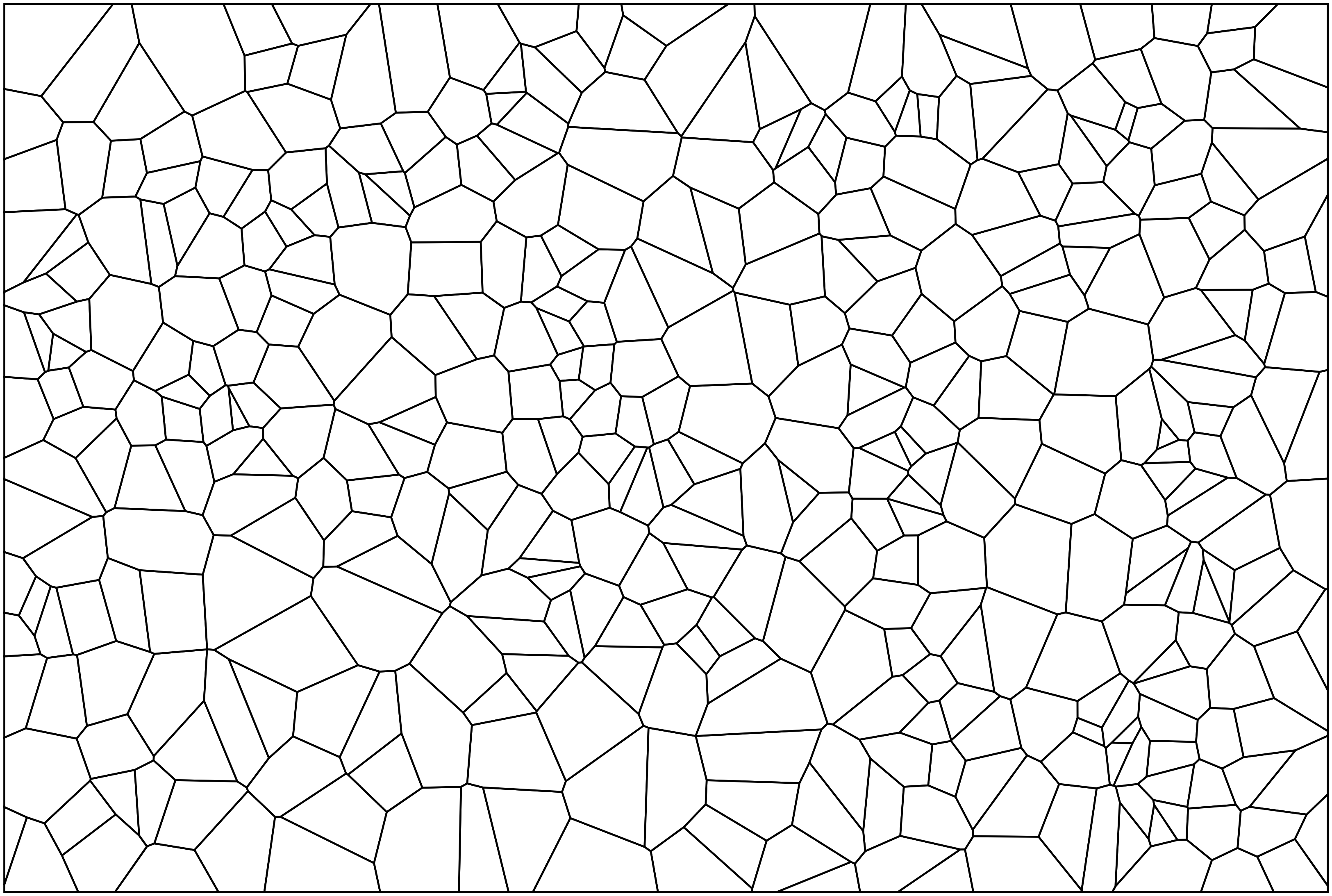
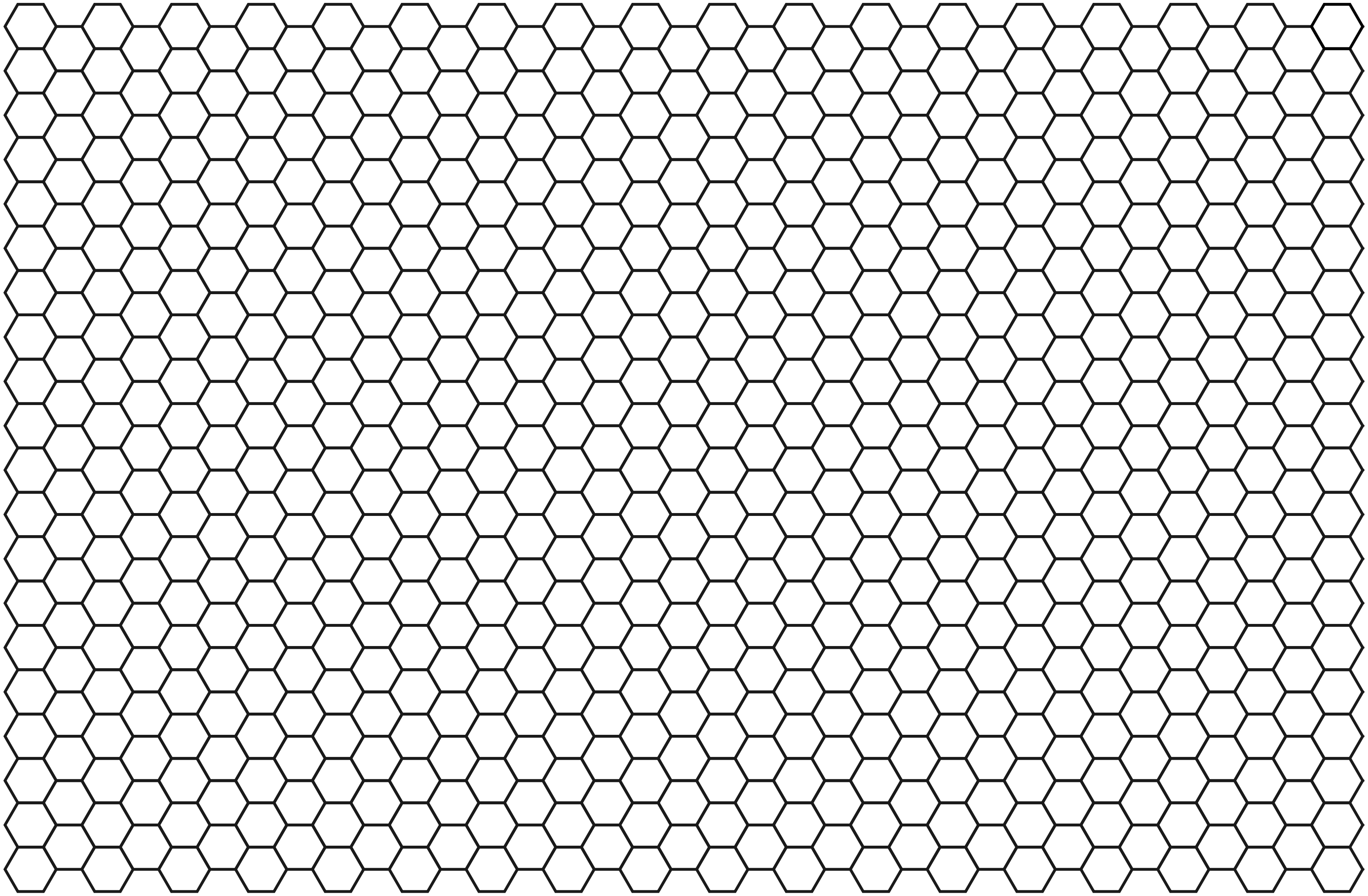
Planar maps with at most six neighbors on average

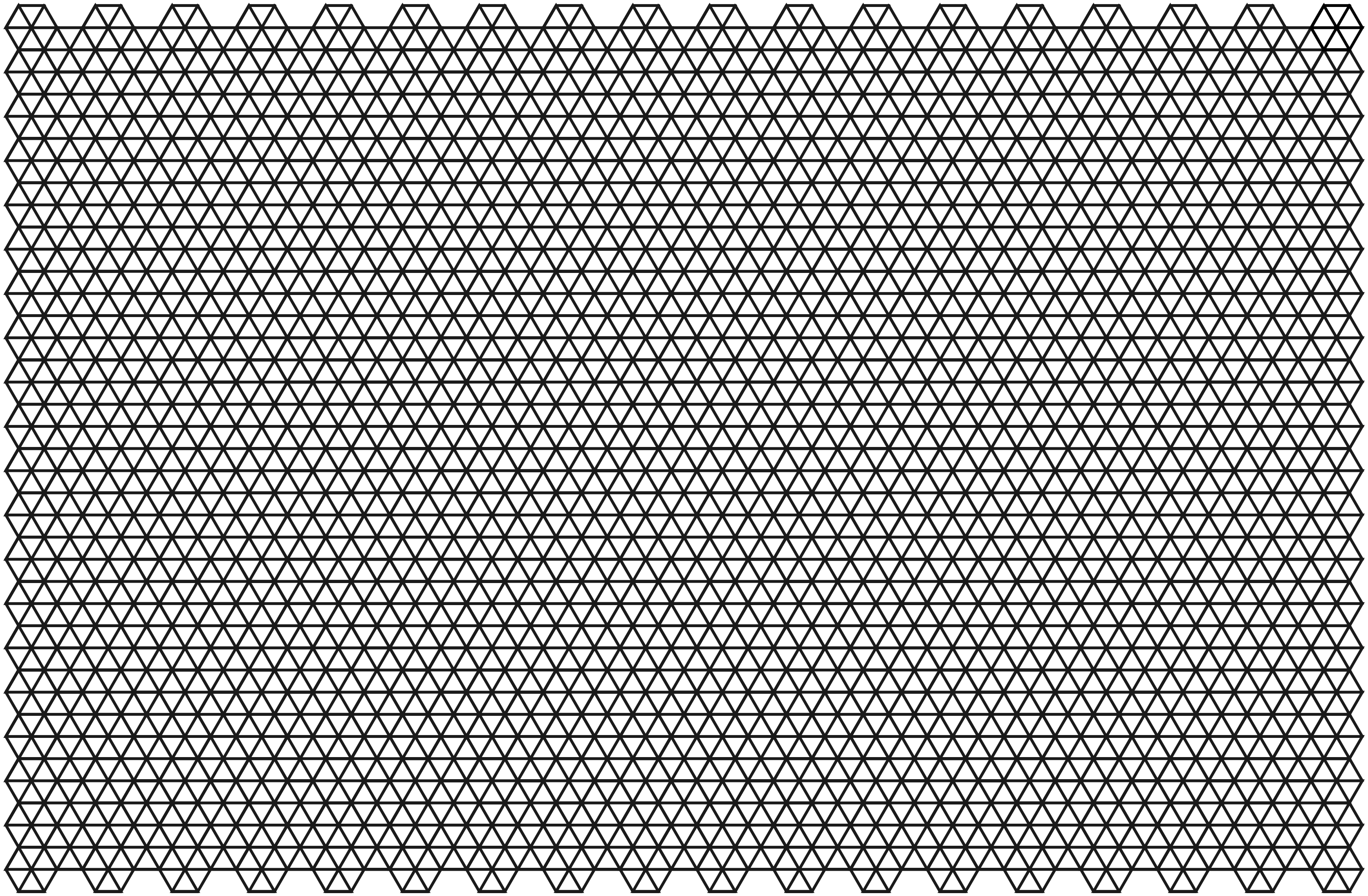
Christopher Bishop, Stony Brook
Dennis Sullivan, Stony Brook
Michael Wigler, Cold Spring Harbor

