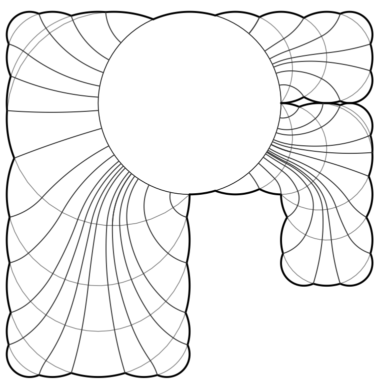
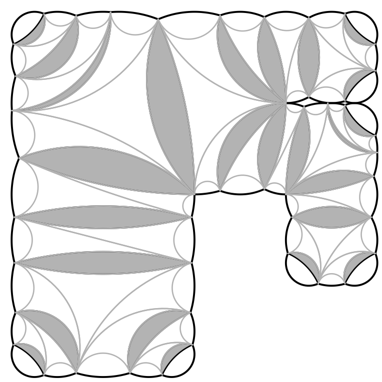
Conformal Mapping in Linear Time

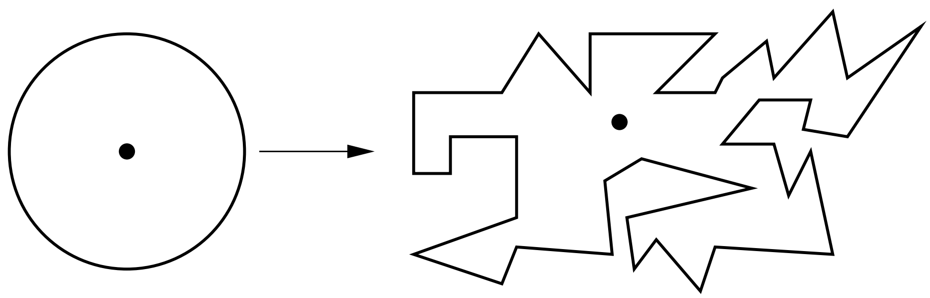
Christopher J. Bishop
SUNY Stony Brook





copies of lecture slides available at
www.math.sunysb.edu/~bishop/lectures

Riemann Mapping Theorem: If Ω is a simply connected, proper subdomain of the plane, then there is a conformal map $f : \Omega \to \mathbb{D}$.



I recently came across "Numerical conformal mapping using cross ratios and Delaunay triangulation" by Driscoll and Vavasis (1998). Thinking about this paper led to:

• 3-D hyperbolic geometry gives way to visualize and compute conformal maps.
• Computational geometry gives time bounds for doing these computations.

Theorem: If $\partial\Omega$ is an n-gon we can compute a $(1+\epsilon)$-quasiconformal map between Ω and $\mathbb{D}$ in time $O(n\log^2\frac{1}{\epsilon}\log\log\frac{1}{\epsilon})$.

Theorem: Suppose $\partial\Omega$ is an n-gon. We can construct points $\mathbf{w}=\{w_1,\ldots,w_n\}\subset\mathbb{T}$ so that:

1. requires at most $C(\epsilon)n$ steps.
2. $d_{QC}(\mathbf{w},\mathbf{z})<\epsilon$.

$\mathbf{z}=f^{-1}(\mathbf{v})$ are conformal prevertices.

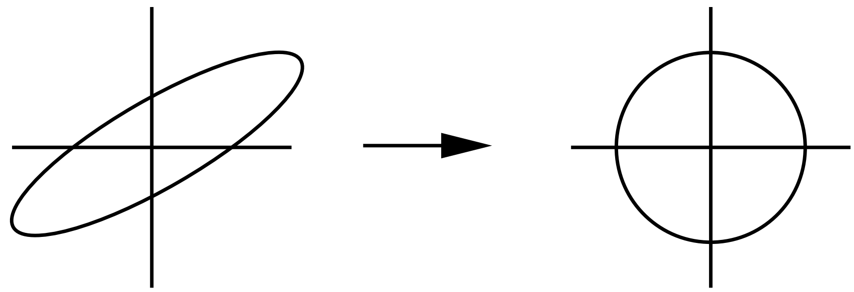
$d_{QC}(\mathbf{w},\mathbf{z})=\inf\{\log K:\exists h\in\mathrm{QC_K},\mathrm{h}(\mathbf{w})=\mathbf{z}\}$.

$\mathrm{QC_K}=K$-quasiconformal maps.

$C(\epsilon)=C+C\log^2\frac{1}{\epsilon}\log\log\frac{1}{\epsilon}$

A mapping is K-quasiconformal if either:

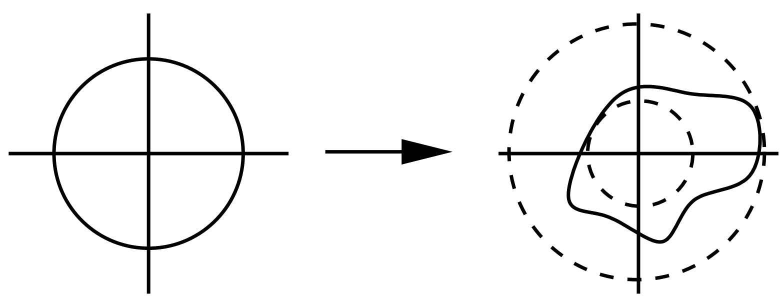
Analytic definition: $|f_{\bar{z}}| \leq \frac{K-1}{K+1}|f_z|$



$$f_z = \frac{1}{2}(f_x - if_y),\ f_{\bar{z}} = \frac{1}{2}(f_x + if_y).$$

Metric definition: For every $x \in \Omega$, $\epsilon > 0$ and small enough $r > 0$, there is $s > 0$ so that

$$D(f(x), s) \subset f(D(x, r)) \subset D(f(x), s(K+\epsilon)).$$



• The map is determined (up to Möbius maps) by

$$\mu_f = f_{\bar{z}}/f_z,$$

For μ with $\|\mu\|_\infty < 1$, there is a f with $\mu_f = \mu$.

• $\mu = 0$ iff f is conformal.

• K-QC maps form a compact family.

• f is a **quasi-isometry** if

$$\frac{1}{A}\rho(x,y) - B \leq \rho(f(x), f(y)) \leq A\rho(x,y) + B.$$

Theorem: $f : \mathbb{T} \to \mathbb{T}$ has a QC-extension to interior iff it has QI-extension (hyperbolic metric).

Proof of theorem is in three steps:

Step 1: Find K-QC $f_0 : \Omega \to \mathbb{D}$.

Step 2: Given $\epsilon < \epsilon_0$ and $(1+\epsilon)$-QC $f_n : \Omega \to \mathbb{D}$ construct $(1 + C\epsilon^2)$-QC map $f_{n+1} : \Omega \to \mathbb{D}$.

If $K < \epsilon_0$ then done. Otherwise need:

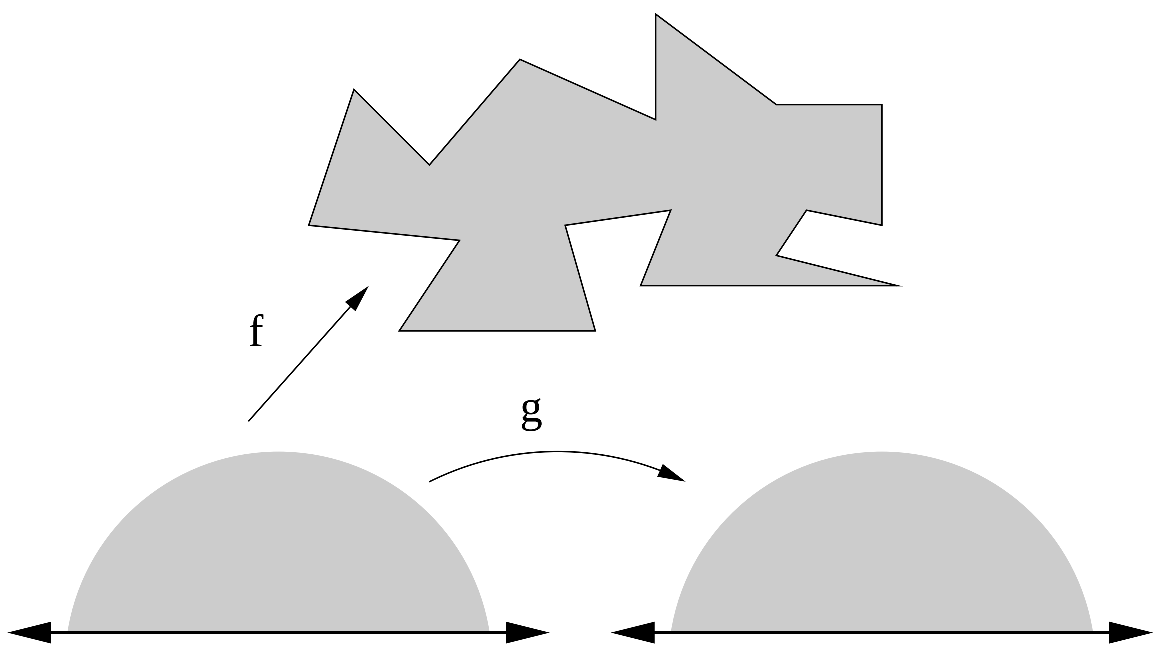
Step 3: Build chain $\mathbb{D} = \Omega_0, \ldots, \Omega_N = \Omega$ with explicit $\sqrt{1 + \epsilon_0}$-QC maps $g_k : \Omega_k \to \Omega_{k+1}$. Find conformal $f_k : \mathbb{D} \to \Omega_k$ by induction.

