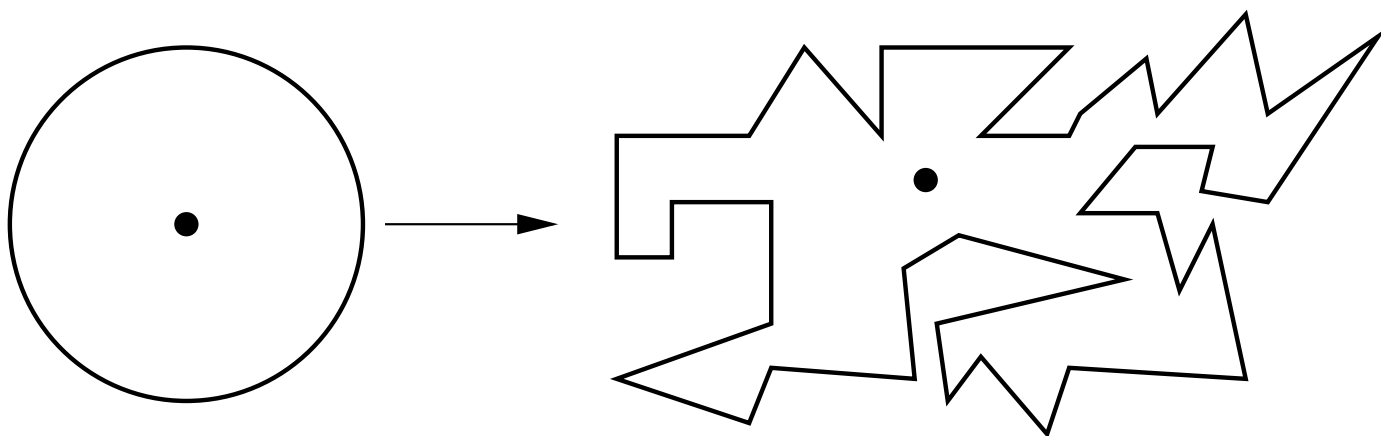


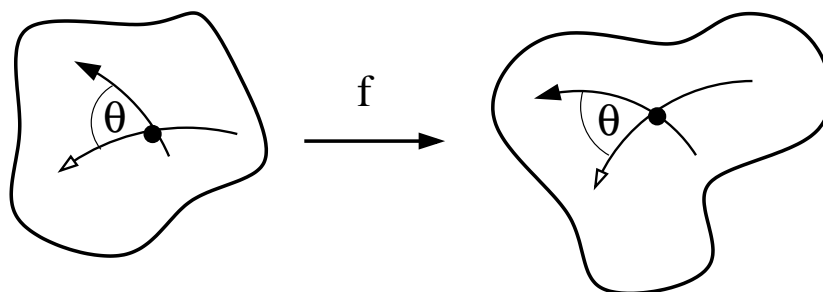
# Conformal Mapping in Linear Time

Christopher J. Bishop  
SUNY Stony Brook

**Riemann Mapping Theorem:** If  $\Omega$  is a simply connected, proper subdomain of the plane, then there is a conformal map  $f : \Omega \rightarrow \mathbb{D}$ .



Conformal = angle preserving



Enough to compute boundary values.

For polygons, enough to compute vertices.

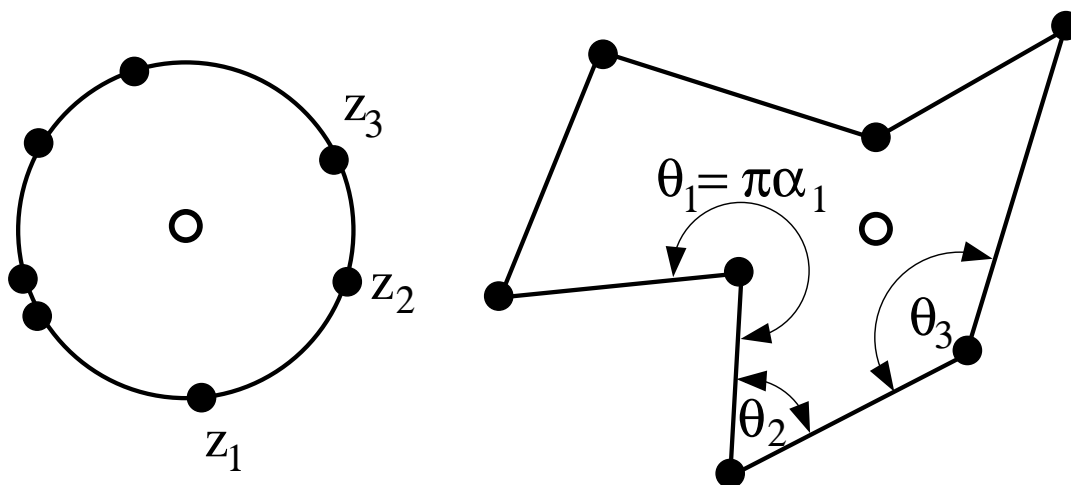
## The Schwarz-Christoffel formula:

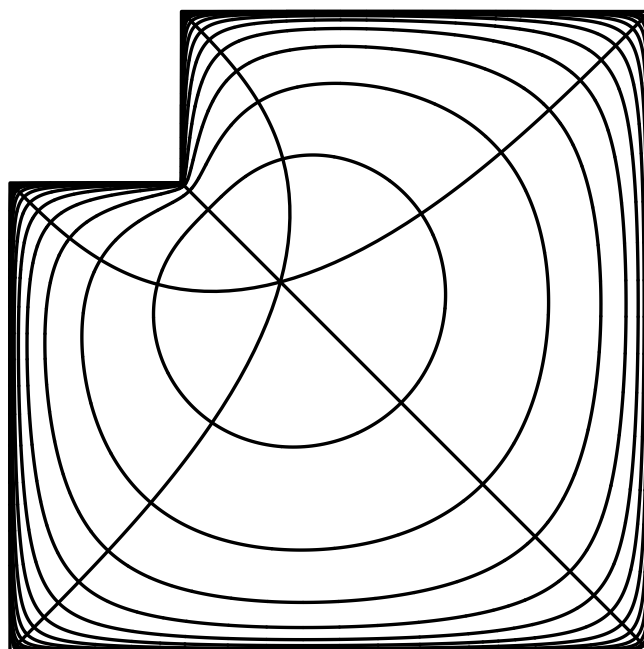
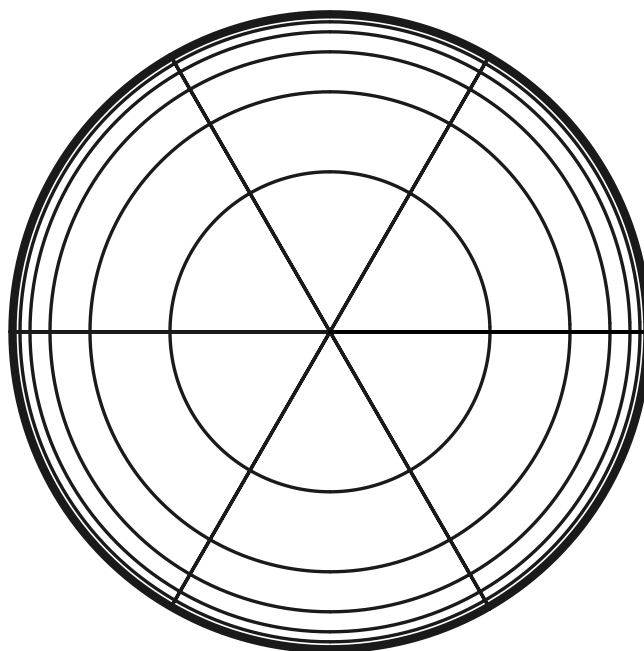
$$f(z) = A + C \int^z \prod_{k=1}^n \left(1 - \frac{w}{z_k}\right)^{\alpha_k - 1} dw.$$

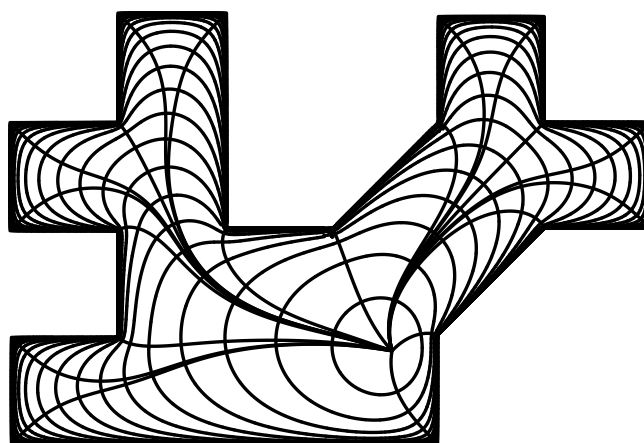
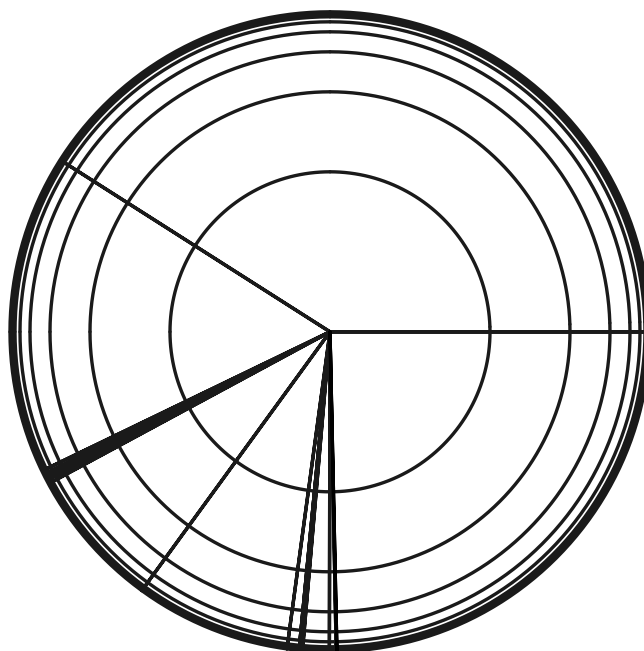
Gives conformal map to polygon.

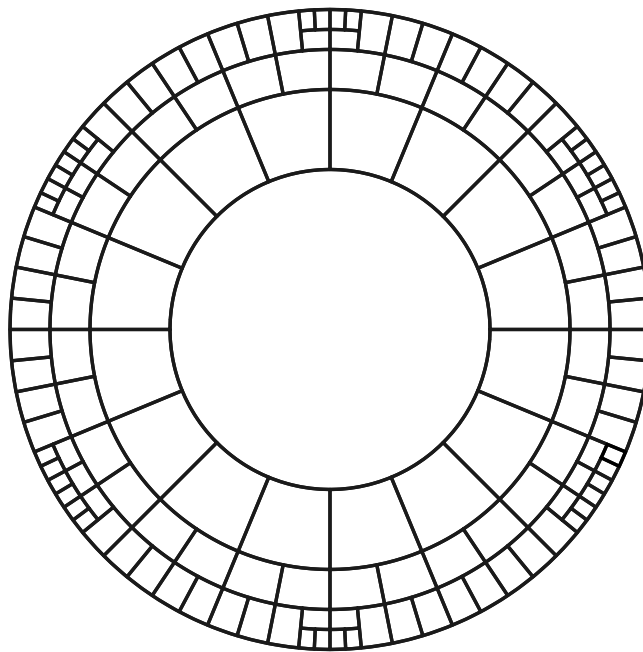
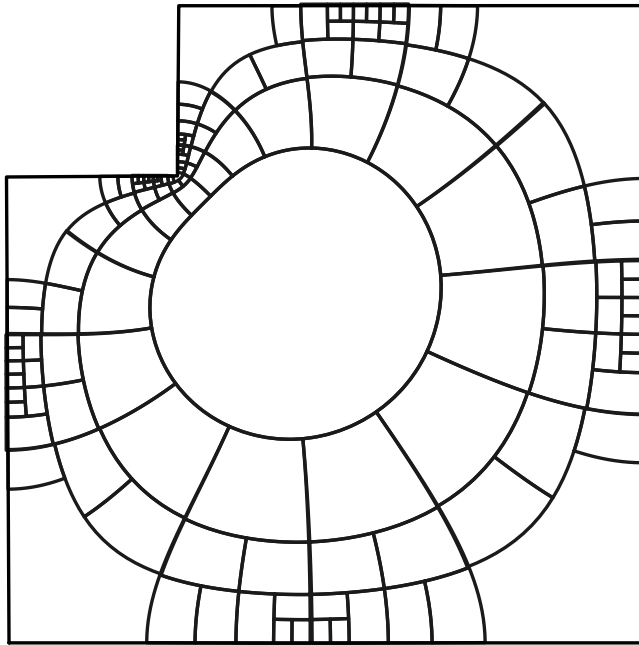
$\pi\alpha_k$ 's are interior angles.

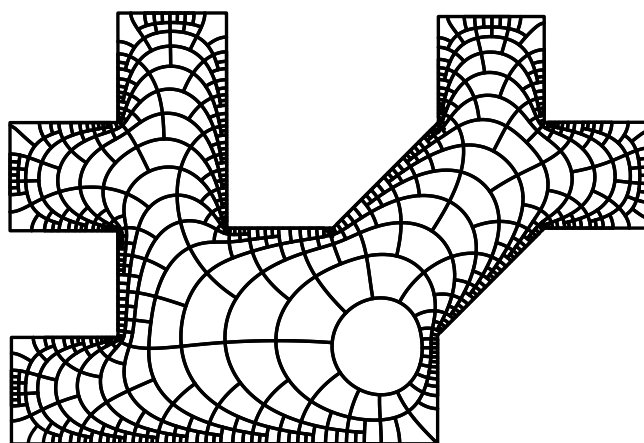
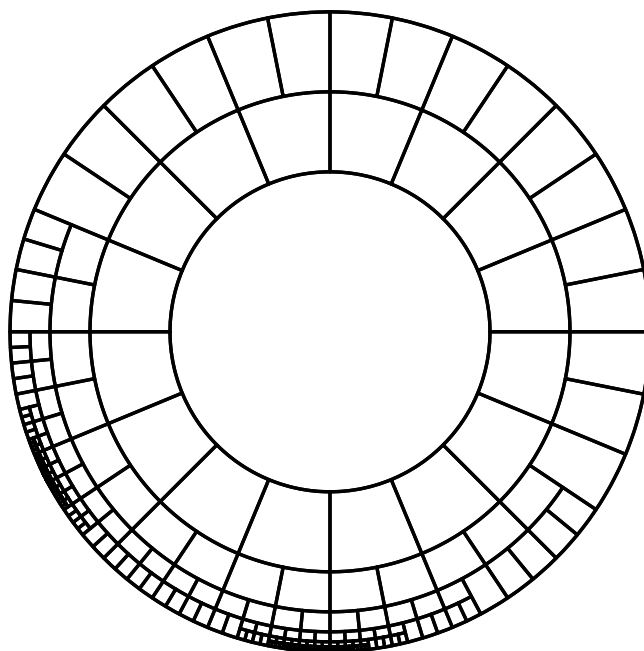
$z_k$ 's map to vertices.