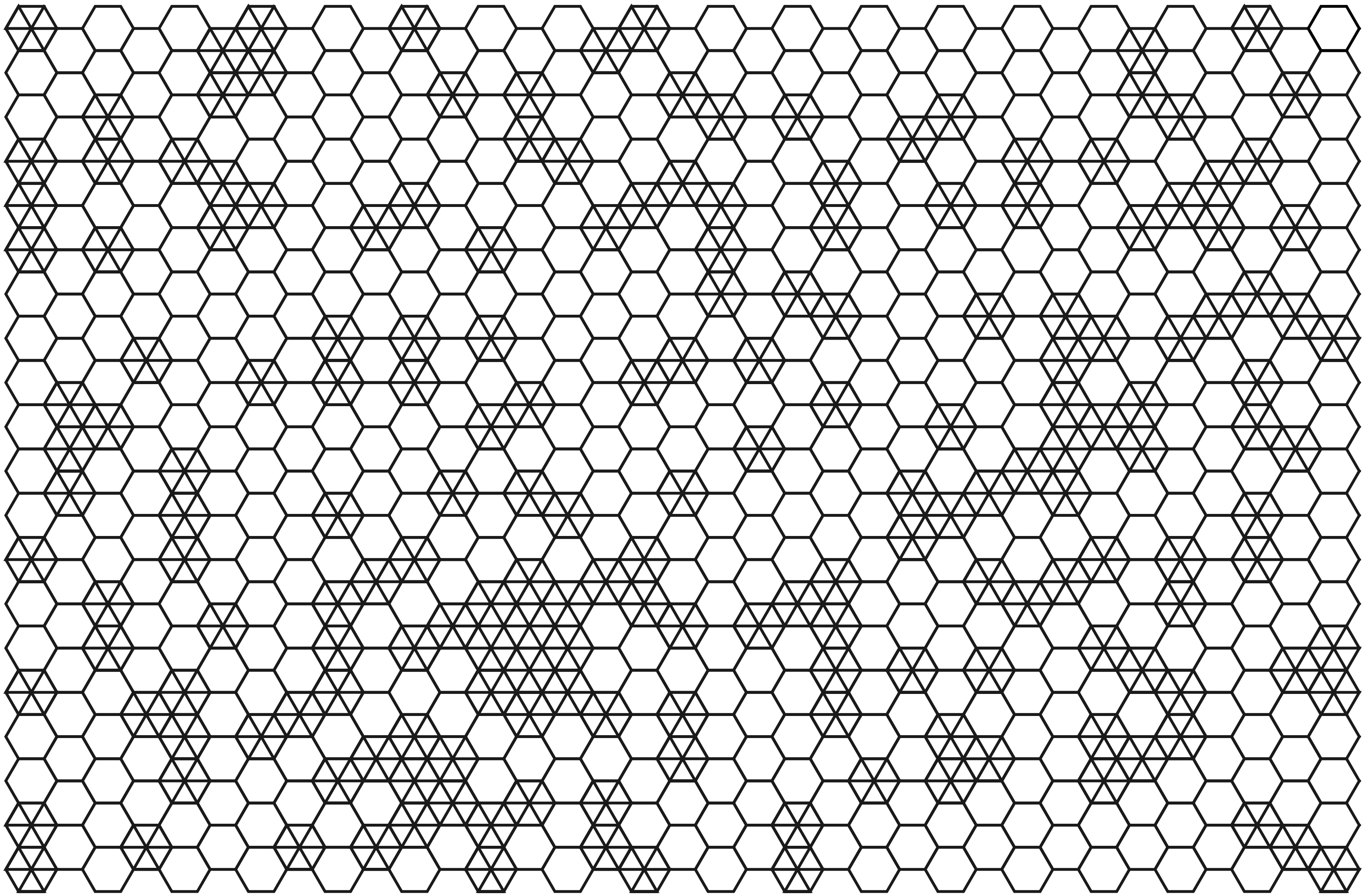
AMS Sectional Meeting – March 19, 2016 Stony Brook NY

www.math.sunysb.edu/~bishop/lectures

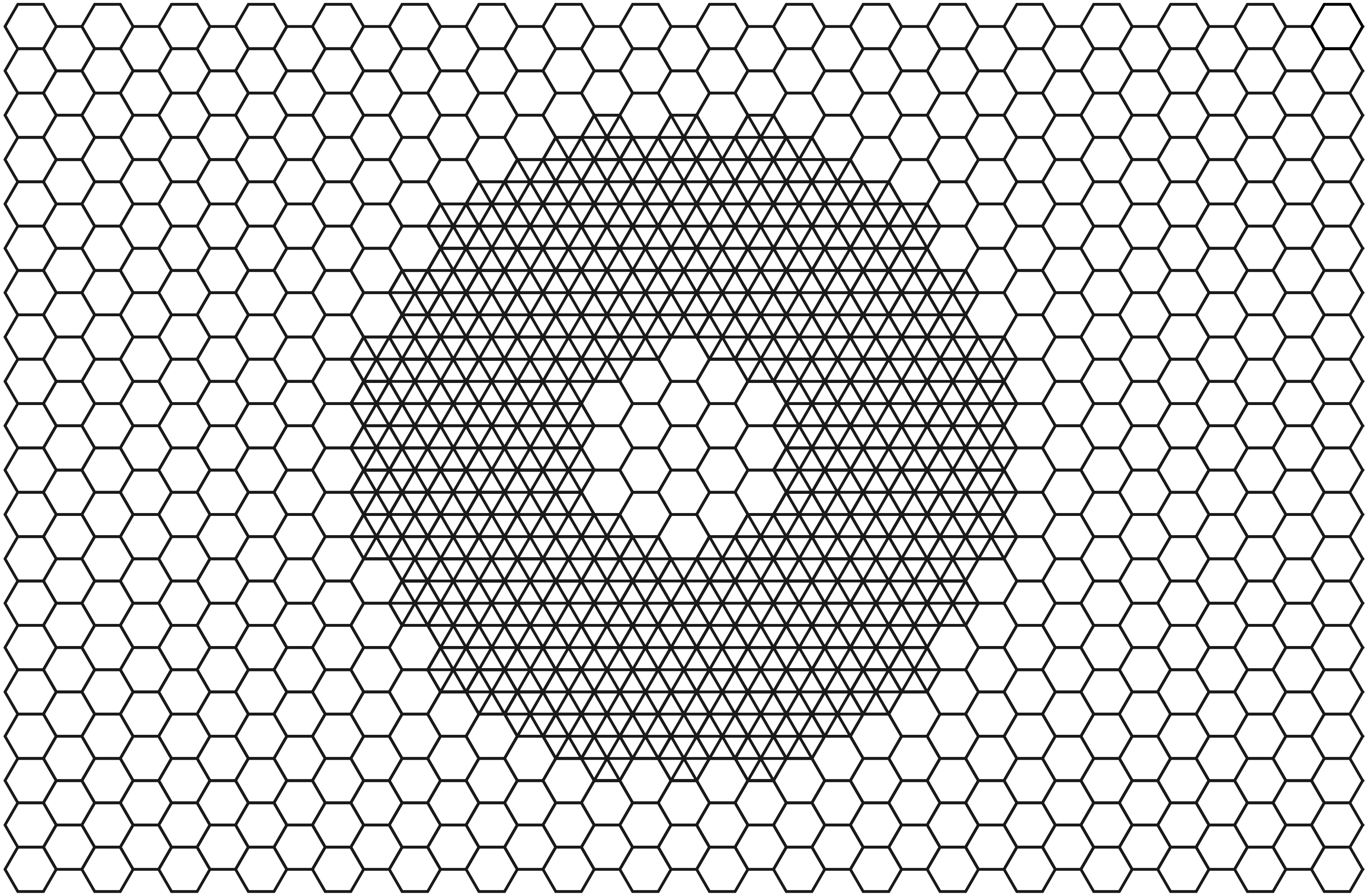




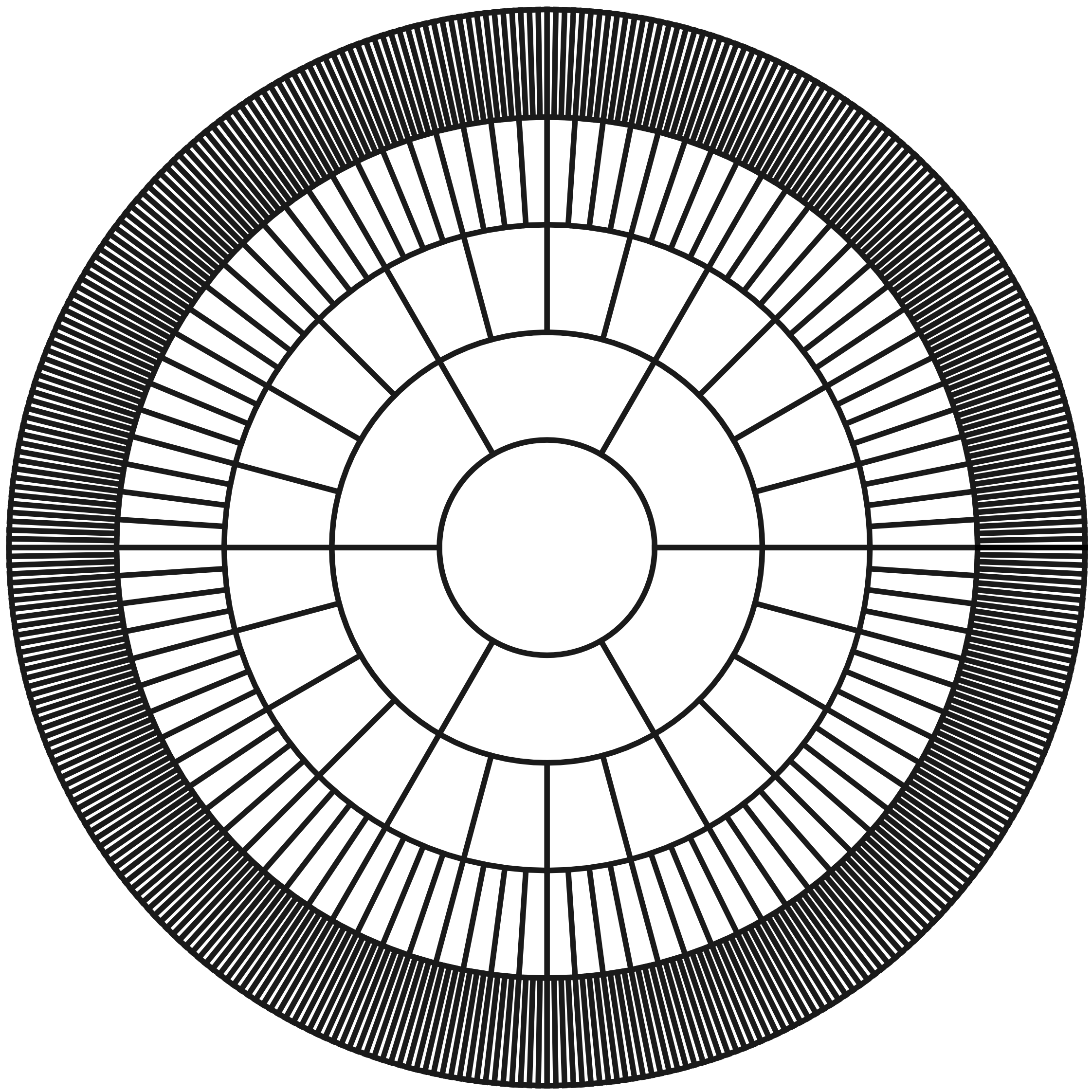




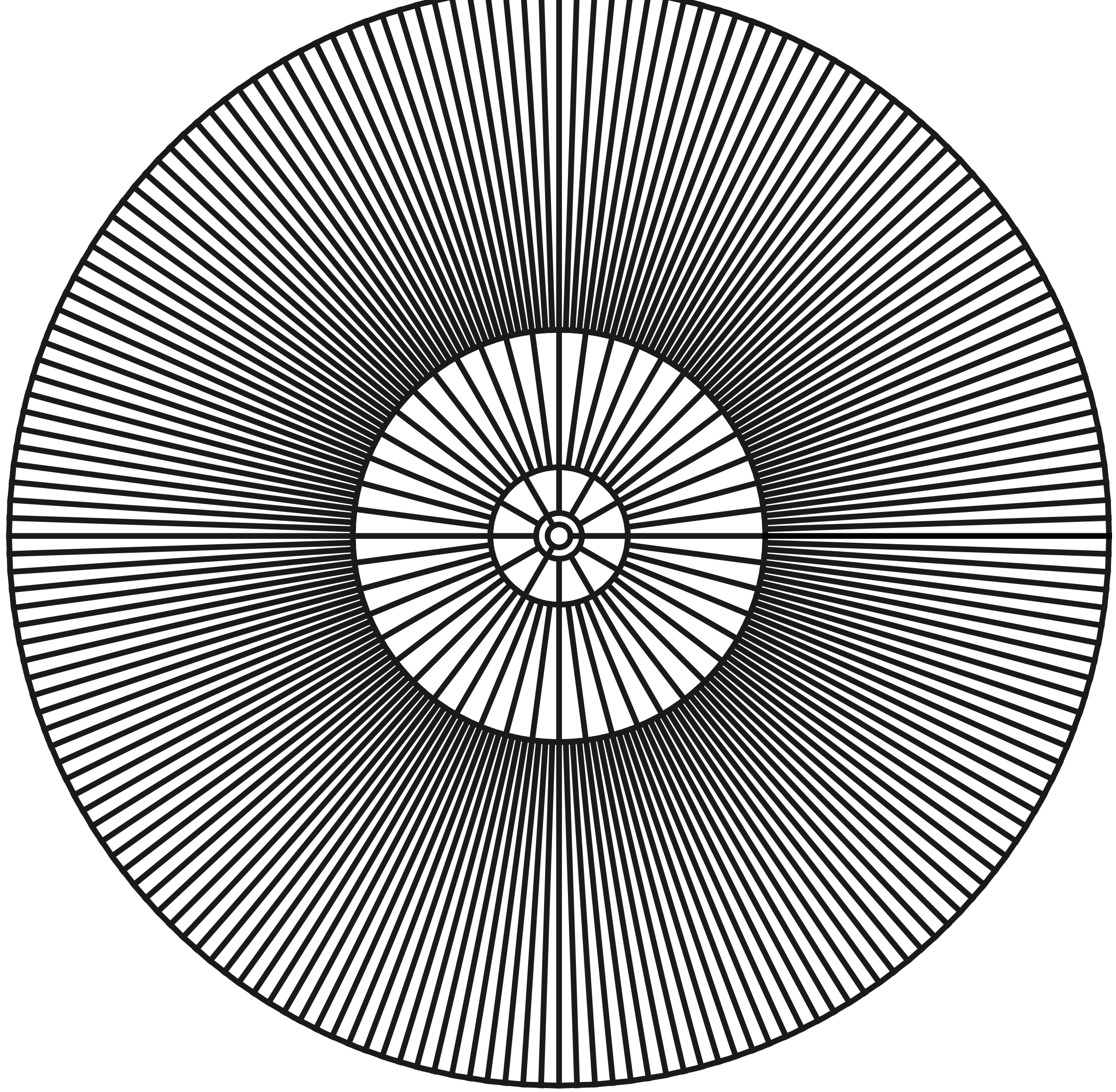
Hexagon $\to$ triangles with probability p, ANS $= \frac{12p+6}{5p+1}$ ($p = 1/4$ shown).