Clearly $f_0 = \text{Id}$. Use $g_0 \circ f_0$ as starting point to iterate to f_1. When within $\sqrt{1 + \epsilon_0}$ of f_1, compose with g_1 and start iterating to f_2. Continue until reach ϵ_0-ball around f_N.

Idea for Step 2: Suppose

$$f : \mathbb{H} \to \Omega, \qquad g : \mathbb{H} \to \mathbb{H}, \qquad \mu_f = \mu_g.$$

Then $f \circ g^{-1} : \mathbb{H} \to \Omega$ is conformal.

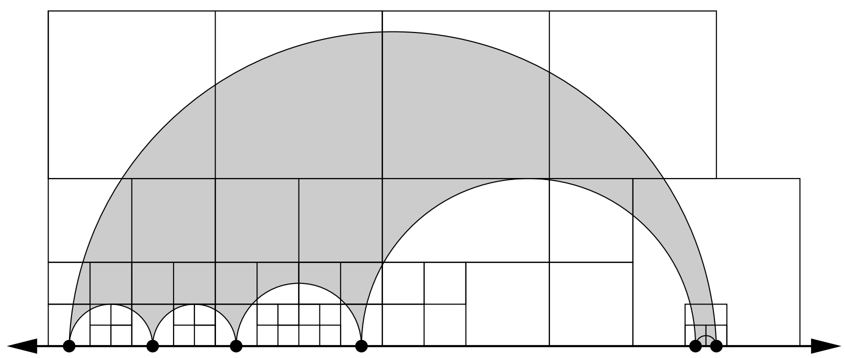


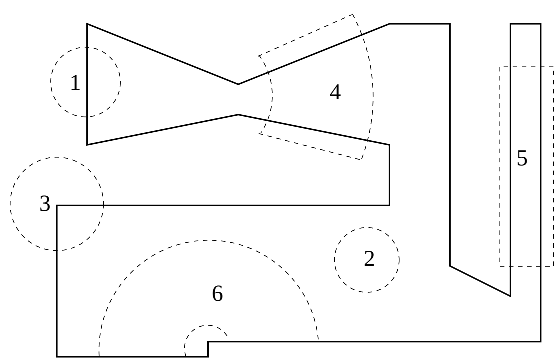
Can't solve Beltrami equation $g_{\bar{z}} = \mu g_z$ exactly in finite time, but can quickly solve

$$g_{\bar{z}} = (\mu + O(\|\mu\|^2))g_z.$$

Then $f \circ g^{-1}$ is $(1 + C\|\mu\|^2)$-QC.

Cut $\mathbb{H}$ into $O(n)$ pieces on which f, f^α or $\log f$ has nice series representation. Need $p = O(|\log \epsilon|)$ terms on each piece to get ϵ accuracy.



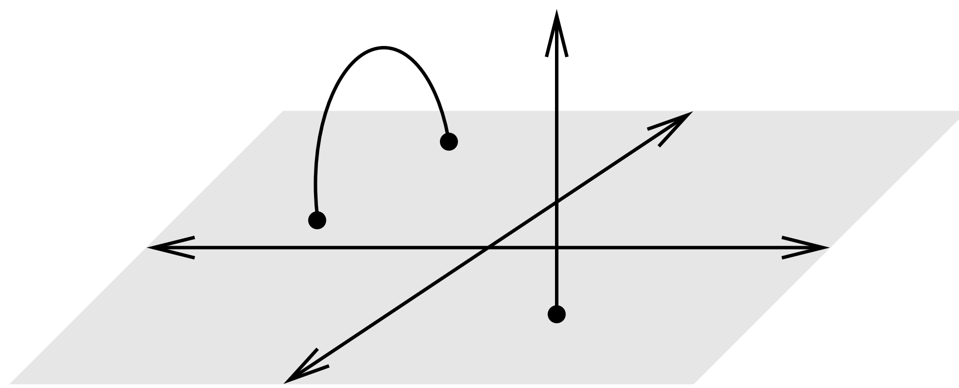


Use partition of unity supported near partition edges to combine expansions. Can compute μ explicitly. Use fast multipole method to approximately solve in time $O(n)$.

Hyperbolic space: Metric on $\mathbb{R}^3_+$,

$$d\rho = |dz|/\mathrm{dist}(z, \mathbb{R}^2).$$

Geodesics are circles or lines orthogonal to $\mathbb{R}^2$.

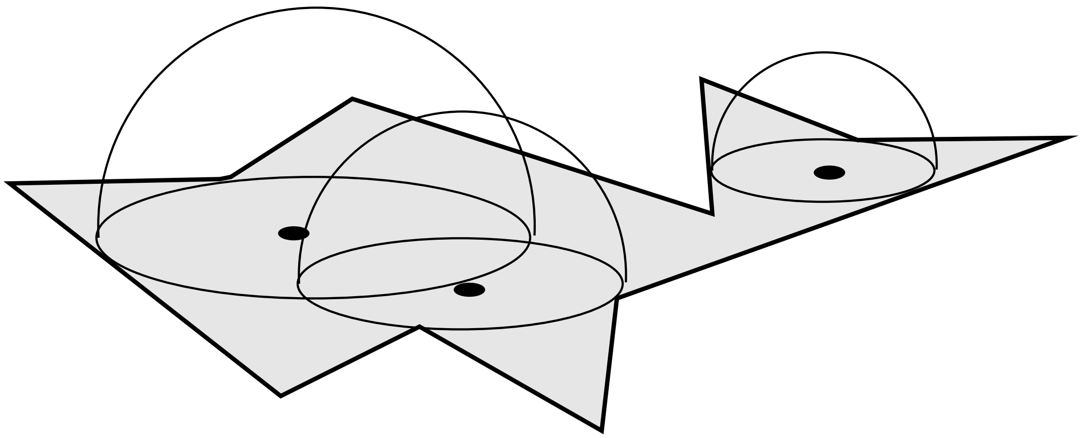


The hyperbolic metric on the disk or ball is

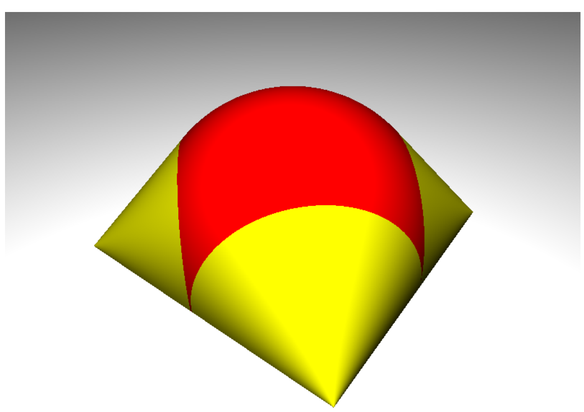
$$d\rho = 2|dz|/(1 - |z|^2).$$

The hyperbolic metric on a simply connected domain plane Ω is defined by transferring the metric on the disk by the Riemann map.

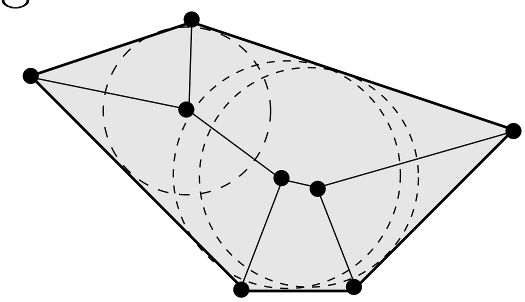
The **dome** of Ω is boundary of union of all hemispheres with bases contained in Ω.