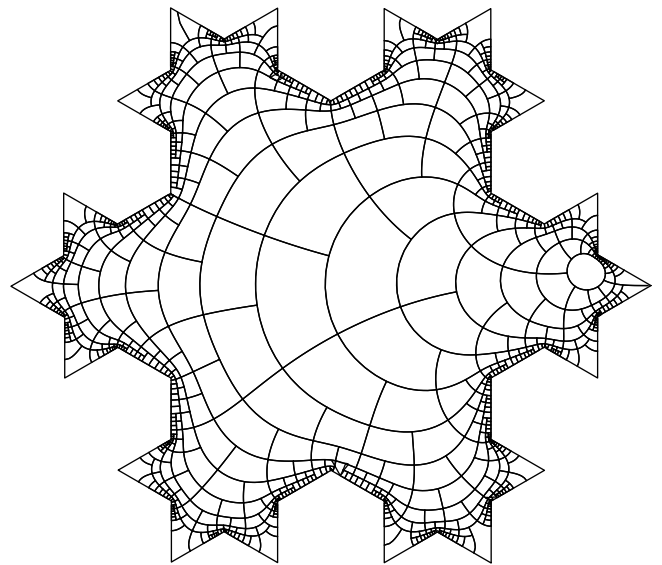
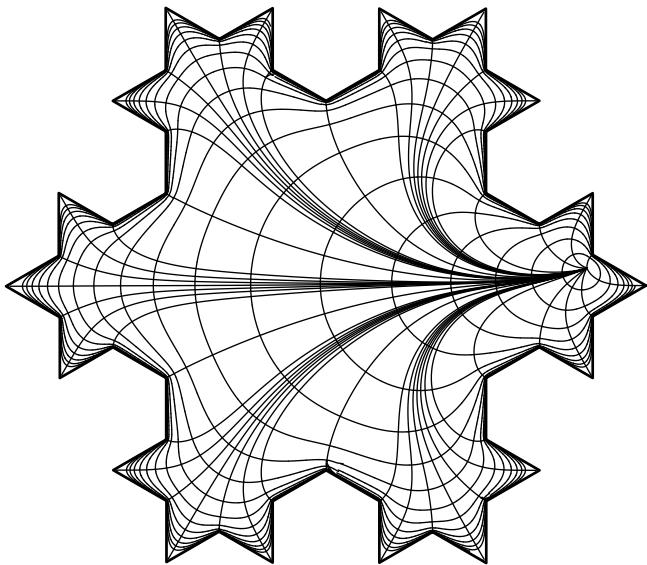
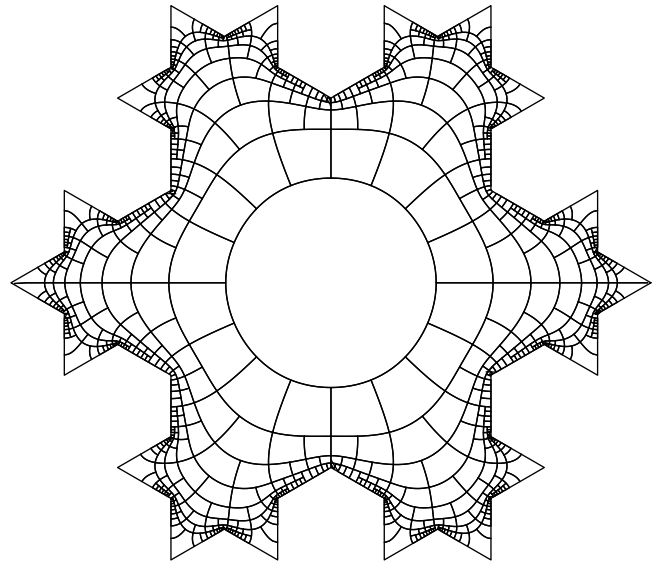
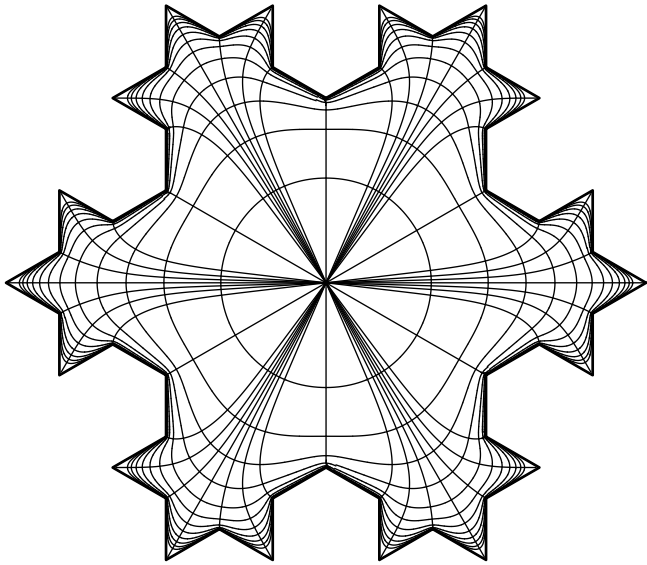




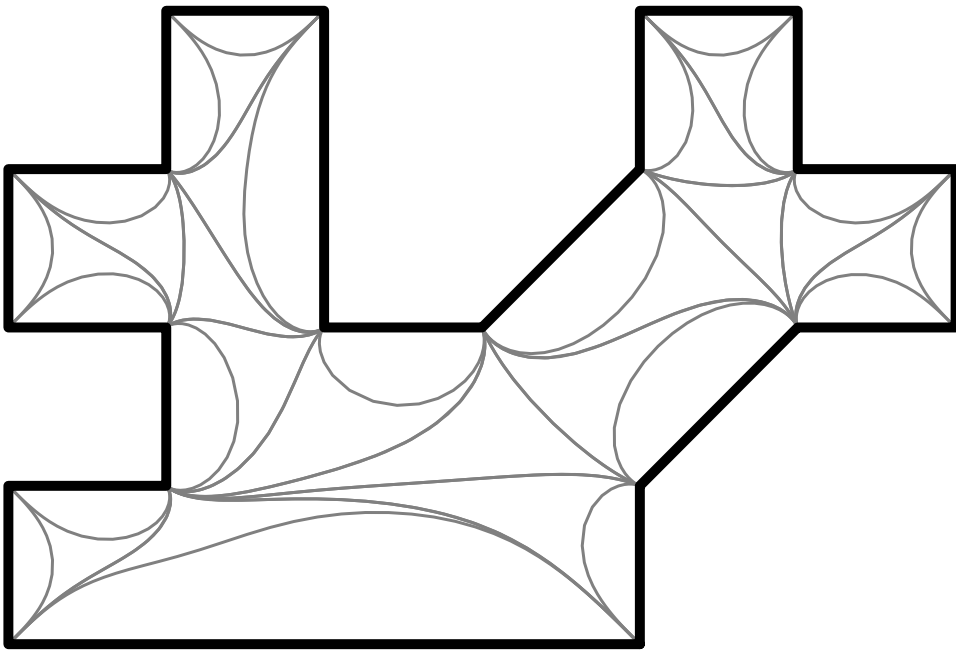
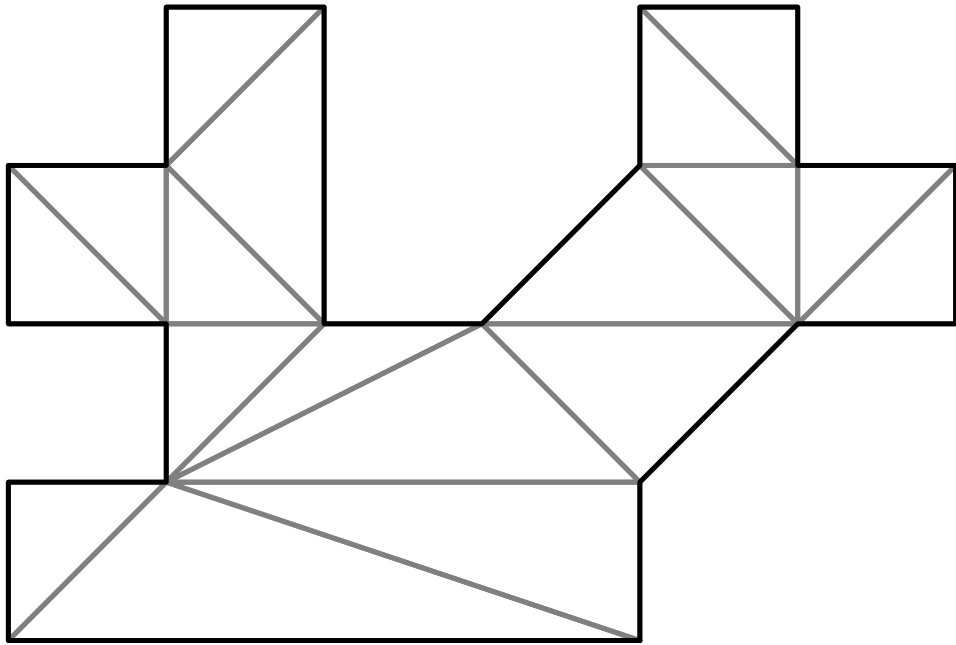








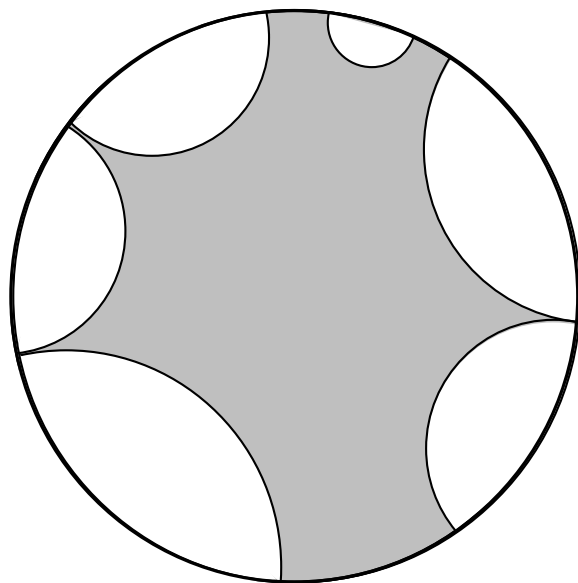
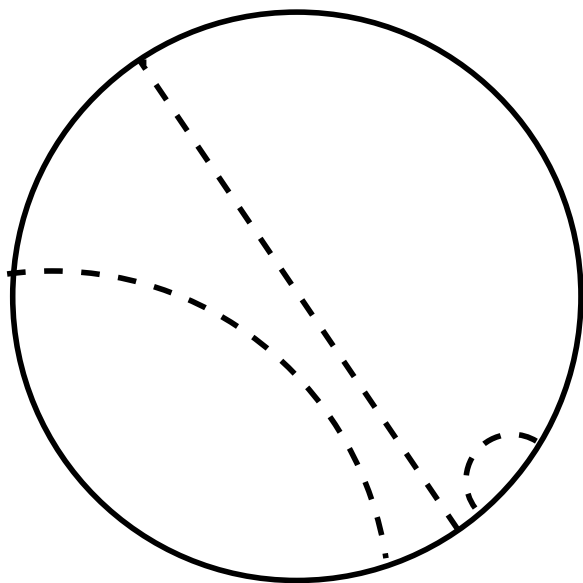


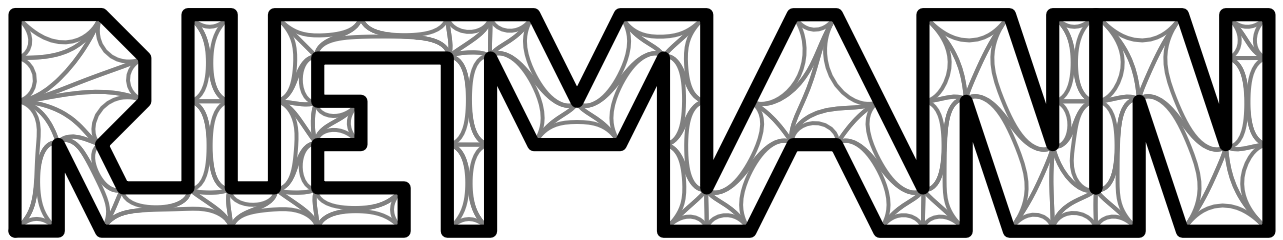
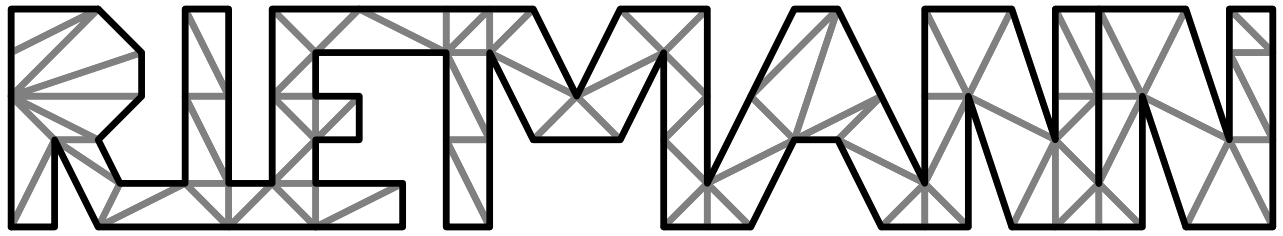


**Hyperbolic metric** on disk given by

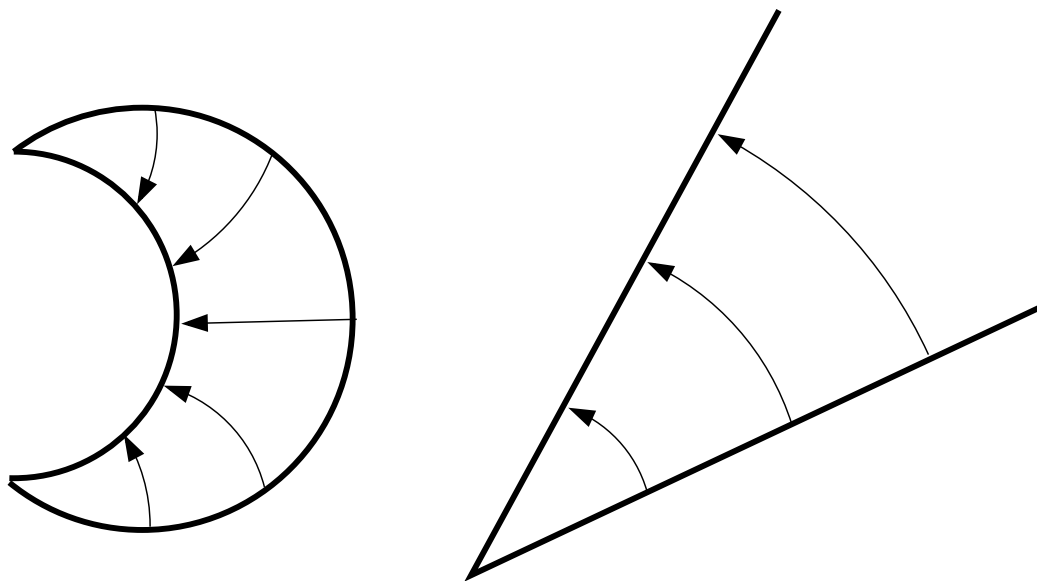
$$d\rho = \frac{ds}{1 - |z|^2} \simeq \frac{ds}{\text{dist}(z, \partial D)}.$$

- Isometries are the Möbius transformations.
- Geodesics are circles perpendicular to boundary.
- Volume grows exponentially





Crescents are bounded by two circular arcs.

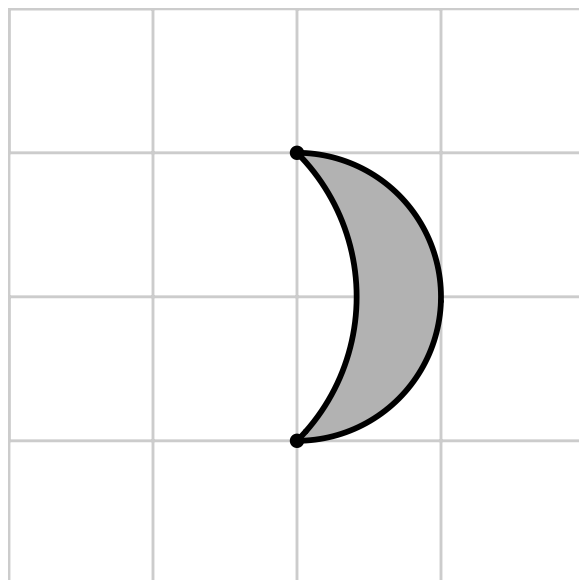


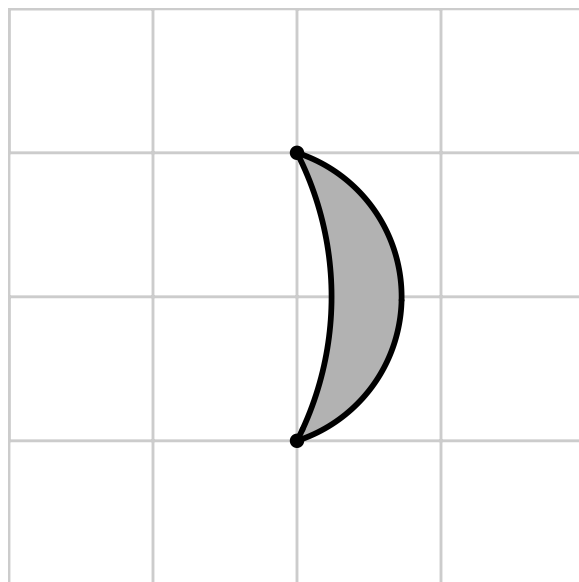
Crescents have foliation into circular arc perpendicular to boundary. Gives mapping from one boundary arc to other.

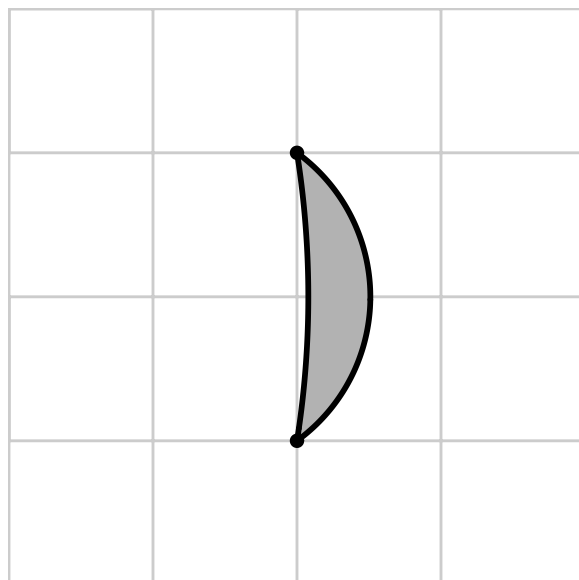
It is Möbius (linear fractional) transformation.

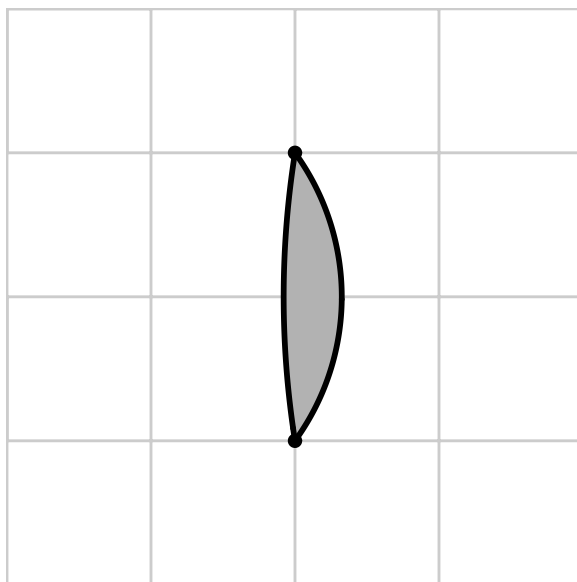
$$z \rightarrow \frac{az + b}{zc + d} \quad \sim \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

The crescent maps are elliptic rotations.