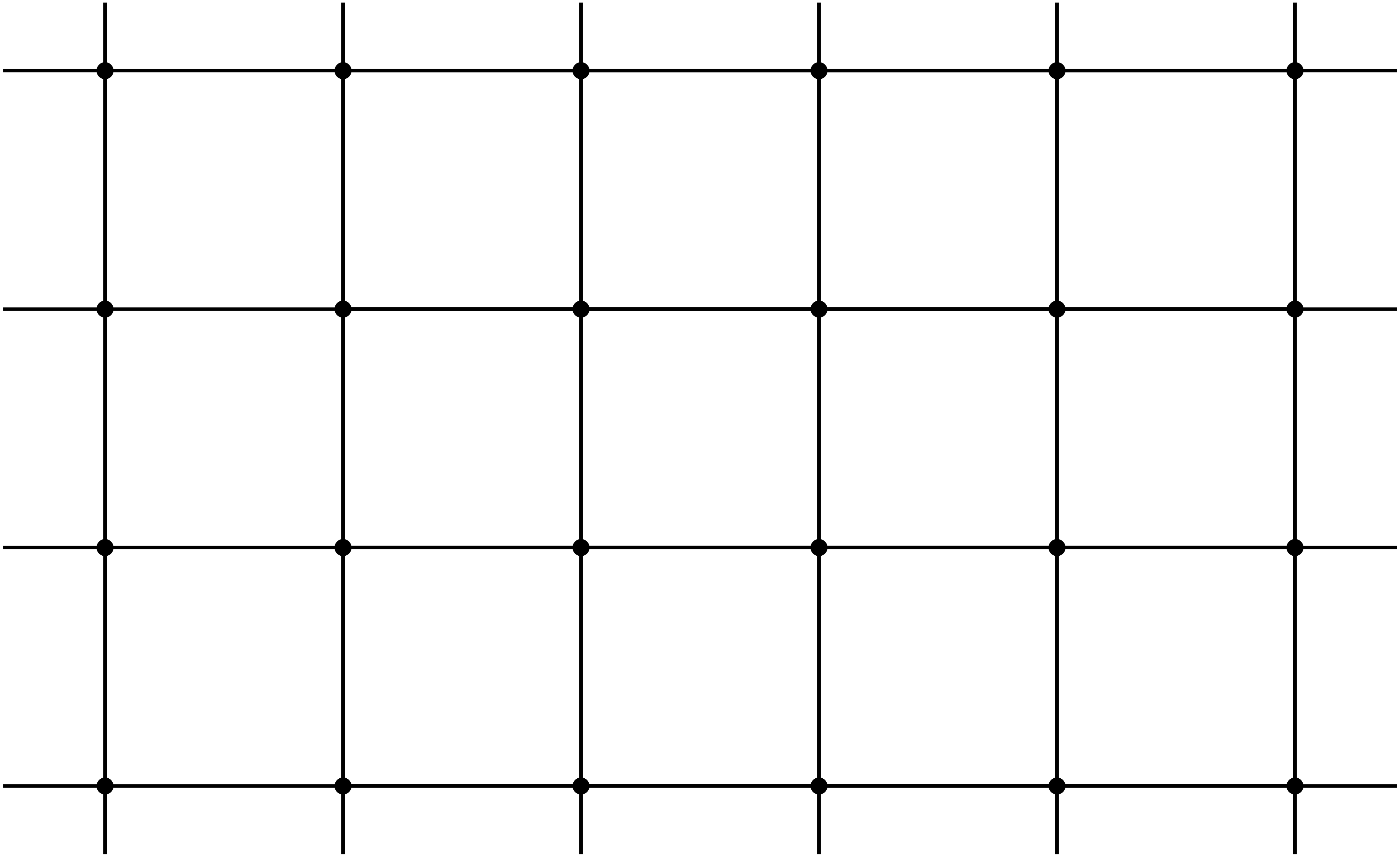
Limit need not exist

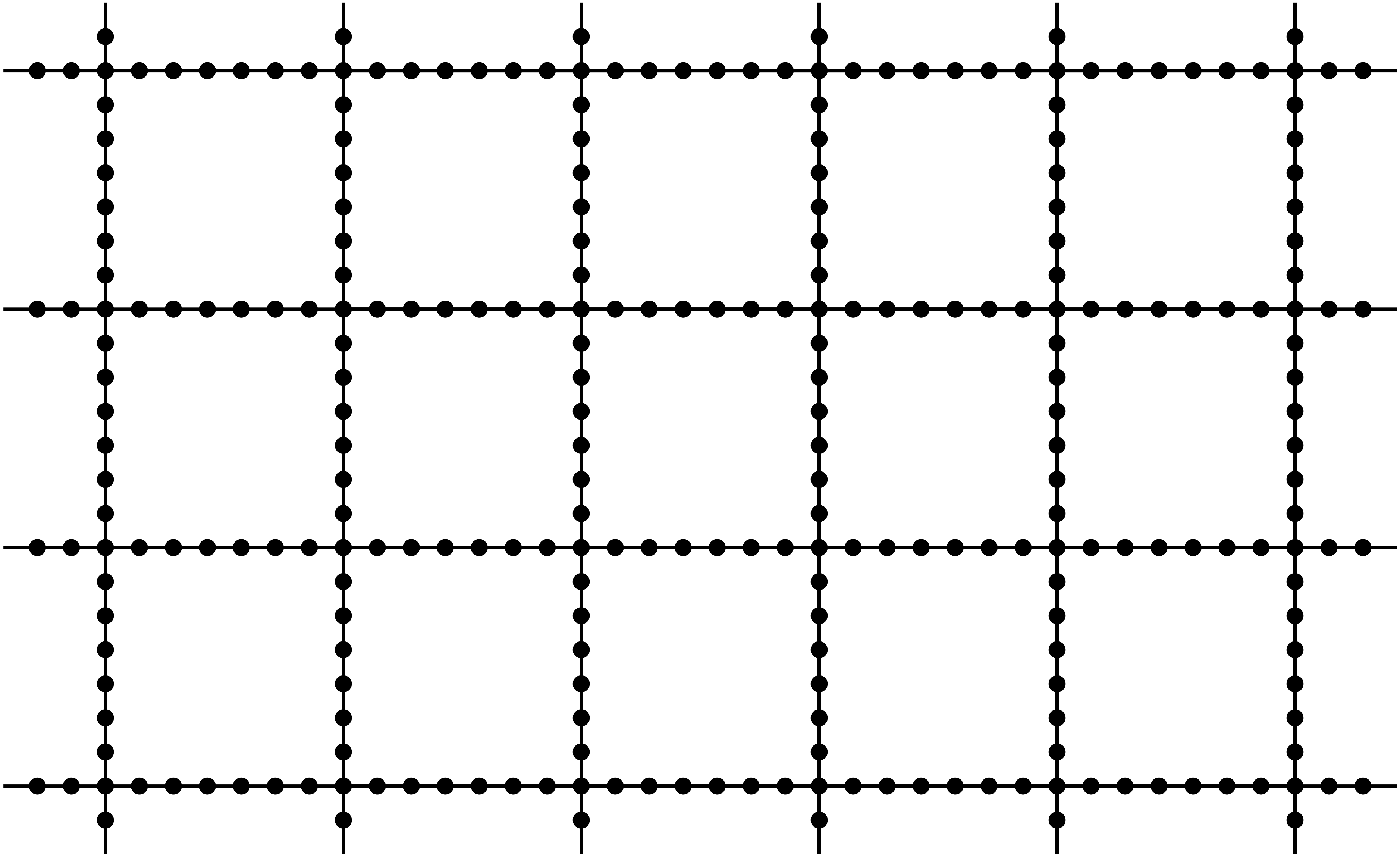


average number sides $\nearrow \infty$, areas $\to 0$, diameters bounded



average number sides $\nearrow \infty$, diameters $\to \infty$, areas bounded below



