OPTIMAL TRIANGULATION OF POLYGONS

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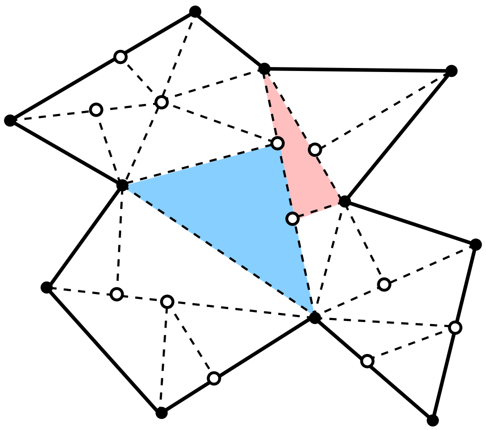
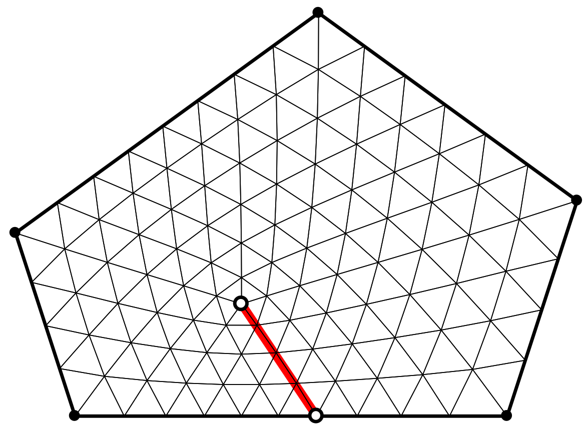
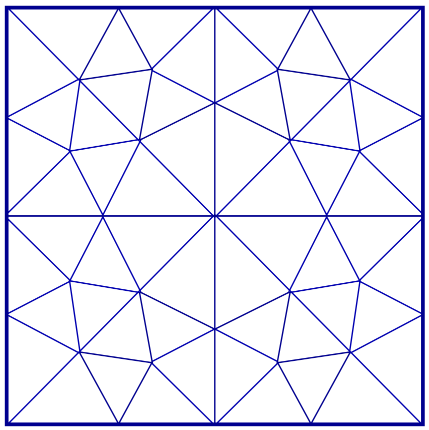
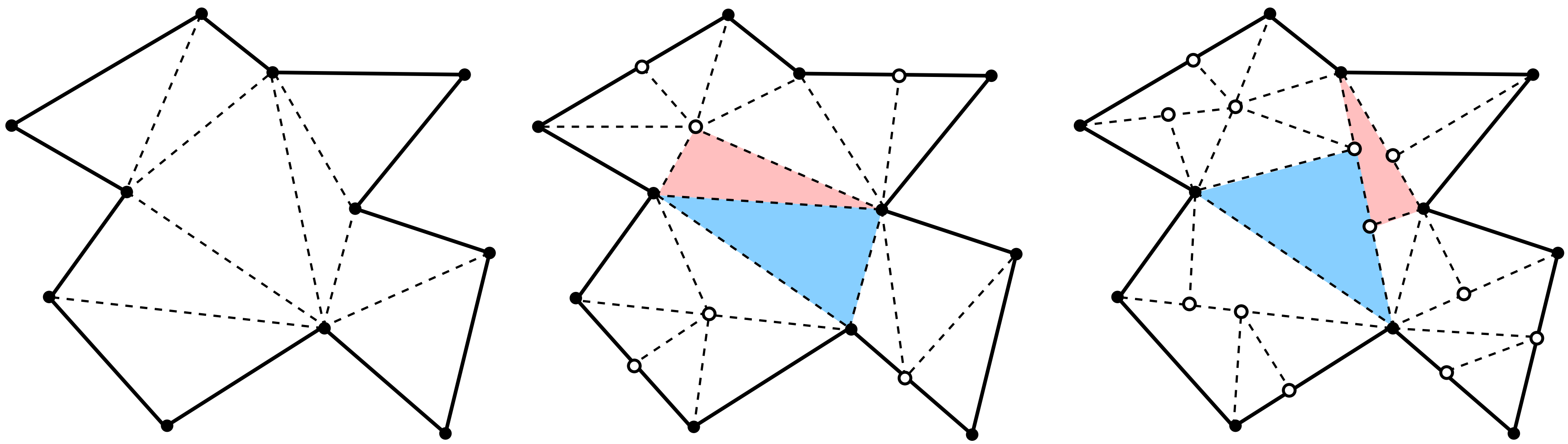






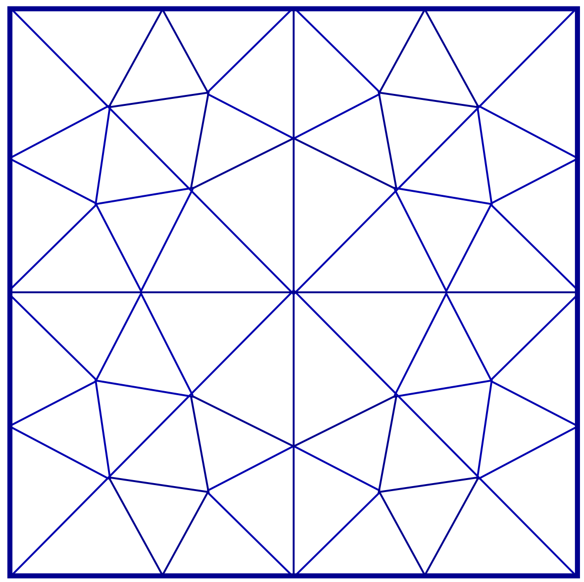
THE PLAN

- Definitions and history
- Necessary conditions
- The theorem and corollaries
- Sketch of proof
- Related results and open problems

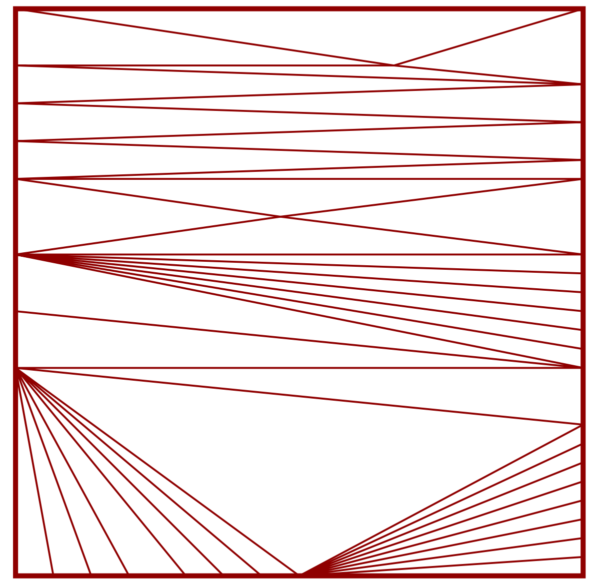


No Steiner Points   With Steiner Points   Dissection

Three types of triangulations



Good



Bad

Goal: make pieces as close to equilateral as possible.

Minimize the maximum angle (compute MinMax angle).

"Good" meshes improve performance of numerical methods.

Defn: acute triangle = all angles $< 90^\circ$.

Defn: nonobtuse triangle = all angles $\leq 90^\circ$.

Defn: ϕ-triangulation = all angles $\leq \phi$.

Defn: $\Phi(P) = \inf\{\phi : P \text{ has a } \phi\text{-triangulation}\}$.

Thm (Burago-Zalgaller, 1960): $\Phi(P) < 90^\circ$ all polygons.

"Every polygon has an acute triangulation."

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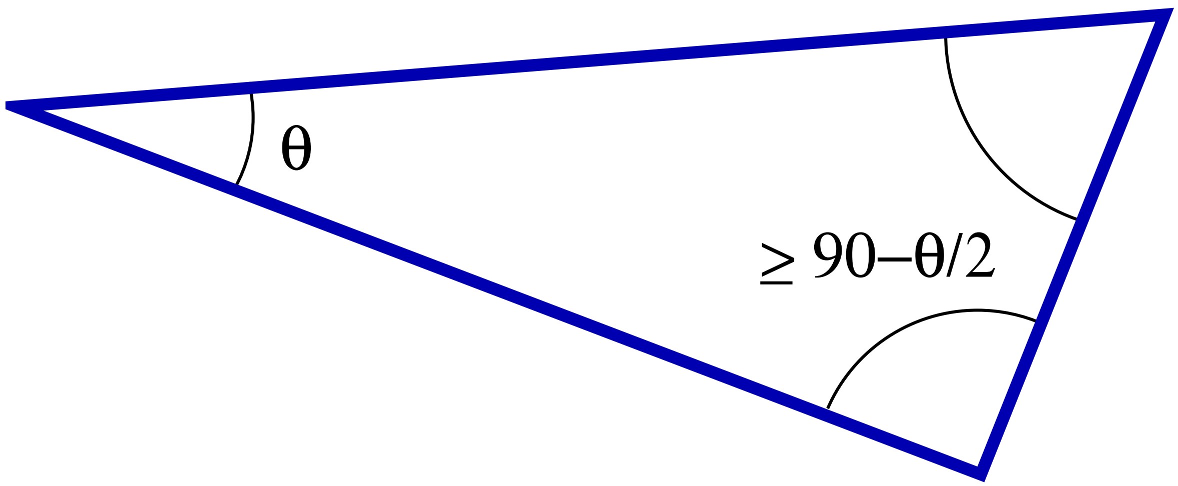
No bound $< 90^\circ$ works for **all** polygons.



Any triangle with an angle $\leq \theta$ also has an angle $\geq 90^\circ - \theta/2$.

Thm (Burago-Zalgaller, 1960): $\Phi(P) < 90^\circ$ all polygons.

"Every polygon has an acute triangulation."

Rediscovered by Baker-Grosse-Rafferty, 1988 (weaker version).

Much work on acute and non-obtuse triangulations by

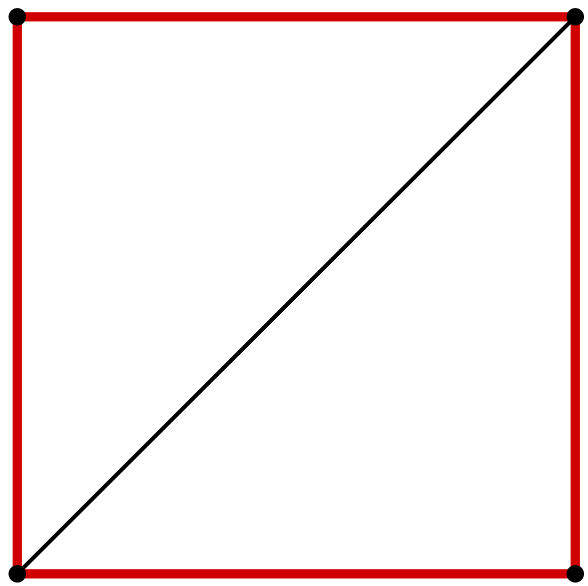
Barth,	Hirani,	Przytycki,	Tan,
Bern,	Itoh,	Ruppert,	Üngör,
Edelsbrunner,	Kopczyński,	Saalfeld,	VanderZee,
Eppstein,	Maehara,	Saraf,	Vavasis,
Erten,	S. Mitchell	Sheffer,	Yuan,
Gilbert,	Pak,	Shewchuk,	Zamfirescu,

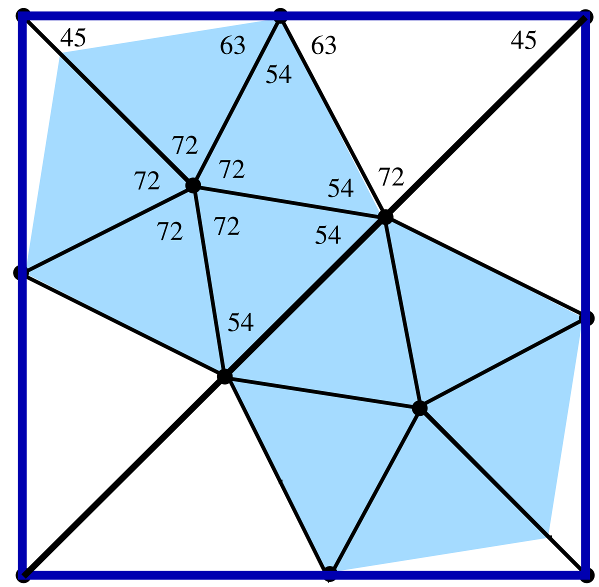
and many others (sorry if I omitted you).

Thm: every n-gon has an acute triangulation of size $O(n)$.

Burago-Zalgaller result first cited in CS literature around 2004.

Steiner points versus no Steiner points.





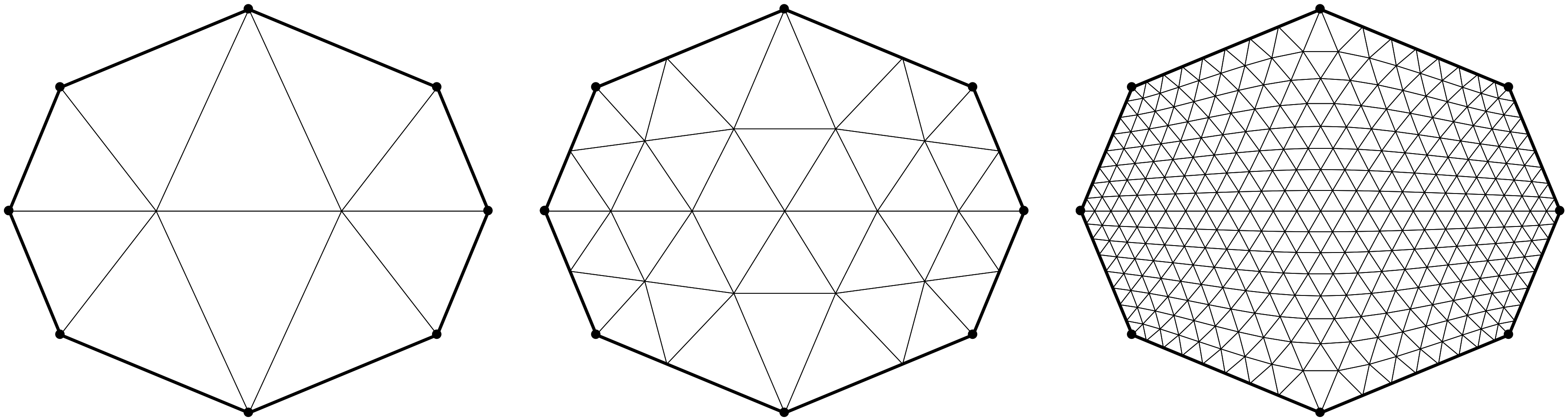
Consider triangulations of a square.

Without Steiner points, 90° is best angle bound (Delaunay triangulation).

Using Steiner points, a 72°-triangulation is possible.

We shall prove later this is best possible.

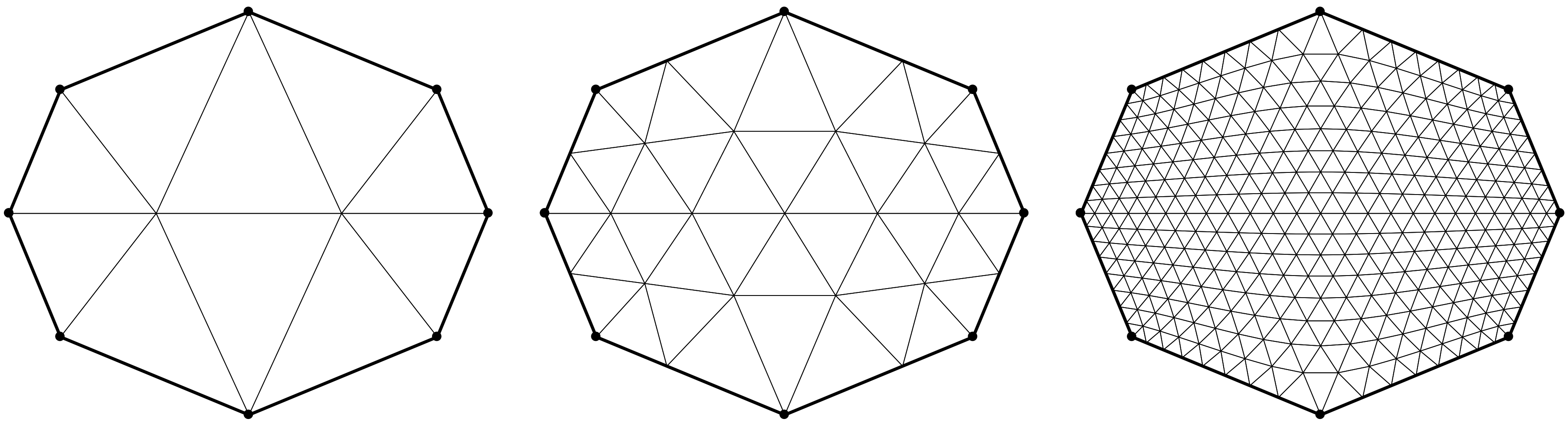
With Steiner points is the optimum attained?



$\theta_{\max} \approx 74.7482$ $\theta_{\max} \approx 70.3590$ $\theta_{\max} \approx 67.8690$

With Steiner points there are infinitely many possibilities.

Not obvious that optimal triangulation exists.