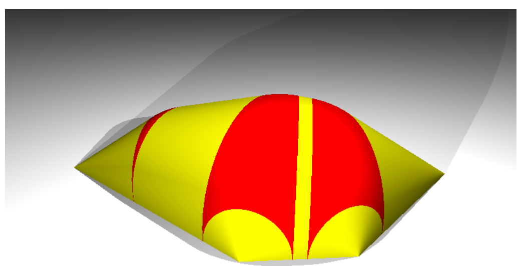


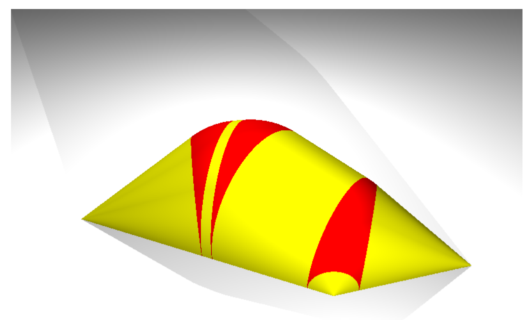
Equals boundary of hyperbolic convex hull of Ω^c. Similar to Euclidean space where complement of closed convex set is a union of half-spaces.



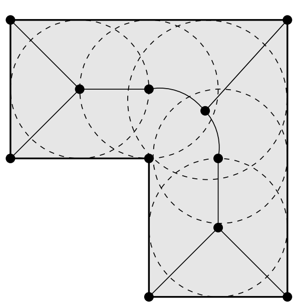
A convex polygon:

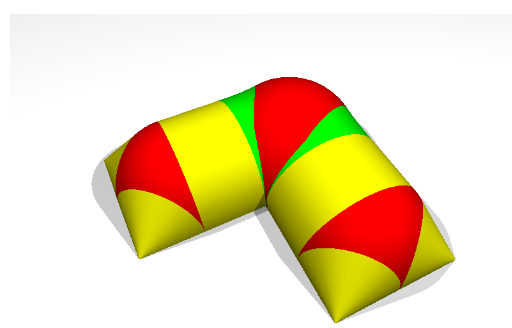






A non-convex polygon:





Each point on Dome(Ω) is on dome of a maximal disk D in Ω. Must have $|\partial D \cap \partial\Omega| \geq 2$. The centers of these disks form the **medial axis**.

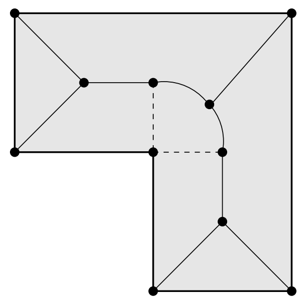
For polygons is a finite tree with 3 types of edges:
- point-point bisectors (straight)
- edge-edge bisectors (straight)
- point-edge bisector (parabolic arc)

For applications see:
www.ics.uci.edu/ eppstein/gina/medial.html+

In CS is attributed to Blum (1967), but Erdös proved dim(MA) = 1 in 1945.

Goggle("medial axis")= 26,300
Goggle("hyperbolic convex hull")= 71

Medial axis is boundary of Voronoi cells:

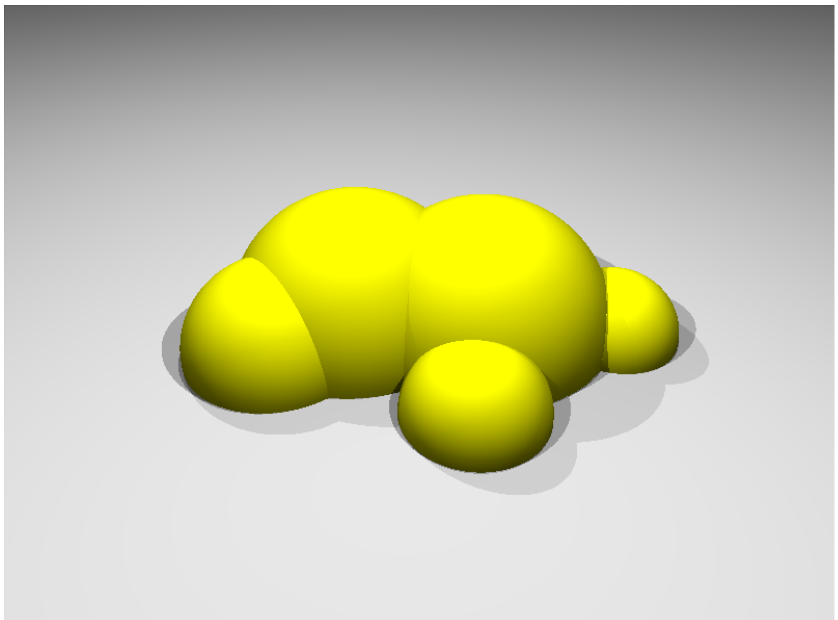


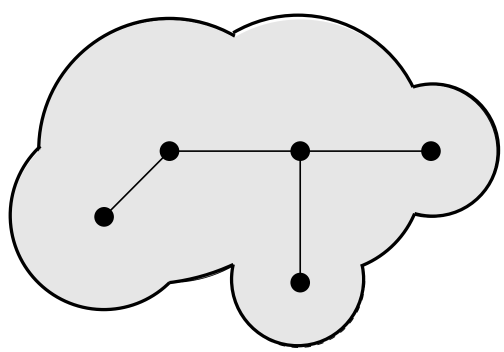
Chin-Snoeyink-Wang (1998) gave $O(n)$ algorithm. Uses Chazelle' theorem (1991): an n-gon can be triangulated in $O(n)$ time.

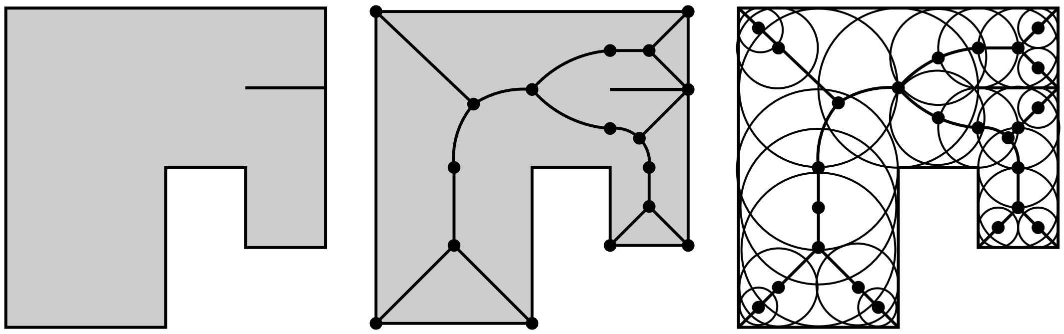
They use this to divide polygon into almost convex regions ("monotone histograms"); compute for each piece (Aggarwal-Guibas-Saxe-Shor, 1989) and merge results.