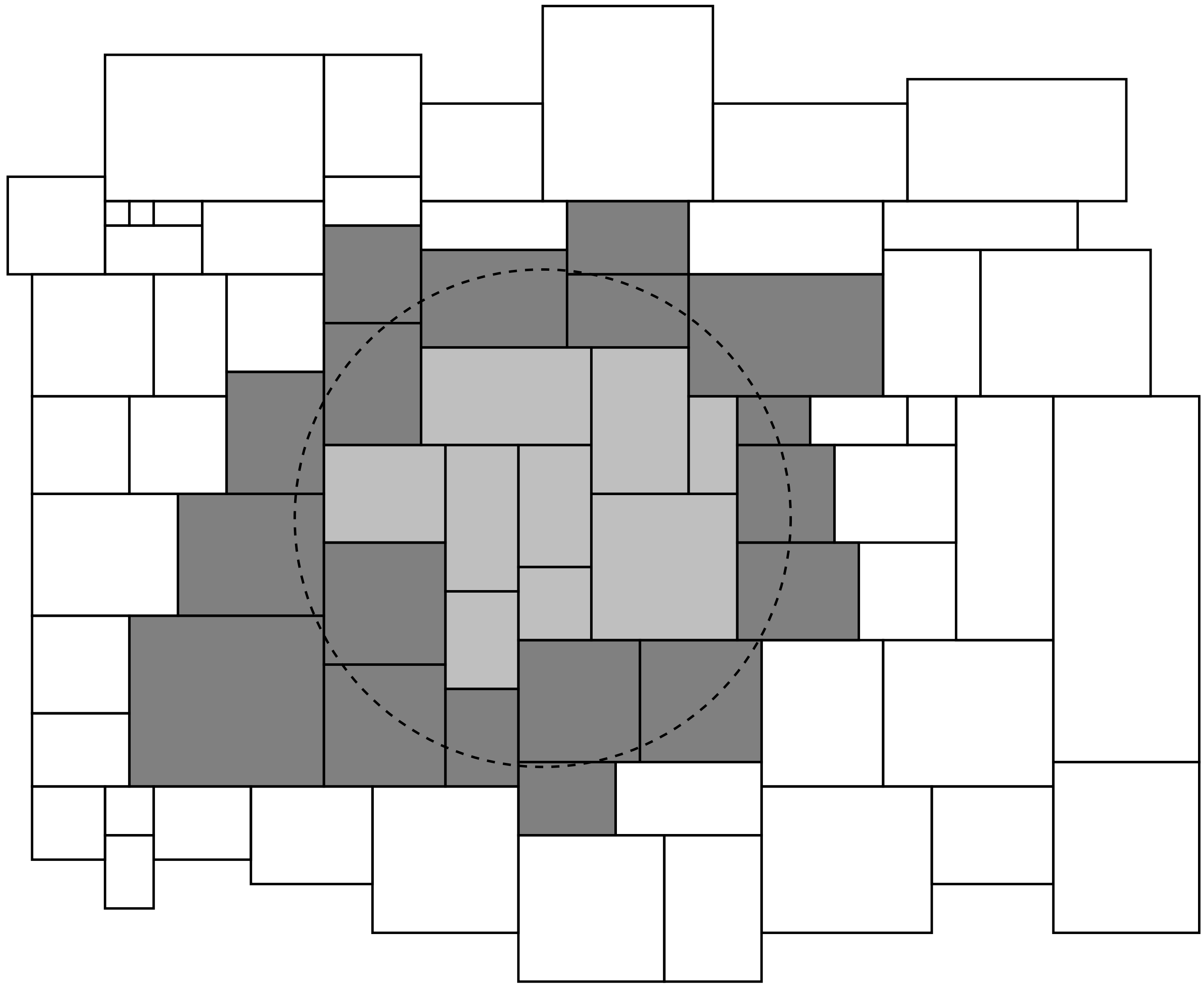


average number sides $\nearrow \infty$, forbid vertices of degree 1 and 2

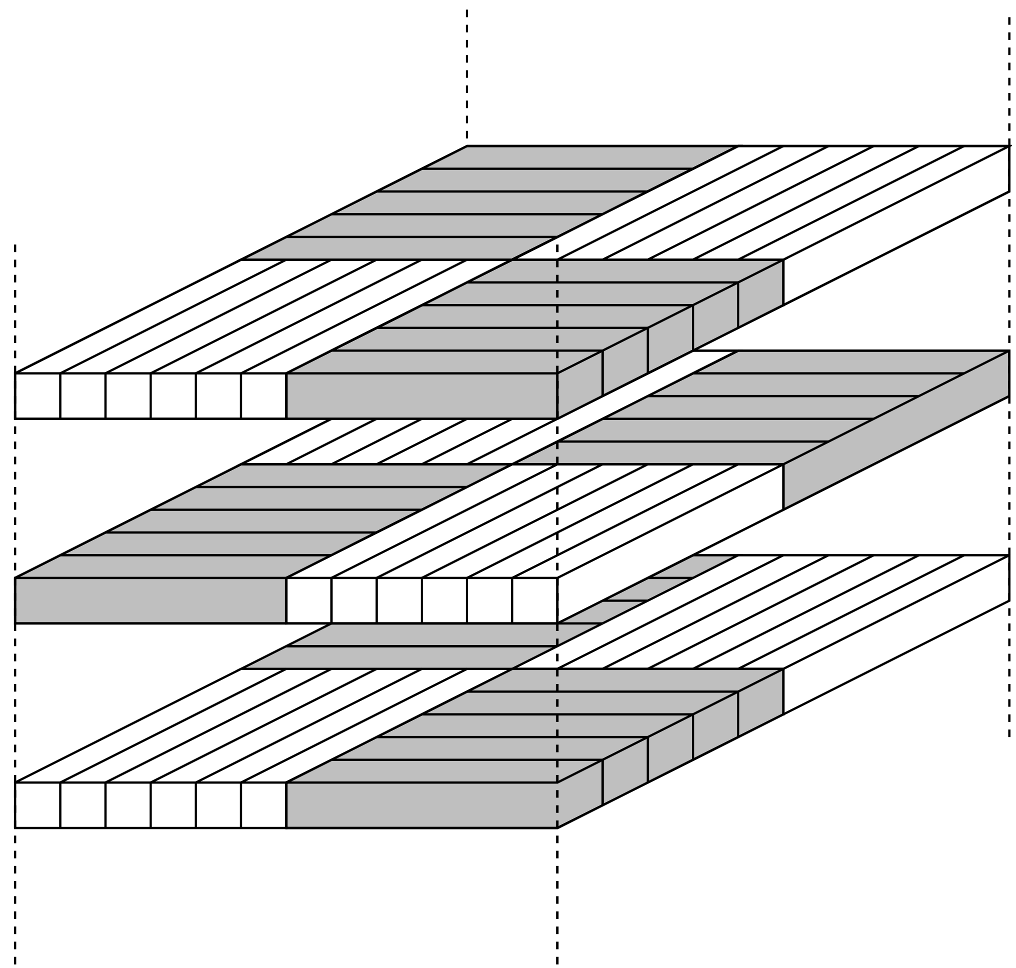
Theorem 1: Suppose faces have areas that are bounded below and diameters that are bounded above, and every vertex had degree ≥ 3. Then

$$\limsup_{t \to \infty} \mathrm{ANS}(t) \leq 6.$$

Average is over all faces contained in circle of radius t.



Average over faces inside/hitting an expanding circle



Theorem false in three dimensions

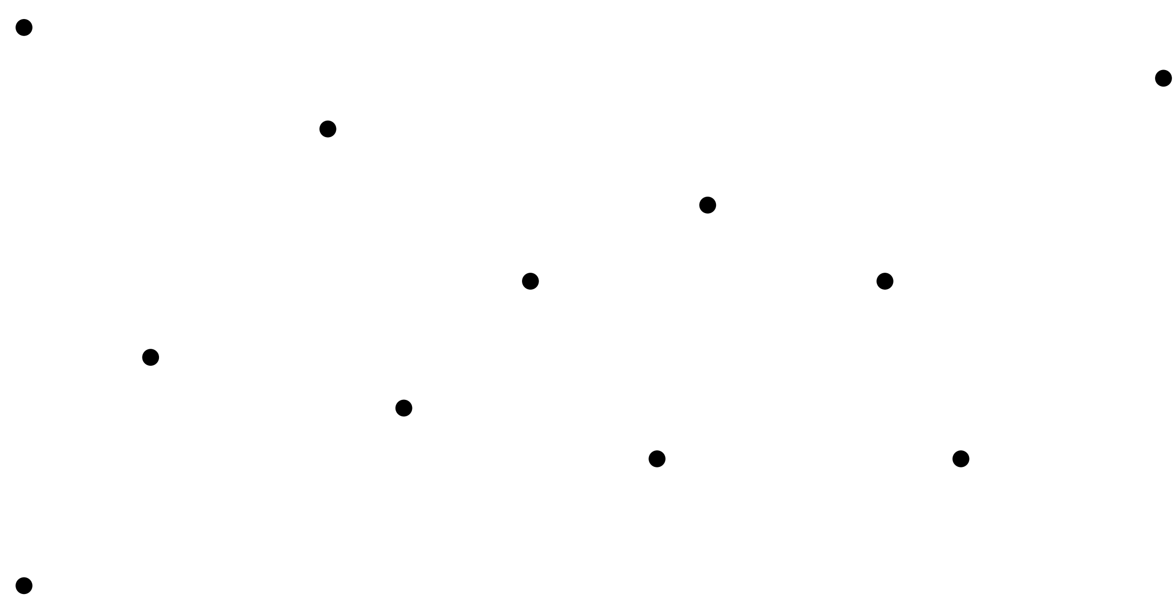
Euler's formula for finite planar graphs: $V - E + F = C + 1$

V = number vertices

E = number of edges

F = number of faces

C = number of components



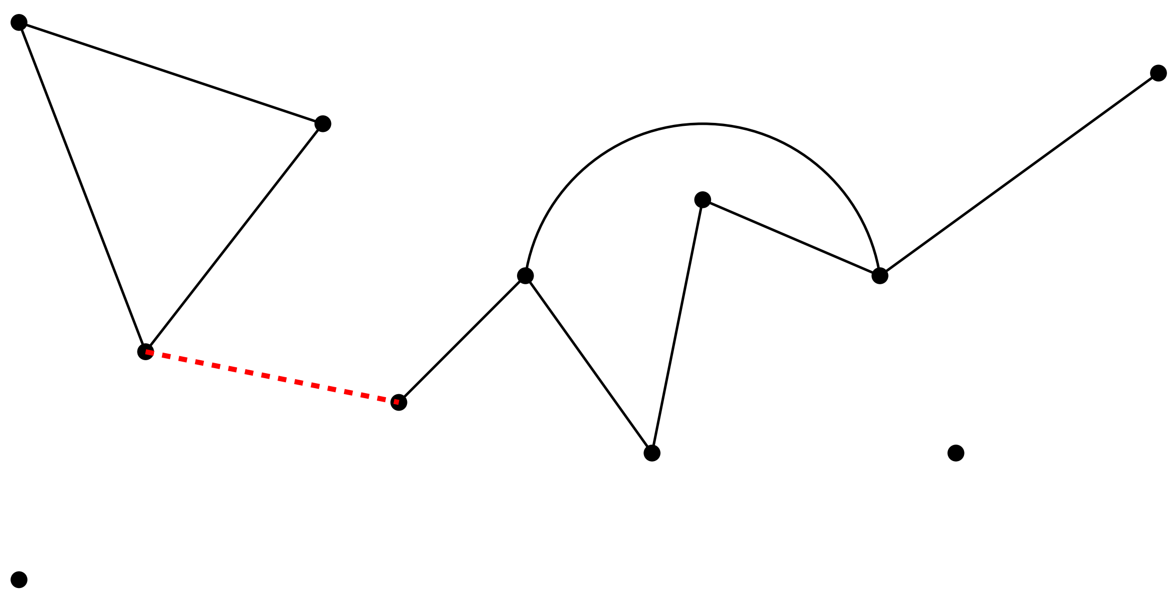
Euler's formula for finite planar graphs: $V - E + F = C + 1$

V = number vertices

E = number of edges

F = number of faces

C = number of components



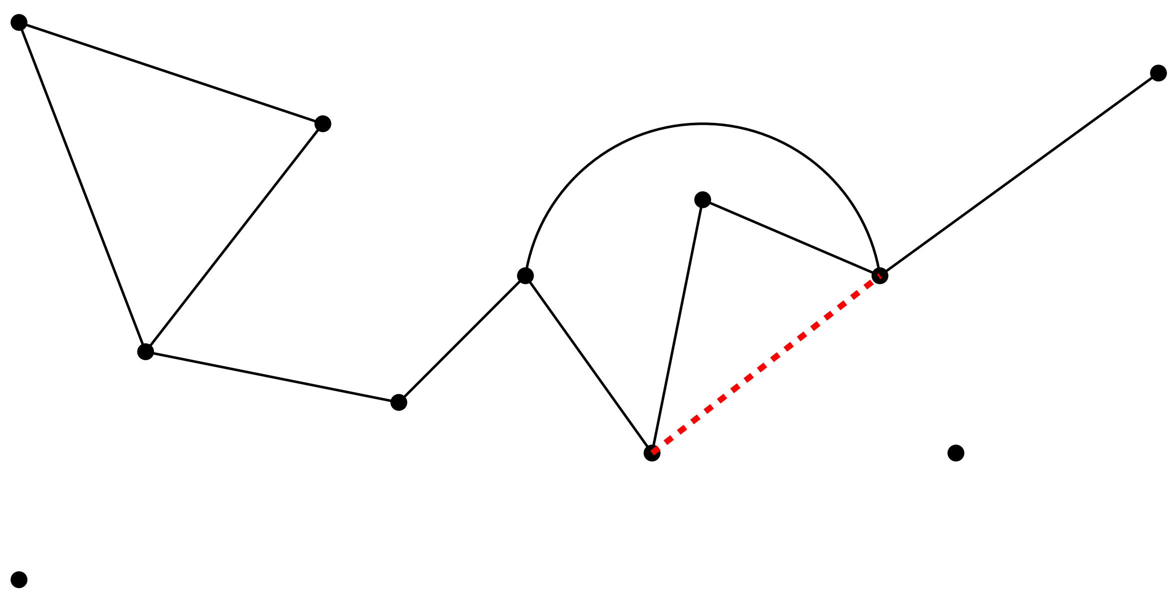
Euler's formula for finite planar graphs: $V - E + F = C + 1$

V = number vertices

E = number of edges

F = number of faces

C = number of components



Sphere Theorem: If H is a finite graph on the 2-sphere and every vertex has degree ≥ 3, then $E < 3F$.

Sphere Theorem: If H is a finite graph on the 2-sphere and every vertex has degree ≥ 3, then $E < 3F$.

Proof: Clearly

$$\sum_{v \in H} \deg(v) = 2E.$$

By assumption, $\deg(v) \geq 3$ for all vertices so $3V \leq 2E$. Hence $V \leq \frac{2}{3}E$.

Sphere Theorem: If H is a finite graph on the 2-sphere and every vertex has degree ≥ 3, then $E < 3F$.

Proof: Clearly

$$\sum_{v \in H} \deg(v) = 2E.$$

By assumption, $\deg(v) \geq 3$ for all vertices so $3V \leq 2E$. Hence $V \leq \frac{2}{3}E$.