$\theta_{\max} \approx 74.7482$ $\theta_{\max} \approx 70.3590$ $\theta_{\max} \approx 67.8690$

With Steiner points there are infinitely many possibilities.

Not obvious that optimal triangulation exists.

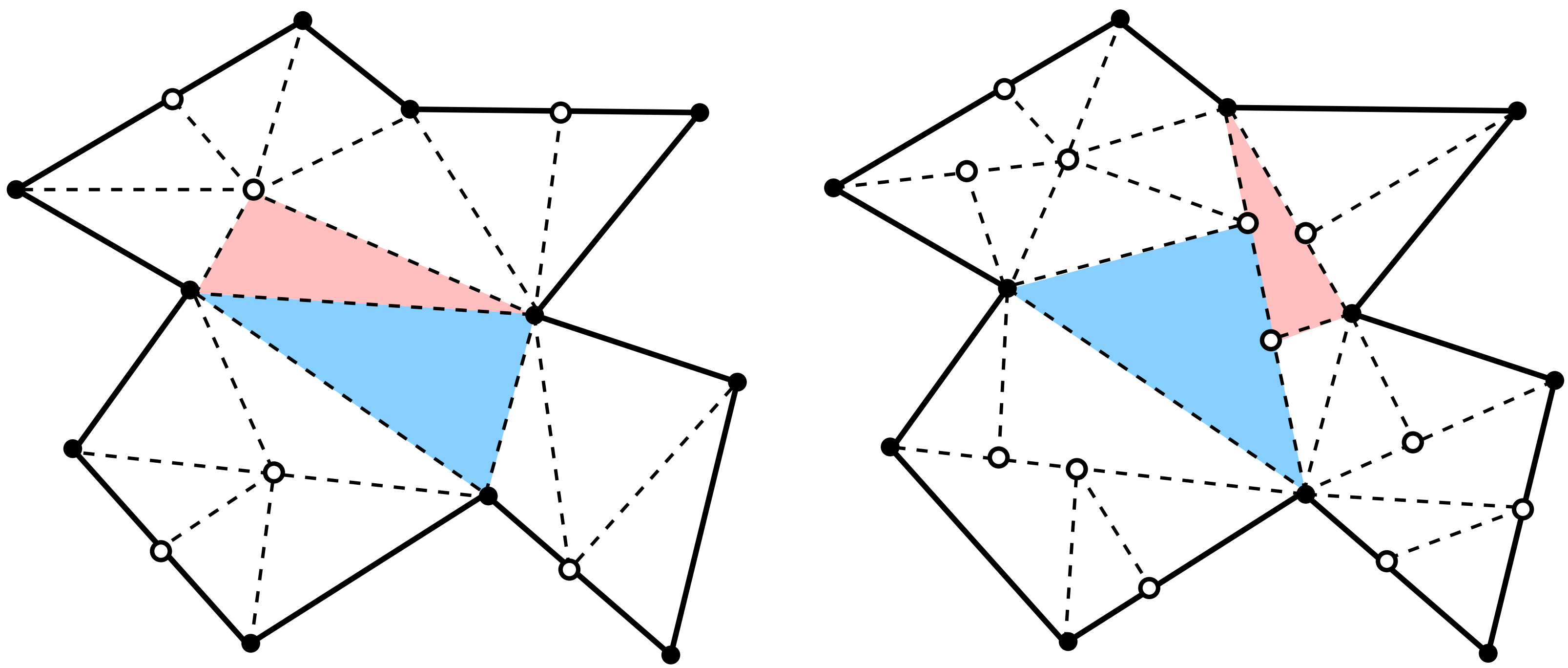
In case above, optimal angle is 67.5° and is attained.

But sometimes, the optimum bound is not achieved.

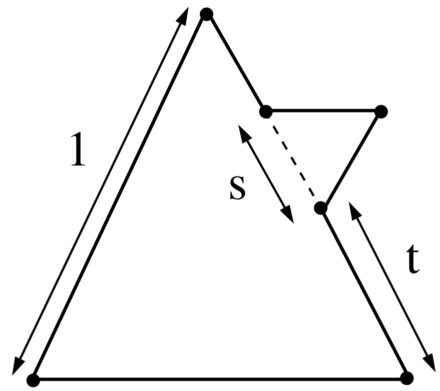
Dissections versus triangulations.

In a triangulation, triangles meet at vertices or full edges.

Dissections are more general, but do they give better angles?



Defn: 60°-polygon = all angles are multiples of 60°.



Polygon has a dissection into two equilateral triangles.

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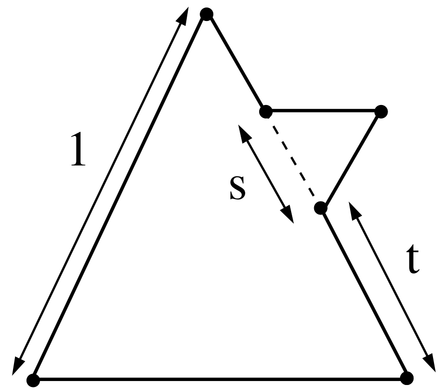
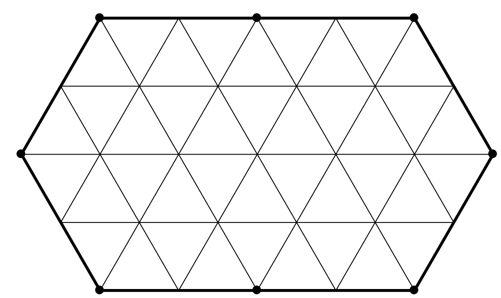


Polygon has a dissection into two equilateral triangles.

Claim: it need not have any equilateral triangulation.

Equilateral triangulation
$\Rightarrow$ all triangles the same size
$\Rightarrow$ edge lengths are integer multiples of triangle length
$\Rightarrow$ s/t is rational



Conclusion: a 60°-dissection exists, but a 60°-triangulation need not.

Lemma: For 60°-polygons $\Phi(P) = 60^\circ$.

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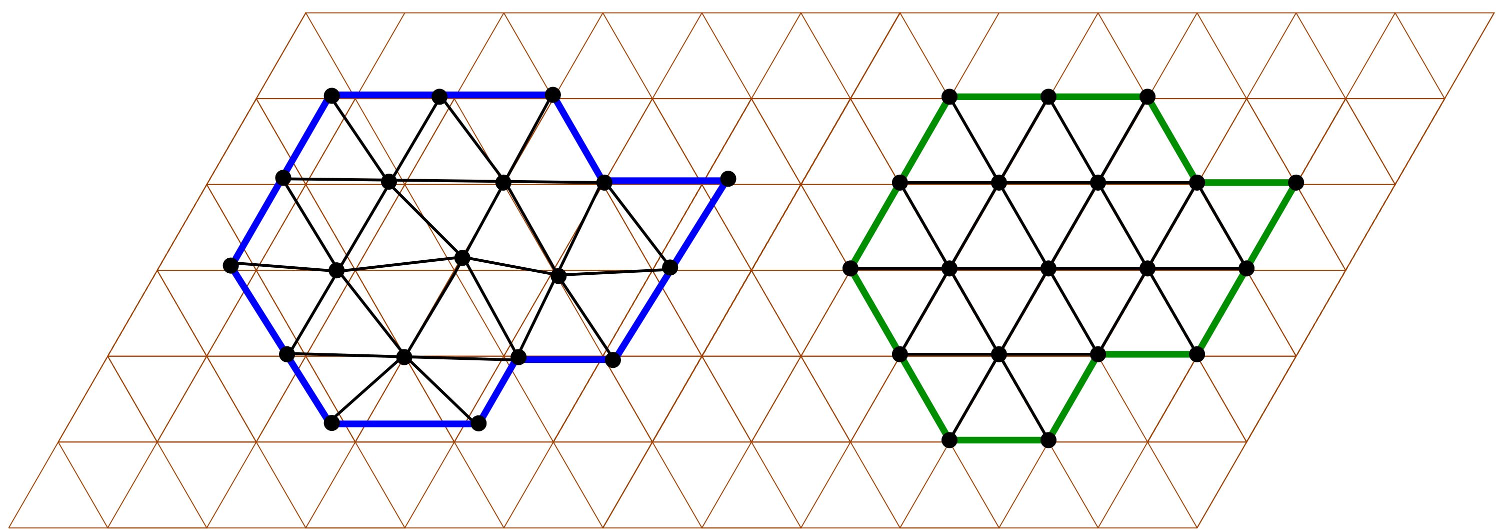
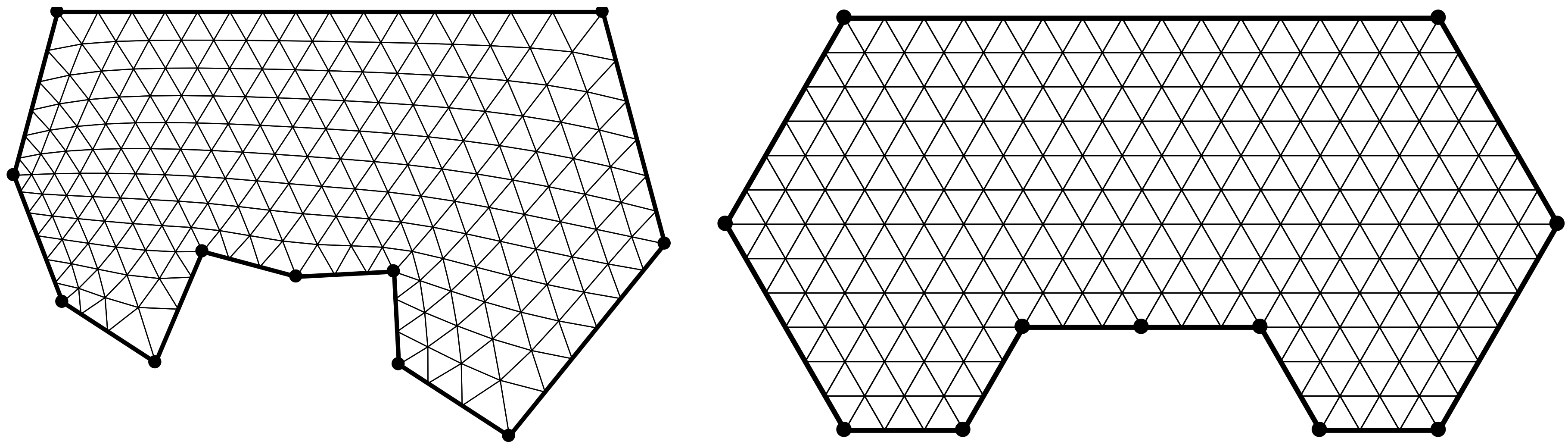
Sketch of Proof: Given P,

- Choose P' near P, with same angles and vertices on equilateral grid.
- Map both conformally to disk. Vertex images almost agree.
- Align vertex images using small QC distortion of disk.
- Get small QC map $P' \to P$. Image triangulation has angles near 60°.



Conclusion: in this case, triangulations = dissections + ϵ.

Main idea: conformal images of 60°-polygons



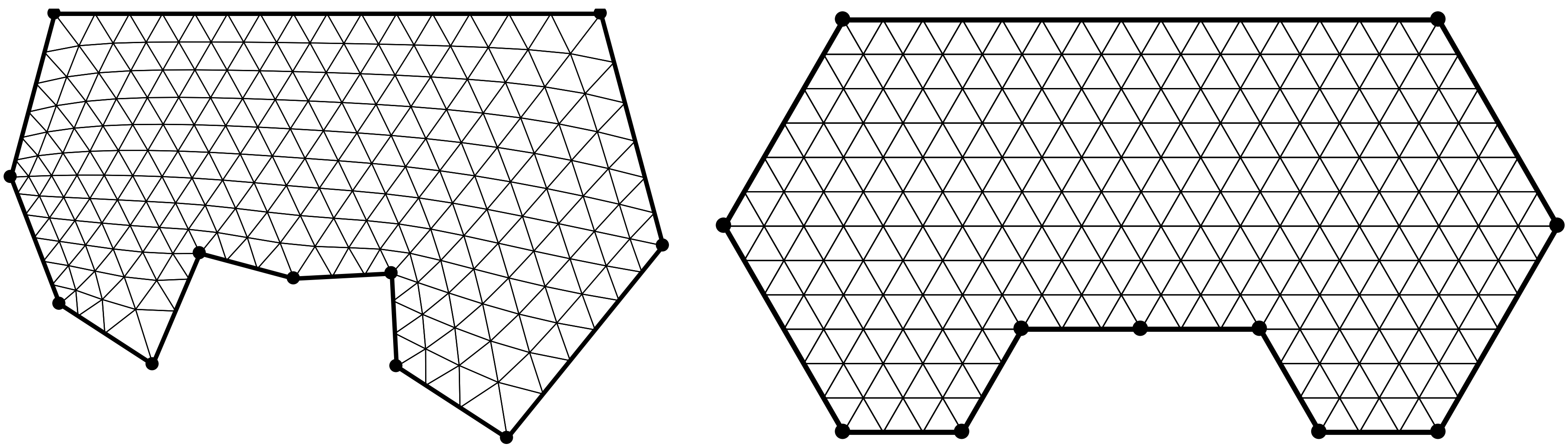
Given P, construct a 60°-polygon P' that "approximates" P.

Conformally map a nearly equilateral triangulation from P' to P.

Conformal = 1-1, holomorphic = preserves angles infinitesimally.

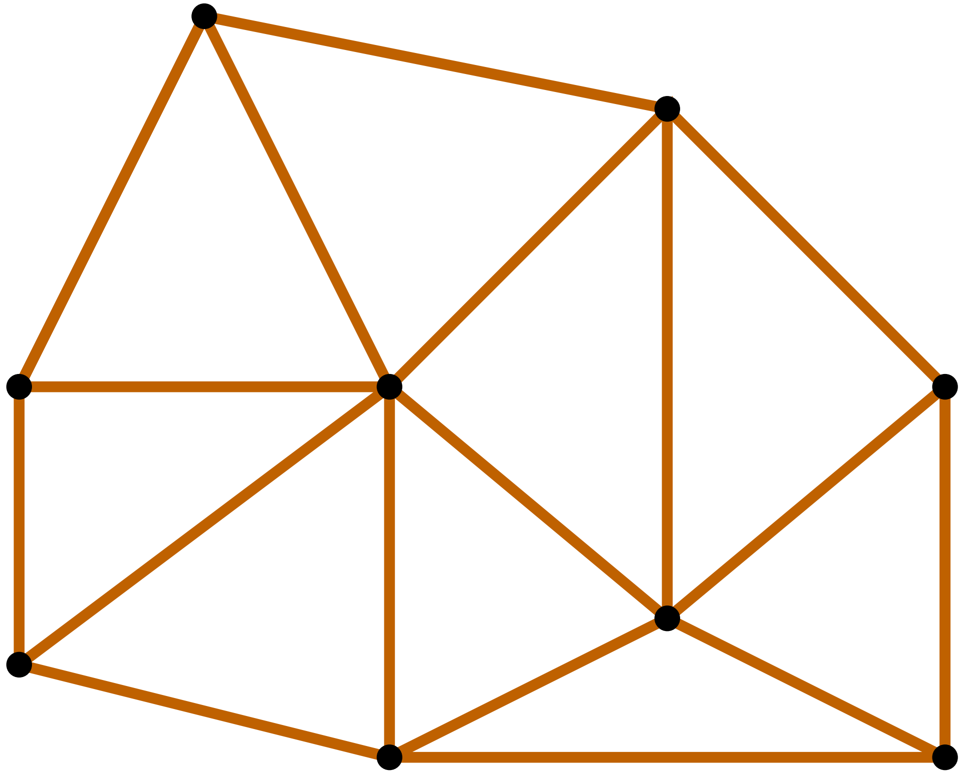
Map only vertices; then connect by segments. (Edge images are curved).

