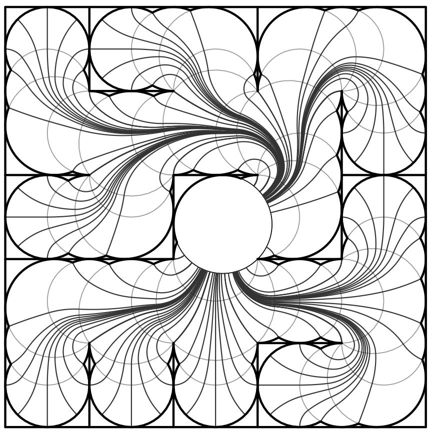
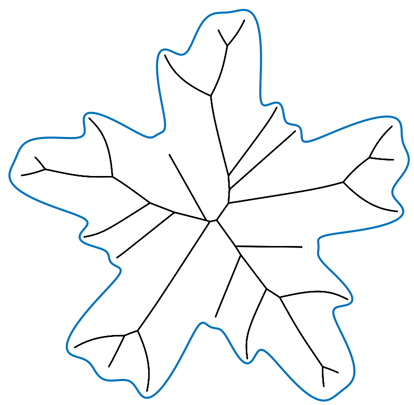
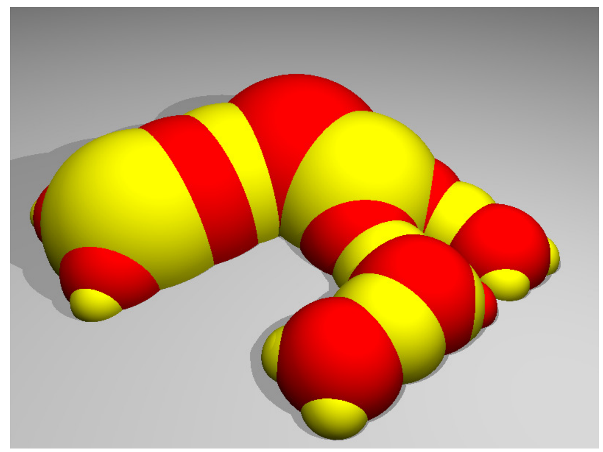
CURVES, TREES & SURFACES: APPLICATIONS TO MAPPING AND MESHING

Christopher Bishop, Stony Brook University

Curves, Trees and Surfaces, Berlin June 2-6, 2025
Celebrating Steffen Rohde's 60th Birthday



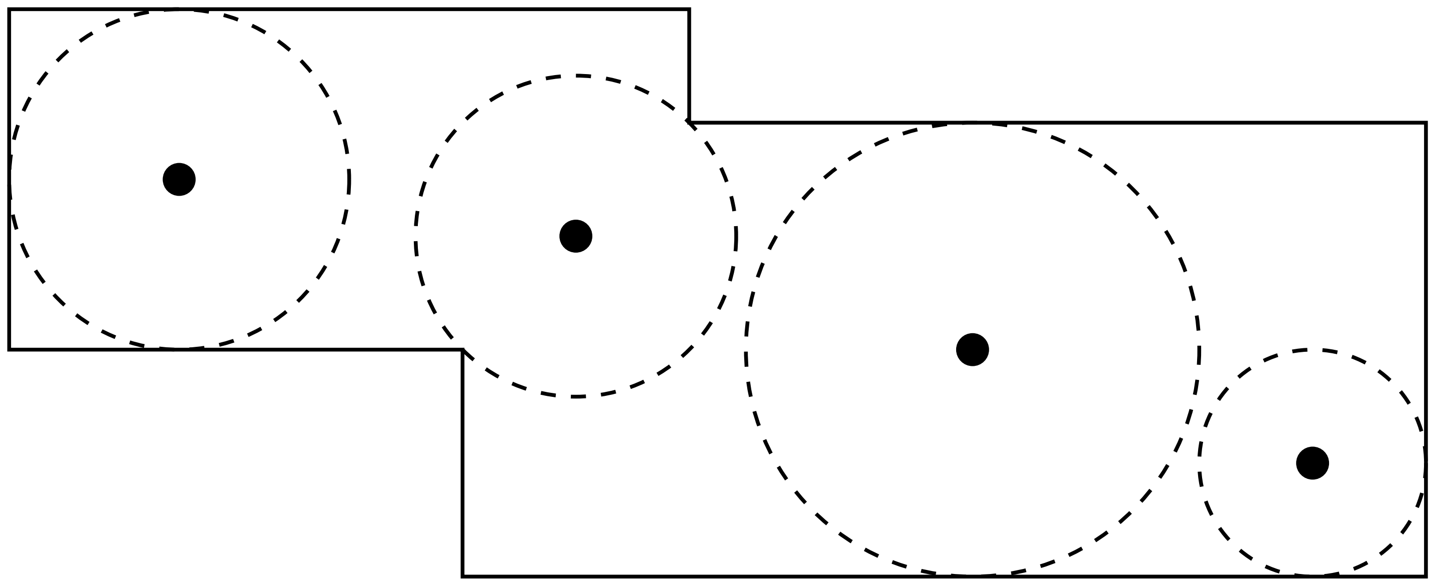




PART I: CURVES ENCODED BY TREES

Medial axis:

centers of disks that hit boundary in at least two points.



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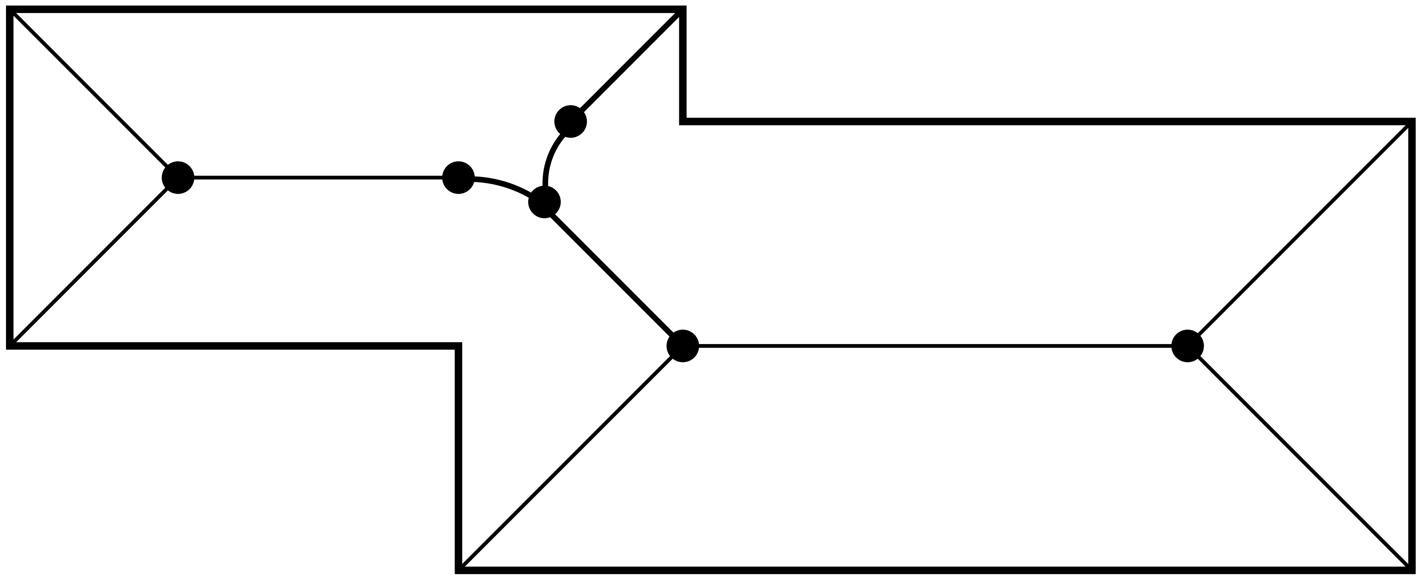
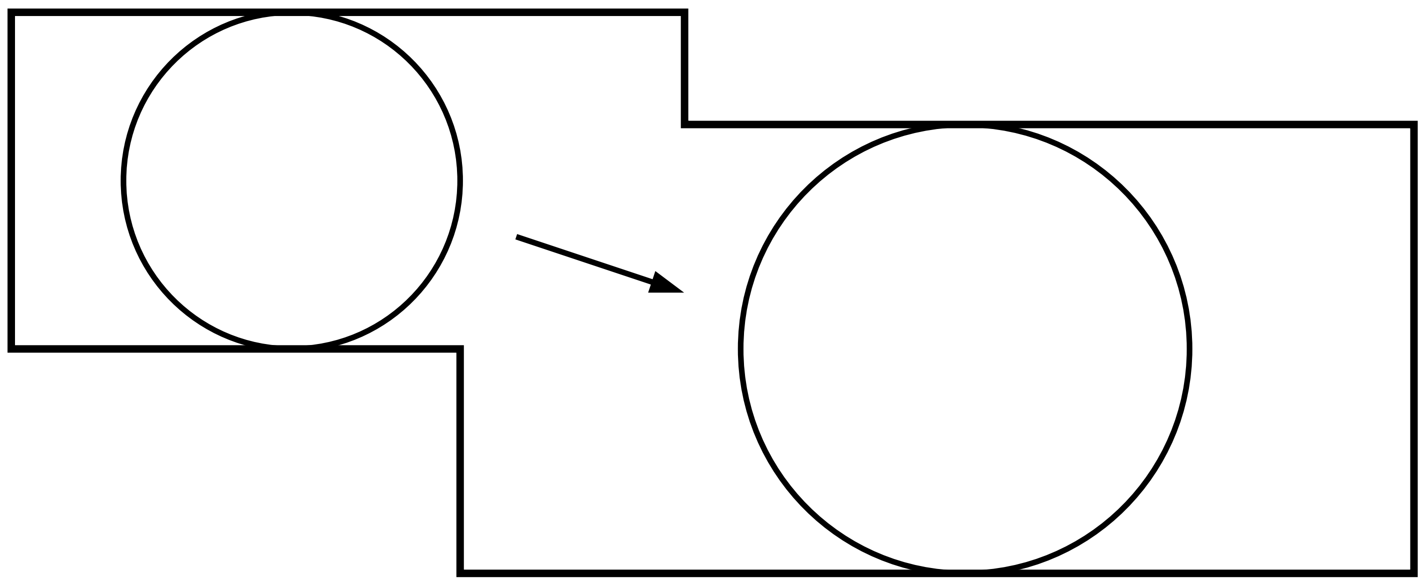


Medial axis of a polygon is a finite tree.

Computable in O(n), Chin-Snoeyink-Wang (1999).

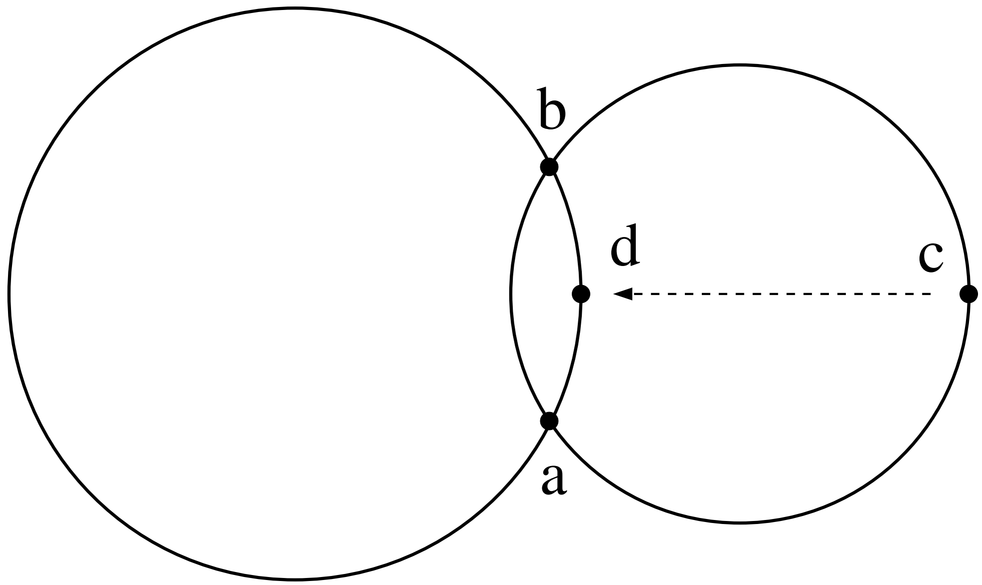
Related to Voronoi diagrams: divides polygon according to nearest edge.

Medial axis:
centers of disks that hit boundary in at least two points.



Claim: there is a "natural" choice of conformal map between any two medial axis disks.

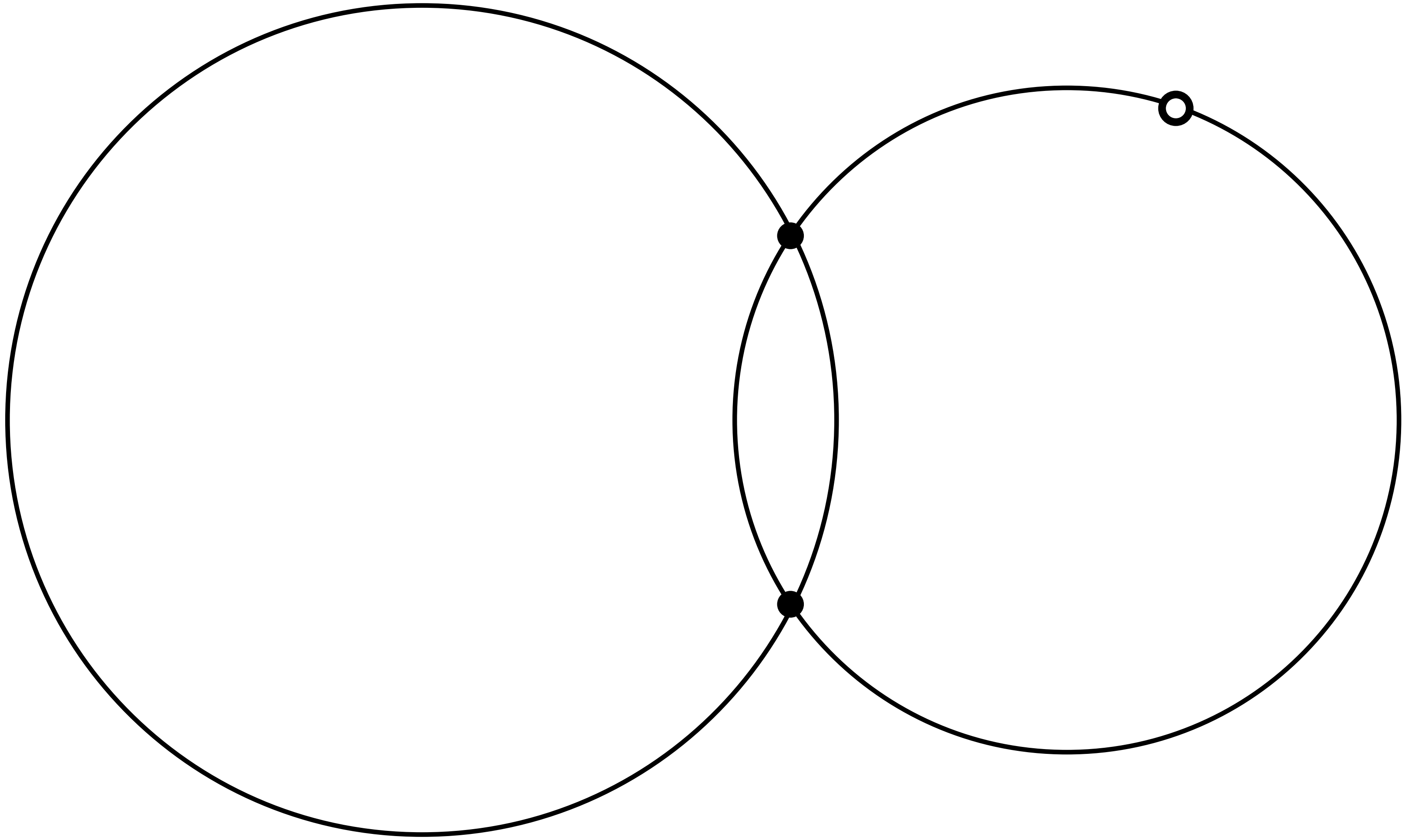
Intersecting circles:

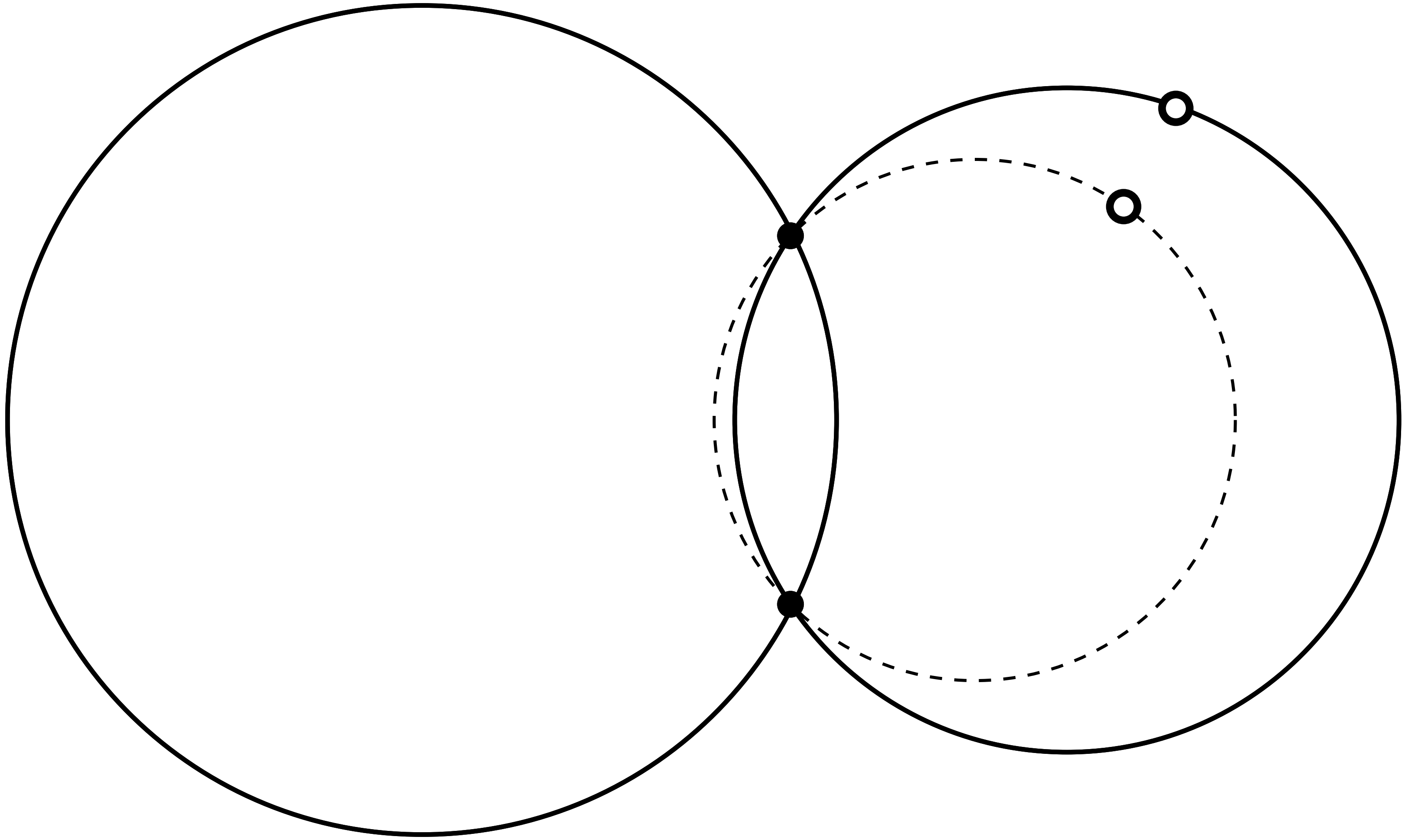


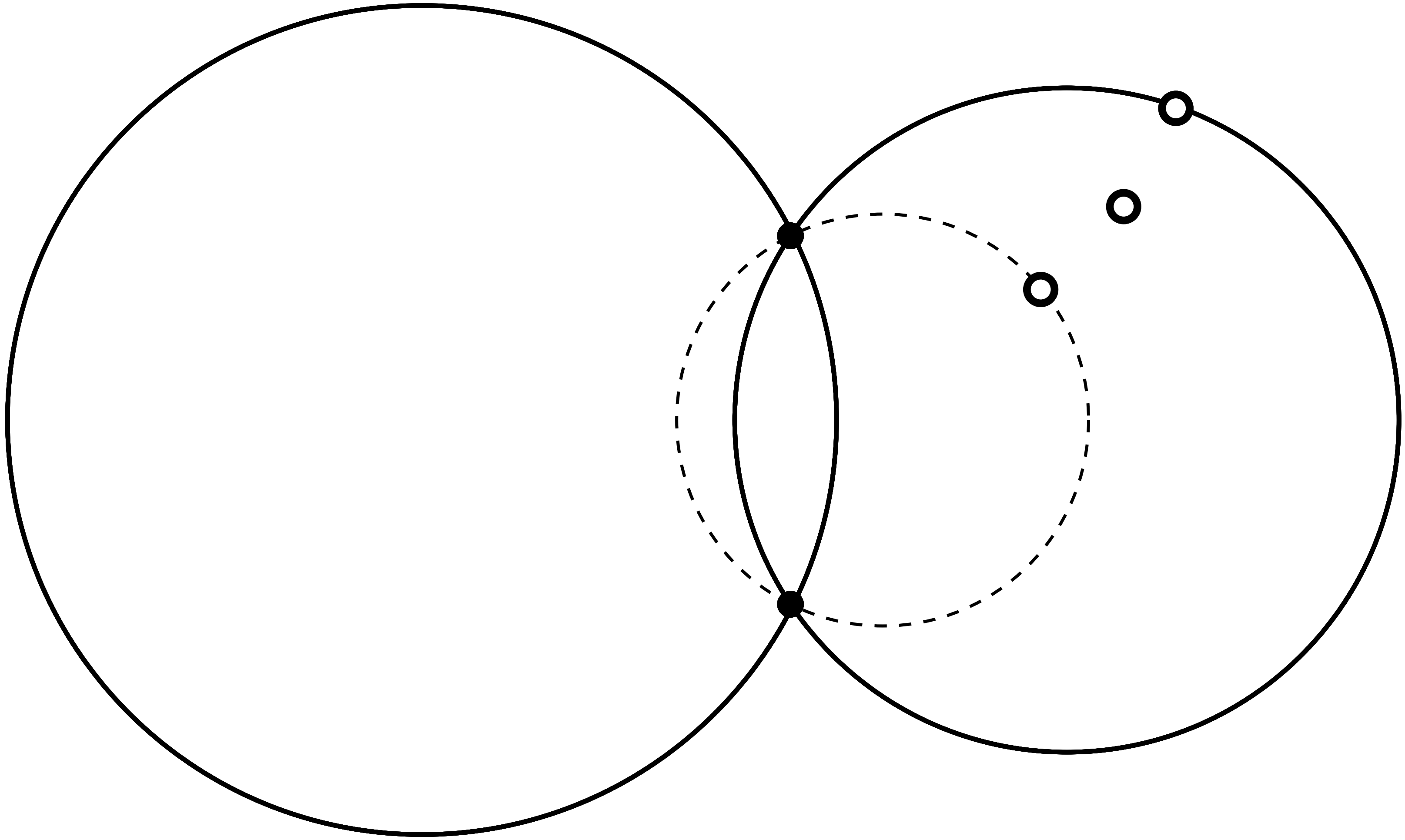
Fix intersection points a, b and map $c \to d$ as shown.

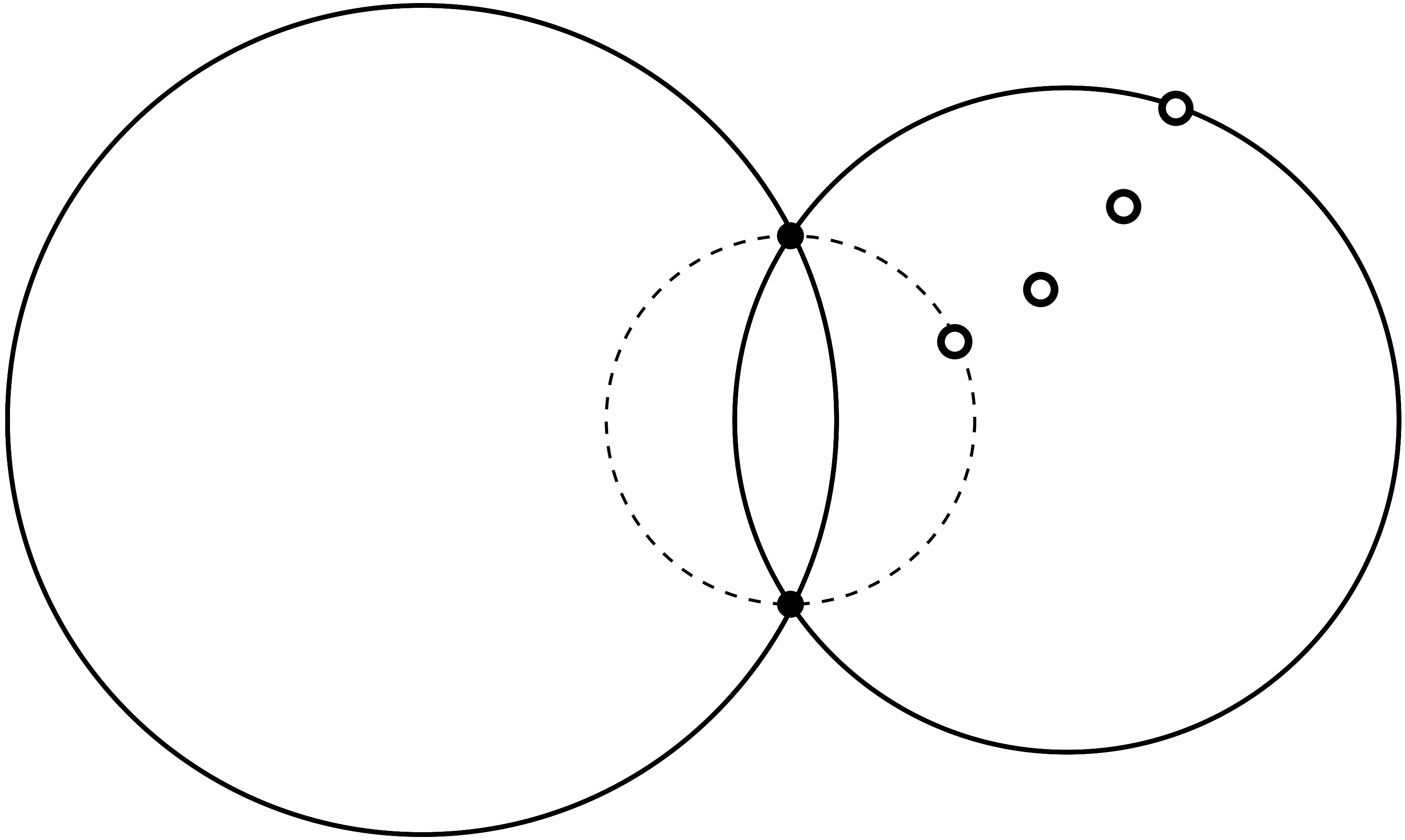
Determines unique Möbius map between disks.

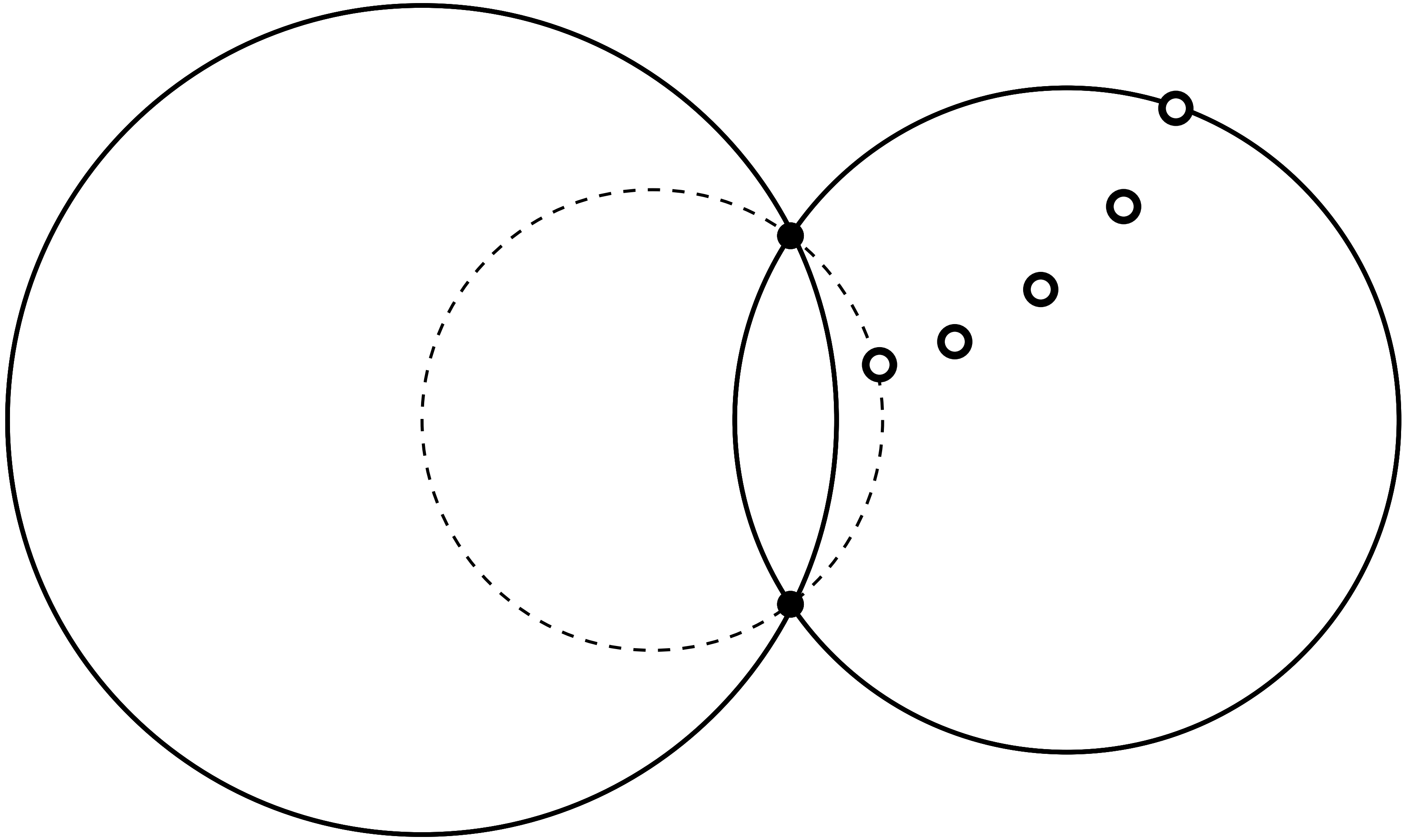
Part of 1-parameter symmetric family fixing a, b.