Plugging this into Euler's formula $V - E + F = C + 1$ gives

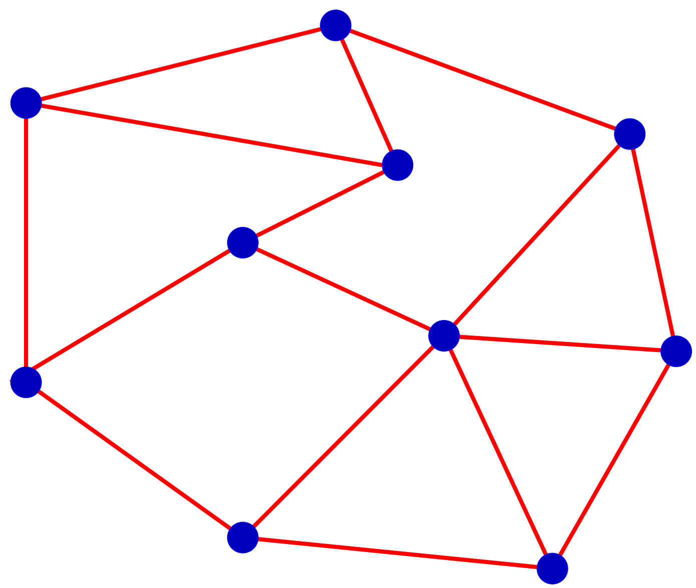
$$-\frac{1}{3}E + F \geq C + 1$$

or

$$E \leq 3F - 3(C + 1) < 3F.$$

Sphere Theorem: If H is a finite graph on the 2-sphere and every vertex has degree ≥ 3, then $E < 3F$.

Corollary: Number of sides $= 2E < 6F$. So for a finite planar graph (all degrees ≥ 3), the average number of sides per face is ≤ 6.



$$E = 16, \quad F = 8, \quad \text{ANS} = 4$$

This is for finite H. What about infinite maps?

Choose a piecewise smooth region R in the plane and let $tR + x$ denote the region dilated by a factor of $t > 0$ and translated by x.

Let $H = H(R, t, x)$ be the sub-map of G consisting of the 2-cells of G that lie inside $tR + x$.

Let ANS(t) be average number of sides over faces in $H(R, t, x)$.

Usually take circle around origin.

Theorem 1: Suppose all faces of G have diameter $\leq D < \infty$, and have area $\geq A > 0$, and that every vertex has degree ≥ 3. Then

$$\limsup_{t \to \infty} \mathrm{ANS}(H(R, t, x)) \leq 6.$$

Theorem 1: Suppose all faces of G have diameter $\leq D < \infty$, and have area $\geq A > 0$, and that every vertex has degree ≥ 3. Then

$$\limsup_{t\to\infty} \mathrm{ANS}(H(R,t,x)) \leq 6.$$

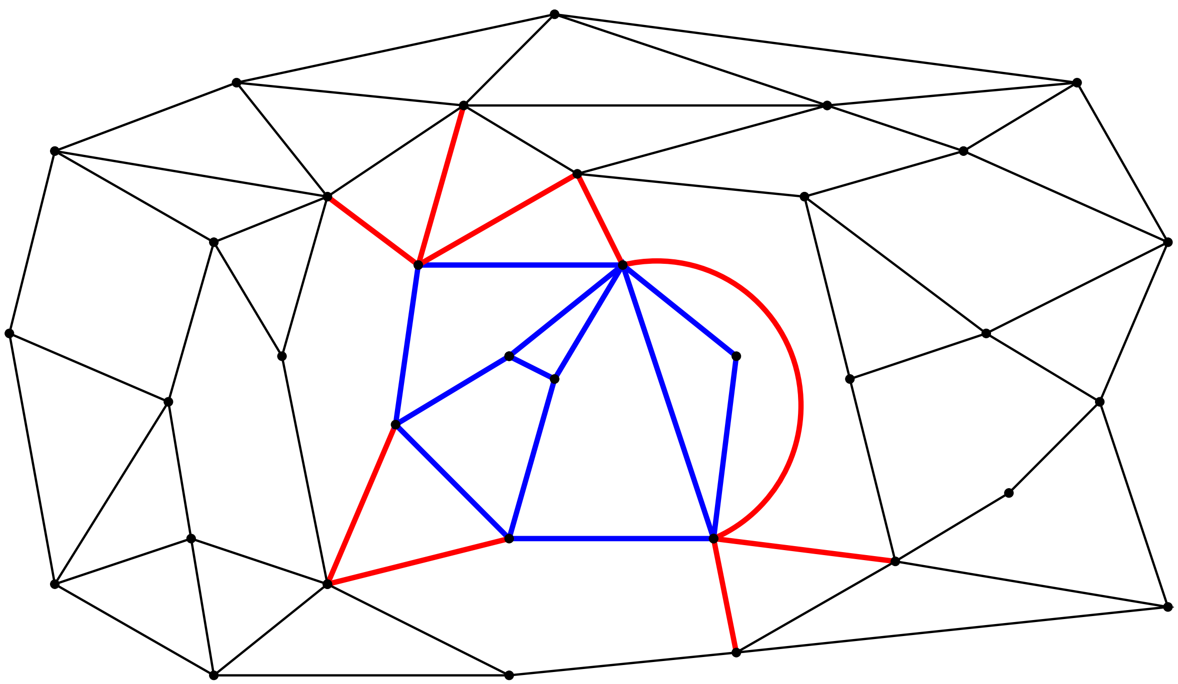
Average is taken over faces contained in $tR + x$.

Same conclusion holds for faces hitting $tR + x$

Really only need R to have non-empty interior and zero area boundary.

Definition: edge boundary: If $H \subset G$, then $\partial_E H$ is set of edges in $G \setminus H$ with at least one endpoint in H.

$$e(H) = \frac{|\partial_E H|}{F}.$$



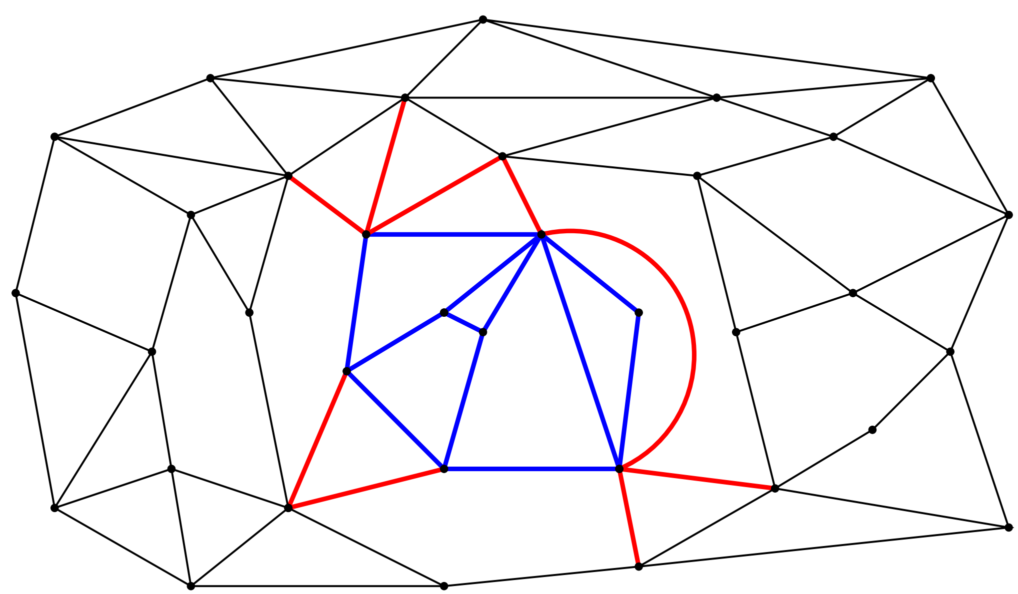
Here $e(H) = 9/5$. Related to Cheeger constant of G.

Lemma 1: If H is a sub-map of G then $\mathrm{ANS}(H) \leq 6 + 4e(H)$.

Proof: Note that

$$2E + |\partial_E H| \leq \sum_{v \in H} \deg_G(v) \leq 2E + 2|\partial_E H|,$$

The left hand inequality would be an equality, except that some edges in $\partial_E H$ might have both endpoints on ∂H.



Lemma 1: If H is a sub-map of G then $\text{ANS}(H) \leq 6 + 4e(H)$.

Proof continued: Every vertex has degree ≥ 3, so

$$3V \leq \sum_{v \in H} \deg_G(v) \leq 2E + 2|\partial_E H|.$$

Divide by 3 and put estimate for V into Euler's formula:

$$(\frac{2}{3}E + \frac{2}{3}|\partial_E H|) - E + F \geq C$$

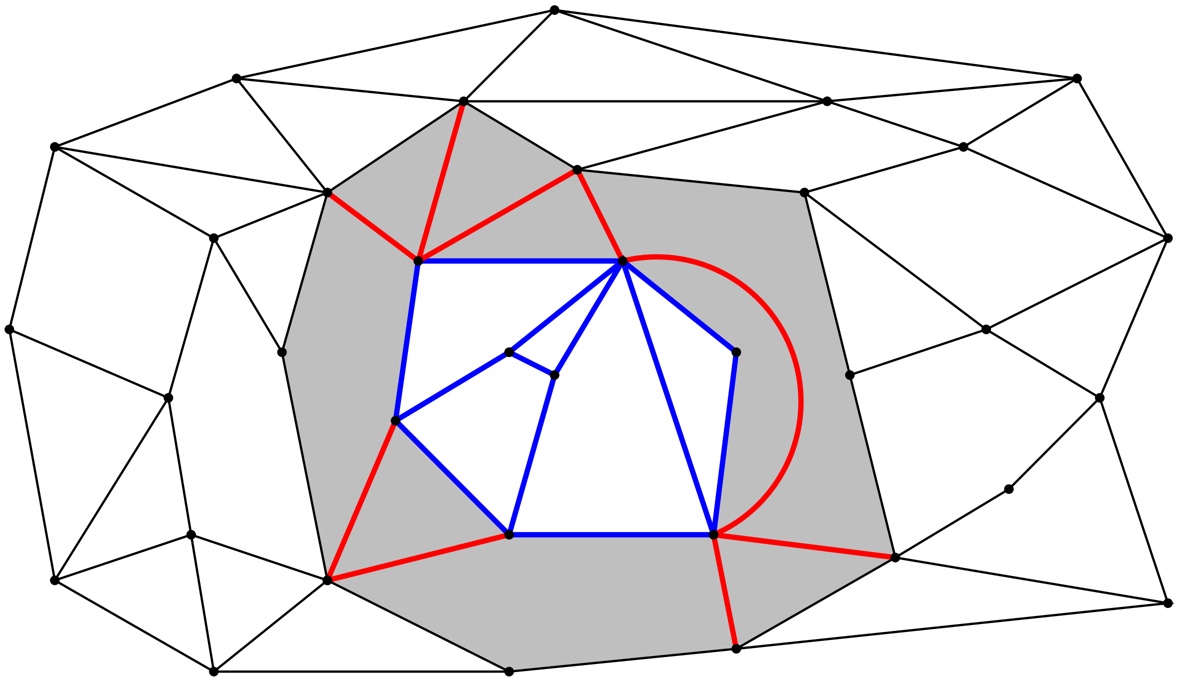
Simplifying gives

$$\frac{E}{F} \leq 3 + 2\frac{|\partial_E H|}{F} - 3\frac{C}{F} \leq 3 + 2e(H).$$