Main idea: conformal images of 60°-polygons



Problems to overcome (among others):

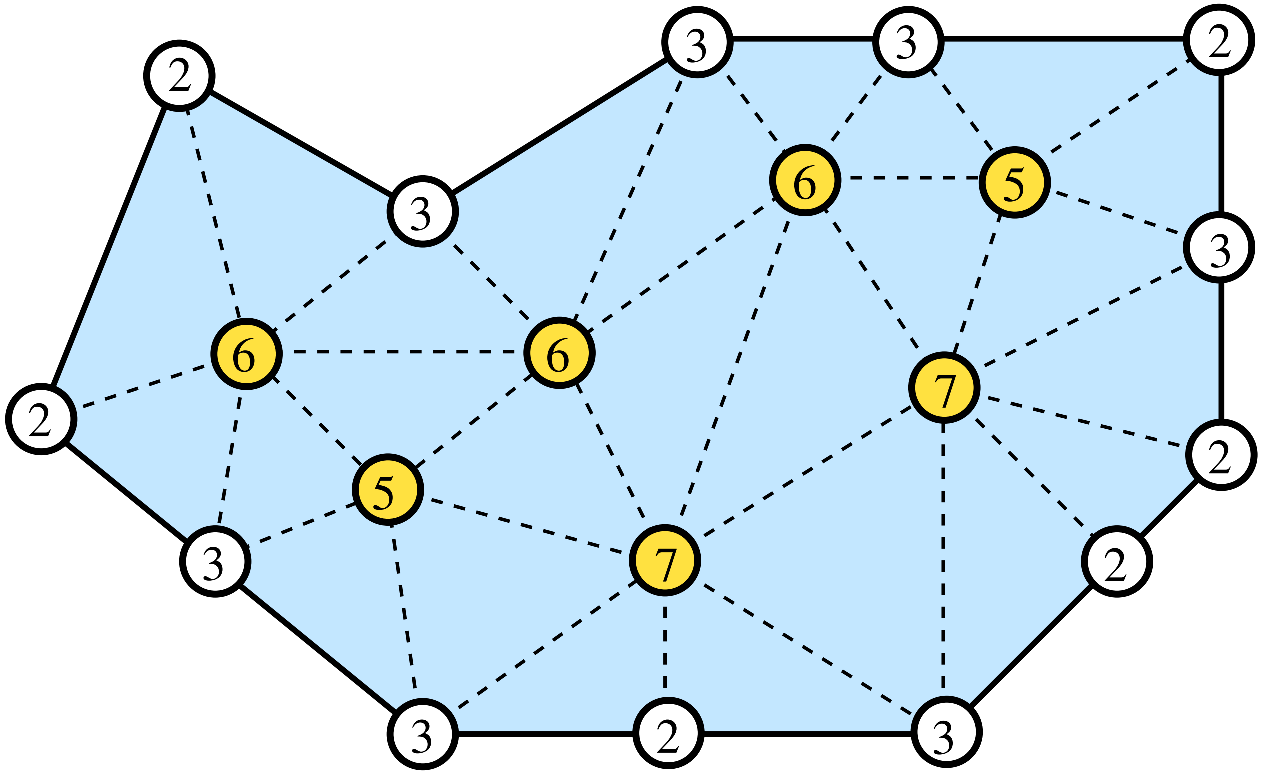
- must map vertices to vertices,
- bound angle distortion at positive scales,
- attain sharp bounds versus approximate them,
- **Euler's formula may force vertices of degree 5 or 7.**



Euler's formula: $F - E + V = 1$

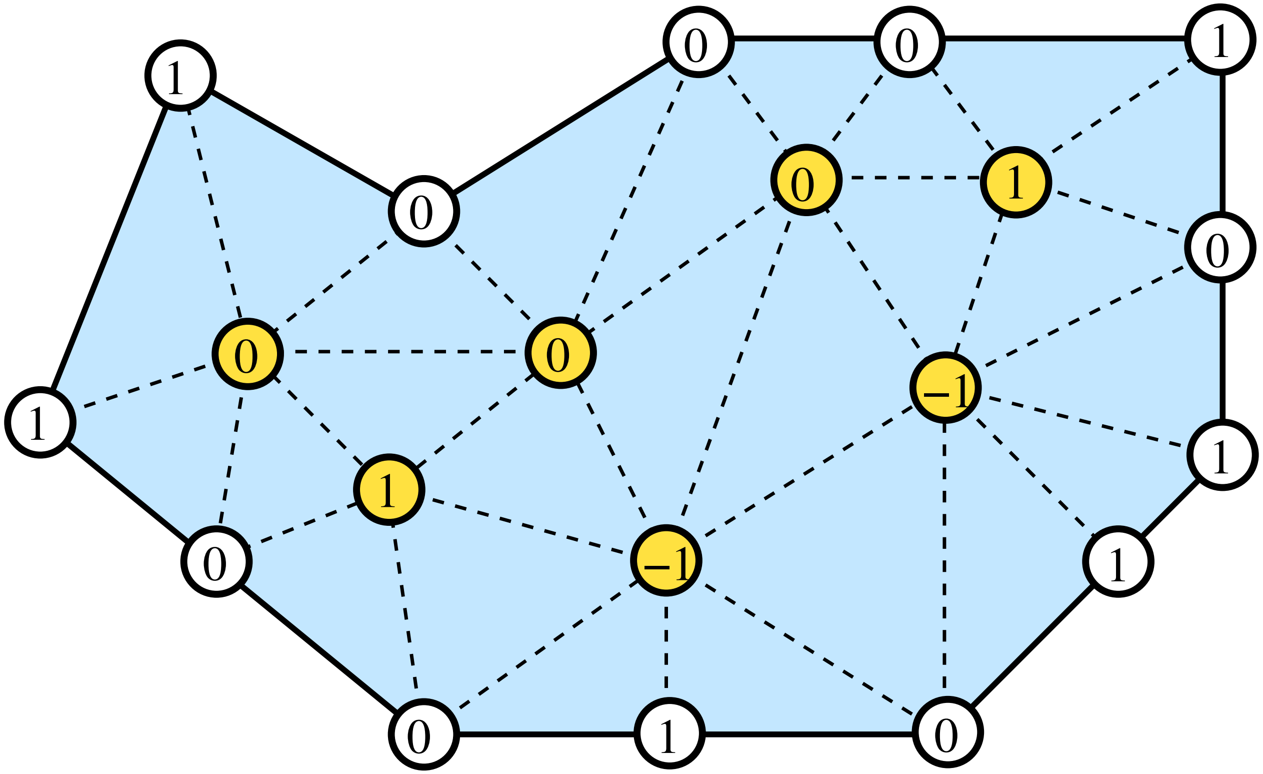
Faces - Edges + Vertices = 1

9 -17+9 = 1



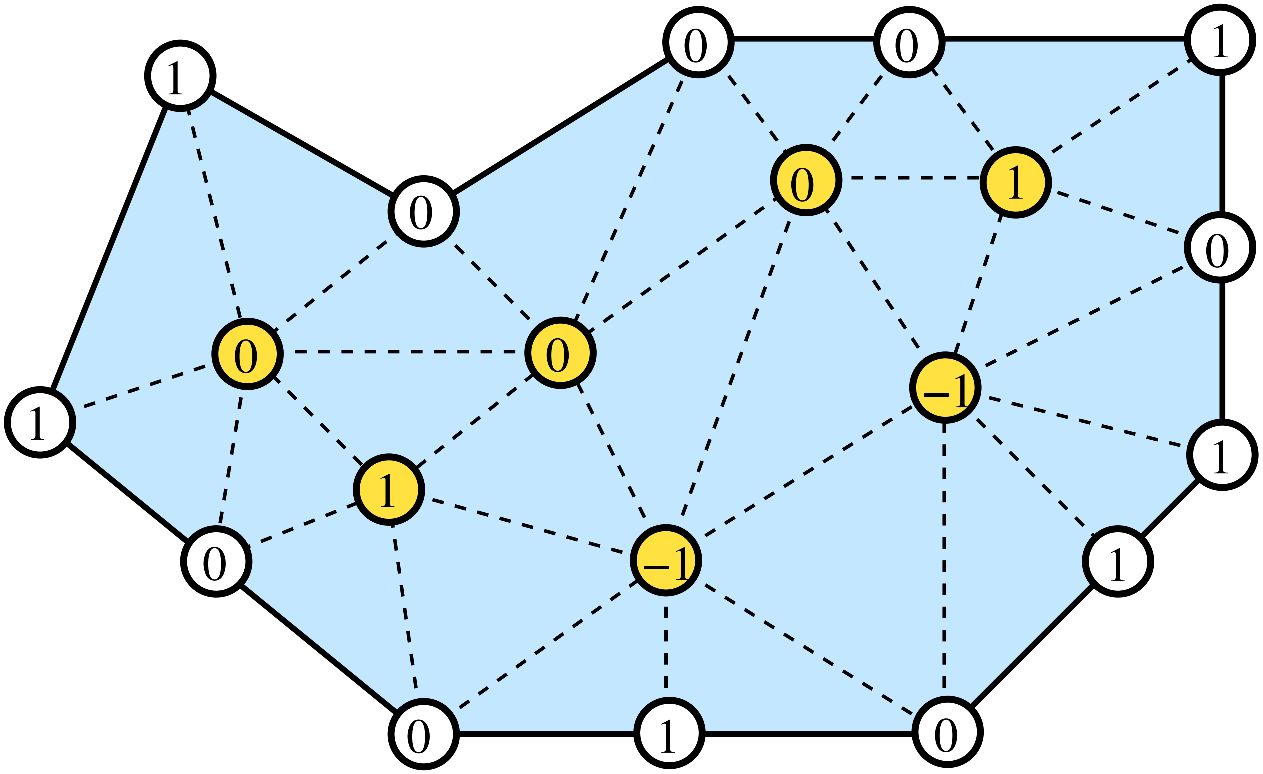
Let $L(v)$ = number of triangles with v as vertex.

In particular, this gives a labeling of P by positive integers.



Curvature of boundary vertex v: $\kappa(v) = 3 - L(v)$.

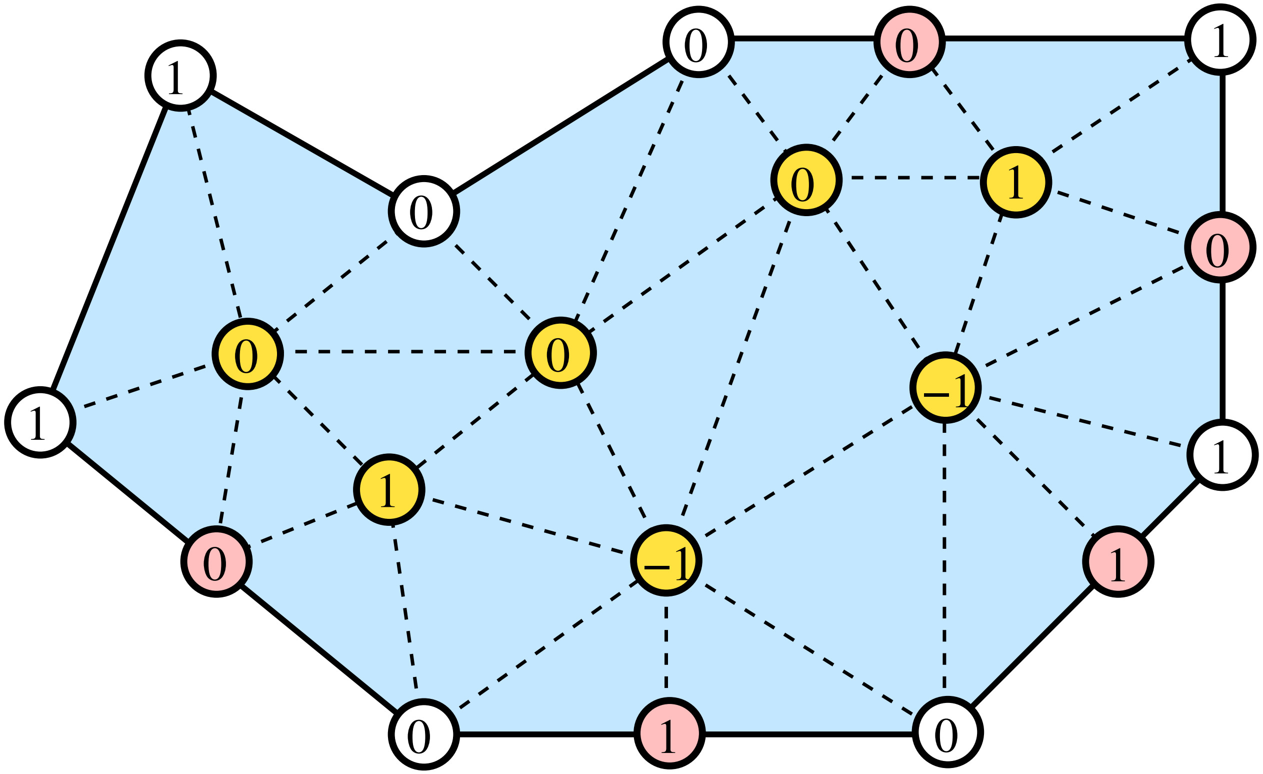
Curvature of interior vertex v: $\kappa(v) = 6 - L(v)$.



Euler's formula can be rewritten to look like Gauss-Bonnet:

$$\sum_{v \in \text{interior}} \kappa(v) = 6 - \sum_{v \in \text{boundary}} \kappa(v)$$

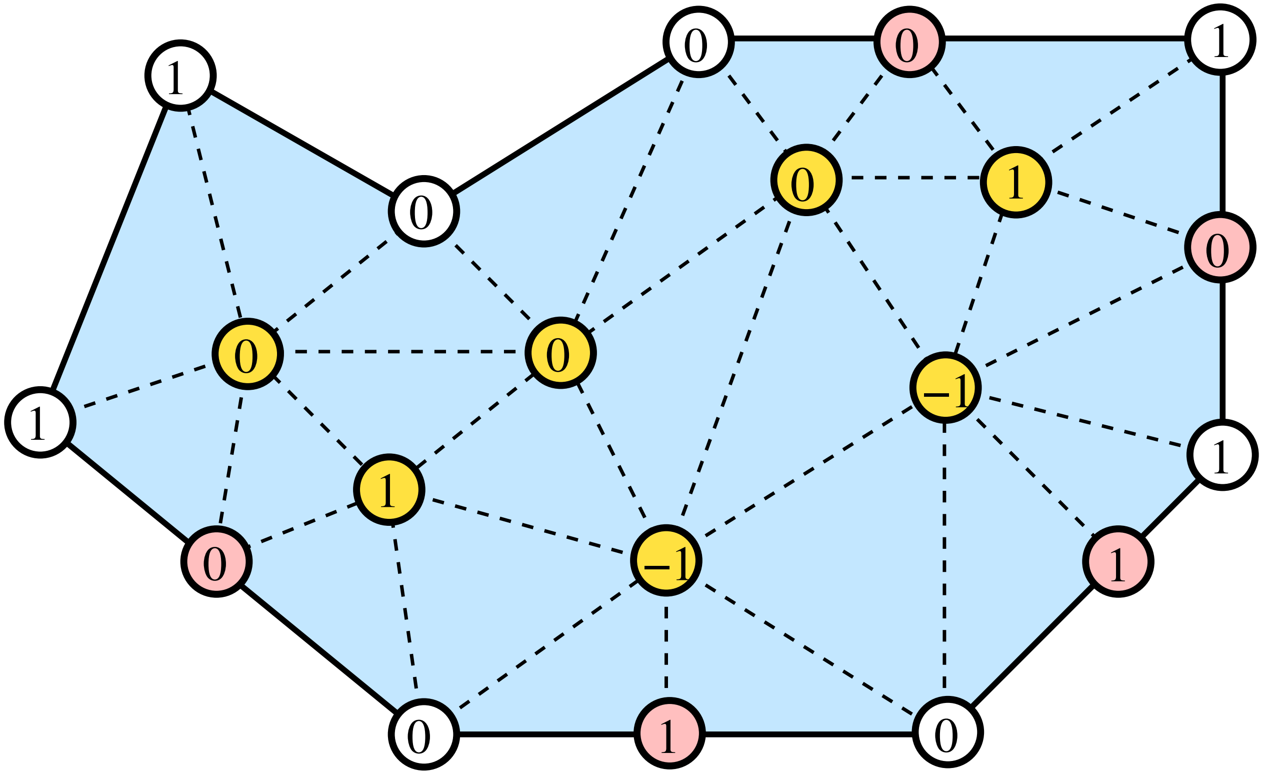
$$\kappa(\mathcal{T}) = 6 - \kappa(\partial\mathcal{T})$$



Define curvature of labeling L of vertices V of P (omit Steiner points):

$$\kappa(L) = 6 - \sum_{v \in P} \kappa(v) = 6 - 3|V| + \sum_{v \in P} L(v).$$

Labelings of ϕ-triangulations have certain curvature restrictions.

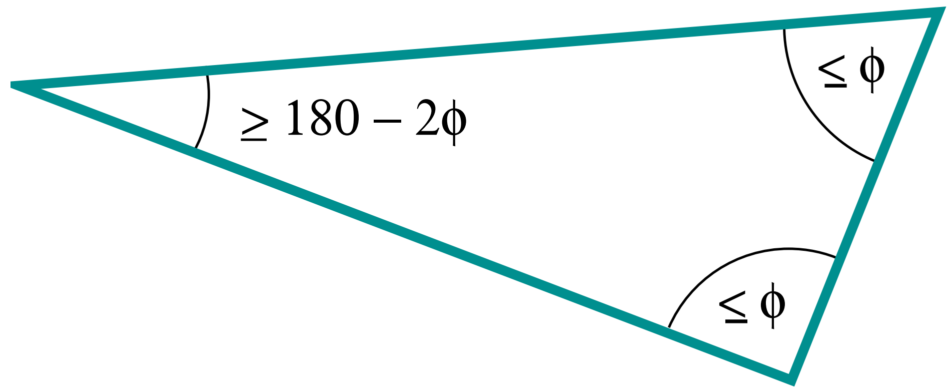


For acute triangulations (angles $< 90°$) we must have

$$\kappa(L) \leq \kappa(\mathcal{T})$$

since omitted boundary Steiner points have $L(v) \geq 3 \Rightarrow \kappa(v) \leq 0$.

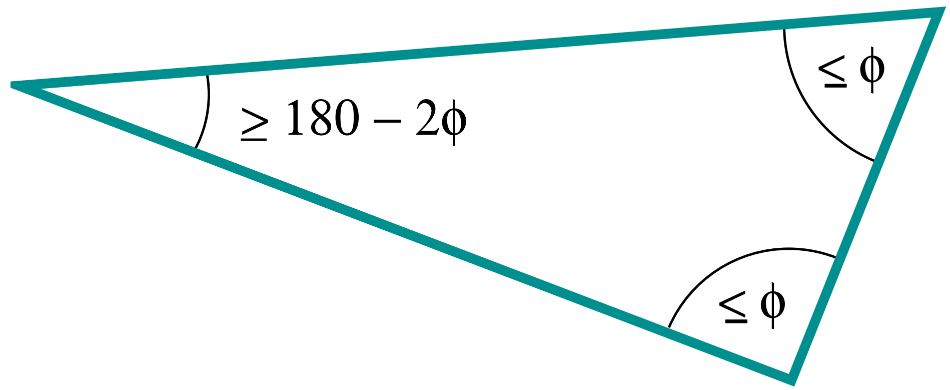
If a triangle has all angles $\leq \phi$, then all angles are $\geq 180^\circ - 2\phi$.



