Conformal Mapping in Linear Time

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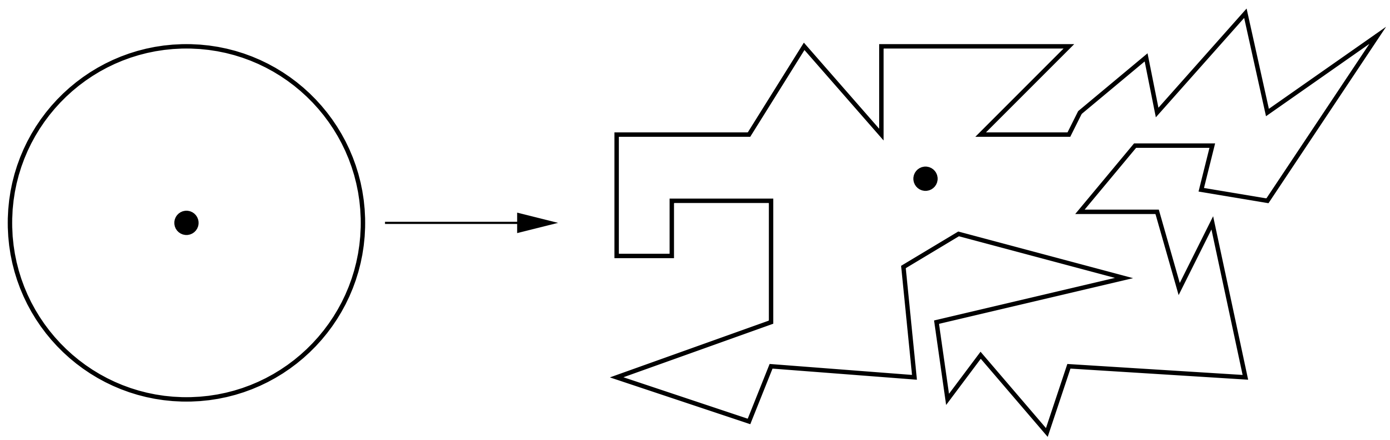
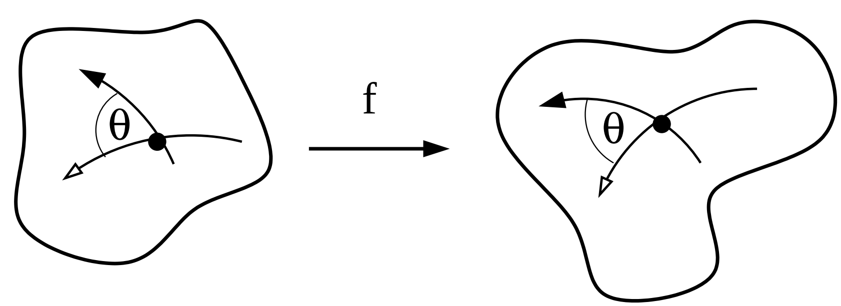
Riemann Mapping Theorem: If Ω is a simply connected, proper subdomain of the plane, then there is a conformal map $f : \Omega \to \mathbb{D}$.



Conformal = angle preserving

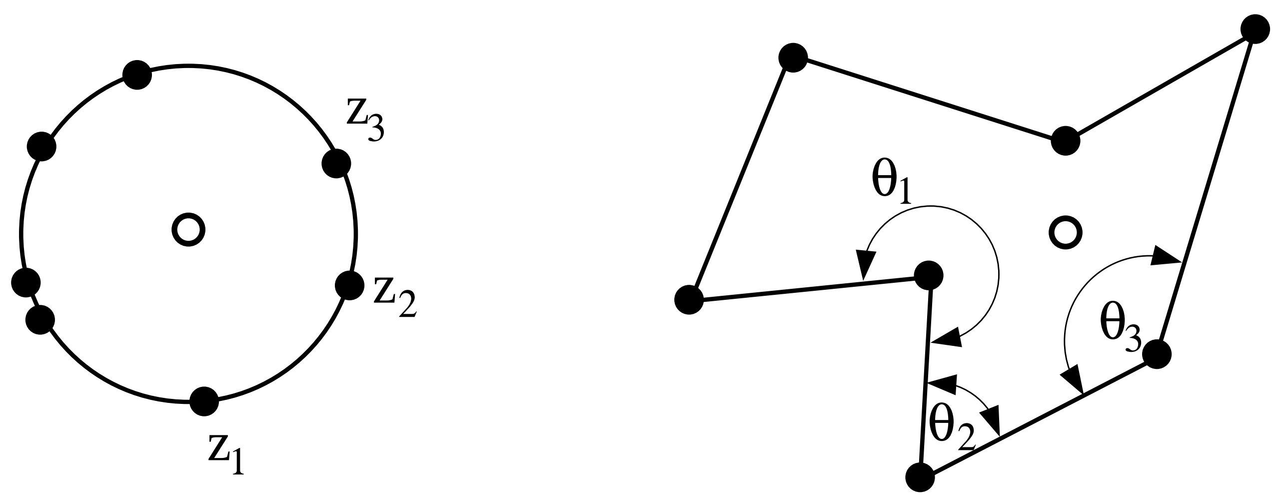


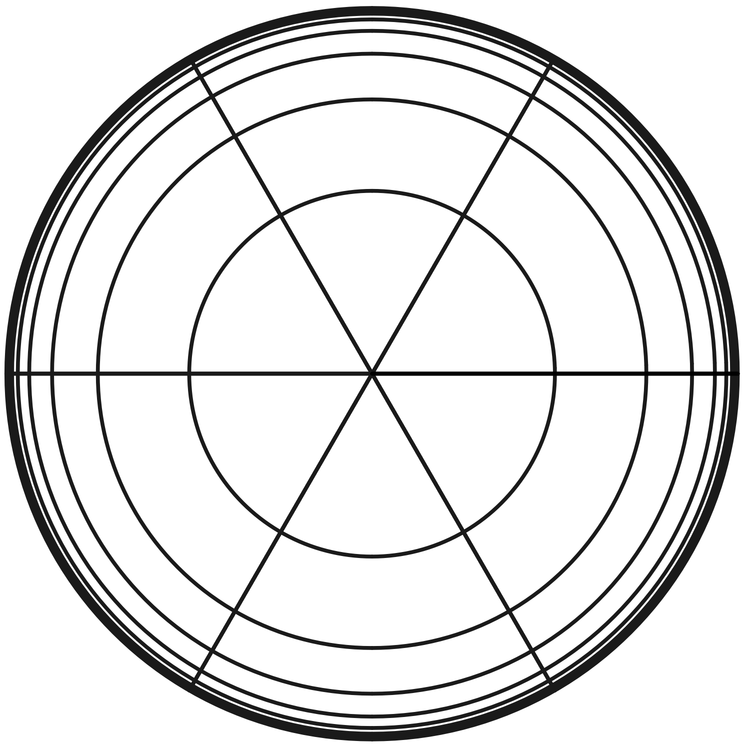
For polygons, enough to compute vertices.

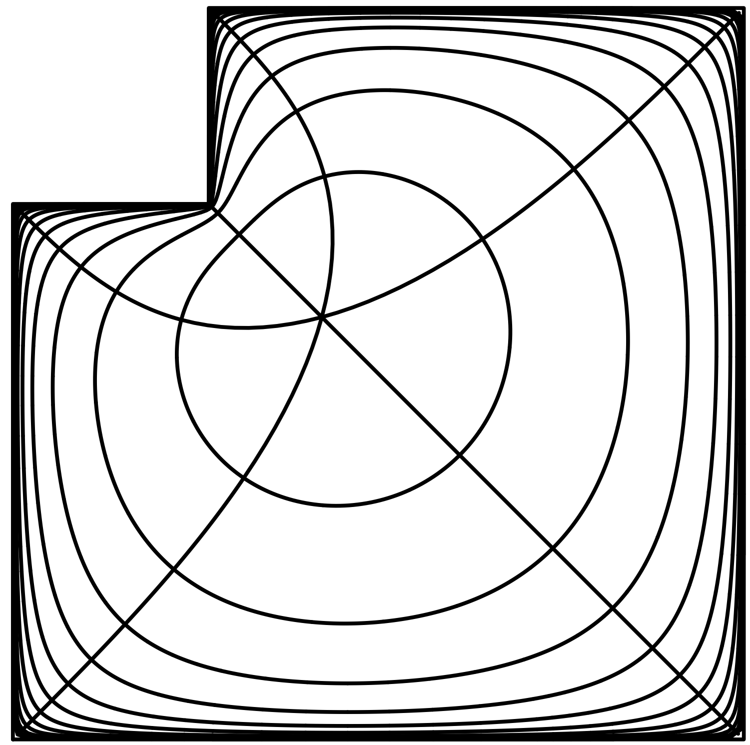
The Schwarz-Christoffel formula:

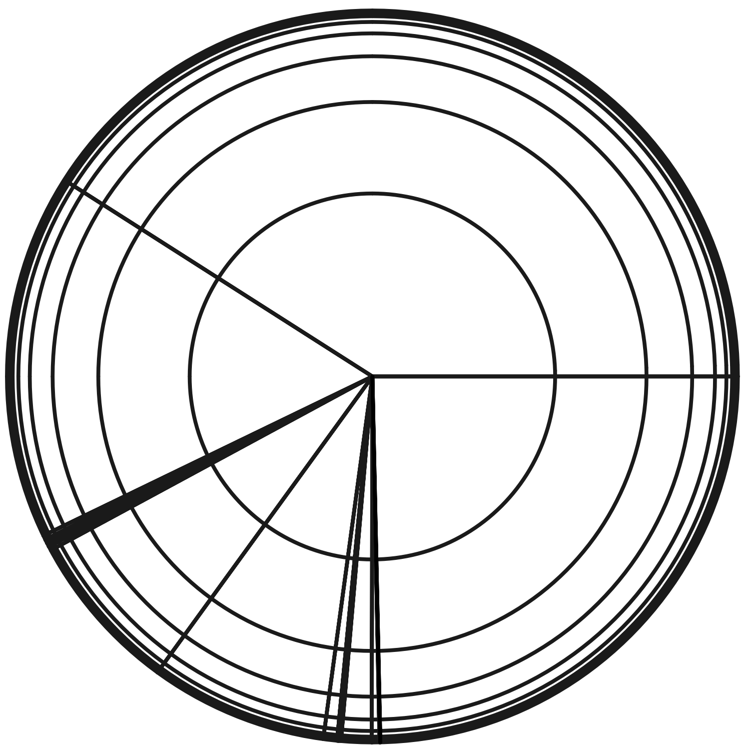
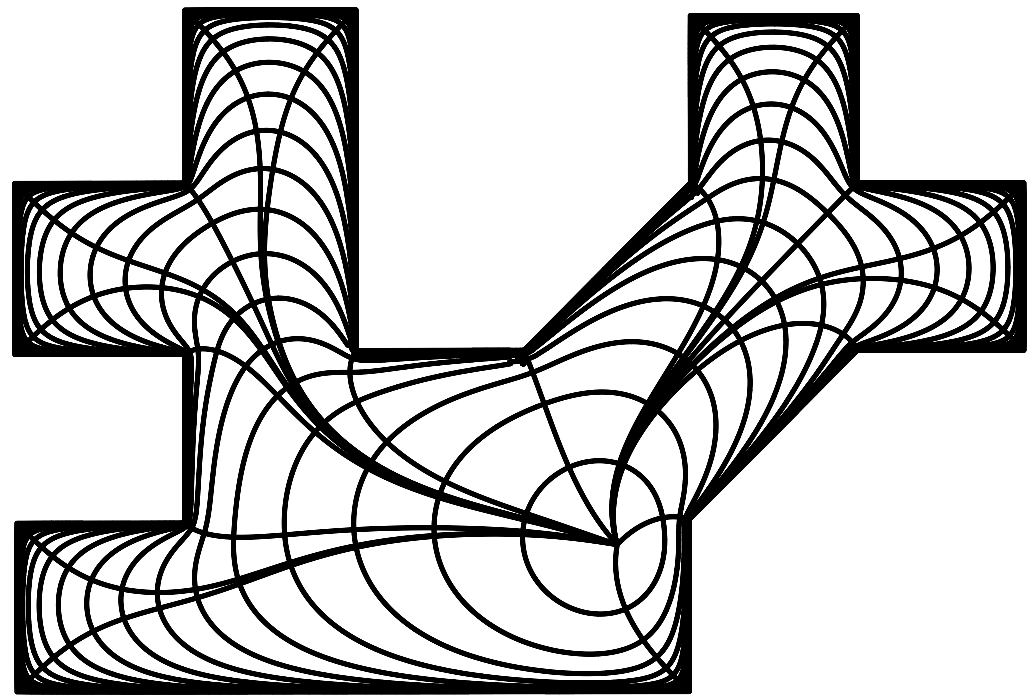
$$f(z) = A + C \int^{z} \prod_{k=1}^{n} (1 - \frac{w}{z_k})^{\alpha_k - 1} dw.$$

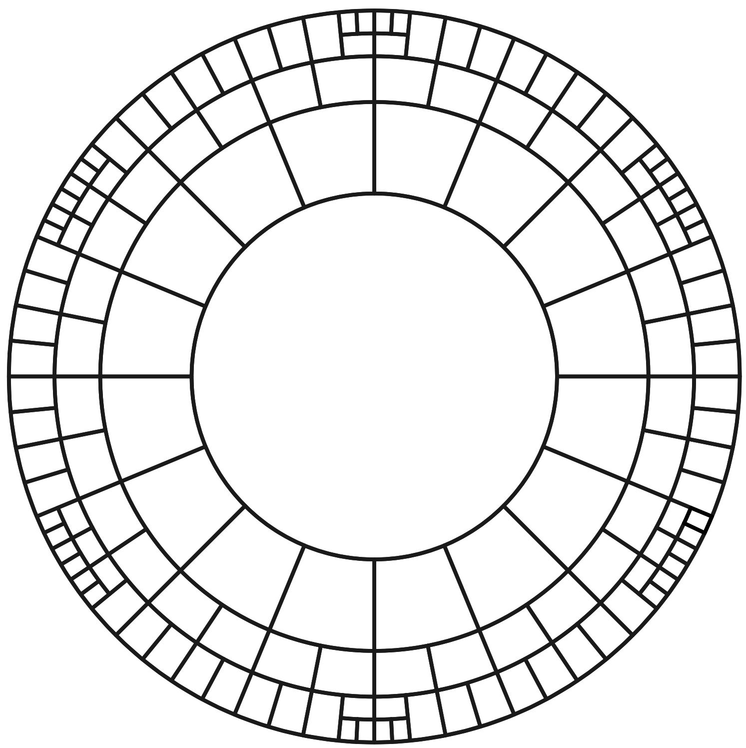
Gives conformal map to polygon.
$\pi\alpha_k$'s are interior angles.
z_k's map to vertices.

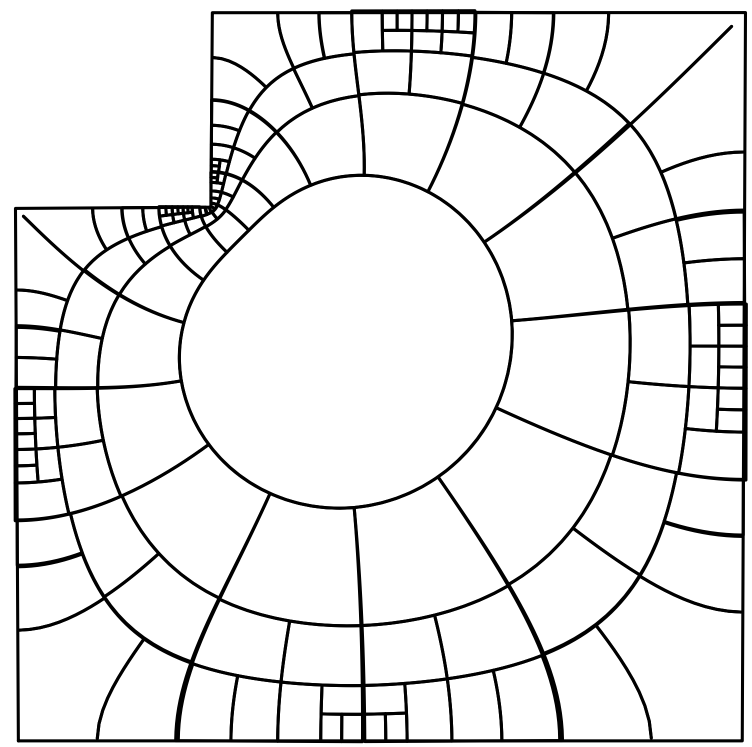


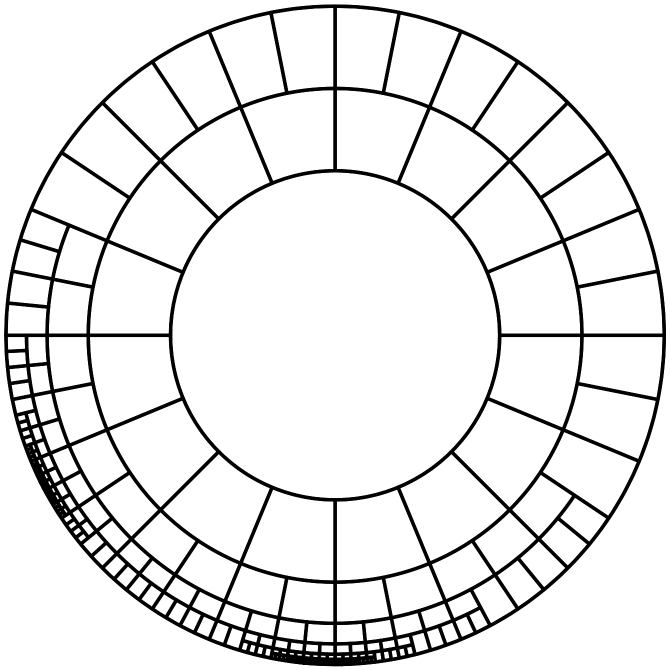
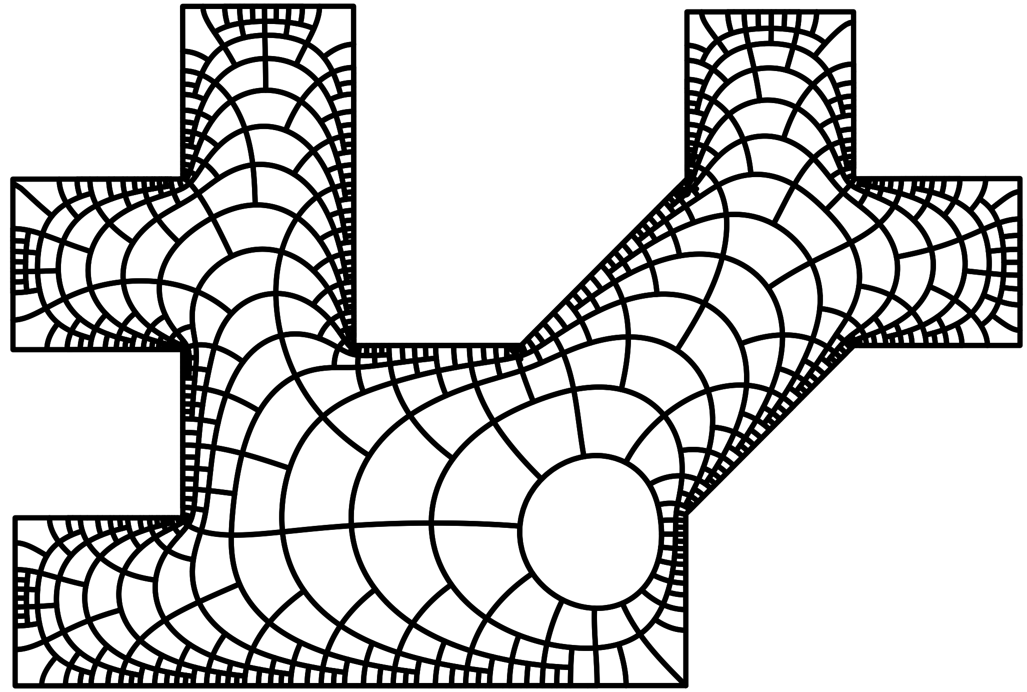