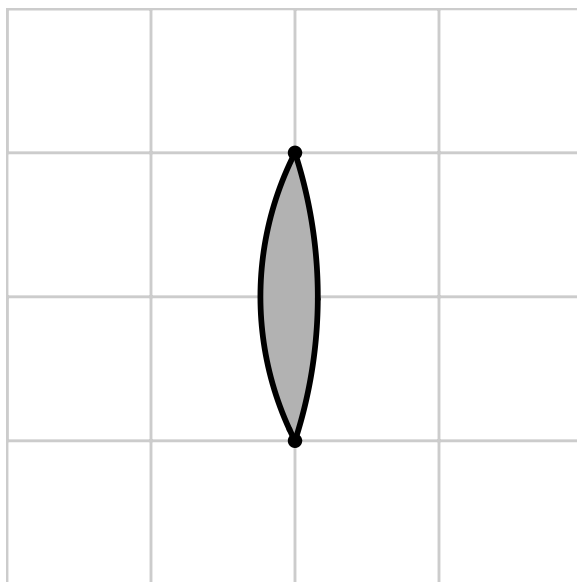


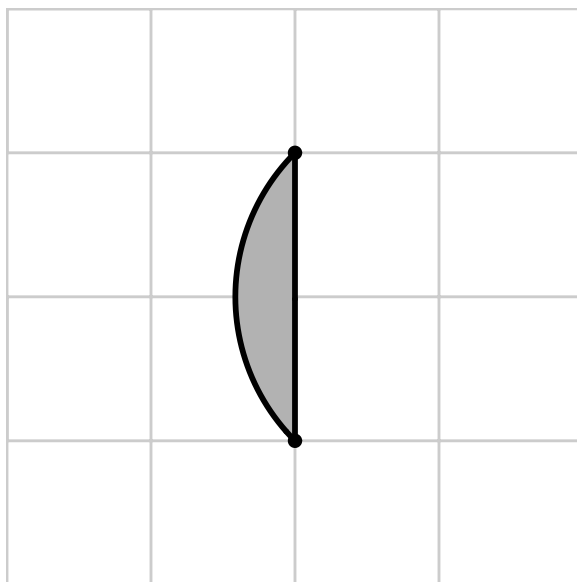


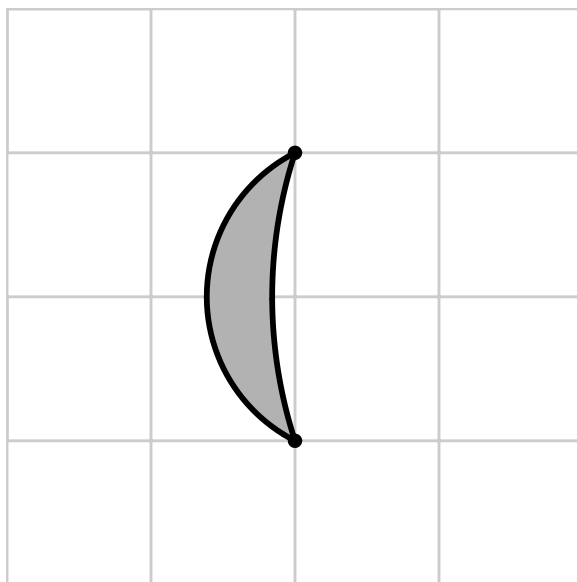


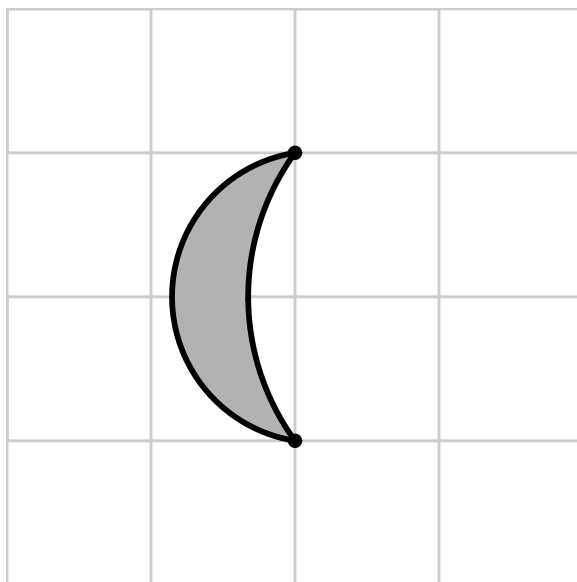


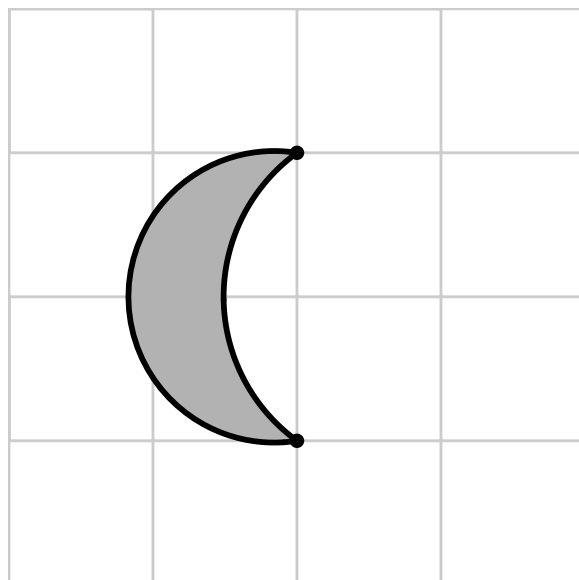


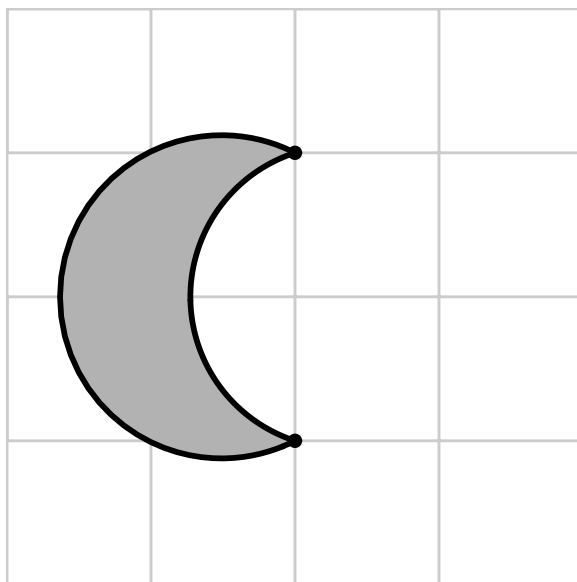


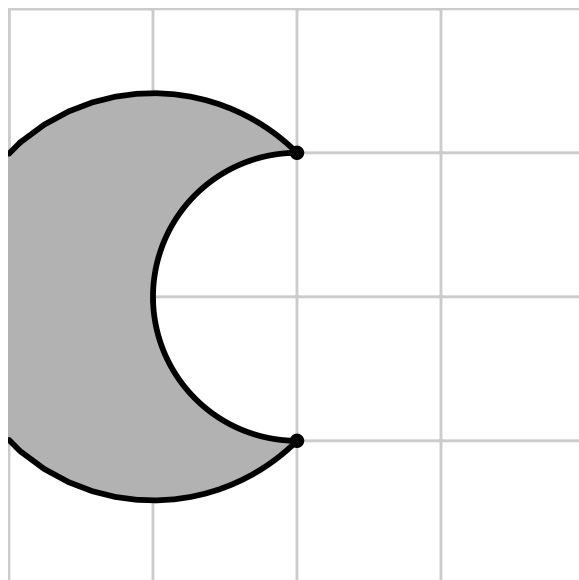


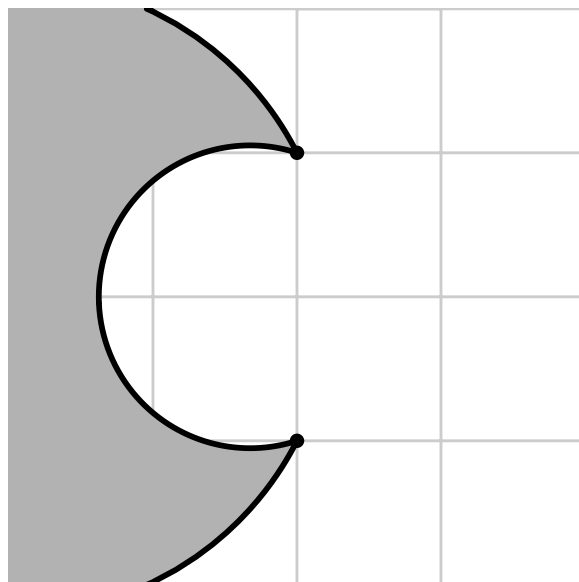




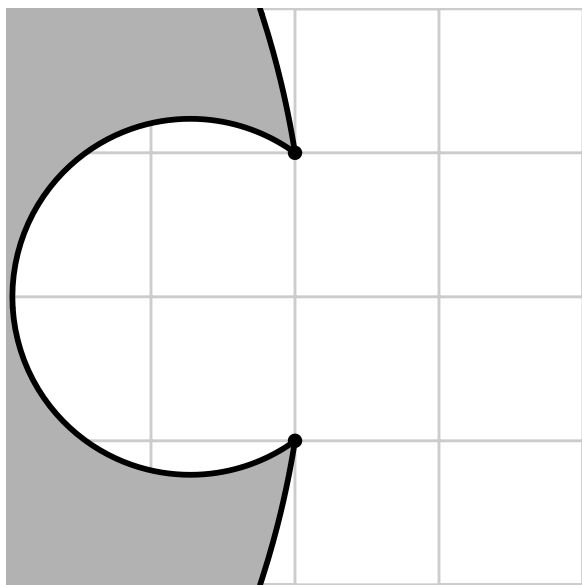


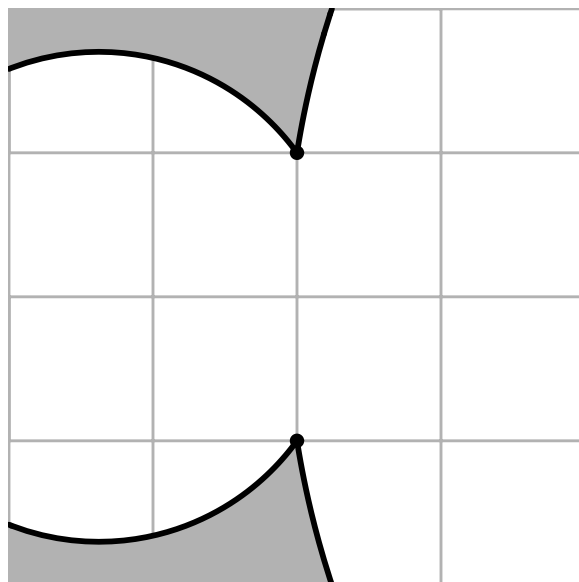


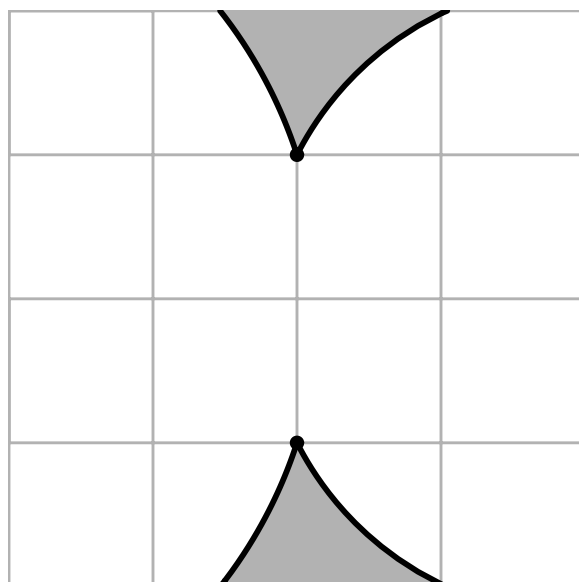


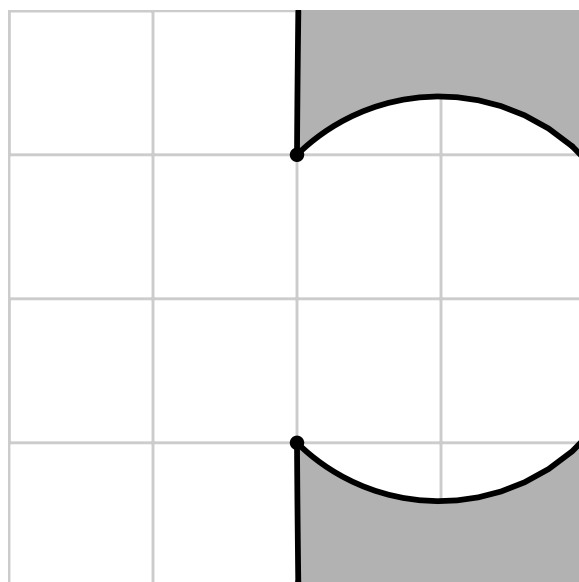


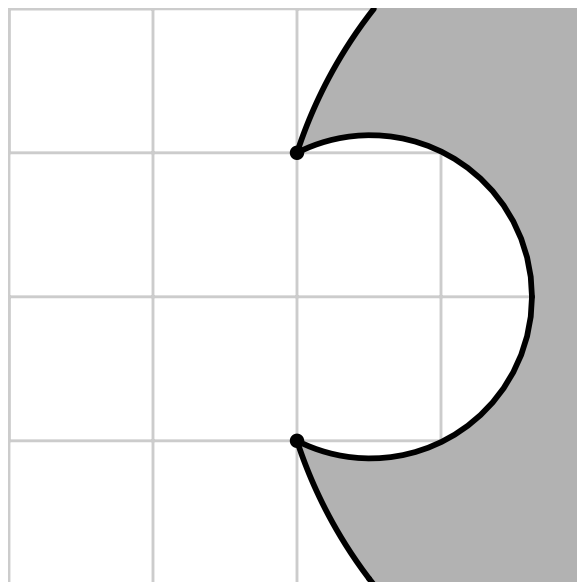


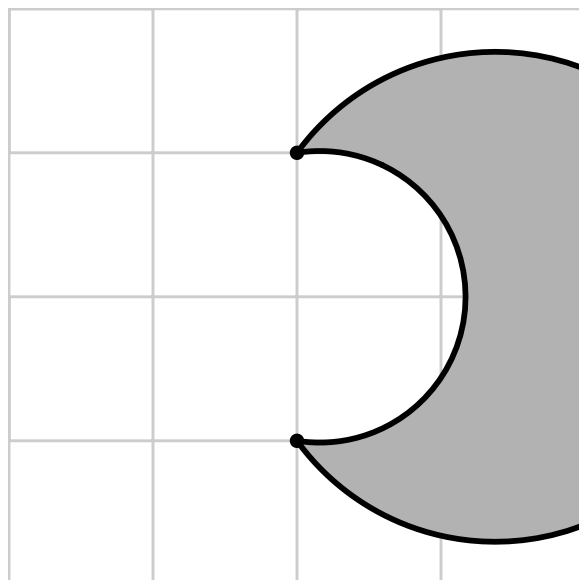


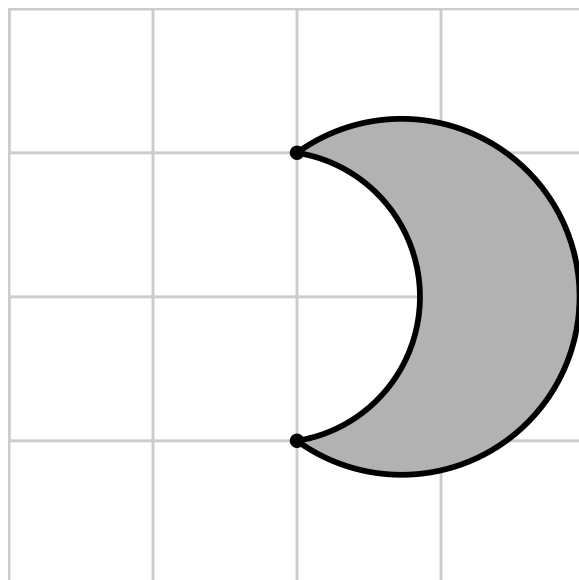


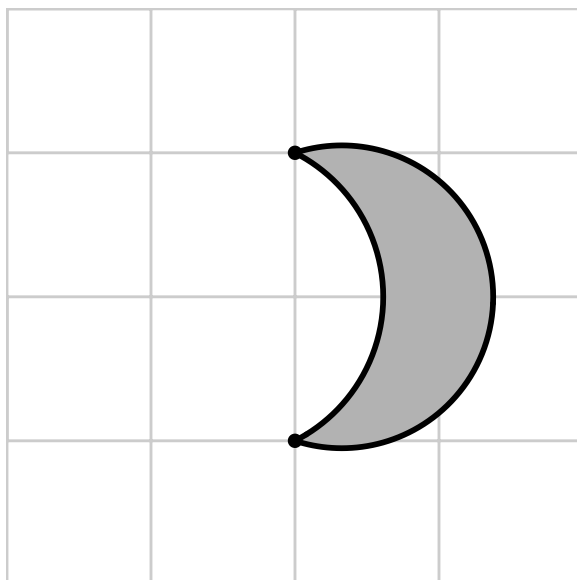




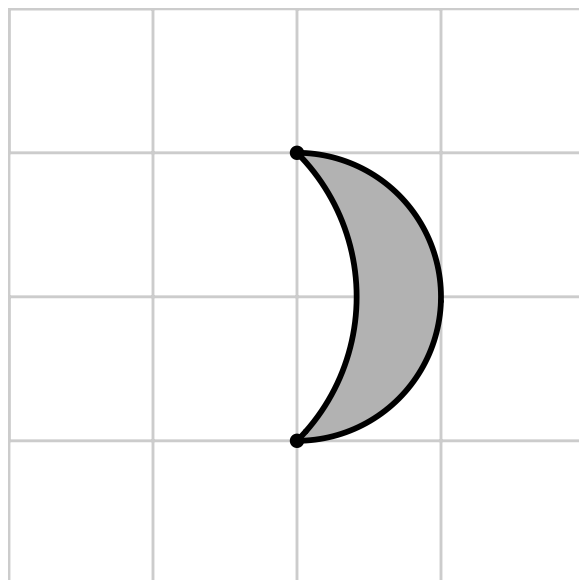




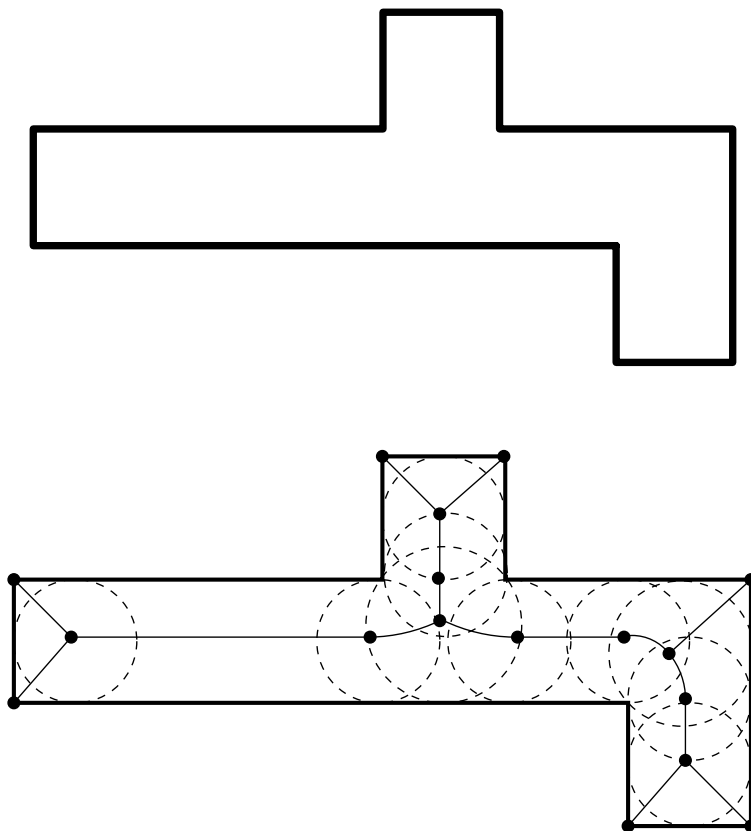








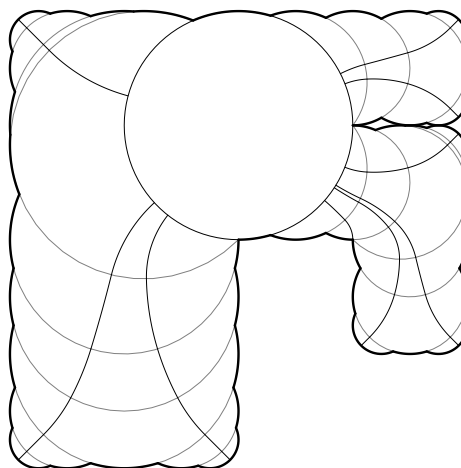
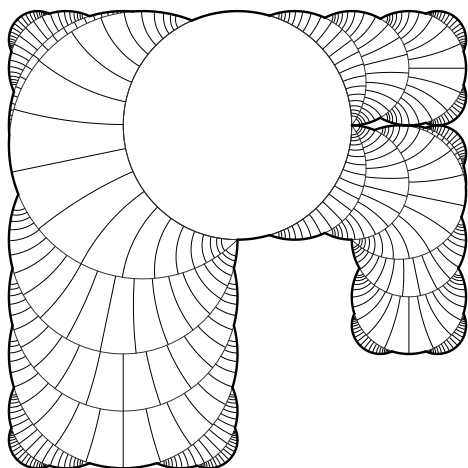
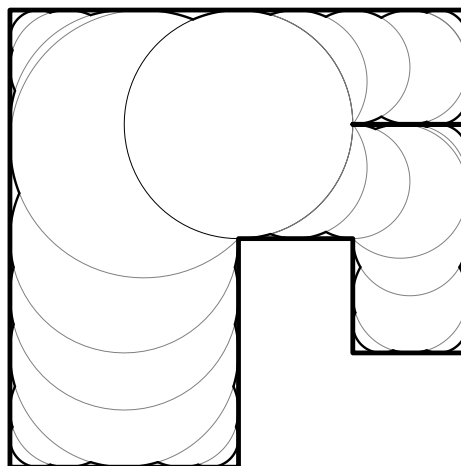
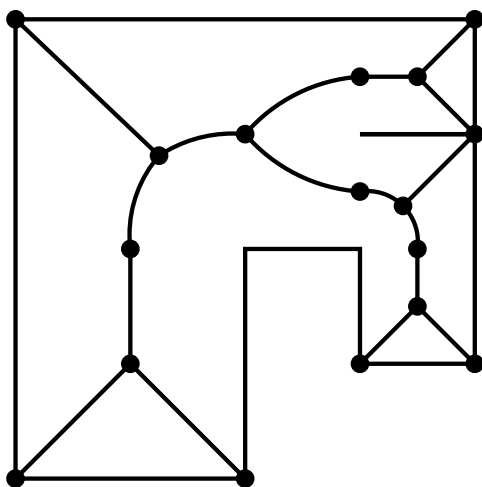
The medial axis consists of centers of disks in  $\Omega$  which touch the boundary in at least two points.



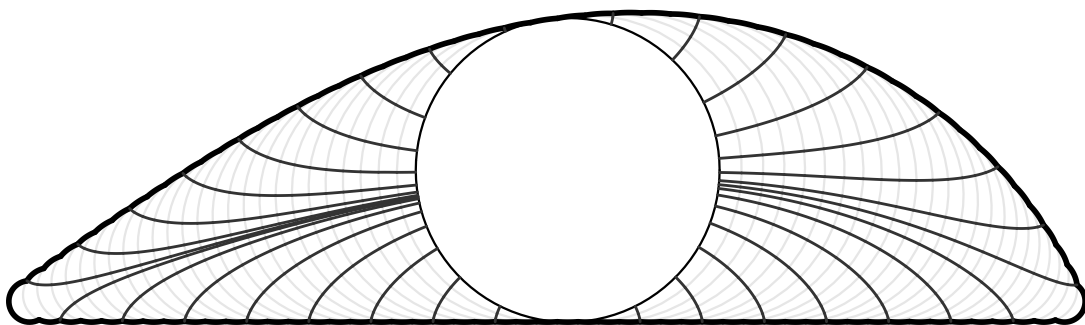
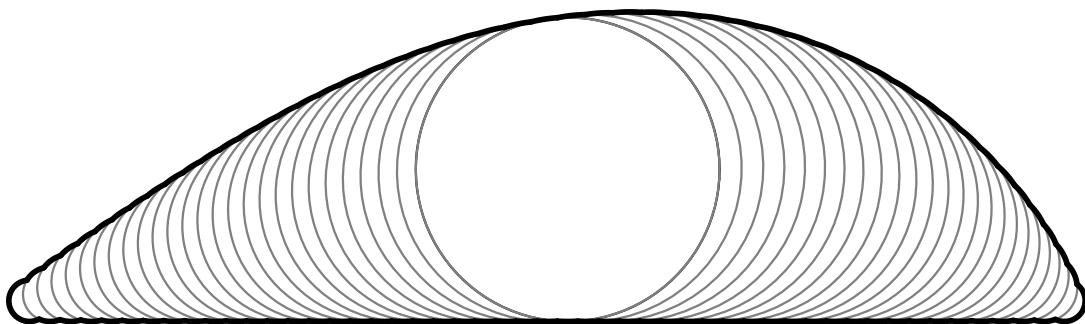
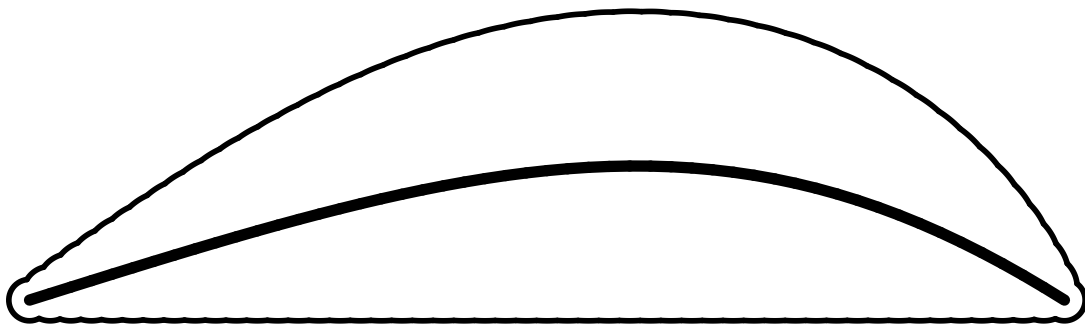
We can use medial axis to approximate a domain by a finite union of disks.

## Fast Almost Riemann Mapping Theorem:

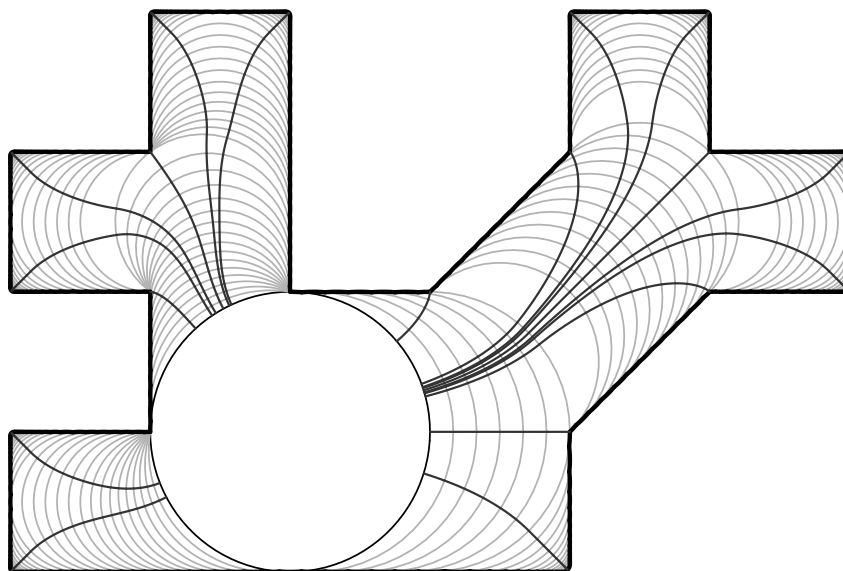
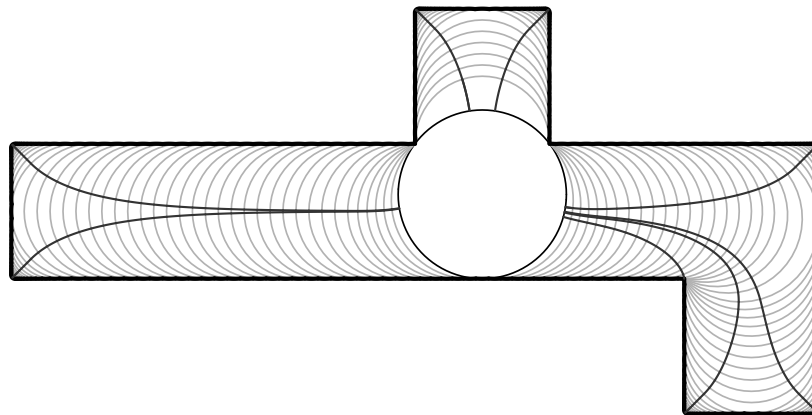