If a ϕ-triangulation has $L(v)$ triangles at vertex $v \in P$ of angle θ_v, then

$$L(v) \cdot (180^\circ - 2\phi) \;\leq\; \theta_v \;\leq\; L(v) \cdot \phi.$$

If a triangle has all angles $\leq \phi$, then all angles are $\geq 180^\circ - 2\phi$.



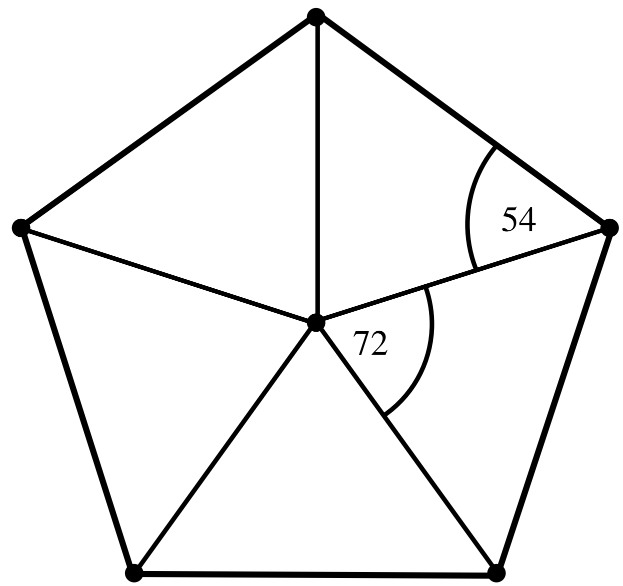
If a ϕ-triangulation has $L(v)$ triangles at vertex $v \in P$ of angle θ_v, then

$$L(v) \cdot (180^\circ - 2\phi) \;\leq\; \theta_v \;\leq\; L(v) \cdot \phi.$$

Defn: A labeling L of P is a **ϕ-labeling** if these inequalities hold, i.e.,

$$\frac{\theta_v}{\phi} \;\leq\; L(v) \;\leq\; \frac{\theta_v}{180^\circ - 2\phi}.$$

By definition, every ϕ-triangulation gives a ϕ-labeling.

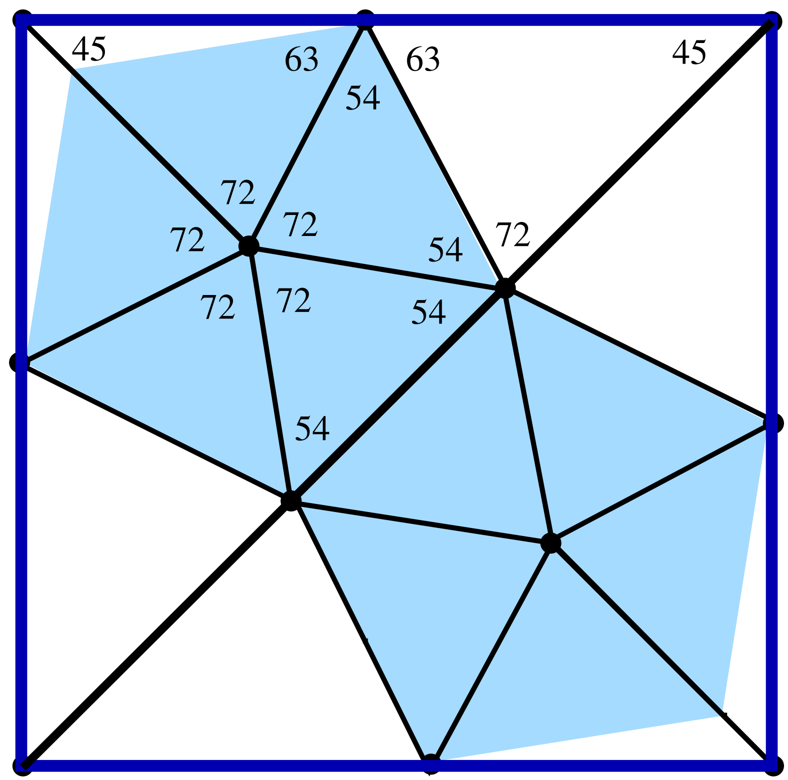


Suppose labeling L corresponds to a ϕ-triangulation. Then

- $\phi < 72^\circ \Rightarrow$ no degree ≤ 5 vertices $\Rightarrow \kappa(L) \leq \kappa(\mathcal{T}) \leq 0$.

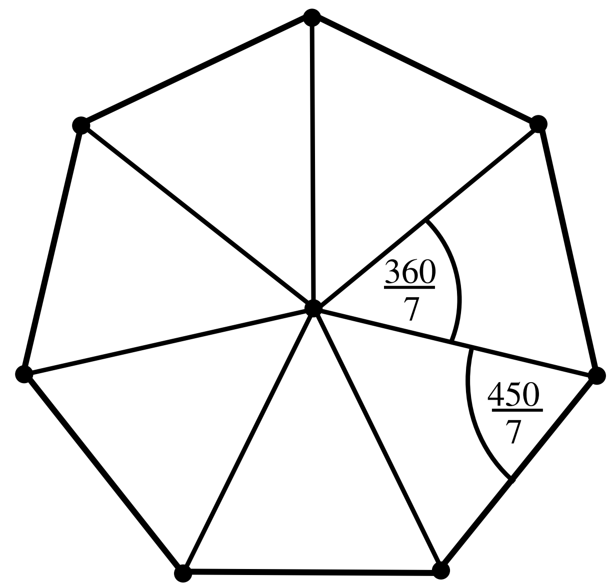
Example: for a square, $\Phi(P) = 72^\circ$.

Get $\Phi(P) \leq 72^\circ$ by explicit construction:



Converse:

- Suppose P has a ϕ-triangulation with $\phi < 72^\circ$.
- Euler $\Rightarrow$ there is a ϕ-labeling L of corners with $\kappa(L) \leq 0$.
- $\Rightarrow \kappa(L) = 6 - \sum \kappa(v) \leq 0$, so $\kappa(v) \geq 2$ for some corner.
- $\Rightarrow \kappa(v) = 3 - L(v) \geq 2$, so $L(v) = 1$.
- $\Rightarrow$ the triangulation has a 90° angle at corner $v \Rightarrow\Leftarrow$.



Suppose labeling L corresponds to a ϕ-triangulation. Then

- $\phi < (450/7)^\circ \Rightarrow$ no degree ≥ 7 interior vertices and only degree 3 Steiner boundary vertices $\Rightarrow \kappa(L) = \kappa(\mathcal{T}) = 0.$

To summarize: if a polygon P has a ϕ-triangulation, then the vertex set has ϕ-labeling L. Moreover, there is a labeling so that

$$\kappa(L) \leq 0 \text{ if } \phi < 72^\circ,$$

$$\kappa(L) = 0 \text{ if } \phi < (450/7)^\circ.$$

These necessary conditions observed by Joseph Gerver in 1984.

Theorem: For $60° < \phi < 90°$, a polygon P has a ϕ-triangulation **iff**

1. $72° \leq \phi < 90°$ and P has a ϕ-labeling L of V_P,
2. $\frac{5}{7} \cdot 90° \leq \phi < 72°$, and P has a ϕ-labeling with $\kappa(L) \leq 0$,
3. $60° < \phi < \frac{5}{7} \cdot 90°$, and P has a ϕ-labeling with $\kappa(L) = 0$.

Theorem: For $60^\circ < \phi < 90^\circ$, a polygon P has a ϕ-triangulation iff

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3. $60^\circ < \phi < \frac{5}{7} \cdot 90^\circ$, and P has a ϕ-labeling with $\kappa(L) = 0$.

Corollary: For $\phi > 60^\circ$, the following are equivalent:
(1) P has a $(\phi + \epsilon)$-triangulation for all $\epsilon > 0$.
(2) P has a ϕ-triangulation.

Equivalent: If P is not a 60°-polygon, then the angle bound $\Phi(P)$ is attained by some triangulation of P.

Theorem: For $60^\circ < \phi < 90^\circ$, a polygon P has a ϕ-triangulation iff

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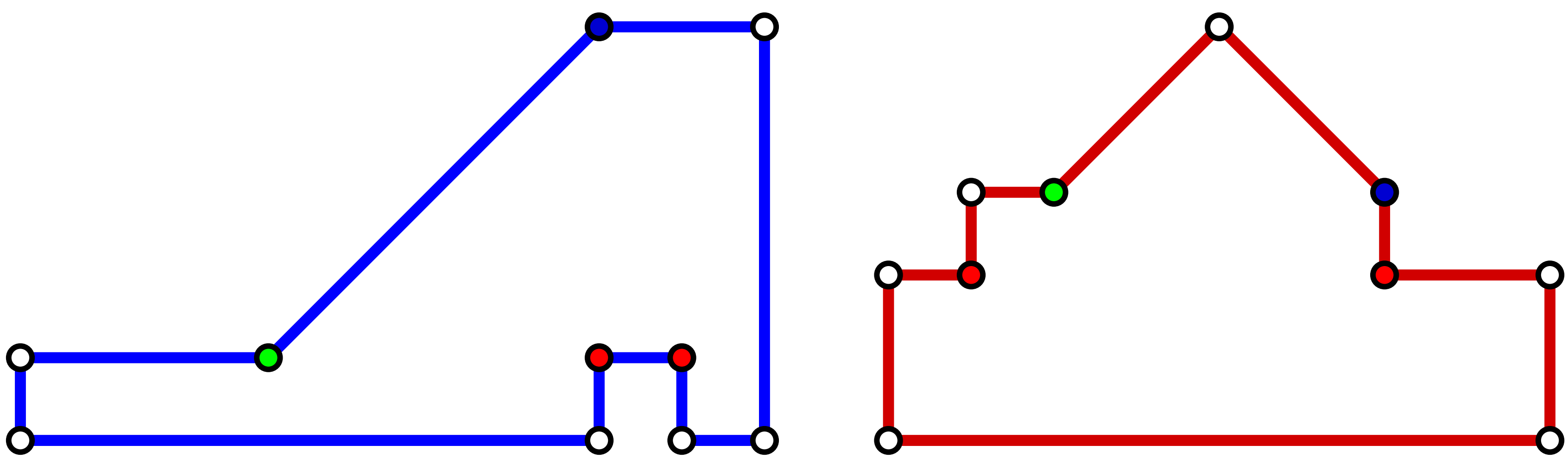
2. $\frac{5}{7} \cdot 90^\circ \leq \phi < 72^\circ$, and P has a ϕ-labeling with $\kappa(L) \leq 0$,
3. $60^\circ < \phi < \frac{5}{7} \cdot 90^\circ$, and P has a ϕ-labeling with $\kappa(L) = 0$.

Cor: $\Phi(P)$ only depends on set of angles. Not order or edge lengths.



Theorem: For $60^\circ < \phi < 90^\circ$, a polygon P has a ϕ-triangulation iff

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3. $60^\circ < \phi < \frac{5}{7} \cdot 90^\circ$, and P has a ϕ-labeling with $\kappa(L) = 0$.

Gerver (1984) proved necessity when P only has ϕ-dissection.

Corollary: For $\phi > 60^\circ$, the following are equivalent:

(1) P has a ϕ-dissection.

(2) P has a ϕ-triangulation.

$\Rightarrow$ **Dissections and triangulations give same angle bound.**

Theorem: For $60^\circ < \phi < 90^\circ$, a polygon P has a ϕ-triangulation iff

1. $72^\circ \leq \phi < 90^\circ$ and P has a ϕ-labeling L of V_P,
2. $\frac{5}{7} \cdot 90^\circ \leq \phi < 72^\circ$, and P has a ϕ-labeling with $\kappa(L) \leq 0$,
3. $60^\circ < \phi < \frac{5}{7} \cdot 90^\circ$, and P has a ϕ-labeling with $\kappa(L) = 0$.

Some geometric consequences of the proof:

Cor: If $\theta_{\text{min}} \geq 36^\circ$, then $\Phi(P) \leq 72^\circ$.

Cor: If $\theta_{\text{min}} \leq 36^\circ$, then $\Phi(P) = 90^\circ - \frac{1}{2}\theta_{\text{min}}$.

Cor: $\Phi(P) = 72^\circ$ for any axis-parallel polygon.

Cor: If $\theta_{\text{min}} \geq 144^\circ$ then $\Phi(P) = 72^\circ$. (This has interior!)

Theorem: For $60^\circ < \phi < 90^\circ$, a polygon P has a ϕ-triangulation iff

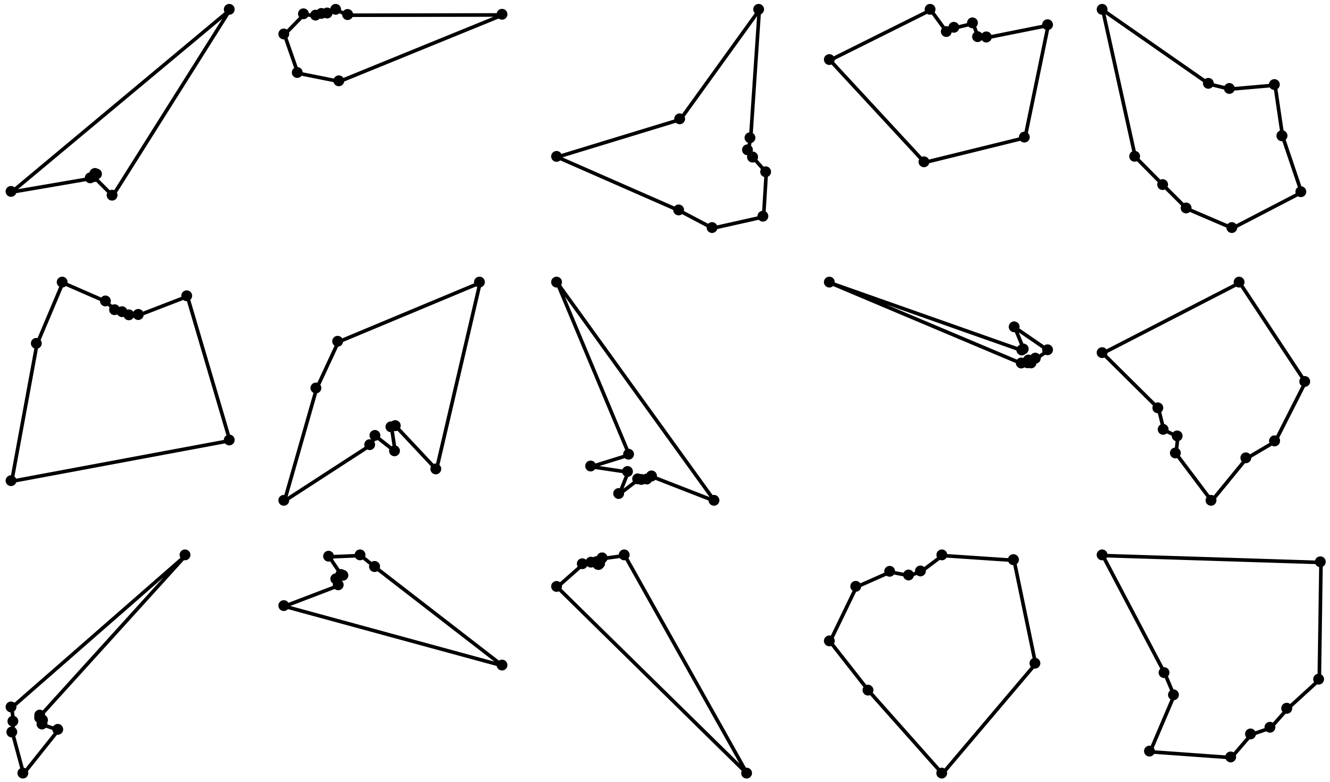
1. $72^\circ \leq \phi < 90^\circ$ and P has a ϕ-labeling L of V_P,
2. $\frac{5}{7} \cdot 90^\circ \leq \phi < 72^\circ$, and P has a ϕ-labeling with $\kappa(L) \leq 0$,
3. $60^\circ < \phi < \frac{5}{7} \cdot 90^\circ$, and P has a ϕ-labeling with $\kappa(L) = 0$.

Cor: Put a topology on n-gons by thinking of them as a subset of $\mathbb{R}^{2n}$.