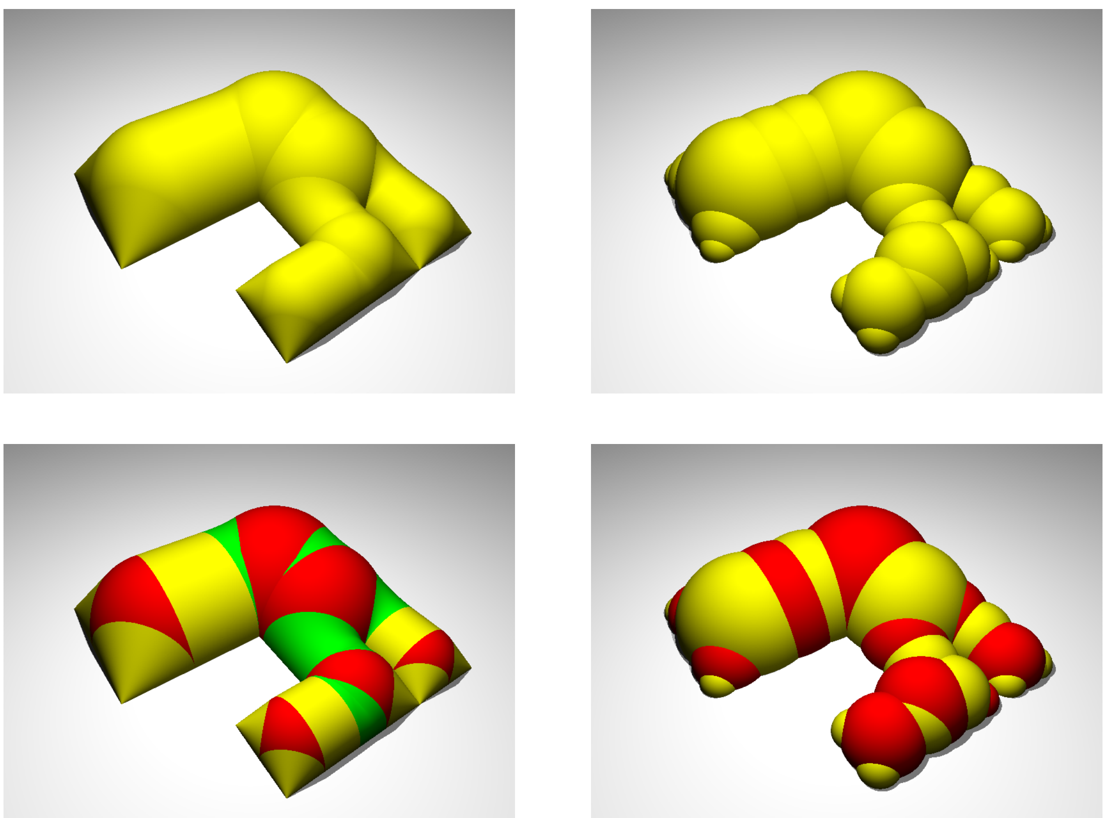
Merge Lemma: Suppose n sites $S = S_1 \cup S_2$ are divided by a line. Then diagram for S can be built from diagrams for S_1, S_2 in time $O(n)$.

Finitely bent domain (= finite union of disks).





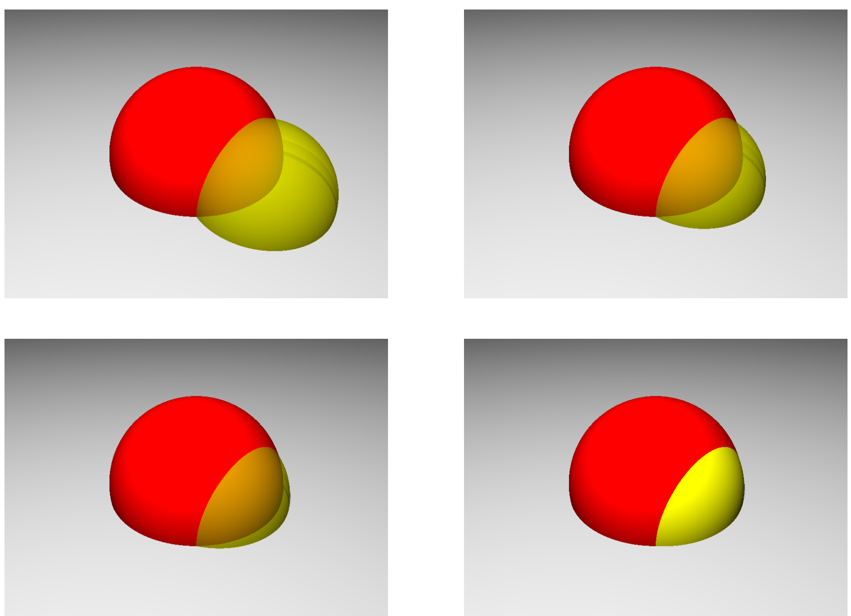




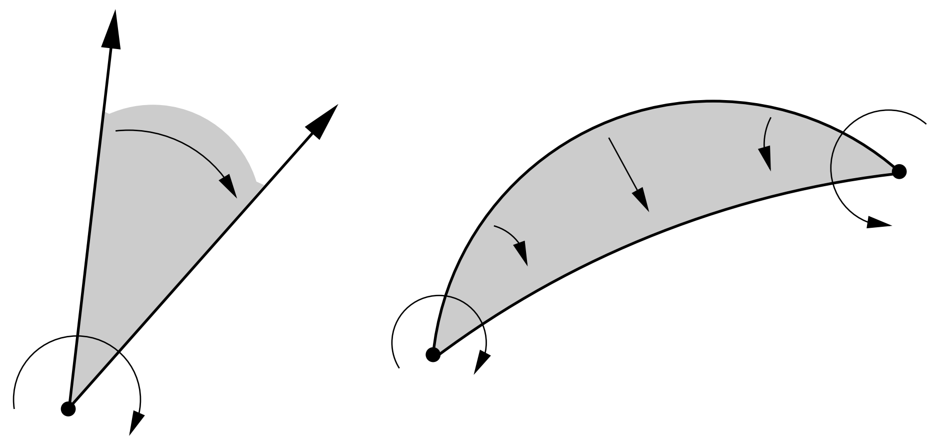
Let ρ_S be the hyperbolic path metric on S.

Theorem (Thurston): There is an isometry ι from (S, ρ_S) to the hyperbolic disk.

For finitely bent domains rotate around each bending geodesic by an isometry to remove the bending (more obvious if vertices are 0 and ∞).

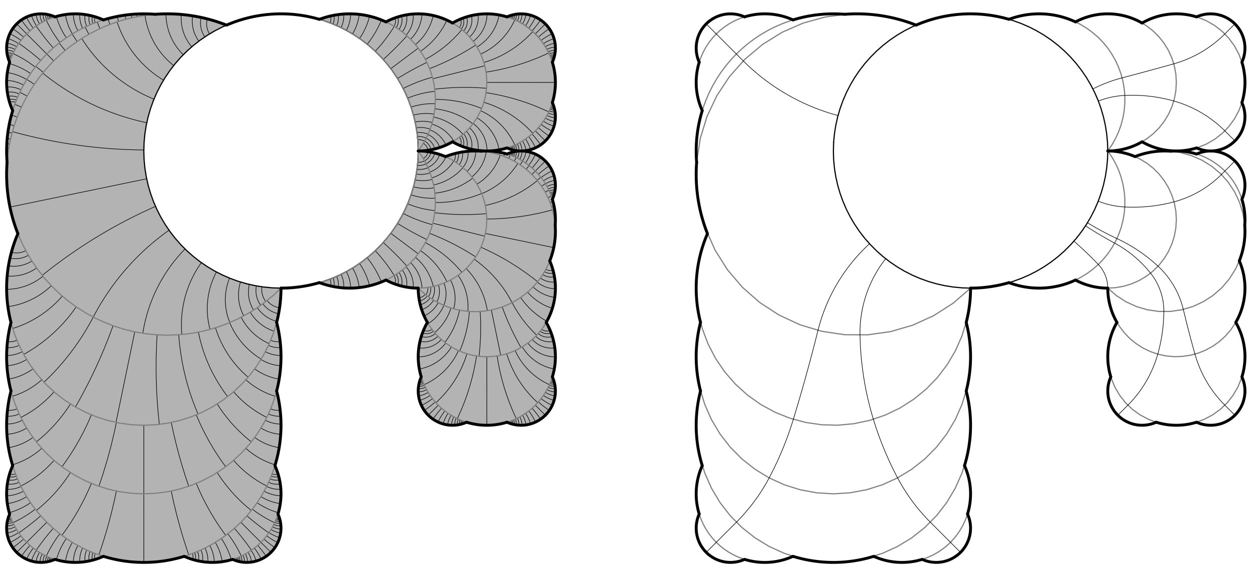


Elliptic Möbius transformation is conjugate to a rotation.



Elliptic transformation determined by fixed points and angle of rotation θ. It identifies sides of a crescent of angle θ: think of flow along circles orthogonal to boundary arcs.

Visualize ι as a flow: Write finitely bent Ω as a disk D and a union of crescents. Foliate crescents by orthogonal circles. Following leaves of foliation in $\Omega \setminus D$ gives $\iota : \partial\Omega \to \partial D$.