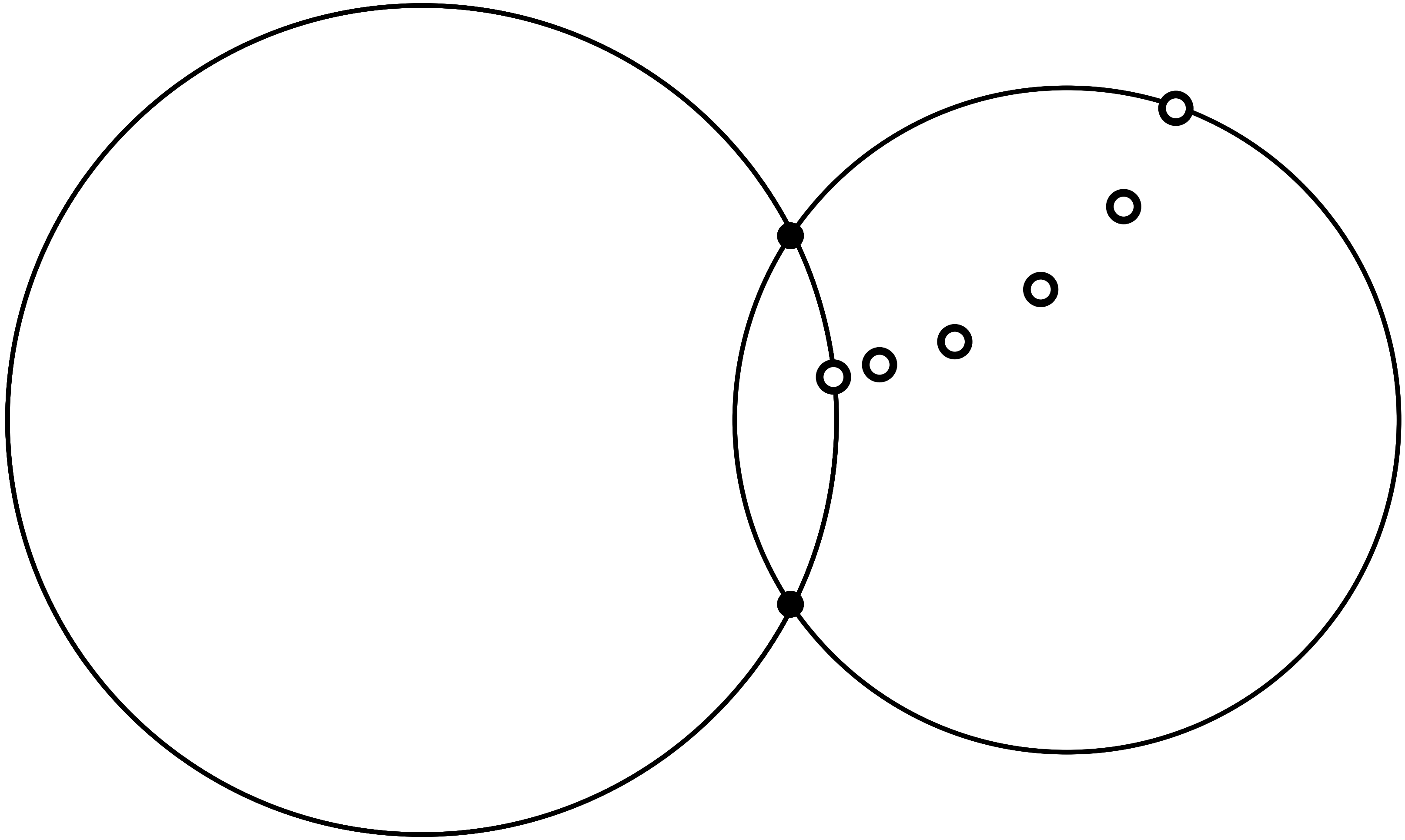


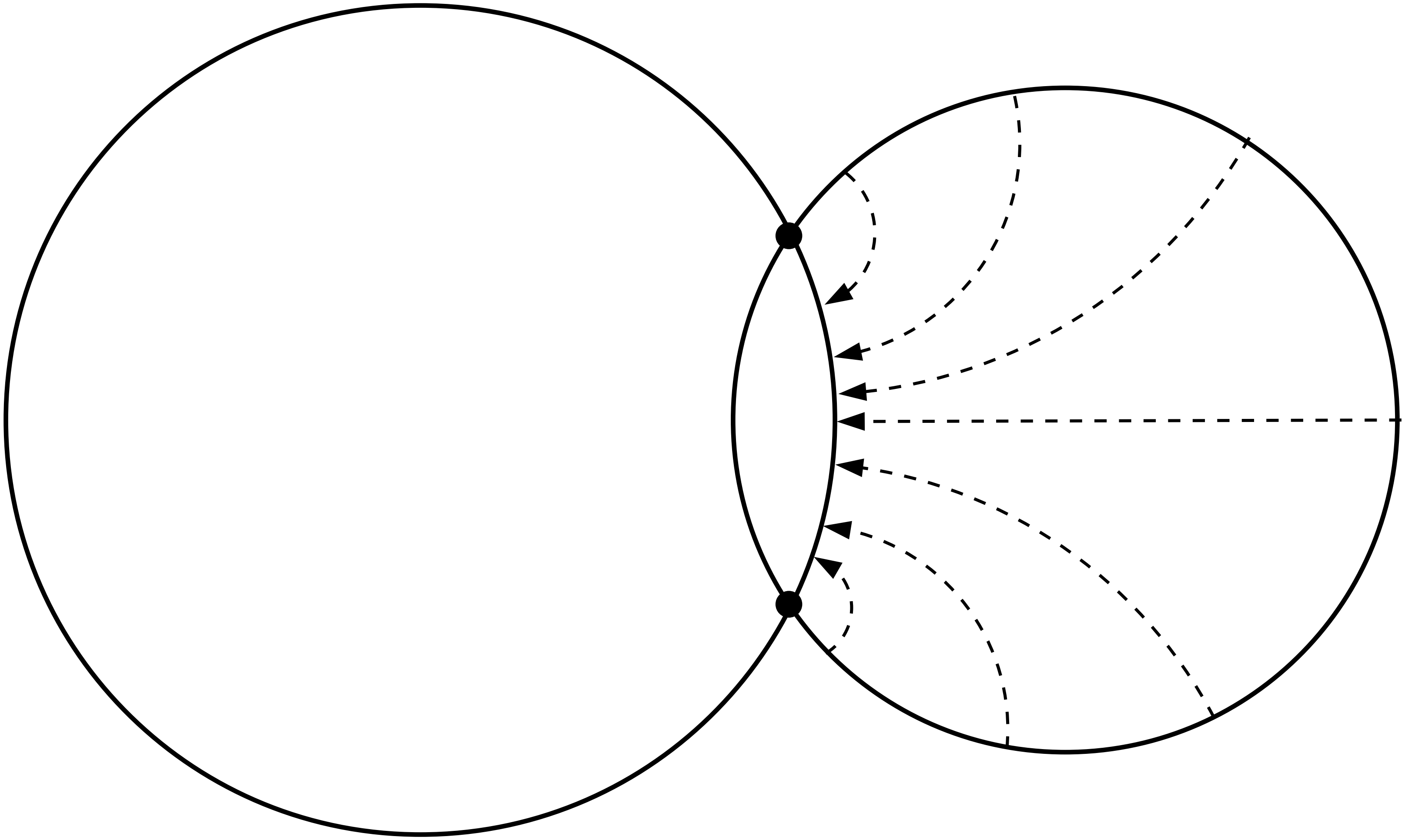






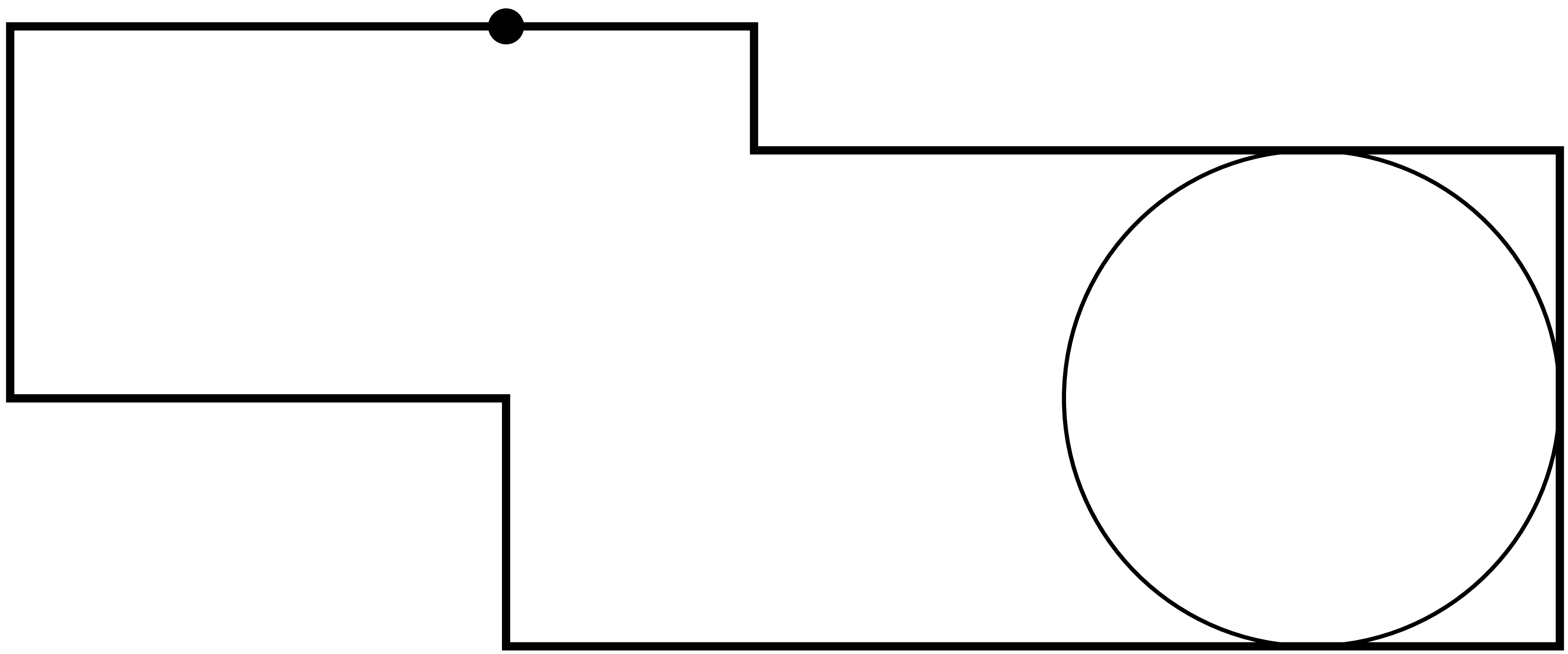




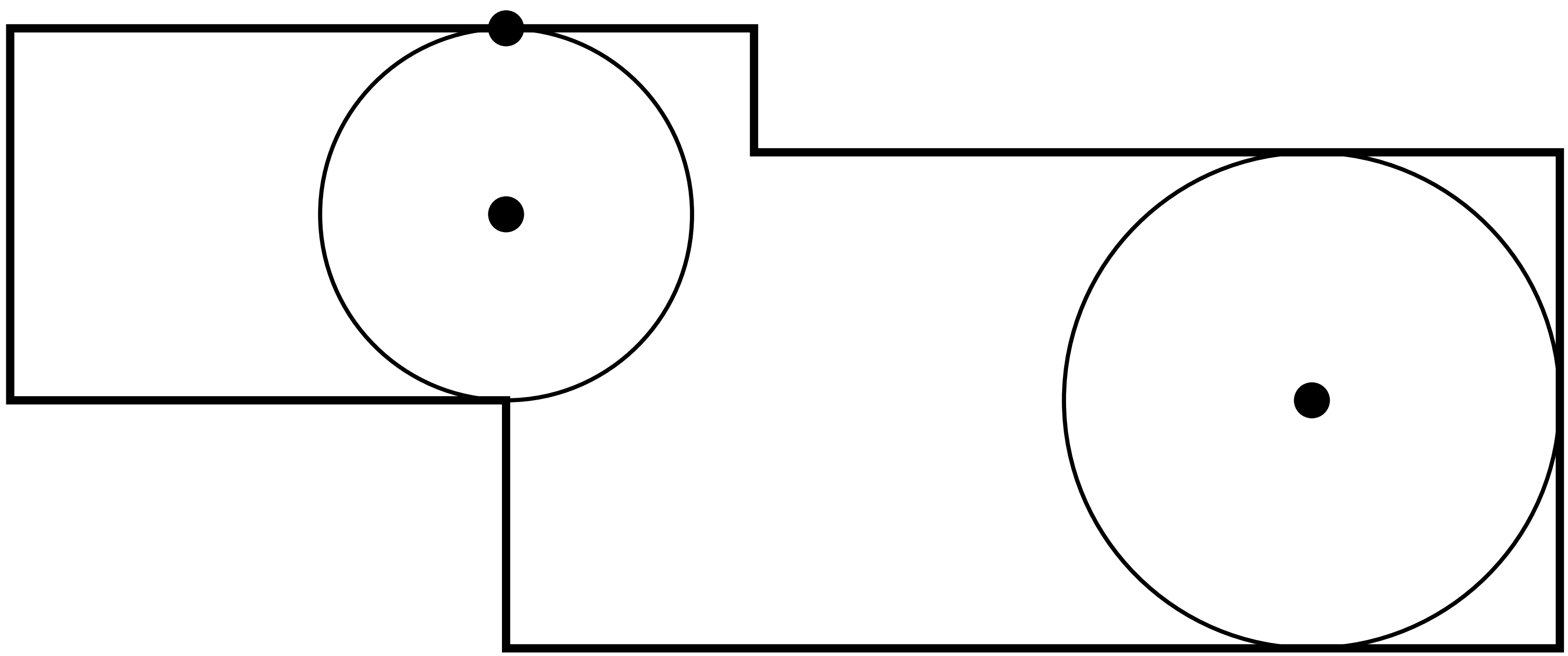


Points follow circular paths, perpendicular to boundary.

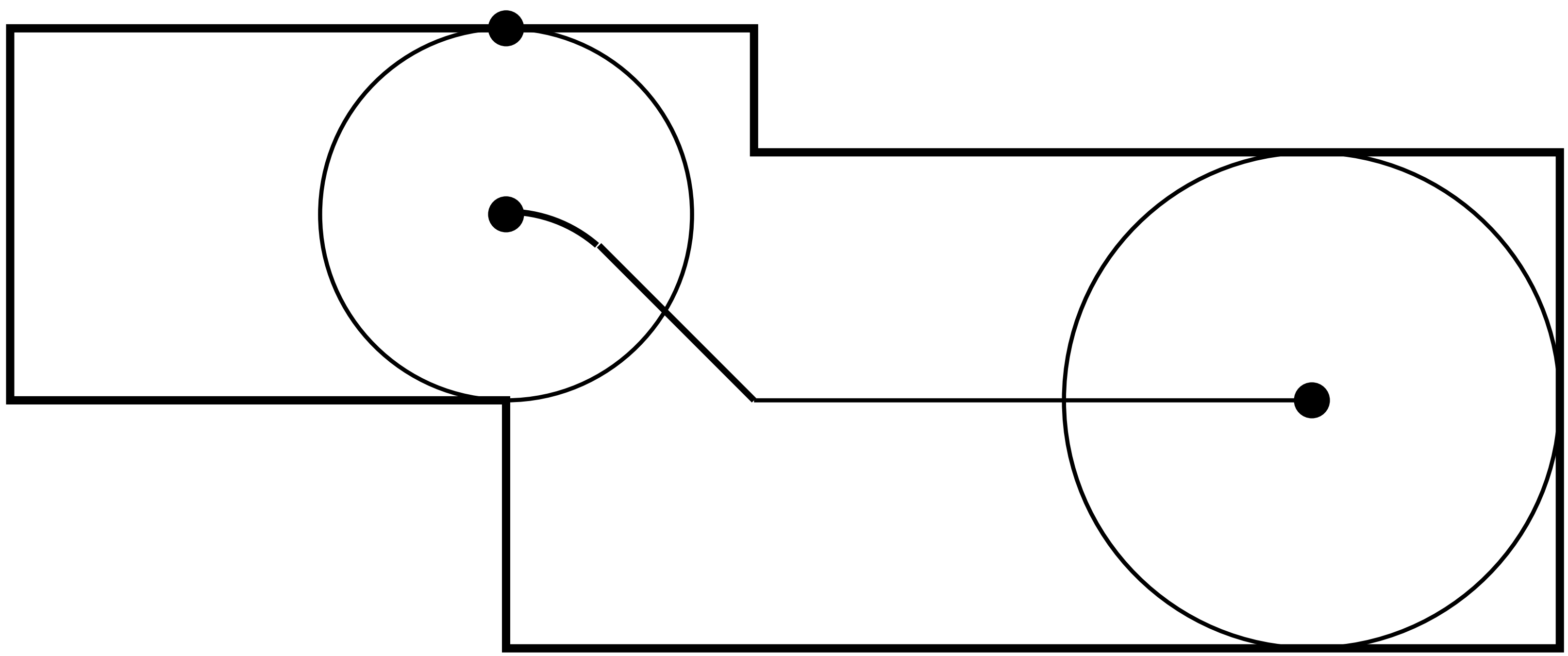
How does this give a map from polygon P to a circle?



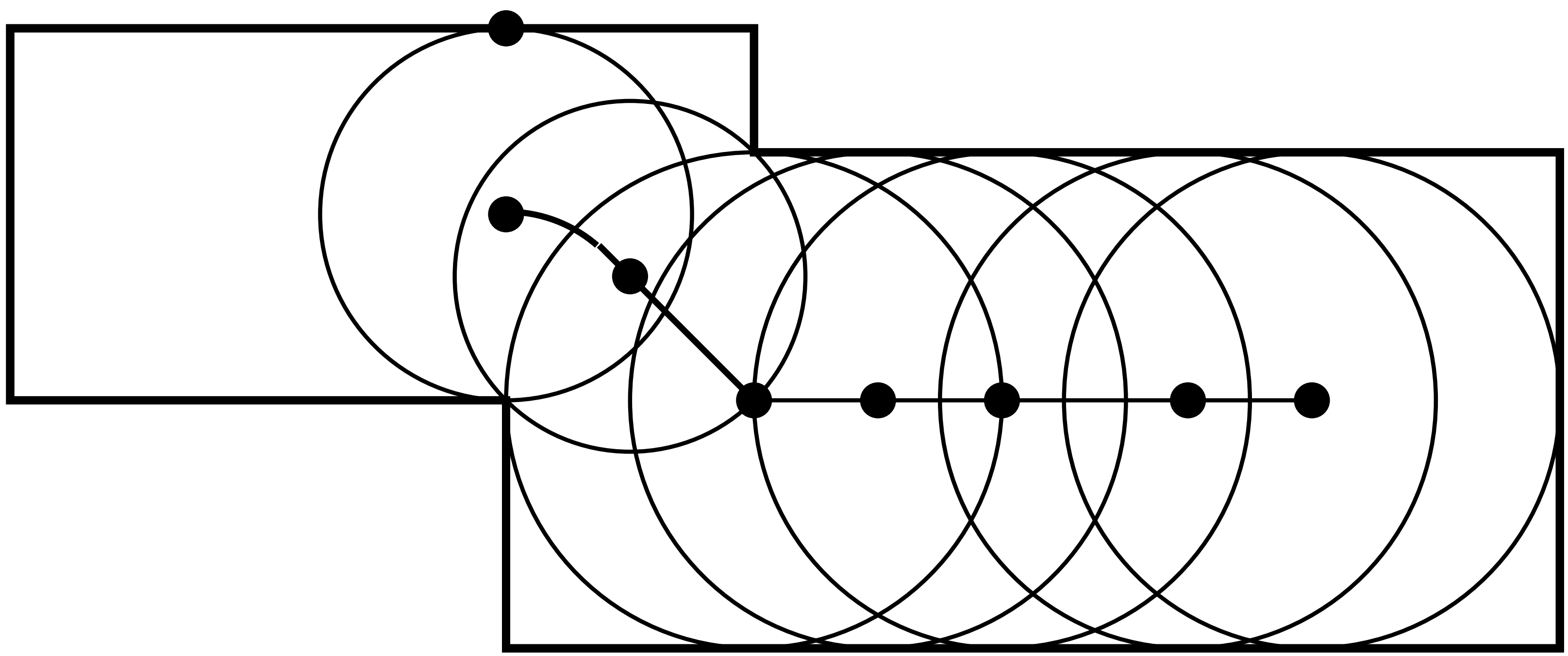
- Fix a “root” MA disk D.

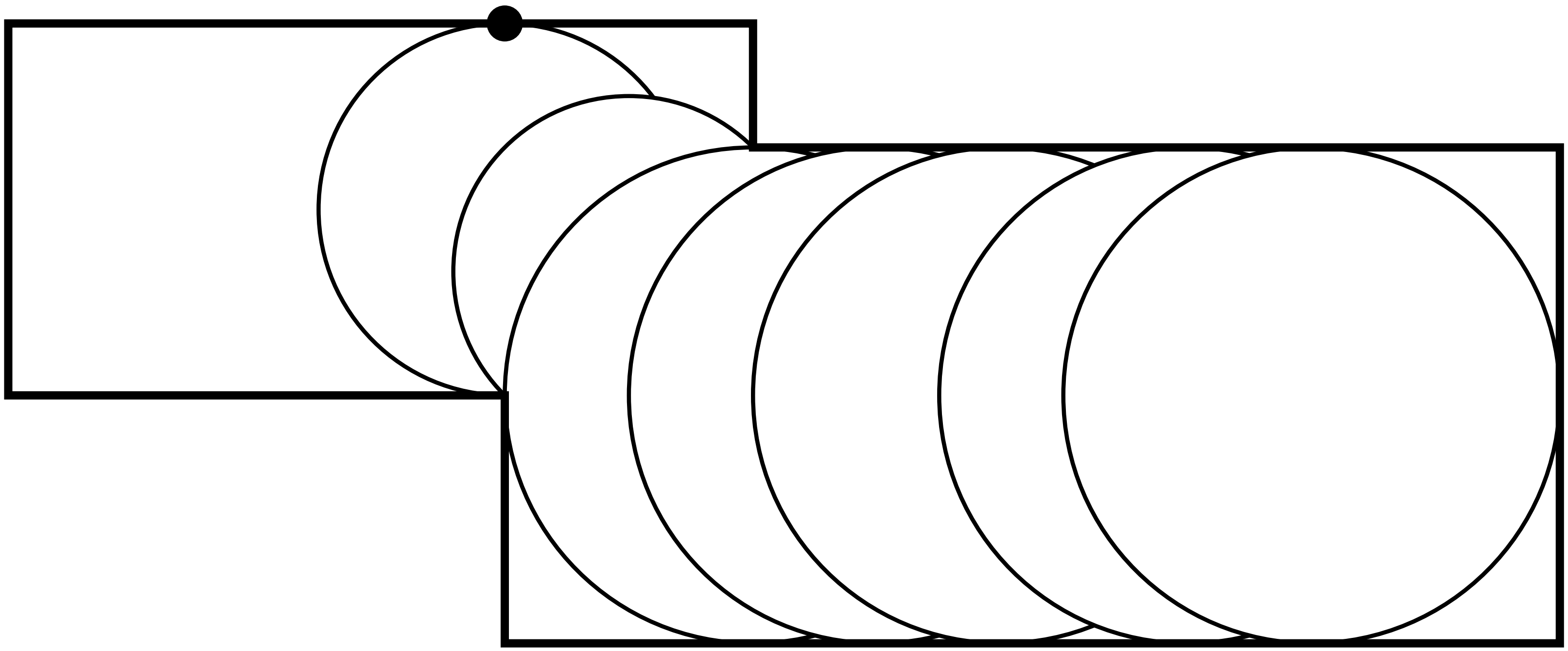


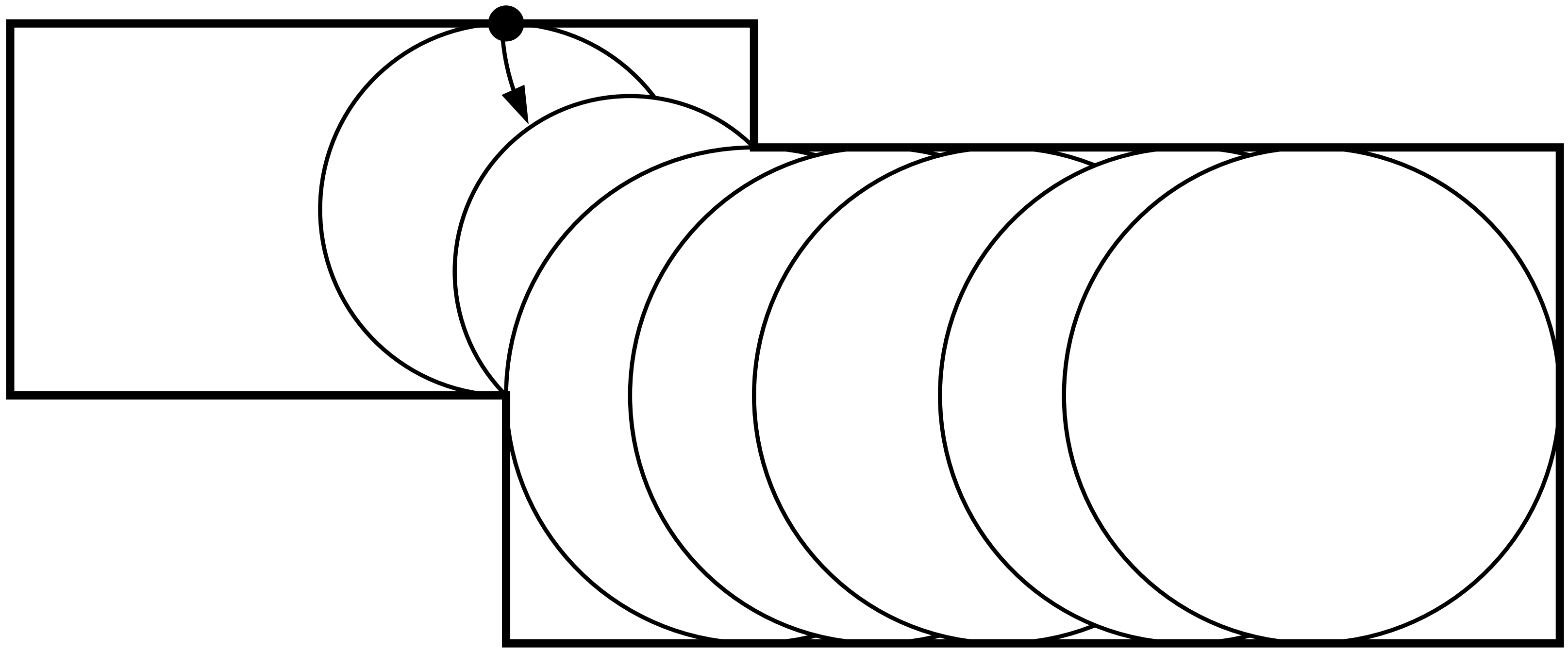
- For any $z \in P$, take MA disk D_z touching z.