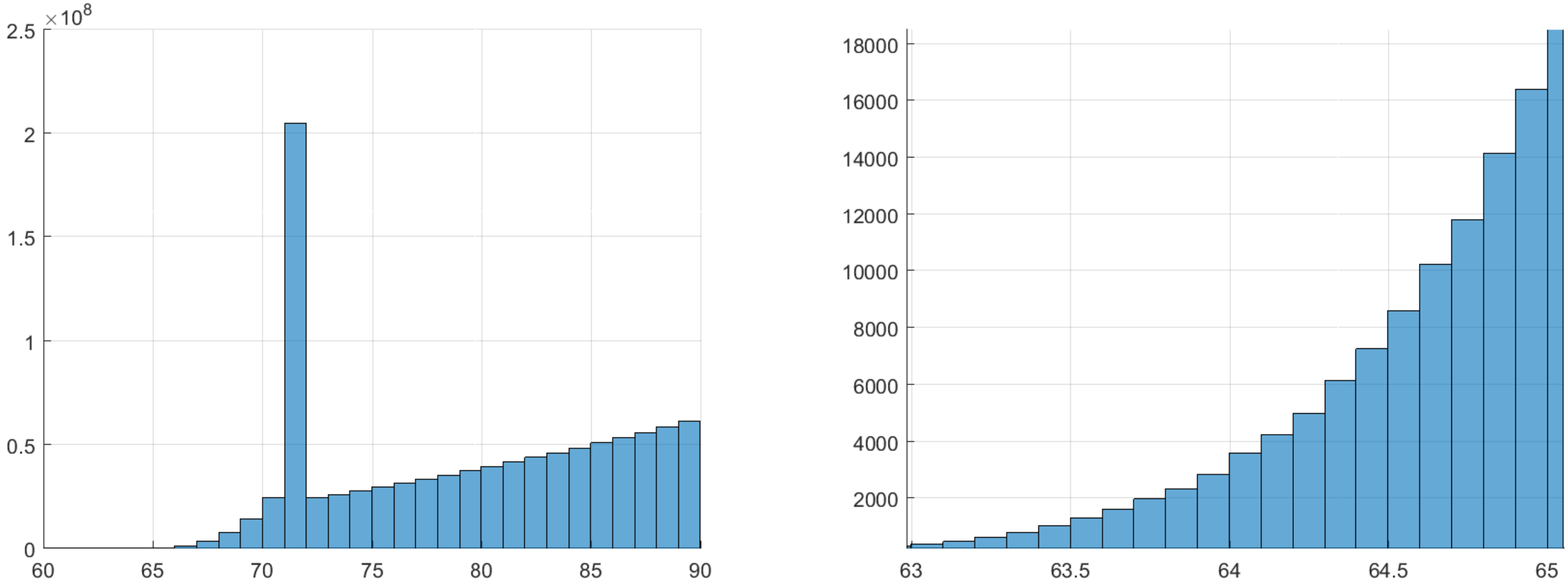
(a) The map $P \to \Phi(P)$ is continuous, so $\{P : \Phi(P) = \phi\}$ is closed.

(b) This set has non-empty interior iff $\phi = \frac{5}{7} \cdot 90^\circ$ or $\phi = 72^\circ$.

(c) Otherwise it has co-dimension ≥ 1.



Generating random 10-gons.

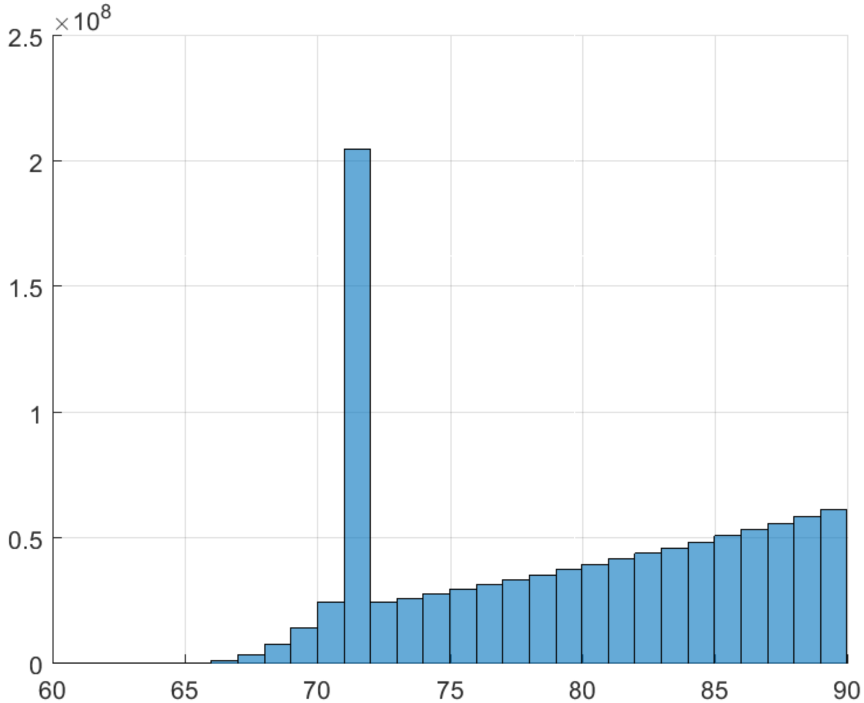


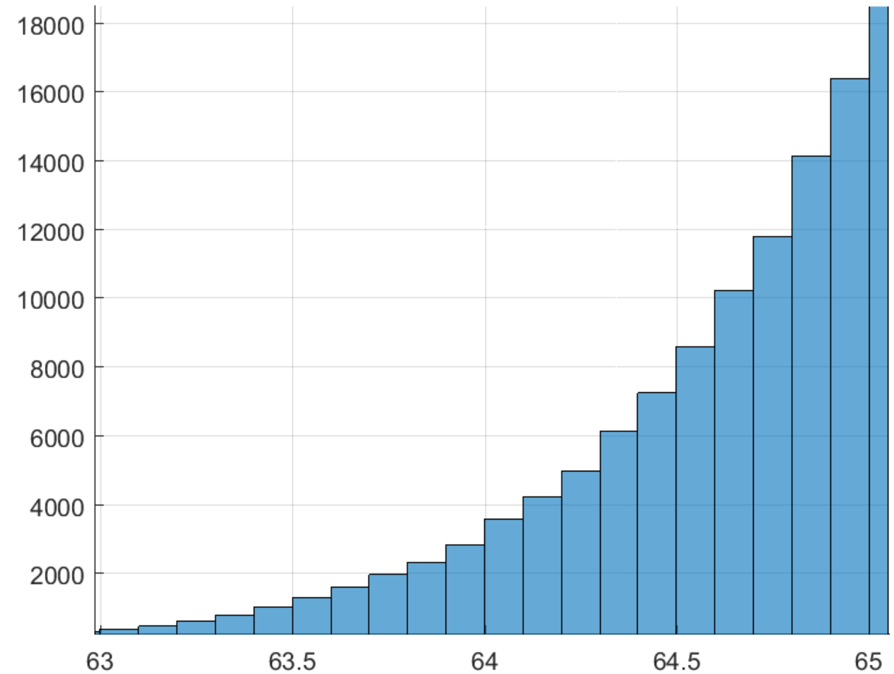
The distribution of optimal upper bounds over 10^9 random samples.

On the left is a histogram based on 1° bins. The spike a 72° is evident.

On the right is an enlargement near 64° using $.1^\circ$ bins.

No spike at $\frac{5}{7} \cdot 90^\circ \approx 64.26^\circ$ is visible. Is $N = 10$ too small?





In these experiments I just chose angles at random (with correct sum).

Didn't choose edge lengths or check for self-intersections.

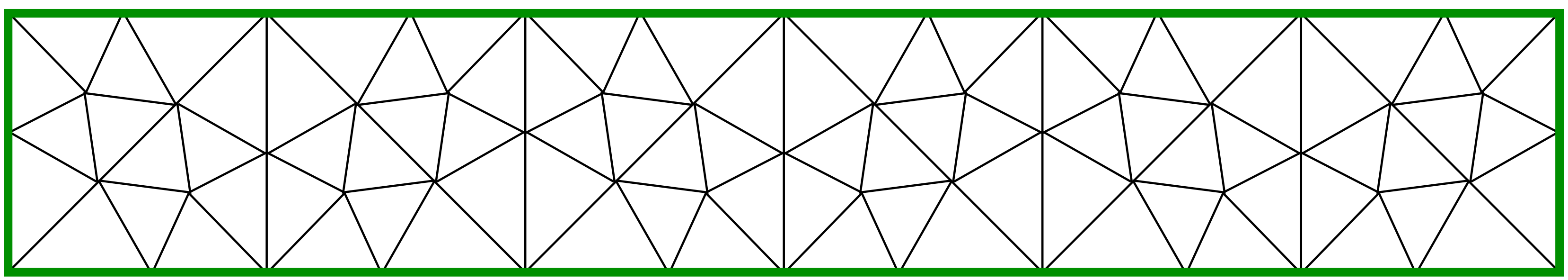
What is better model for random polygons?

Theorem: For $60^\circ < \phi < 90^\circ$, a polygon P has a ϕ-triangulation iff

1. $72^\circ \leq \phi < 90^\circ$ and P has a ϕ-labeling L of V_P,
2. $\frac{5}{7} \cdot 90^\circ \leq \phi < 72^\circ$, and P has a ϕ-labeling with $\kappa(L) \leq 0$,
3. $60^\circ < \phi < \frac{5}{7} \cdot 90^\circ$, and P has a ϕ-labeling with $\kappa(L) = 0$.

Cor: For an N-gon $\Phi(P)$ can be computed in time $O(N)$.

However, $1 \times R$ rectangle needs $\gtrsim R$ triangles.



$\Rightarrow$ no bound for number of triangles in terms of N.

Sketch of $O(N)$ computation of $\Phi(P)$, $N =$ number of vertices:

Find $\theta_{\min}$ in time $O(N)$.

If $\theta_{\min} \le 36^\circ$ then $\Phi(P) = 90^\circ - \theta_{\min}/2$. Done.

If $\theta_{\min} \ge 144^\circ$ then $\Phi(P) = 72^\circ$. Done.

If P is a 60° degree polygon then $\Phi(P) = 60^\circ$. Done.

Can now assume $36^\circ < \theta_{\min} < 144^\circ$ and $60^\circ < \Phi(P) \le 72^\circ$.

After some work, computing $\Phi(P)$ reduces to finding two numbers:

$$\phi_\infty = \inf\{\phi : \exists\ \phi\text{-labeling } L\},$$

$$\phi_0 = \inf\{\phi : \exists\ \phi\text{-labeling } L \text{ with } \kappa(L) = 0\}$$

Computing ϕ_∞ is easy: for v find the minimum ϕ so that either

$$\frac{\theta_v}{\phi} \quad \text{or} \quad \frac{\theta_v}{180 - 2\phi}$$

is an integer; this takes $O(1)$ work per vertex.

Then take maximum of these results ($O(N)$ work).

To compute ϕ_0, we rewrite it as

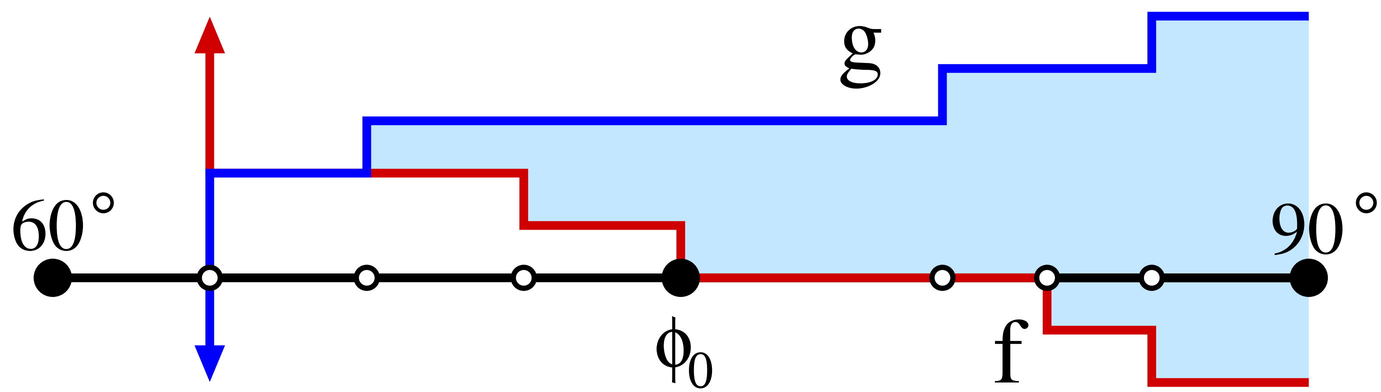
$$\phi_0 = \inf\{\phi : f(\phi) \leq 0 \text{ and } g(\phi) \geq 0\}$$

where f, g are the monotone step functions:

$$f(\phi) = \min \kappa(L) = 6 - 3|V| + \sum_{v \in P} \inf\{k : 180 - 2\phi \leq \frac{\theta_v}{k} \leq \phi\}$$

$$g(\phi) = \max \kappa(L) = 6 - 3|V| + \sum_{v \in P} \sup\{k : 180 - 2\phi \leq \frac{\theta_v}{k} \leq \phi\}$$

Note $f \leq g$, f = decreasing, g = increasing.



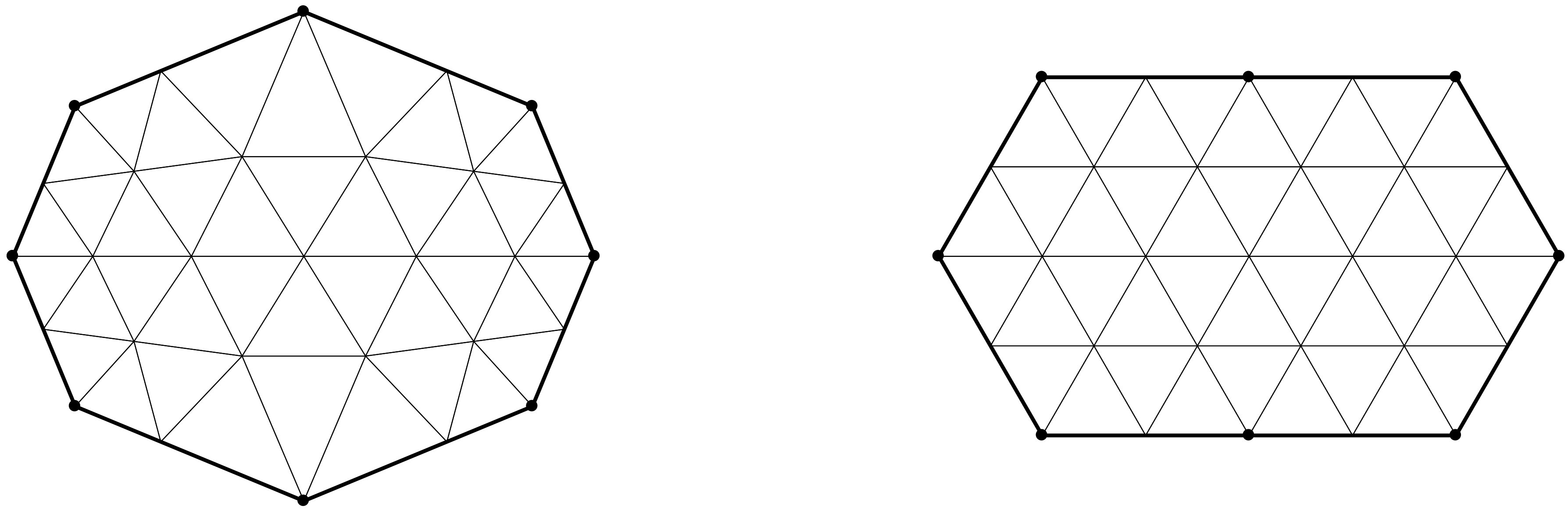
$O(N)$ jump points to check, $O(N)$ work per point $= O(N^2)$ work?

No, we can find $\phi_0 \in J$ in time $O(N)$ as follows:

- Find smallest, largest elements of J. Evaluate f, g.
- Find median of J by median-of-medians algorithm. Evaluate f, g.
- Decide if ϕ_0 is $\geq$ or $\leq$ median. Delete half of J.
- Repeat last two steps until ϕ_0 is found.
- Monotonicity implies new evaluations only use remaining points.

$\Rightarrow$ Work diminishes geometrically. Total is $O(N)$.

Idea behind proof of main theorem: conformal maps

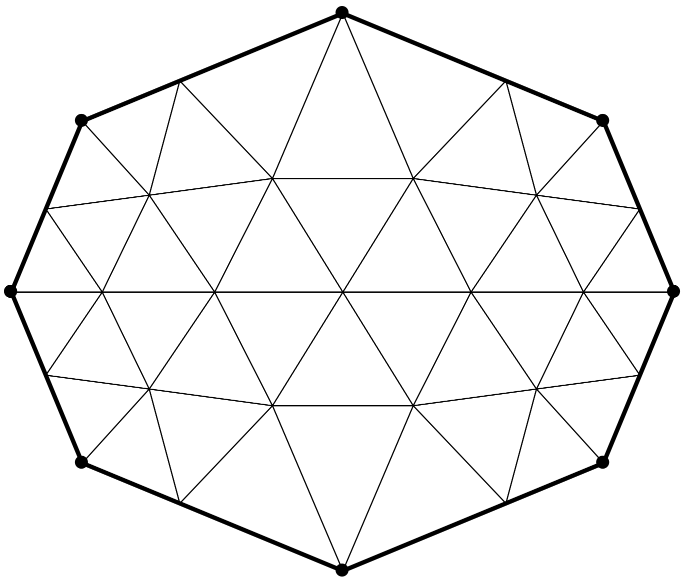


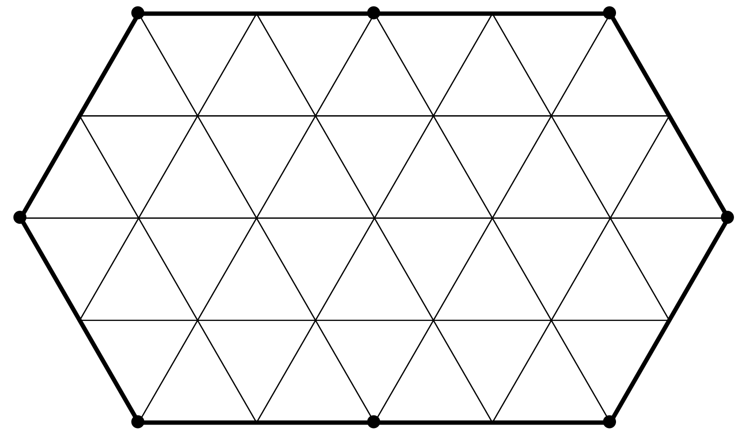
Given P with angles $\{\theta_k\}$, approximate by 60°-polygon P', angles $\{\psi_k\}$.