Can construct a map from  $n$ -gon  $\Omega$  to disk in  $O(n)$  time and is “close to” the conformal map.



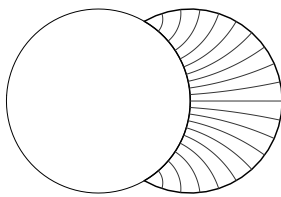
Similar flow for any simply connected domain.



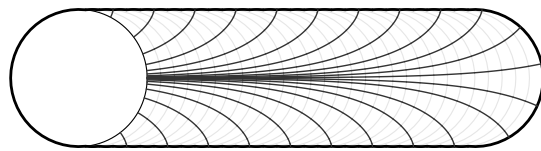
Explicit formulas for polygons. Images of all  $n$  vertices computable in  $O(n)$ .



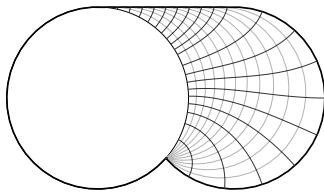
Flow defines Möbius map between medial axis disks. Explicit formula for edges of medial axis using disks (center, radius) and edge type.



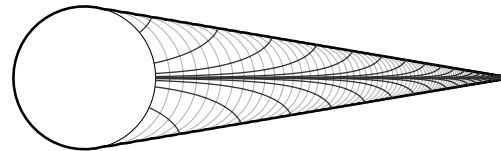
point-point



edge-edge



point-edge

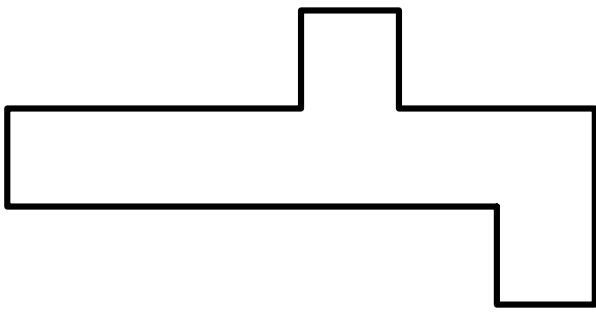


convex vertex

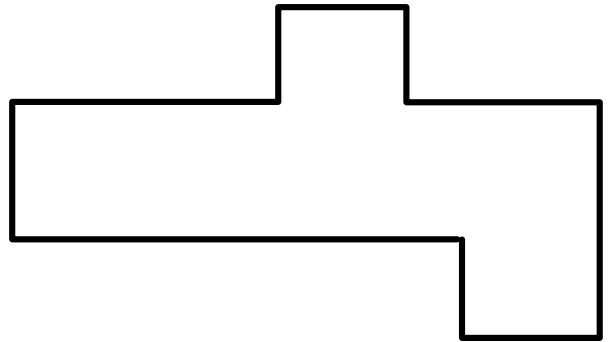
From medial axis can compute  $\text{iota}$  in  $O(n)$ . Medial axis computable in  $O(n)$  by result of Chin, Snoeyink and Wang.

Plug  $\iota$  parameters into SC formula. Should get a good approximation to original polygon.

Plug iota parameters into SC formula. Should get a good approximation to original polygon.



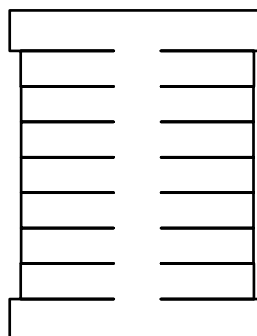
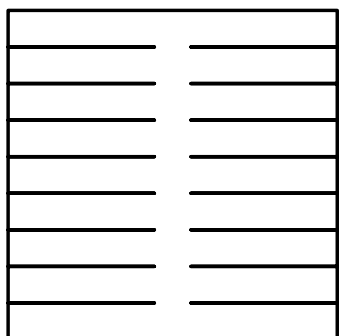
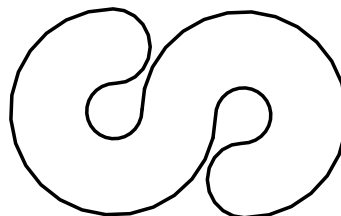
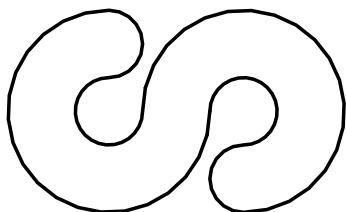
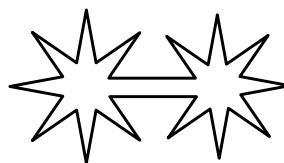
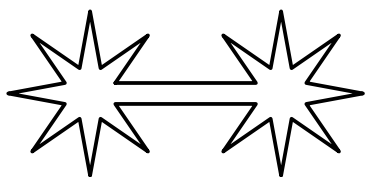
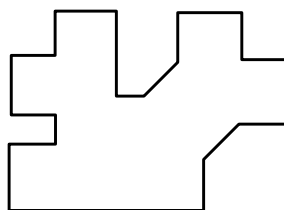
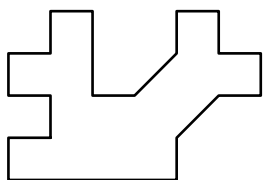
## Target Polygon



SC-image with iota  
parameters

MA flow gives “formula” for SC-parameters  $\{z_k\}$ .