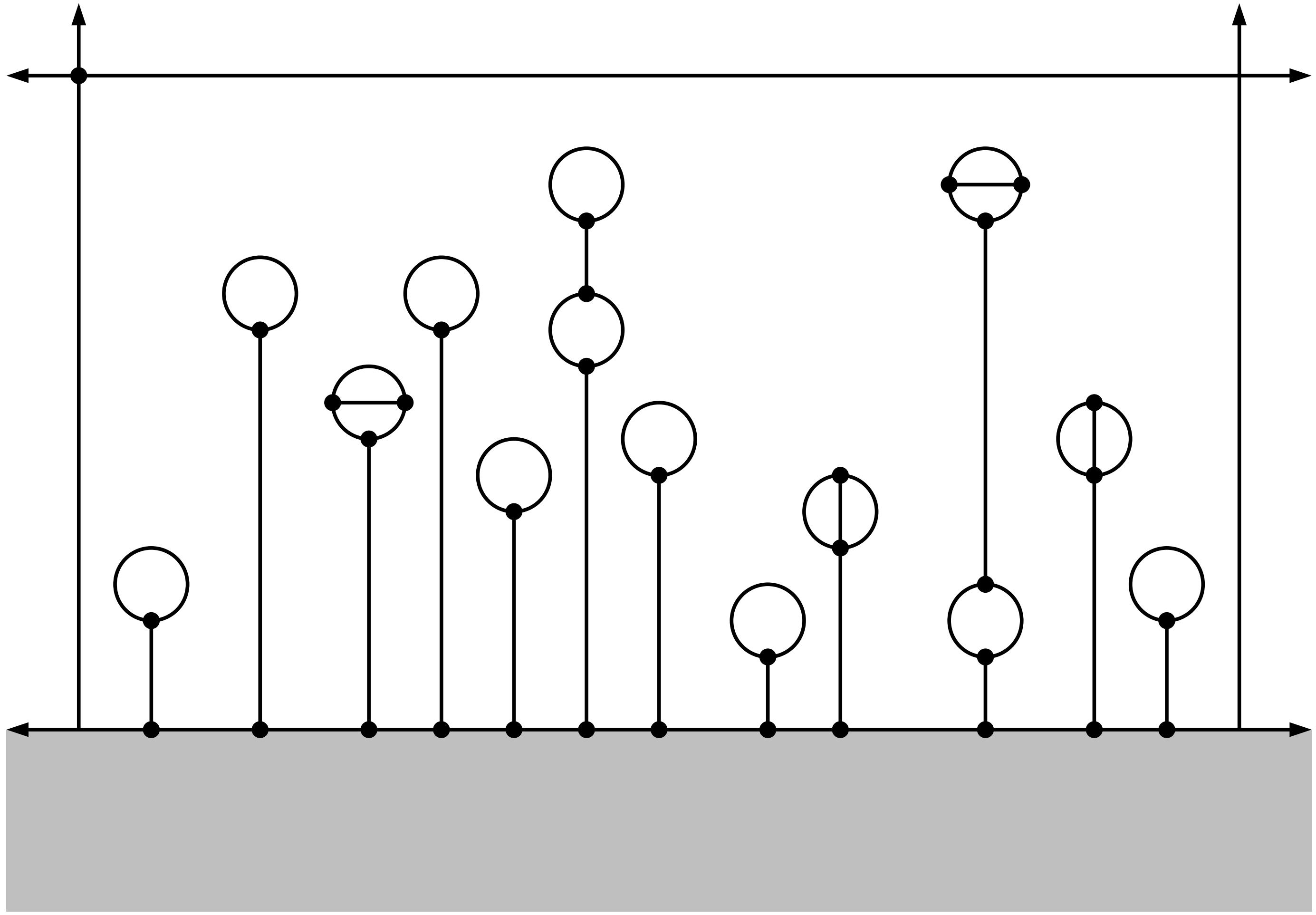
Hence

$$\text{ANS}(H) \leq 2E/F \leq 6 + 4e(H).$$

Definition: the face boundary: $\partial_F H$ is the set of faces in $G \setminus H$ that touch H.



Easy to see $|\partial_F H| \leq |\partial_E H|$. We need converse direction.



One adjacent face, many adjacent edges

Lemma 2: Suppose the faces of G have diameters $\leq D < \infty$ and H is a sub-map. Let N be the number of faces that lie inside a $3D$-neighborhood of ∂H. Then $|\partial_E H| \leq 3N$.

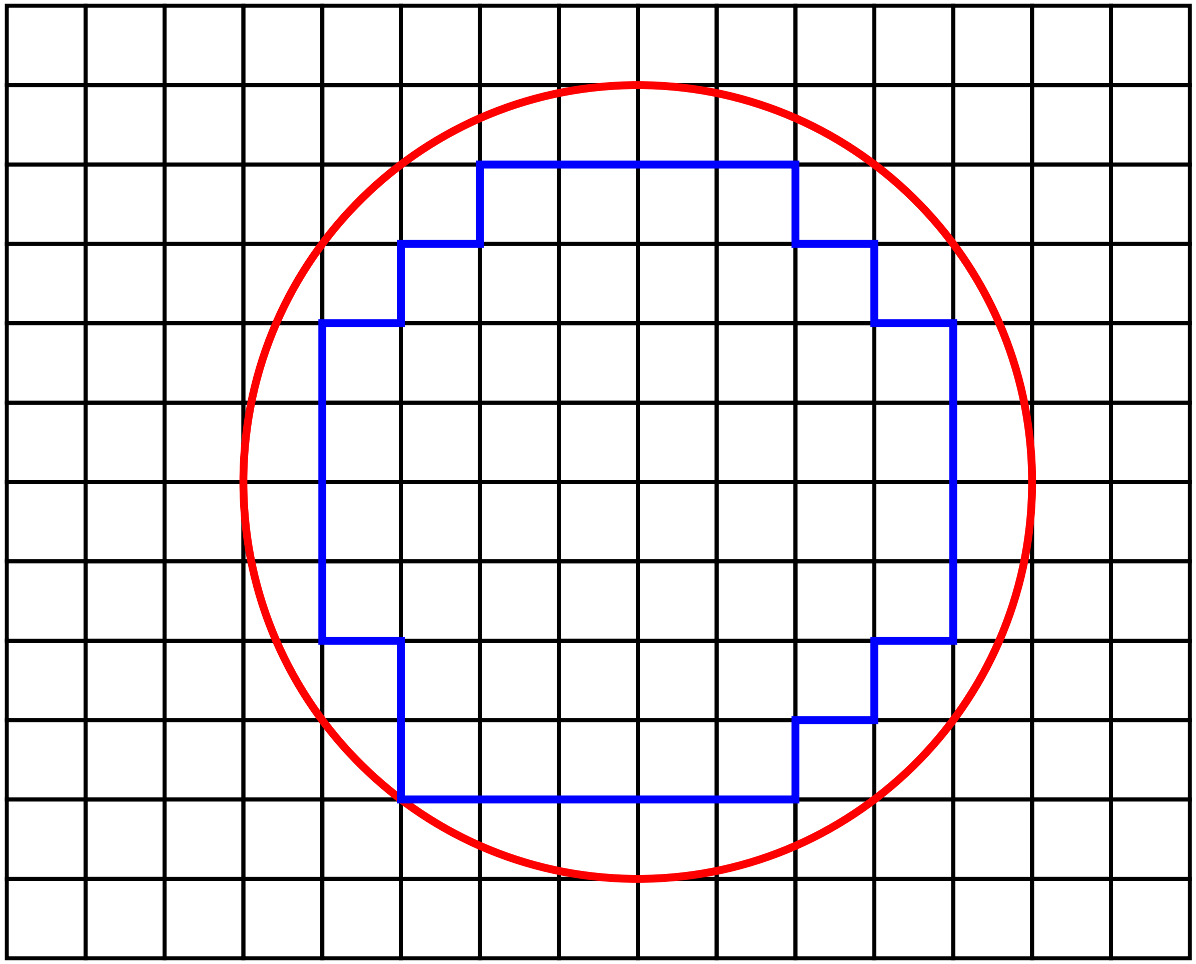
Lemma 2: Suppose the faces of G have diameters $\leq D < \infty$ and H is a sub-map. Let N be the number of faces that lie inside a $3D$-neighborhood of ∂H. Then $|\partial_E H| \leq 3N$.

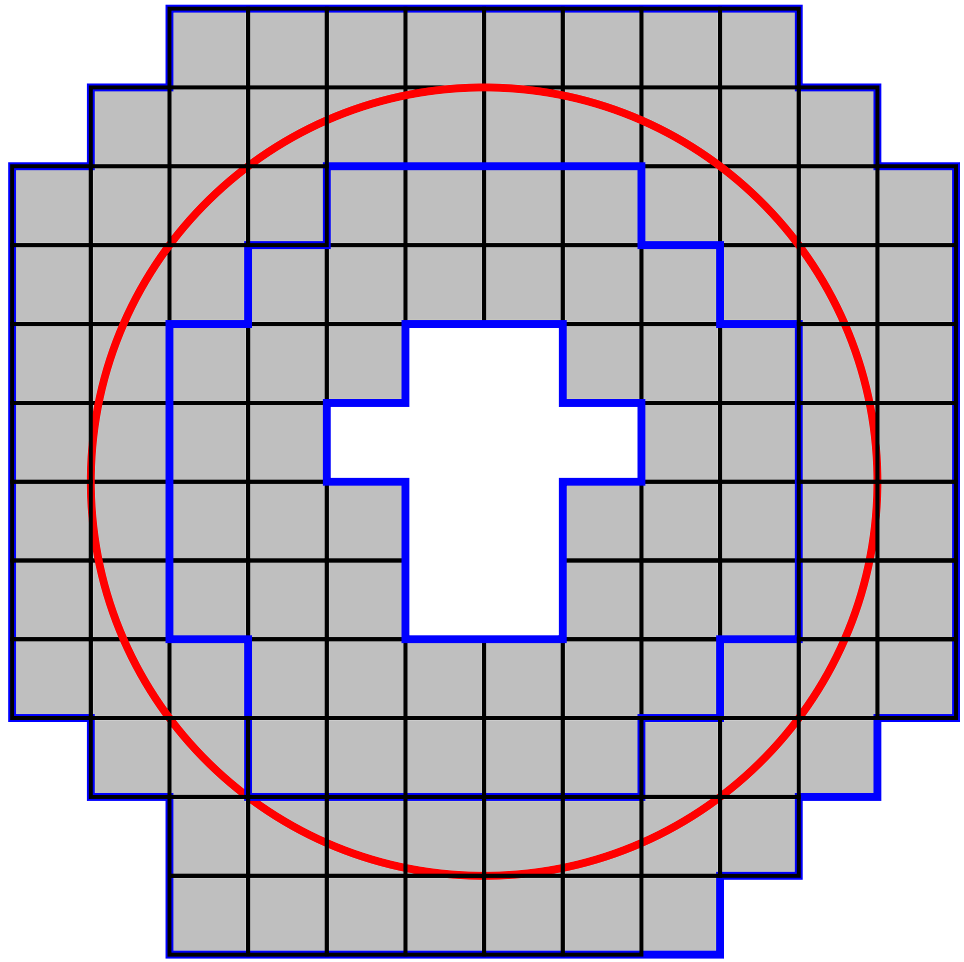
Proof:

Let G' be the finite graph on the sphere consisting of the faces of G that lie within the $2D$-neighborhood of ∂H, together with their edges and vertices.

Check that edges of G' include all edges in $\partial_E H$.

Each face of G' is either a face of G or contains a face of G that is within $3D$ of ∂H. Thus number of faces of G' is at most N.





Lemma 2: Suppose the faces of G have diameters $\leq D < \infty$ and H is a sub-map. Let N be the number of faces that lie inside a $3D$-neighborhood of ∂H. Then $|\partial_E H| \leq 3N$.

Proof continued:

Define G'' by removing any vertices of degree 2 from G'; combine edges.

G' and G'' have same number of faces, $\leq N$.

If $e \in \partial_E H$, endpoints have degree ≥ 3 in G' (any edge touching e in G is in G'). So e is an edge of G''.

By Sphere Theorem, edges in G'' bounded by three times faces in G''.

Hence $|\partial_E H| \leq 3N$.

Theorem 1: Suppose all faces of G have diameter $\leq D < \infty$, and have area $\geq A > 0$, and that every vertex has degree ≥ 3. Then

$$\limsup_{t \to \infty} \mathrm{ANS}(H(R, t, x)) \leq 6.$$

Theorem 1: Suppose all faces of G have diameter $\leq D < \infty$, and have area $\geq A > 0$, and that every vertex has degree ≥ 3. Then

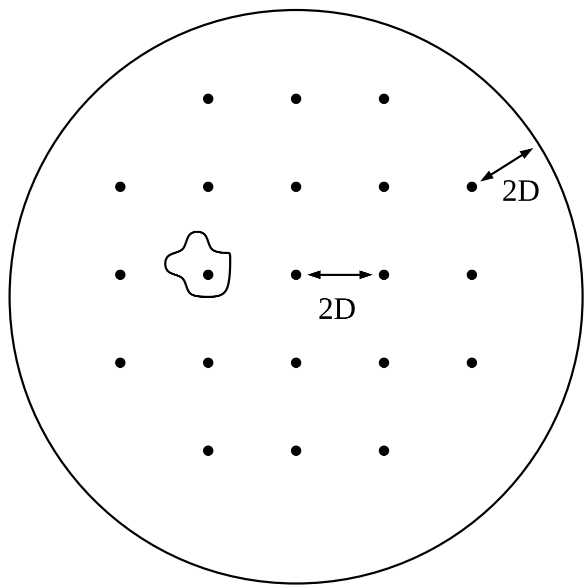
$$\limsup_{t\to\infty} \mathrm{ANS}(H(R,t,x)) \leq 6.$$

Proof: Only need show $e(H(R,t,x)) \to 0$.

Theorem 1: Suppose all faces of G have diameter $\leq D < \infty$, and have area $\geq A > 0$, and that every vertex has degree ≥ 3. Then

$$\limsup_{t \to \infty} \mathrm{ANS}(H(R,t,x)) \leq 6.$$

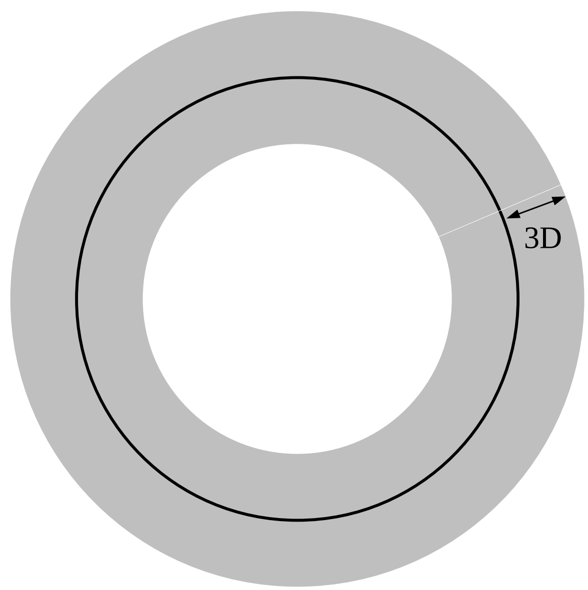
Proof continued: The number of faces in $H(R,t,x)$ is $\geq ct^2$.