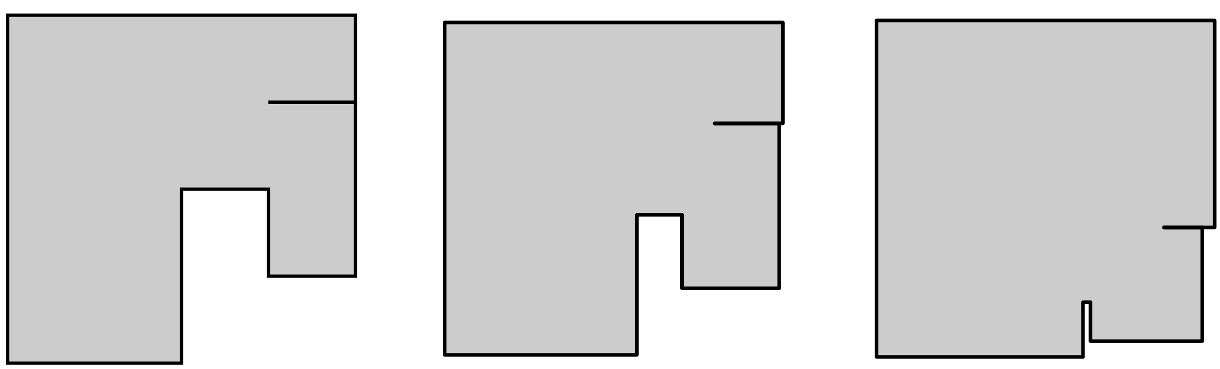
Has continuous extension to interior: identity on disk and collapses orthogonal arcs to points.

- ι has K-QC extension to interior.
- ι can be evaluated at n points in time $O(n)$.

The Schwarz-Christoffel formula gives the Riemann map onto a polygonal:

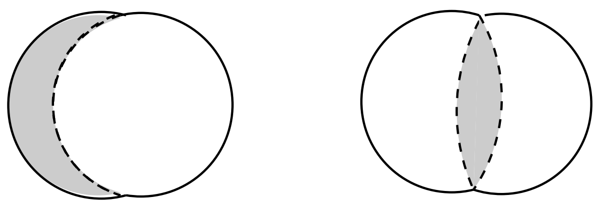
$$f(z) = A + C \int^{z} \prod_{k=1}^{n} (1 - \frac{w}{z_k})^{\alpha_k - 1} dw.$$

α's are known (interior angles) but z's are not (preimages of vertices).

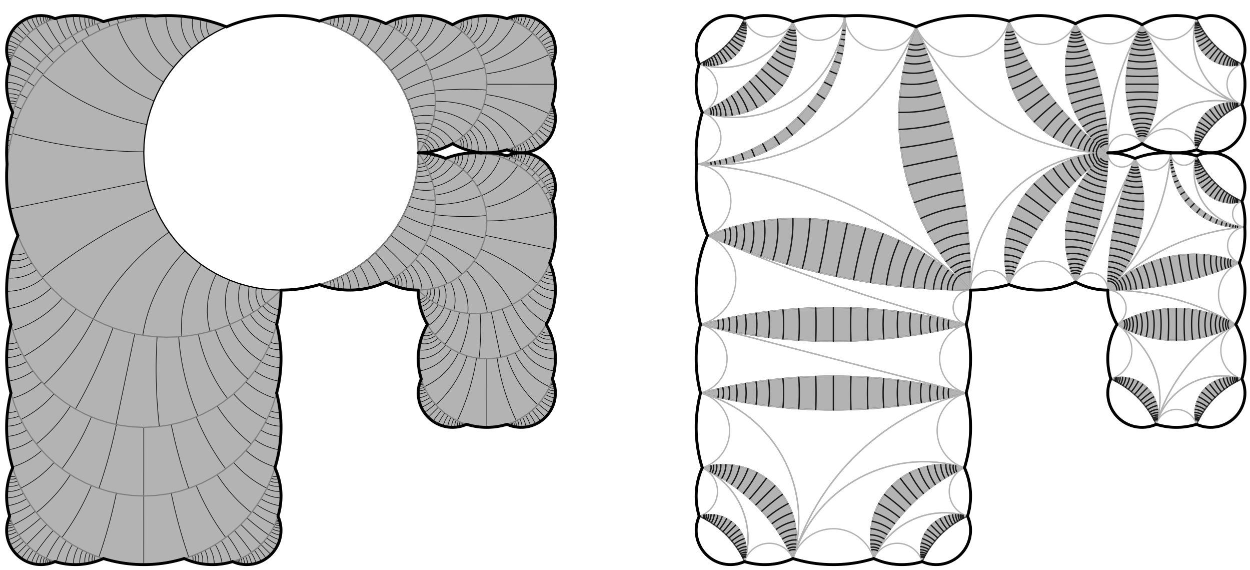


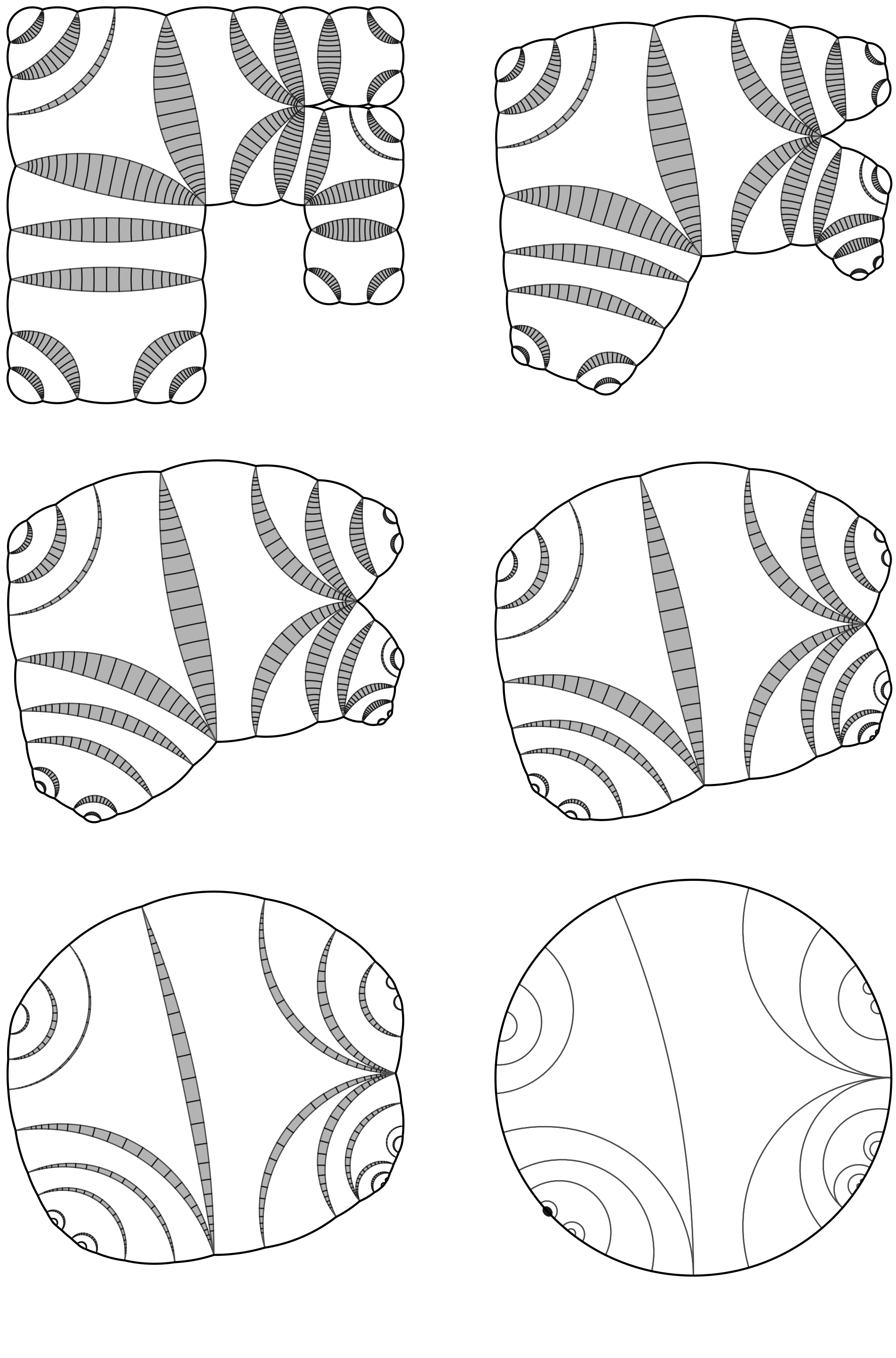
If we plug in ι-images of vertices we almost get the correct polygon (center). Using uniformly spaced points is clearly worse (right).