Rounding to nearest multiple of 60° often doesn't work: need

$$\sum \psi_k = (N-2) \cdot 180^\circ = \sum \theta_k.$$

Idea behind proof of main theorem: conformal maps



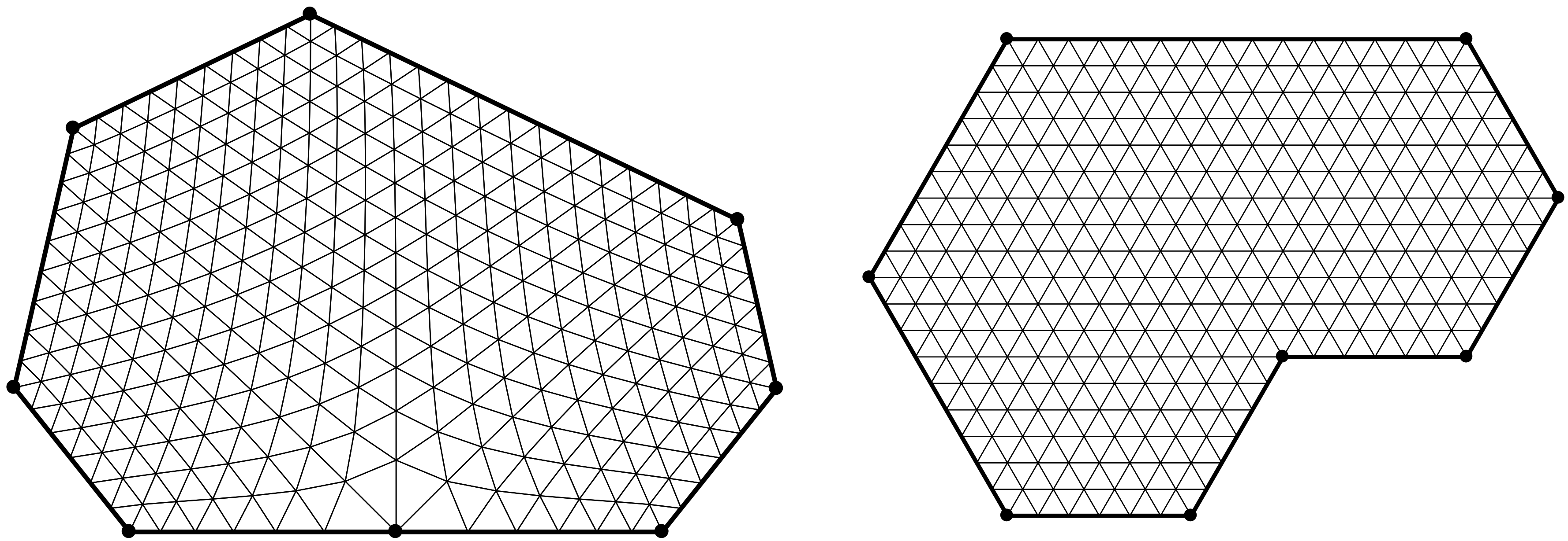


After angles $\{\psi_k\}$ are chosen, P' is built using Schwarz-Christoffel formula:

$$F(z) = A + C \int^z \prod_{k=1}^{n} (1 - \frac{w}{z_k})^{(\theta_k/\pi)-1} dw.$$

$F : \mathbb{D} \to P$ where $\{\theta_k\}$ = angles, $\{\mathbf{z}_k\}$ = vertex pre-images on unit circle.
P' has same z_k-parameters as P, new angles = $\{\psi_k\}$.

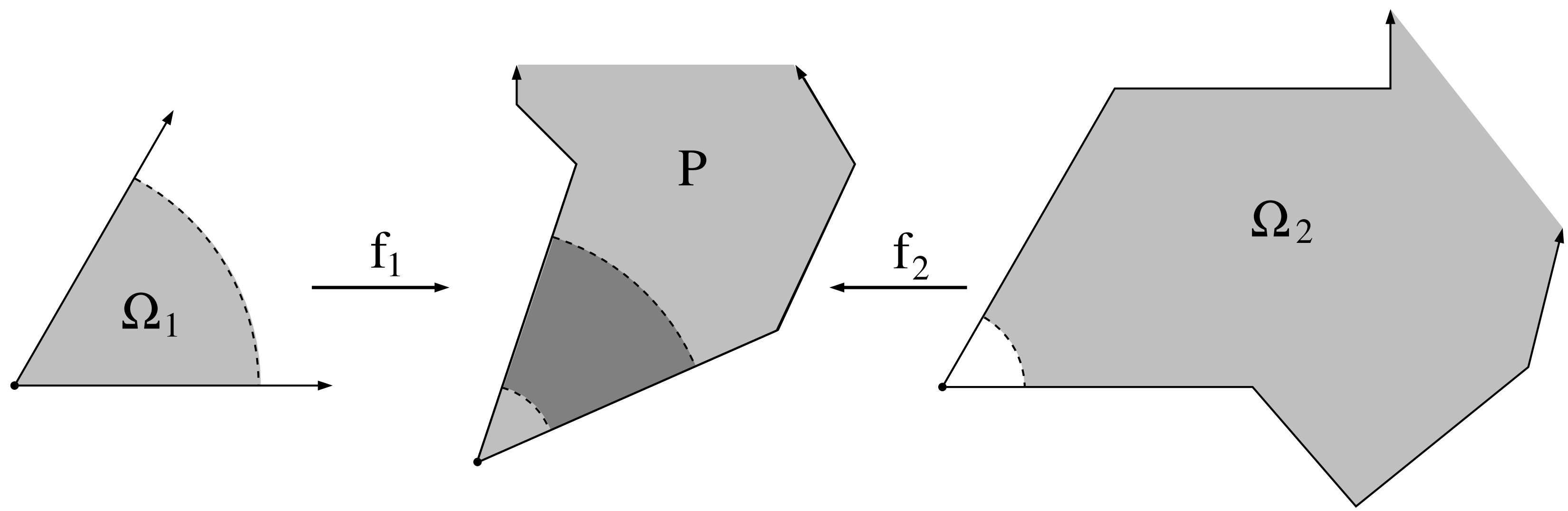
Idea behind main theorem: conformal maps



Map a (nearly) equilateral triangulation of P' to P.

Can prove worst angle distortion is at vertices $= \theta_k/\psi_k$.

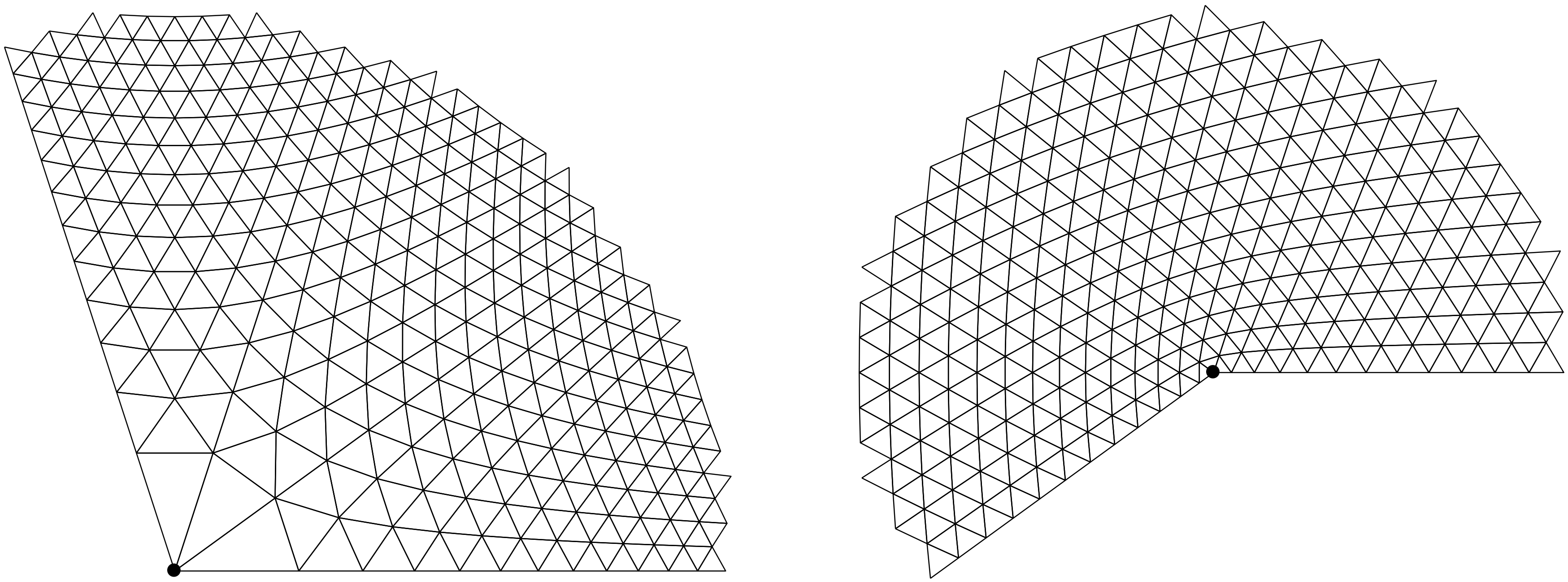
Extra work needed to ensure angle bound attained, not just approximated.



Triangulation of a sector can be computed with power map.

Away from corners, all angles are close to 60° (Koebe's theorem).

In between, we use quasiconformal interpolation to define triangulation.



Images of equilateral triangulation of half-plane under power map.

"Worst" angles near vertex. Tends towards 60° away from vertex.

But,...

This approach gives a triangulation with only degree 6 interior vertices.

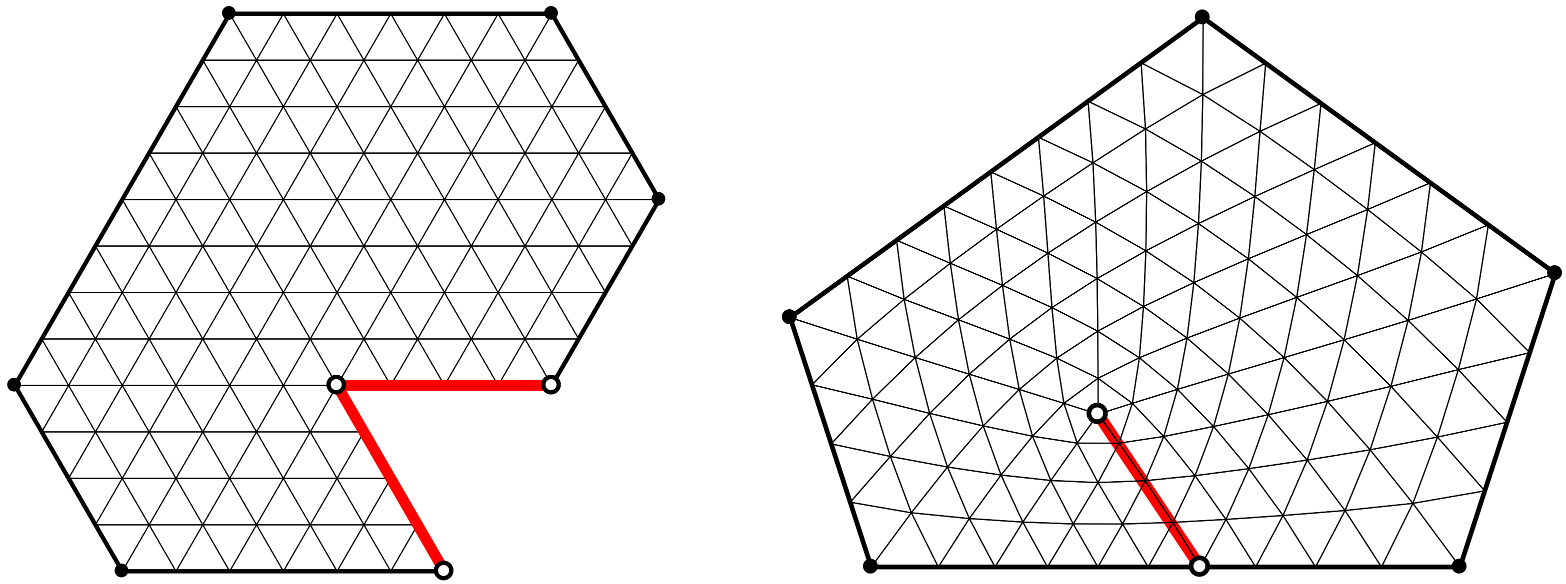
Only works when P has a zero curvature ϕ-labeling.

Let L minimize $|\kappa(L)|$ over ϕ-labelings of P.

If $\kappa(L) > 0$ there must be interior vertices of degree ≤ 5.

If $\kappa(L) < 0$ there must be interior vertices of degree ≥ 7.

Creating degree 5 vertices by folding:

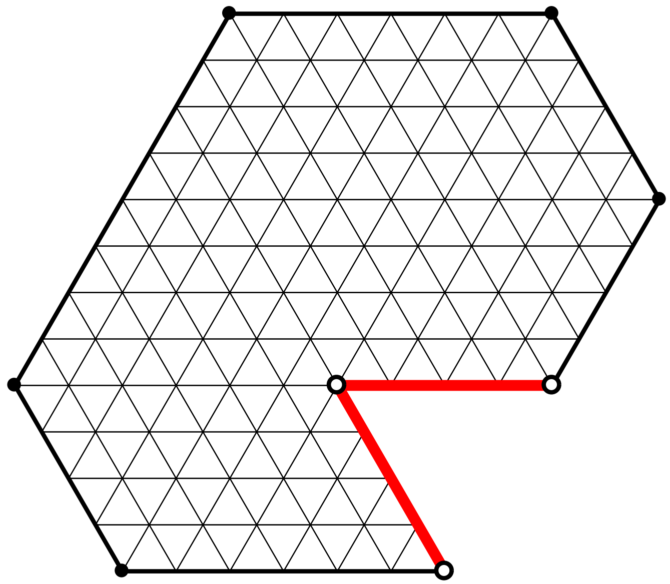


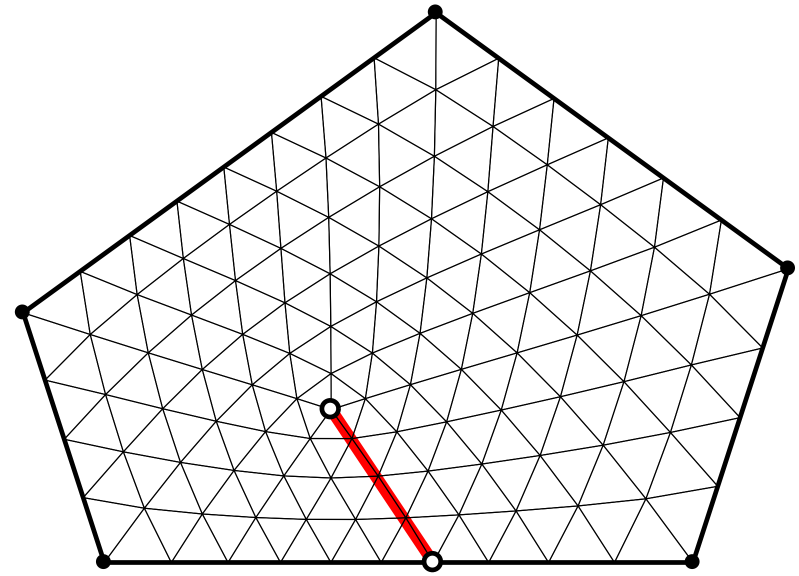
f maps P' to P with a slit removed; identifies boundary segments.

A degree 5 interior vertex is created.

But is this really a triangulation of P?