Theorem 1: Suppose all faces of G have diameter $\leq D < \infty$, and have area $\geq A > 0$, and that every vertex has degree ≥ 3. Then

$$\limsup_{t \to \infty} \mathrm{ANS}(H(R, t, x)) \leq 6.$$

Proof continued: Area within $3D$ of ∂H is $O(t \cdot D)$, so $|\partial_E H| = O(tD/A)$.



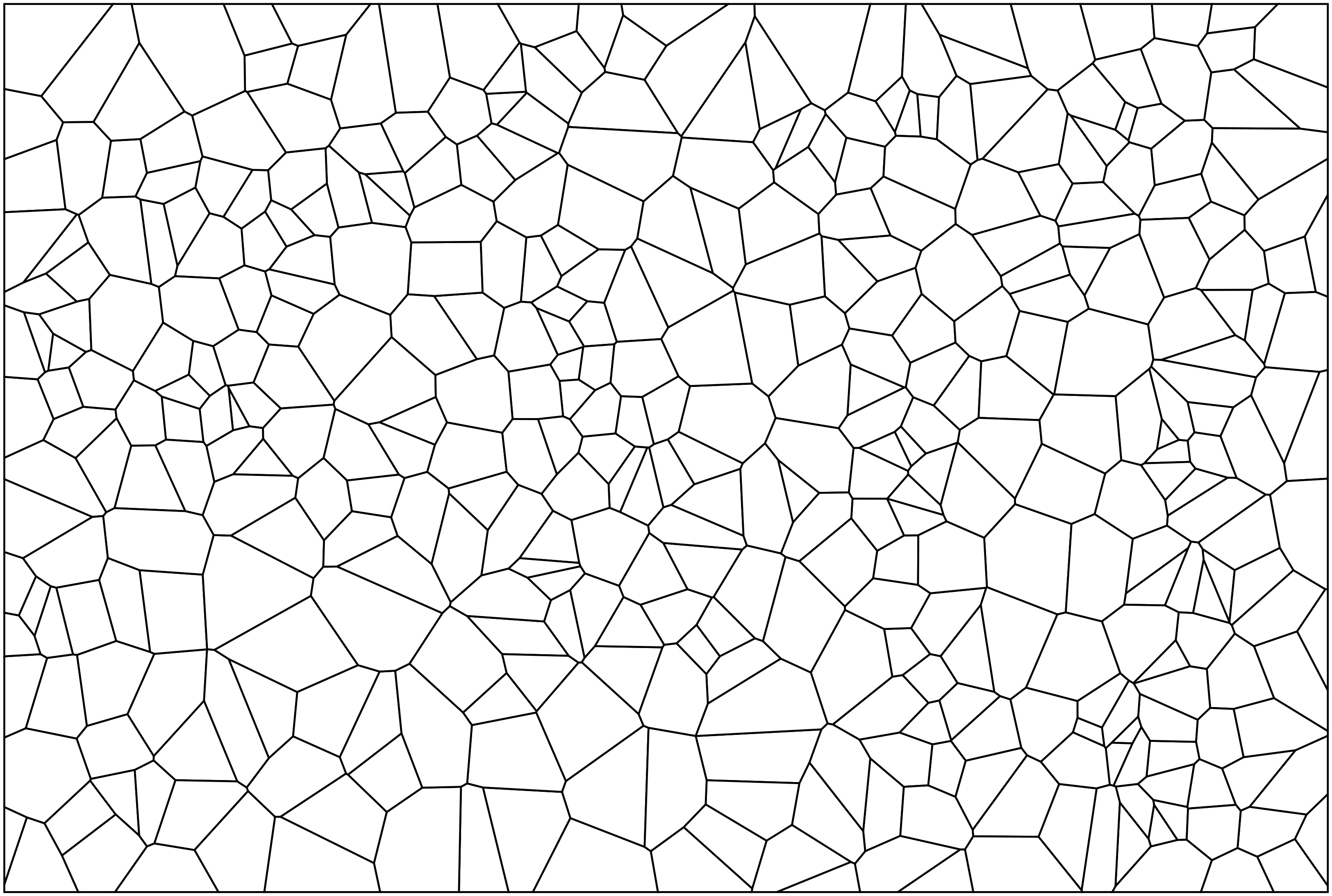
Theorem 1: Suppose all faces of G have diameter $\leq D < \infty$, and have area $\geq A > 0$, and that every vertex has degree ≥ 3. Then

$$\limsup_{t \to \infty} \mathrm{ANS}(H(R, t, x)) \leq 6.$$

Proof continued: Hence $e(H) = O(t/t^2) = O(1/t)$.

Theorem 2: If G satisfies the area lower bound and diameter upper bouned and every vertex has degree 3, then

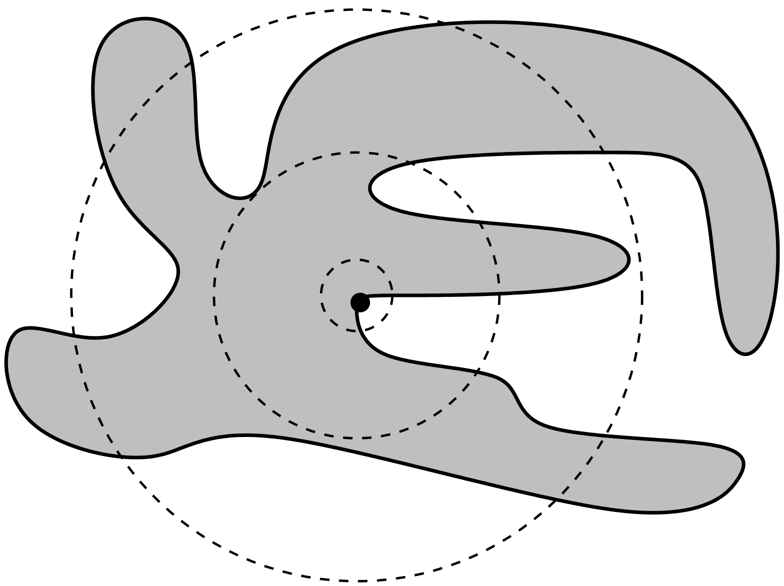
$$\lim_{t \to 0} \mathrm{ANS}(t) = 6.$$



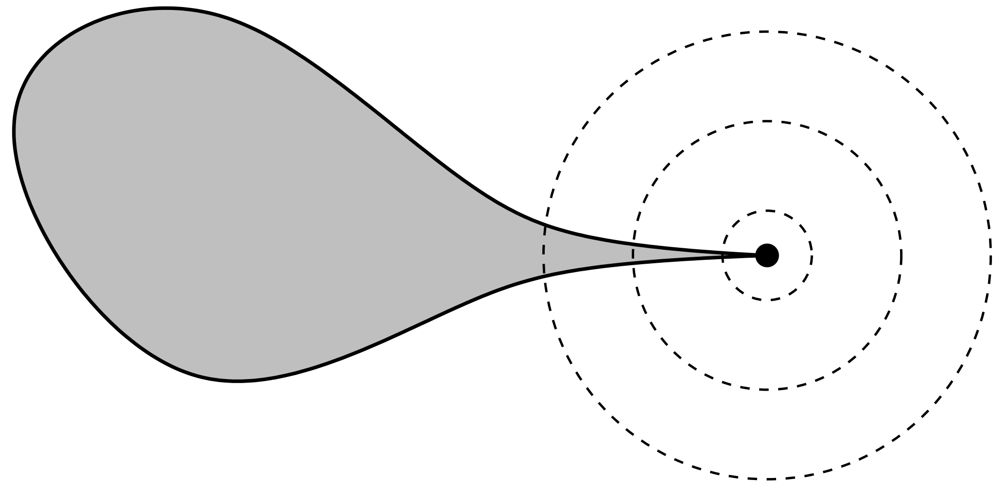
Voronoi diagram of Poisson point process

Ω is δ-thick if for all $0 < r < \mathrm{diam}(\Omega)$ and all $x \in \partial\Omega$ we have

$$\mathrm{area}(\Omega \cap D(x, r)) \geq \delta r^2.$$



Thick



Not Thick

True if Ω is convex, with a δ that depends on the aspect ratio.

Holds for quasidisks.

Theorem 3: Suppose G is a planar map and every face is δ-thick, for some $\delta > 0$. Then there is a nested, increasing sequence of sub-maps $\{H_n\}$ so that $\lim_{n\to\infty} \mathrm{ANS}(H_n) \leq 6$.

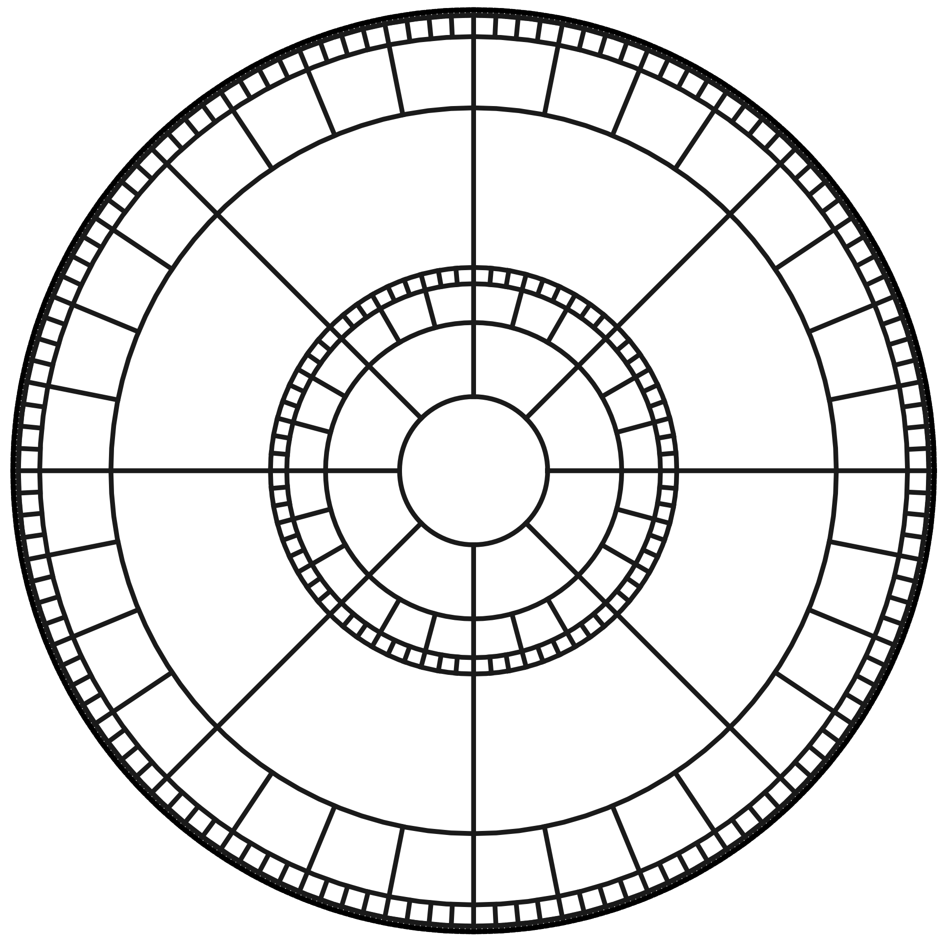
Theorem 3: Suppose G is a planar map and every face is δ-thick, for some $\delta > 0$. Then there is a nested, increasing sequence of sub-maps $\{H_n\}$ so that $\lim_{n\to\infty} \text{ANS}(H_n) \leq 6$.

Idea of proof:

If $\liminf > 6$, then $e(H(t)) > \epsilon > 0$ for all $t > t_0$.

We can show this forces $|\partial_F H(t)| \nearrow \infty$ in a finite time.