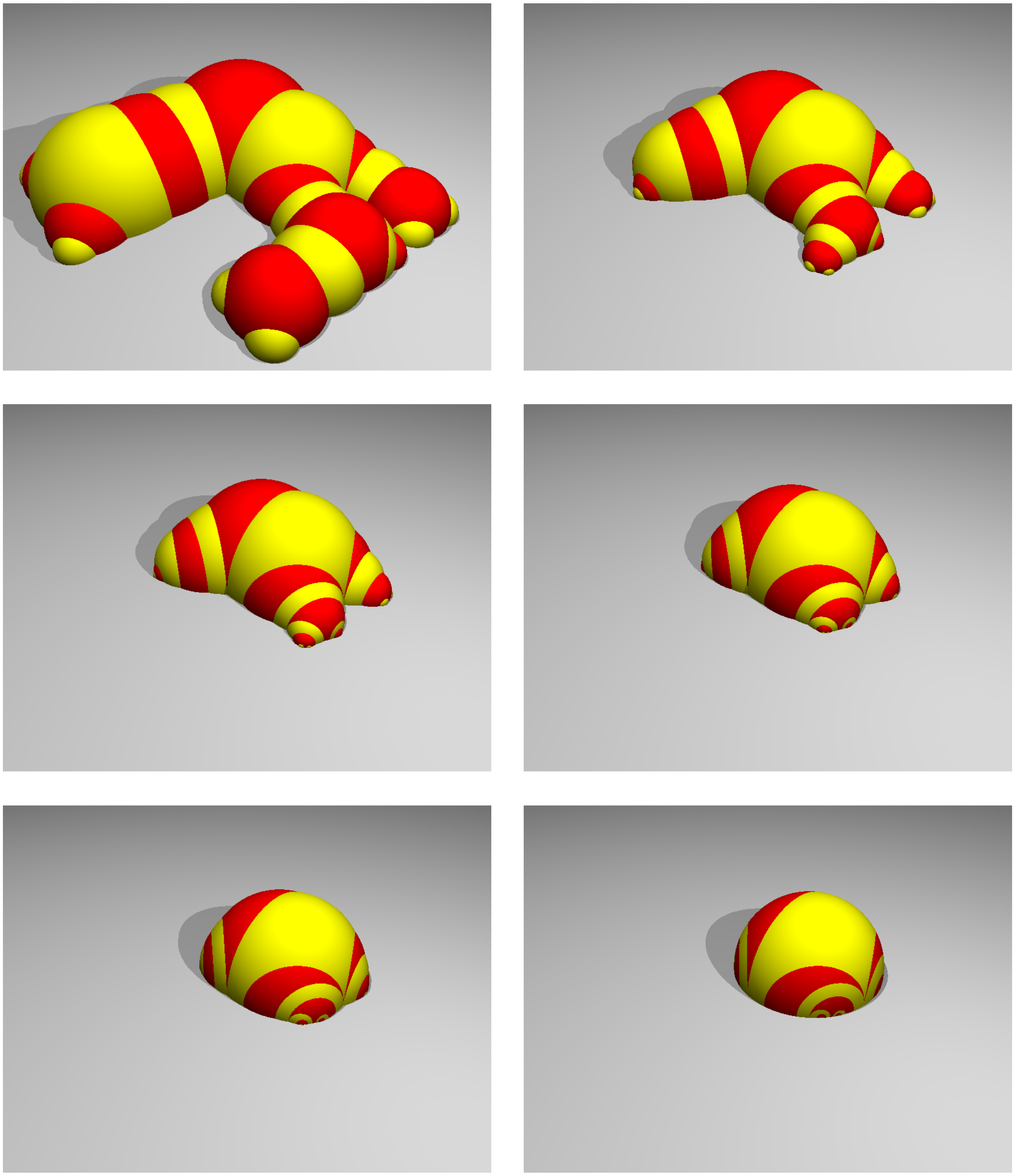
There are at least two ways to decompose a finite union of disks using crescents (with same angles and vertices in both cases).

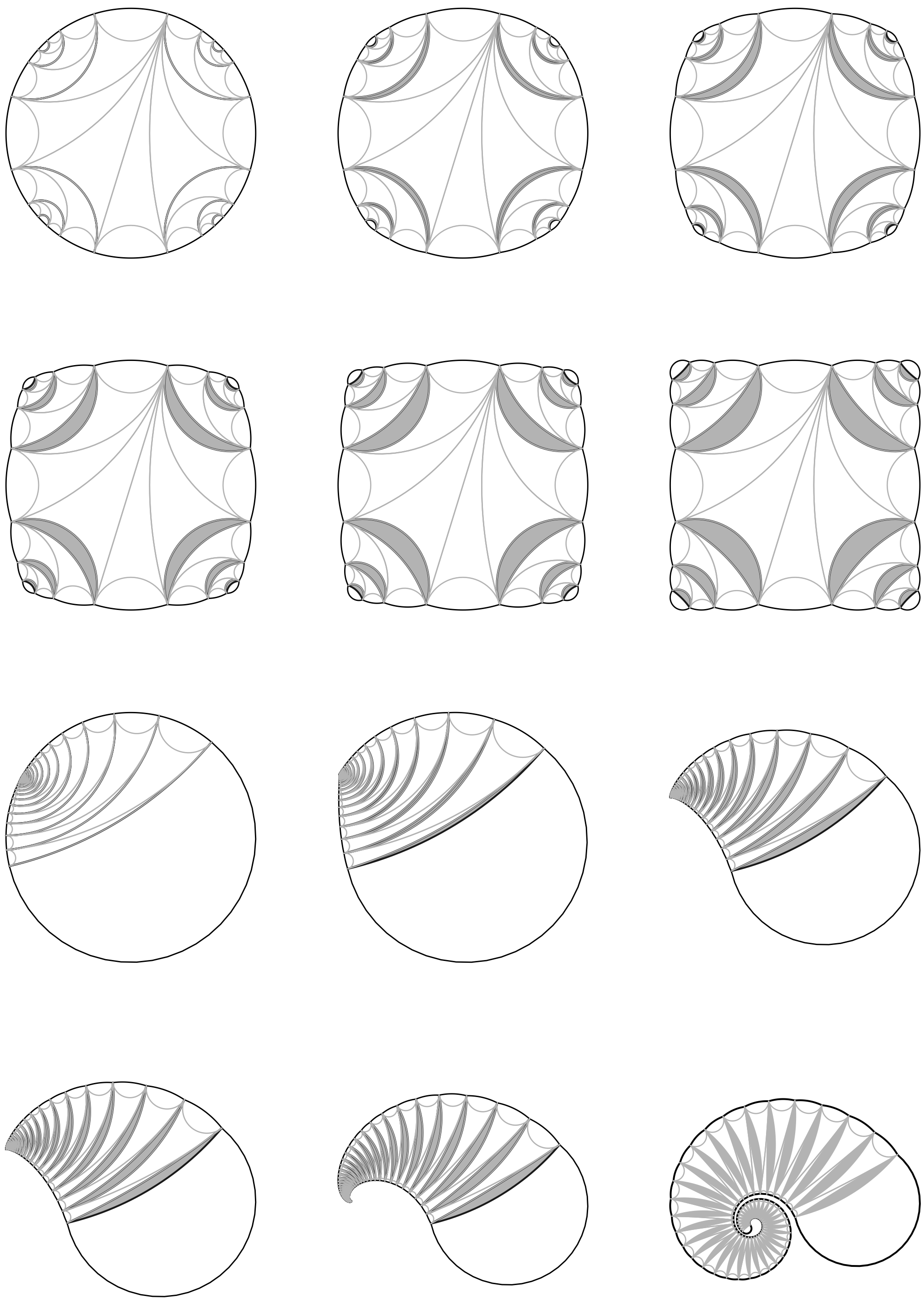


We call these **tangential** and **normal** crescents. A finitely bent domain can be decomposed with either kind of crescent.