Creating degree 5 vertices by folding:



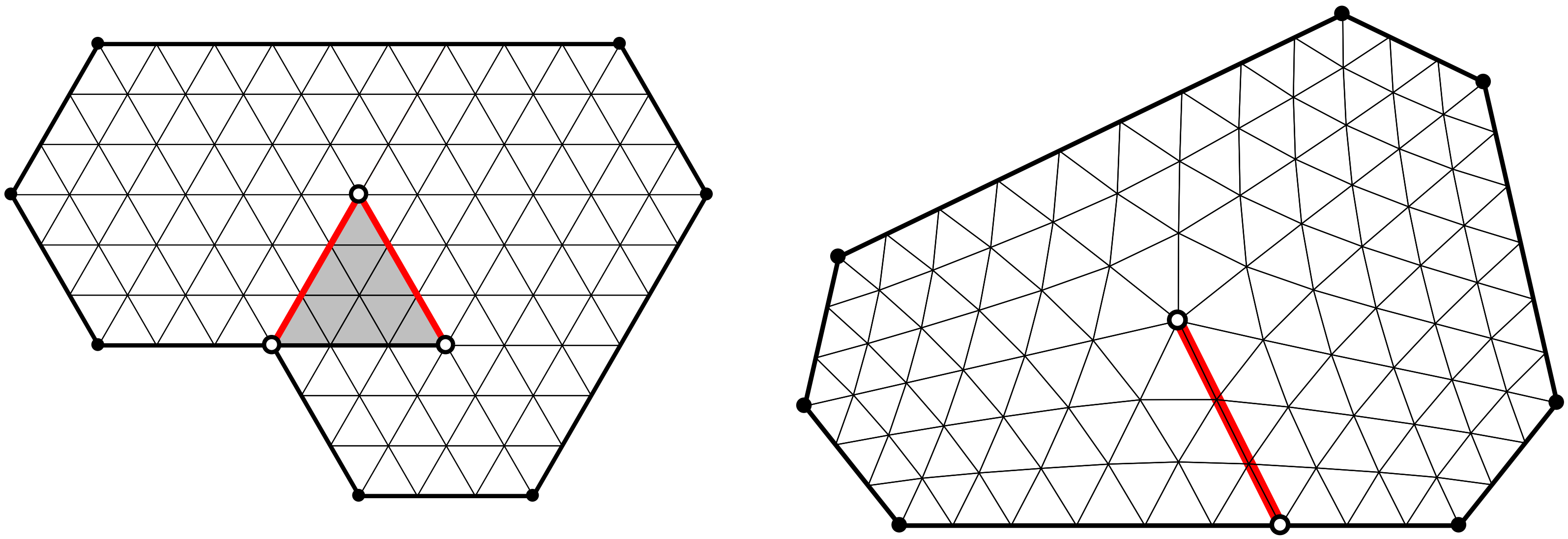


Technical difficulty: Image triangulations must match up across slit.

Matching occurs if $|f'(w)| = |f'(z)|$ whenever $f(w) = f(z)$.

Differential equation can be solved explicitly (= conformal welding).

Solution gives a curved slit (tangent changes by 3° above).



Creating a degree 7 vertex requires P' to be Riemann surface.

All non-zero curvature cases can be handled.

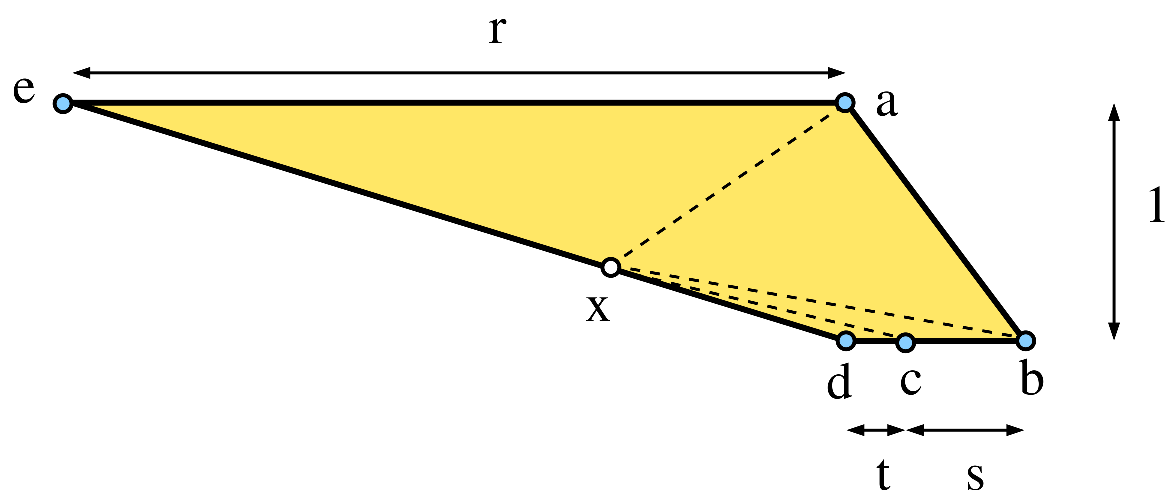
$\Rightarrow$ interior vertices are all degree six with $|\kappa(L)|$ exceptions

Open problems:

- Minimal weight Steiner triangulations.
- How large are optimal triangulations?
- Surfaces and solids.
- Triangulations of PSLGs (planar straight line graphs).

We saw a triangulation achieving MinMax angle usually exists.

A **minimal weight Steiner triangulation** (MWST) minimizes total edge length. It need not exist ($t \ll s \ll 1 \ll r$):

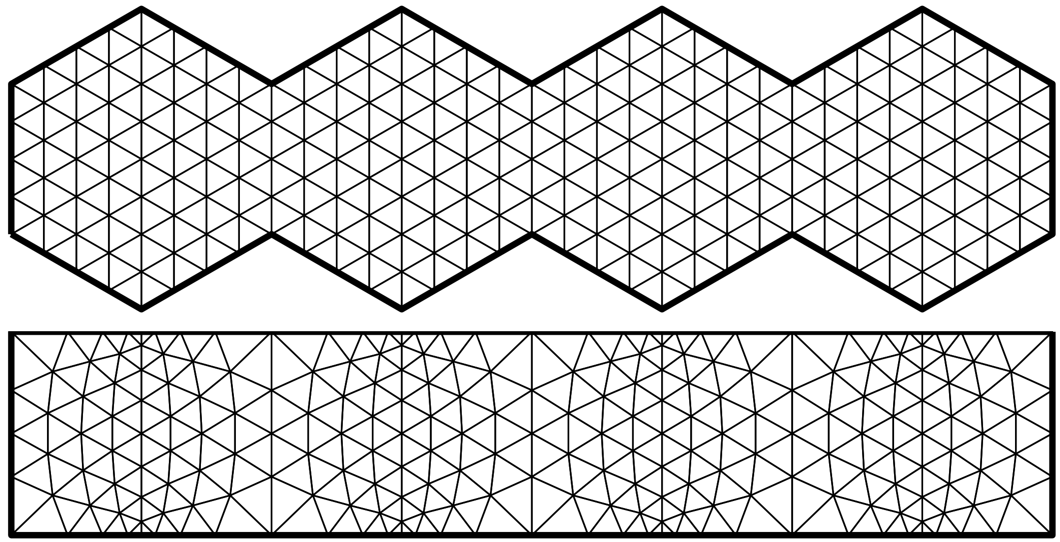


Question: Does a MWST exist for polygons in general position?

Without Steiner points, finding a MWT is NP-hard for point sets (Mulzer-Rote 2008) and $O(n^3)$ for polygons (Gilbert 1979, Klincsek 1980).

O(optimal) approximation of MWST is possible (Eppstein, 1994)

How many triangles does MaxMin solution need?



Proof of theorem gives exponentially many triangles for $1 \times R$ rectangle.

But good choice of $60°$-polygon P' above gives $O(R)$ triangles.

Estimate smallest number of triangles needed for general P?

Thick-thin decomposition of polygons may help.

Is exact minimum NP-hard to compute?

Main question in 3 dimensions:

Does every polyhedron have an acute triangulation of polynomial size?

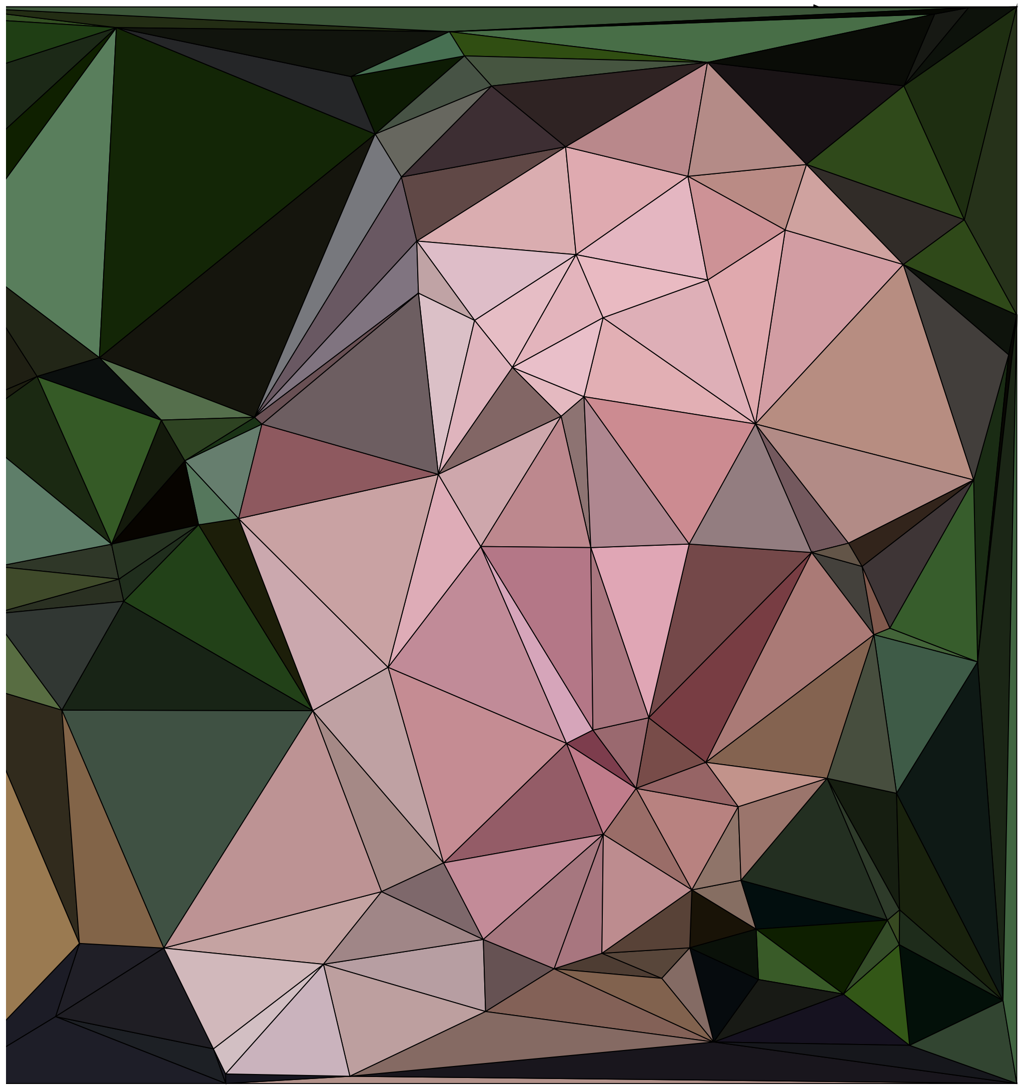
triangulation = tetrahedralization with dihedral angles $< 90°$.

Acute triangulation exists for unit cube $[0, 1]^3$: 1370 tetrahedra.

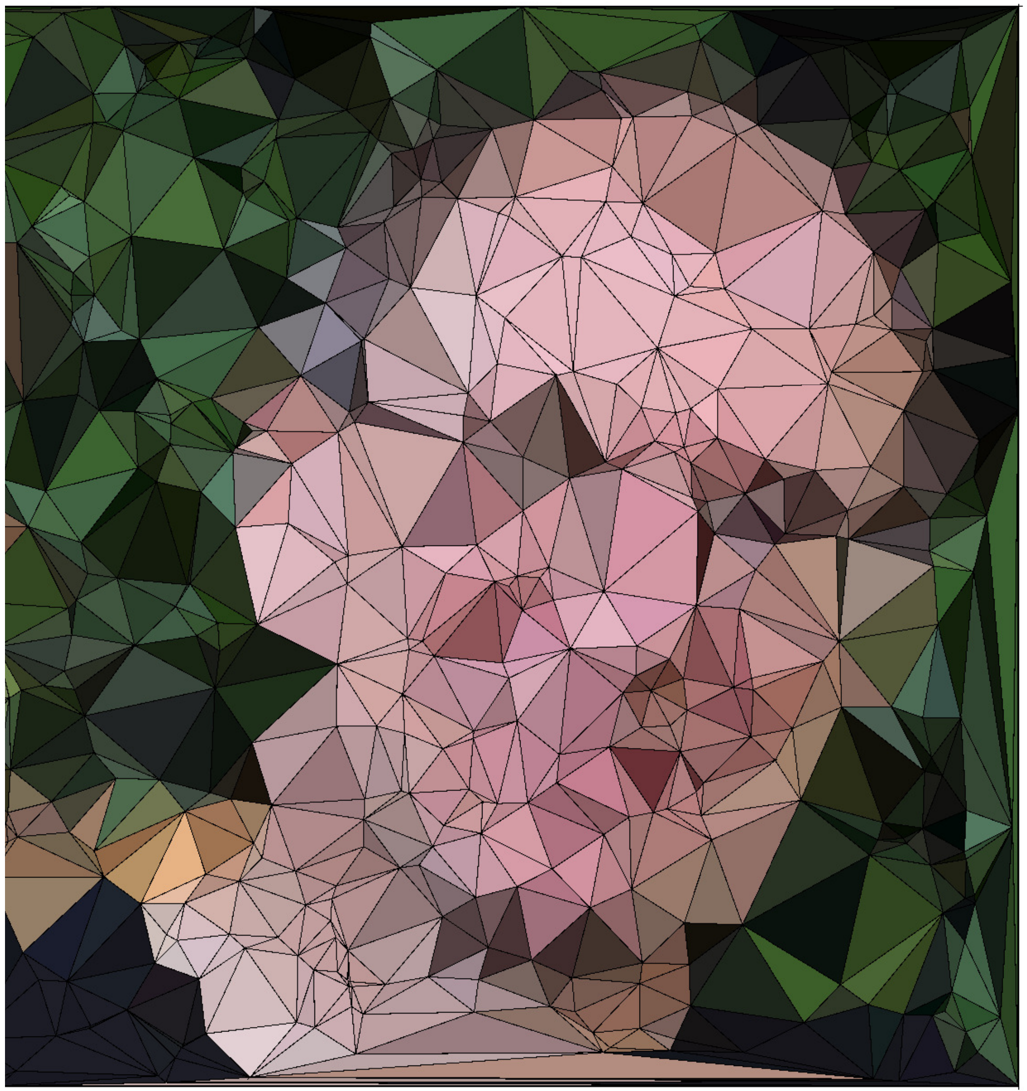
No acute triangulation of cube in $\mathbb{R}^n$, $n \geq 4$.

Kopczynski-Pak-Przytycki 2009, VanderZee-Hirani-Zharnitsky-Guoy 2010.

Do polyhedral surfaces have polynomial sized acute triangulations?



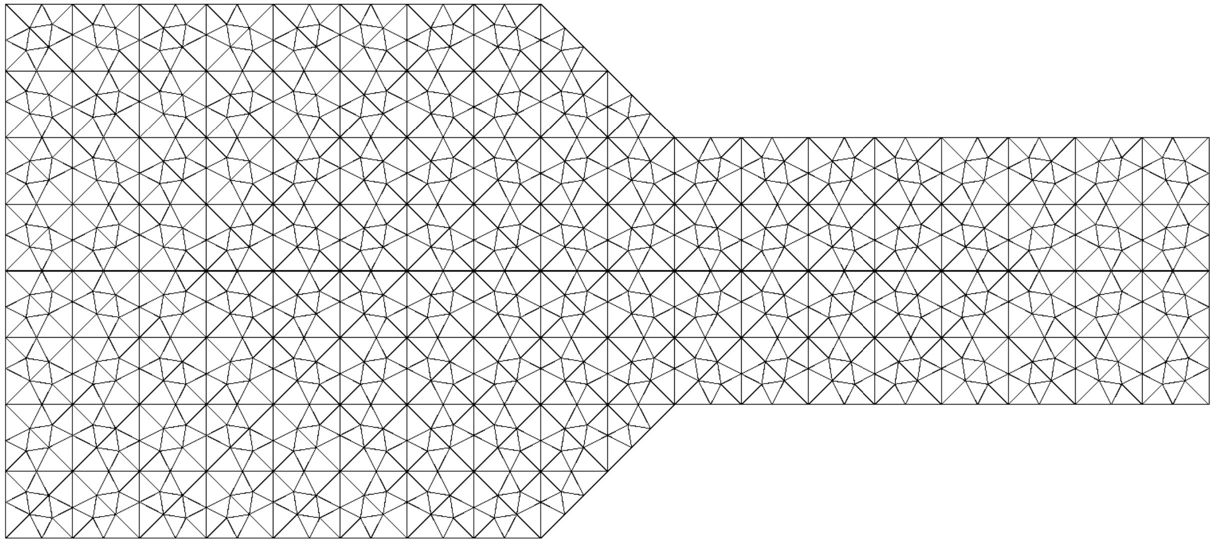
Mario, triangulated (N=100)