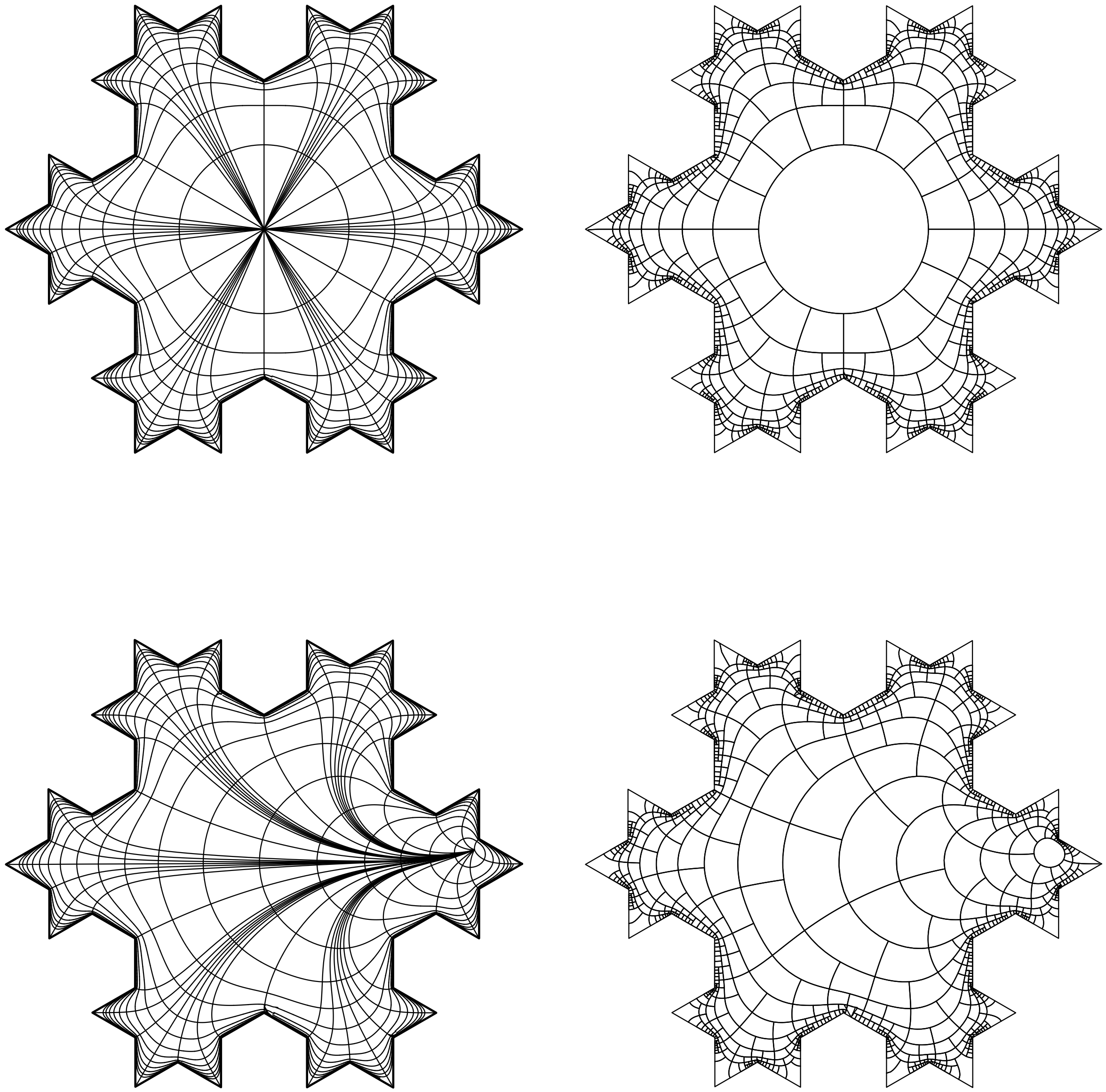


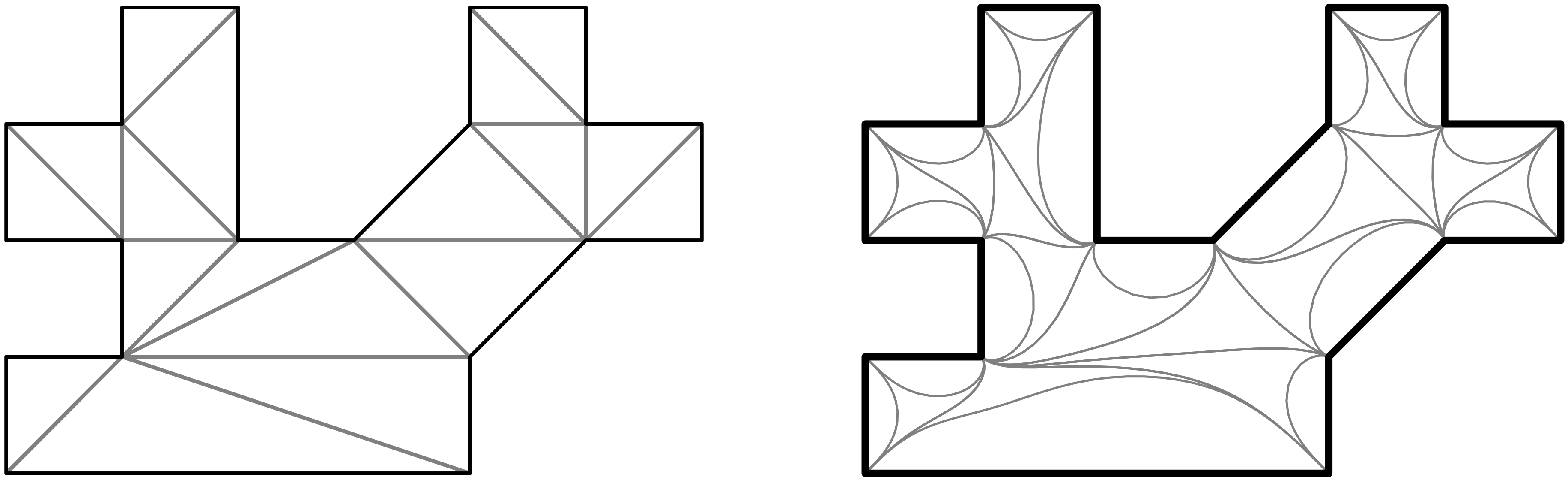







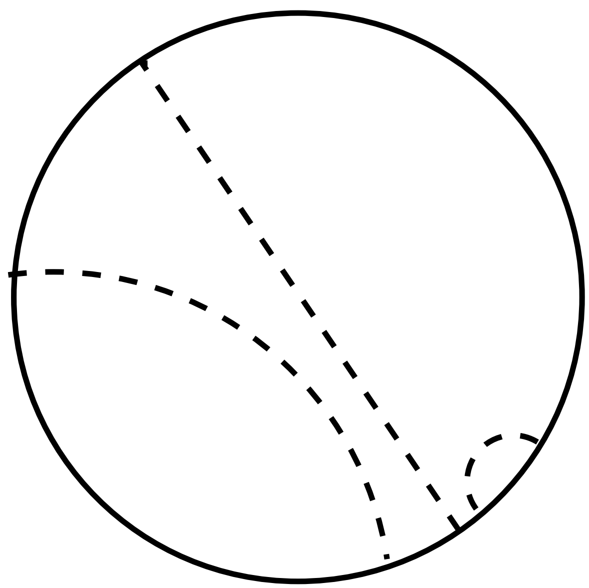


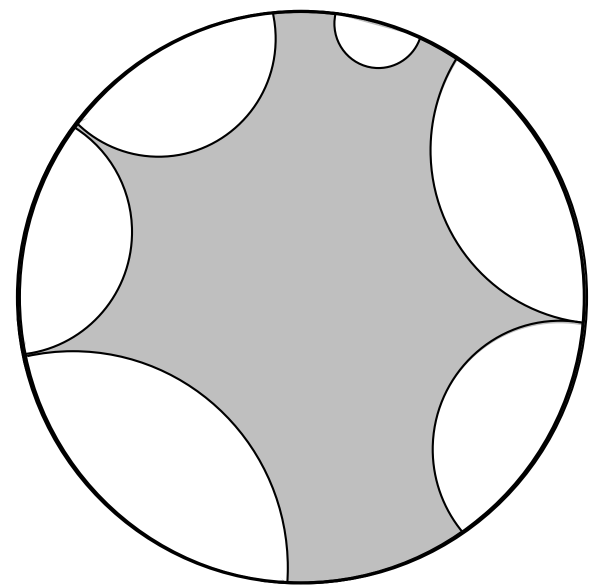


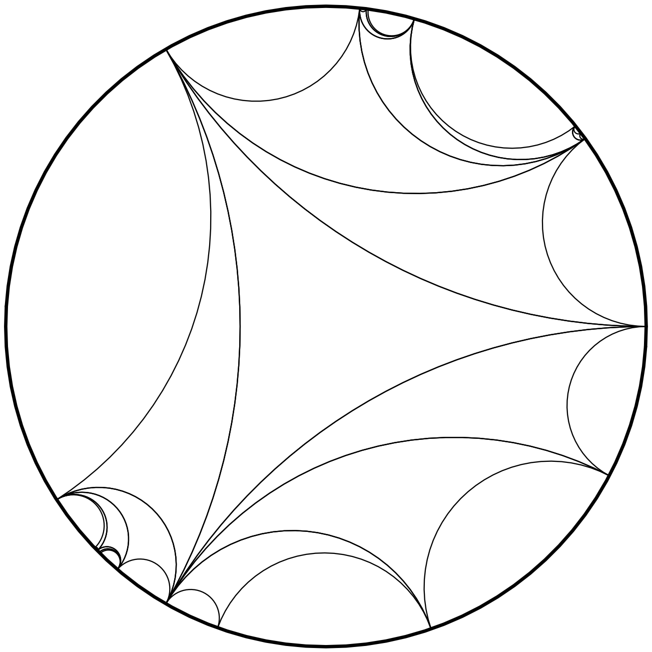
Hyperbolic metric on disk given by

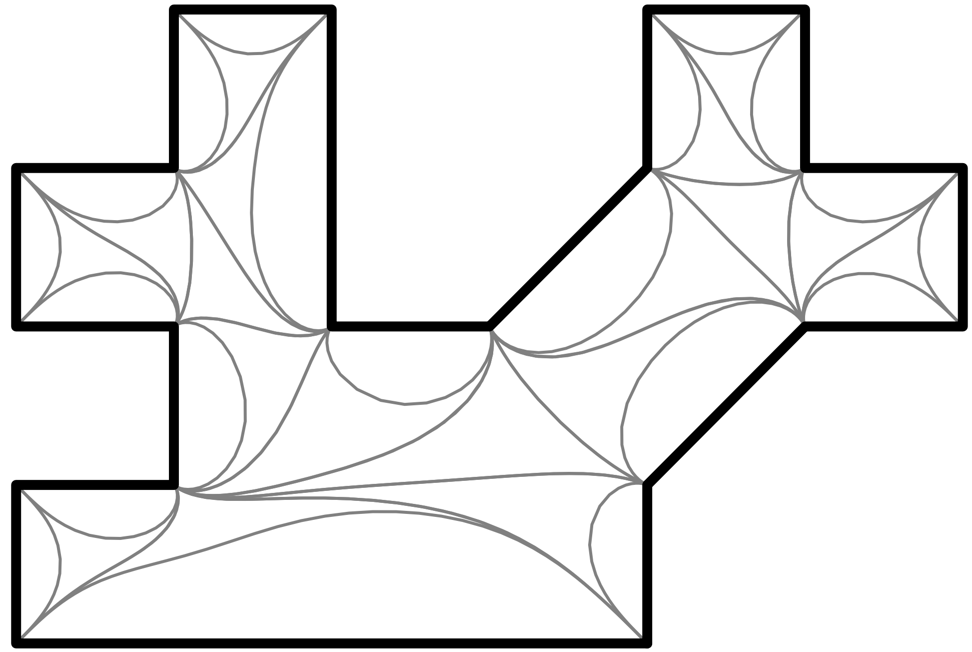
$$d\rho = \frac{ds}{1-|z|^2} \simeq \frac{ds}{\mathrm{dist}(z, \partial D)}.$$

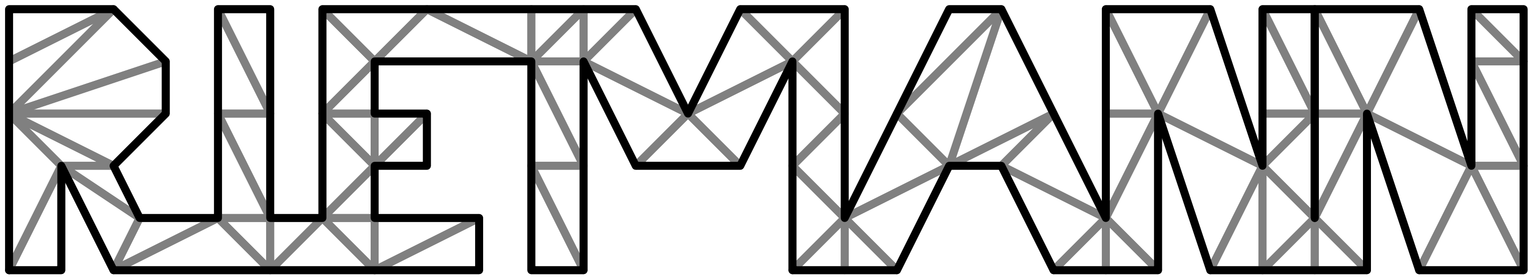
- Geodesics are circles perpendicular to boundary.
- Shaded region is hyperbolically convex.
- Metric transfers via conformal maps to other domains.

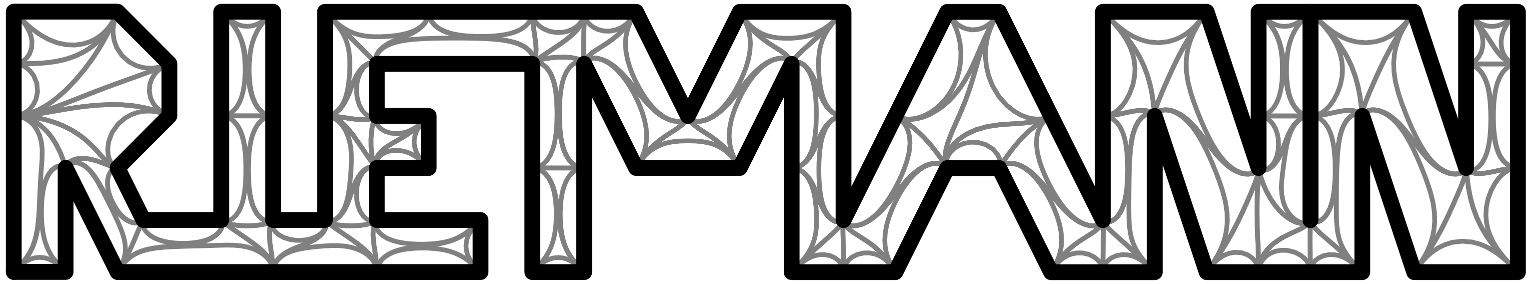









RIEMANN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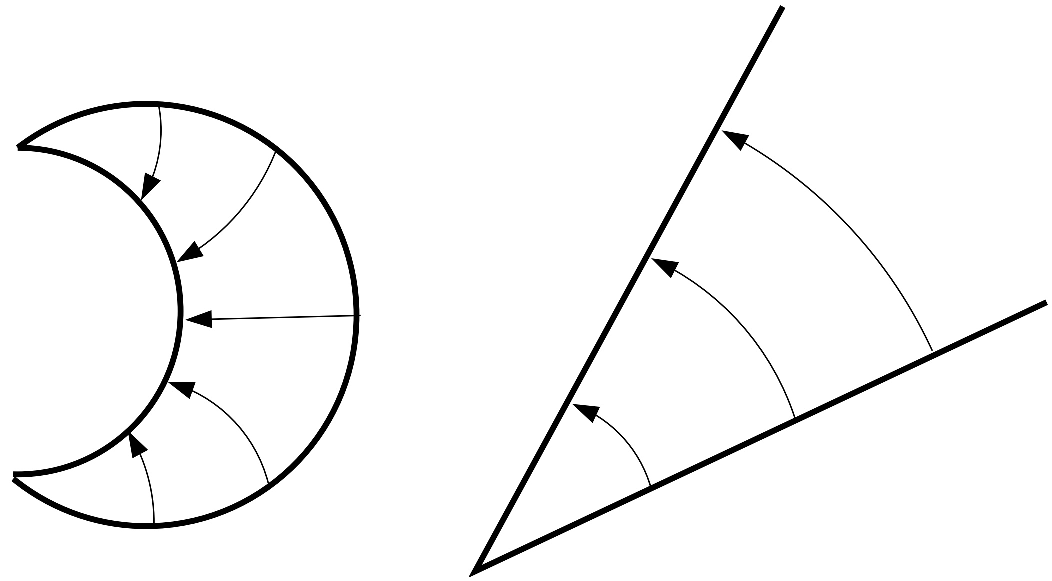
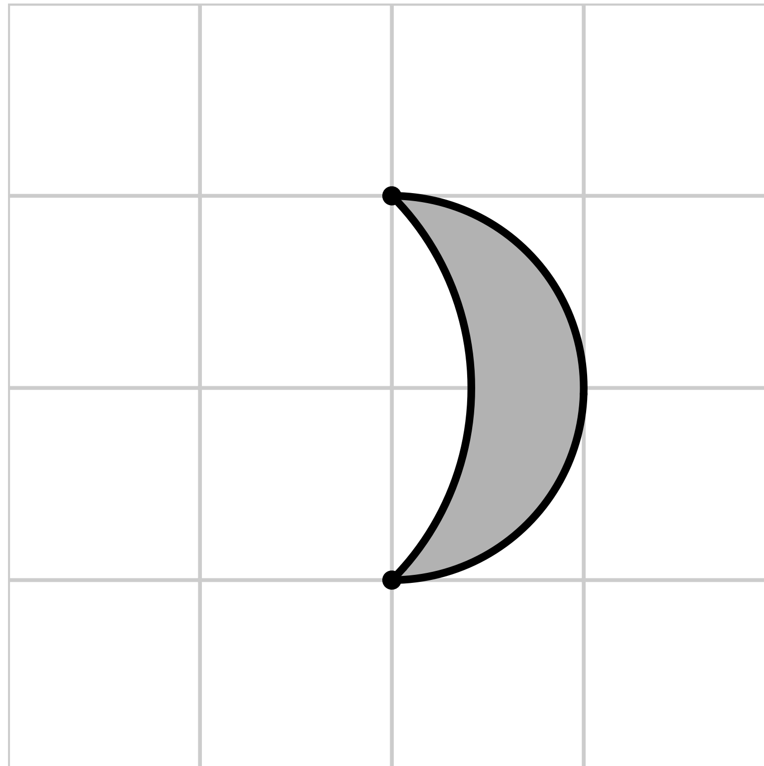
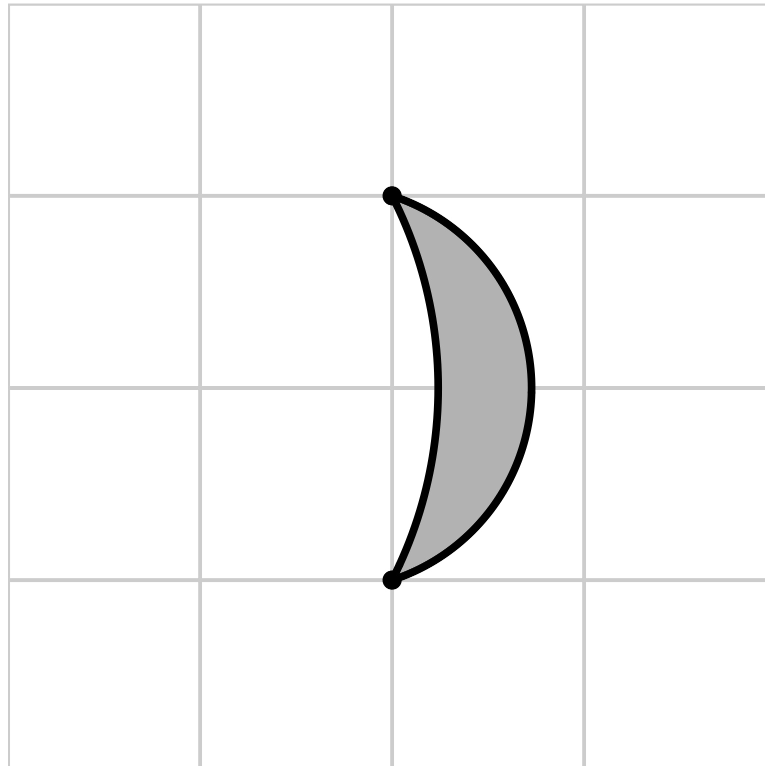
Crescents are regions bounded by two circular arcs.

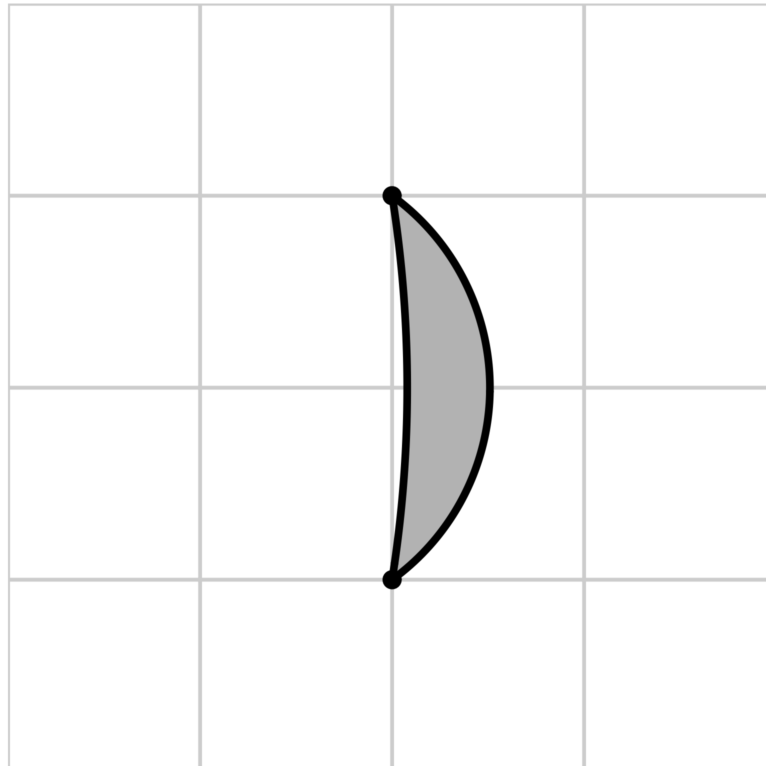


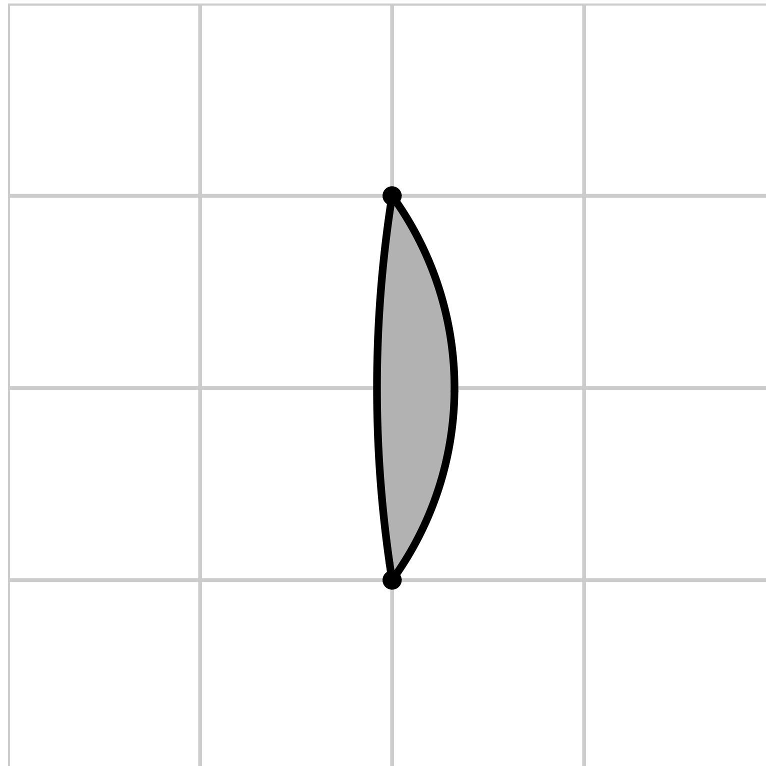
Crescents have foliation into circular arcs perpendicular to boundary. Gives mapping from one boundary arc to other. It is Möbius (linear fractional) transformation.