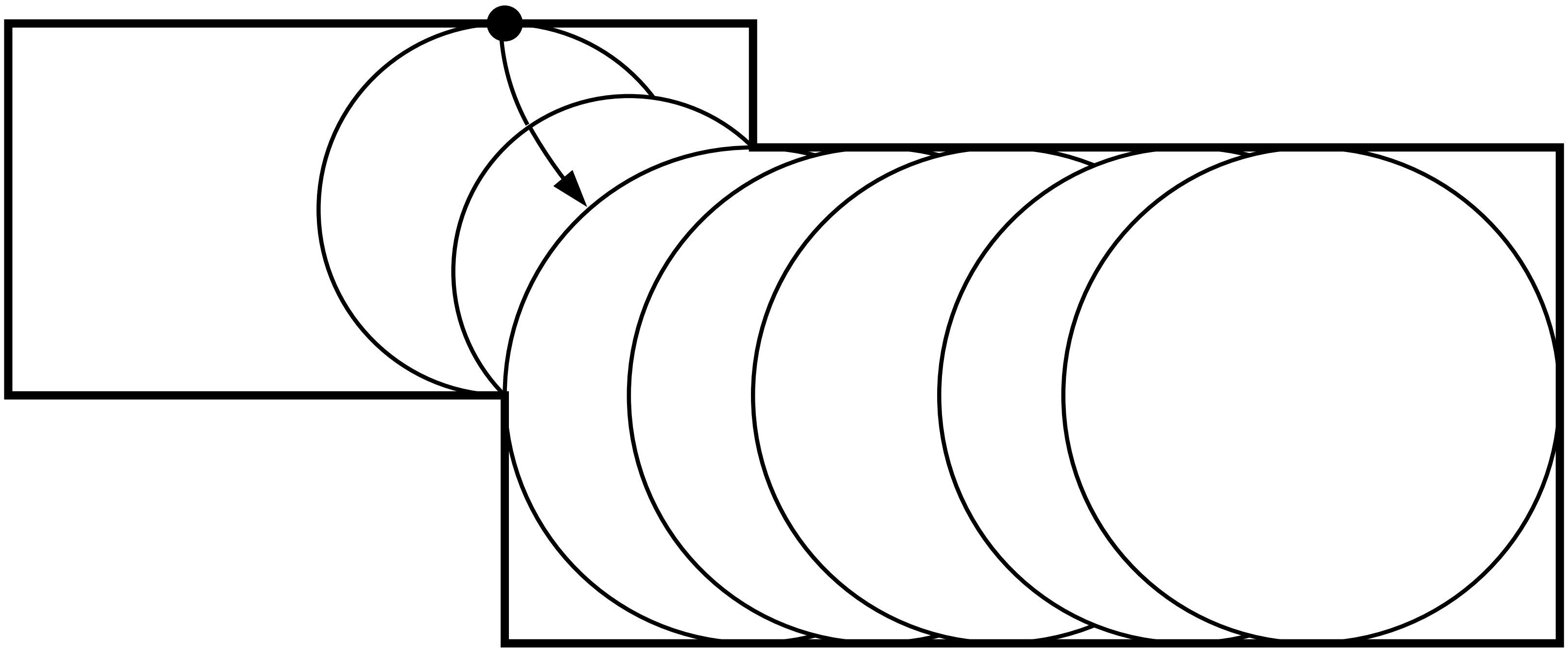


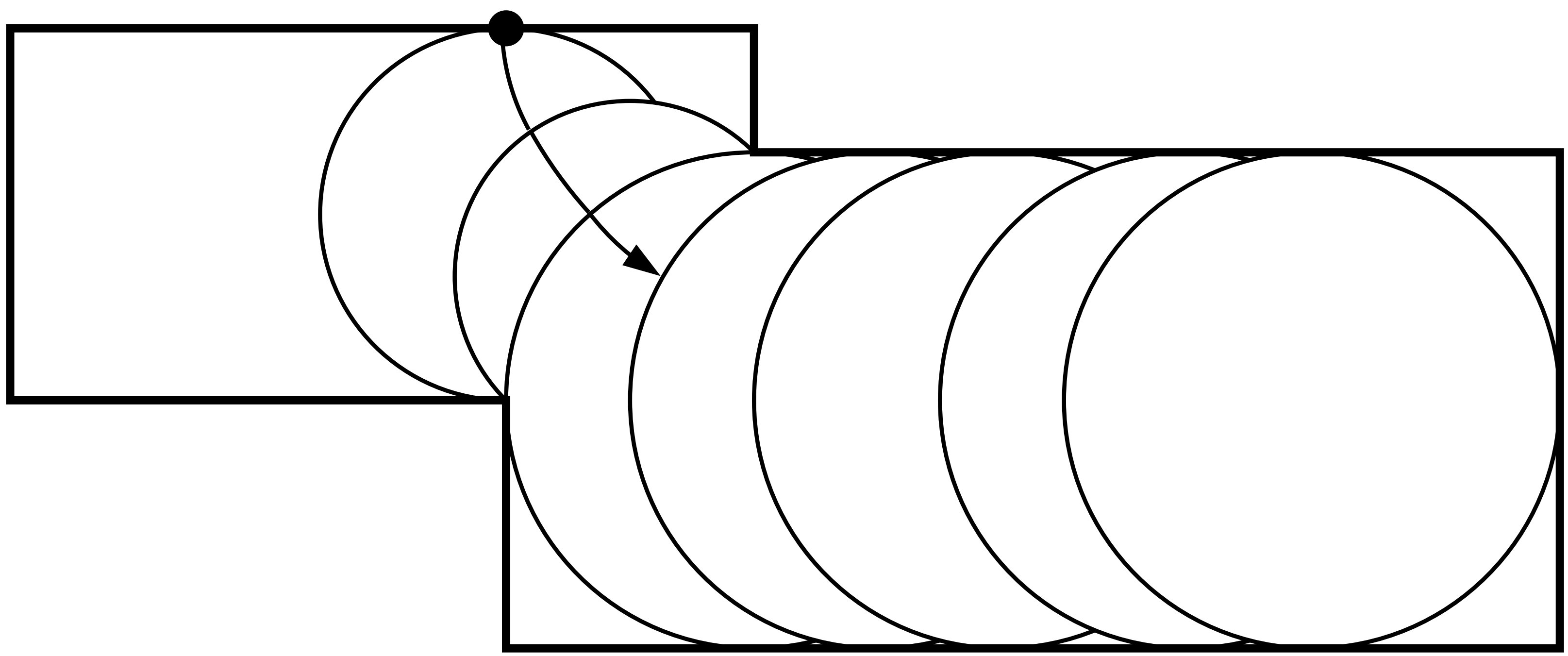
- Connect D_z to D on MA.

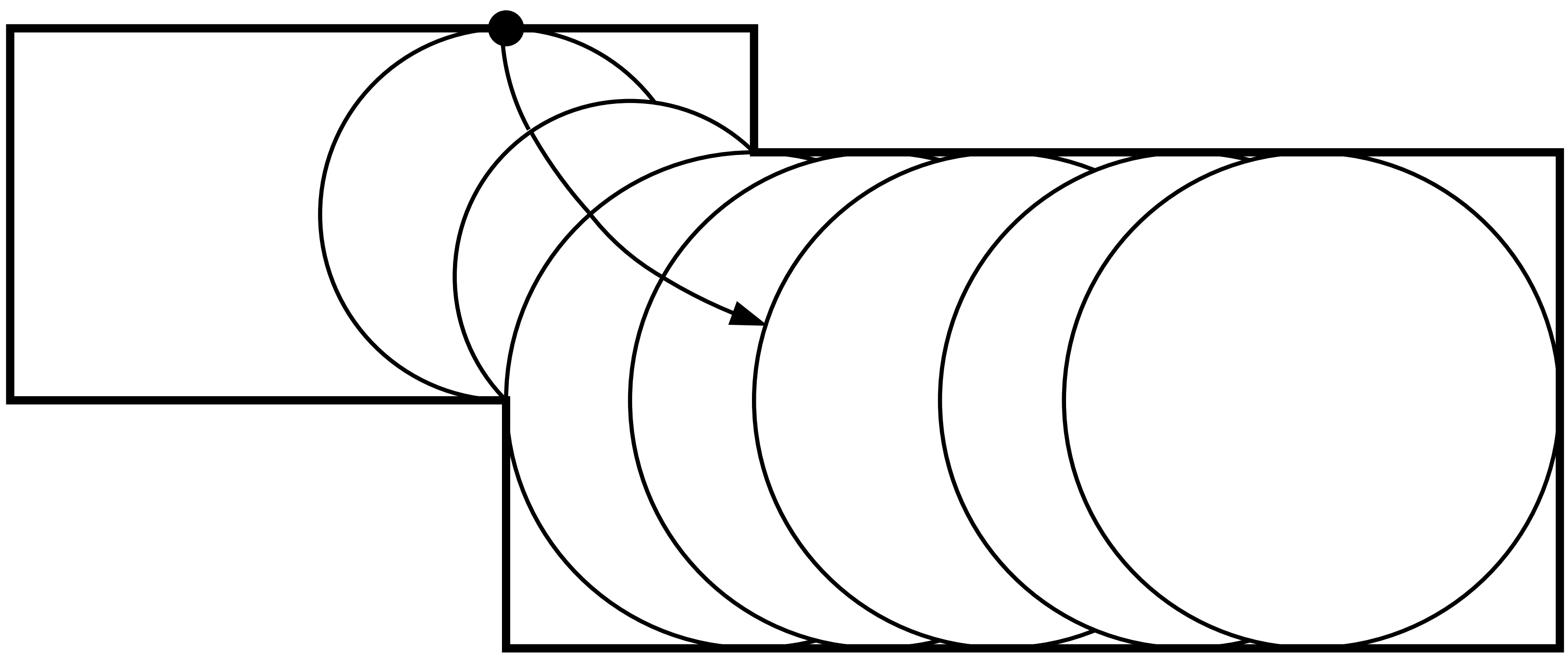


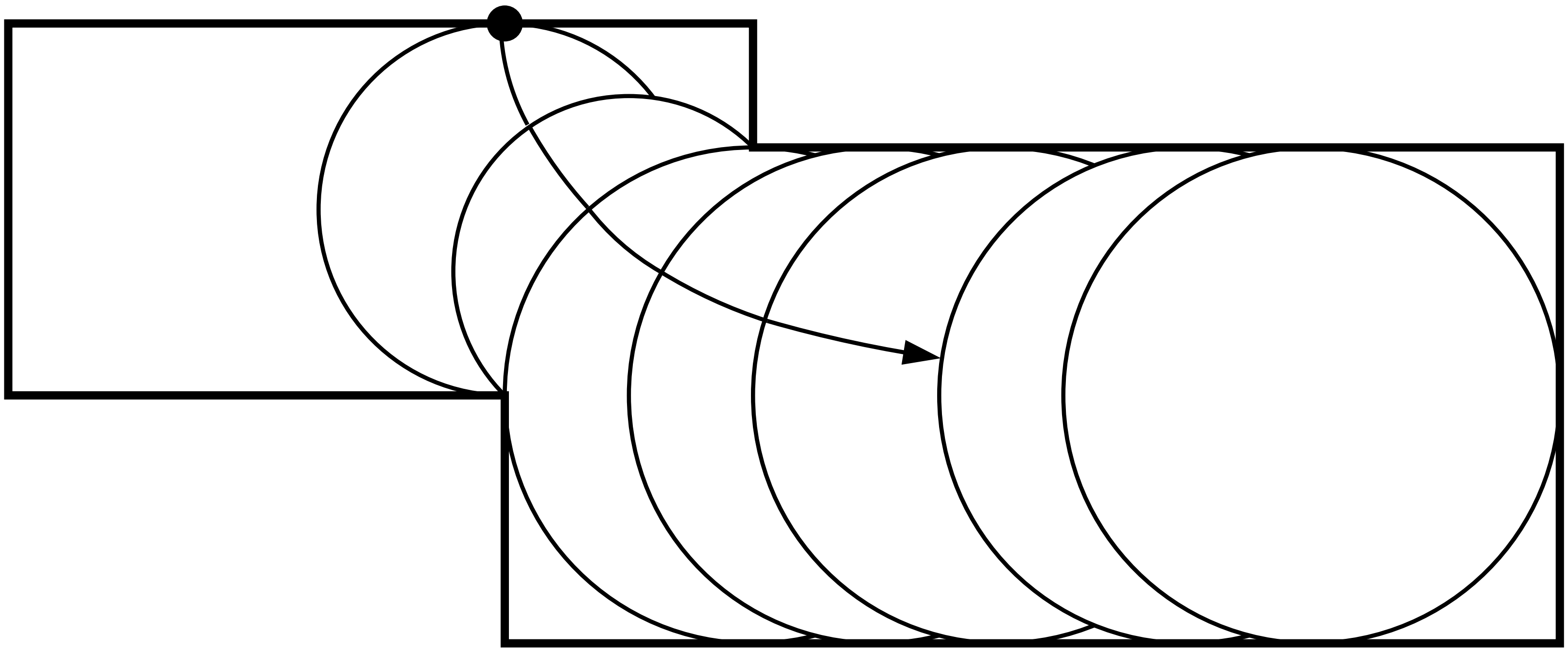


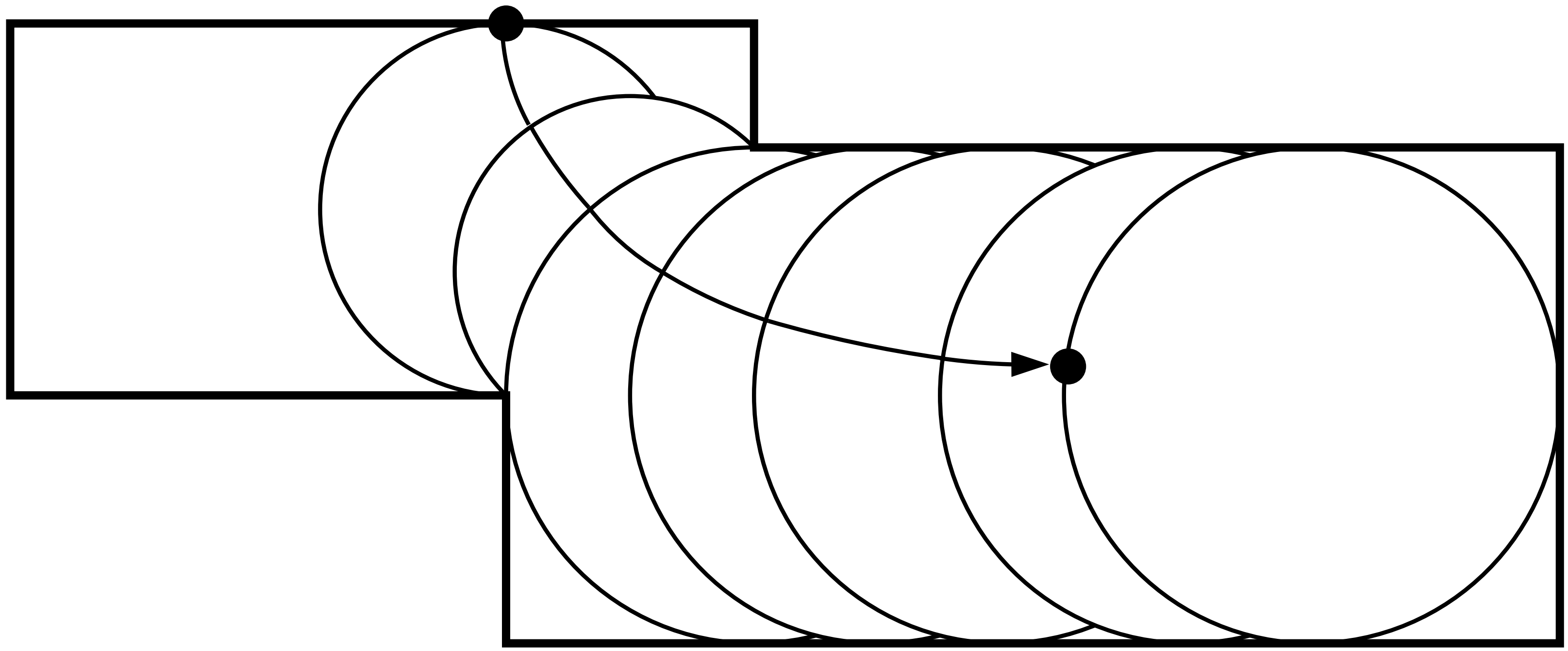


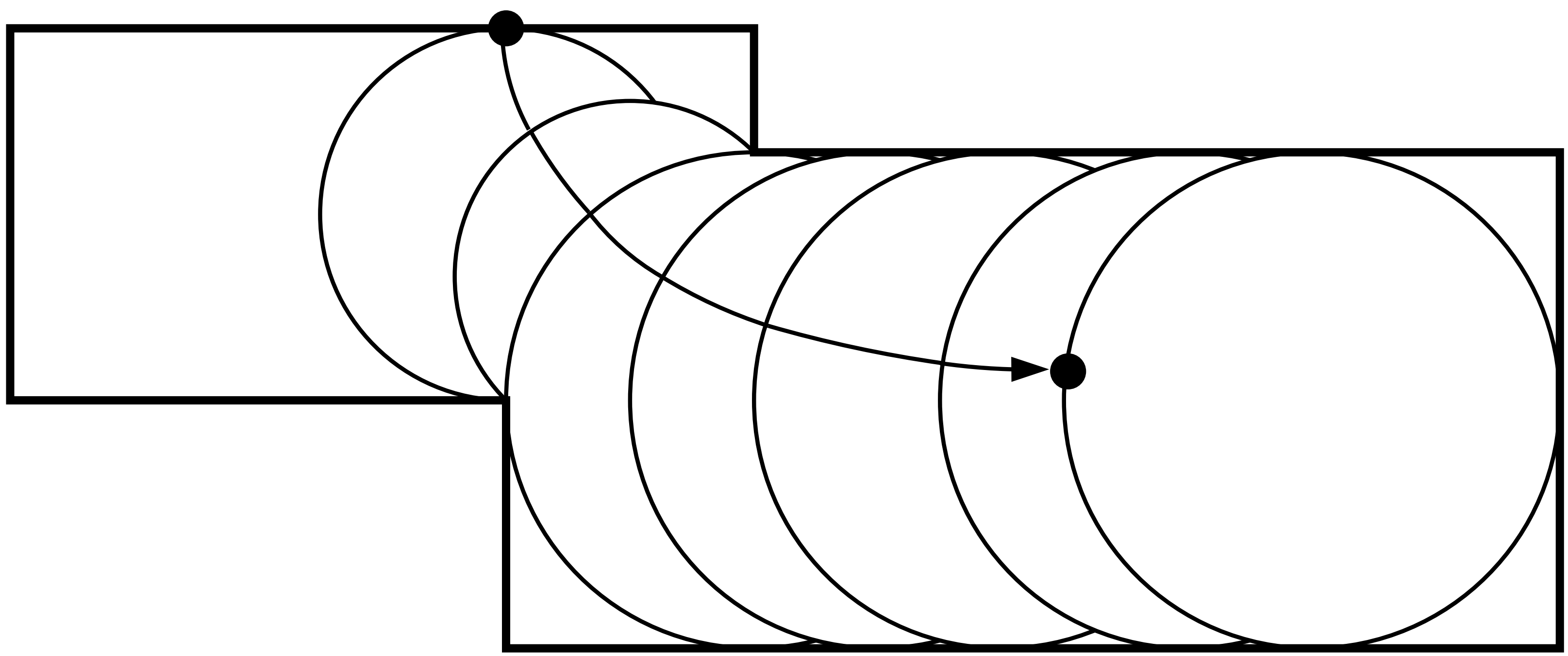










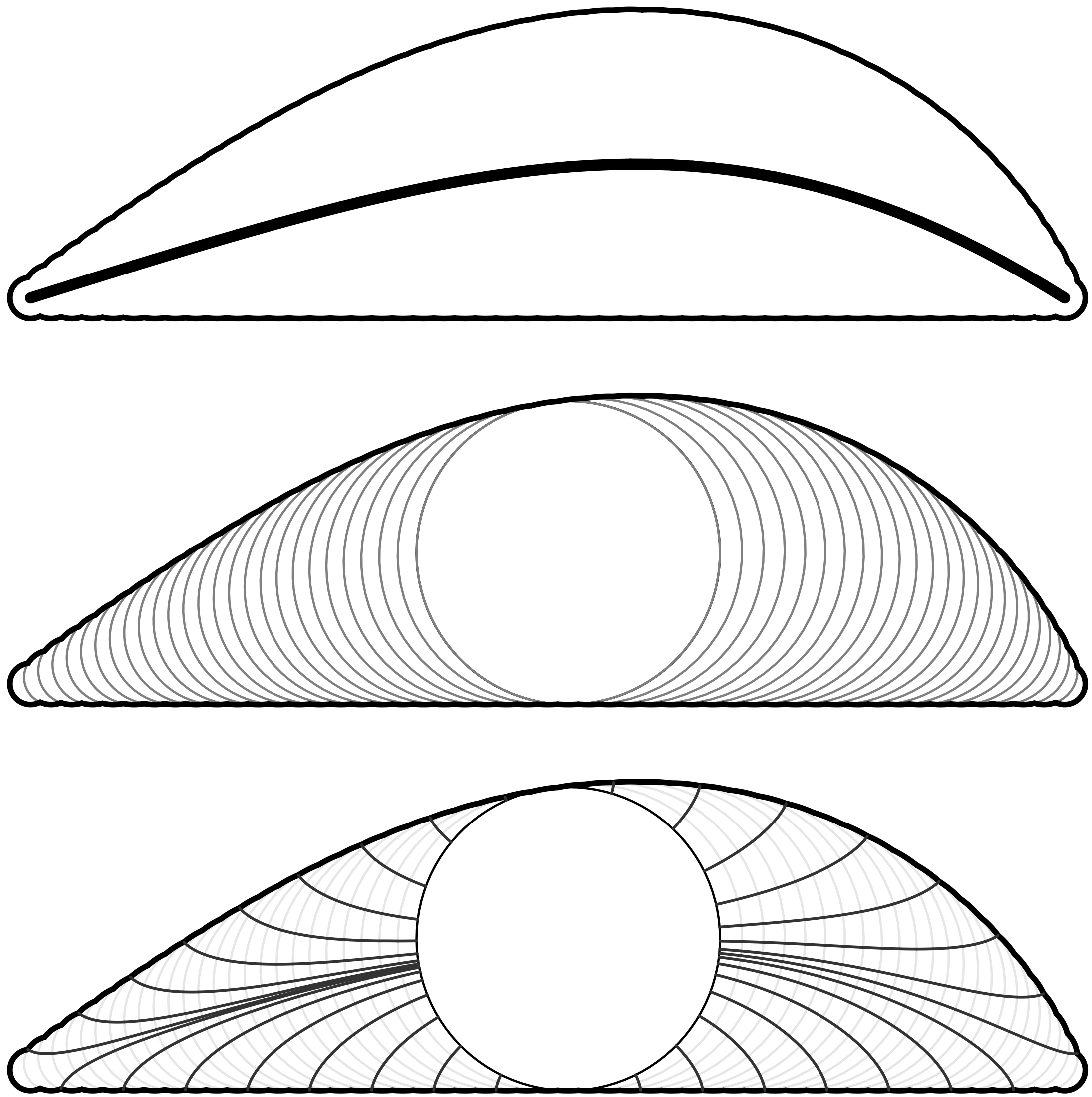


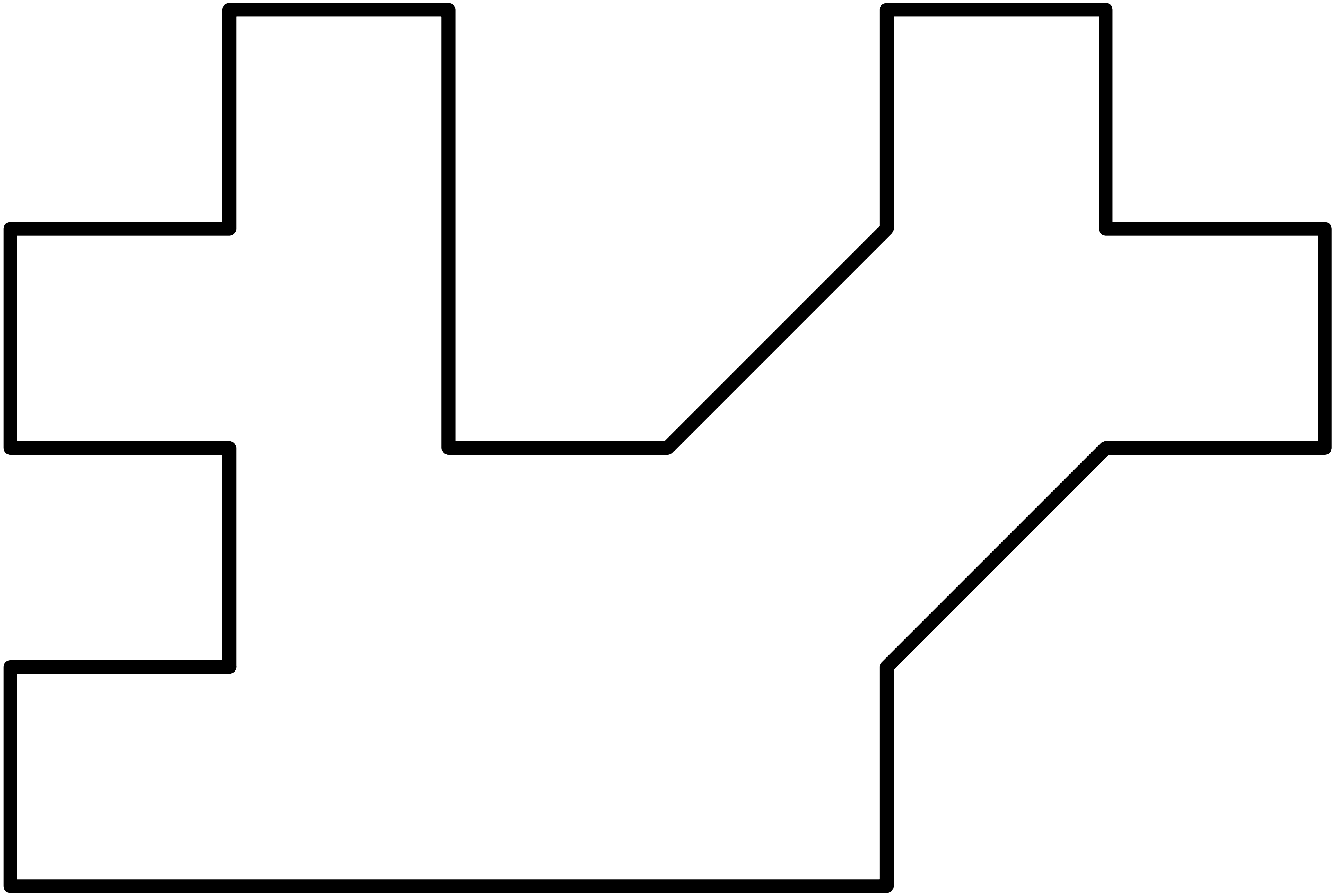
We discretize only to draw picture.

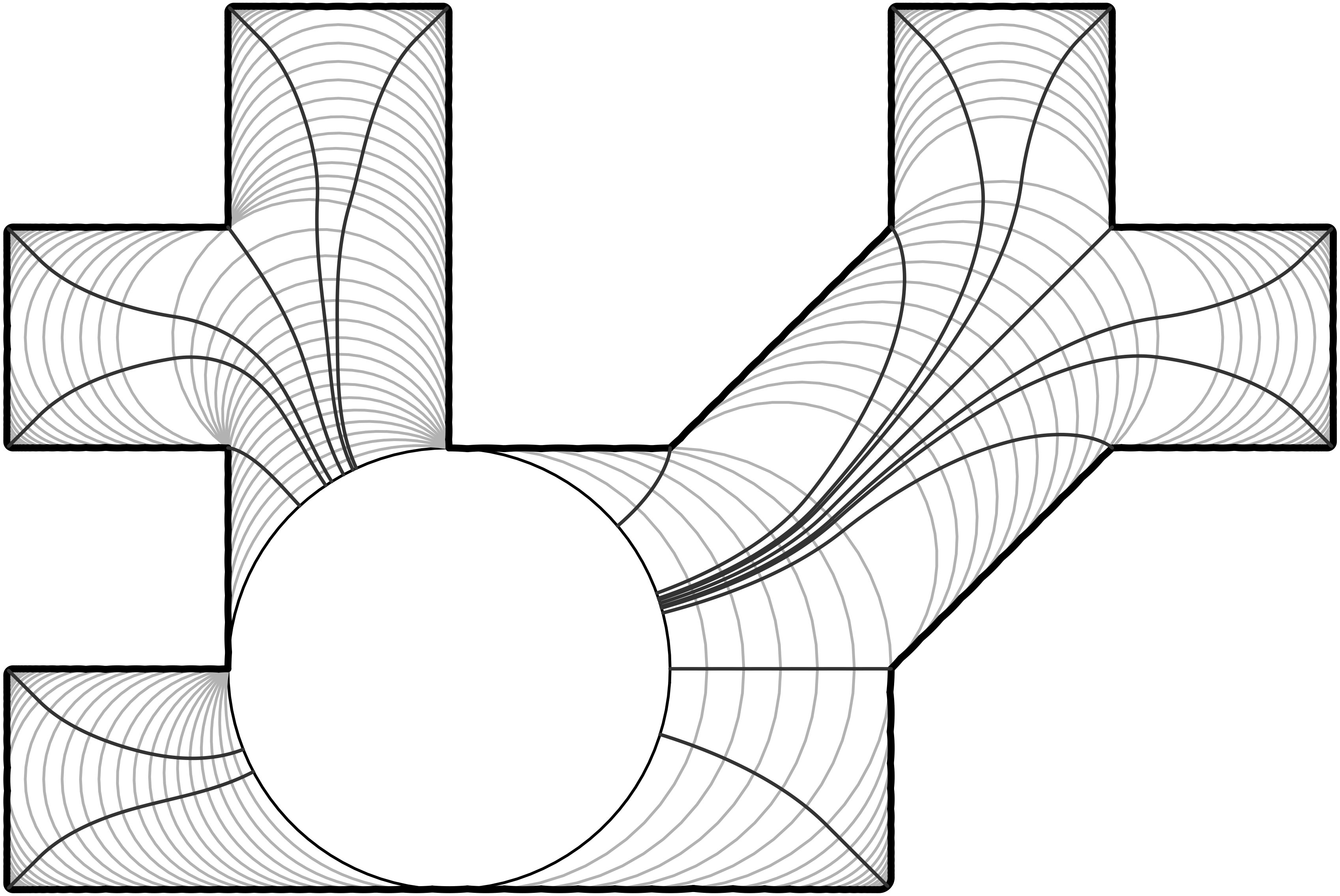
Limiting map has **formula** in terms of medial axis.