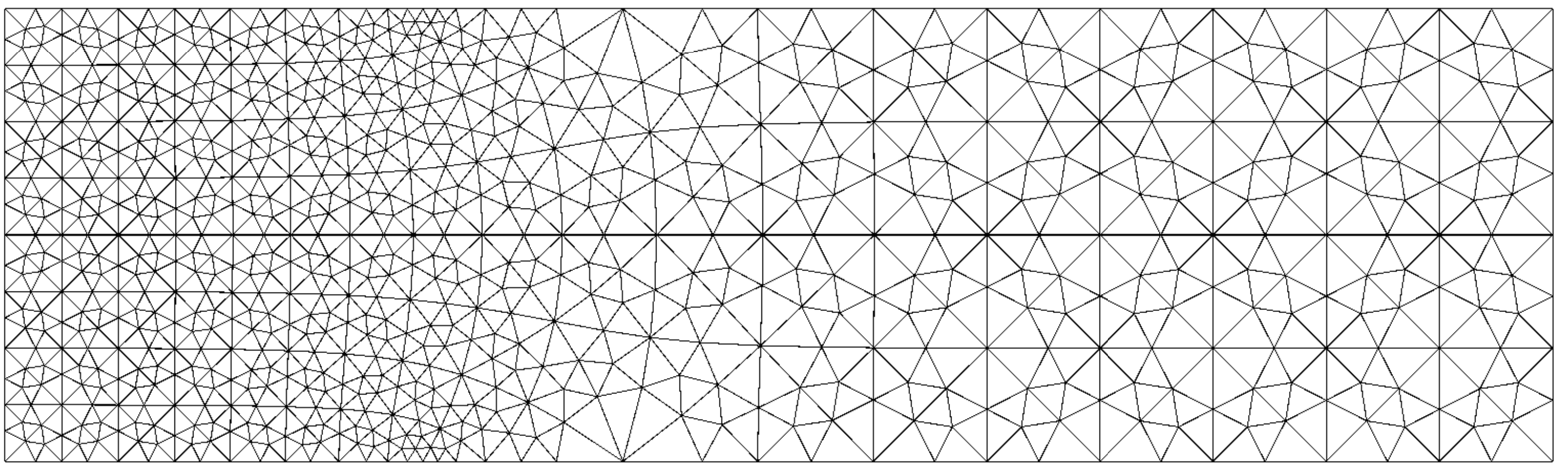


Mario, triangulated (N=500)

Mario, triangulated (N=1000)

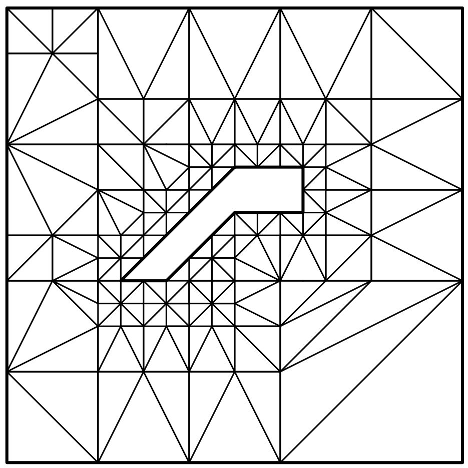
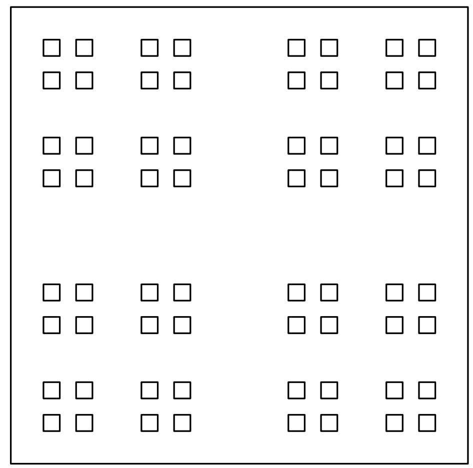
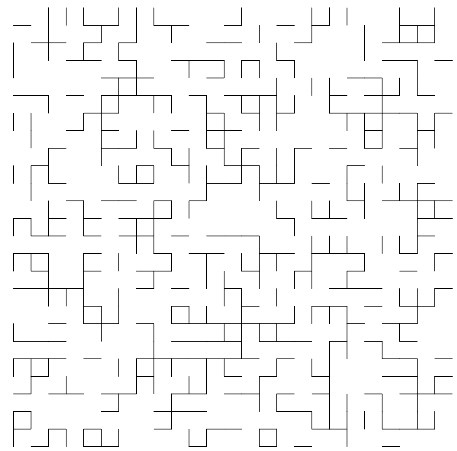
Happy Birthday Mario!





Thanks for listening. Questions?

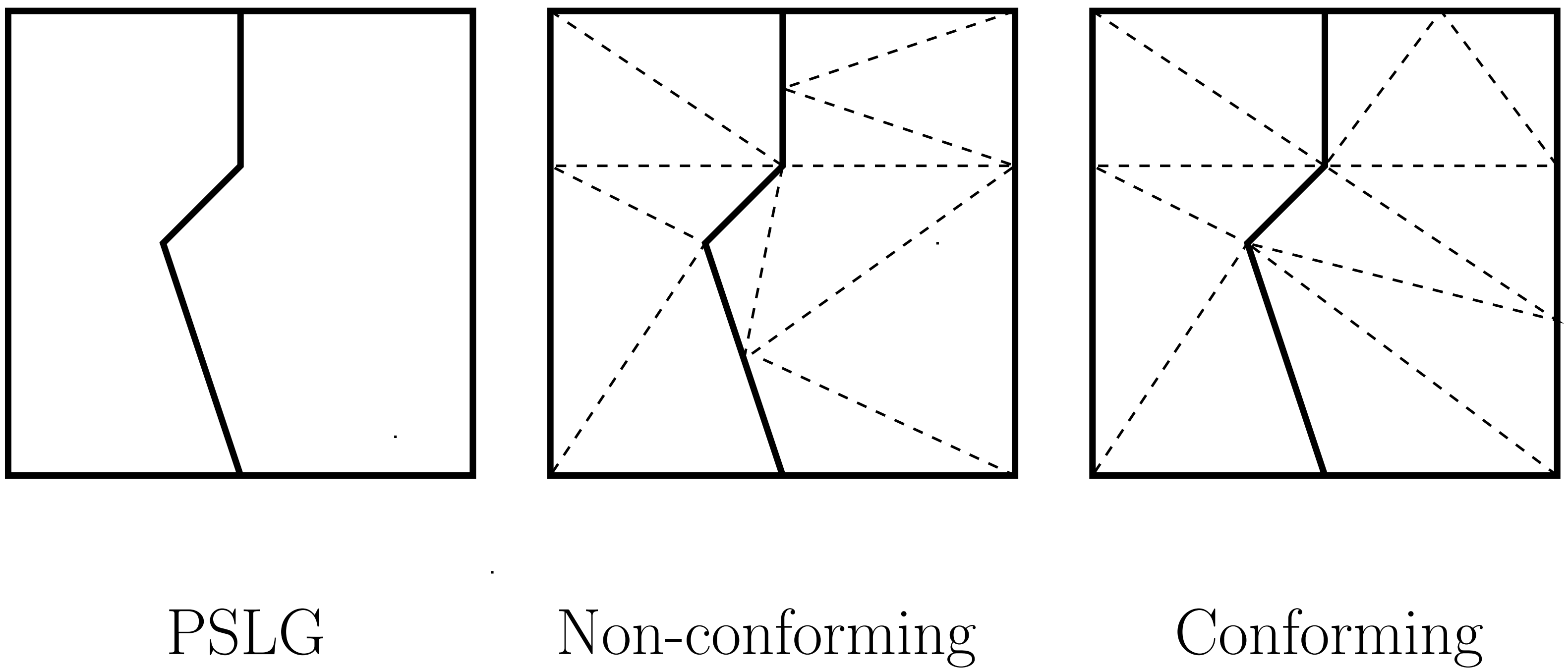
A **planar straight line graph** Γ (or **PSLG**) is finite union of points V and a collection of disjoint edges E with endpoints among these points.

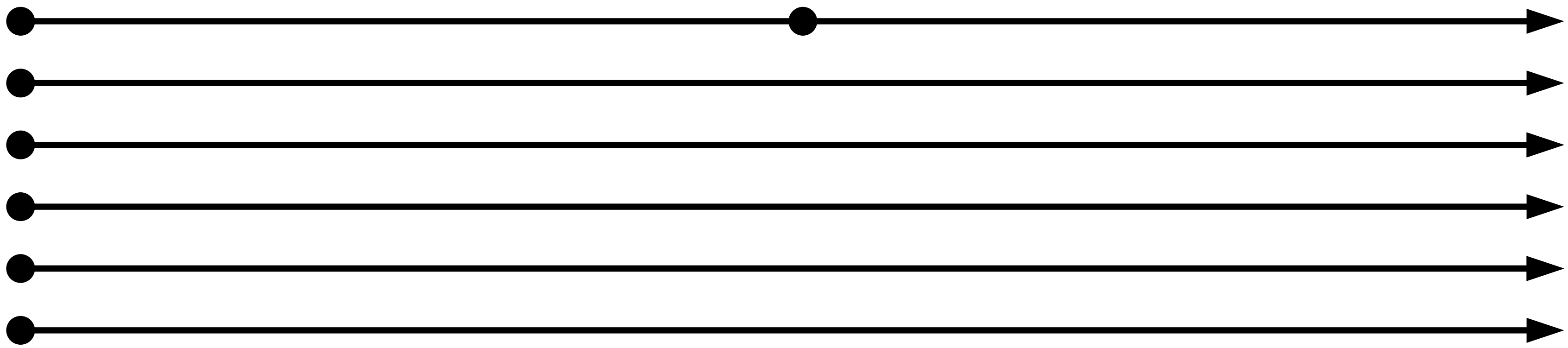
Generally let $n = |V|$ be the number of vertices.

A simple polygon is a PSLG where edges form a closed cycle.

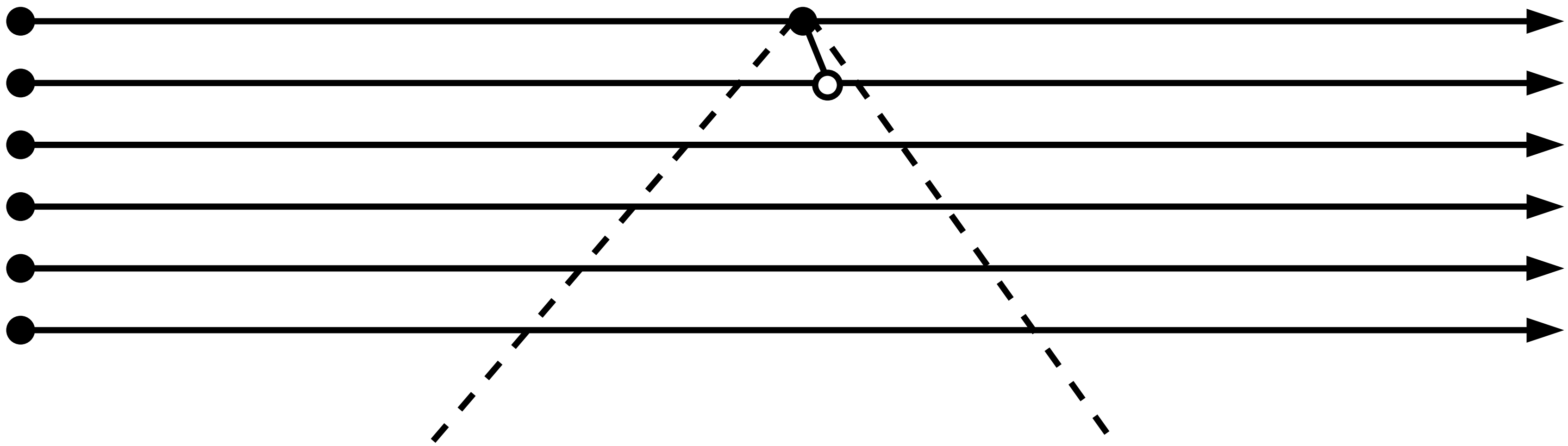
A **conforming triangulation** of a PSLG is a triangulation of each face, consistent across edges of the PSLG.



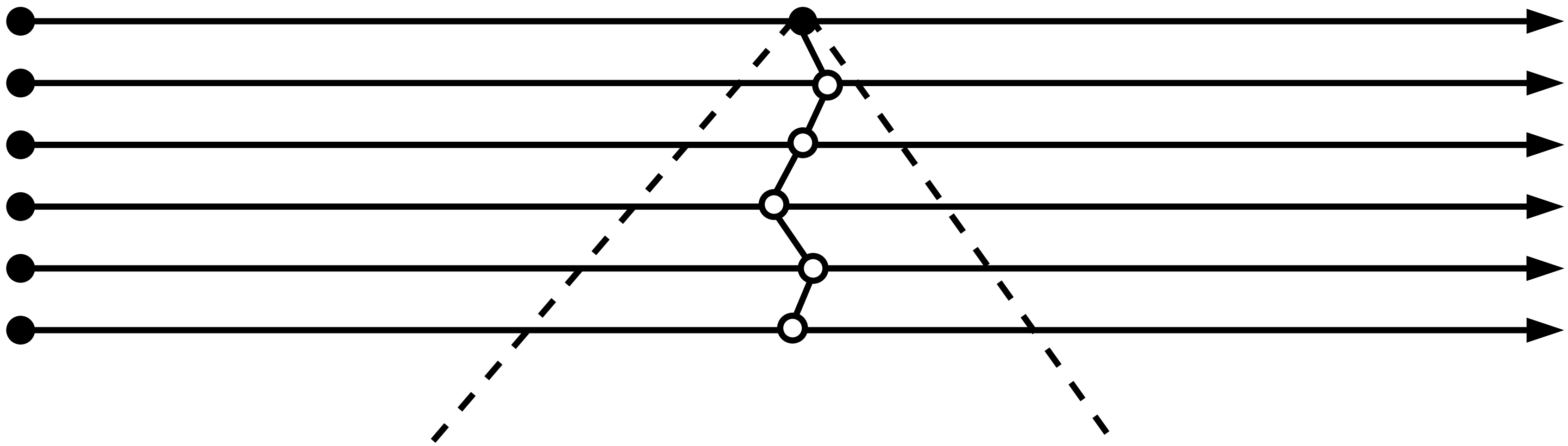
NOT = Non-Obtuse Triangulation = all angles $\leq 90°$.



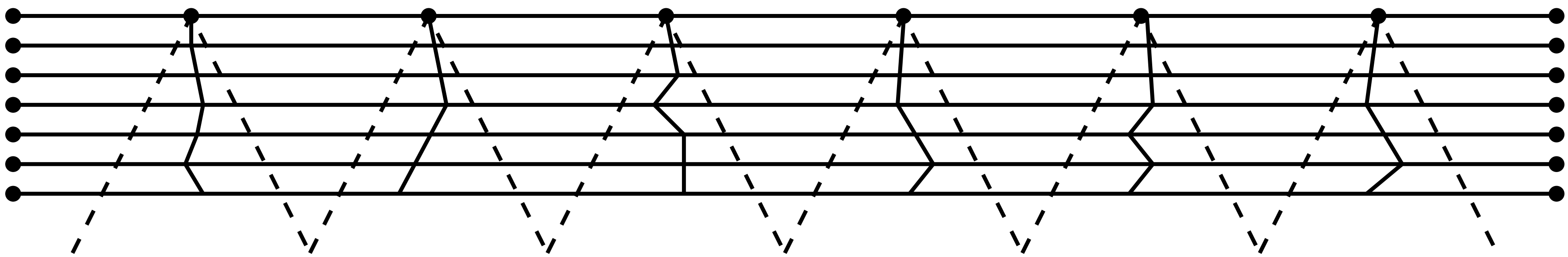
Consider a NOT for this PSLG (or any angle bound $< 180^\circ$).



An edge must leave the vertex with the 90° wedge.



Iterating shows many new vertices, edges are needed.



A NOT for this PSLG needs $\gtrsim n^2$ triangles.

Burago-Zalgaller, 1960: Every PSLG has an NOT (no size bound).

S. Mitchell, 1993: Every PSLG has a $157.5°$-triangulation, size $O(n^2)$.

Tan, 1996: Every PSLG has a $132°$-triangulation, size $O(n^2)$.

NOT-Thm (B. 2018): Every PSLG has a NOT with $O(n^{2.5})$ elements.

Improves $O(n^3)$ for Delaunay triangulation by Edelsbrunner, Tan (1993).

First polynomial bound for NOTs.

Proof uses a "discrete closing lemma" for flows.

Problems for PSLGs Γ:

- **NOT Conj:** Every PSLG has a NOT with $O(n^2)$ elements.
- Compute $\Phi(\Gamma)$ = MinMax angle for conforming triangulation of Γ.
- When is minimum MinMax angle attained?
- Give bounds on $\Phi(\Gamma)$ in terms of minimum angle $\theta_{\min}$ in Γ.

Problems for PSLGs Γ:

- **NOT Conj:** Every PSLG has a NOT with $O(n^2)$ elements.
- Compute $\Phi(\Gamma) =$ MinMax angle for conforming triangulation of Γ.
- When is minimum MinMax angle attained?
- Give bounds on $\Phi(\Gamma)$ in terms of minimum angle $\theta_{\min}$ in Γ.

Best result so far: there is a $\theta_0 > 0$ so that

$$\Phi(\Gamma) \leq 90^\circ - \min(\theta_0, \theta_{\min})/2.$$

Uses compactness argument: θ_0 not explicit.

Flows associated to triangulations:

- In-circle divides any triangle into three sectors and a central triangle.
- Sectors are foliated by circular arcs centered at vertices.
- Propagating cusp points defines flow lines on triangulation.