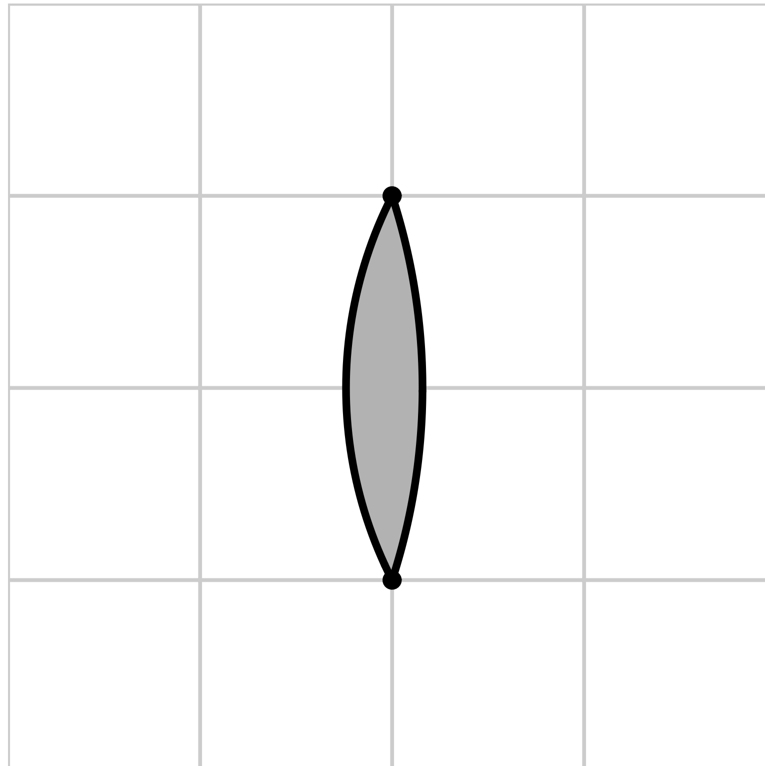
$$z \to \frac{az+b}{zc+d} \qquad \sim \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