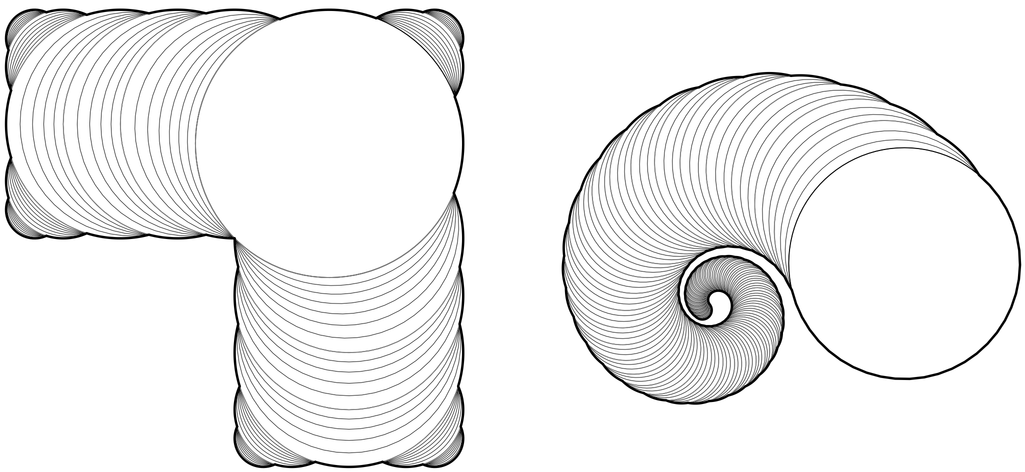




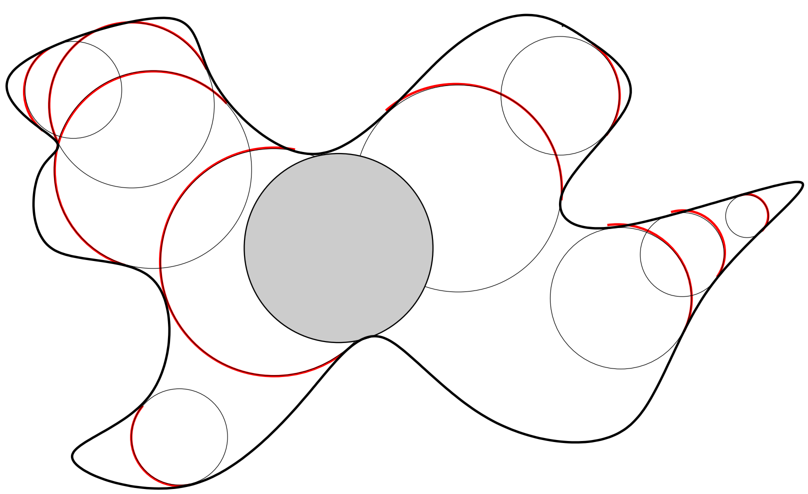




Foliate $\Omega \setminus D$ by arcs of medial axis disks and follow orthogonal flow:



Medial axis foliation and orthogonal flow make sense for any simply connected domain.



Theorem: Collapsing normal crescents gives hyperbolic quasi-isometry $R : \Omega \to \mathbb{D}$.

Corollary: ι has a K-QC extension to interior.

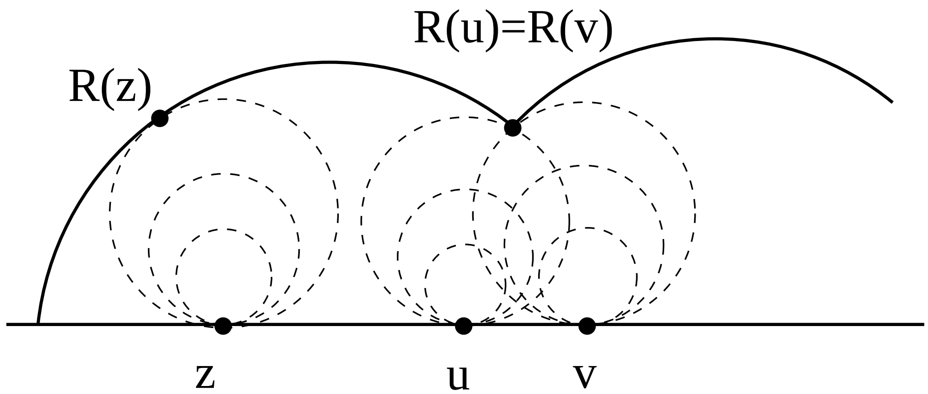
Corollary (Sullivan, Epstein-Marden): There is a K-QC map $\sigma : \Omega \to S_\Omega$ so that $\sigma = \mathrm{Id}$ on $\partial\Omega = \partial S$.

Result comes from hyperbolic 3-manifolds. If Ω is invariant under Möbius group G, $M = \mathbb{R}^3_+/G$ is hyperbolic manifold,

$$\partial_\infty M = \Omega/G, \quad \partial C(M) = \mathrm{Dome}(\Omega)/G.$$

Thurston conjectured $K = 2$ is possible. Best known upper bound is $K < 7.82$.

Nearest point retraction $R : \Omega \to \mathrm{Dome}(\Omega)$**:** Expand ball tangent at $z \in \Omega$ until it hits a point $R(z)$ of the dome.



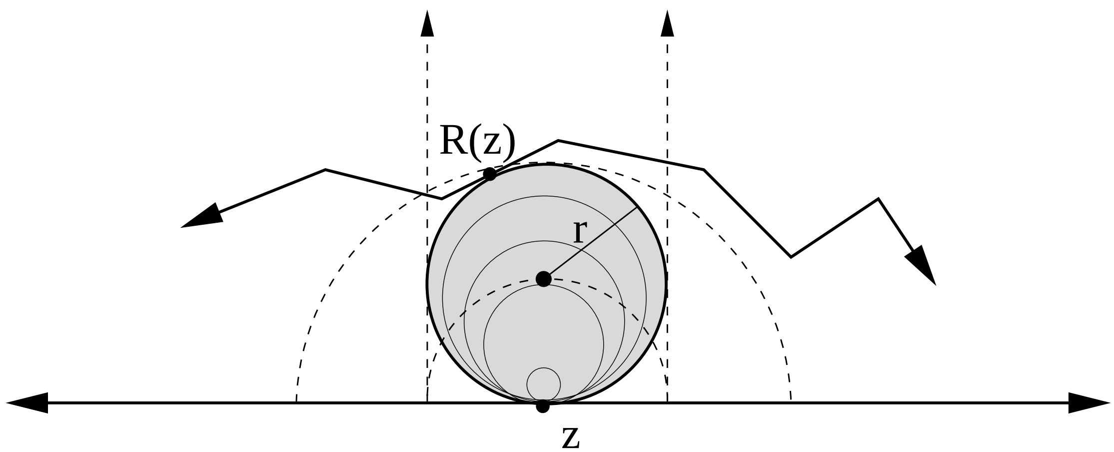
$$\text{normal crescents} = R^{-1}(\text{bending lines})$$
$$\text{gaps} = R^{-1}(\text{faces})$$

collapsing crescents = nearest point retraction

Suffices to show nearest point retraction is a quasi-isometry. This follows from three easy facts.

Fact 1: If $z \in \Omega$, $\infty \notin \Omega$,

$$r \simeq \mathrm{dist}(z, \partial\Omega) \simeq \mathrm{dist}(R(z), \mathbb{R}^2) \simeq |z - R(z)|.$$



Fact 2: R is Lipschitz.

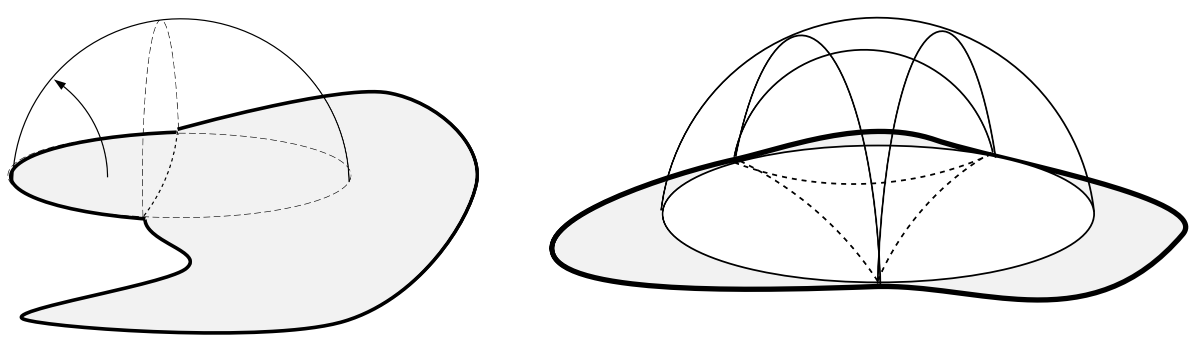
Ω simply connected $\Rightarrow$

$$d\rho \simeq \frac{|dz|}{\mathrm{dist}(z, \partial\Omega)}.$$

$z \in D \subset \Omega$ and $R(z) \in \mathrm{Dome(D)} \Rightarrow$

$$\mathrm{dist}(z, \partial\Omega)/\sqrt{2} \leq \mathrm{dist}(z, \partial D) \leq \mathrm{dist}(z, \partial\Omega)$$

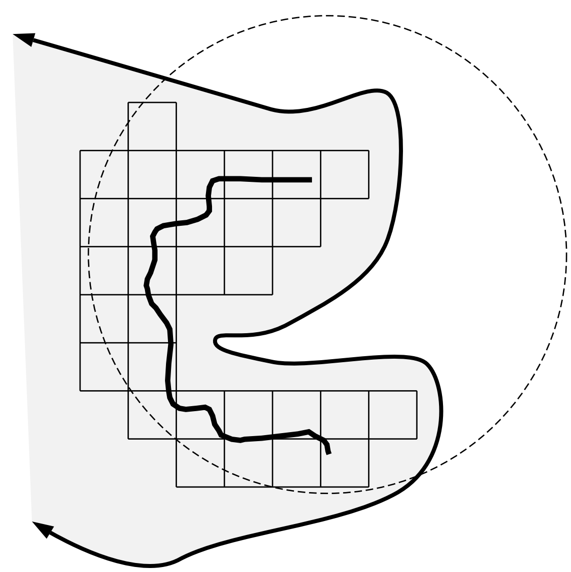
$$\Rightarrow \quad \rho_\Omega(z) \simeq \rho_D(z) = \rho_{\mathrm{Dome}}(R(z)).$$



Fact 3: $\rho_S(R(z), R(w)) \leq 1 \Rightarrow \rho_\Omega(z, w) \leq C$.