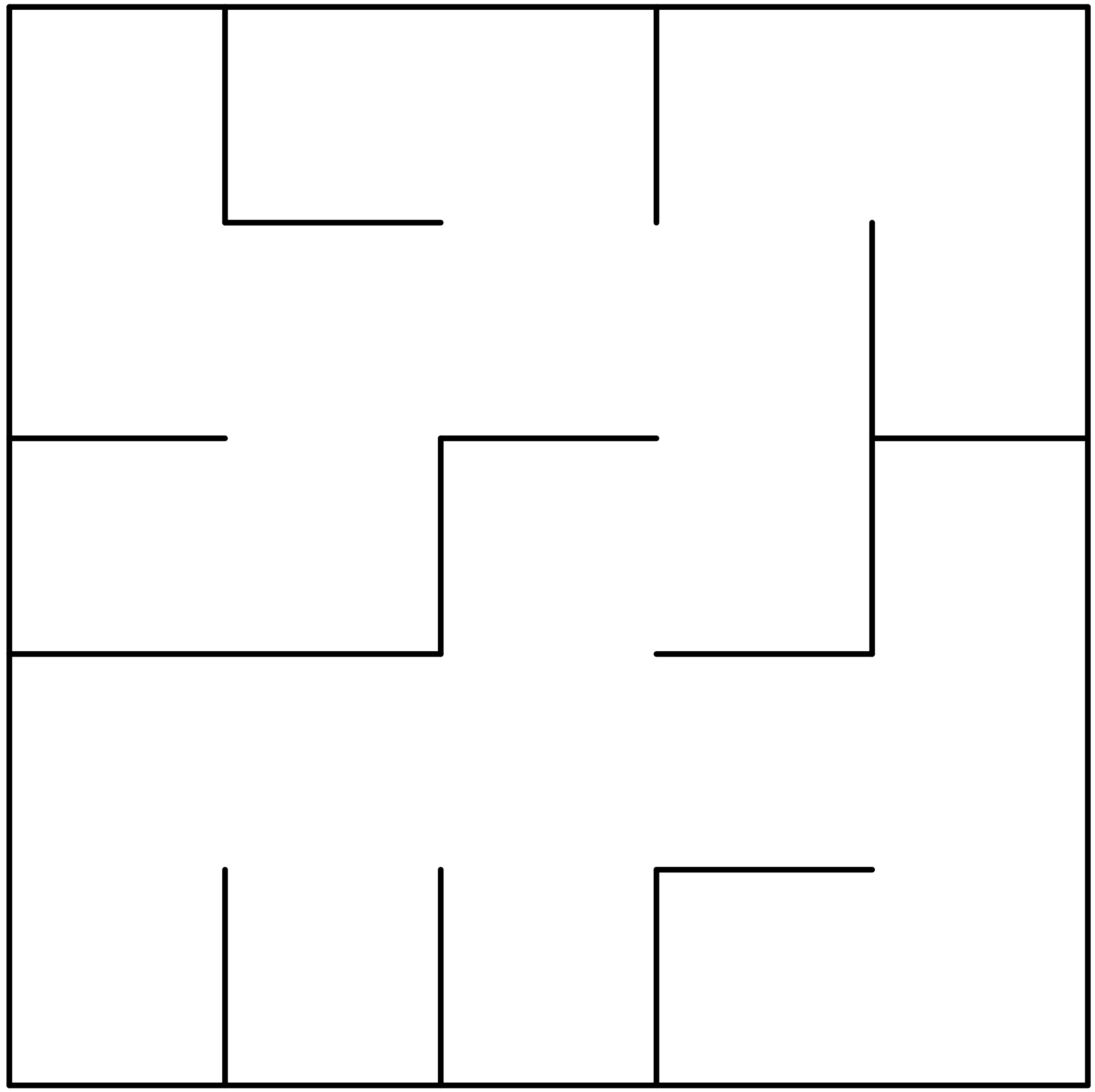
Called “iota map” or “medial axis map”.

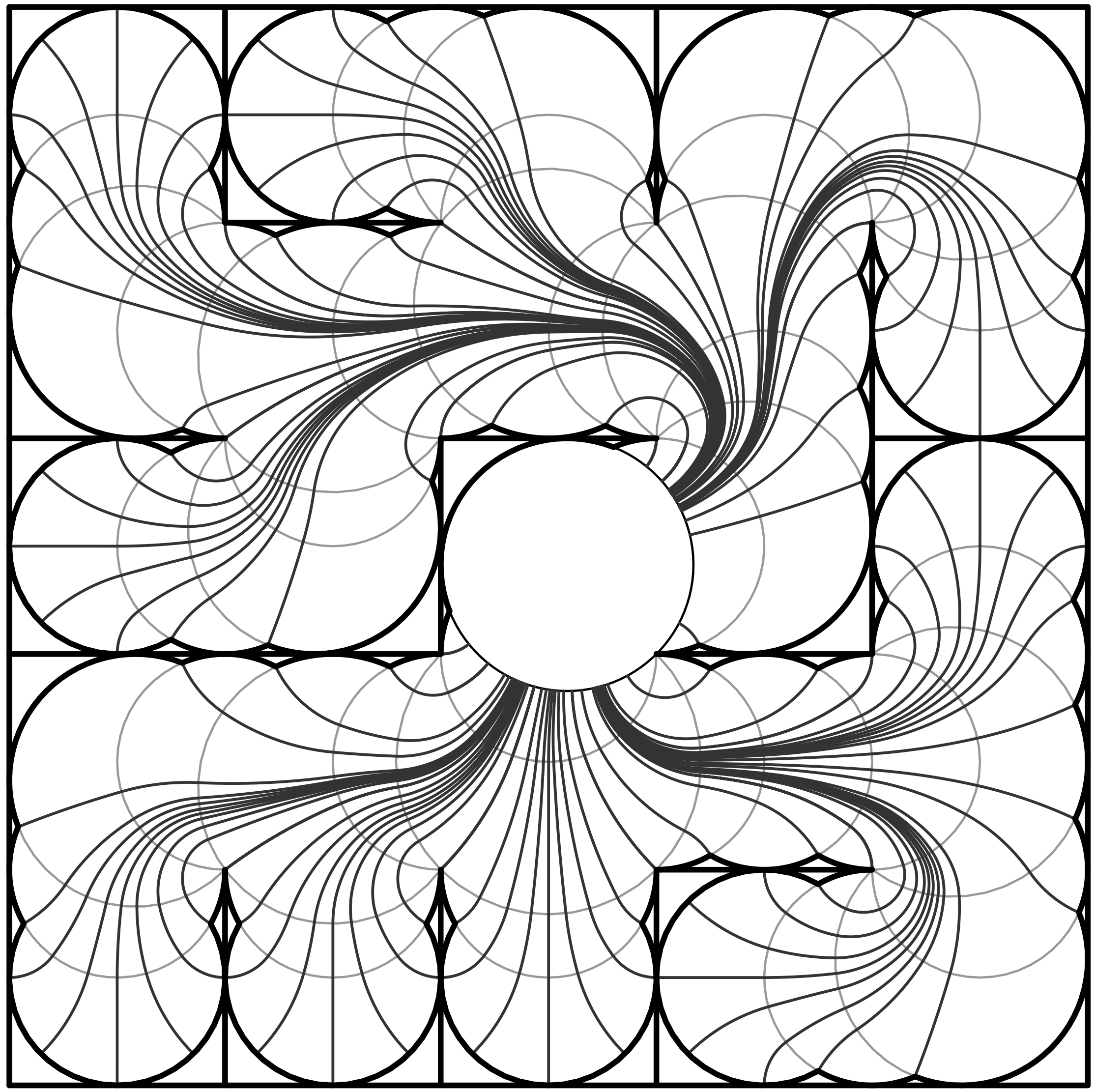
Similar flow for any simply connected domain.





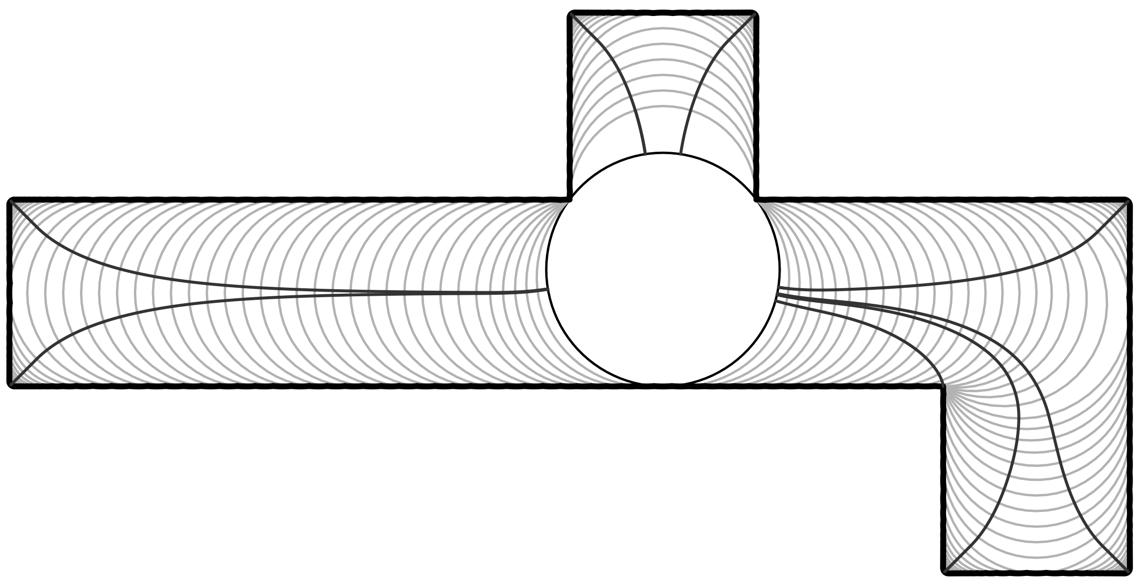






Theorem: Mapping all n vertices takes $O(n)$ time.

Uses linear time computation of MA (Chin-Snoeyink-Wang) and book-keeping with cross ratios. Uses famous (and difficult) theorem of Chazelle.



How close is medial axis map to conformal map?

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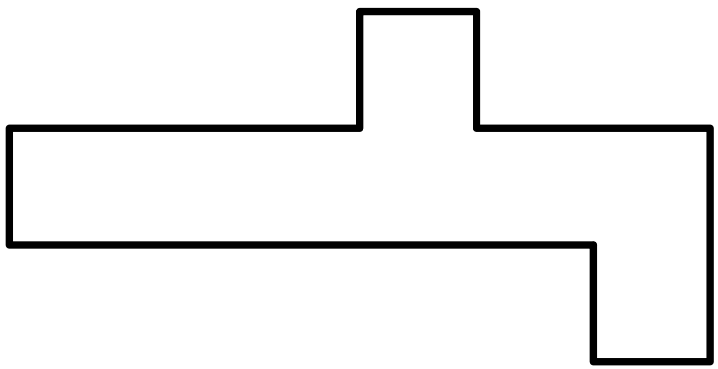
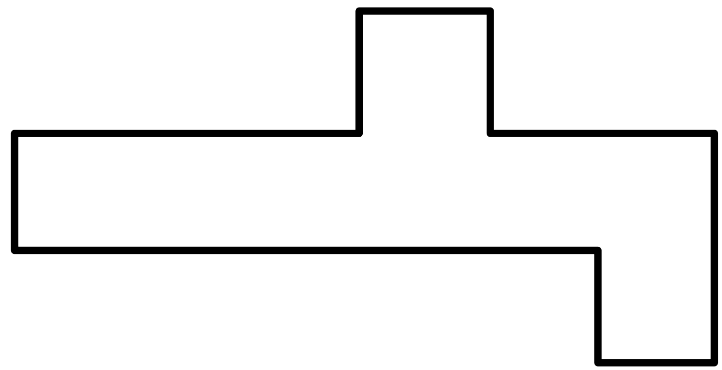
Use “MA-parameters” in Schwarz-Christoffel formula.



Target Polygon

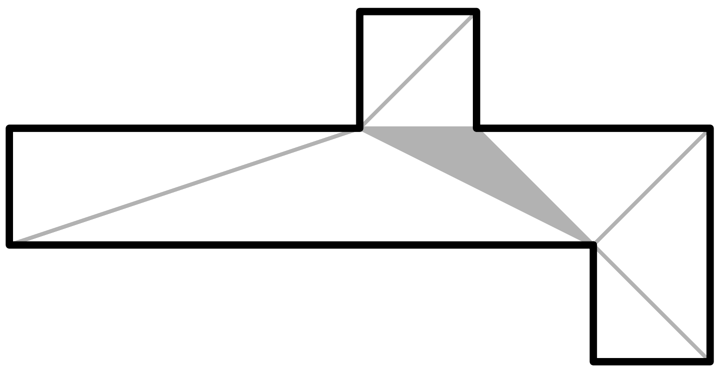


MA Parameters

Looks pretty close.

How close is medial axis map to conformal map?

Use "MA-parameters" in Schwarz-Christoffel formula.



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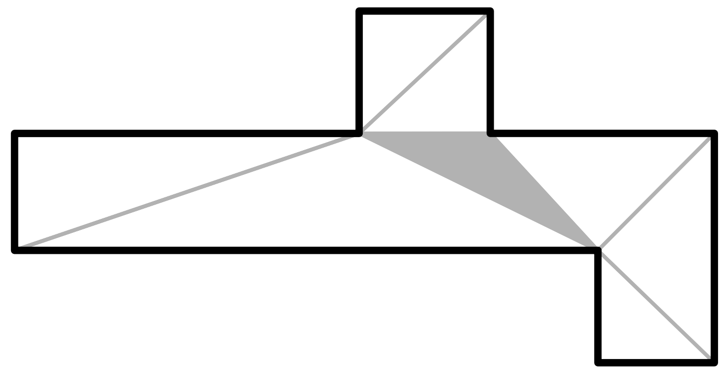


MA Parameters

Triangulate and use affine maps to estimate QC error.

The most distorted triangle is shaded. Here $K = 1.24$.

Theorem: Iota map always gives QC-error $K < 8$.

E.g., iota map has 8-QC extension $\Omega \to \mathbb{D}$.

Theorem: Iota map always gives QC-error $K < 8$.

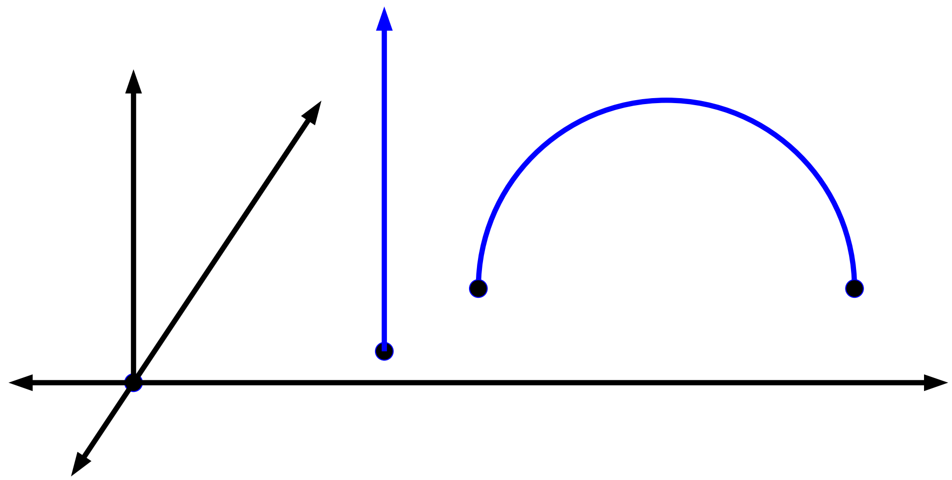
Optimal K is unknown, but is > 2.1 (Epstein and Markovic).

$K = 2$ would have implied the Brennan conjecture.

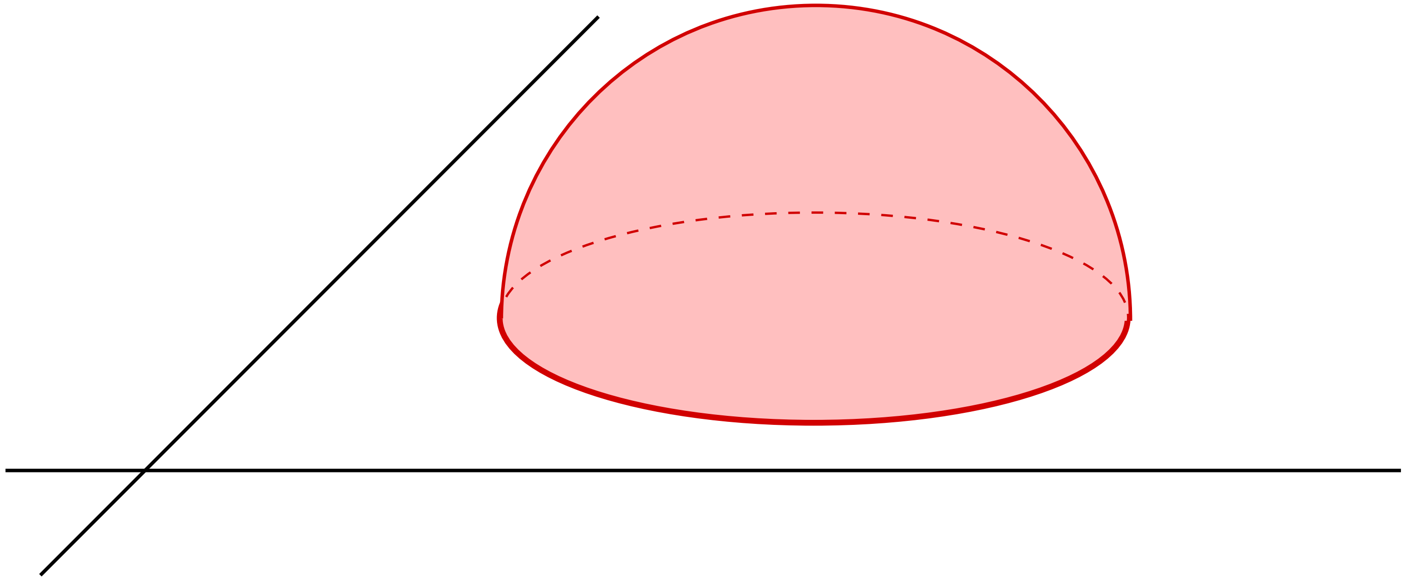
This is actually a theorem about surfaces in hyperbolic 3-space.

PART II: SURFACES ENCODED BY TREES

In the upper half-space $\mathbb{R}^3_+ = \{(x, y, t) : t > 0\}$, metric is $d\rho = ds/2t$.



Geodesics in $\mathbb{R}^3_+$ are vertical rays or semi-circles perpendicular to $\mathbb{R}^2$.

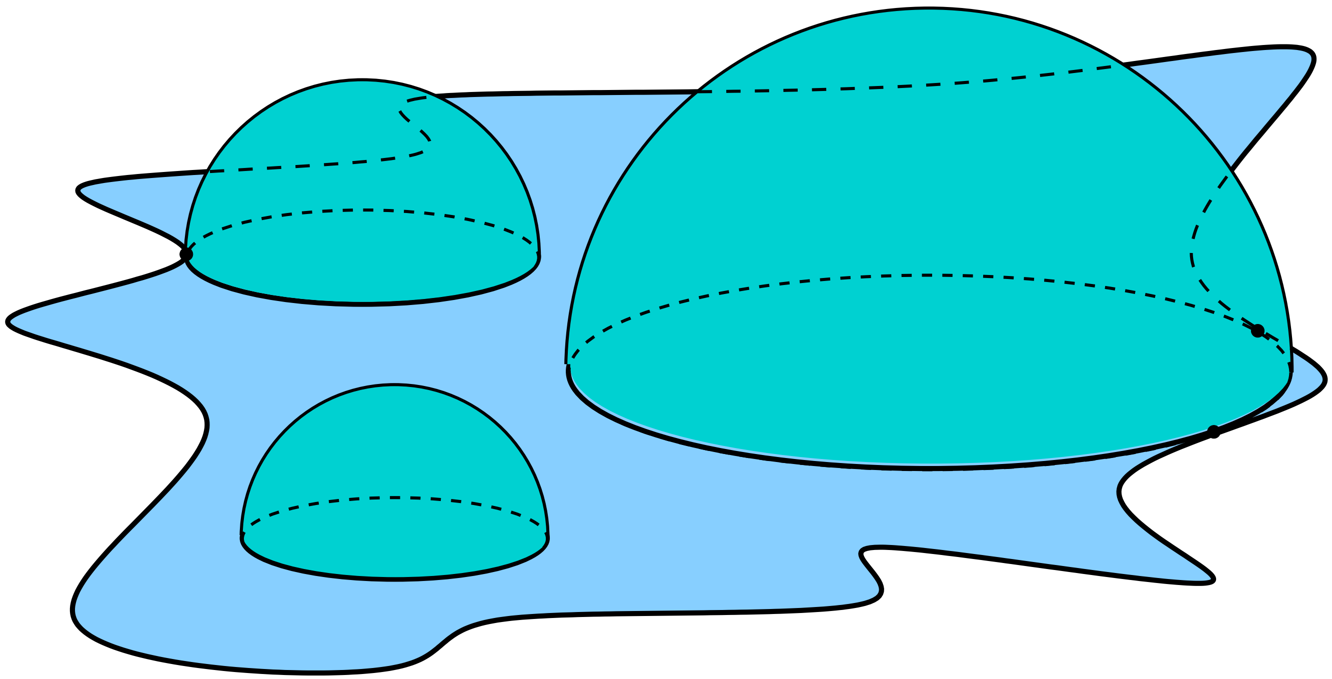


In $\mathbb{R}^3_+$, a hyperbolic half-space = hemisphere.

Given $\Omega \subset \mathbb{R}^2$, compute hyperbolic convex hull its complement.

Easier to visualize the complement of convex hull = union of hemispheres.

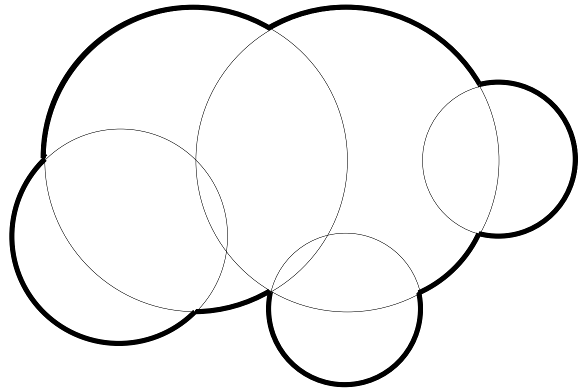
Dome(Ω) is union of hemi-spheres with base disks in Ω.

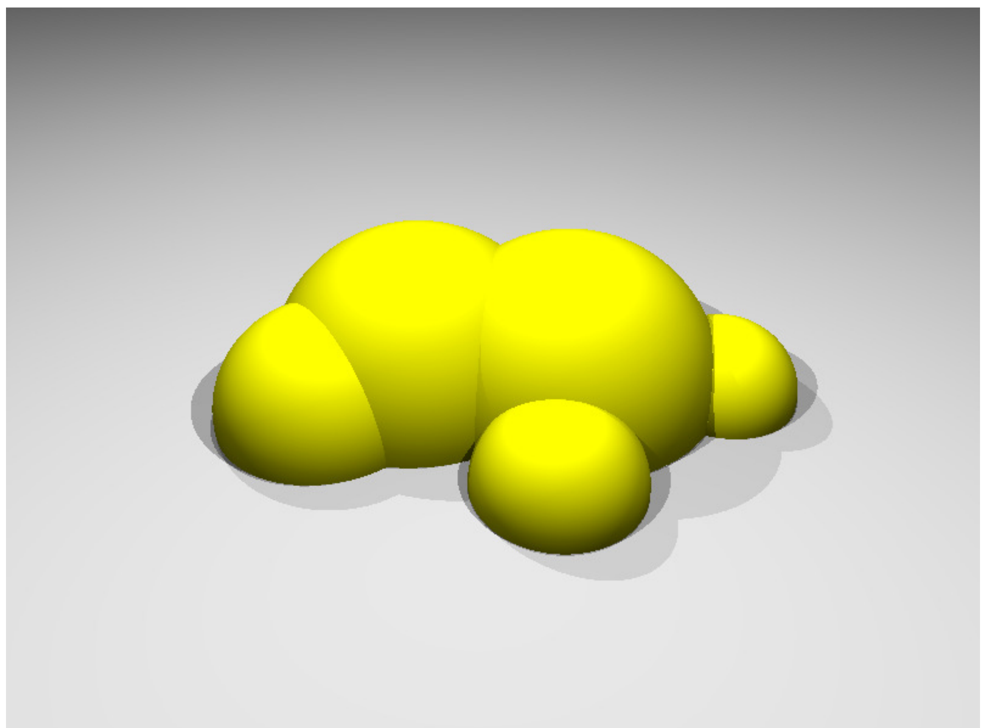


Region above dome is intersection of half-spaces, hence convex.

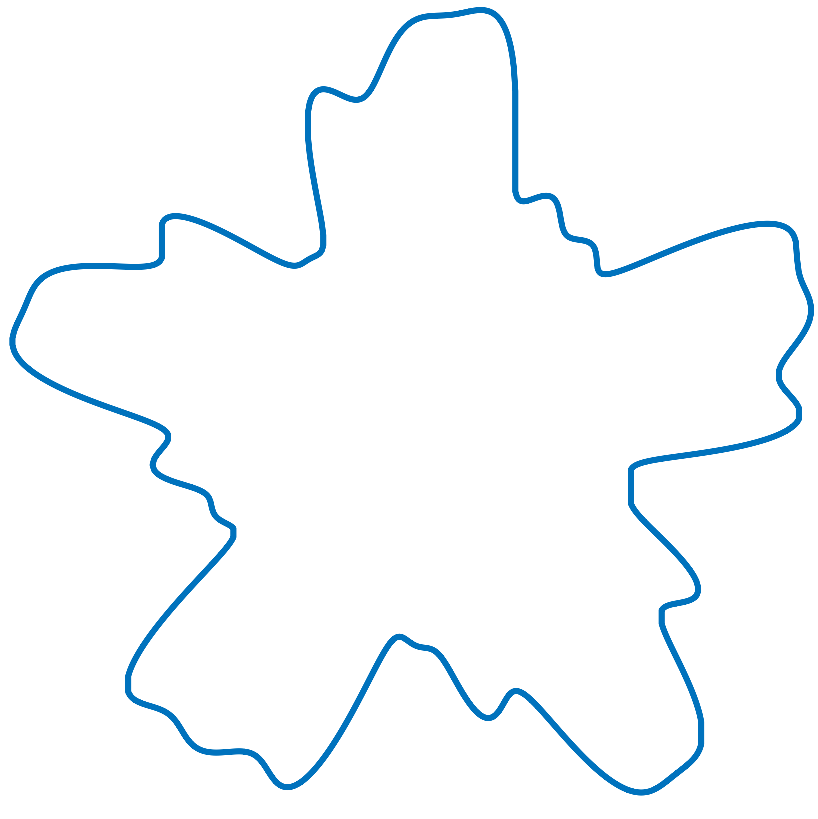
It is hyperbolic convex hull of complement of Ω.

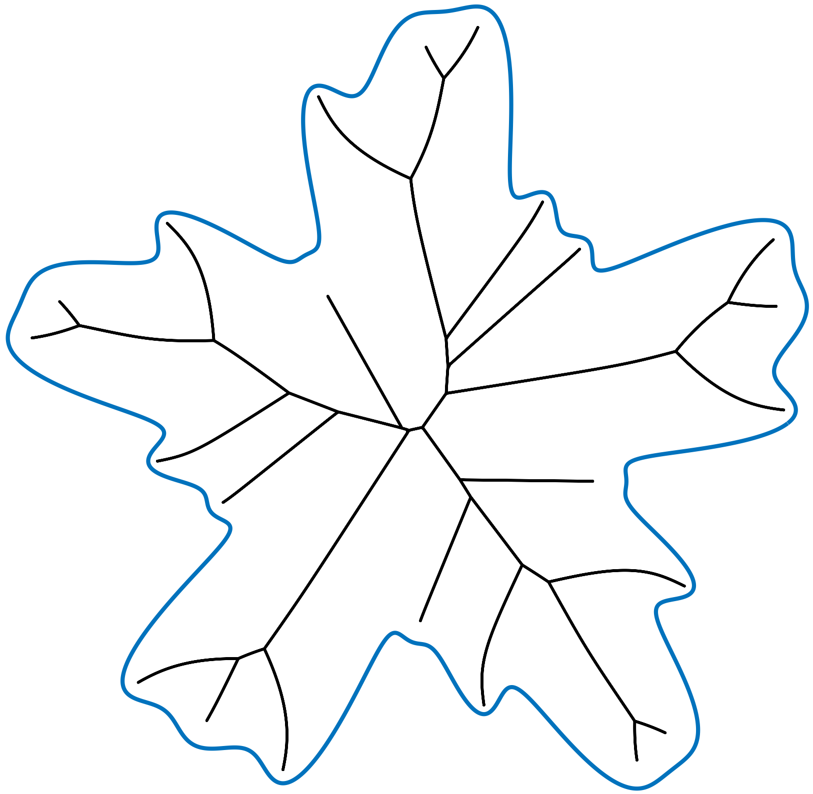
Upper boundary S of dome is a surface in $\mathbb{R}^3_+$ with $\partial S = \partial\Omega$.