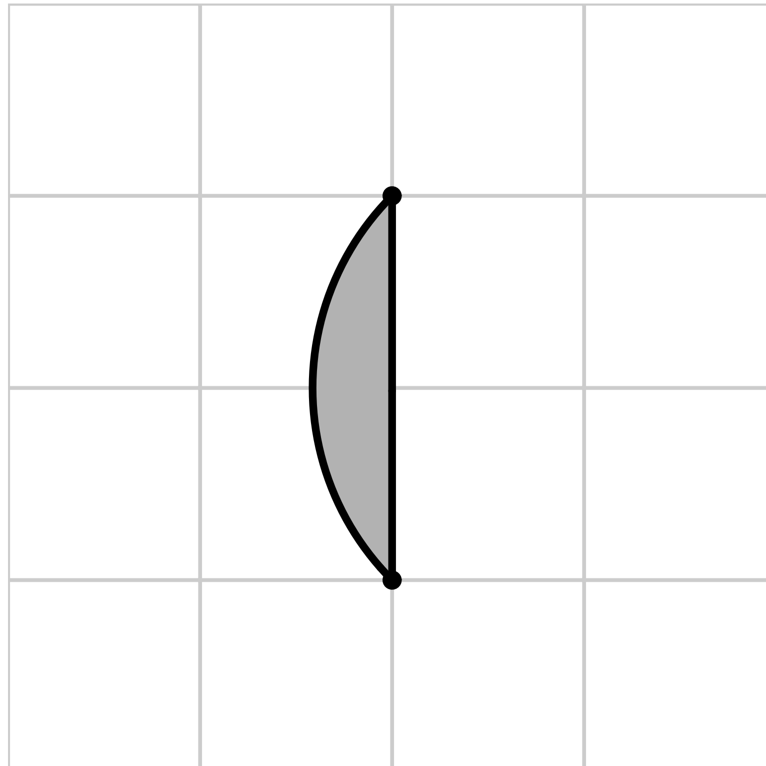


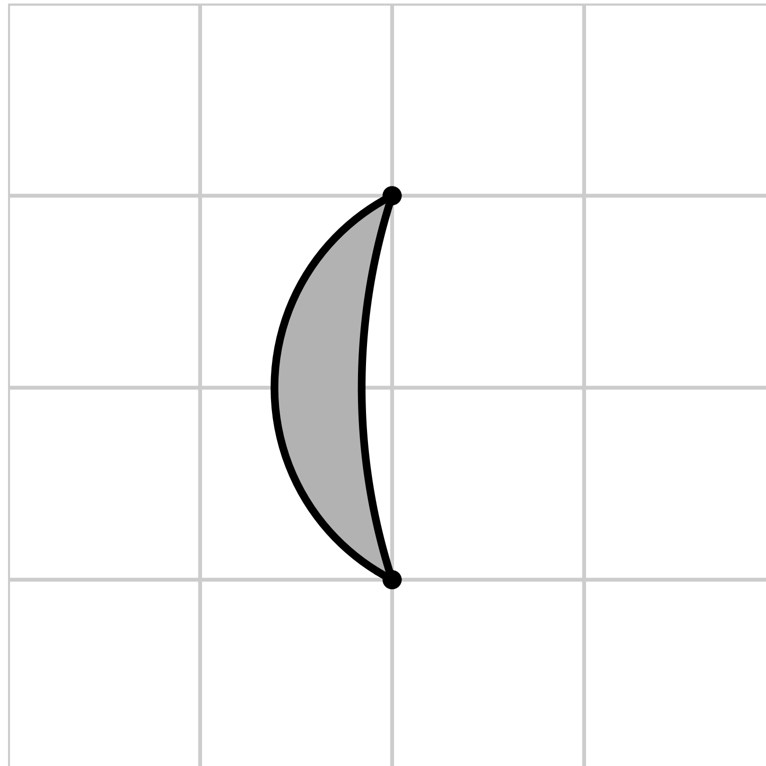


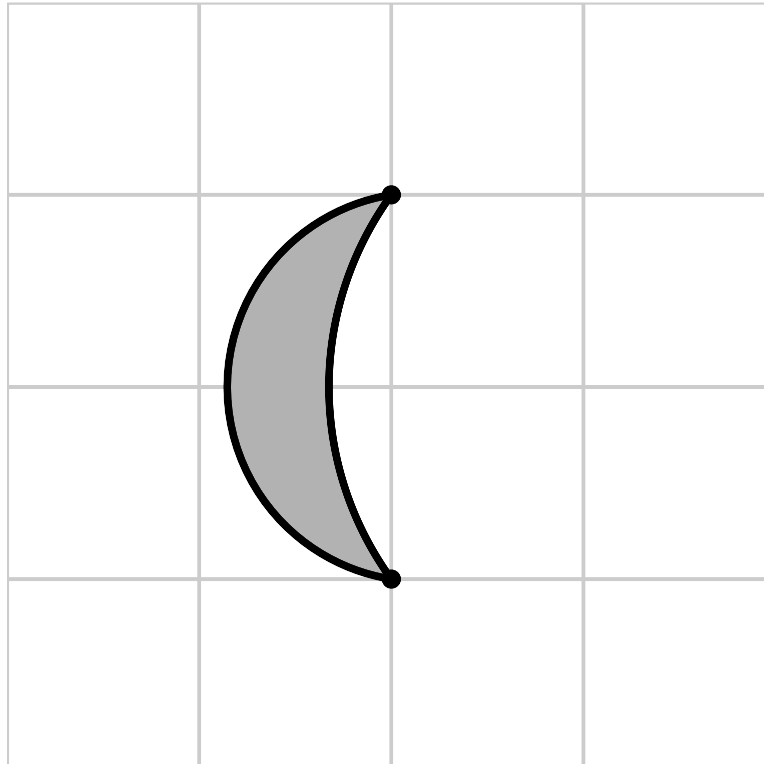


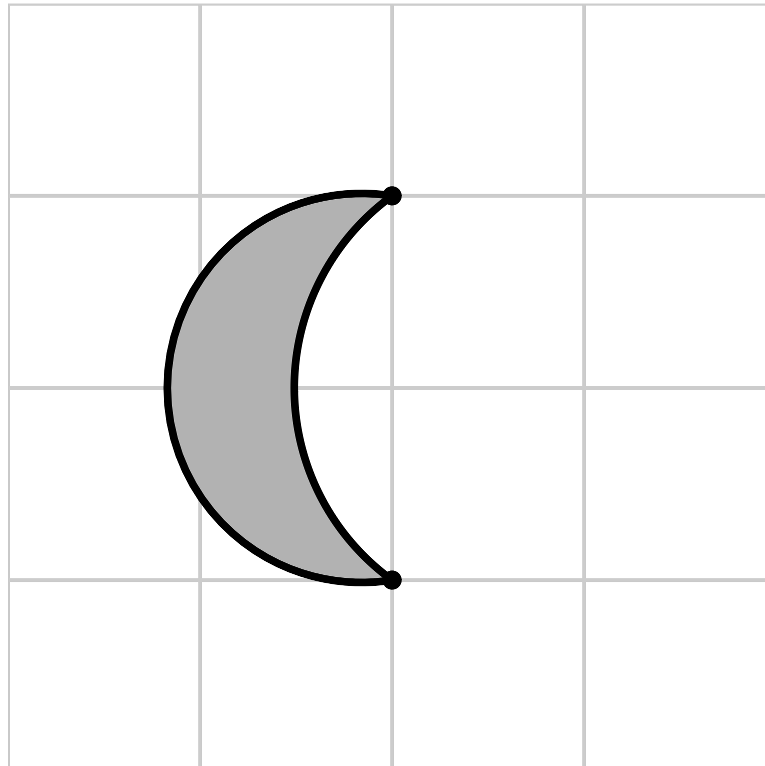


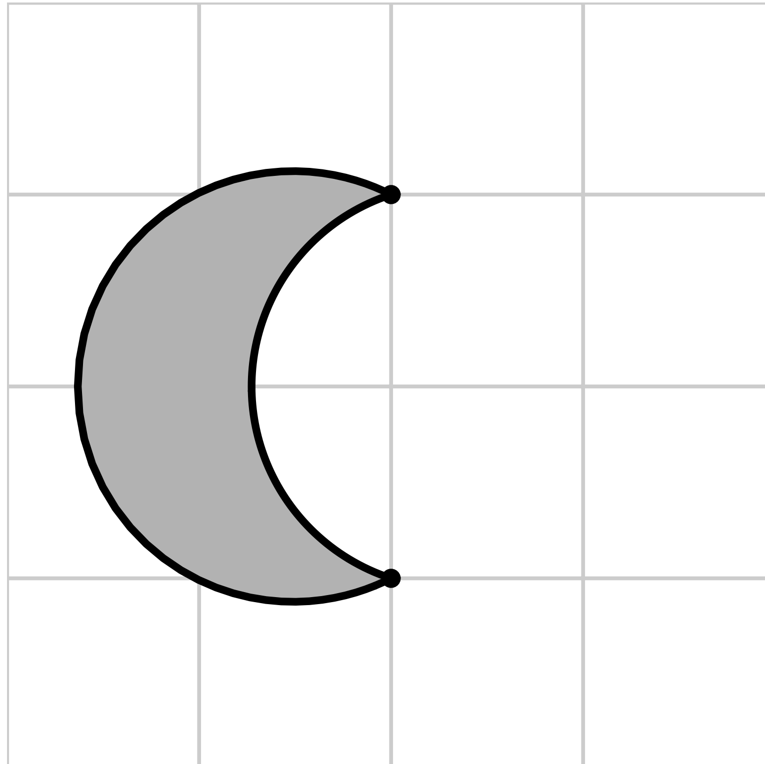


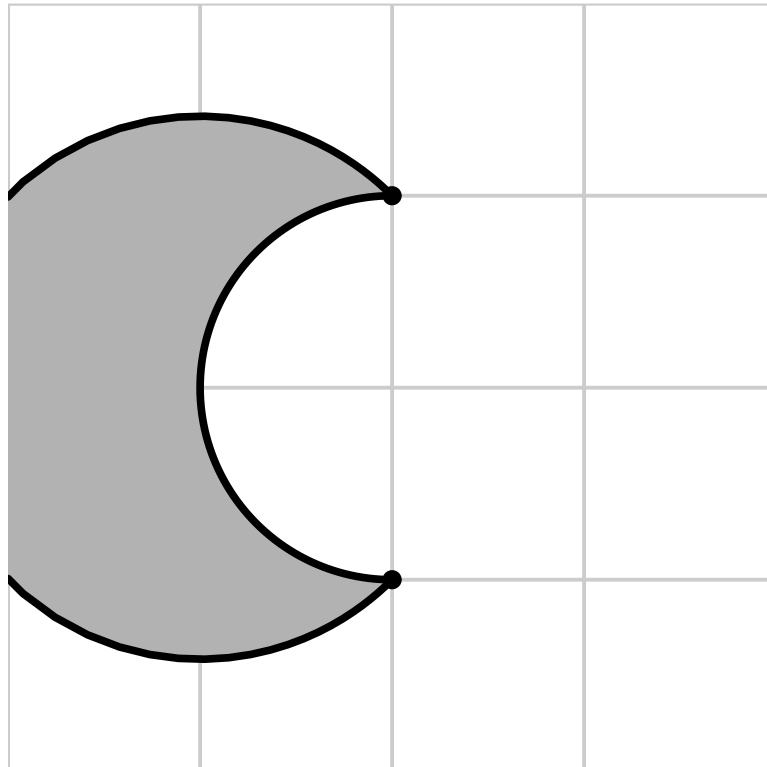


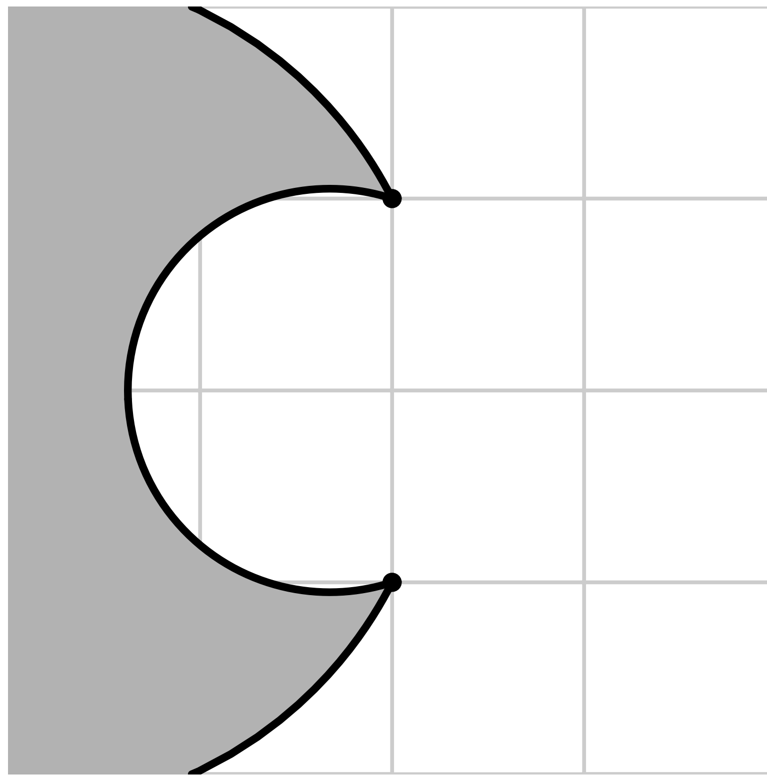


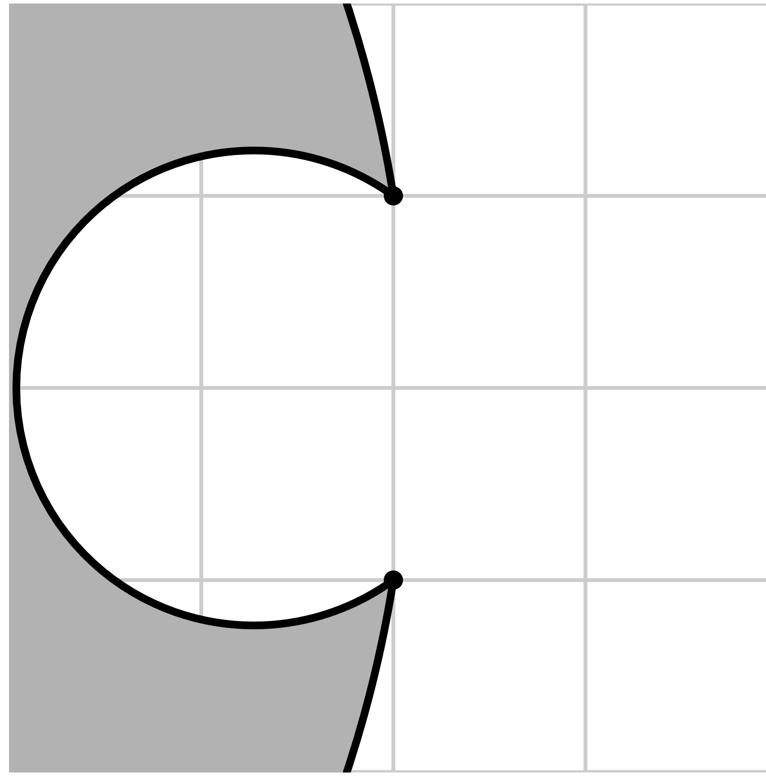


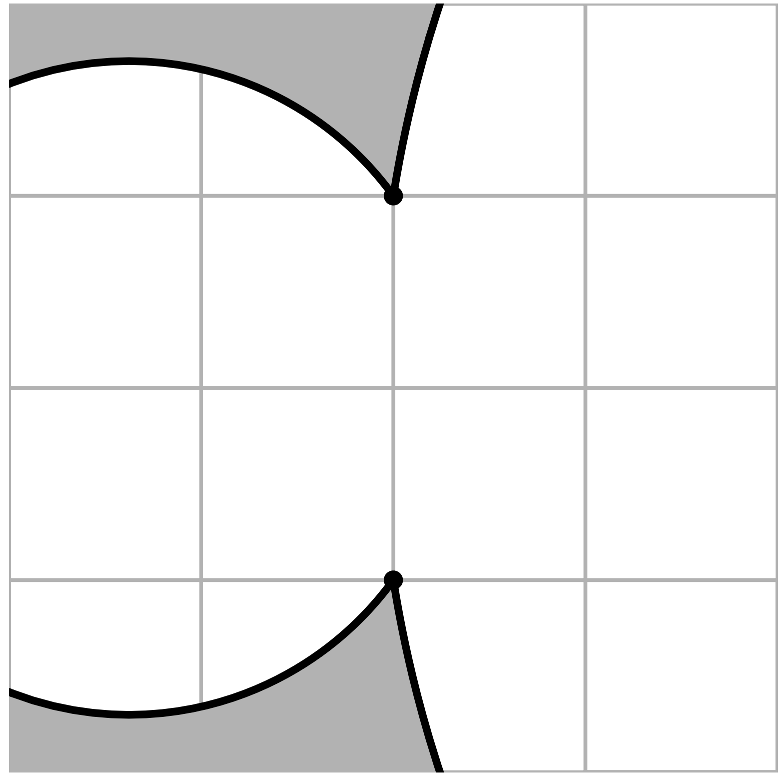


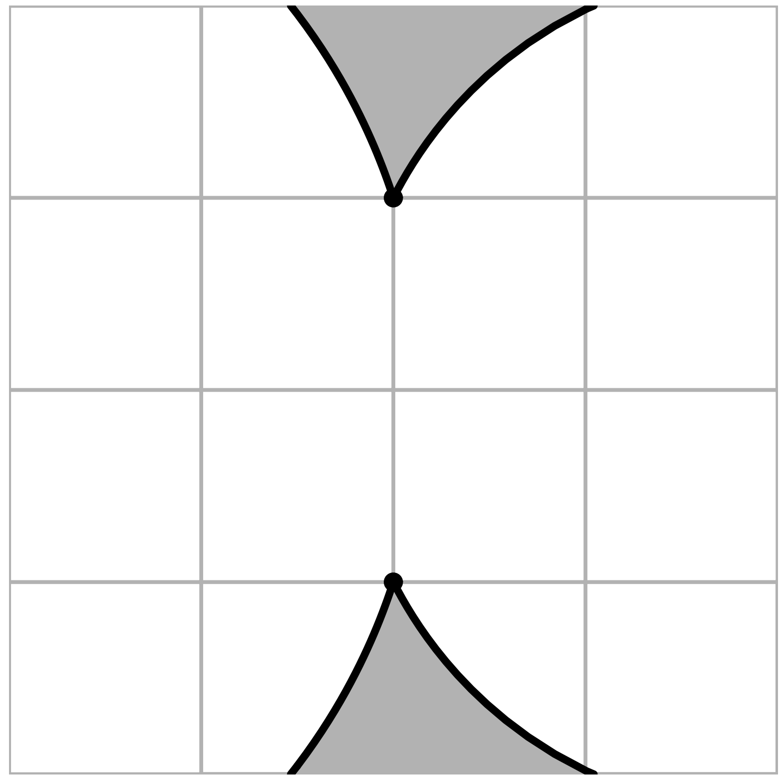


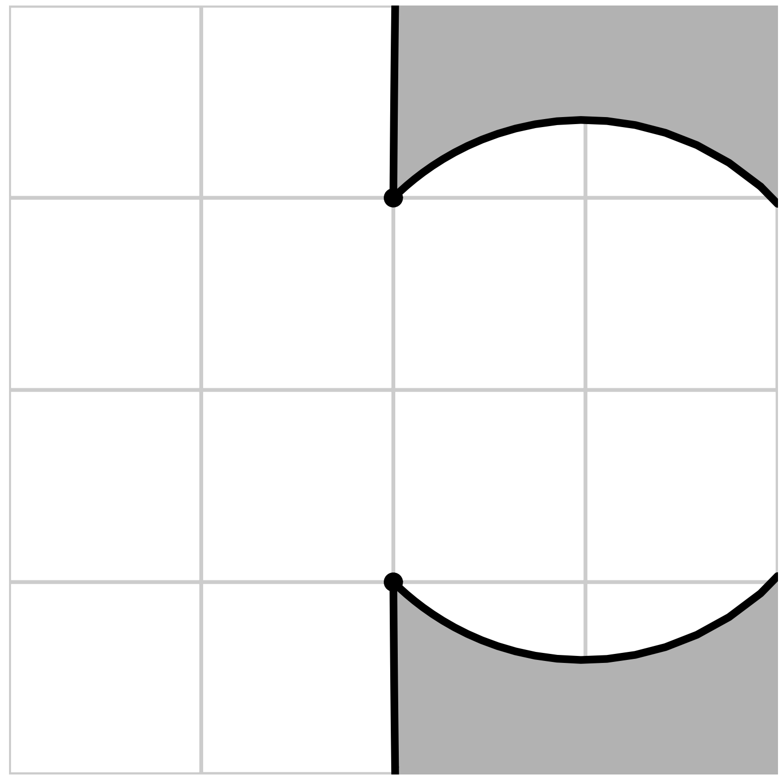


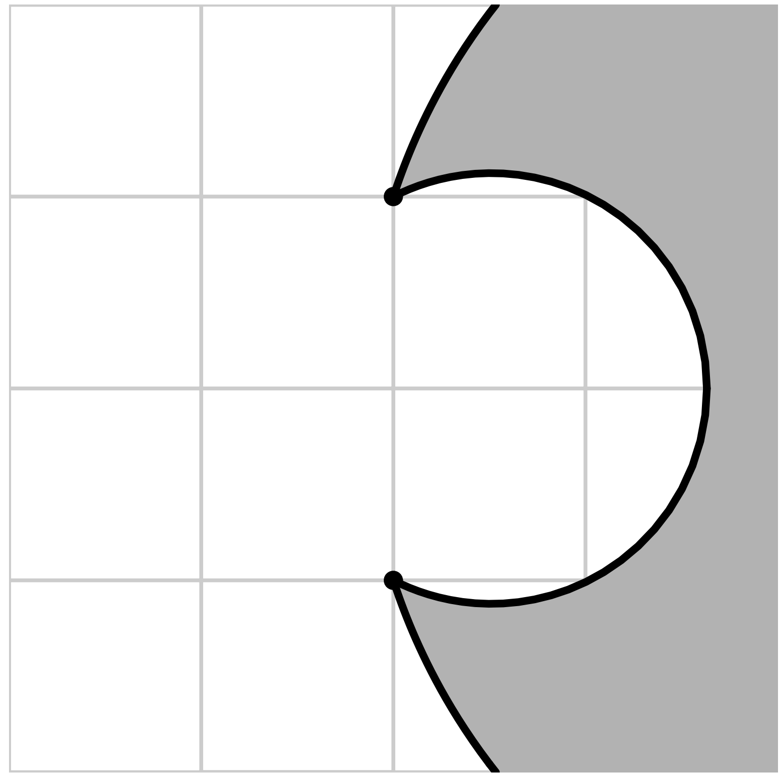


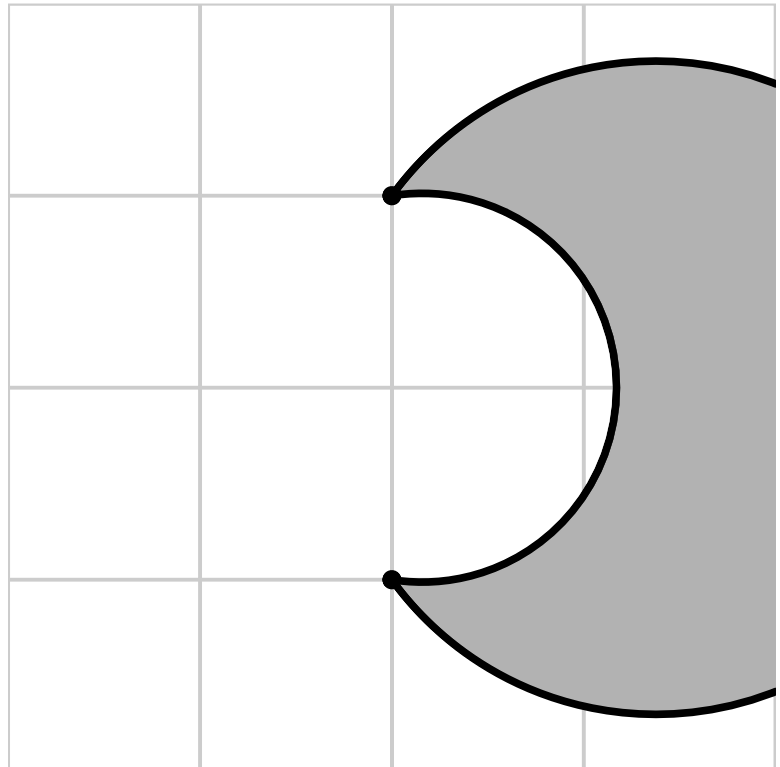


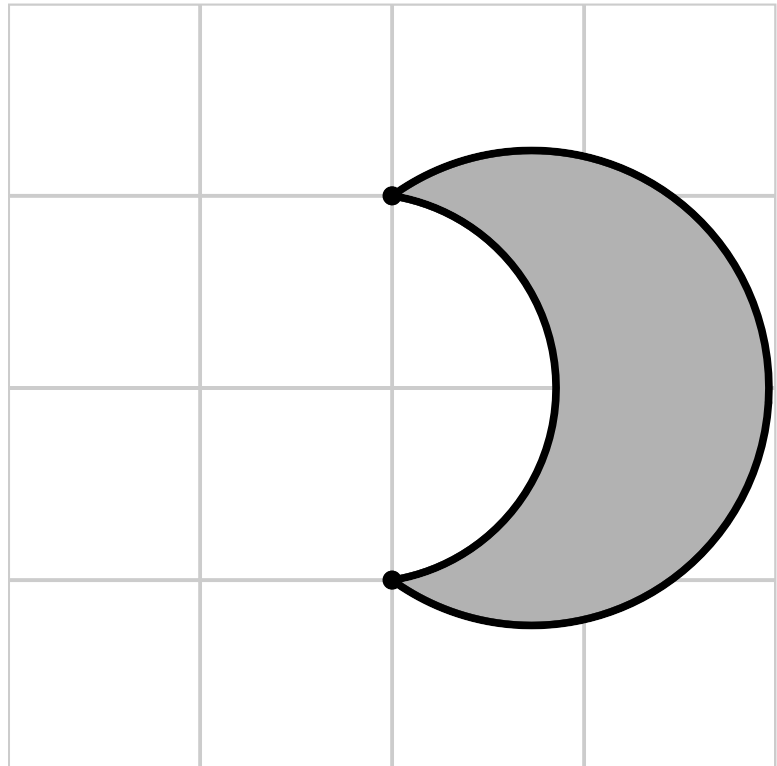


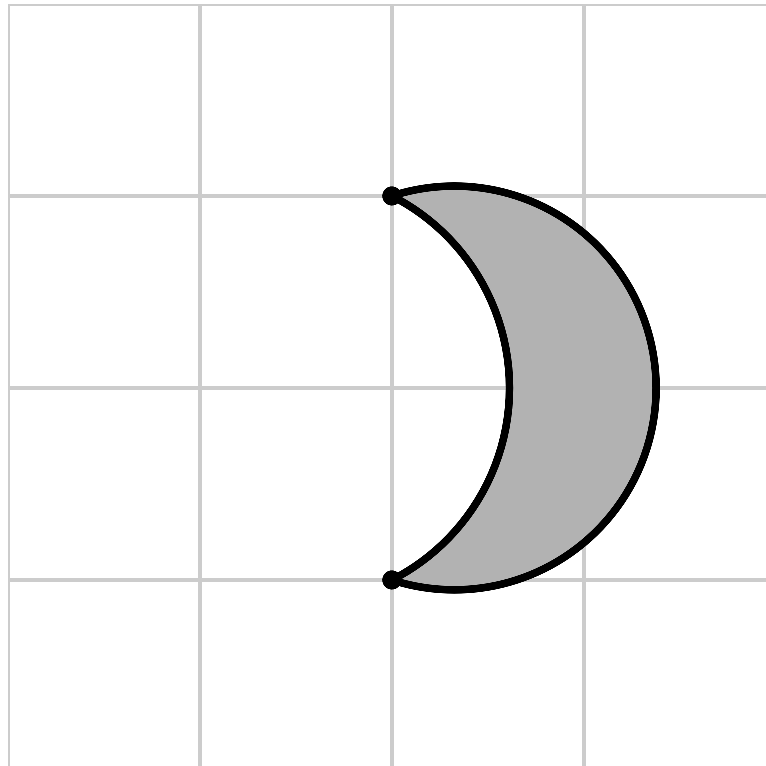


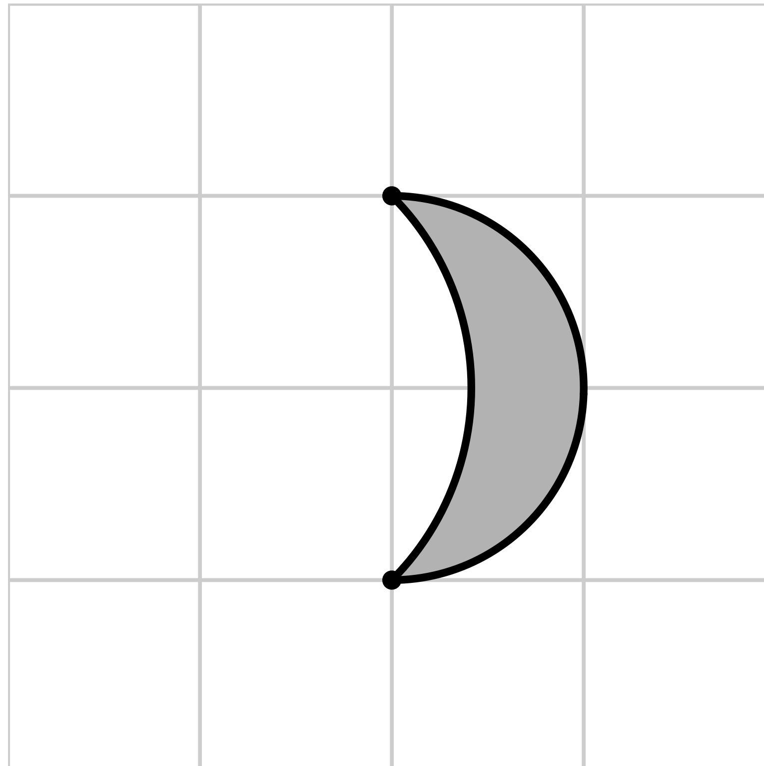




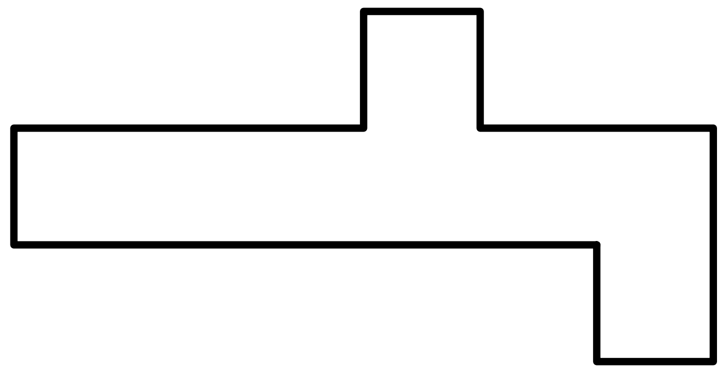


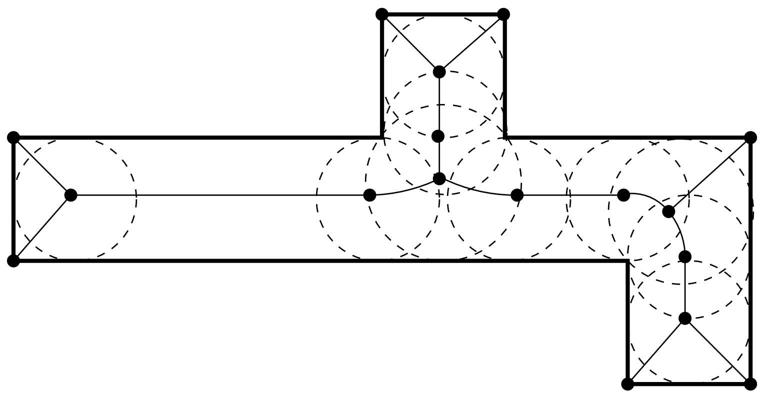






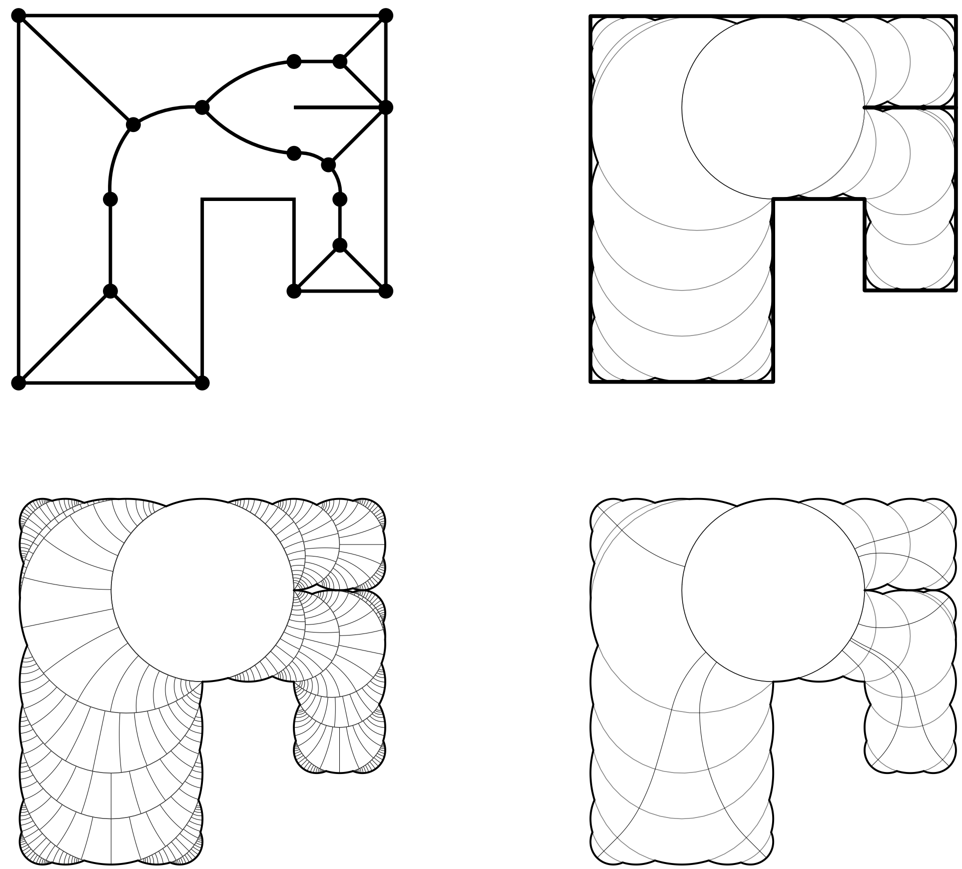
The **medial axis** consists of centers of disks in Ω which touch the boundary in at least two points.