This contradicts local finiteness of G.



$$\limsup = \infty, \ \liminf \leq 6.$$

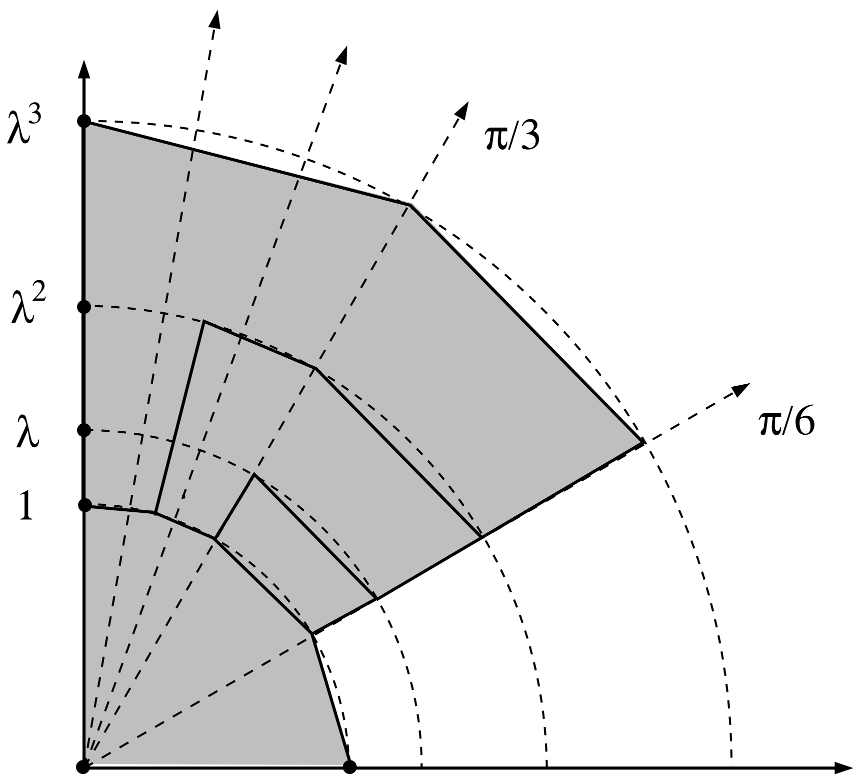
Aspect Ratio$(K) = \inf R/r$ where $D(x, r) \subset K \subset D(y, R)$.

Is there a "6" theorem if faces have uniformly bounded aspect ratios?

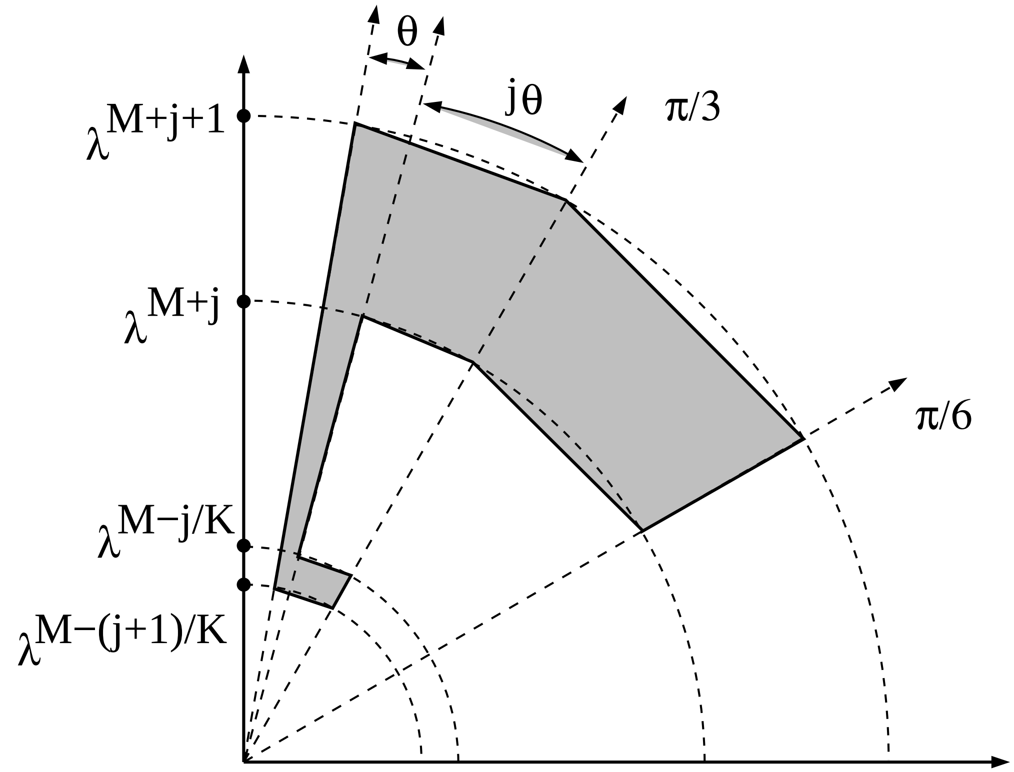
Aspect Ratio$(K) = \inf R/r$ where $D(x,r) \subset K \subset D(y,R)$.

Is there a "6" theorem if faces have uniformly bounded aspect ratios?

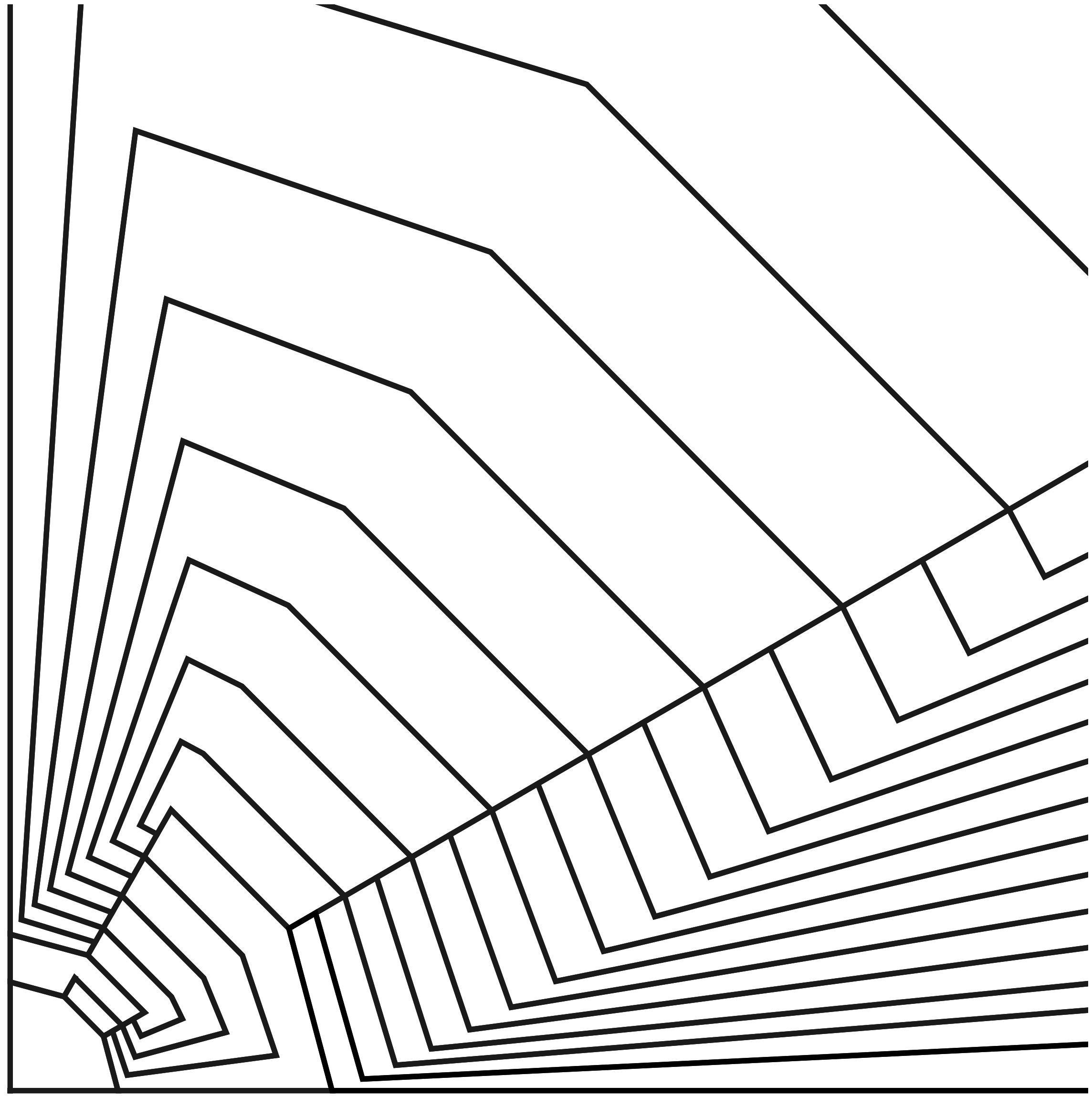
No.



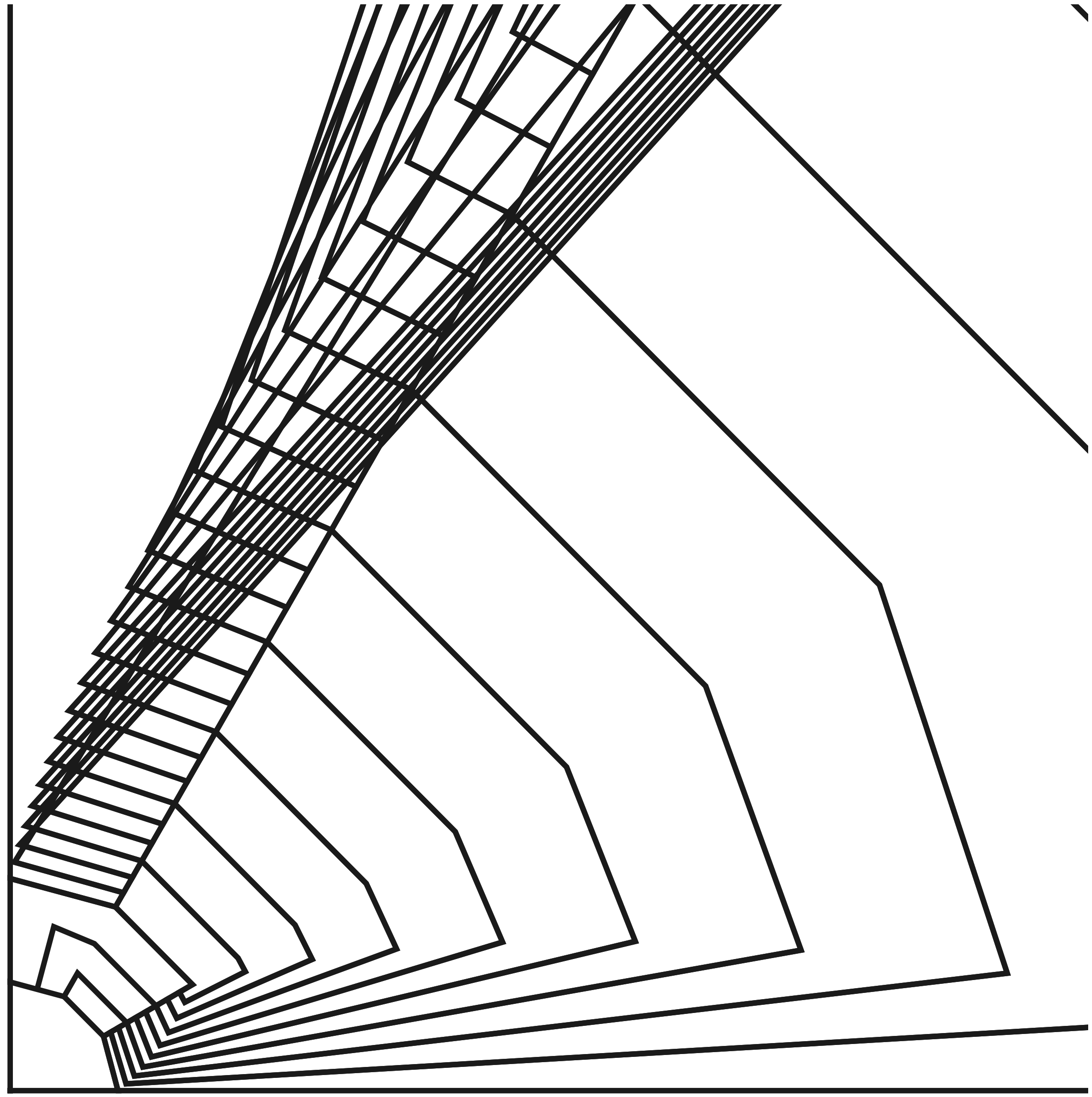
λ^3
λ^2
λ
1
$\pi/3$
$\pi/6$



θ
$j\theta$
λ^{M+j+1}
λ^{M+j}
$\lambda^{M-j/K}$
$\lambda^{M-(j+1)/K}$
$\pi/3$
$\pi/6$



Out-radius/in-radius is bounded, but number of neighbors $\nearrow \infty$



Out-radius/in-radius is bounded, but number of neighbors $\nearrow \infty$.