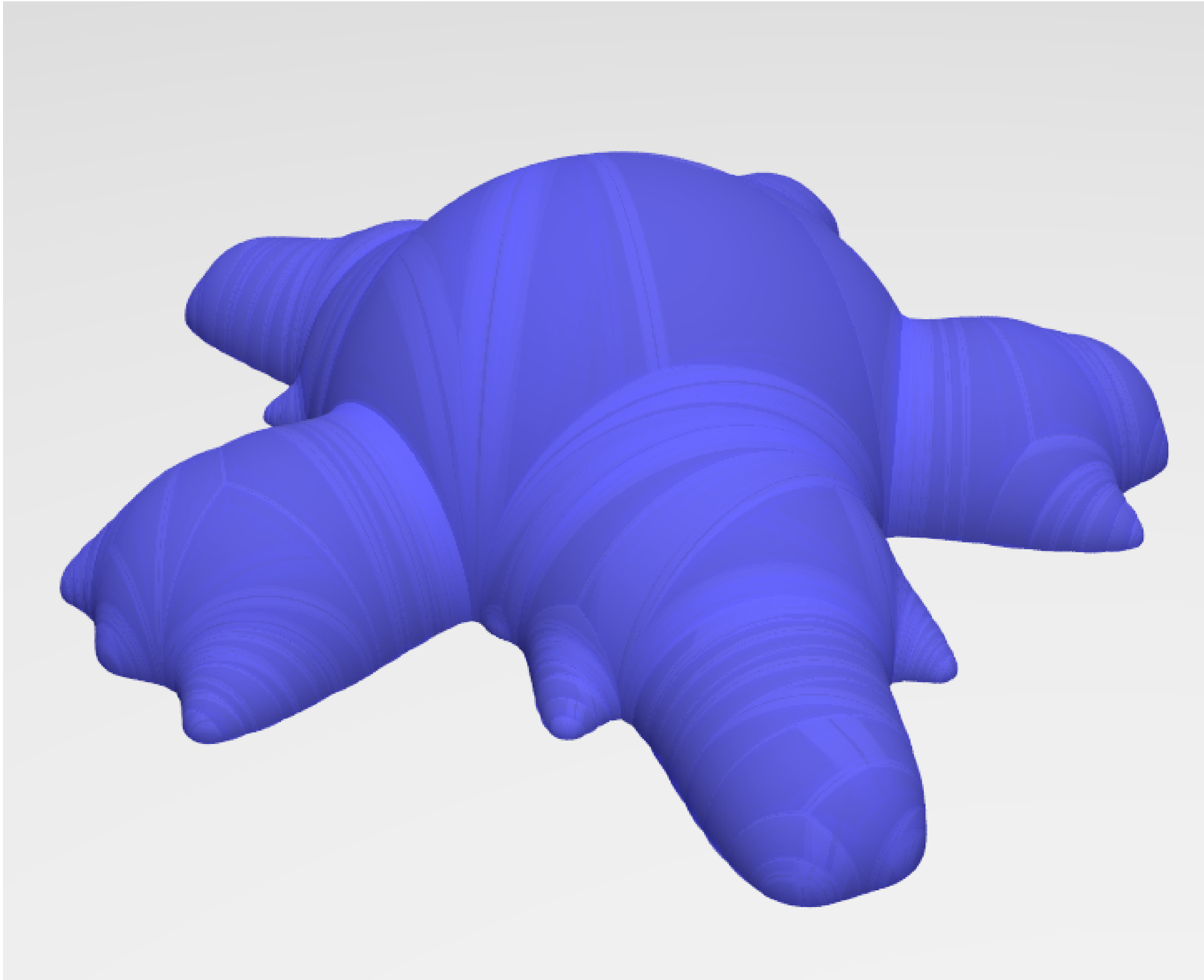


Finite dome





The medial axis. Dome is union over medial axis disks.
In some sense, “Dome = Medial axis”.



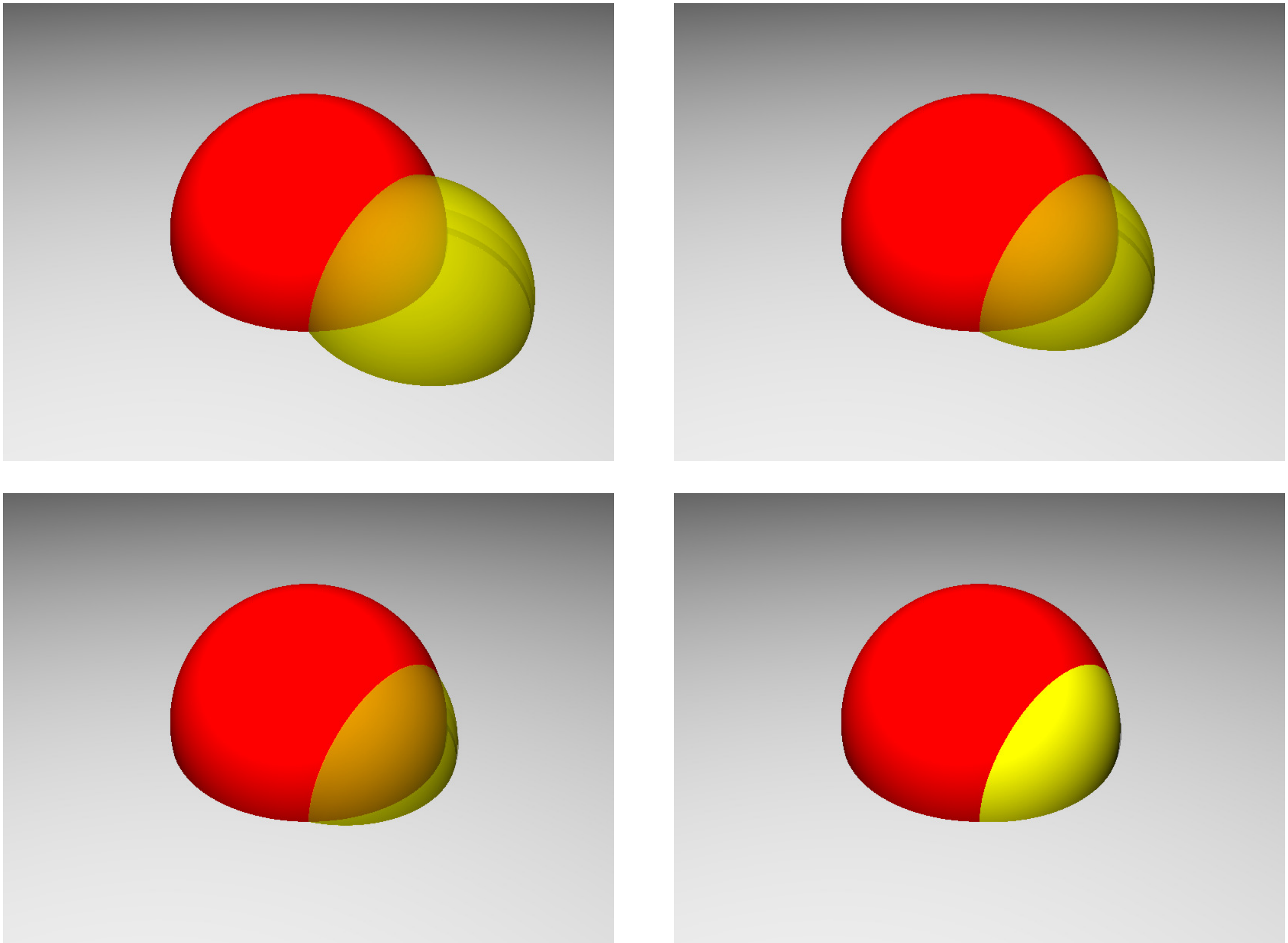
Thm: Simply connected domes are isometric to hyperbolic disk.

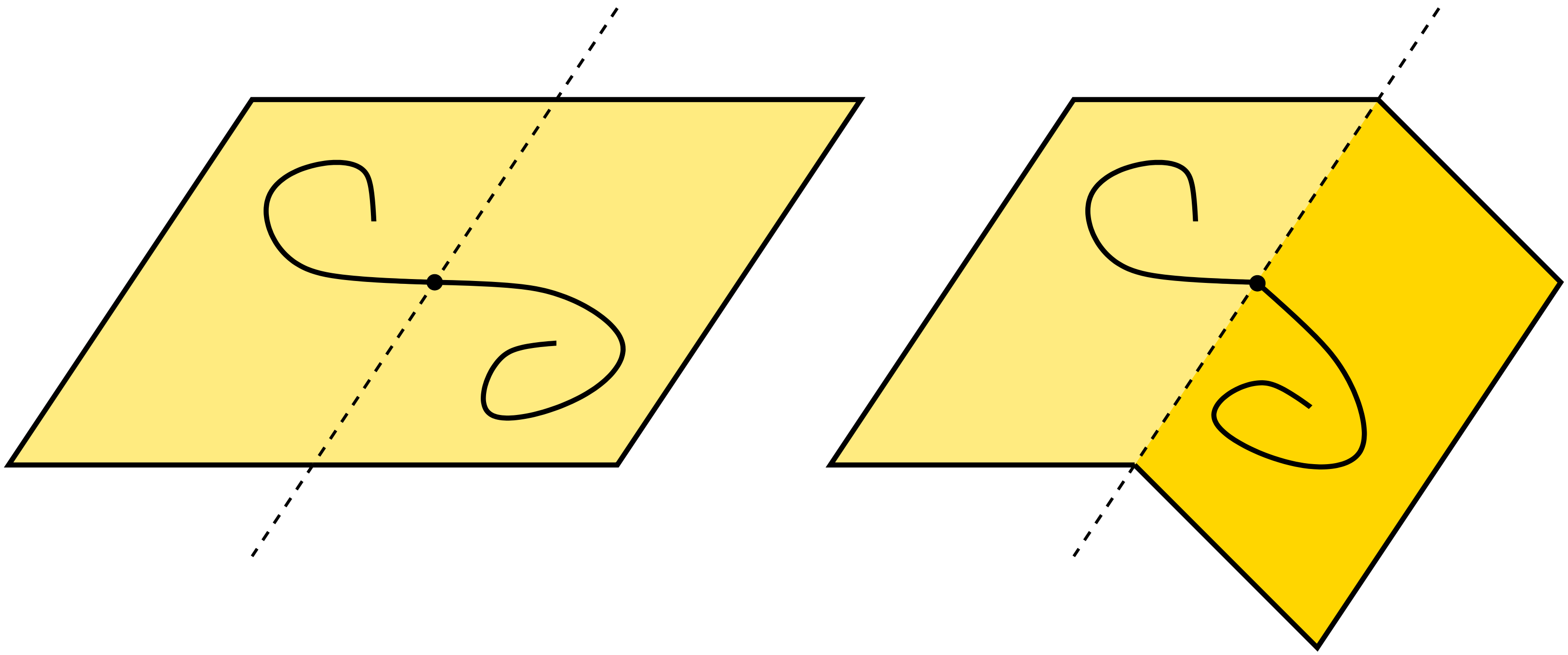
Here we give dome the hyperbolic path metric.

- Prove for finite unions of disks.
- Every dome is a limit of finite domes.
- Limit of isometries is an isometry.

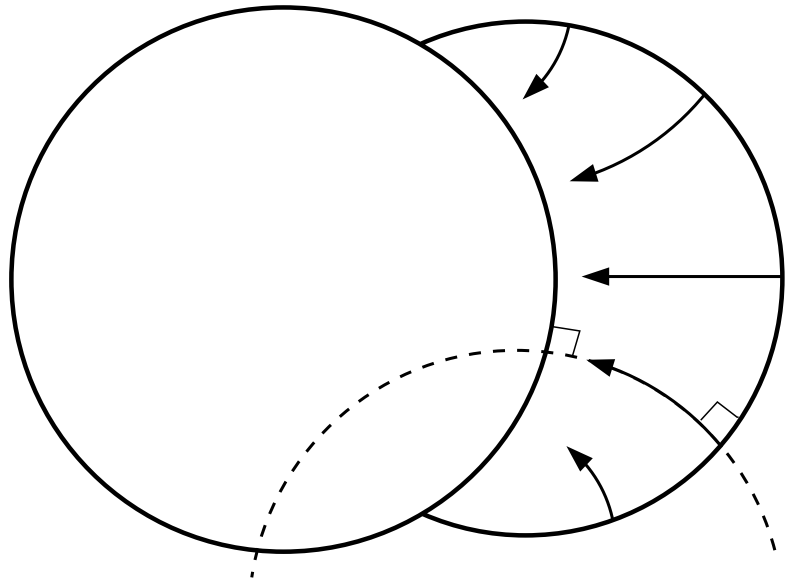
Isometry on boundary Γ defines a map Γ to circle.

Every dome has conformal map to disk by “flattening”.



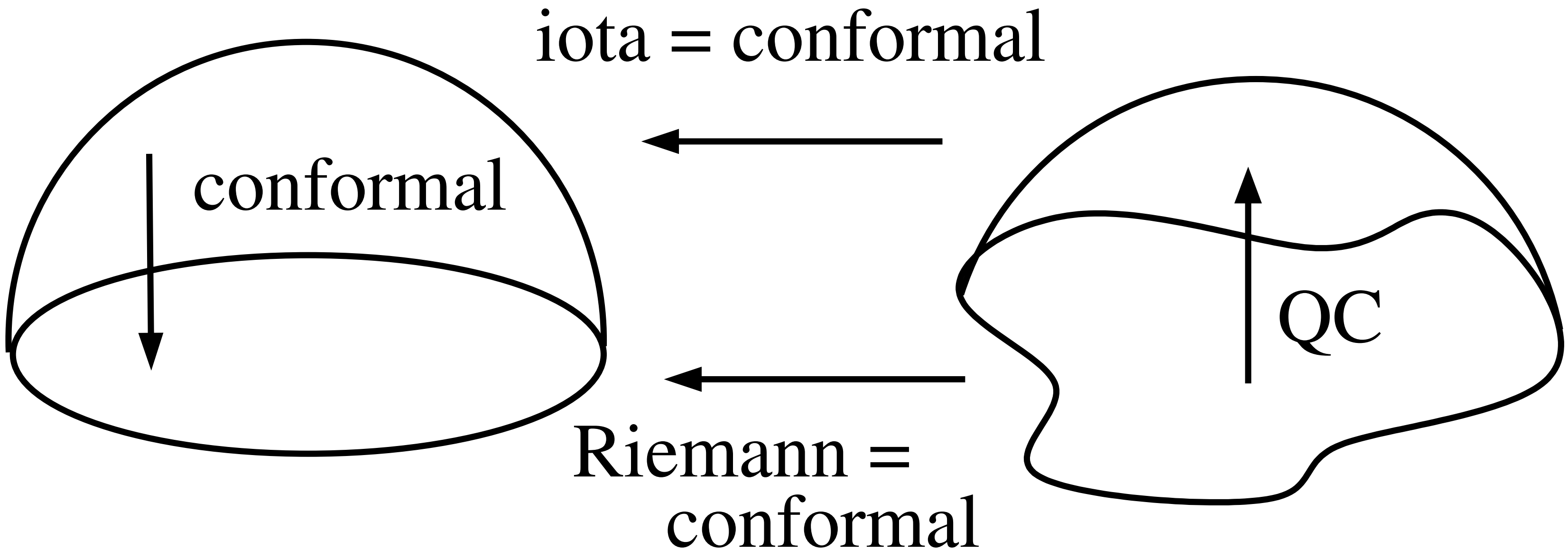


Folding plane along geodesic does not change length.
Pleated surface (folded along disjoint geodesics) = Flat plane



Medial axis map = boundary of flattening map (iota)

= boundary of conformal map of dome to hemisphere

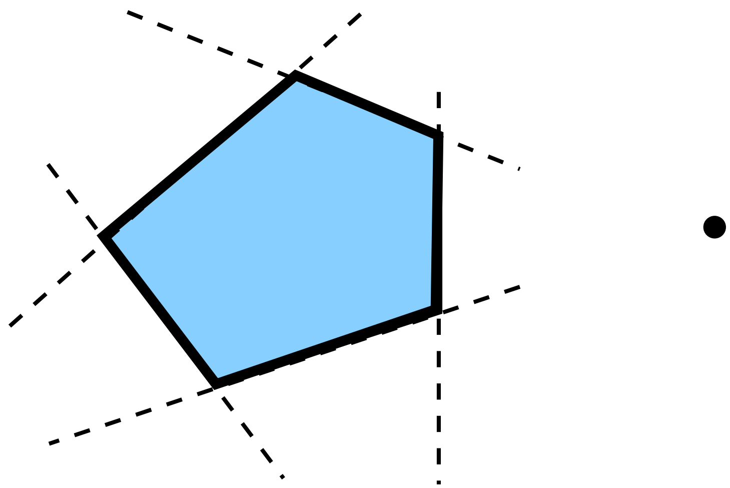


Iota = conformal from dome to disk.

Medial axis flow = boundary values of iota

Claim: There is QC map base → dome fixing boundary pointwise.

Nearest point map in $\mathbb{R}^n$ is Lipschitz.



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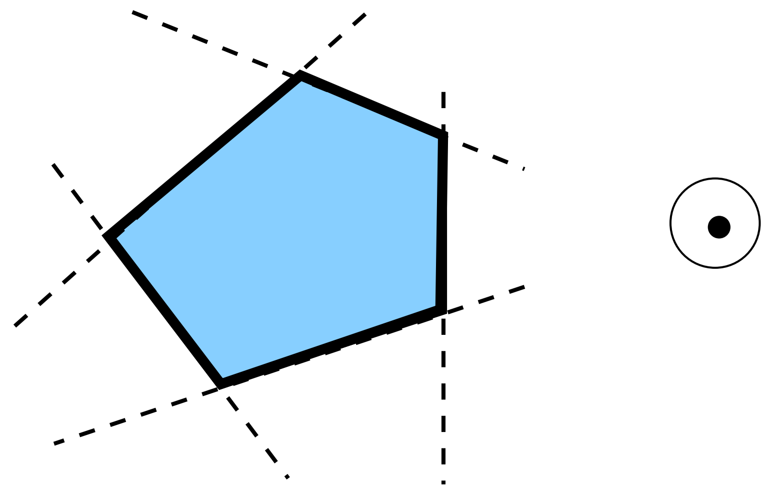

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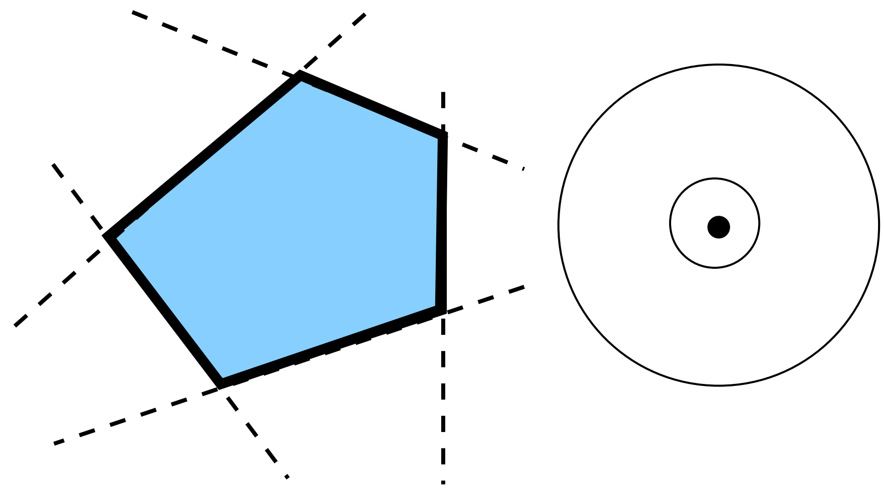

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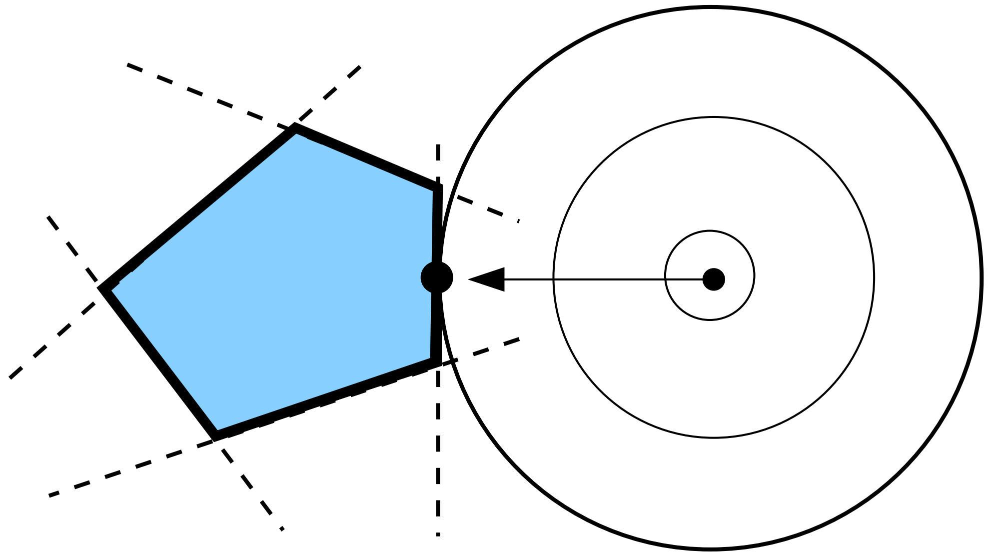

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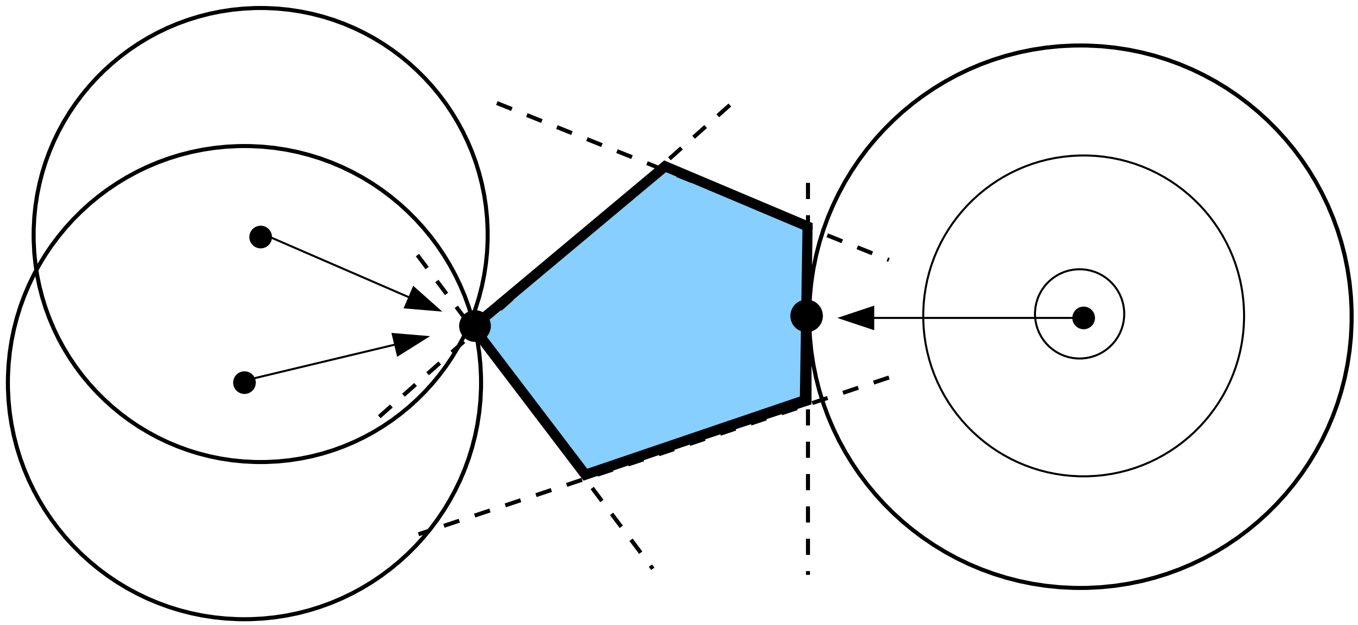
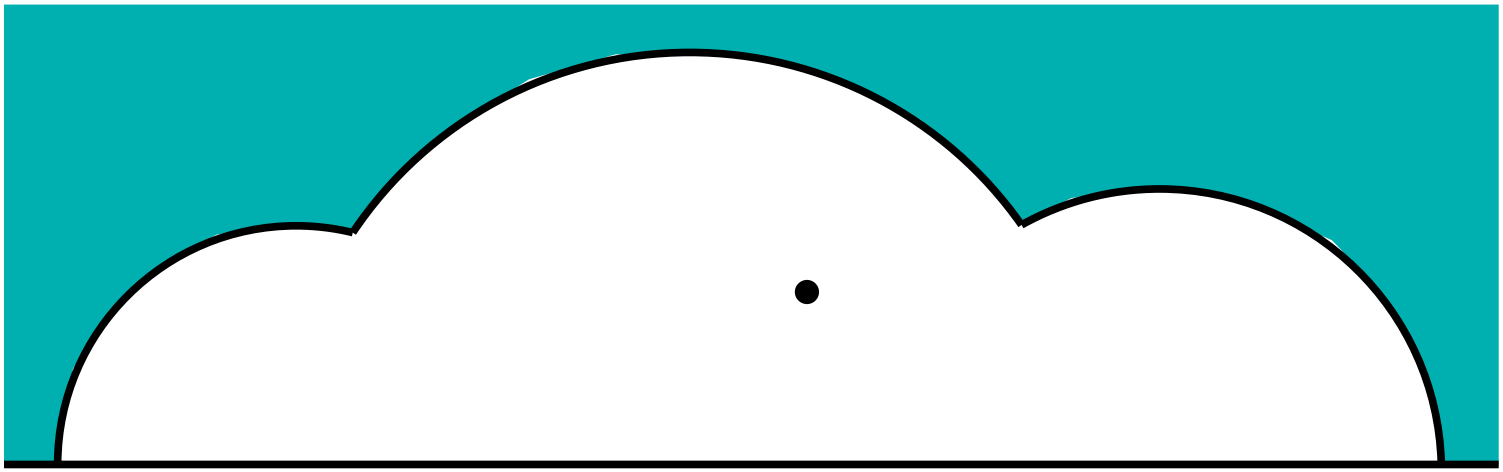


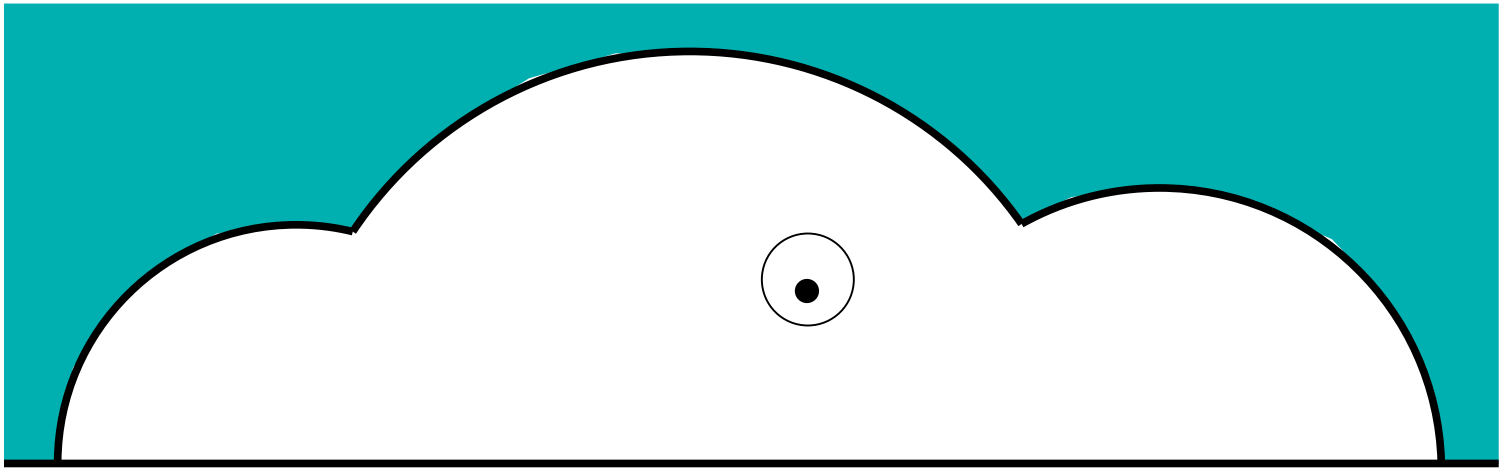


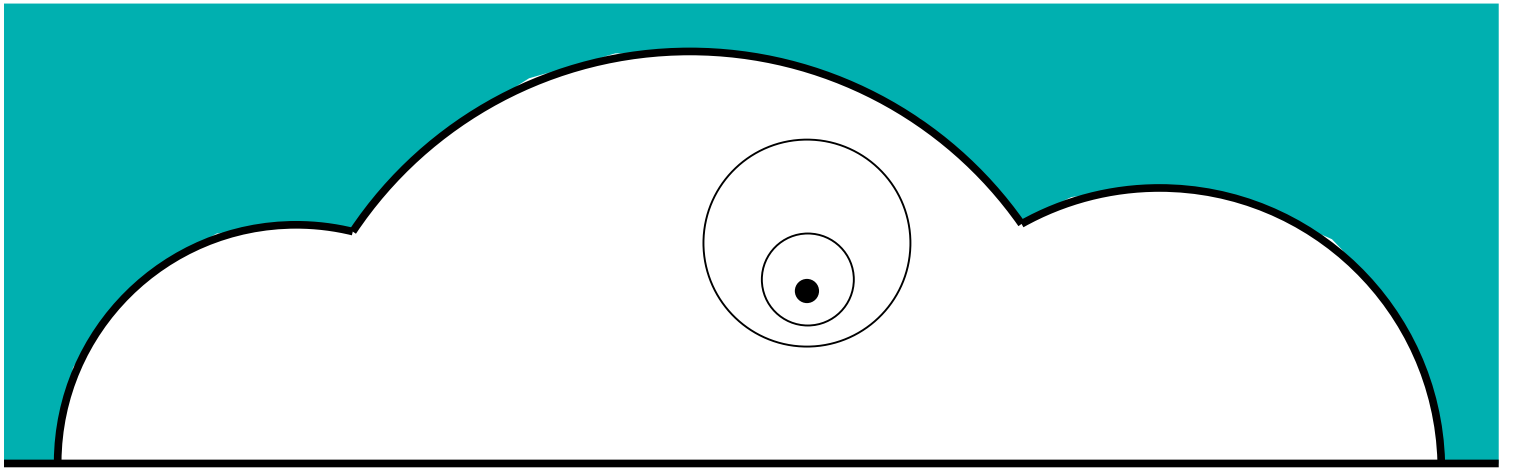
Region below dome is union of hemispheres

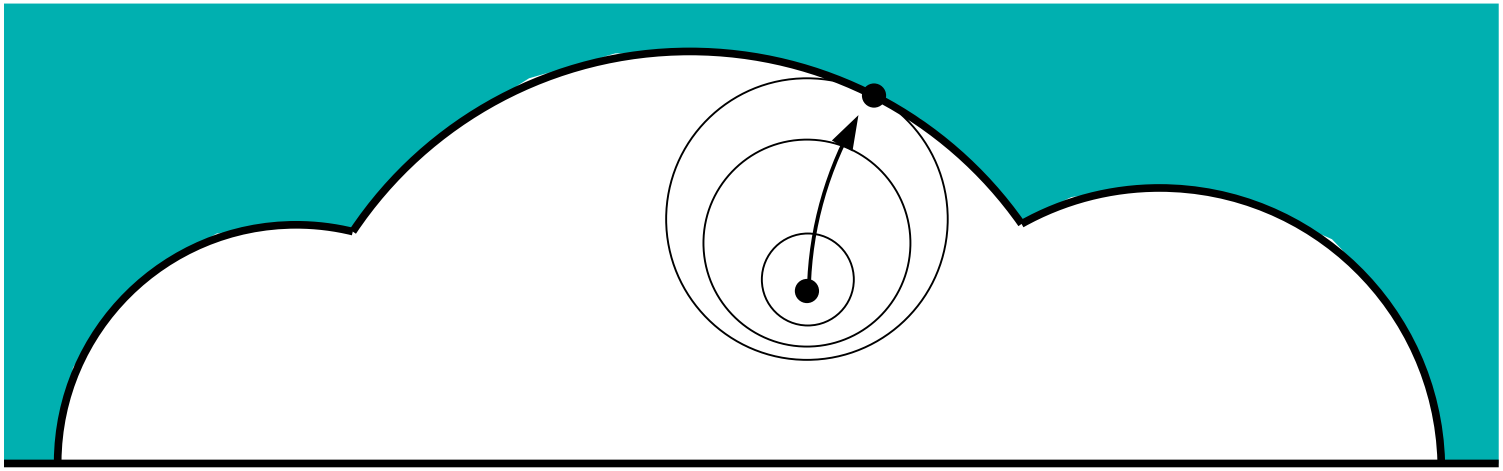
Hemispheres = hyperbolic half-spaces.