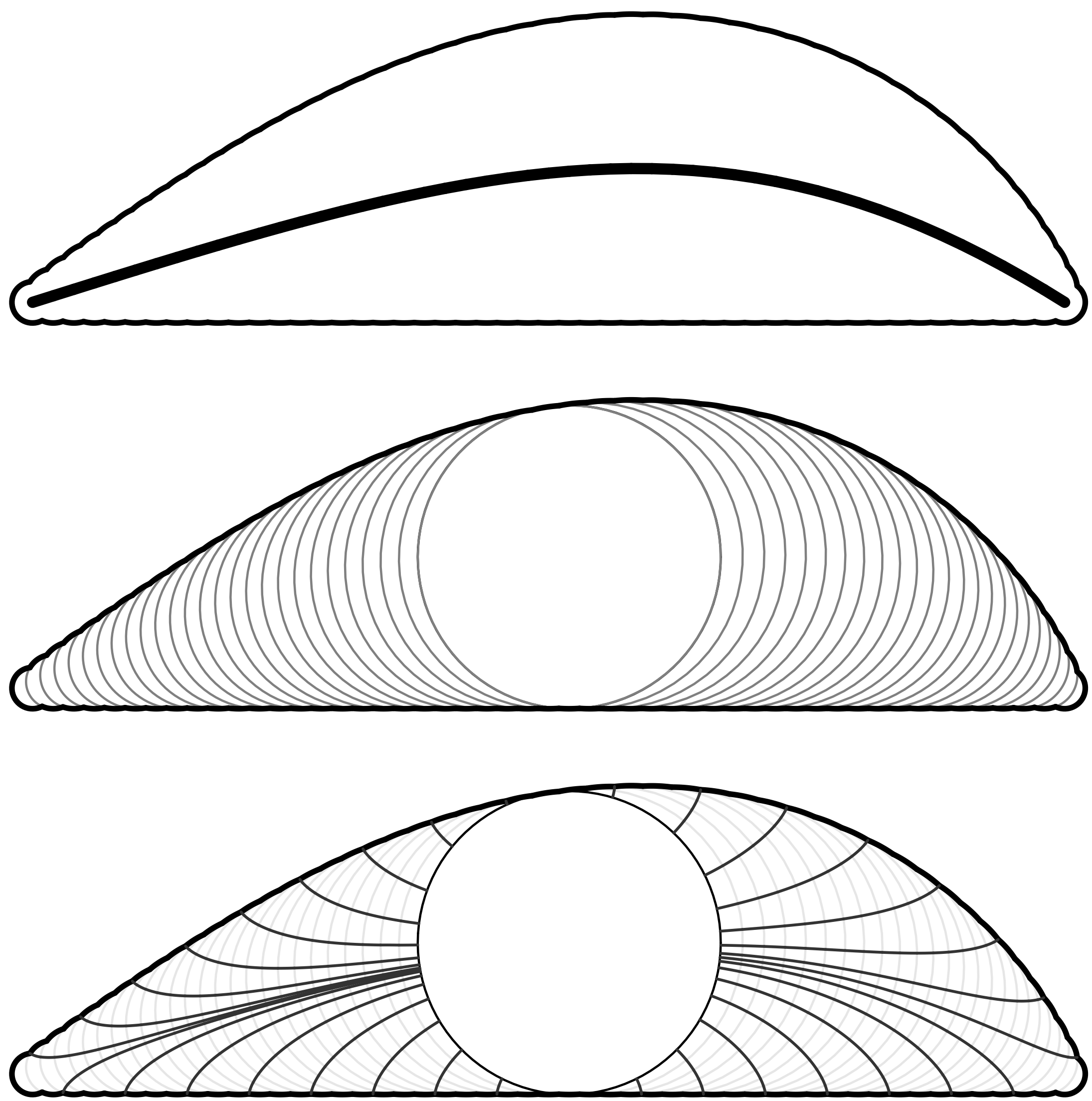


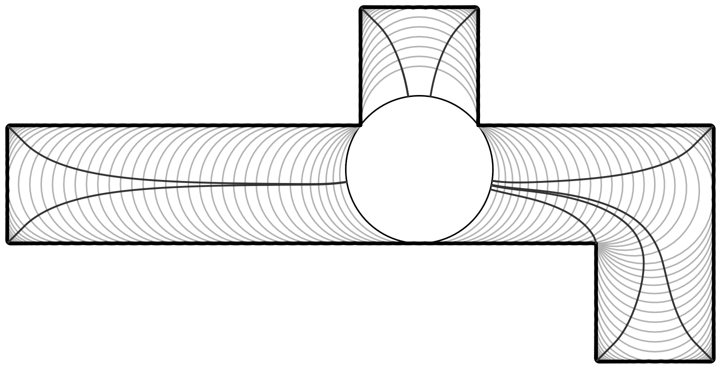
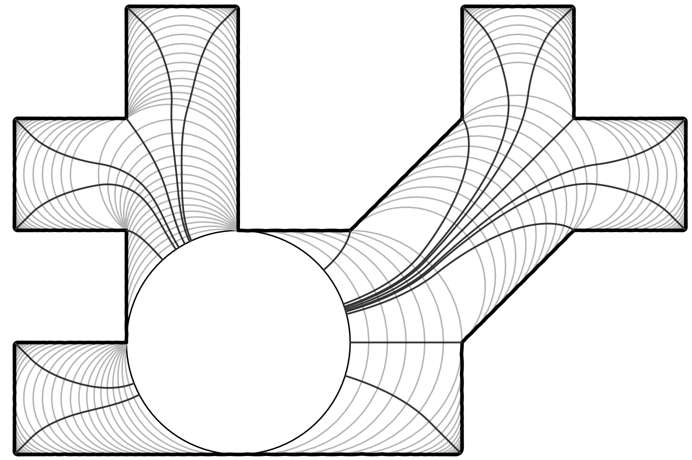
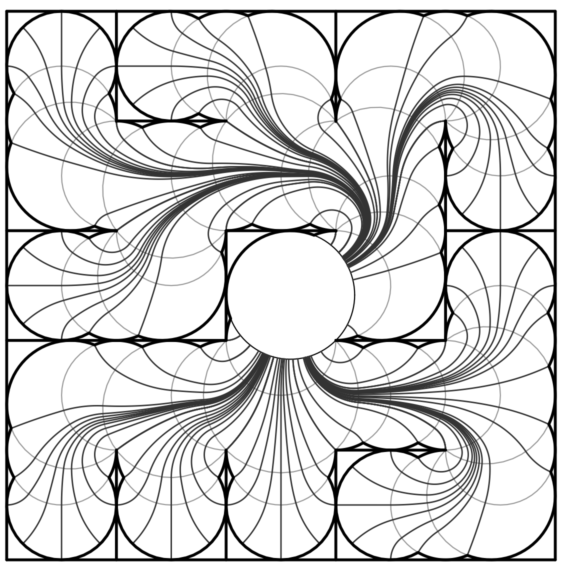
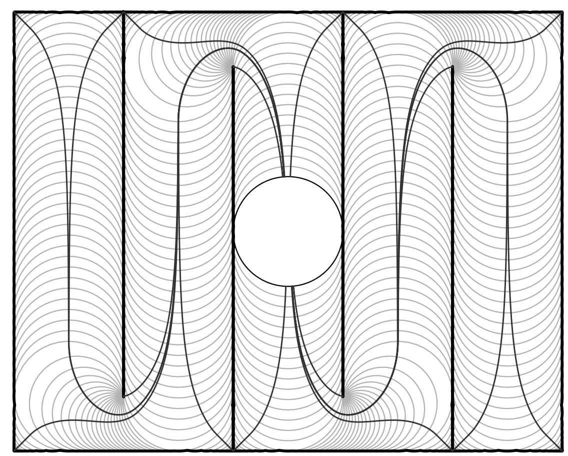
Discretize MA to approximate by a finite union of disks.

Fast Almost Riemann Mapping: Can construct a map from n-gon Ω to disk in $O(n)$ time and is "close to" the conformal map.

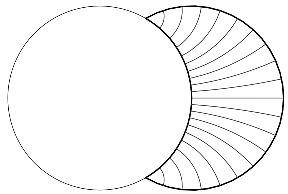


Similar flow for any simply connected domain.

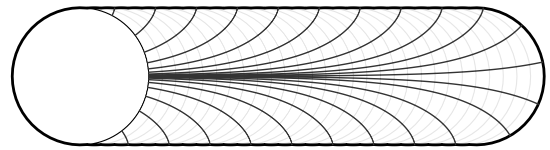


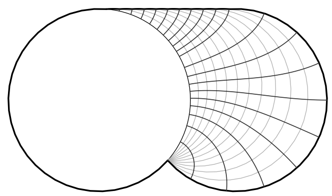
Flow defines Möbius map between medial axis disks. Explicit formula for edges of medial axis.



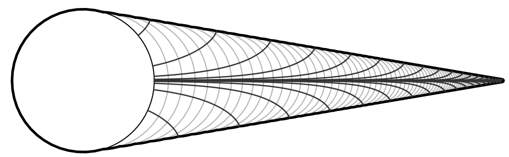
point-point



edge-edge



point-edge



convex vertex

Can compute iota from MA in $O(n)$. MA computable in $O(n)$ due to Chin, Snoeyink and Wang.

Plug iota parameters into SC formula. Should get a good approximation to original polygon.

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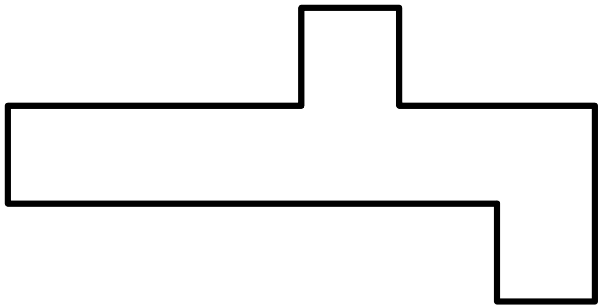
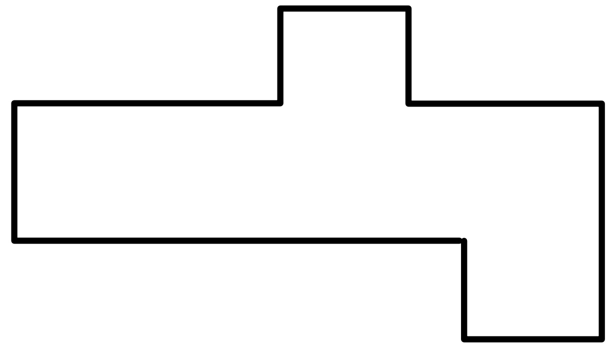
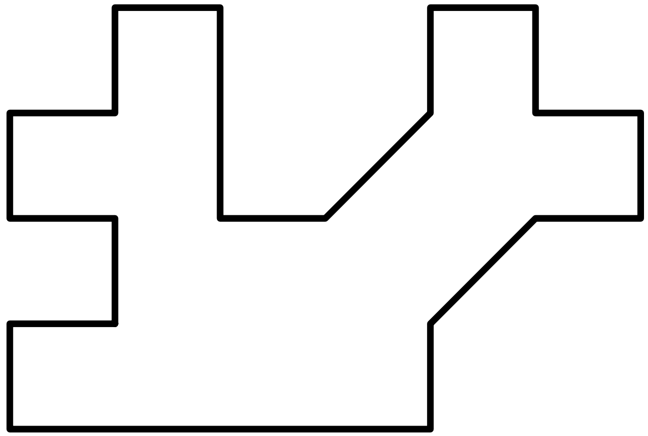


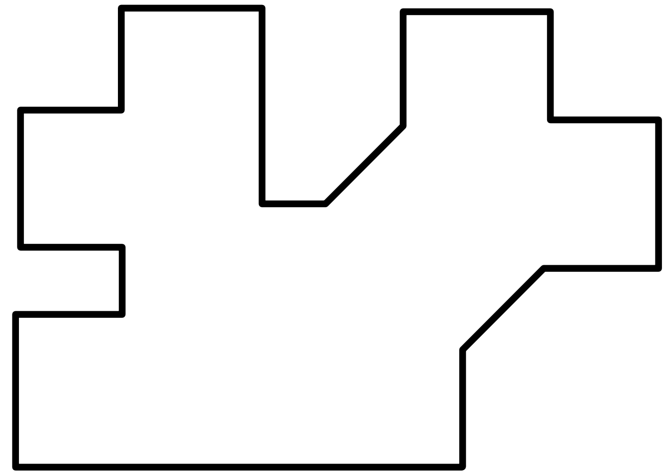
Target Polygon

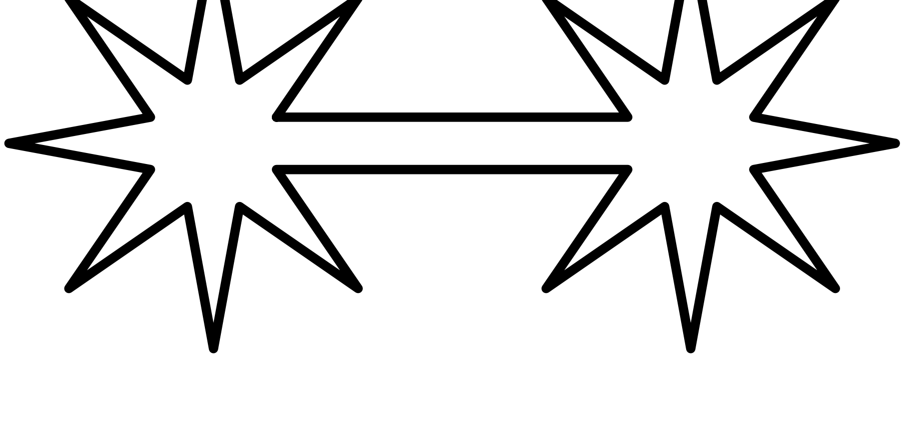


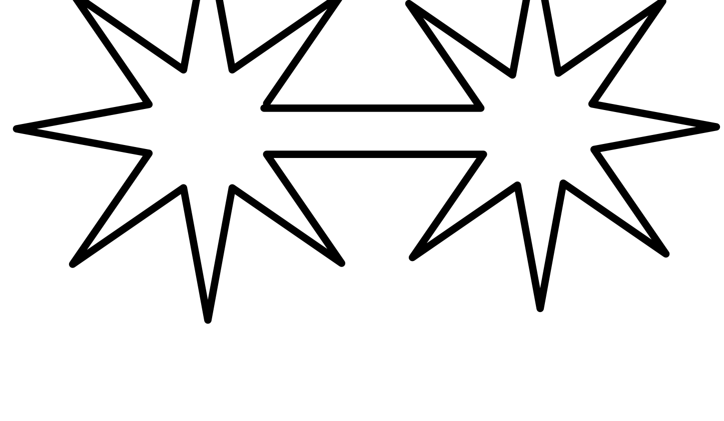
SC-image with iota parameters

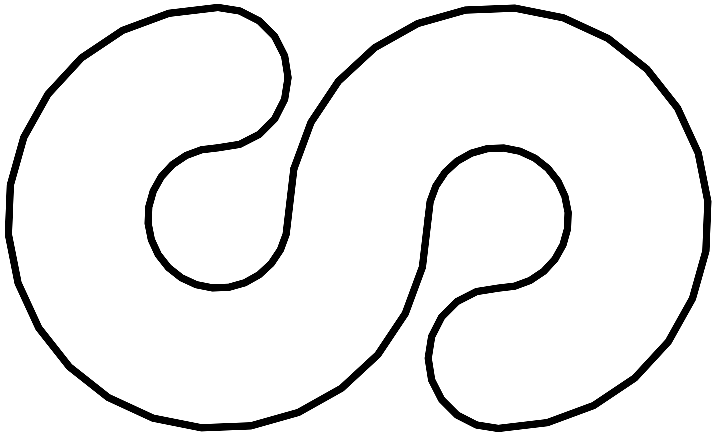
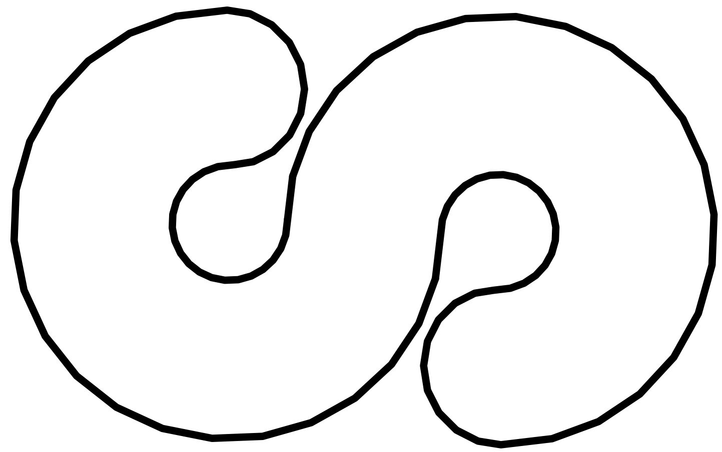
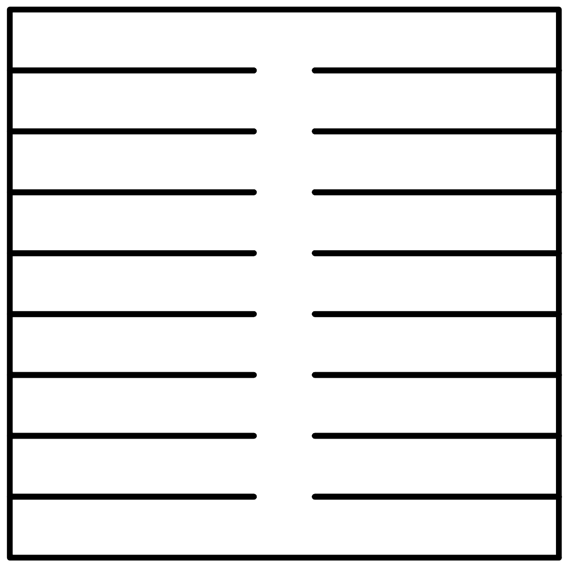
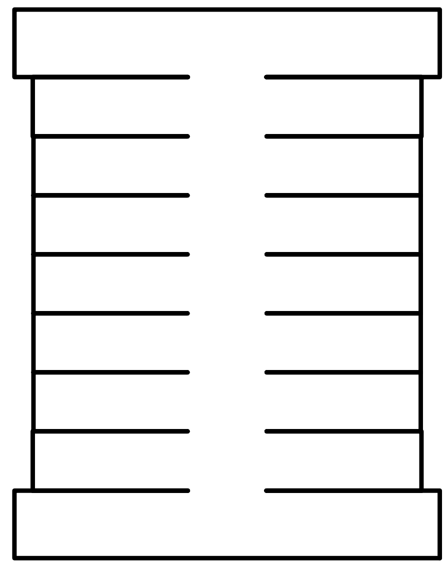
MA flow gives "formula" for SC-parameters $\{z_k\}$.









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Because iota is the boundary map of a conformal map for a surface with the same boundary as Ω.

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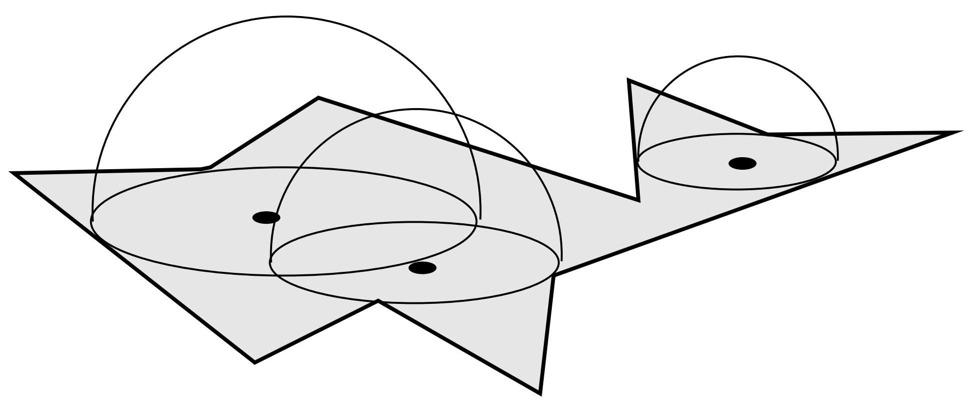
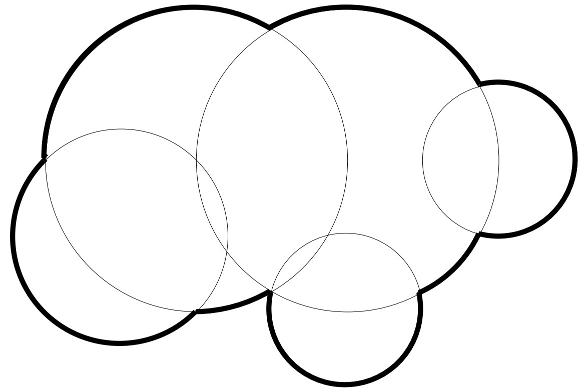
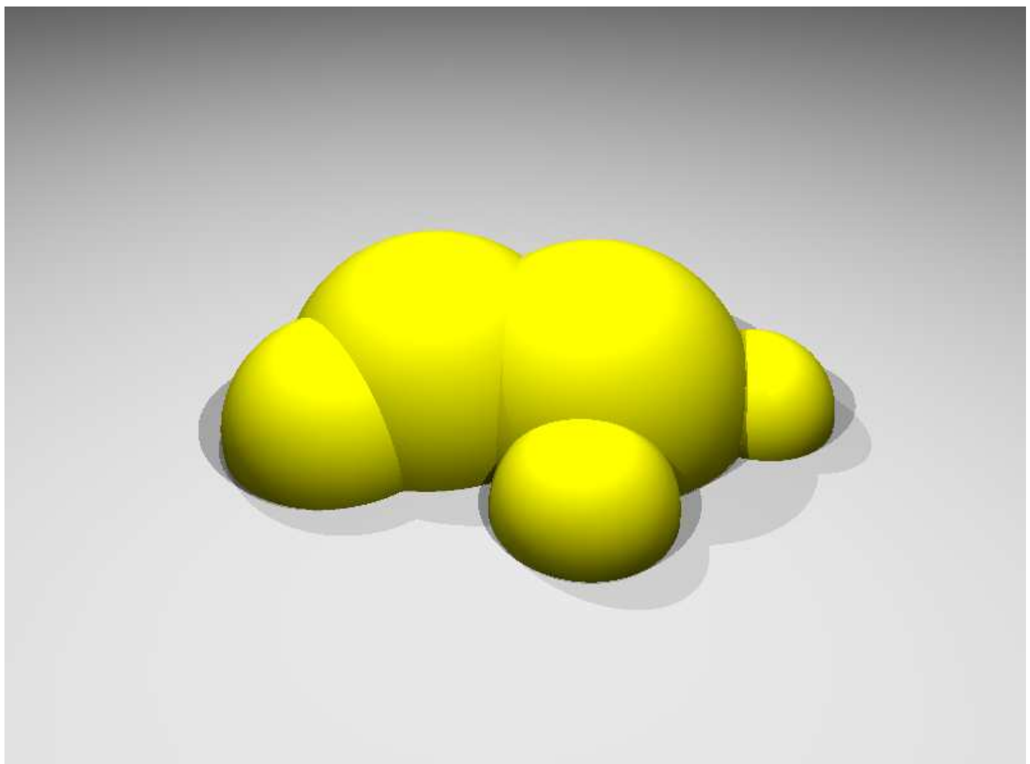
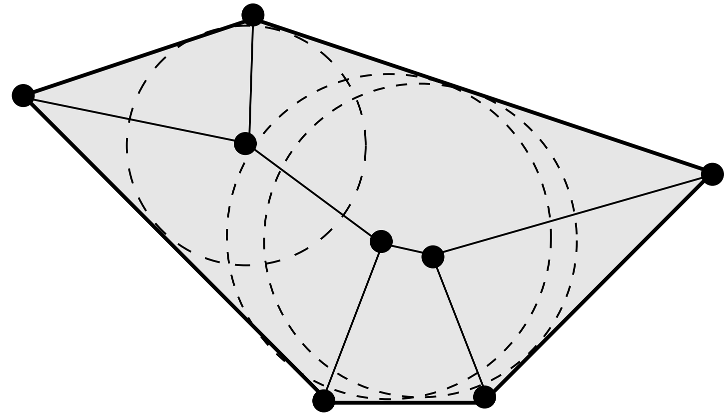
Take union of all hemispheres whose bases lie inside Ω. Upper envelope is the "dome" of Ω.

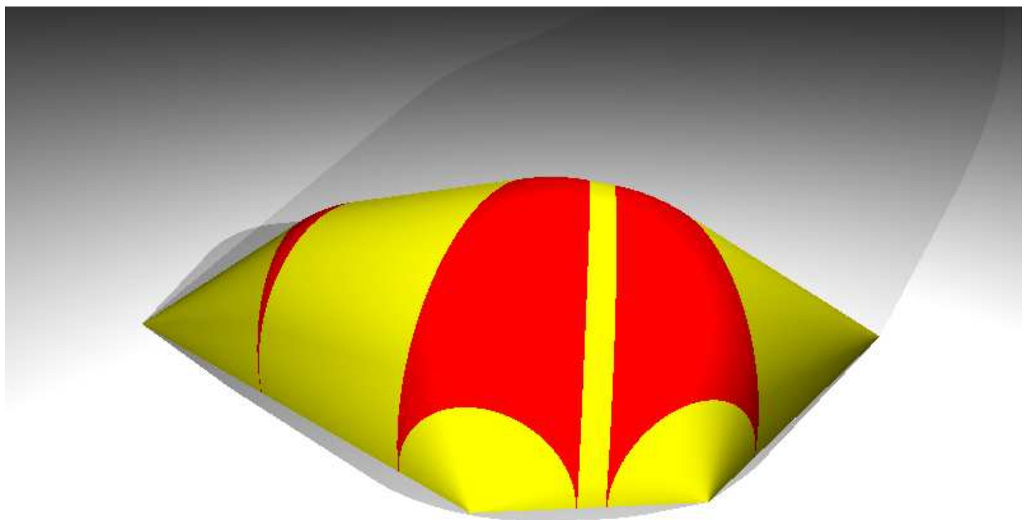


(This is hyperbolic convex hull of Ω^c.)

