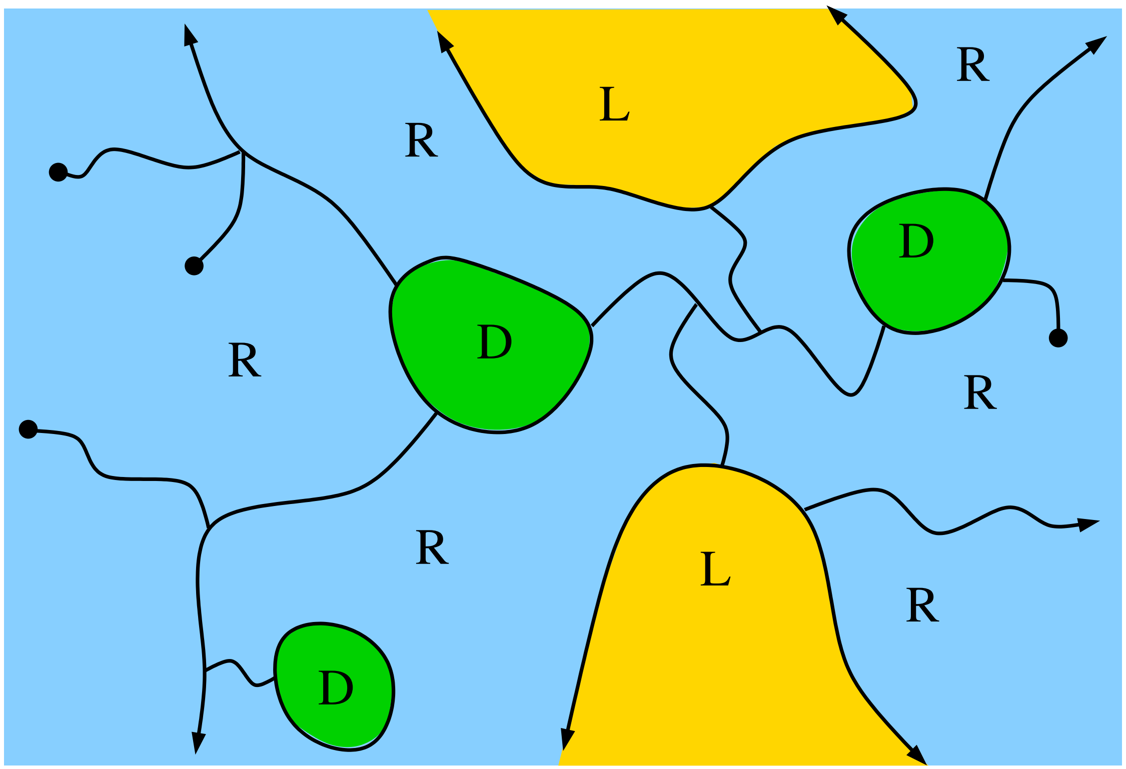


We assume all images have length $\geq \pi$.

Need positive lower bound; actual value usually not important.

Components are "thinner" than half-plane near ∞.

More generally: can replace “nice trees” by “nice graphs”:



R = unbounded domains ($F = e^{\tau(z)}$, previous case)

D = bounded Jordan domains ($F = (z-a)^n$, high degree critical points)

L = unbounded Jordan domains ($F = e^{-\tau(z)}$, finite asymptotic values)

Can specify singular values in last two cases **exactly**.

Some applications of the QC-folding idea:

Thm: There are transcendental entire functions with a bounded singular set that have wandering domains (impossible for finite singular sets.)

Thm: There are transcendental entire functions whose Julia sets have Hausdorff dimension 1 (always ≥ 1 by Baker, 1975).

Thm (B-Pilgrim): Any continuum in the plane can be approximated by the Julia set of a post-critically finite polynomial.

Thm (B-Lazebnik): Any self-map of any discrete set in $\mathbb{C}$ can be approximated by the post-critical dynamics of some meromorphic function.

Thm (B-Eremenko-Lazebnik): Any Euler graph in the plane (all degrees even) can be approximated by a homeomorphic rational leminiscate (level set $\{|r| = 1\}$).

THANKS