Target Polygon

iota

Why does  $\iota$  approximate the Riemann Map?

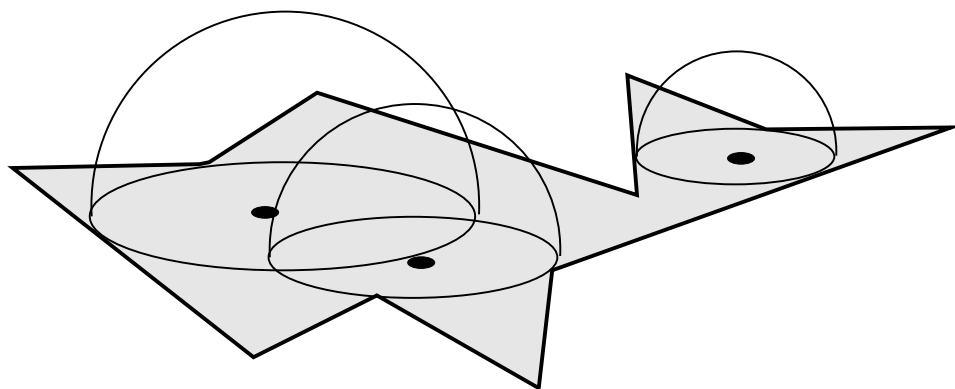
Why does  $\iota$  approximate the Riemann Map?

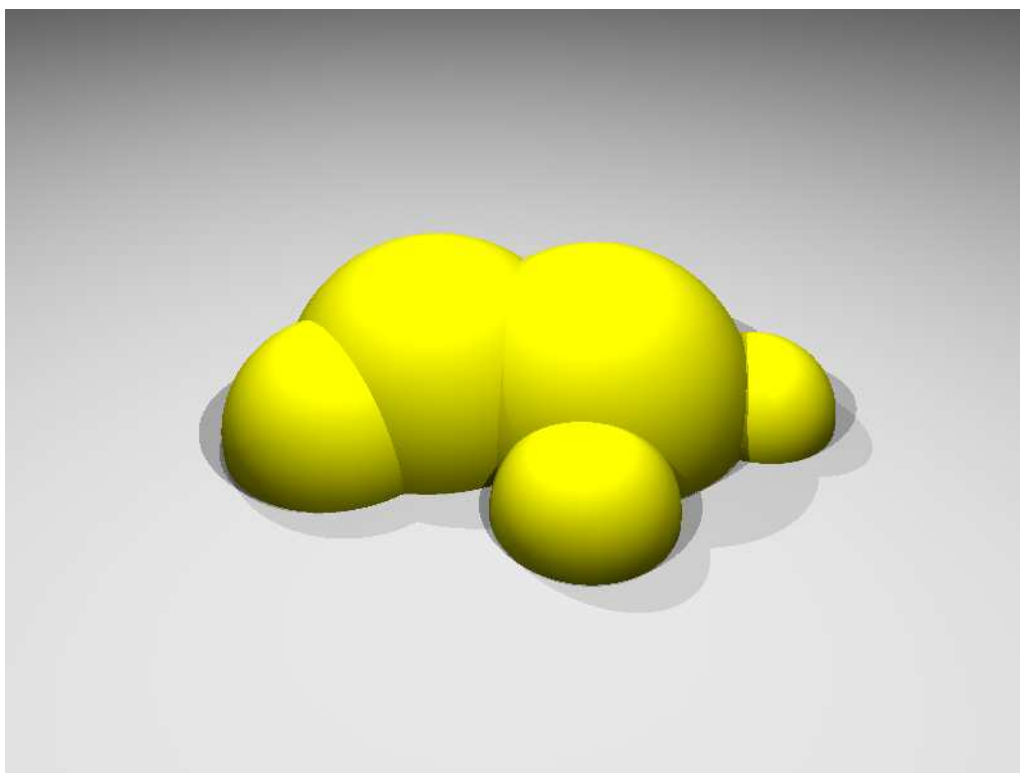
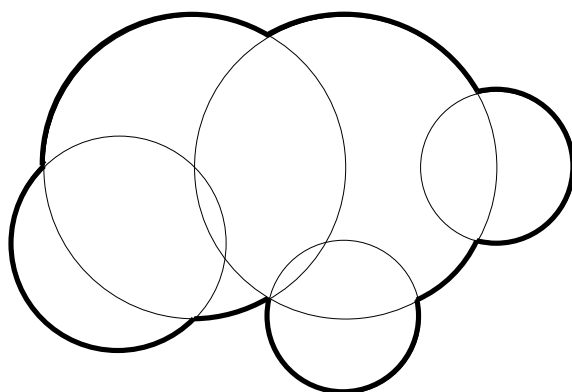
Because  $\iota$  is the boundary map of a conformal map for a surface with the same boundary as  $\Omega$ .

Why does  $\iota$  approximate the Riemann Map?

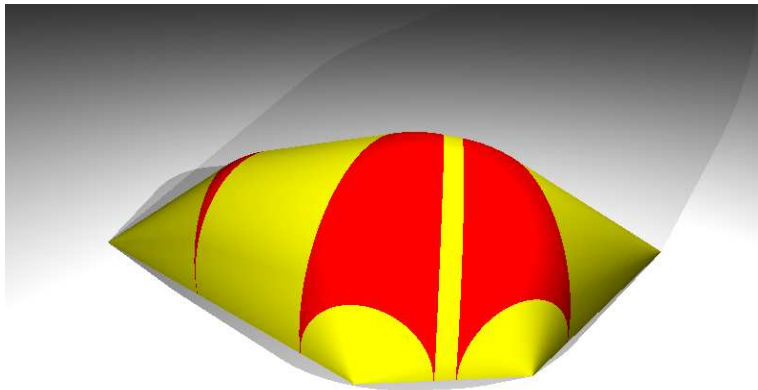
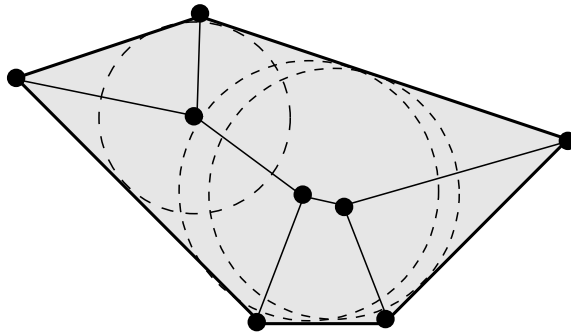
Because  $\iota$  is the boundary map of a conformal map for a surface with the same boundary as  $\Omega$ .

Take union of all hemispheres whose bases lie inside  $\Omega$ . Upper envelope is the “dome” of  $\Omega$ .



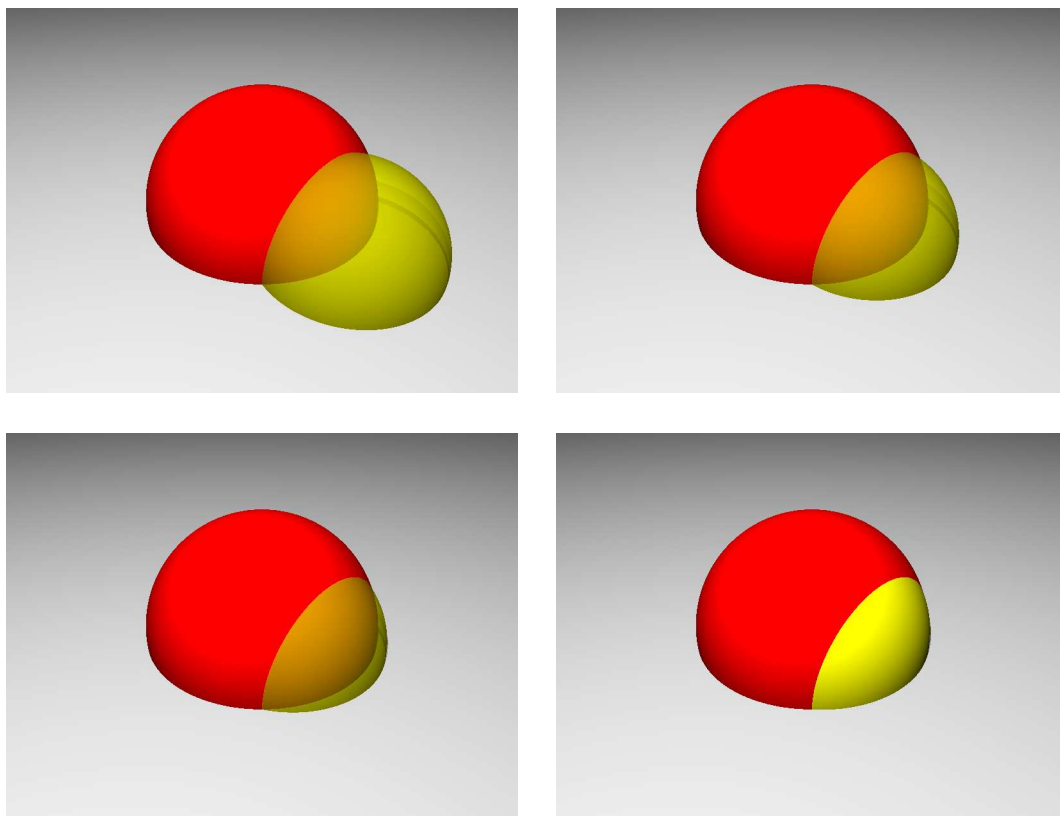


Finite union of disks, finitely bent dome

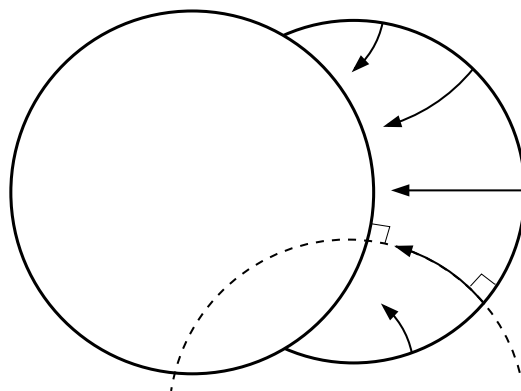


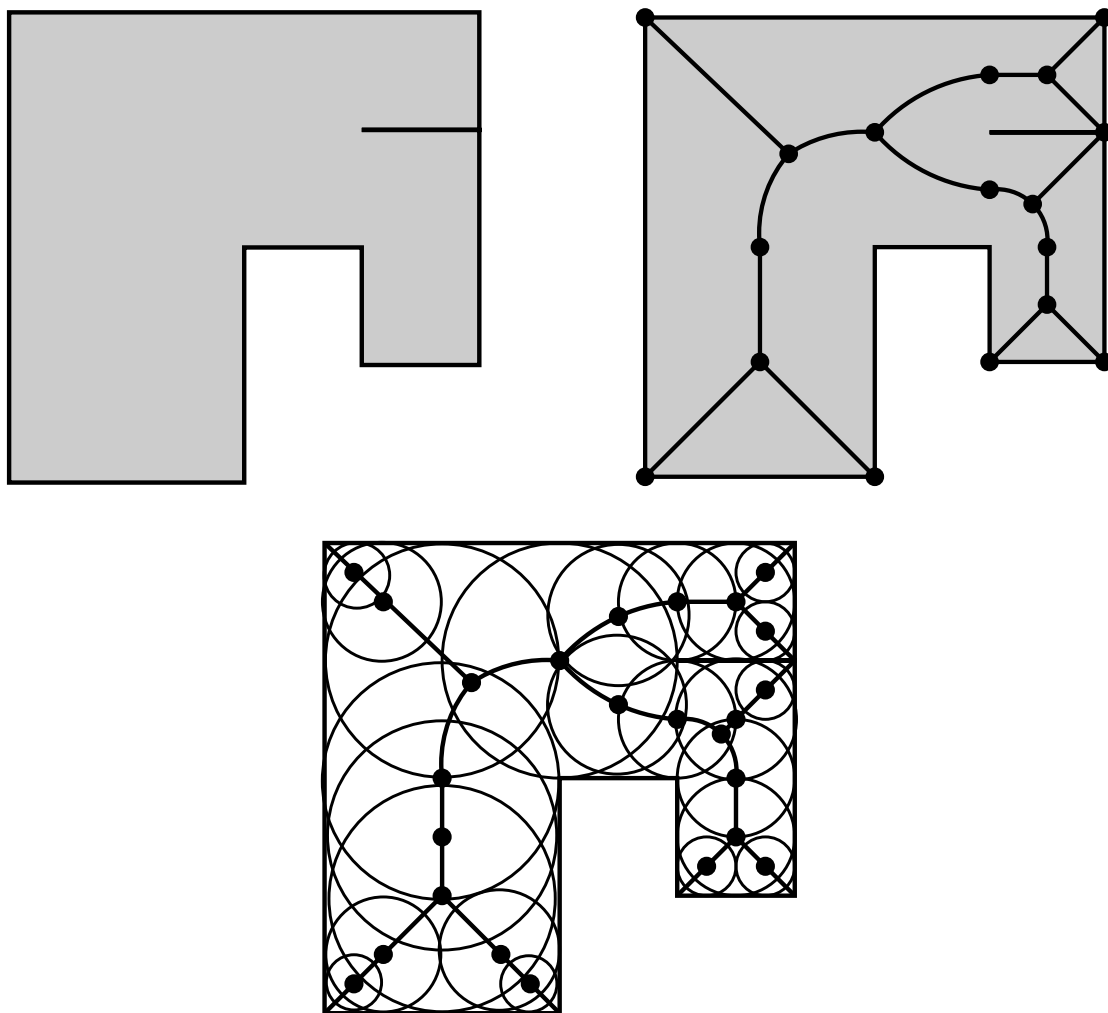
Only need to use medial axis disks.

It is easy to map any dome conformally to a disk.



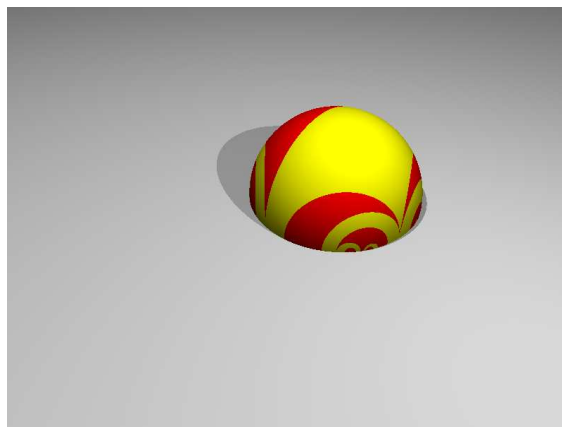
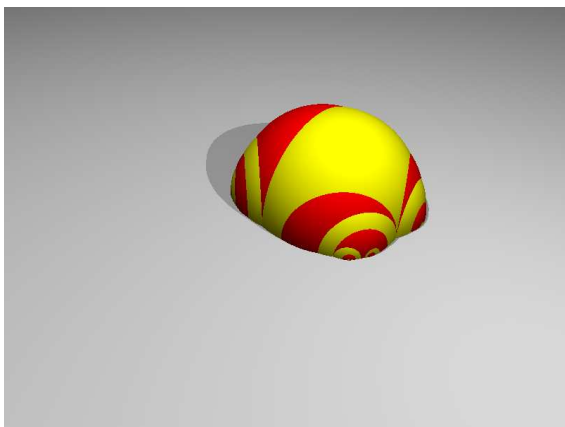
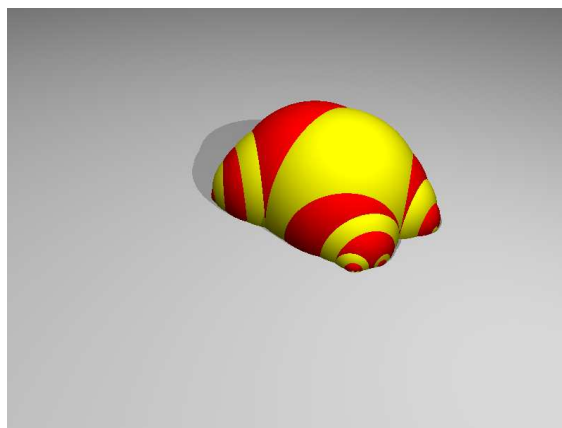
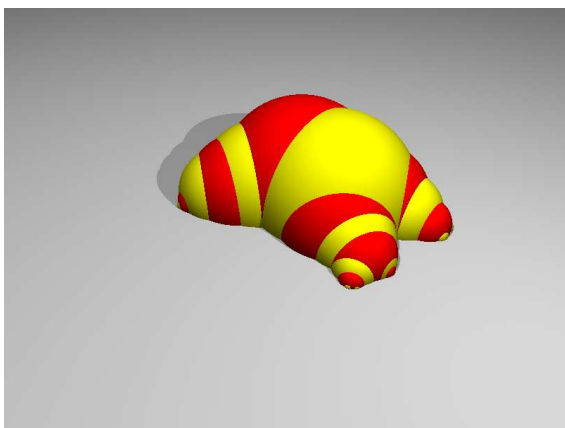
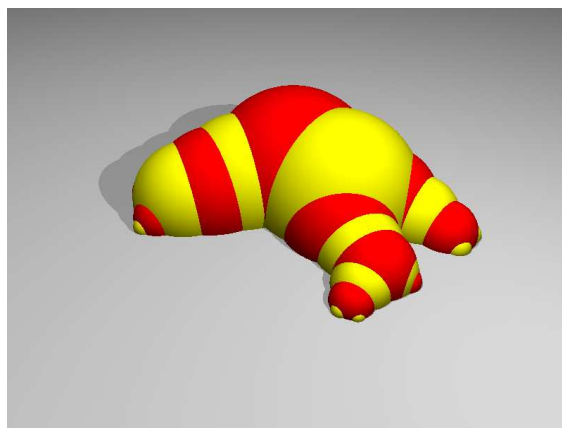
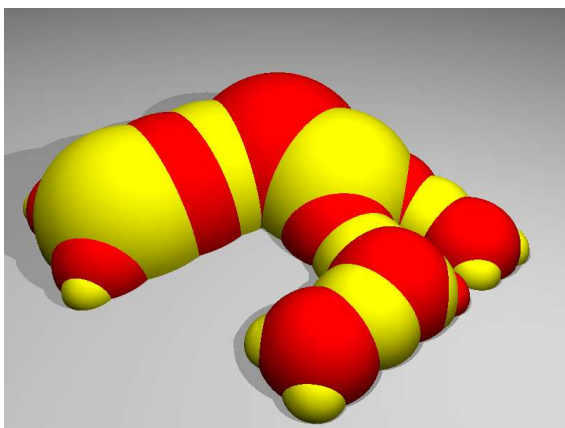
On base, foliate crescent by orthogonal arcs.