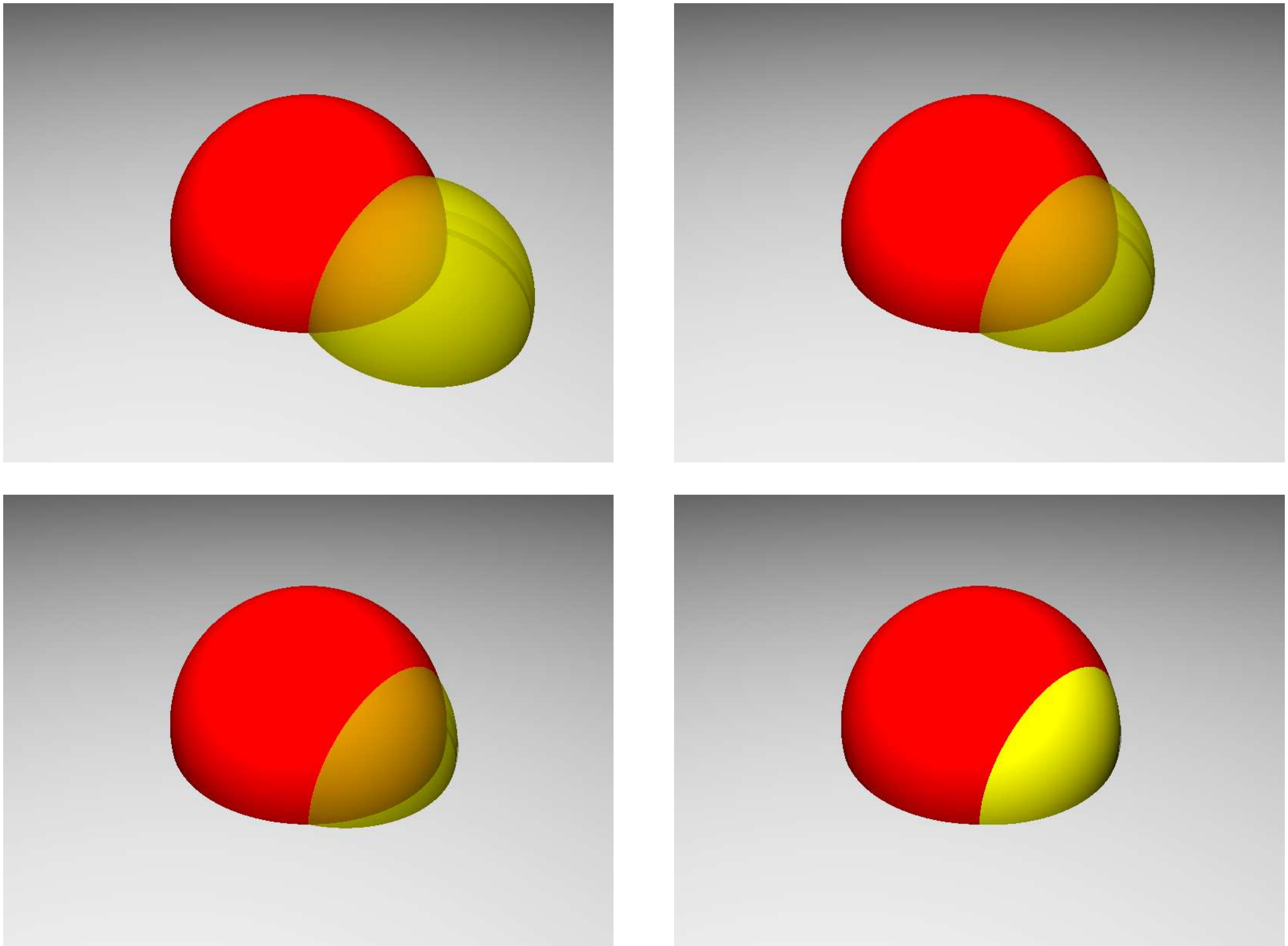




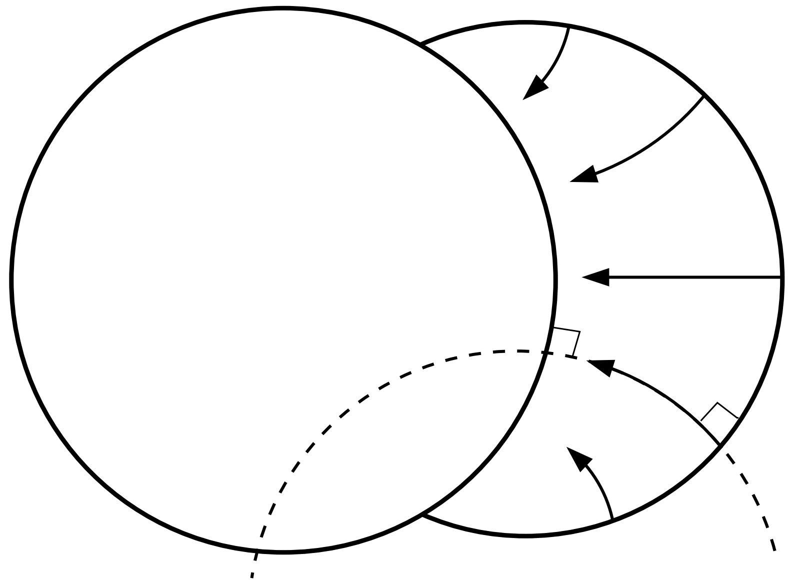


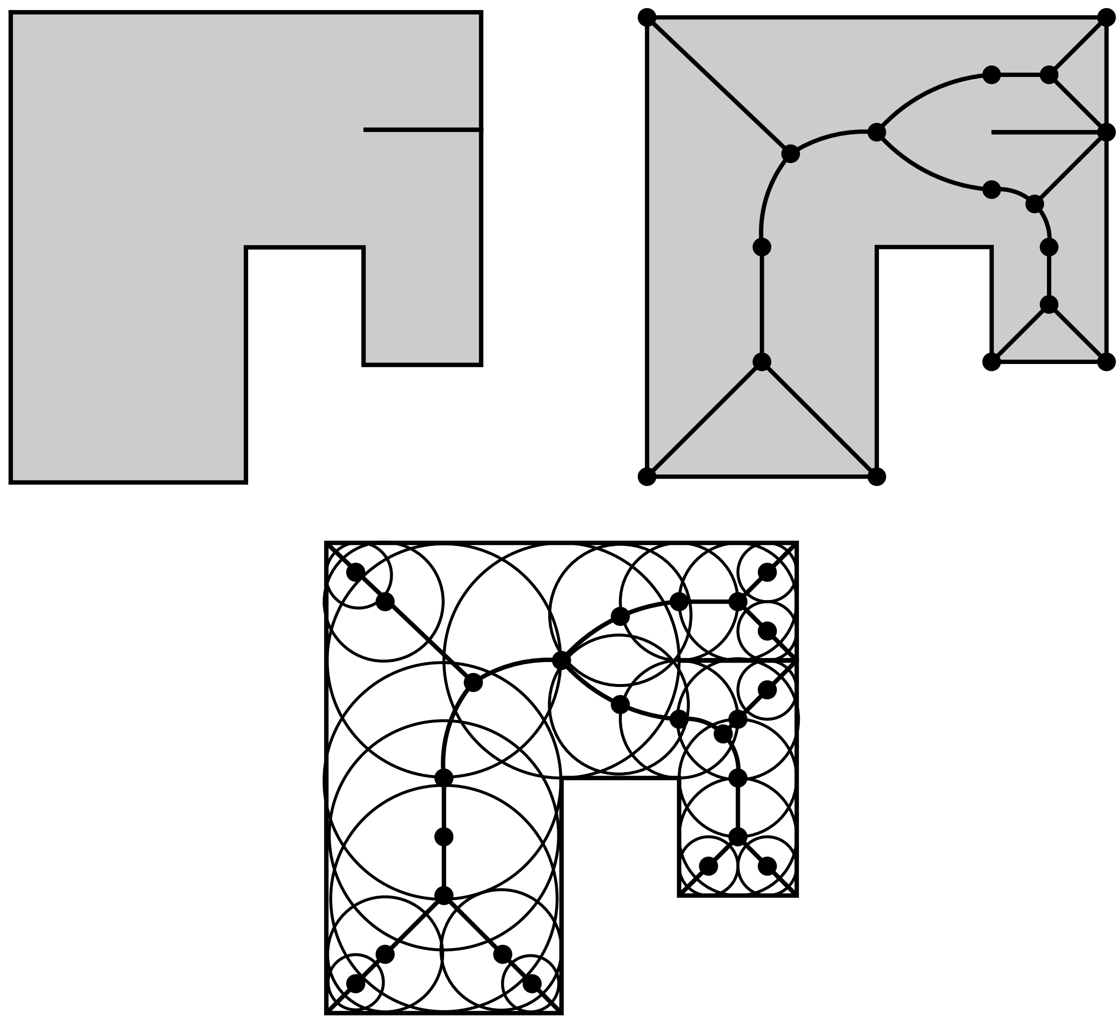


It is easy to map any dome conformally to a disk.

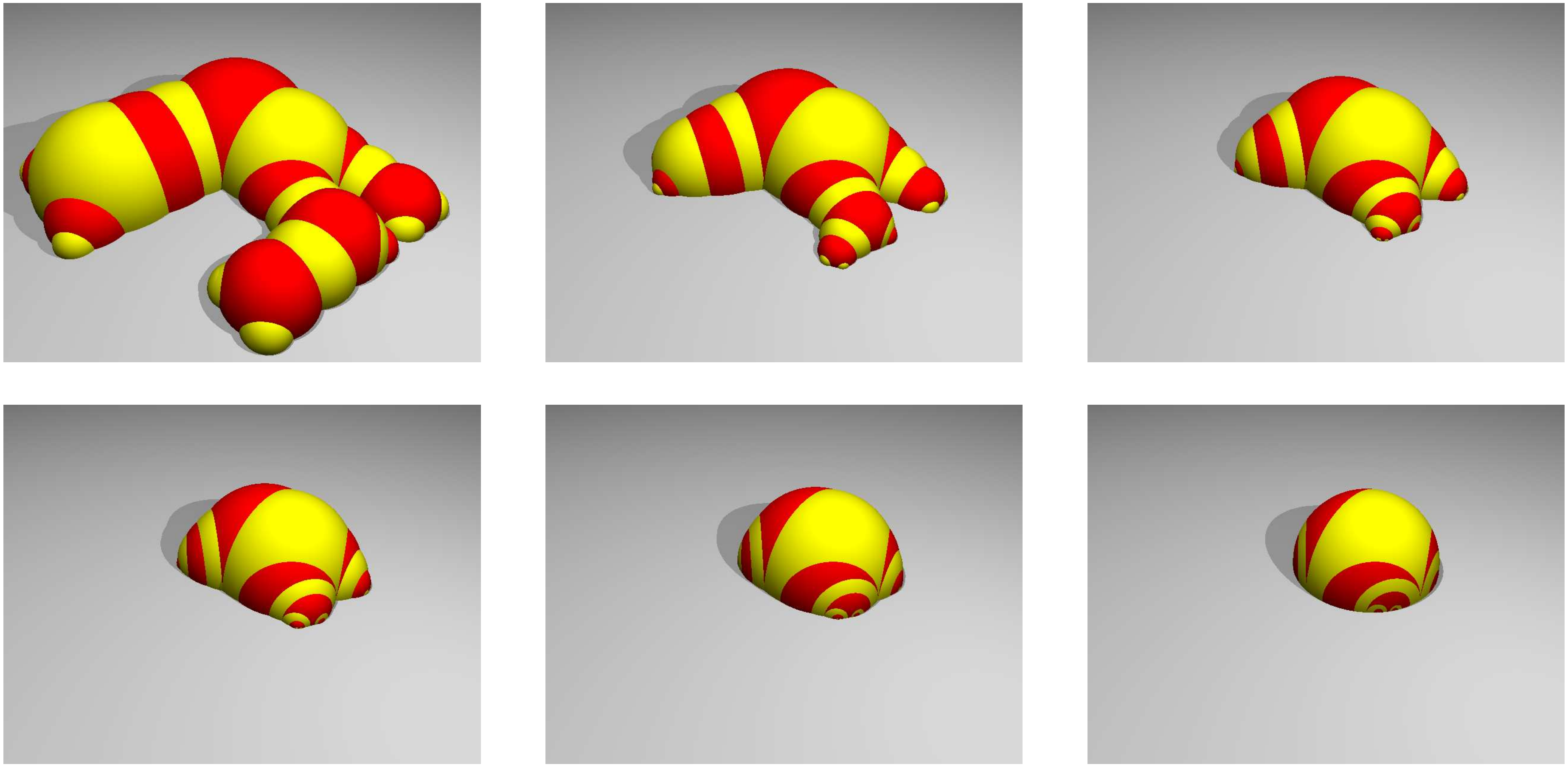


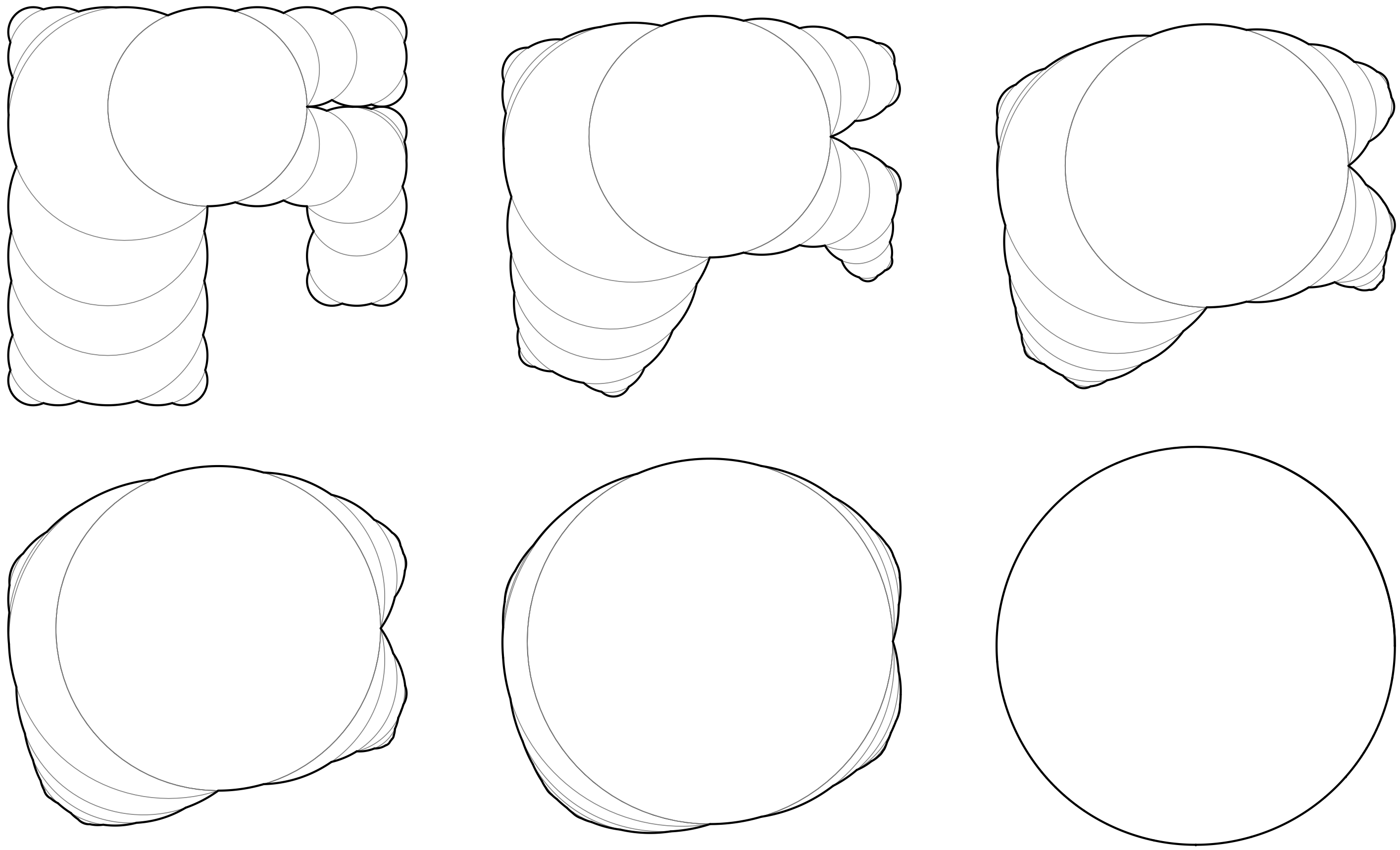
On base, foliate crescent by orthogonal arcs. Points in plane move along these arcs.





A polygon, medial axis, approximation by disks.



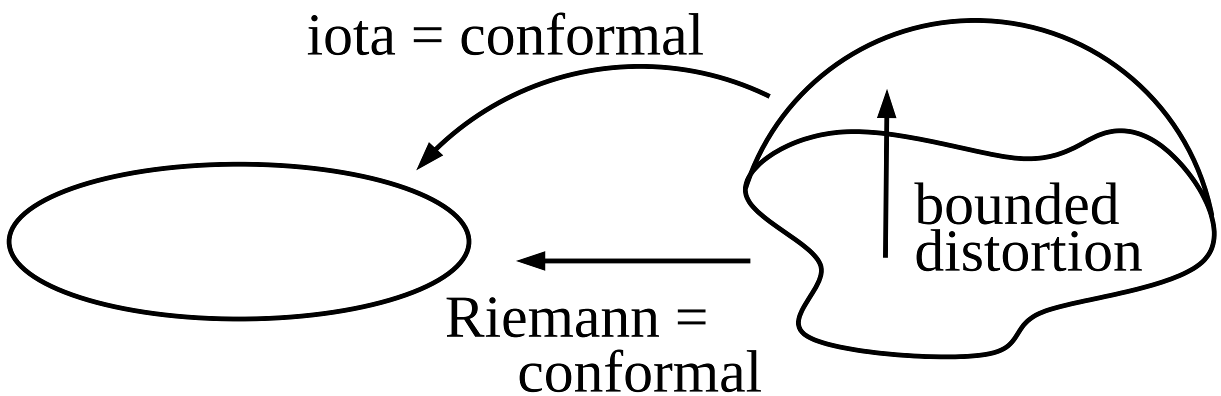


Angle scaling family

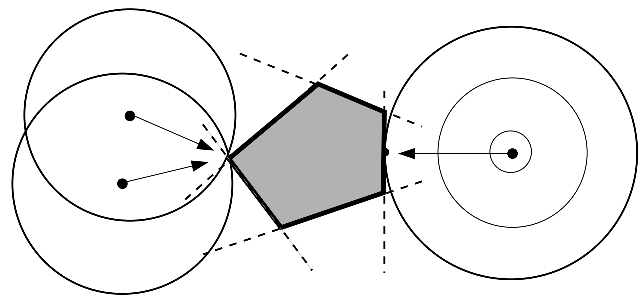
Iota = conformal map from dome to disk.

Riemann = conformal from base to disk

From base to dome = Sullivan's convex hull theorem

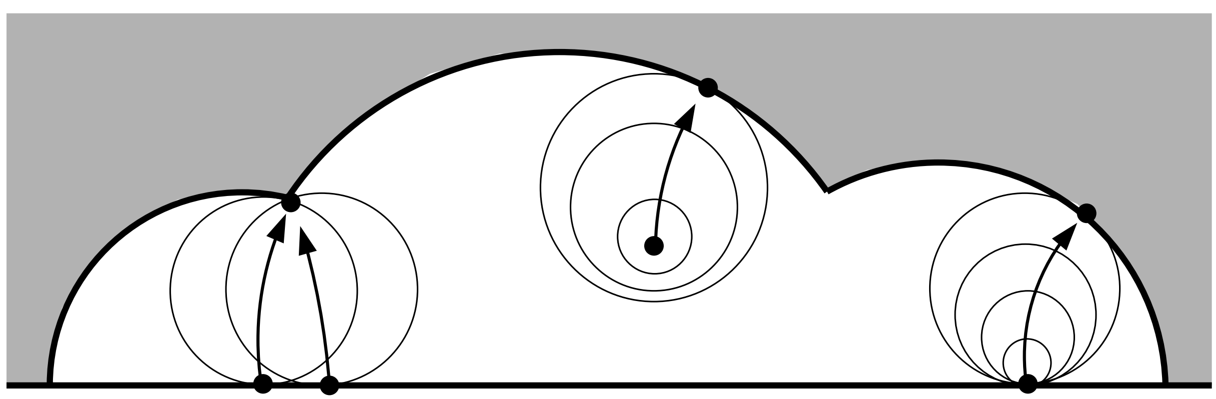


Nearest point map in $\mathbb{R}^n$ is Lipschitz.



Nearest point retraction in hyperbolic space extends to map $R : \Omega \to S =$ Dome and is a quasi-isometry (Dennis Sullivan, David Epstein and Al Marden, C. Bishop)

$$\frac{1}{A}\rho_\Omega(x,y) - B \le \rho_S(R(x), R(y)) \le A\rho_\Omega(x,y).$$



P

P^2

$P =$ hyperbolic geometry, University of Warwick

$P^2 =$ computational geometry, UC Irvine

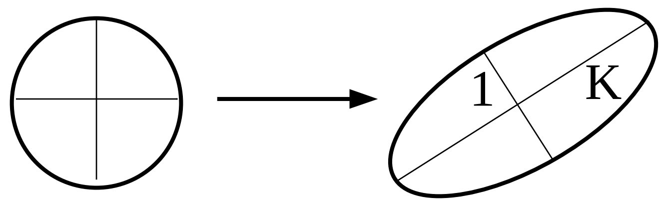
Fast Mapping Theorem:

Given an n-gon we can compute an ϵ-conformal map $f : \mathbb{D} \to \Omega$ in time $O(n \log \frac{1}{\epsilon} \log\log \frac{1}{\epsilon})$.