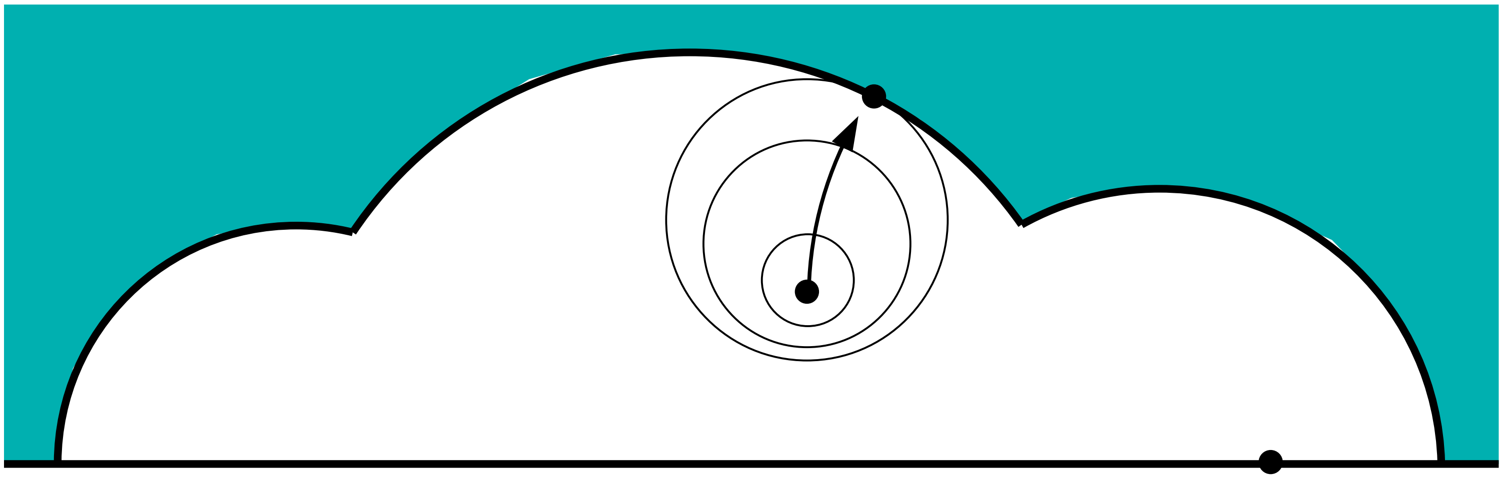
Region above dome is hyperbolically convex.

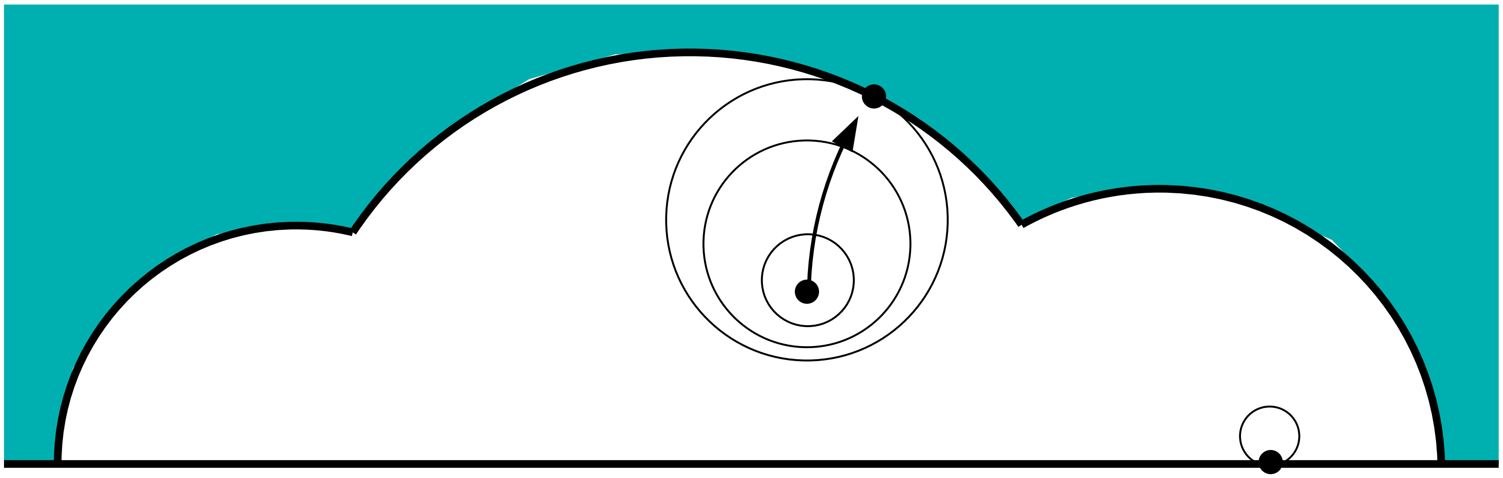
Consider nearest point retraction onto this convex set.

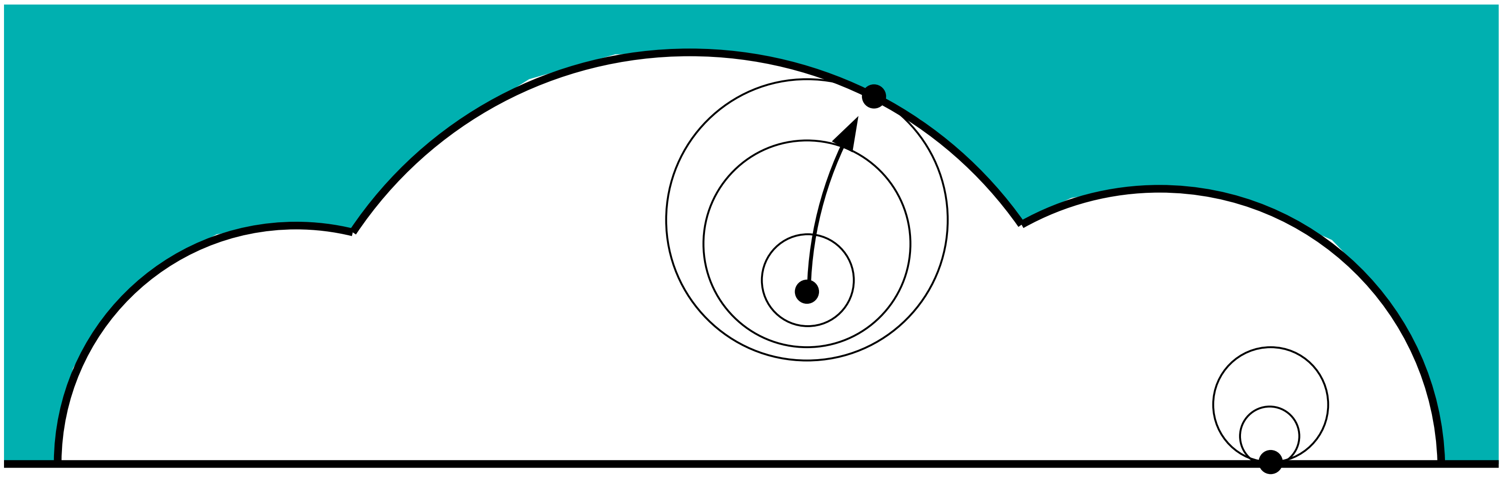


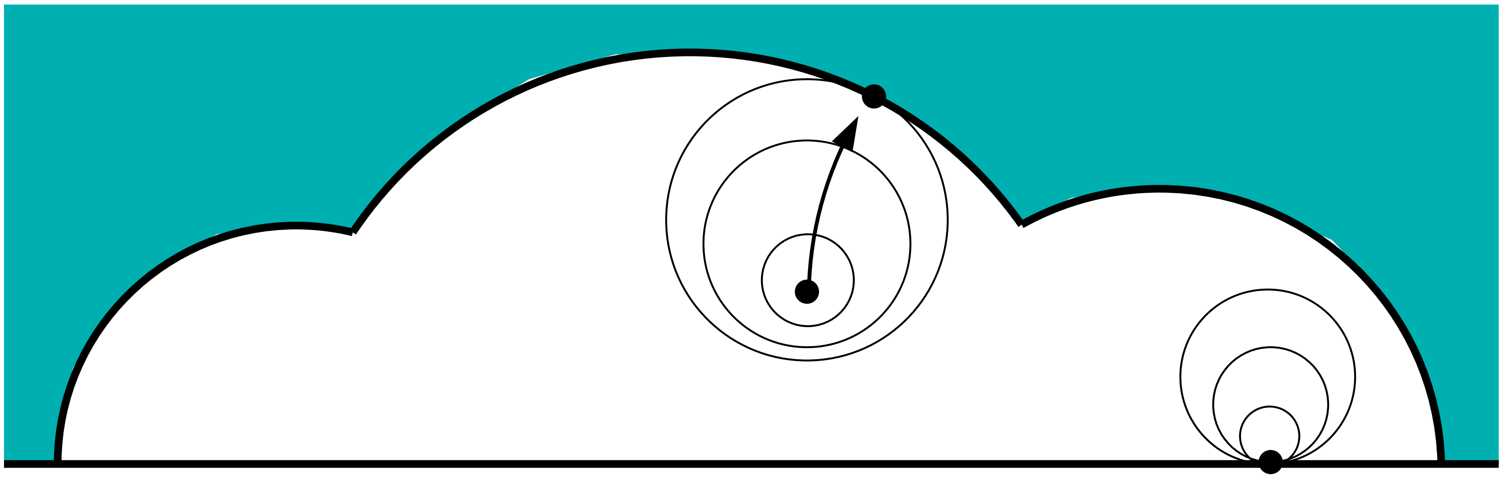


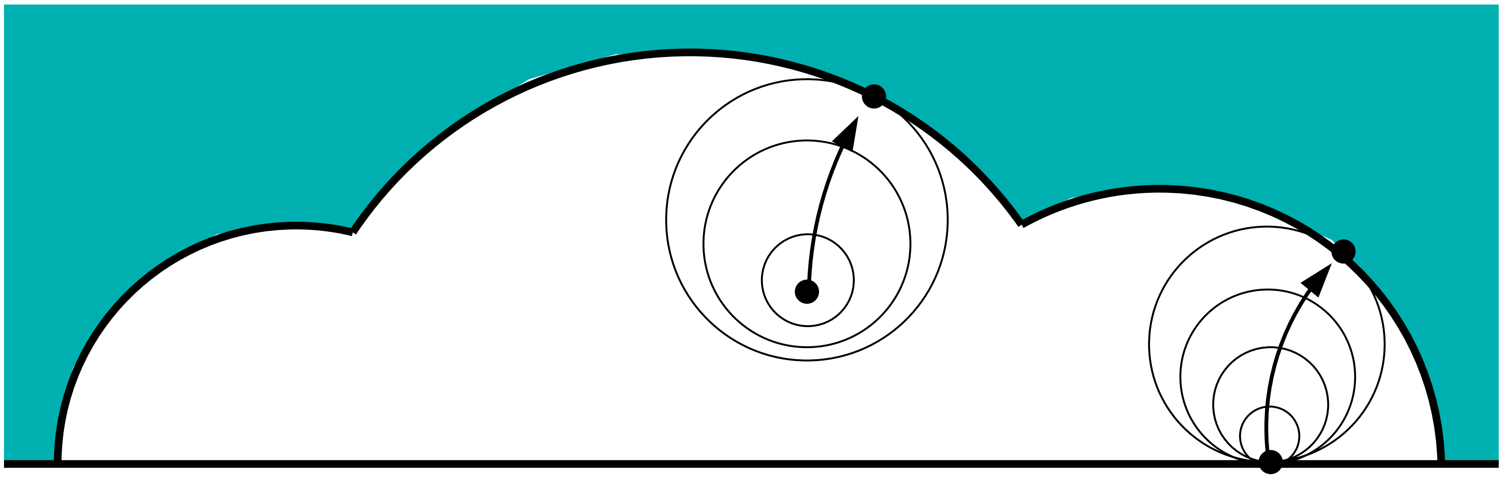


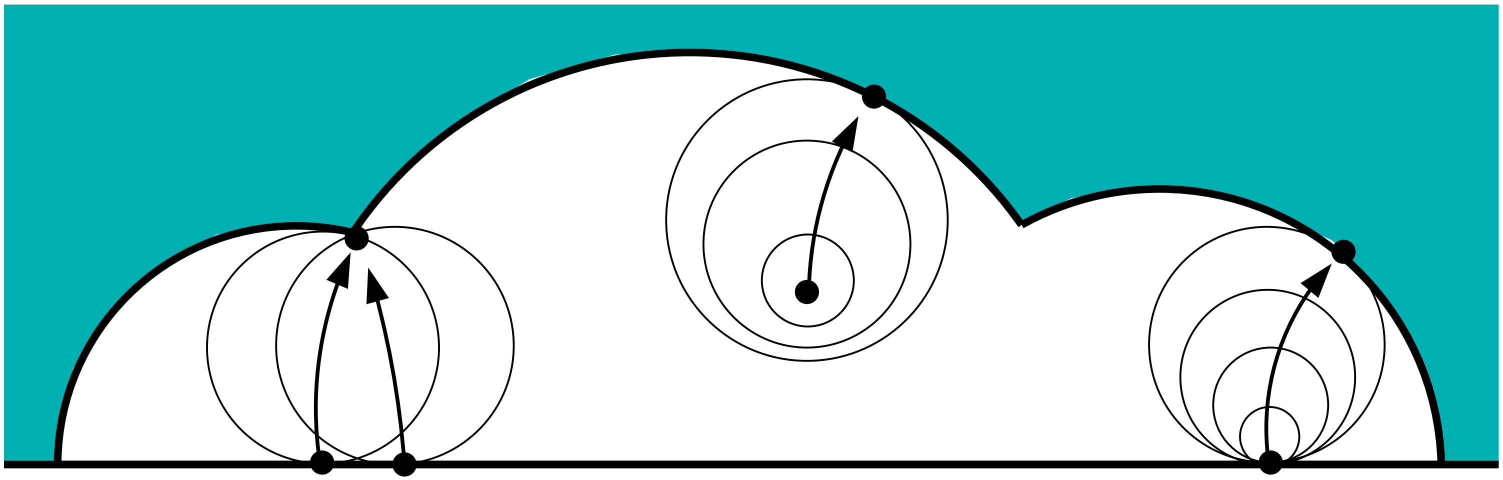




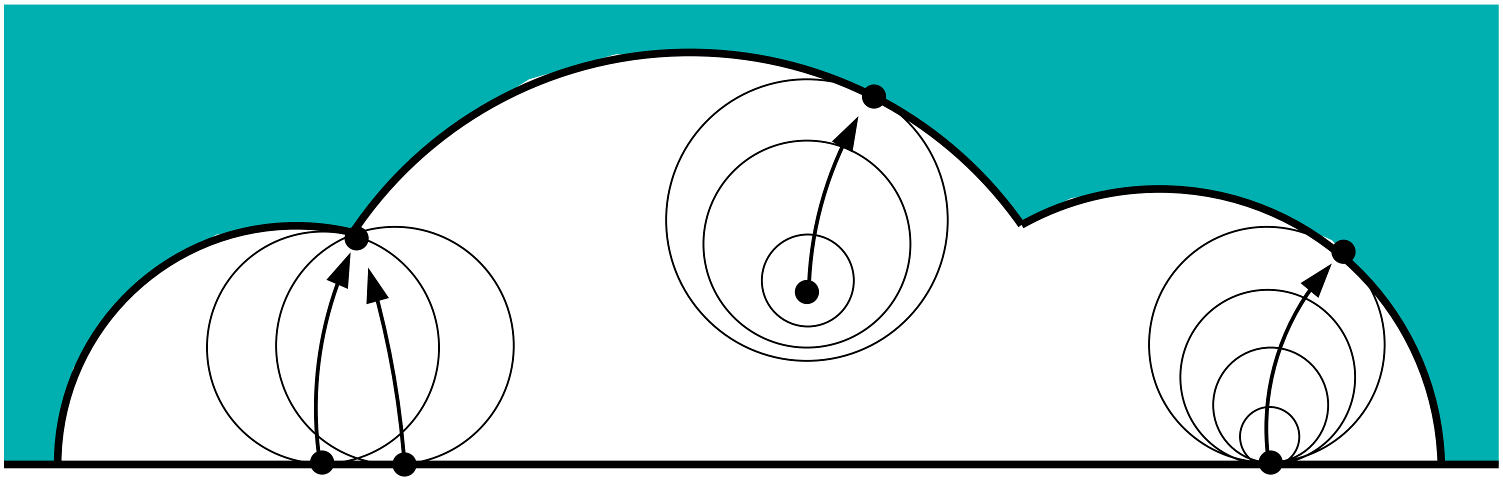








Need not be a homeomorphism, but . . .

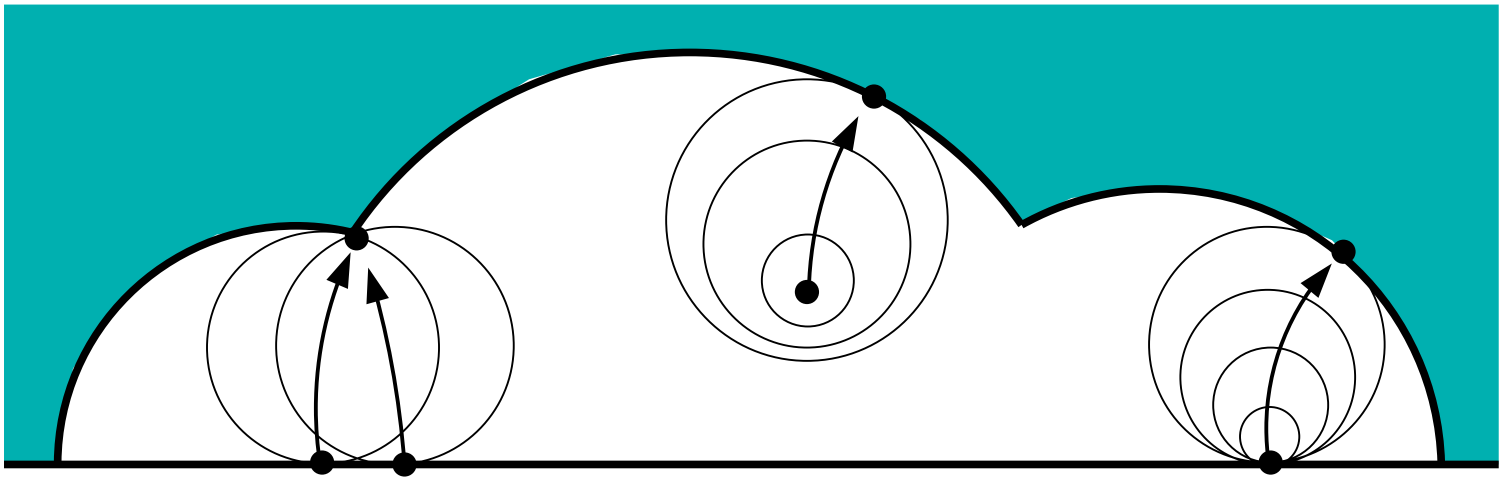


Need not be a homeomorphism, but it is a **quasi-isometry**

$$\frac{1}{A} \leq \frac{\rho(R(x), R(y))}{\rho(x, y)} \leq A, \quad \text{if } \rho(x, y) \geq B.$$

i.e., R is bi-Lipschitz on large scales.

Metrics are hyperbolic metrics on Ω and S.



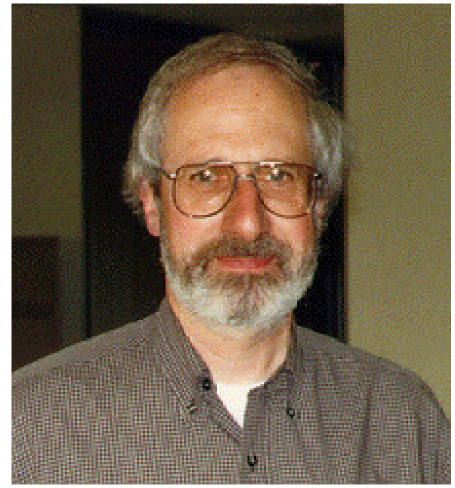
“Smoothing” gives K-QC map fixing boundary points.

Sullivan’s convex hull theorem: K is independent of domain.

Dennis Sullivan, David Epstein and Al Marden, C.B.

Dennis Sullivan

David Epstein



Al Marden

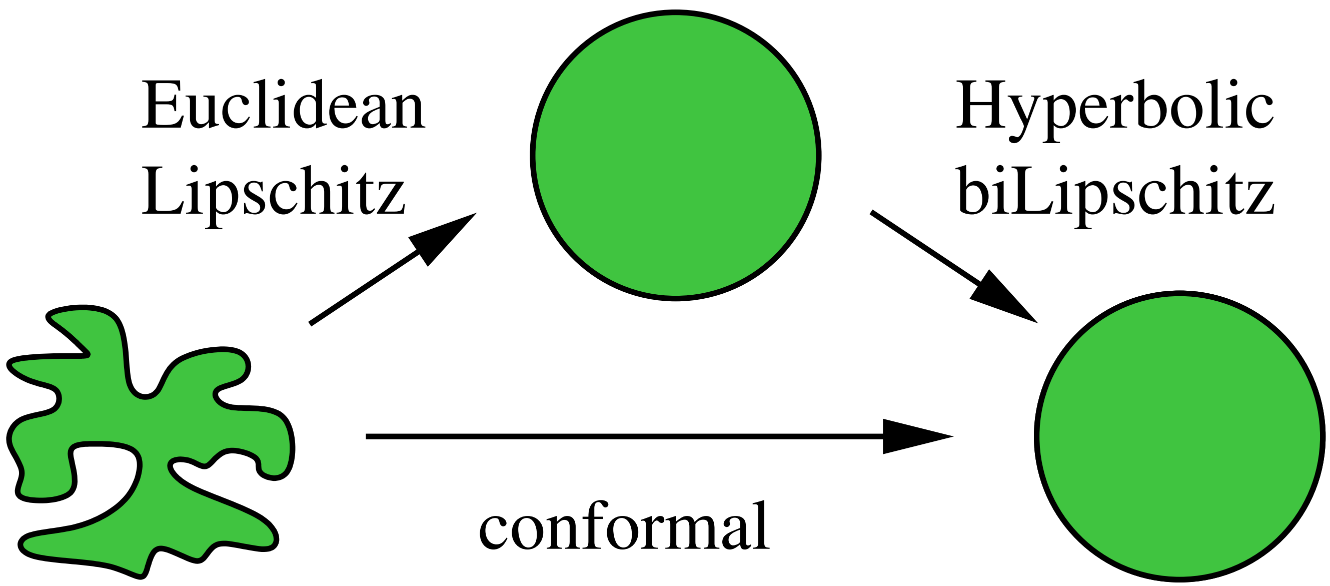
Dennis Sullivan proved this assuming invariance under a group of Möbius transformations. This was used by William Thurston to prove certain 3-manifolds have a hyperbolic metric.

Epstein and Marden extended to general simply connected Ω. $K \approx 85$.

Best value unknown, but $2.1 < K < 7.82$.

Application: factorization: Riemann map $f = h \circ g$ where

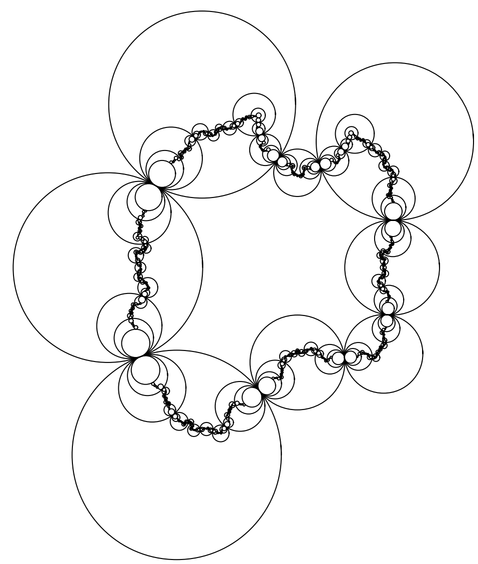
- $g : \Omega \to \mathbb{D}$ is Lipschitz in Euclidean path metrics,
- $h : \mathbb{D} \to \mathbb{D}$ is biLipschitz in hyperbolic metric



Cor: Any simply connected domain can be mapped 1-1, onto a disk D by a contraction for the internal path metric.

Application: Bowen's dichotomy: G is a discrete group of Möbius transformations acting on disk is **divergent** if quotient Riemann surface has no Green's function (Brownian motion is recurrent).

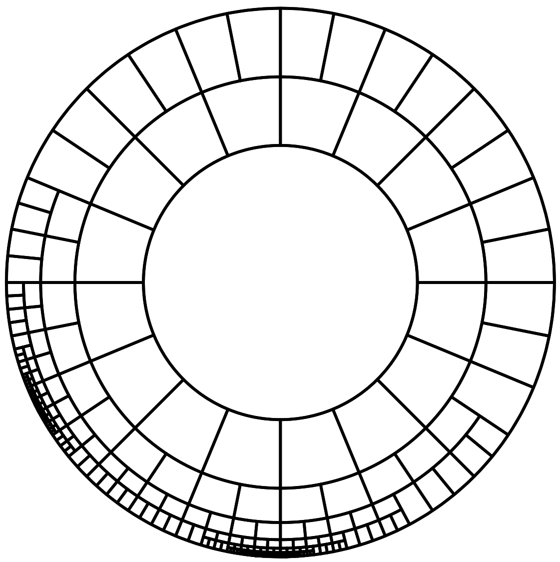
Theorem: Any quasiconformal deformation of divergence group G has a a limit set that is either a circle of a curve with Hausdorff dimension > 1.

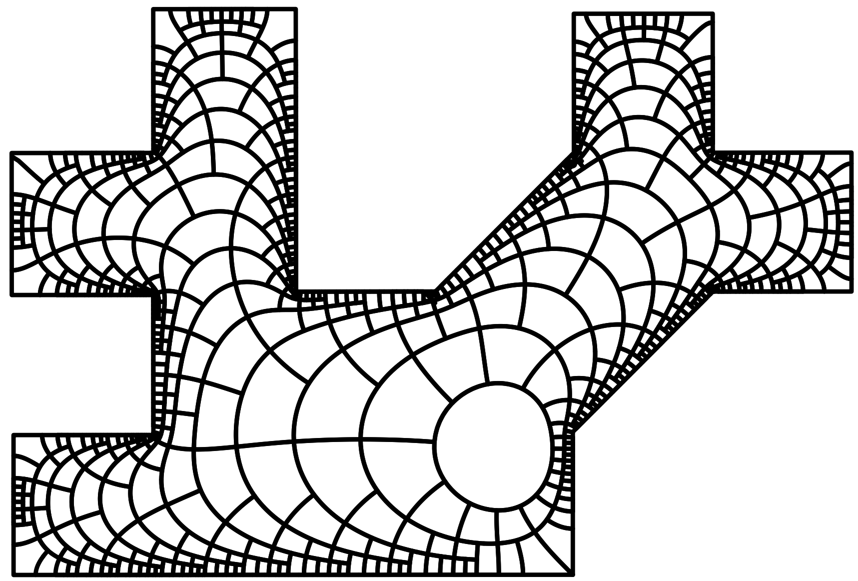


Due to Rufus Bowen for co-compact groups, Dennis Sullivan for co-finite groups. Fails for any non-divergence group (Astala-Zinsmeister).

Application: fast conformal mapping.

Theorem: Can compute ϵ-conformal map onto n-gon in time $C_\epsilon \cdot n$.