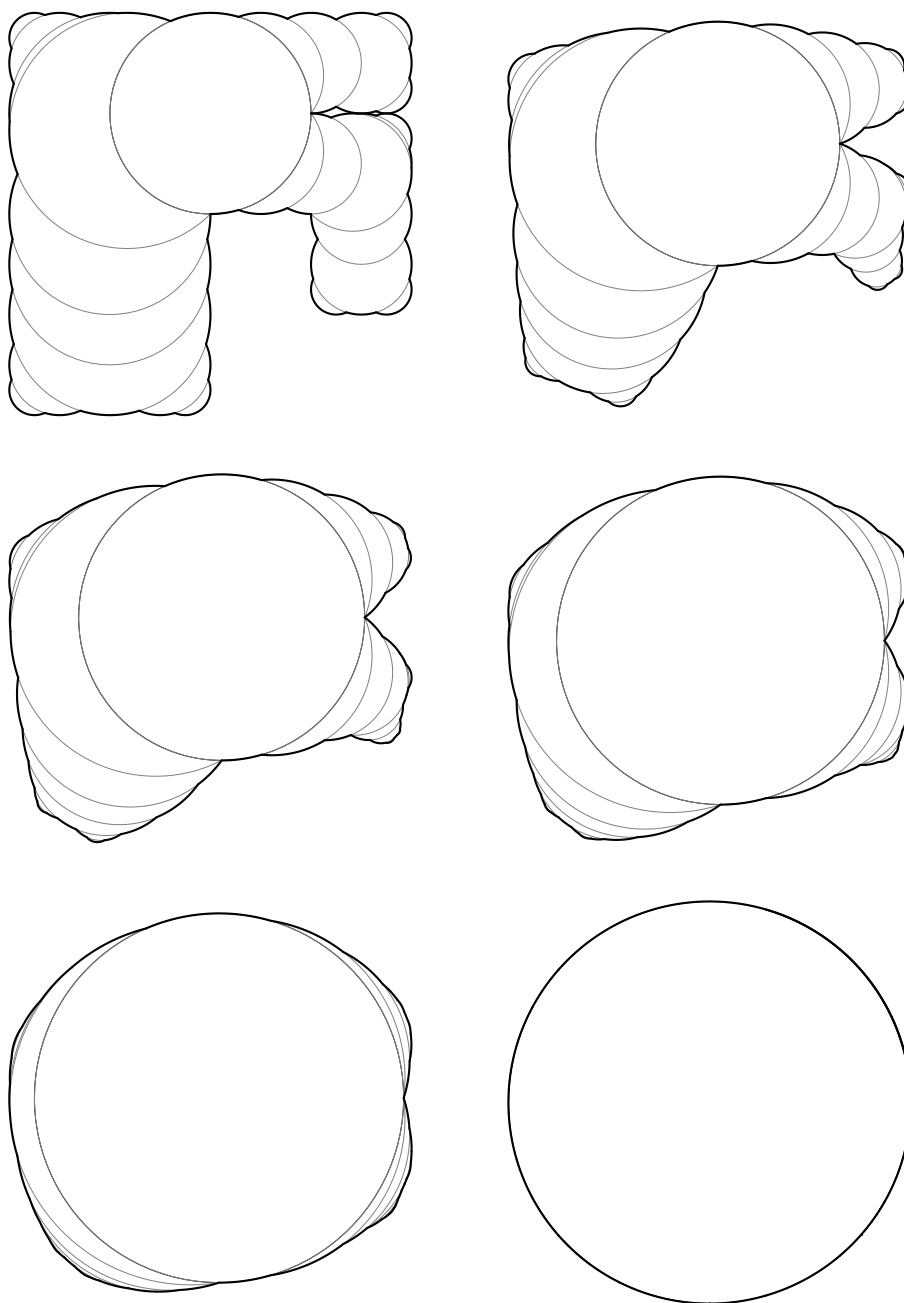




A polygon, medial axis, approximation by disks.





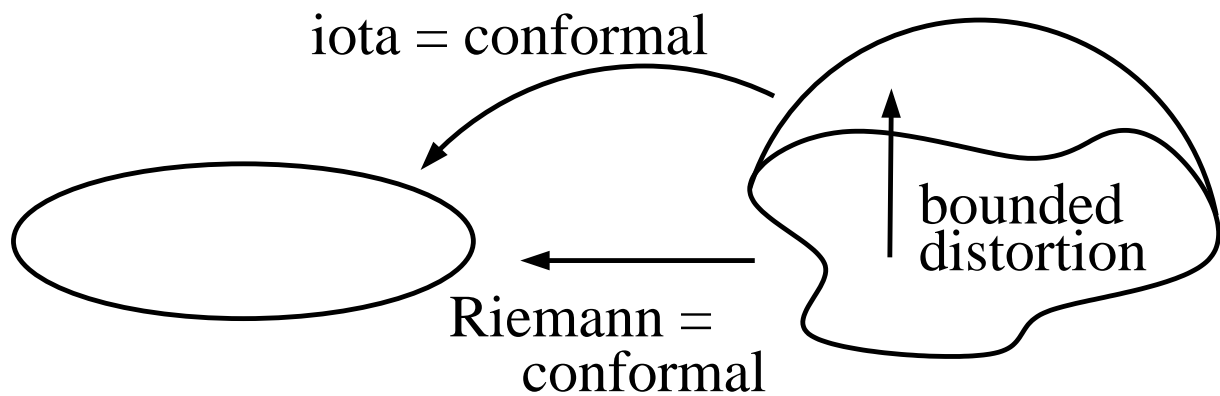


Angle scaling family

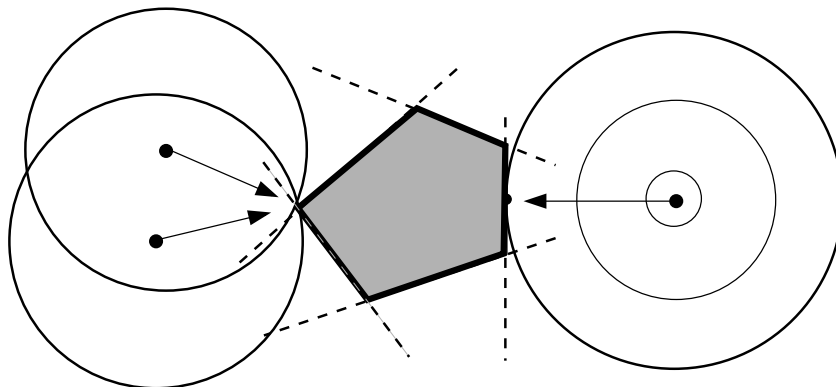
Iota = conformal map from dome to disk.

Riemann = conformal form base to disk

From base to dome = ?

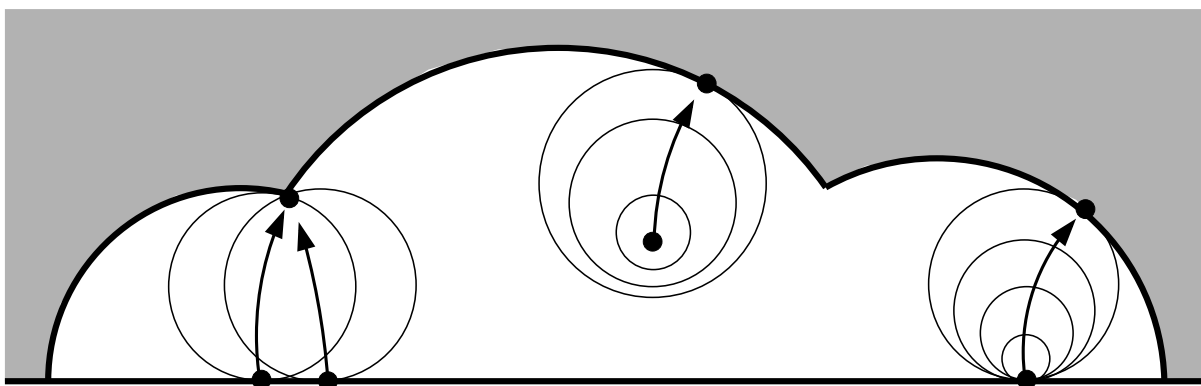


Nearest point map in  $\mathbb{R}^n$  is Lipschitz.



Nearest point retraction in hyperbolic space extends to boundary and is a quasi-isometry

$$\frac{1}{A}\rho(x, y) - B \leq \rho(R(x), R(y)) \leq A\rho(x, y).$$



Dennis Sullivan, David Epstein and Al Marden,  
C. Bishop



$P$



$P^2$

$P$  = hyperbolic geometry, University of Warwick

$P^2$  = computational geometry, UC Irvine

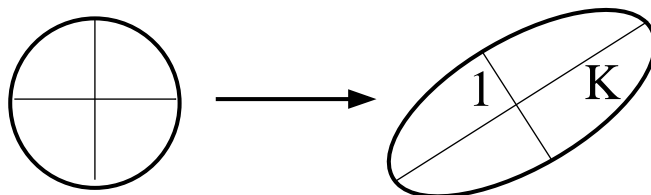
## Fast Mapping Theorem:

Given an  $n$ -gon we can compute an  $\epsilon$ -conformal map  $f : \mathbb{D} \rightarrow \Omega$  in time  $O(n \log \frac{1}{\epsilon} \log \log \frac{1}{\epsilon})$ .

$\epsilon$ -conformal means angles are distorted by  $< \epsilon$ .

(Really means  $(1 + \epsilon)$ -quasiconformal.)

$K$ -quasiconformal means tangent map sends circles to ellipses of eccentricity  $\leq K$ .



For smooth maps let  $\mu = g_{\bar{z}}/g_z$  where

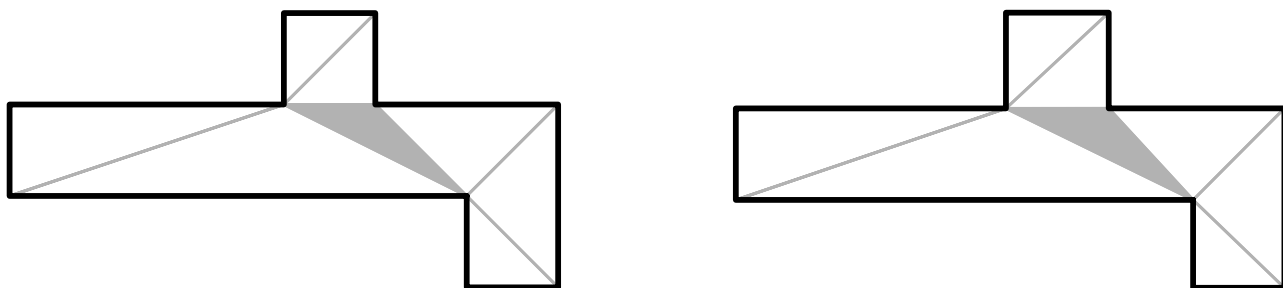
$$g_{\bar{z}} = \frac{1}{2}(g_x + ig_y), \quad g_z = \frac{1}{2}(g_x - ig_y).$$

Then  $|\mu| < 1$  and

$$K = \sup_{\mathbb{D}} \frac{1 + \mu}{1 - \mu}.$$

$K = 1$  gives conformal map.

Check SC-parameters by triangulating image polygon and map affinely to target. Compute  $K$  for each triangle and take maximum.



The most distorted triangle is shaded. We can bound QC distance from solution even though we don't know what the solution is.

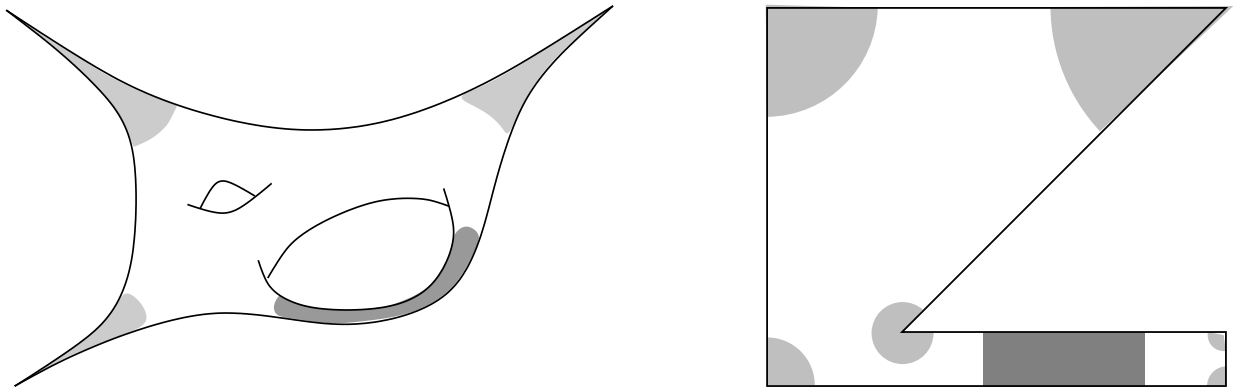


## **Ideas in proof of FMT:**

- Thick and thin parts of polygons
- $O(n)$  domain decomposition
- Newton's type iteration via fast multipole
- Angle scaling

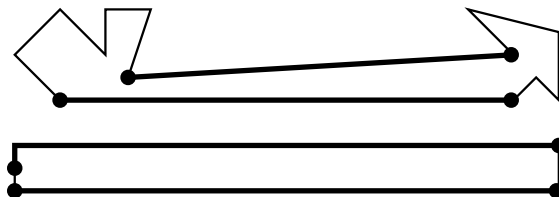
## Thick and Thin parts

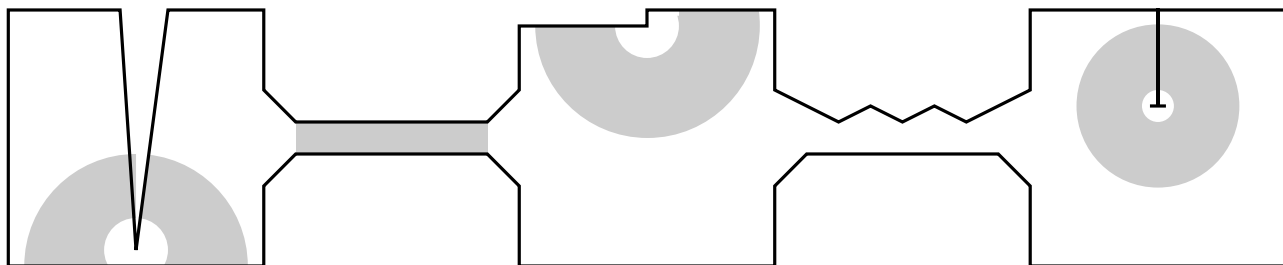
An  $\epsilon$ -**thin part** of a surface is a union of non-trivial loops of length  $\leq \epsilon$  (parabolic/hyperbolic).