ϵ-conformal means angles are distorted by $< \epsilon$.

(Really means $(1+\epsilon)$-quasiconformal.)

K-quasiconformal means tangent map sends circles to ellipses of eccentricity $\leq K$.



For smooth maps let $\mu = g_{\bar{z}}/g_z$ where

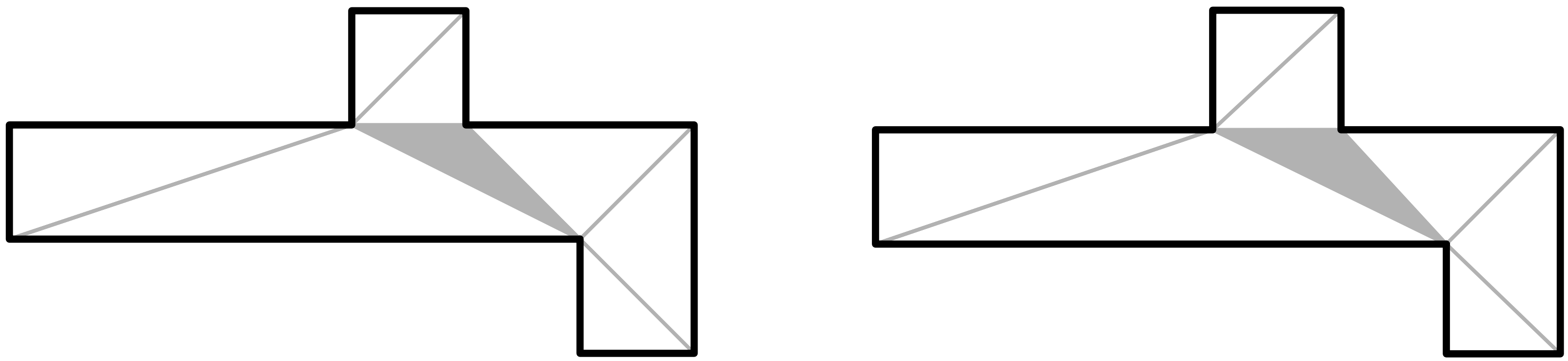
$$g_{\bar{z}} = \frac{1}{2}(g_x + i g_y), \qquad g_z = \frac{1}{2}(g_x - i g_y).$$

Then $|\mu| < 1$ and $K = \sup_{\mathbb{D}} \frac{1+\mu}{1-\mu} \geq 1$.

$\mu = 0$ or $K = 1$ gives conformal map.

FARM: iota map $\partial\Omega \to \mathbb{T}$ has a K_0 QC extension to interiors with K_0 independent of Ω.

Check SC-parameters by triangulating image polygon and map affinely to target. Compute K for each triangle and take maximum.



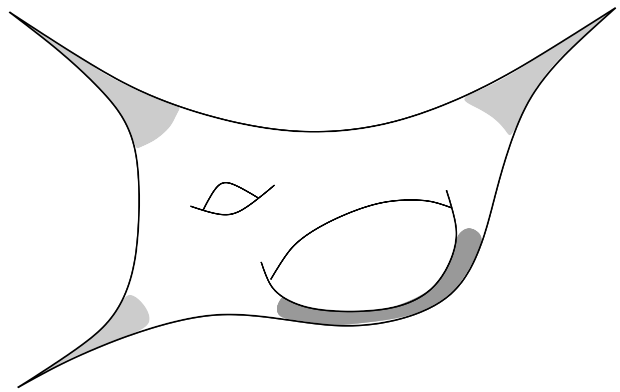
The most distorted triangle is shaded. Here $K = 1.244$. We can bound QC distance from solution even though we don't know what the solution is.

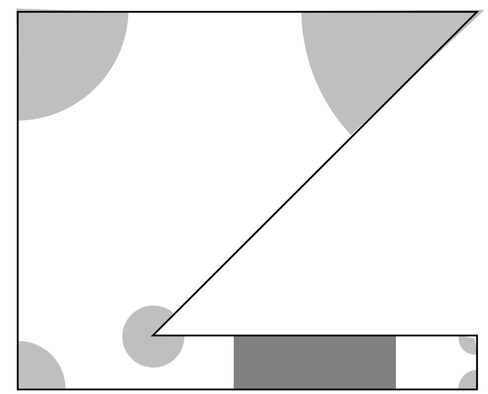
Ideas in proof of FMT:

- Thick and thin parts of polygons
- $O(n)$ domain decomposition
- Newton's type iteration via fast multipole
- Angle scaling

Thick and Thin parts

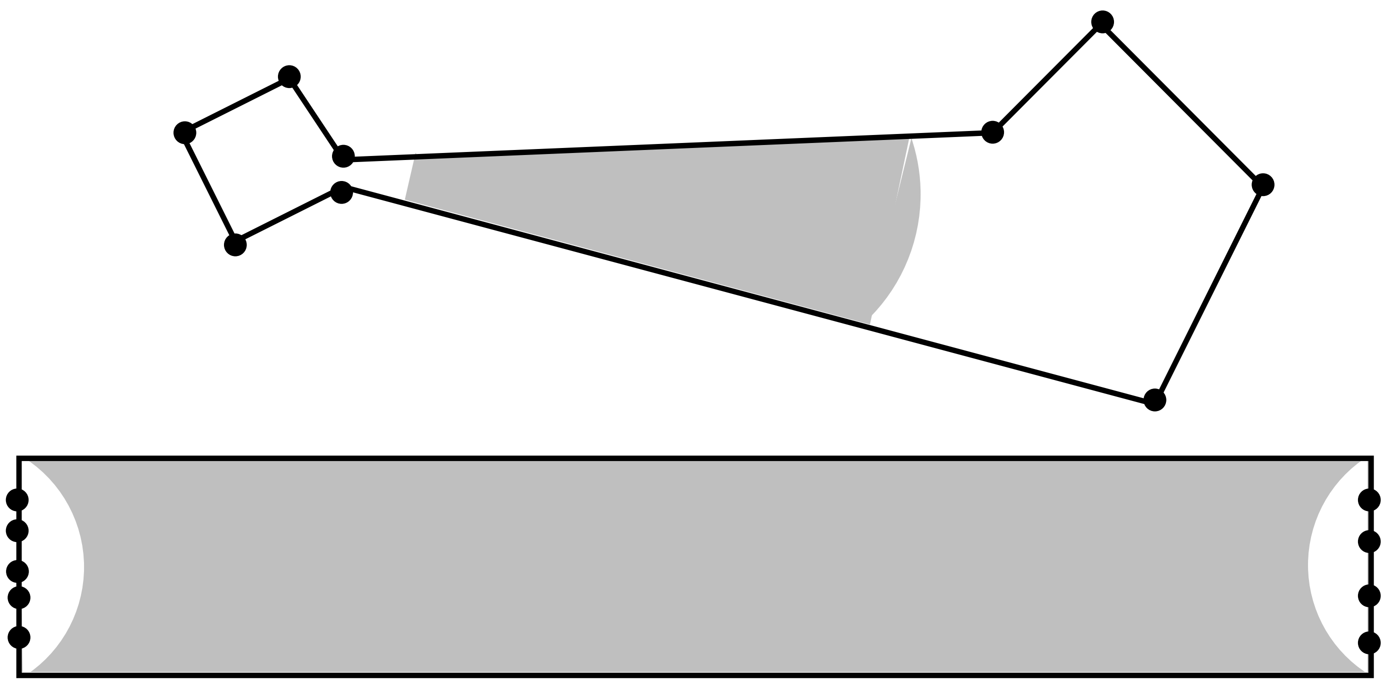
An ϵ**-thin part** of a surface is a union of non-trivial loops of length $\leq \epsilon$ (parabolic/hyperbolic).

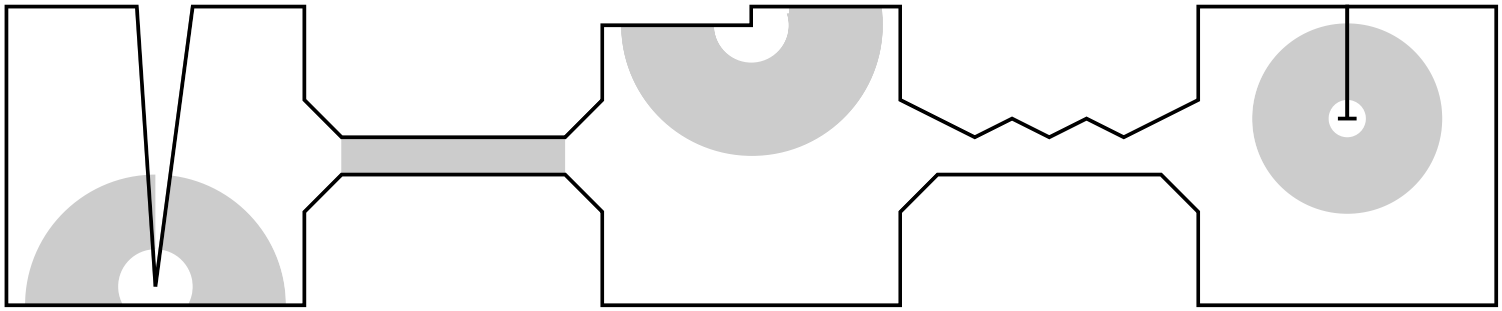




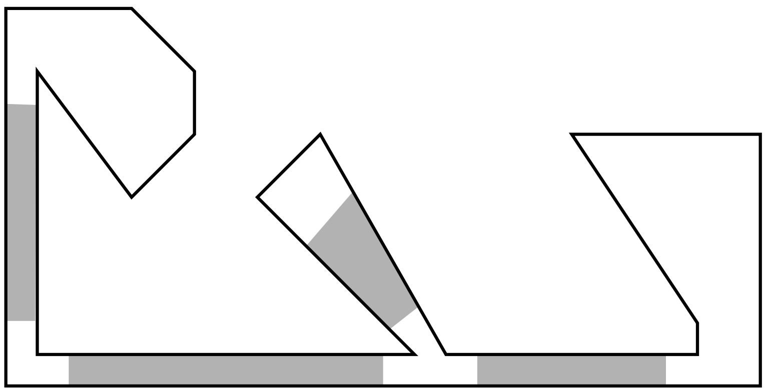
An ϵ-thin part of a polygon is sub-region with two edges on $\partial\Omega$ whose extremal distance is $\leq \epsilon$.

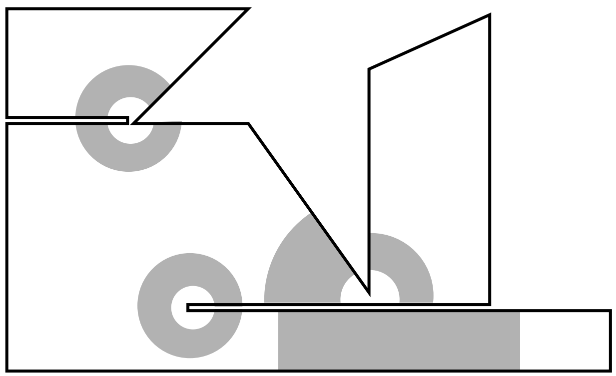
Ω can be mapped to $1 \times \epsilon$ rectangle with two sides of $\partial\Omega$ covering the long sides of rectangle.



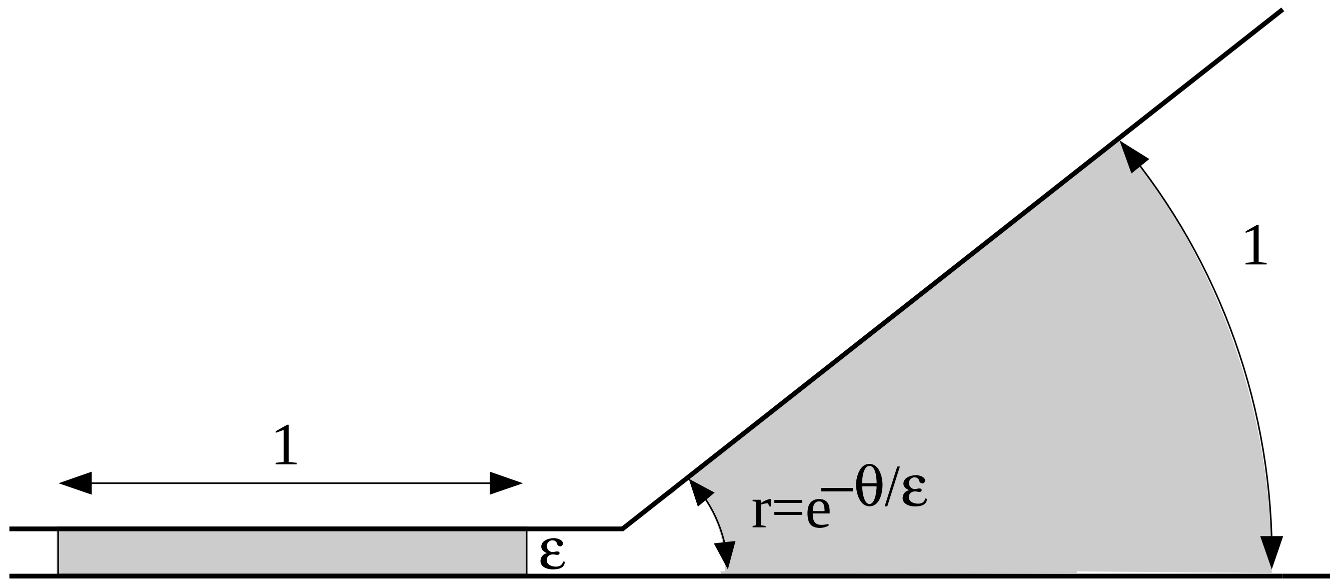


Right channel is not thin because edges on top have length comparable to width of channel.



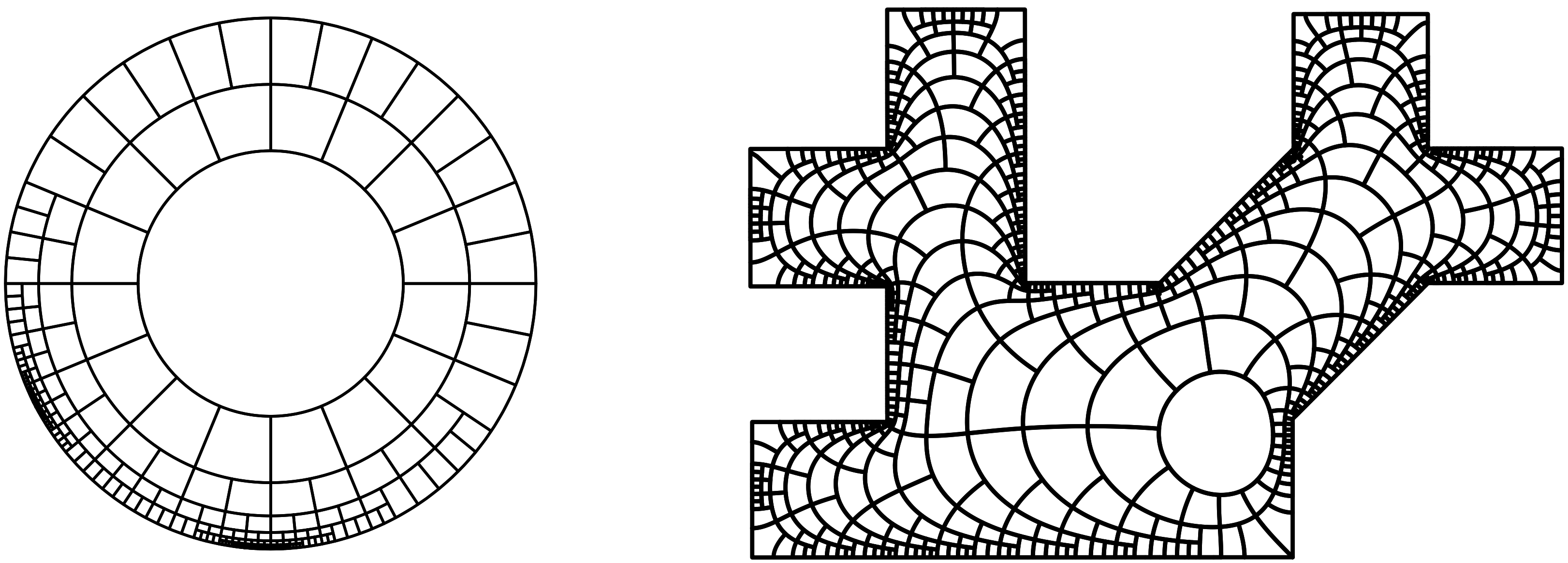


Previous figure not to scale. Proof uses $\epsilon \ll 1$ Two ends of a thin part can be very different scales.



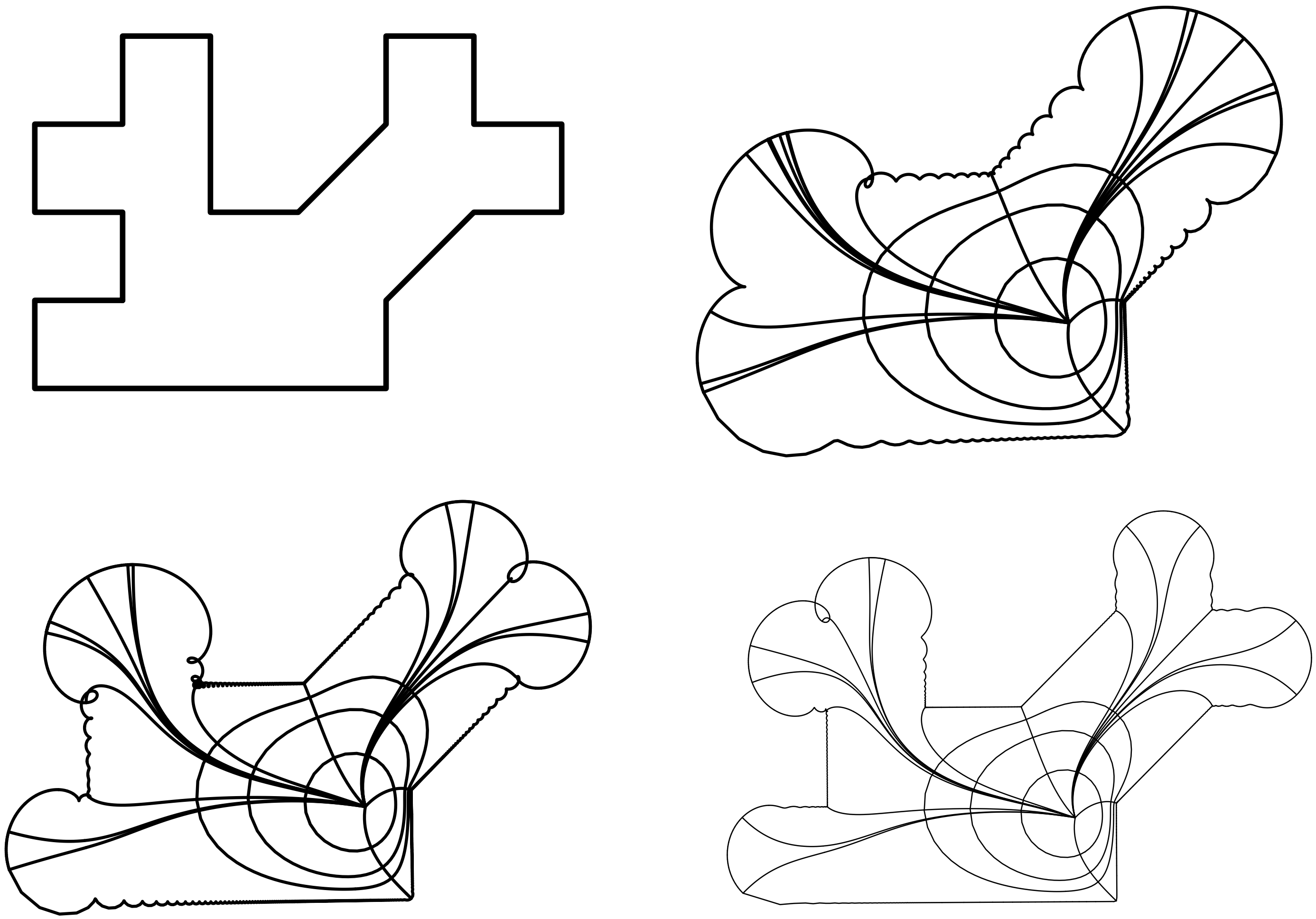
Think $\epsilon = .01$, $r = e^{-75}$.

Disk Decomposition: Break disk into $O(n)$ pieces, and use $p = |\log \epsilon|$ term power series on each piece. Time $O(np \log p)$ allows $O(1)$ FFT's per vertex.