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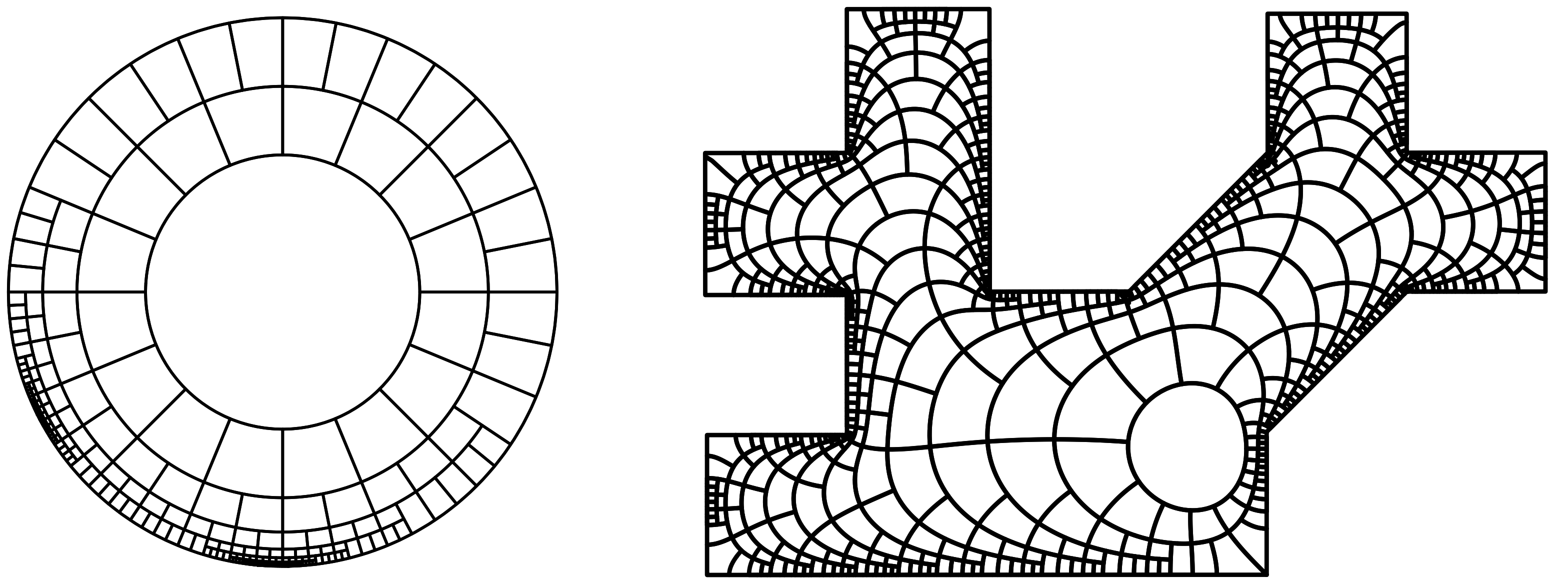
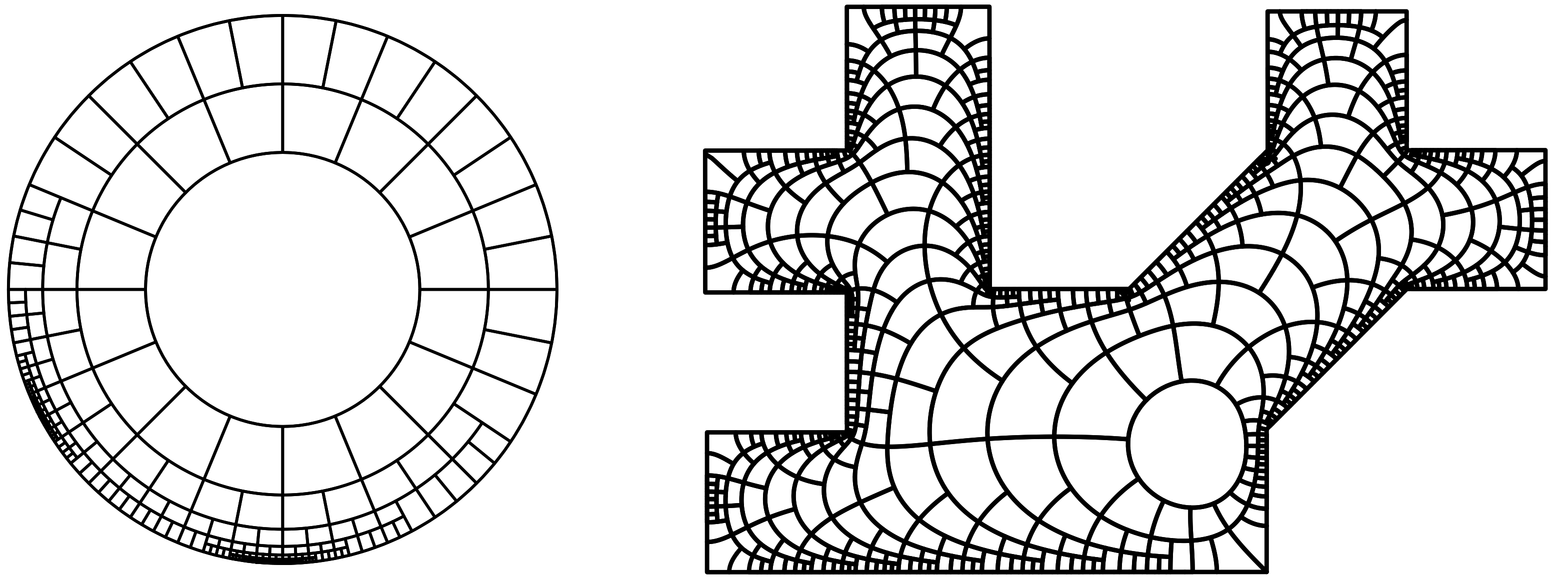


ϵ-conformal $= 1 + \epsilon$ quasiconformal. $\quad C_\epsilon = O(\log \frac{1}{\epsilon} \log\log \frac{1}{\epsilon})$.

Data held as $O(n)$ Laurent series of length $p = \log \frac{1}{\epsilon}$.

Bottleneck is doing $O(1)$ FFTs per vertex of polygon.



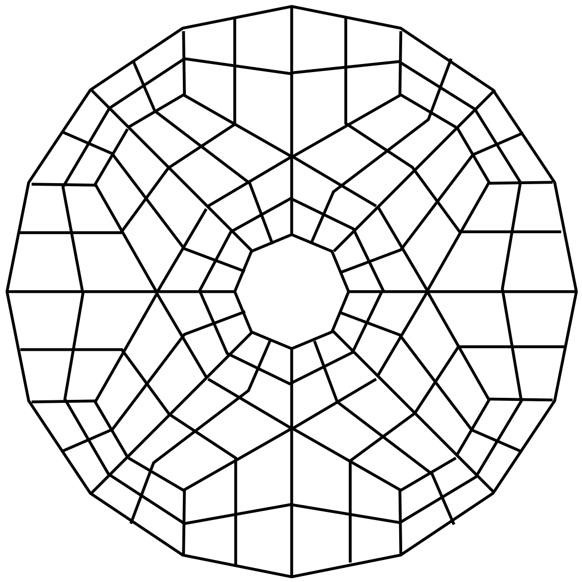
Idea is to use iota map as starting point of a Newton-like iteration.

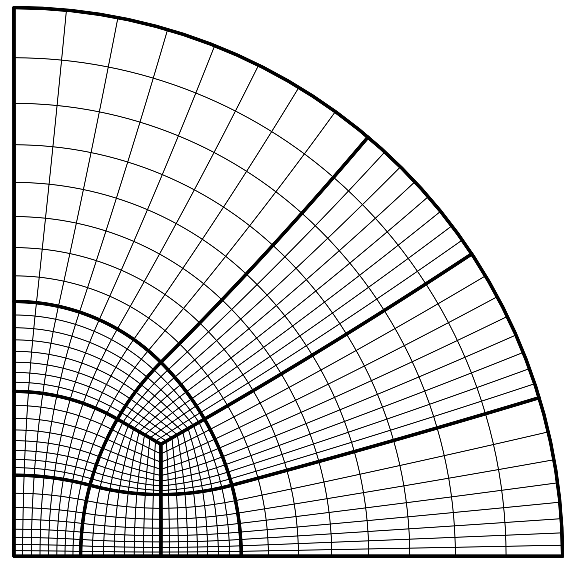
Approximates non-linear Beltrami equation linear D-bar equation.

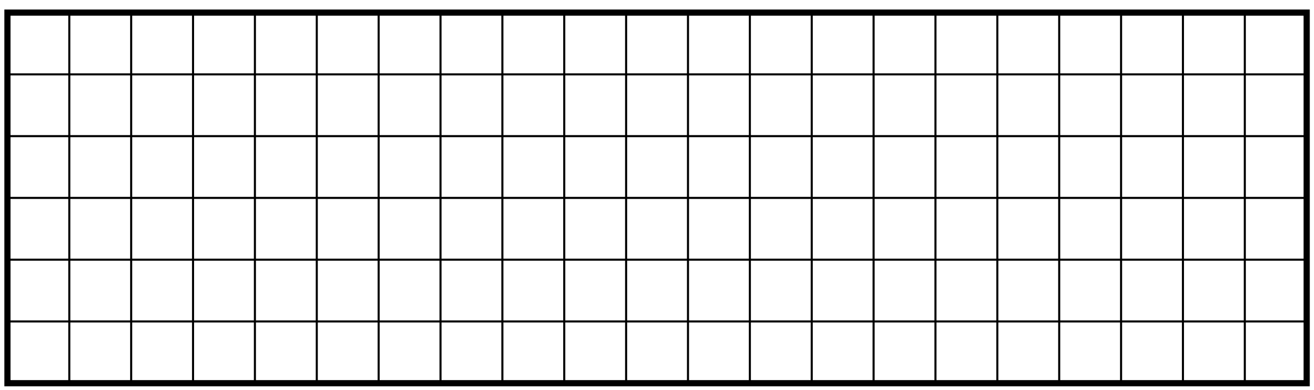
QC constant decreases quadratically.

Can estimate number of steps needed to get ϵ-accuracy.

Application: Quadrilateral meshes

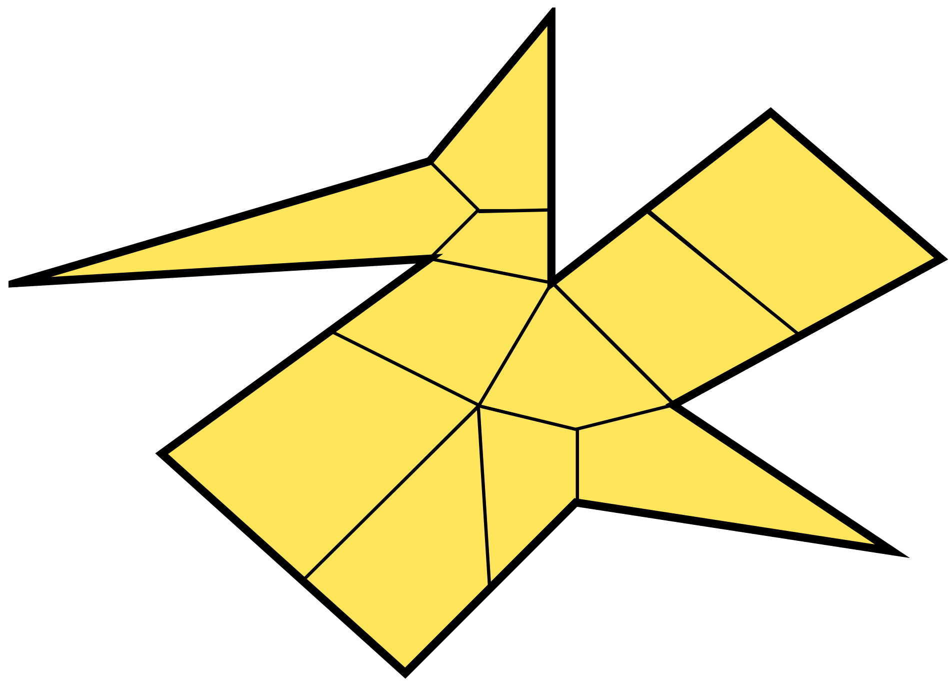


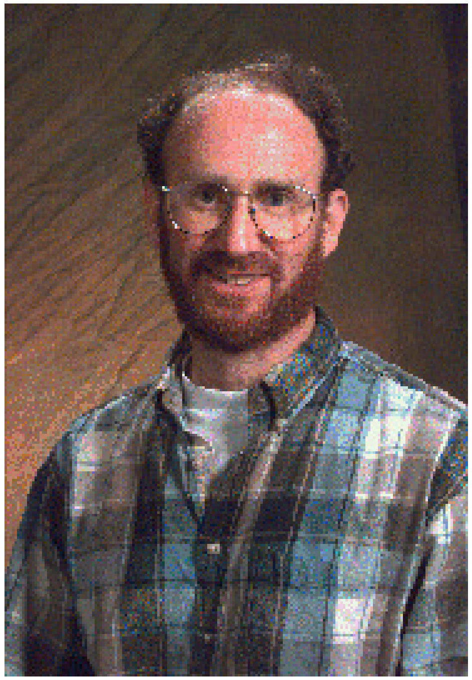




Marshall Bern and David Eppstein (2000) proved:

- n-gons have $O(n)$ quad-mesh with angles $\leq 120°$.
- $O(n \log n)$ work.
- Regular hexagon (and Euler's formula) shows $120°$ is sharp.





Marshall Bern

David Eppstein

David Epstein

David Eppstein

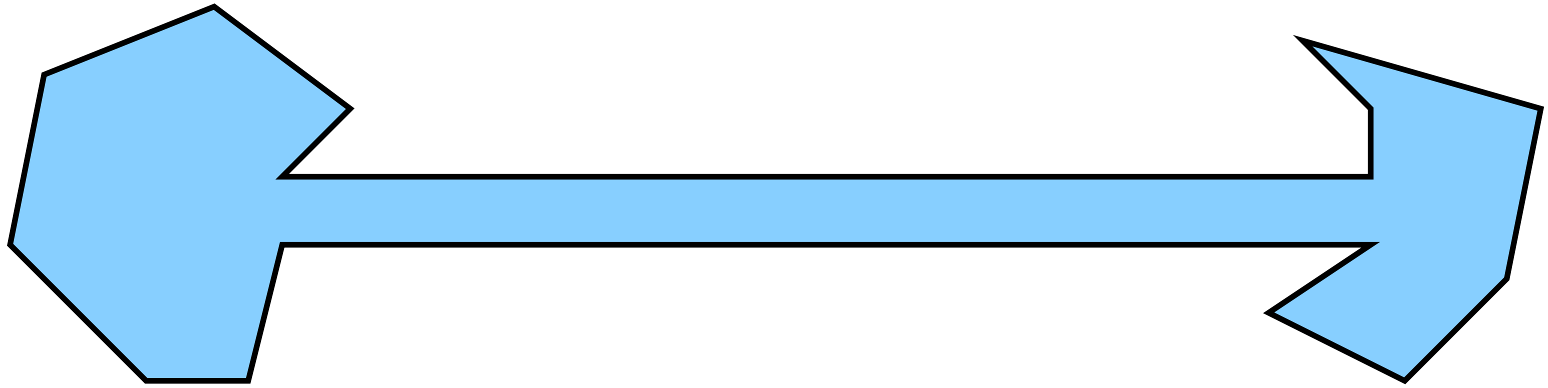
$P =$ hyperbolic geometry, University of Warwick

$P^2 =$ computational geometry, UC Irvine

Theorem (B.): Every n-gon has $O(n)$ quad-mesh with all angles $\leq 120^\circ$ and new angles $\geq 60^\circ$. $O(n)$ work.

Original angles $< 60^\circ$ remain unchanged. 60° is sharp.

Proof uses conformal mapping, plus an idea from hyperbolic manifolds: **thick/thin decompositions**.

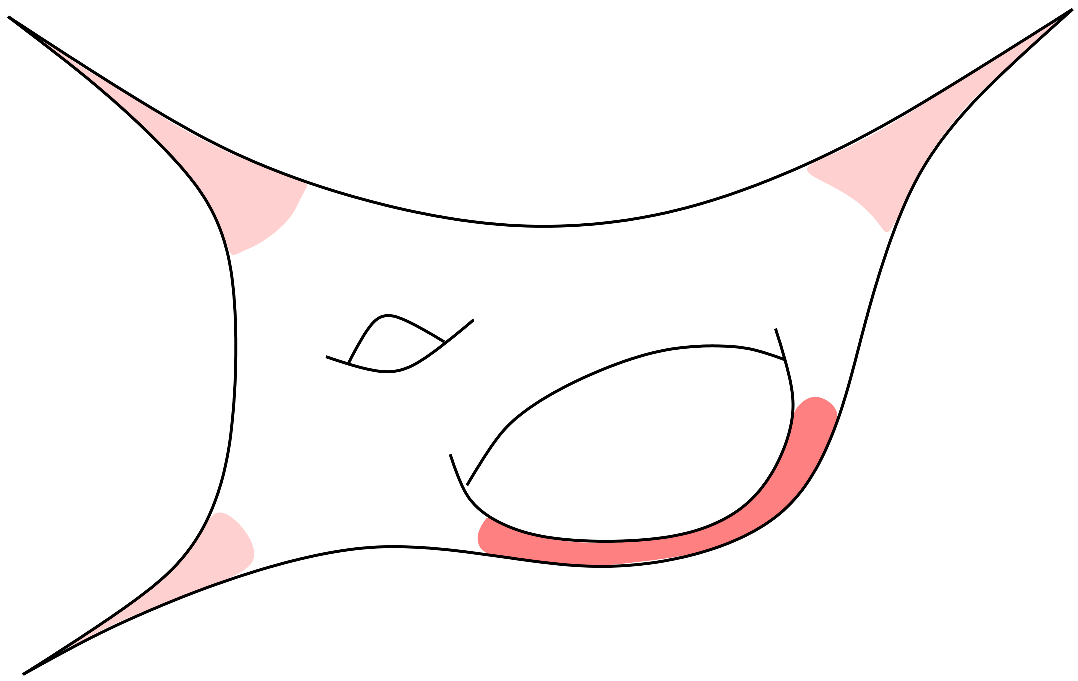


Long, narrow channels require long, narrow quadrilaterals.

Must find all such channels in $O(n)$ time.

Use idea from hyperbolic manifolds: **thick/thin decompositions**.

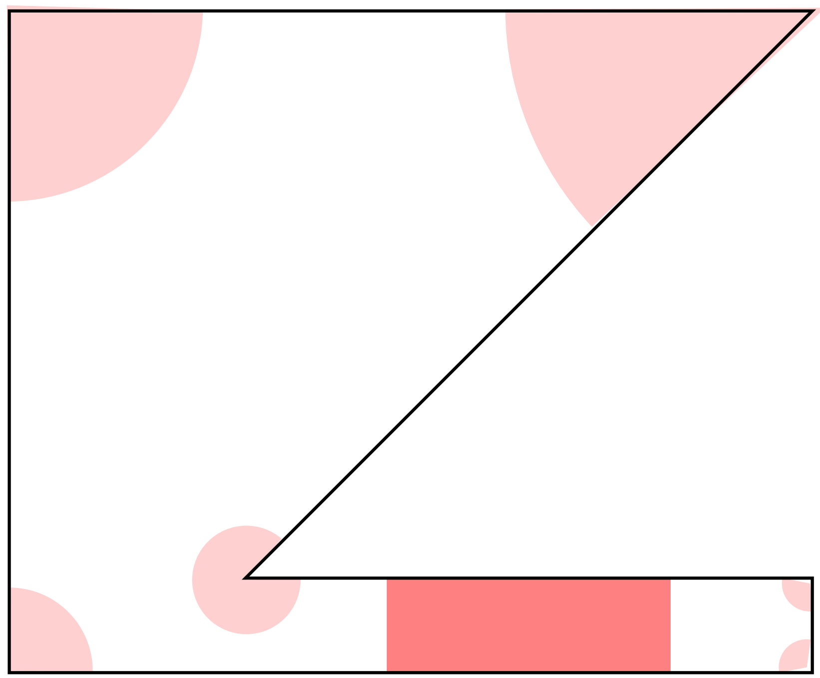
Surface **thin part** is union of short non-trivial loops.



parabolic = puncture, hyperbolic = handle

Thick and Thin parts of a polygon

Thin parts: associated to certain pairs of edges.

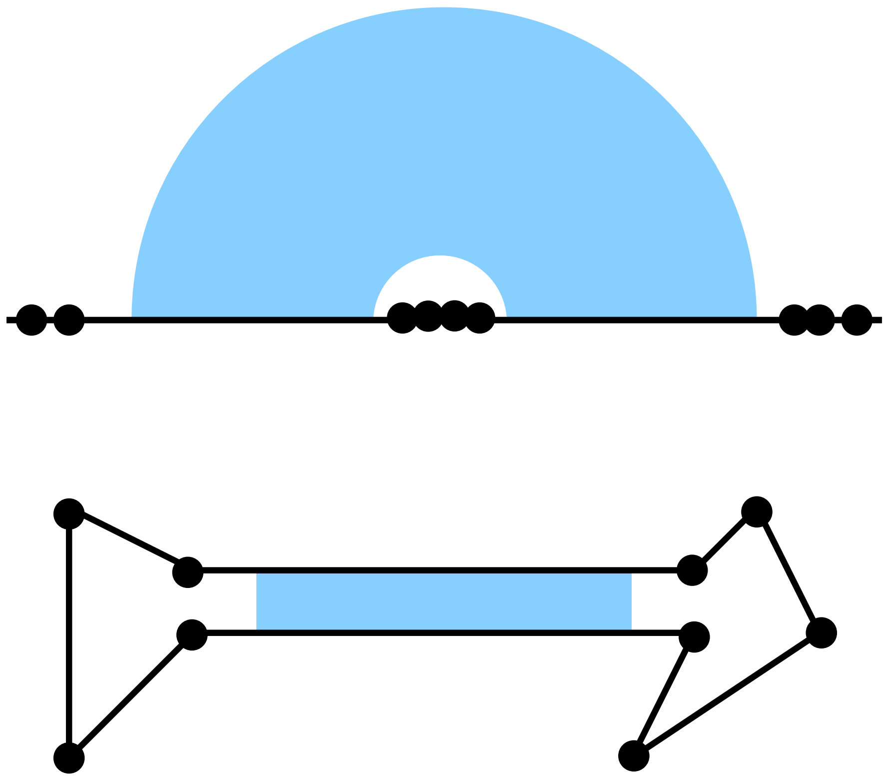


Parabolic = adjacent edges, Hyperbolic = non-adjacent edges

Rough idea: sides I, J so $\mathrm{dist}(I, J) \ll \min(|I|, |J|)$.

Thick parts = remaining components (white)

Thick/thin parts can be computed in linear time by computing conformal preimages (or using iota map).

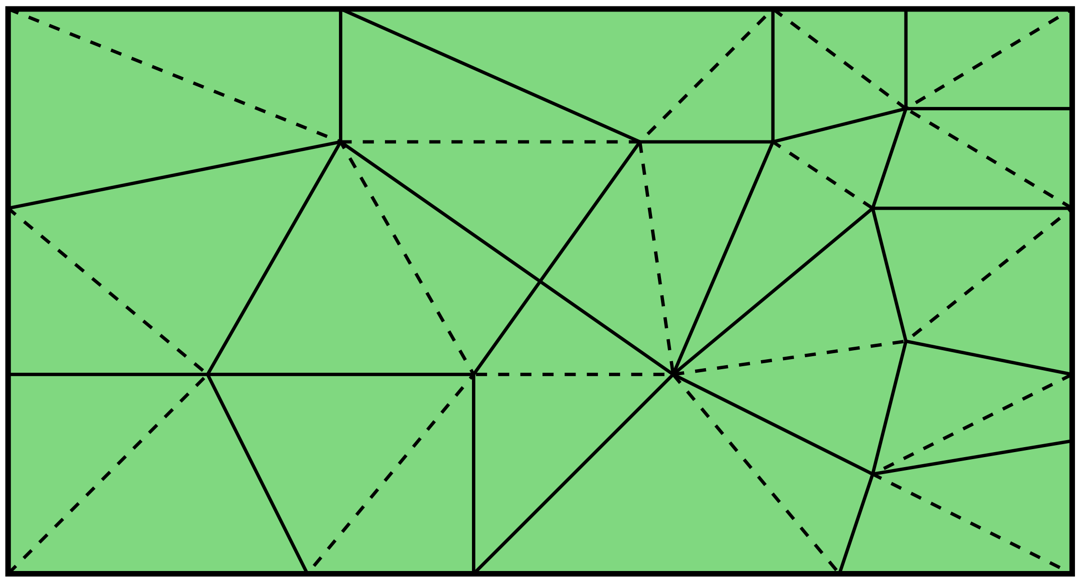


Each thin part creates two widely-separated clusters of preimages on circle. Can found quickly using medial axis.

PART III: OPTIMAL TRIANGULATION

Cor of Part I: Every polygon has a $O(n)$ 120°-triangulation.

Proof: add diagonals to quad-mesh.



Better results known: every polygon has a 90°-triangulation.

= **NOT** = **N**on**O**btuse **T**triangulation

- Acute triangulation always possible (no bound): Burago, Zalgaller 1960.
- Rediscovered: Baker, Grosse, Rafferty, 1988.
- $O(n)$ for points sets: Bern, Eppstein, Gilbert 1990
- $O(n^2)$ for polygons: Bern, Eppstein, 1991
- $O(n)$ for polygons: Bern, S. Mitchell, Ruppert, 1994
- nonobtuse $\Rightarrow$ acute refinement, comparable complexity Maehara 2002.

See also Yuan 2005, Saraf 2009.

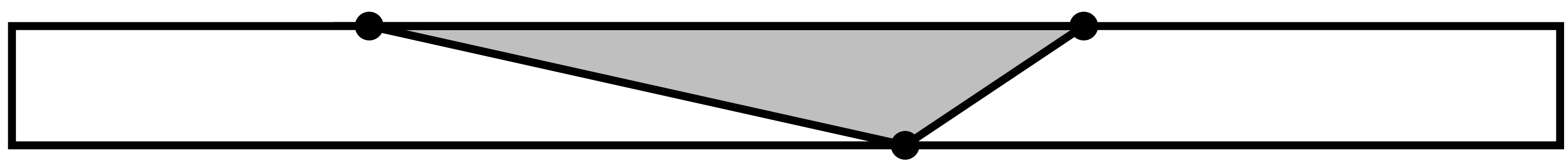
Can we do better? Lower angle bound? Improve 90° upper bound?

Answer depends on complexity.

Complexity bound $\Rightarrow$ no lower angle bound.

For $1 \times R$ rectangle.

$$\text{number of triangles} \gtrsim R \times (\text{smallest angle})$$



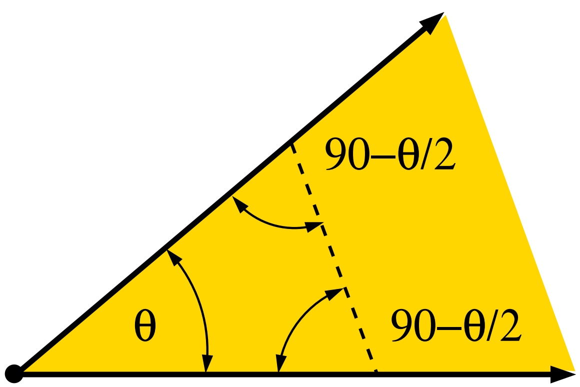
If we bound number of triangles in terms of number of vertices, no uniform lower angle bound is possible.

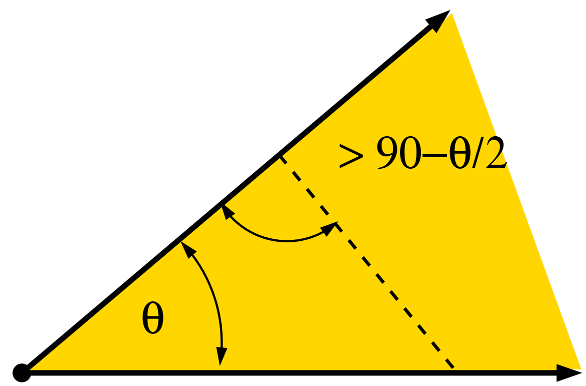
But what if we don't care about complexity bound?

Optimal angle bound depends on P:

If P has angle θ at v, any triangle $v \in T \subset P$ has angle $\leq \theta$.

Opposite angles sum to $\geq 180° - \theta$, so one is $\geq 90° - \theta/2$.





Remarkably, for small θ this is the **only restriction**.

For general polygons, we have (among other results):

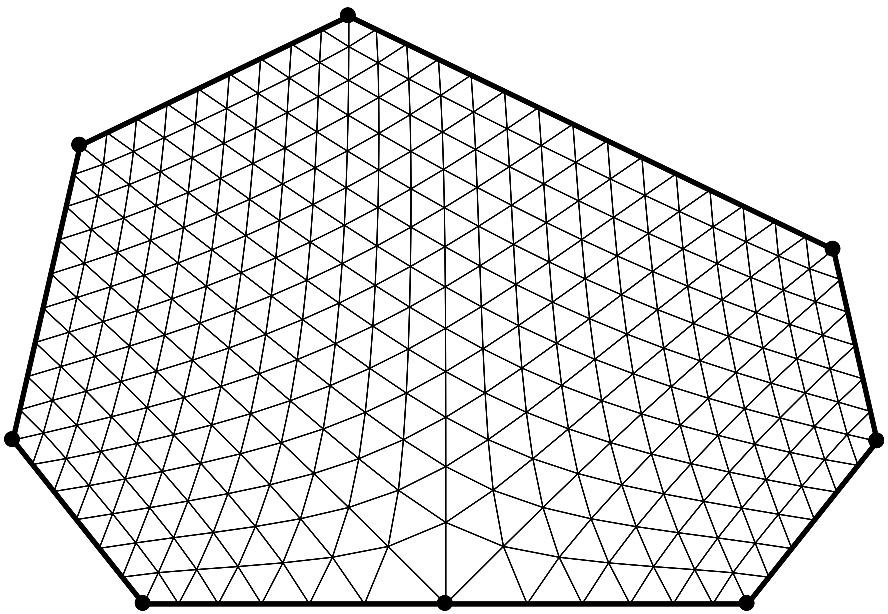
Theorem: The optimal angle bound for an n-gon can be computed in time $O(n)$.