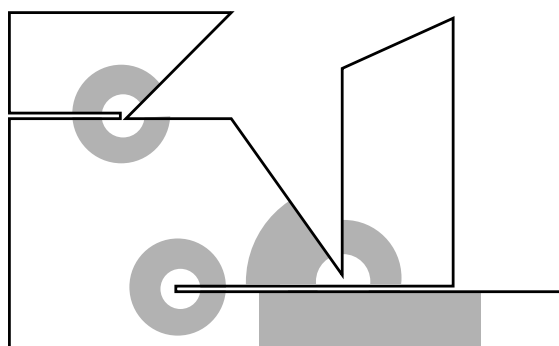
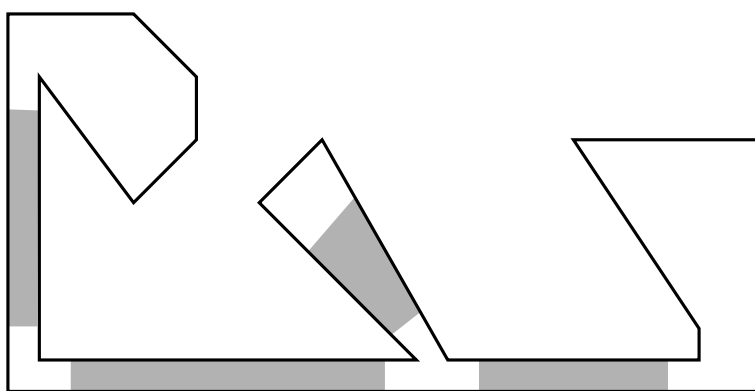
An  $\epsilon$ -thin part of a polygon is sub-region with two edges on  $\partial\Omega$  whose extremal distance is  $\leq \epsilon$ .

$\Omega$  can be mapped to  $1 \times \epsilon$  rectangle with two sides of  $\partial\Omega$  covering the long sides of rectangle.

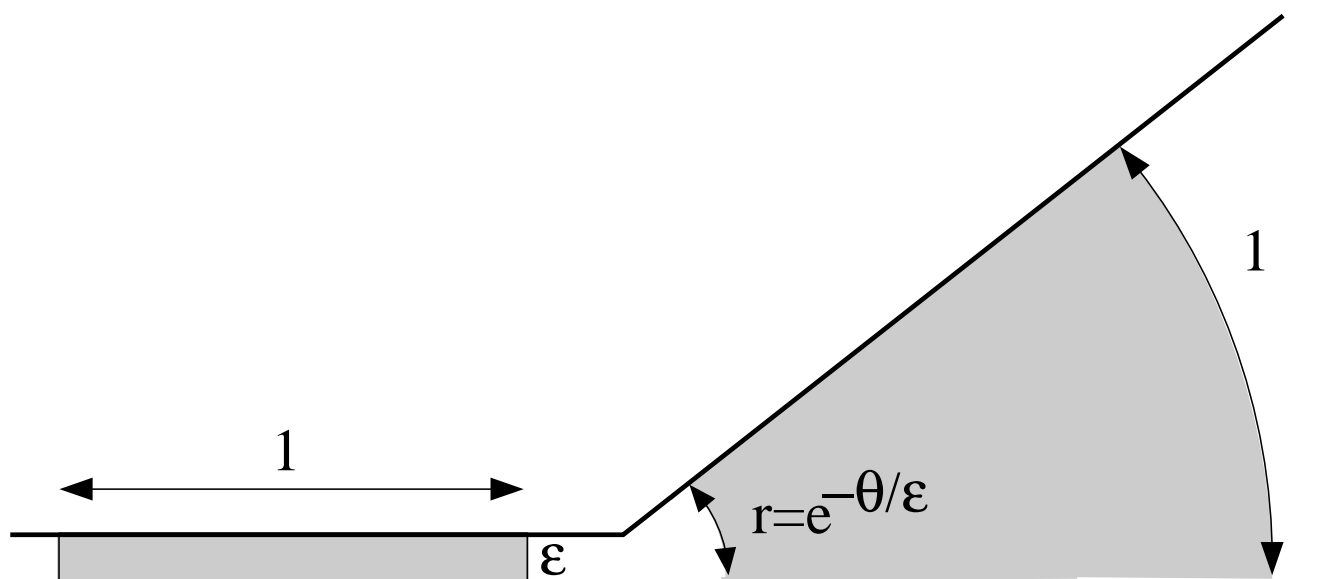




Right channel is not thin because edges on top have length comparable to width of channel.

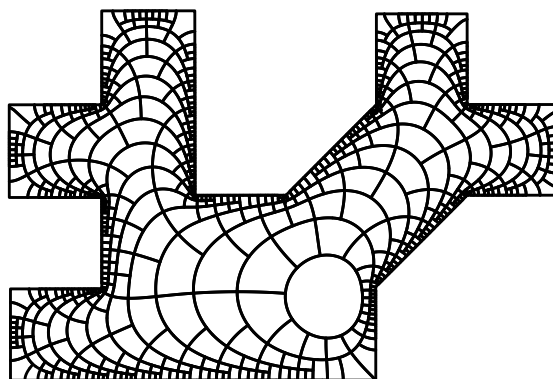
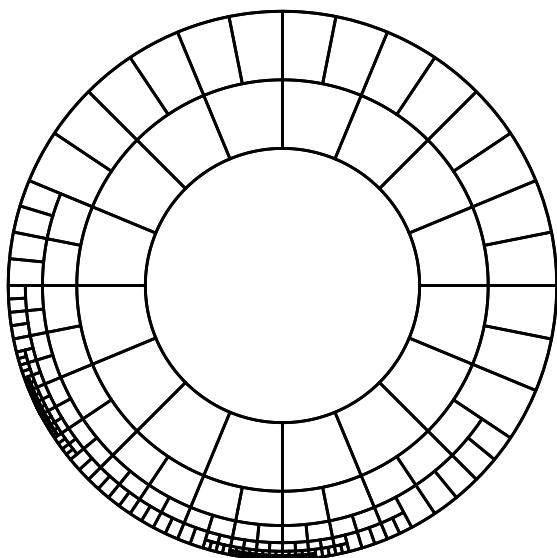


Previous figure not to scale. Proof uses  $\epsilon \ll 1$  (think  $\epsilon = .01$ ) and usually the two ends of a thin part are a very different scales.



## Domain Decomposition:

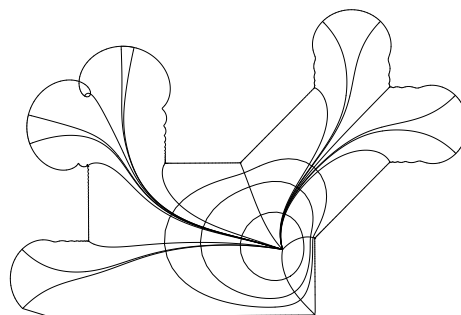
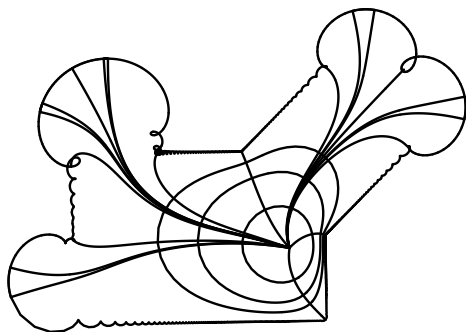
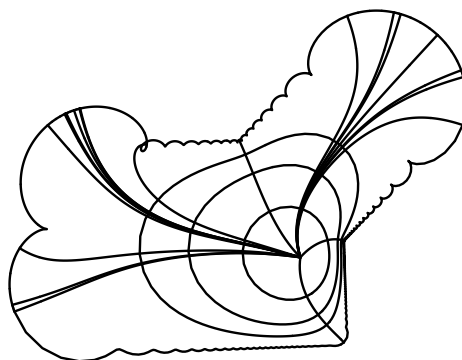
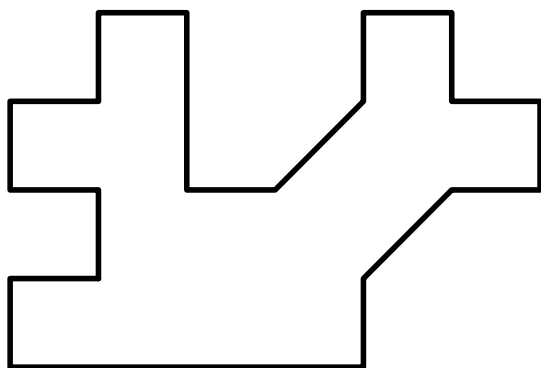
Break disk into  $O(n)$  pieces, and represent map by  $p = |\log \epsilon|$  term power series on each piece. Time  $O(np \log p)$  allows  $O(1)$  FFT's per vertex.



Here  $p = 5$ .

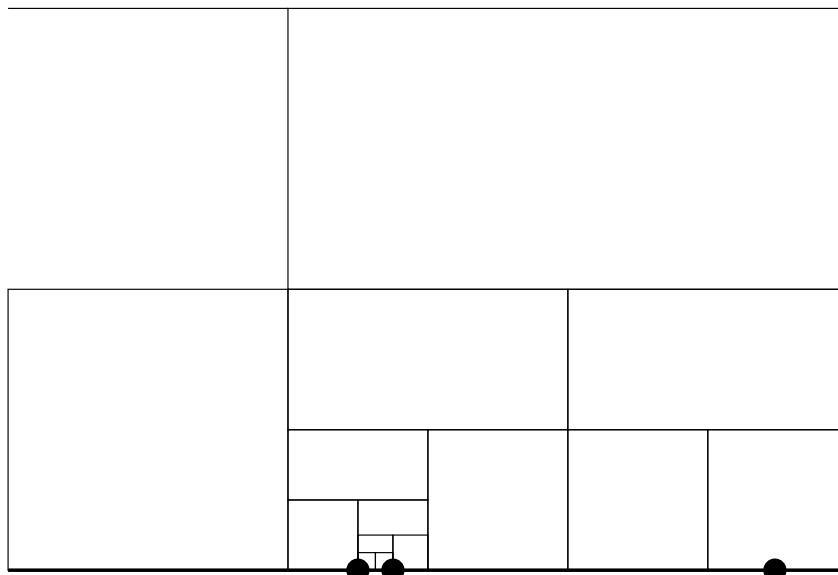
Representation is  $\epsilon$ -map if series on adjacent pieces agree to within  $\epsilon$  (hyperbolic metric).

Using single power series gives bad approximation:  
100, 500, 2500 terms.

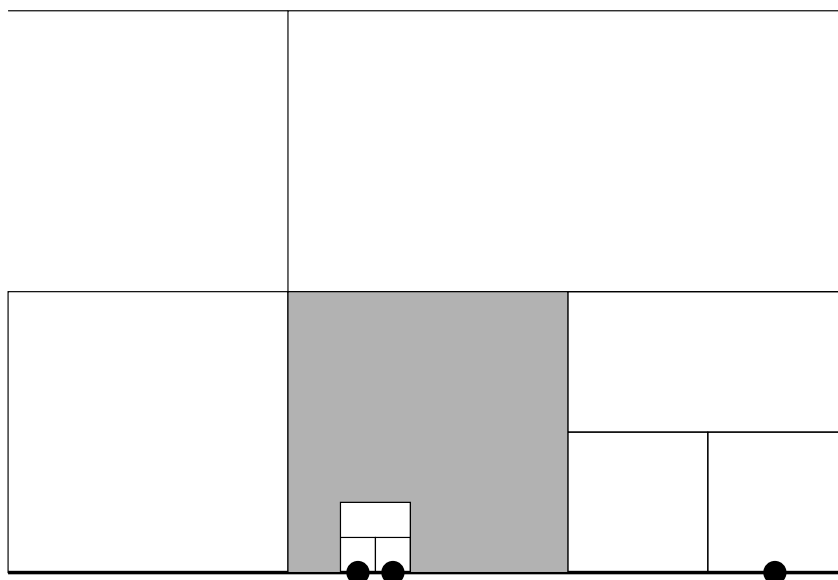


Representation depends on choice of  $n$ -tuple on  $\mathbb{T}$   
(the guessed parameters).

Keep dividing until each box contains  $\leq 1$  point.



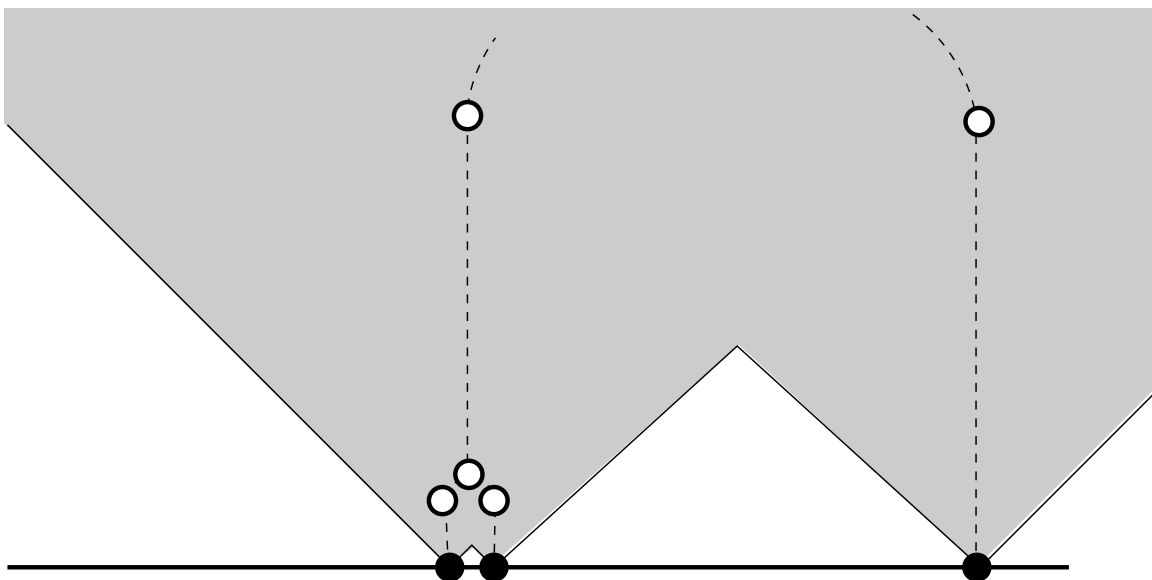
But this can give  $\gg n$  boxes if points too close.



Replace a stack of boxes by a single arch and use a Laurent series instead of a power series.

Half-plane can be decomposed into  $O(n)$  boxes and  $O(n)$  arches. Why?

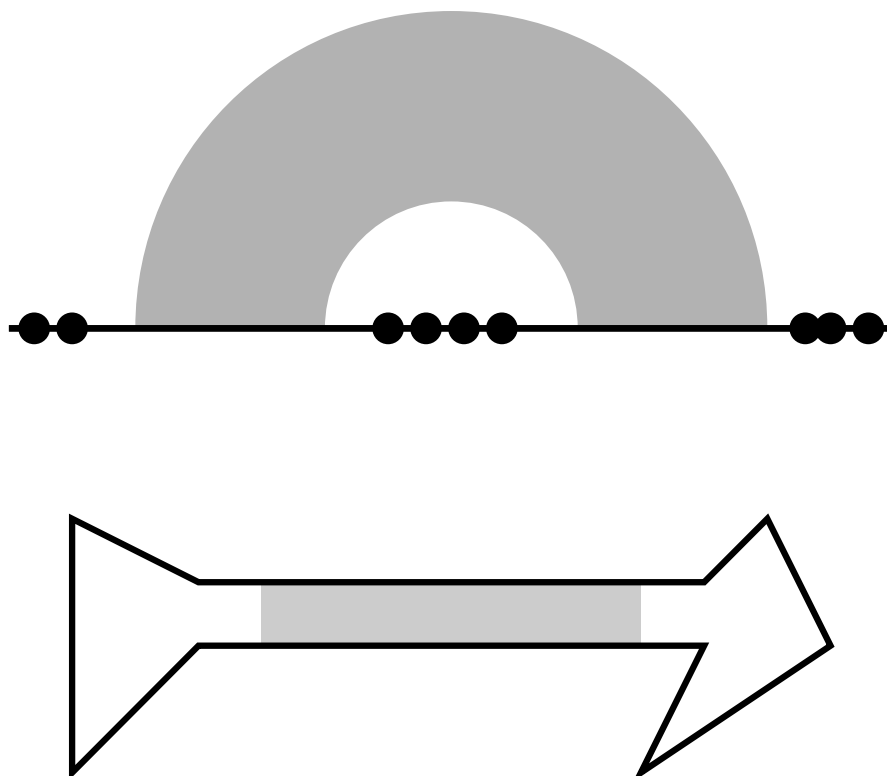




We define sawtooth region with vertices at  $n$ -tuple. Compute medial axis. Edge-edge bisectors are vertical. Arches correspond to long edge-edge bisectors (length  $\geq A$  in hyperbolic metric.) There are  $O(n)$  such edges, so  $O(n)$  arches.

Number of remaining boxes is  $O(n)$  by hyperbolic geometry. (can associate each box to fixed area of hyperbolic convex hull of boundary points and hull has hyperbolic area  $O(n)$ ).

Arches correspond to thin parts. Thus  $O(n)$  thin parts; computable in  $O(n)$  time.



**Beltrami equation:** Given complex function  $\mu$  on  $\mathbb{D}$  with  $\sup |\mu| < 1$ , we can find  $g : \mathbb{D} \rightarrow \mathbb{D}$  so that  $g_{\bar{z}} = \mu g_z$  where

$$g_{\bar{z}} = \frac{1}{2}(g_x + ig_y), \quad g_z = \frac{1}{2}(g_x - ig_y).$$

If  $f : \mathbb{D} \rightarrow \Omega$  and  $\mu = f_{\bar{z}}/f_z$  then  $f \circ g^{-1} : \mathbb{D} \rightarrow \Omega$  is conformal.

There is infinite series of convolution operators that gives exact solution. Can only find approximate solution in finite time.