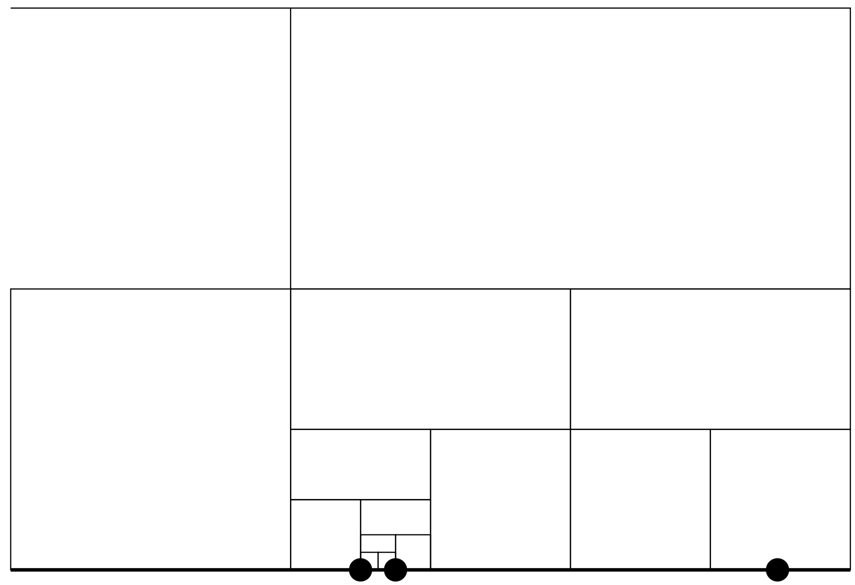
Here $p = 5$. Representation is ϵ-conformal if series on adjacent pieces agree to within ϵ (in certain metric).

Using single power series gives bad approximation: 100, 500, 2500 terms.

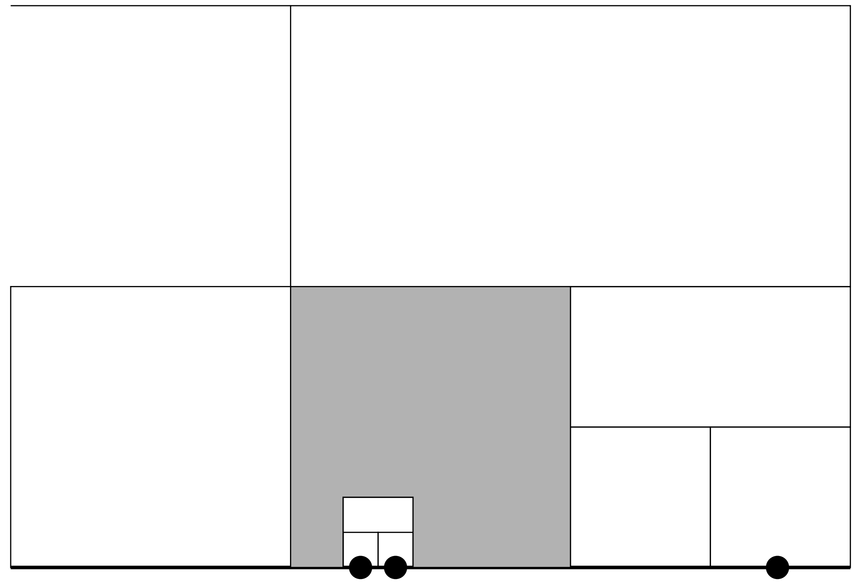


Decomposition depends on choice of n-tuple on $\mathbb{T}$.

Keep dividing until each box contains ≤ 1 point.

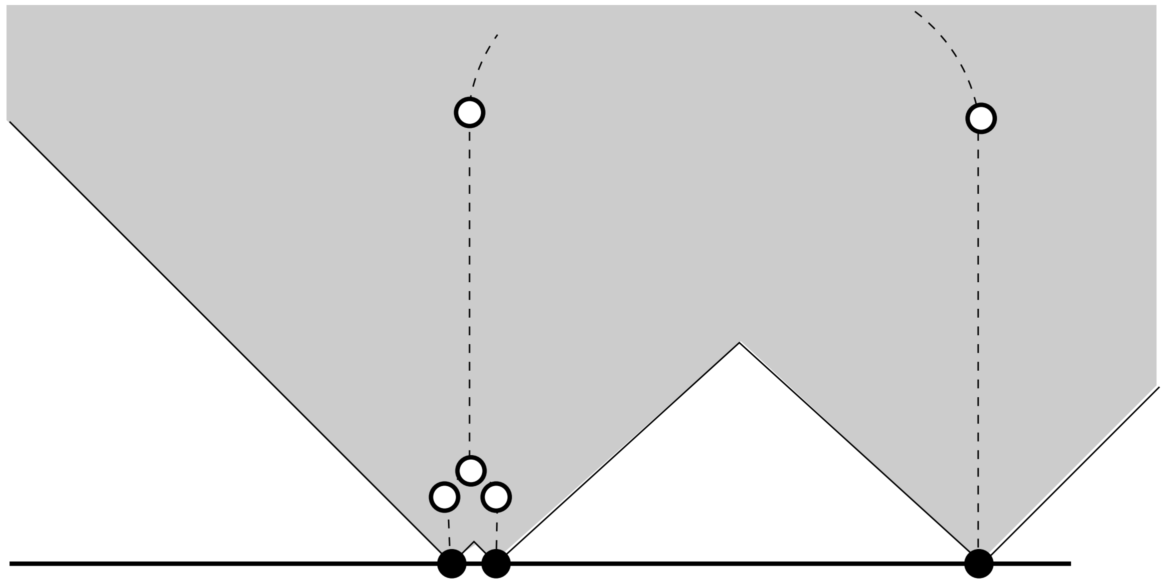


But this can give $\gg n$ boxes if points cluster.



Replace a stack of boxes by a single arch and use a Laurent series instead of a power series.

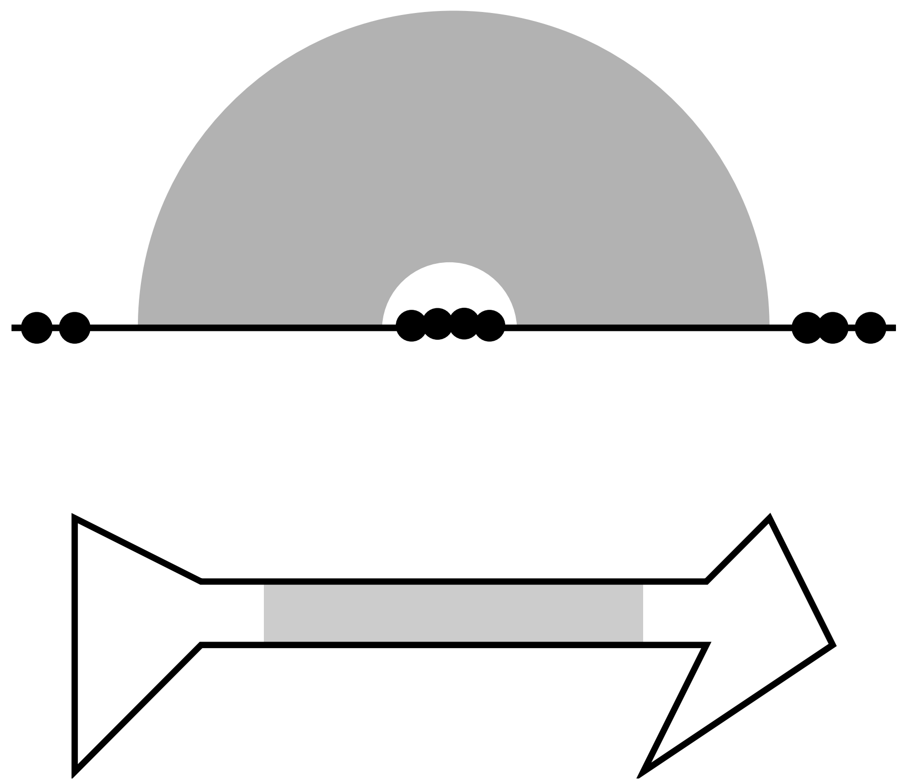
Decomposition has $O(n)$ boxes and $O(n)$ arches. Why?



Define sawtooth region with vertices at n-tuple. Compute medial axis. Edge-edge bisectors are vertical.

Arches correspond to edge-edge bisectors of length $\geq A$ in hyperbolic metric. There are $O(n)$ such edges.

Arches correspond to hyperbolic thin parts.



Beltrami equation:

Given $f : \mathbb{D} \longrightarrow \Omega$ let $\mu = f_{\bar{z}}/f_z$.

Find $g : \mathbb{D} \longrightarrow \mathbb{D}$ so that $g_{\bar{z}}/g_z = \mu$. Then $f \circ g^{-1} : \mathbb{D} \longrightarrow \Omega$ is conformal.

There is infinite series of convolution operators that gives exact solution.

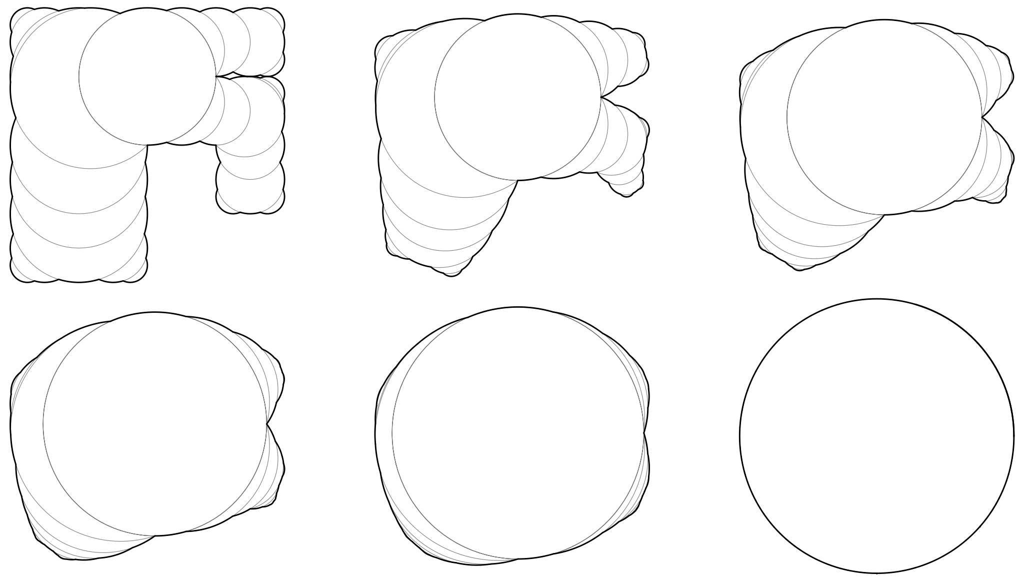
Newton's method: Approximate solution converts an ϵ-map into an $O(\epsilon^2)$-map in $O(n\frac{1}{\epsilon} \log\log \frac{1}{\epsilon})$.

Uniform radius of convergence: iteration works if $\epsilon < \epsilon_0$, independent of n and Ω.

If we knew $K_0 < \epsilon_0$ then iota map could use it as starting point of iteration and be done.

But we don't know this, so have to lead iteration to correct answer in baby steps.

Angle Scaling:

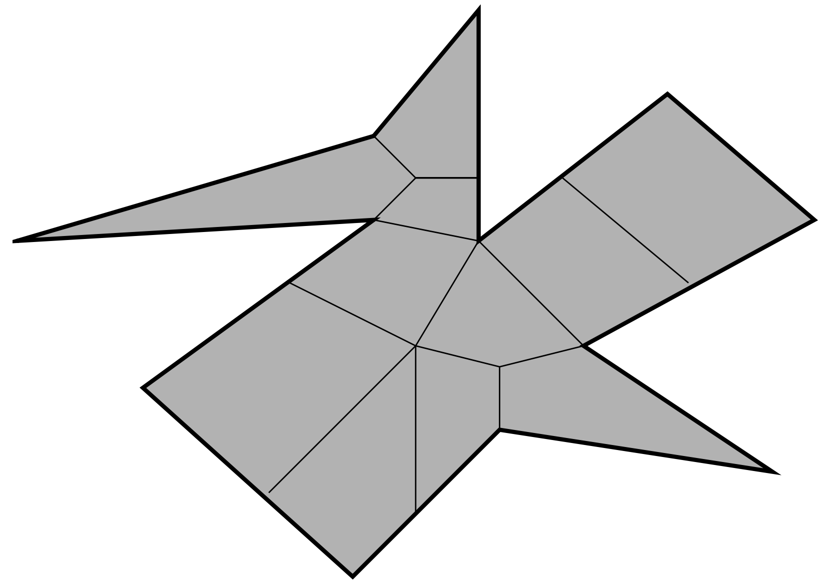


Discretize angle scaling family so gaps are smaller than ϵ_0. Need only $O(1)$ steps.

Use (almost) conformal map onto one domain as initial point for iteration for next domain.

Start with identity map on disk and get ϵ_0-map onto Ω after $O(n)$ work. Iterate $\log\log\frac{1}{\epsilon}$ times to get ϵ-map.

Bern and Eppstein (1997): Any n-gon has a quadrilateral mesh with angles $\leq 120°$. At most $O(n)$ points are added. Runs in $O(n \log n)$.



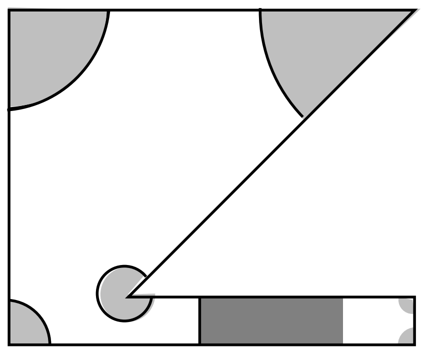
Theorem: Any n-gon has a $O(n)$ quadrilateral mesh so that all new angles are between $60°$ and $120°$ and which can be computed in $O(n)$.

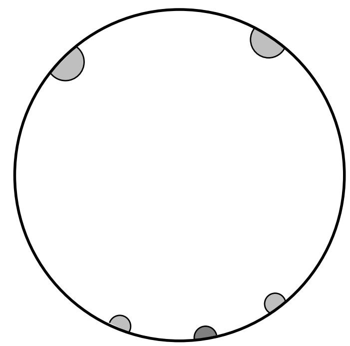
Uses fast mapping theorem, thick-thin decompositions, hyperbolic geometry.

Bern-Eppstein showed angle bounds are best possible.

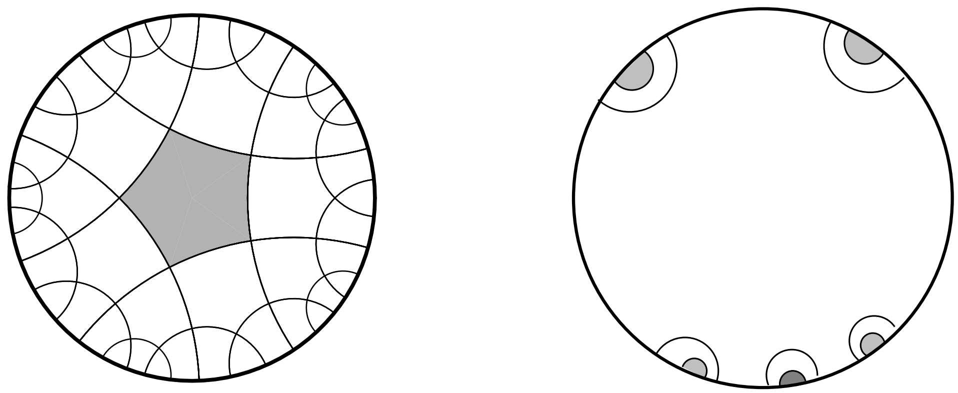
Idea of proof:

Divide polygon into thick and thin parts. Thick parts look piecewise smooth with 90° angles. Map to disk minus hyperbolic half-planes.

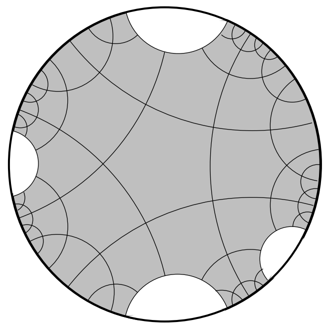


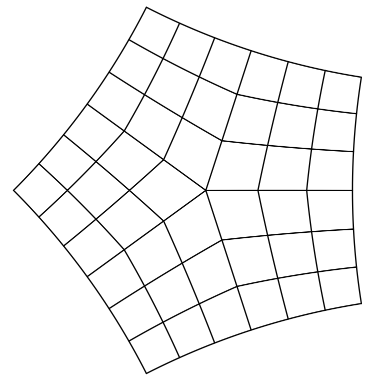


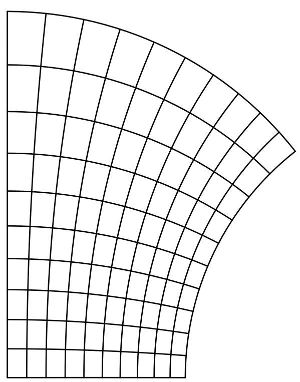
Disk has tesselation by hyperbolic right pentagons. Finite approximation divides disk into pentagons, triangles and quadrilaterals.

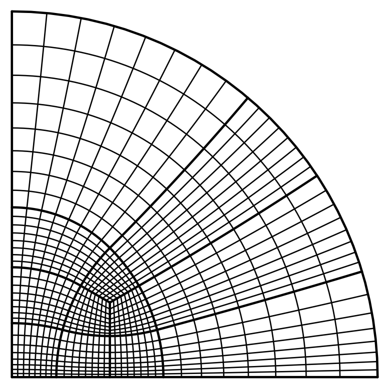


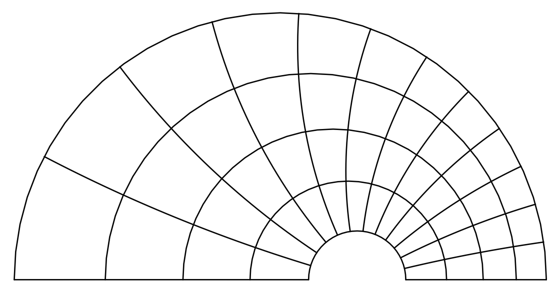
Each piece can be meshed consistently.





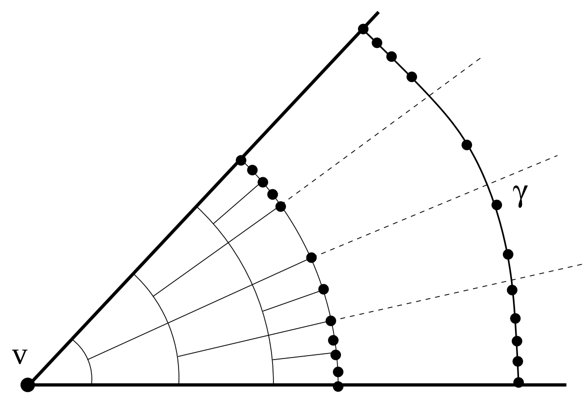


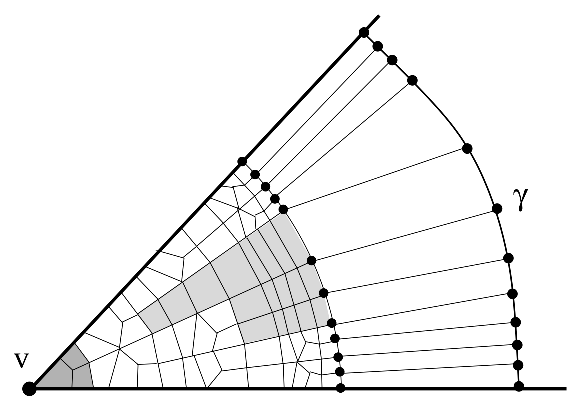




Mesh disk, map to polygon, “snap” curves to line segments. Then mesh thin parts and connect meshes.

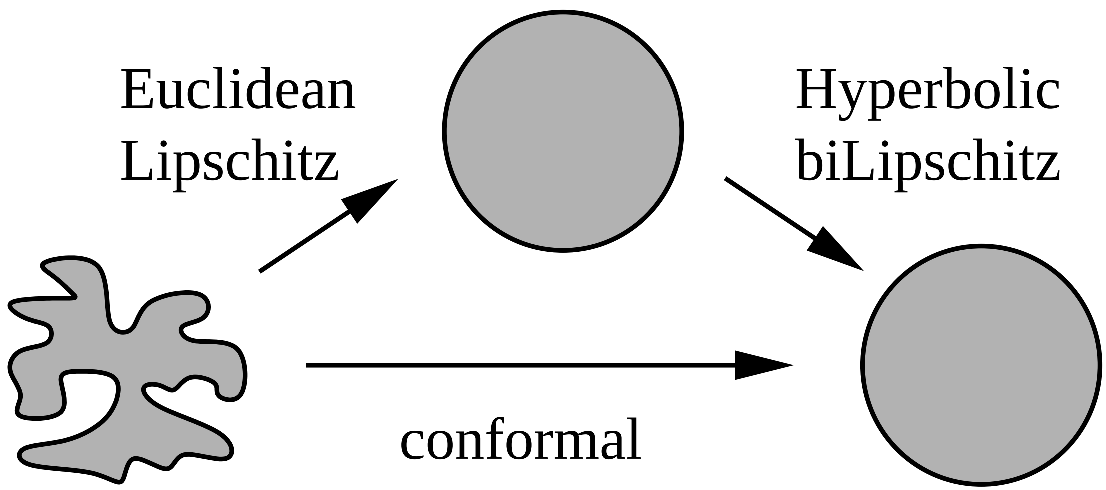
Final step is to mesh the thin parts and connect these to meshing of thick part





The factorization theorem: For any simply connected Ω with inradius ≥ 1, a Riemann map can be factored as $f = h \circ g : \Omega \to \mathbb{D} \to \mathbb{D}$ where

- g is locally Lipschitz (Euclidean metrics)
- h is biLipschitz (hyperbolic metric)



Cor: Any simply connected plane can be mapped to a disk by a homeomorphism which is contracting for the internal path metric.