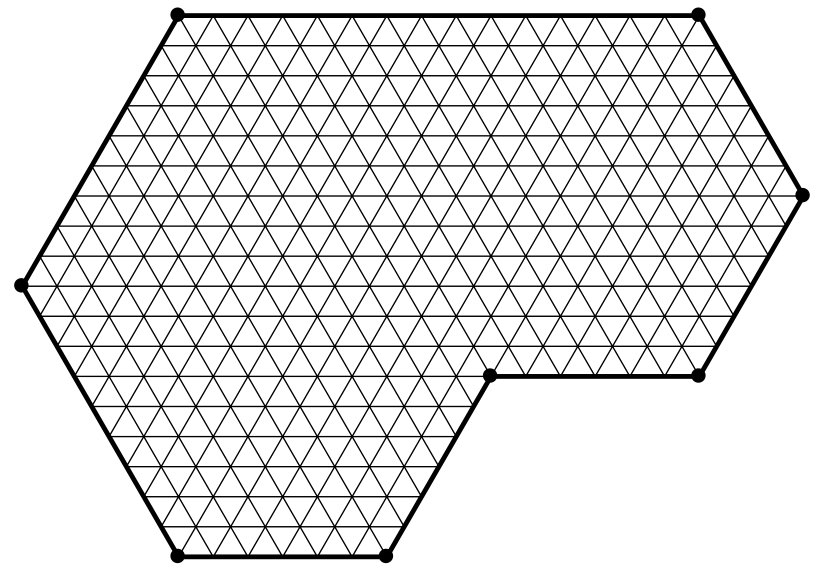
Theorem: The optimal bound is attained by a finite triangulation, except for $60°$-polygons with some irrational length ratio.

Theorem: The optimal bound only depends on the set of angles, not their order or the edge lengths.

Theorem: If P has minimum interior angle θ then it has a triangulation with all angles $\leq \max(72°, 90° - \theta/2)$.

Idea behind main theorem: conformal maps





Given P build a "model" $60°$-polygon P'.

Map equilateral triangulation of P' to P (vertices $\to$ vertices).

Can prove worst angle distortion is at vertices $= \theta_k/\psi_k$.

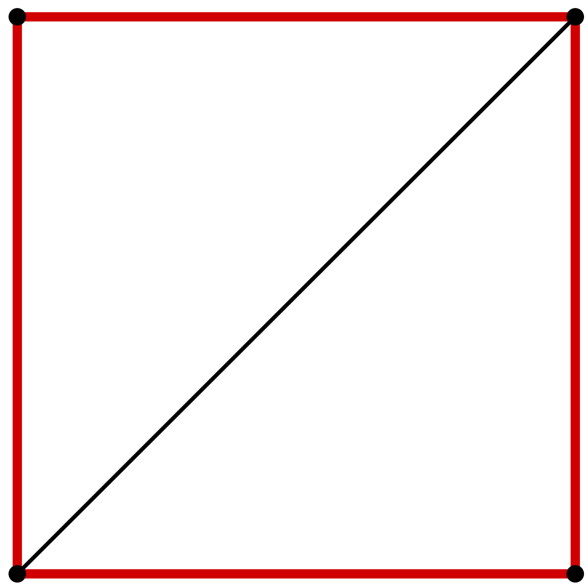
Must show angle bound attained, not just approximated.

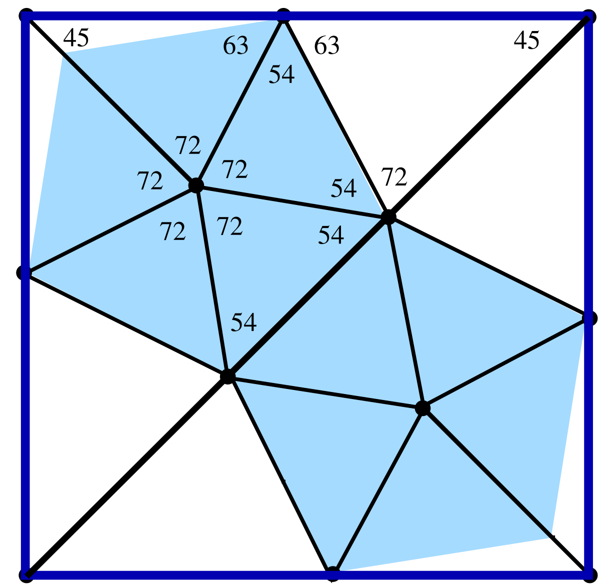
But, ...

This approach gives a triangulation with only degree 6 interior vertices.

Some cases require interior vertices of degree ≤ 5 (e.g., the square).

In other cases there must be interior vertices of degree ≥ 7.





Without Steiner points, 90° is best angle bound (Delaunay triangulation).

Explicit example shows a 72°-triangulation is possible.

A degree five vertex is needed to attain this.

Converse:

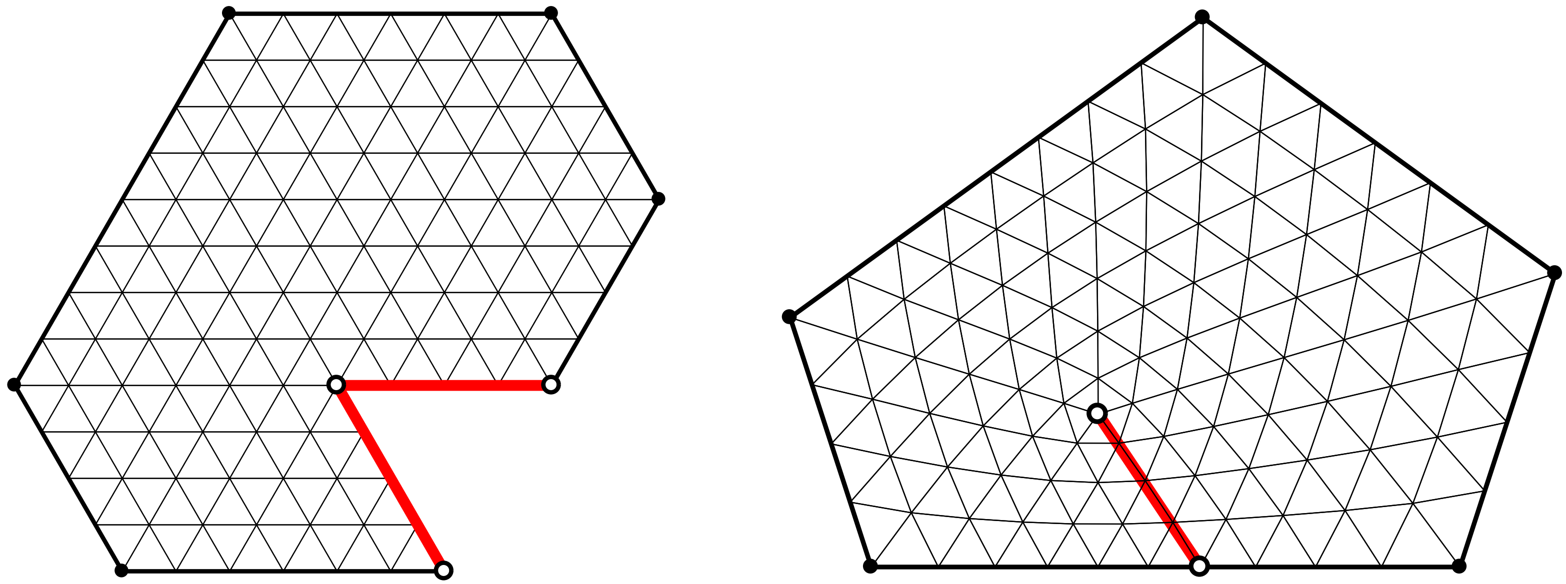
• Suppose P has a triangulation with all angles $< 72^\circ$.

• Then every interior vertex has degree ≥ 6, every edge vertex has degree ≥ 3 and every corner has degree ≥ 2.

• Reflecting across square gives topological triangulation of sphere where every vertex has degree ≥ 6, except for four vertices that have degree ≥ 4.

• This violates Euler's formula $F - E + V = 2$ for 2-sphere.

Creating degree 5 vertices by folding:

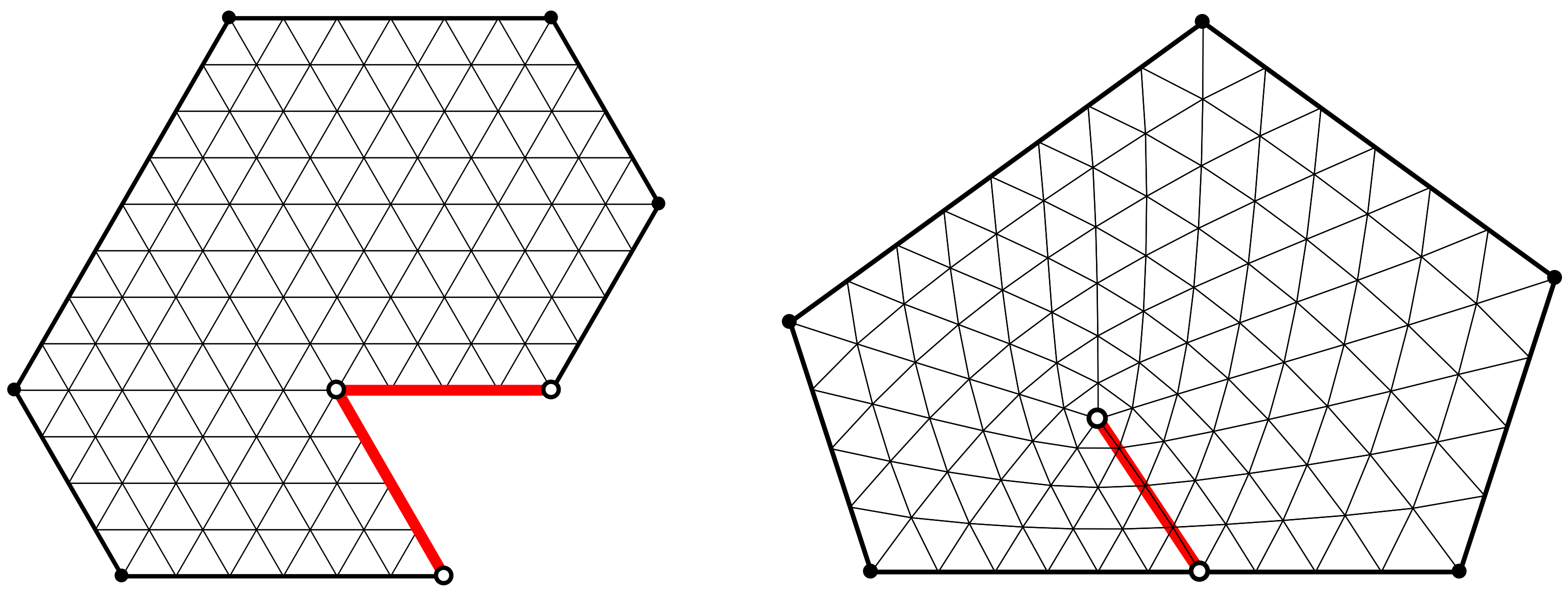


f maps P' to P with a slit removed; identifies boundary segments.

A degree 5 interior vertex is created.

But is this really a triangulation of P?

Creating degree 5 vertices by folding:

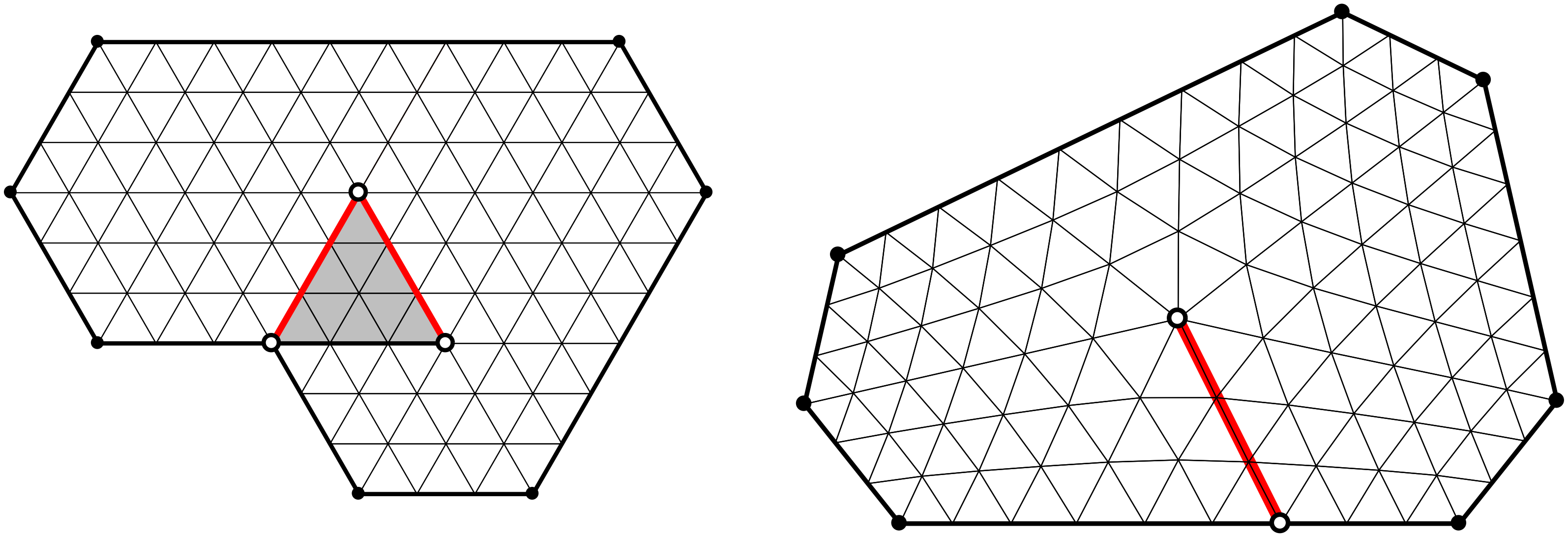


Technical difficulty: Image triangulations must match up across slit.

Matching occurs if $|f'(w)| = |f'(z)|$ whenever $f(w) = f(z)$.

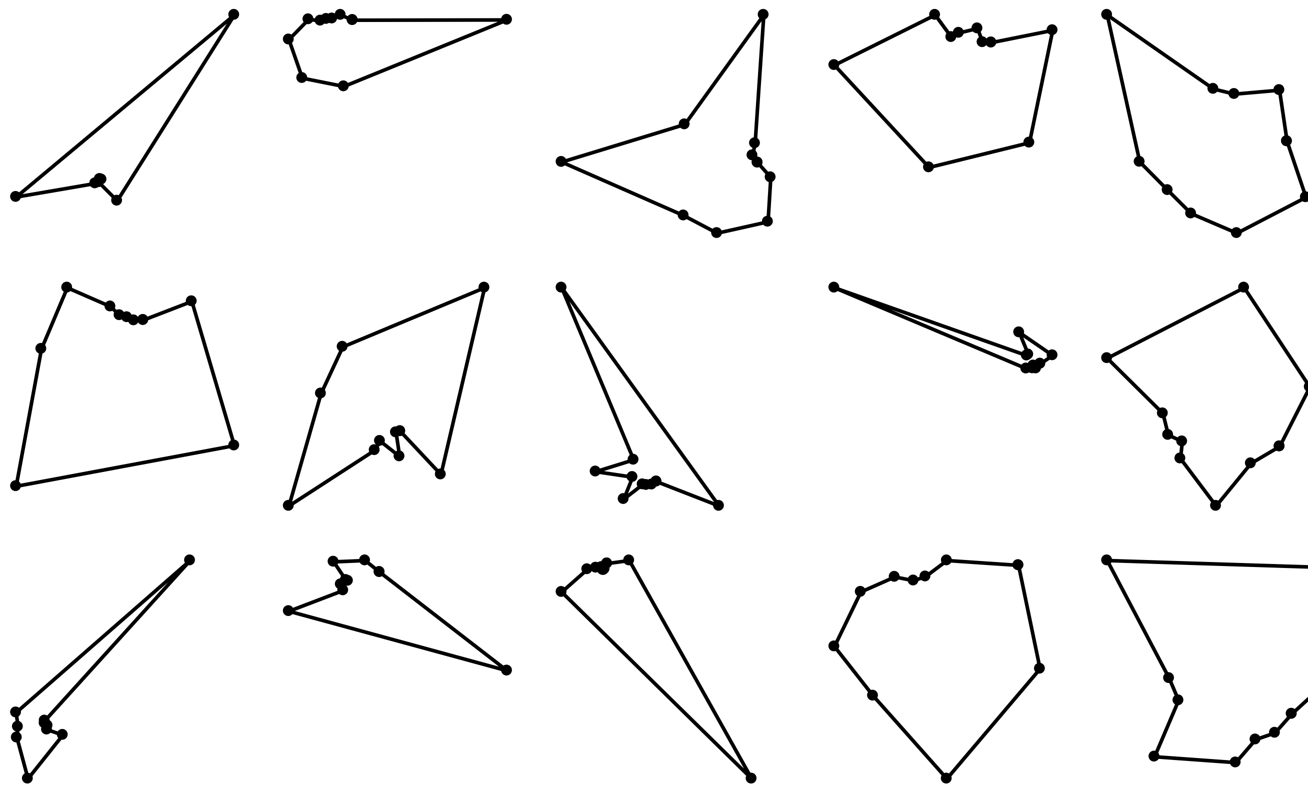
Differential equation can be solved explicitly (= conformal welding).

Solution gives a curved slit (tangent changes by 3° above).



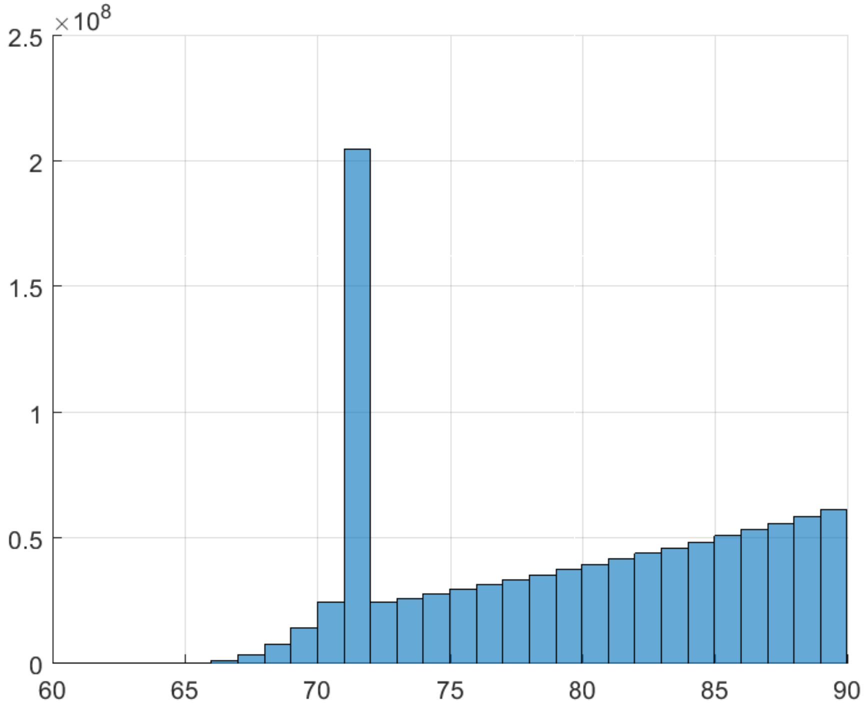
Creating a degree 7 vertex requires P' to be Riemann surface.

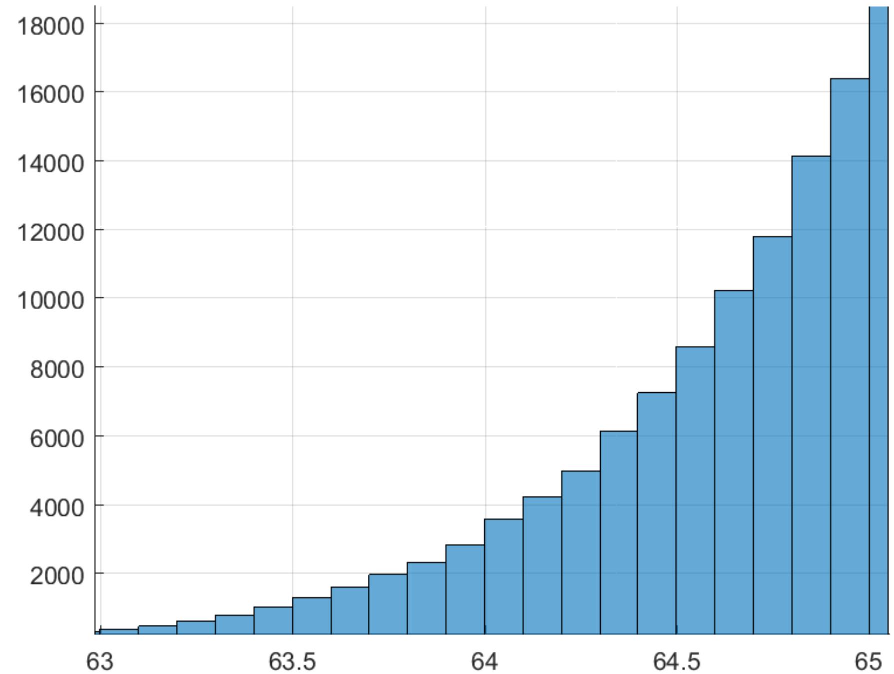
$\Rightarrow$ allows us to handle all cases



Generating random 10-gons.

Computed optimal triangulation bound for 10^9 examples.

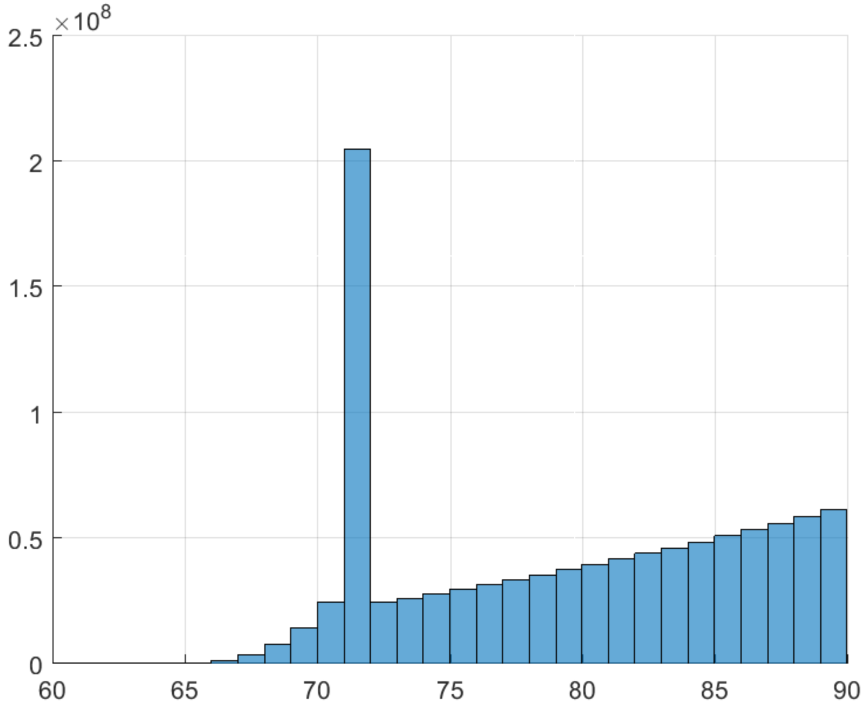


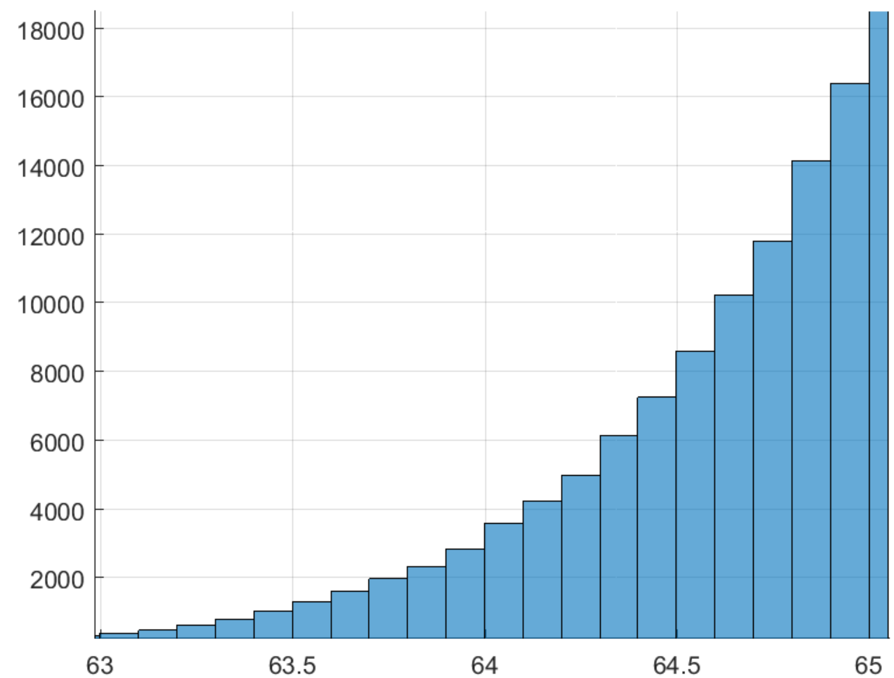


On the left is a histogram based on 1° bins. The spike at 72° is evident.

Simply n-gons form subset of $\mathbb{R}^{2n}$

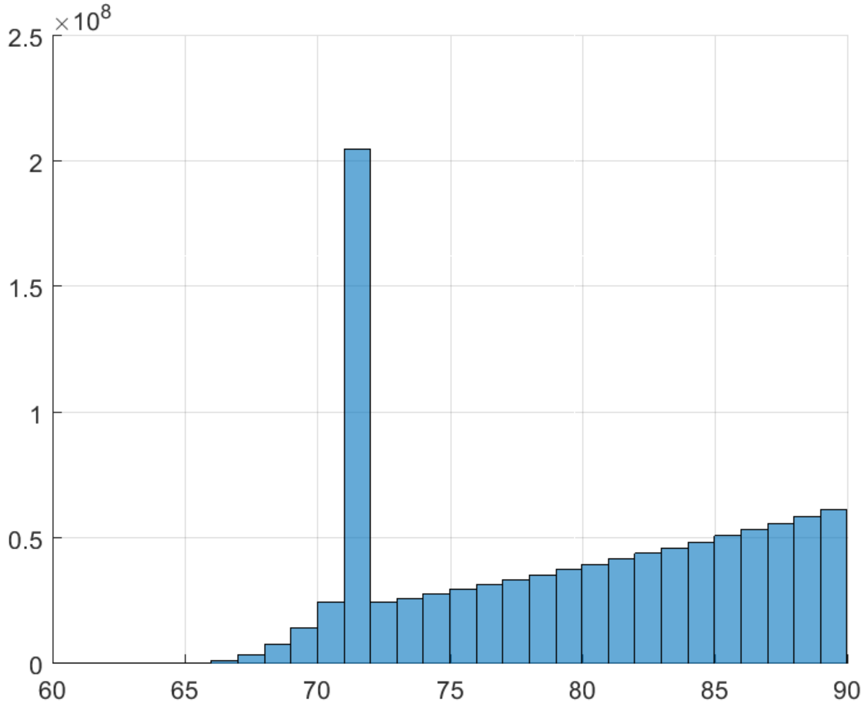
Set of polygons attaining bound 72° or $\frac{5}{7} \cdot 90^\circ \approx 64.26^\circ$ has interior.

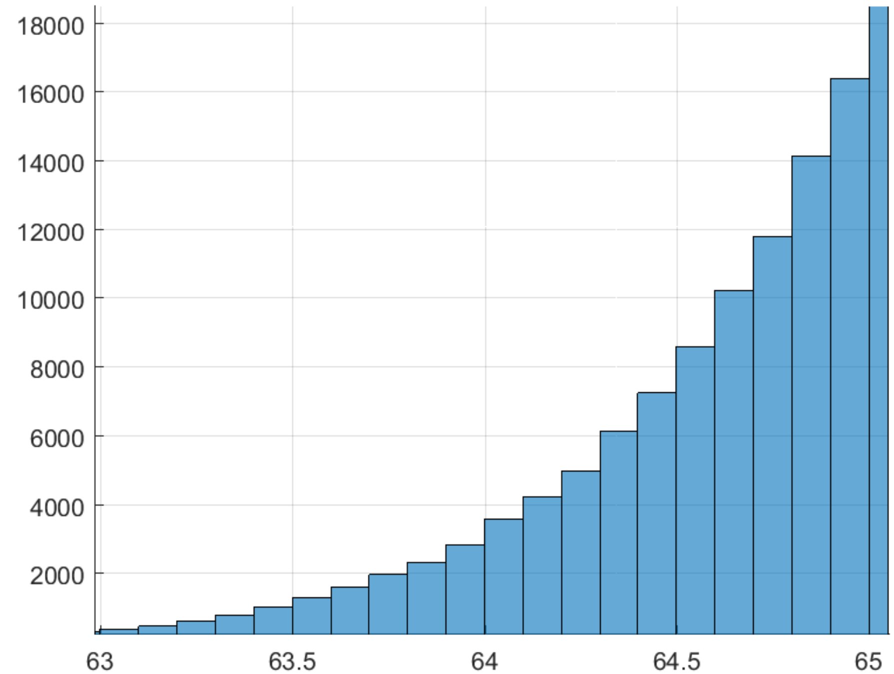




Others angles give measure zero sets.

No spike near 64° is visible. Is $N = 10$ too small?





In these experiments I just chose angles at random (with correct sum).

Didn't choose edge lengths or check for self-intersections.

What is better model for random polygons?


Happy BirthDay
Steffen
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