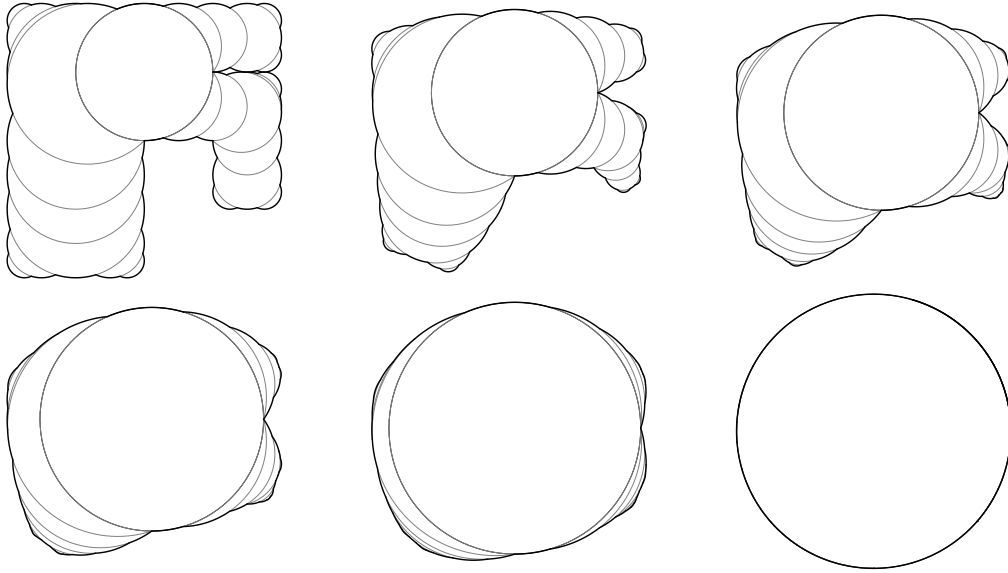
**Newton's method:** Approximate solution converts an  $\epsilon$ -map into an  $O(\epsilon^2)$ -map in time  $O(np \log p)$ ,  $p = |\log \epsilon|$ .

**Uniform radius of convergence:** iteration works if  $\epsilon < \epsilon_0$ , independent of  $n$  and  $\Omega$ .

If we knew  $\iota$  map was a  $\epsilon_0$ -map then we could use it as starting point of iteration and be done.

But we don't know this, so have to lead iteration to correct answer in baby steps.

## Angle Scaling:



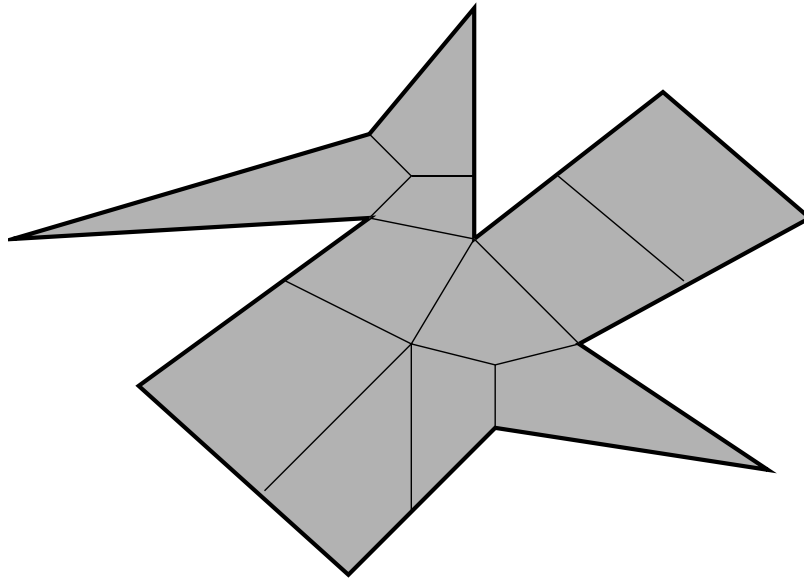
Discretize angle scaling family so gaps are smaller than convergence radius.

Use almost conformal map onto one domain as initial point for iteration for next domain.

Start with identity map on disk and end with map onto  $\Omega$ .

$O(n)$  work to get  $\epsilon_0$ -map onto  $\Omega$ . Iterate  $\log \log \frac{1}{\epsilon}$  times to get  $\epsilon$ -map.

**Bern and Eppstein (1997):** Any  $n$ -gon has a quadrilateral mesh with angles  $\leq 120^\circ$ . At most  $O(n)$  points are added. Runs in  $O(n \log n)$ .

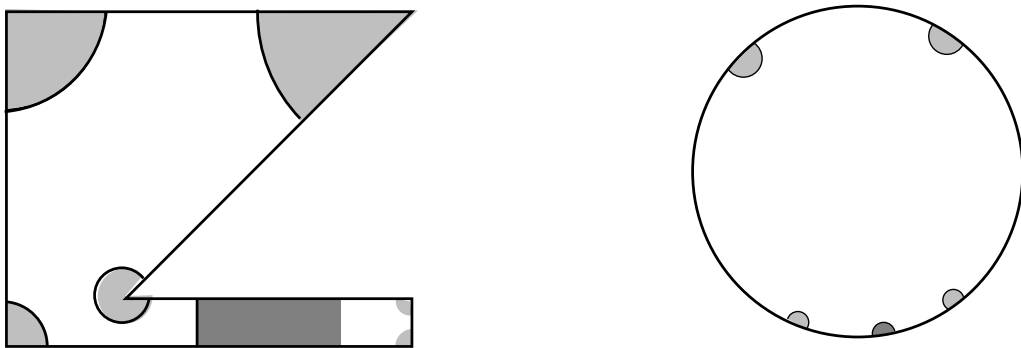


**Theorem:** Any  $n$ -gon has a  $O(n)$  quadrilateral mesh so that all new angles are between  $60^\circ$  and  $120^\circ$  and which can be computed in  $O(n)$ . Angle bounds are best possible.

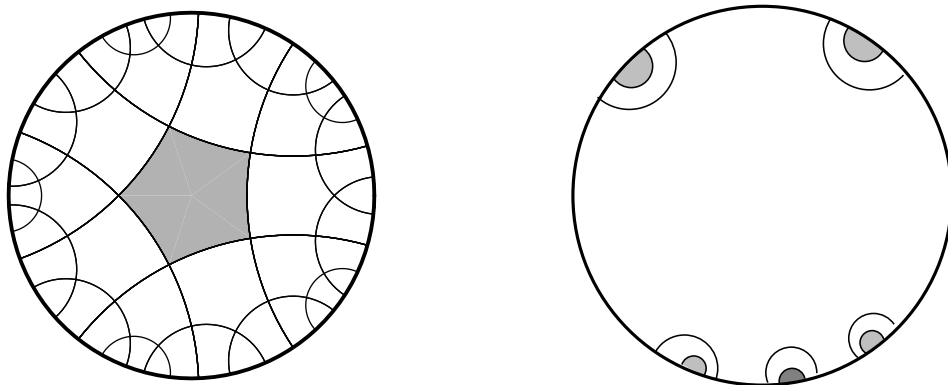
Uses fast mapping theorem, thick-thin decompositions, hyperbolic geometry.

## Idea of proof:

Divide polygon into thick and thin parts. Thick parts look piecewise smooth with  $90^\circ$  angles. Map to disk minus hyperbolic half-planes.

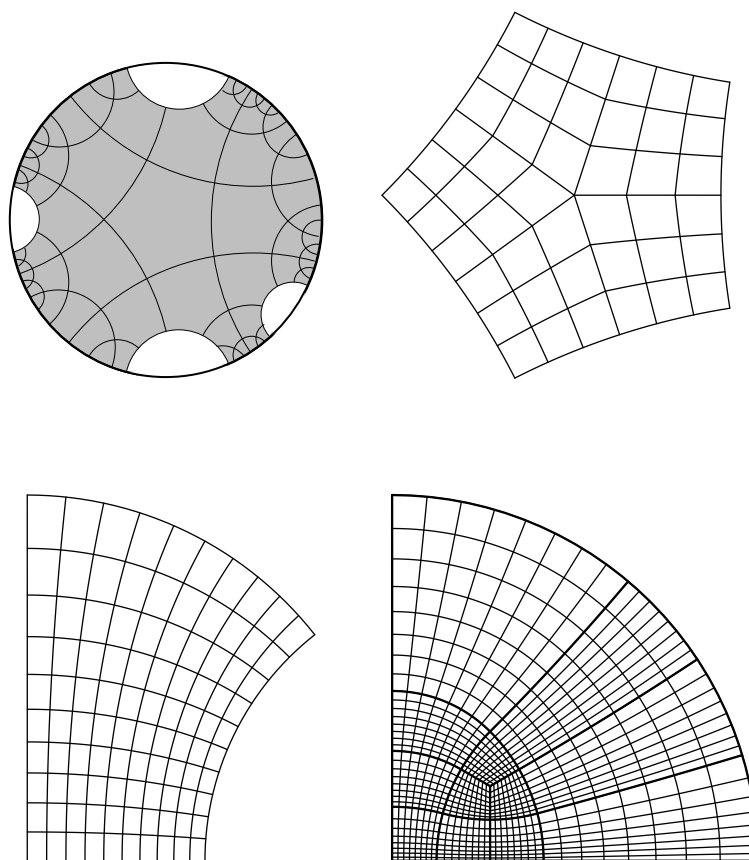


Disk has tessellation by hyperbolic right pentagons. Finite approximation divides disk into pentagons, triangles and quadrilaterals.



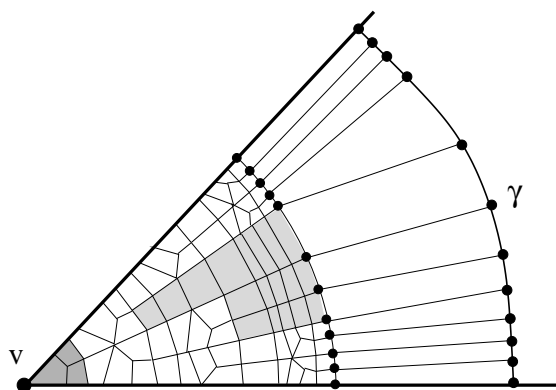
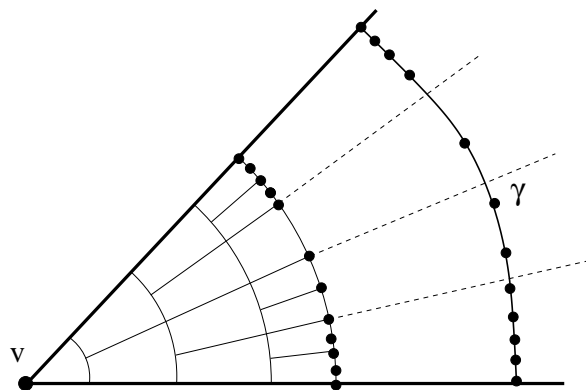


Each piece can be meshed consistently.



Mesh disk, conformal map to polygon, “snap” curves to line segments. This meshes thick parts. Then mesh thin parts and connect meshes.

Final step is to mesh the thin parts and connect these to meshing of thick part

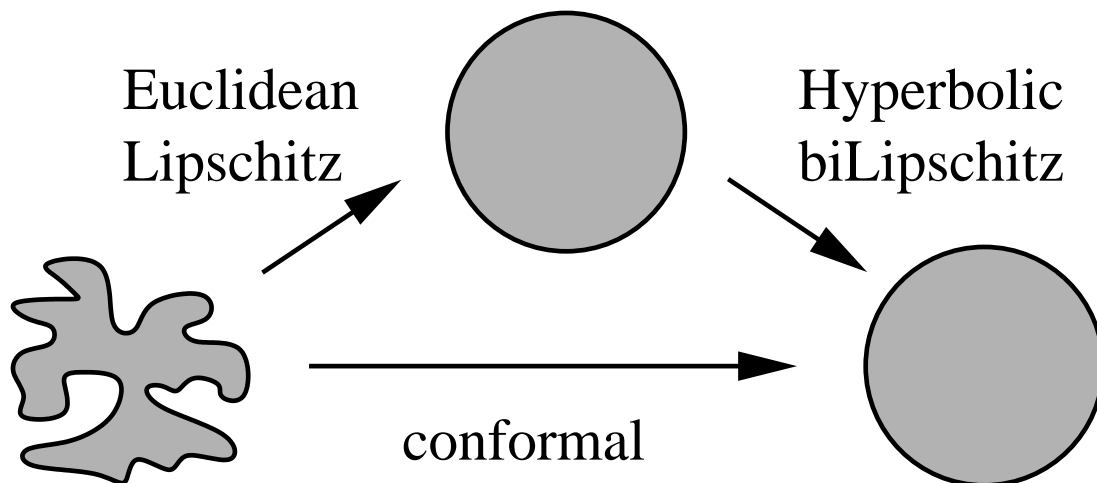


Medial Axis flow decreases boundary length. Thus

**The factorization theorem:** For any simply connected  $\Omega$  with inradius  $\geq 1$ , a Riemann map can be factored as  $f = h \circ g : \Omega \rightarrow \mathbb{D} \rightarrow \mathbb{D}$  where

- $g$  is locally Lipschitz (Euclidean metrics)
- $h$  is biLipschitz (hyperbolic metric)

The bounds are independent of the domain.



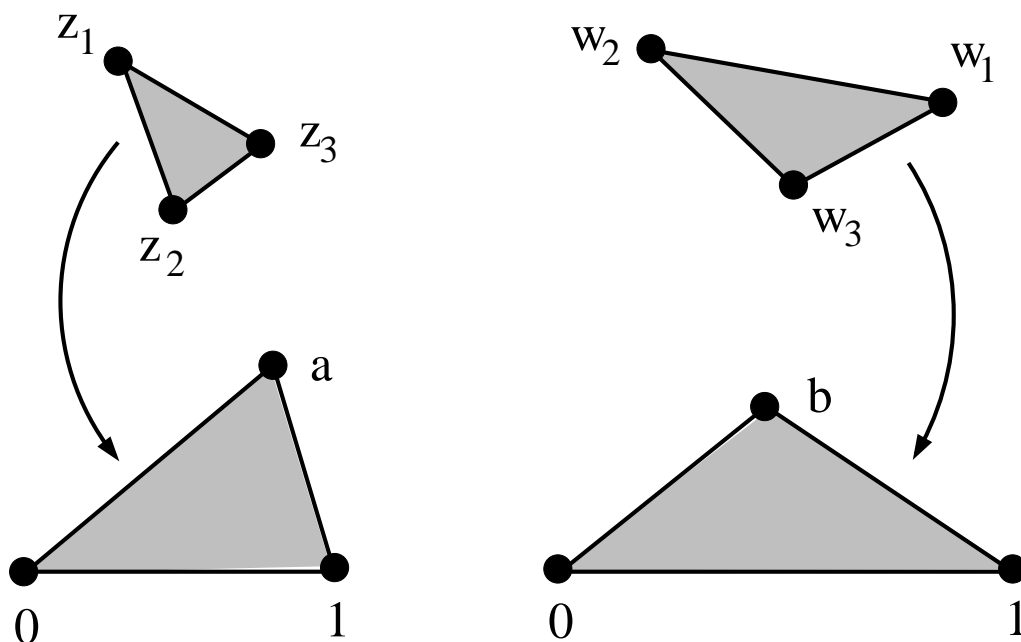
**Cor:** Any simply connected plane can be mapped to a disk by a homeomorphism which is contracting for internal path metric.

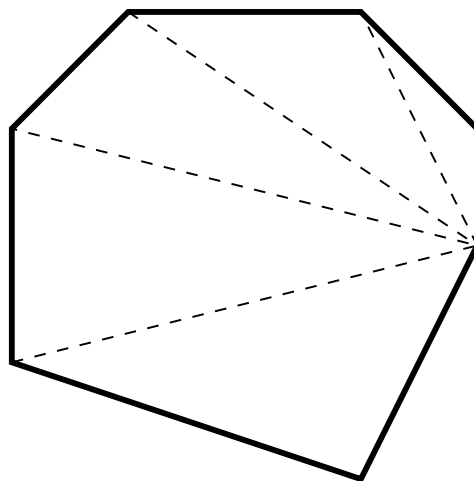
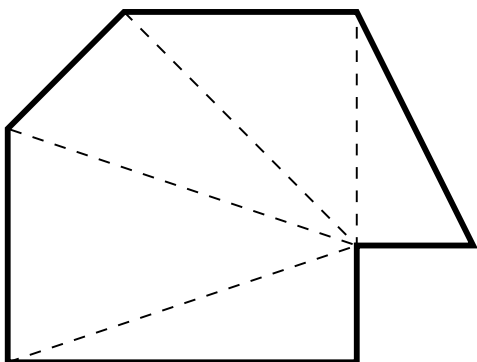
Can measure degree of distortion. For a triangles with vertices  $\{z_1, z_2, z_3\}$ ,  $\{w_1, w_2, w_3\}$ :

$$\mu = \frac{a - b}{b - \bar{a}}, \quad K = \frac{\mu + 1}{\mu - 1}$$

$$a = \frac{z_3 - z_1}{z_2 - z_1}, \quad b = \frac{w_3 - w_1}{w_2 - w_1}$$

This number measure minimum  $K$  so we can map first triangle to second by  $F \circ G \circ H$  where  $F, H$  are similarities and  $G : (x, y) \rightarrow (Kx, y)$ . triangles.





$$\mu = \max\{2.15501, 1.33333, 1.37016, 1.64039, 4.26556\}$$