Suppose $\operatorname{dist}(R(z), \mathbb{R}^2) = r$ and γ is geodesic from z to w.

$$
\begin{aligned}
\Rightarrow \quad & \operatorname{dist}(\gamma, \mathbb{R}^2) \simeq r \\
\Rightarrow \quad & \operatorname{dist}(R^{-1}(\gamma), \partial\Omega) \simeq r, \\
& \quad R^{-1}(\gamma) \subset D(z, Cr) \\
\Rightarrow \quad & \rho_\Omega(z, w) \leq C
\end{aligned}
$$

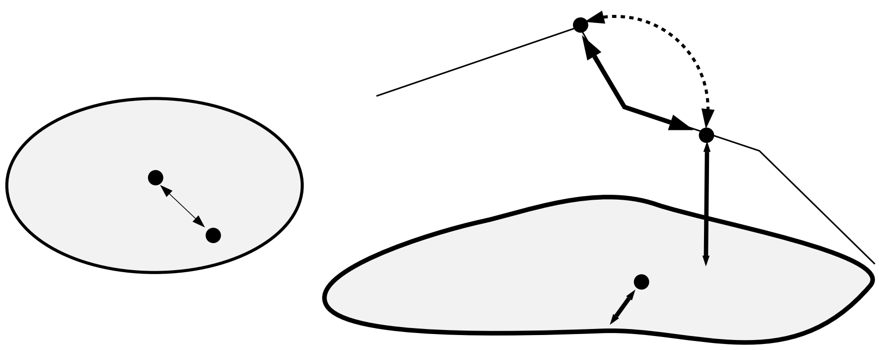


Moreover, $g = \iota \circ \sigma : \Omega \to \mathbb{D}$ is locally Lipschitz. Standard estimates show

$$|g'(z)| \simeq \frac{\text{dist}(g(z), \partial\mathbb{D})}{\text{dist}(z, \partial\Omega)}.$$

Use Fact 1

$$\begin{aligned}
\text{dist}(z, \partial\Omega) &\simeq \text{dist}(\sigma(z), \mathbb{R}^2) \\
&\simeq \exp(-\rho_{\mathbb{R}^3_+}(\sigma(z), z_0)) \\
&\gtrsim \exp(-\rho_S(\sigma(z), z_0)) \\
&= \exp(-\rho_D(g(z), 0)) \\
&\simeq \text{dist}(g(z), \partial D)
\end{aligned}$$



Corollary: Every simply connected domain can be mapped to the disk by a QC Lipschitz homeomorphism (w.r.t. internal path metric).

Corollary: Any quasicircle can be mapped to circle by Lipschitz QC mapping of plane.

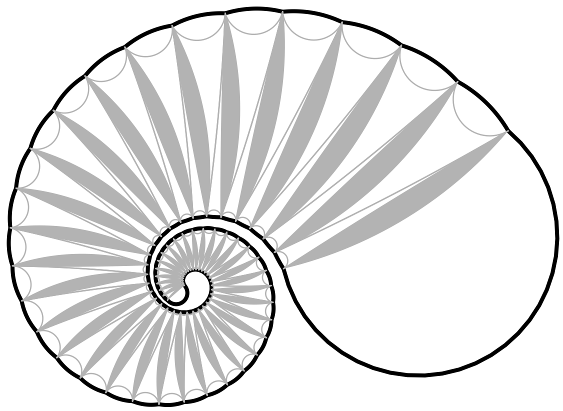
Corollary: $f : D \to \Omega$ conformal implies $f = g \circ h$ where $h : \mathbb{D} \to \mathbb{D}$ is K-QC and $|g'| > \epsilon > 0$.

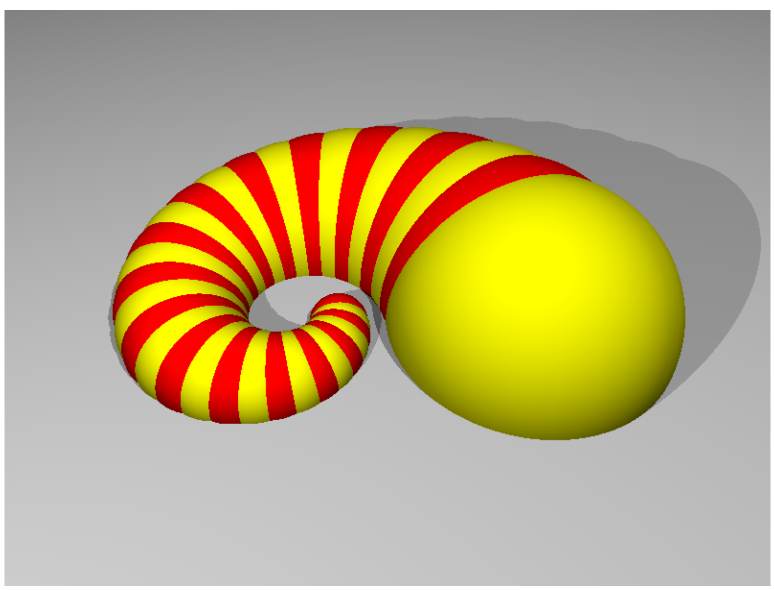
Indeed $|g'(tz)| \leq C|g'(z)|$, $0 \leq t \leq 1$

Astala $\Rightarrow$ if h is 2-QC then $h' \in \text{weak} - \mathrm{L}^4$.

Corollary: $K = 2 \Rightarrow$ Brennan's conjecture.

But, Epstein and Markovic showed $K > 2.1$ for some log spirals.





Still some avenues for further investigation. Write

$$g : \mathbb{D} \to \Omega, \quad h : \mathbb{D} \to \mathbb{D}, \quad g \circ h^{-1} \text{conformal}$$

• g' not just bounded below, but tends to ∞ where h' goes to 0.

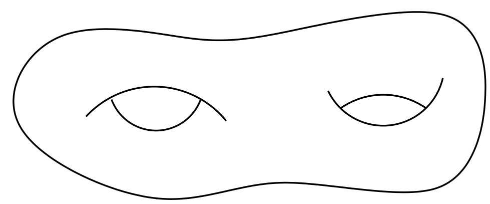
• Use path of domains. Estimate derivative of weak L^4 norm along path. Remains finite until $t = 1$?

• Want to show $\int_{|h'|>\lambda|g'|} |h'|^2 dxdy < C/\lambda^2$.

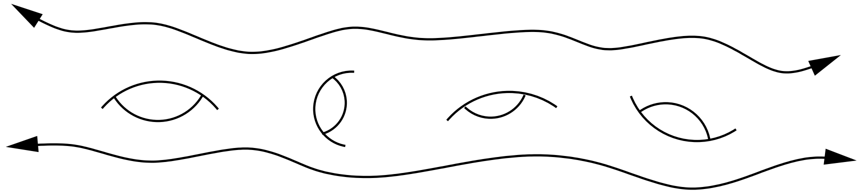
• g and h are solutions of Beltrami equation which are orthogonal in some sense (h moves tangential to unit circle, g moves normal to to circle). Is there analog to estimates for Hilbert transform?

Application to Kleinian groups:

Bowen's Dichotomy: If Ω is simply connected and $R = \Omega/G$ is compact Riemann surface then either $\partial\Omega =$ circle, or $\dim(\partial\Omega) > 1$.



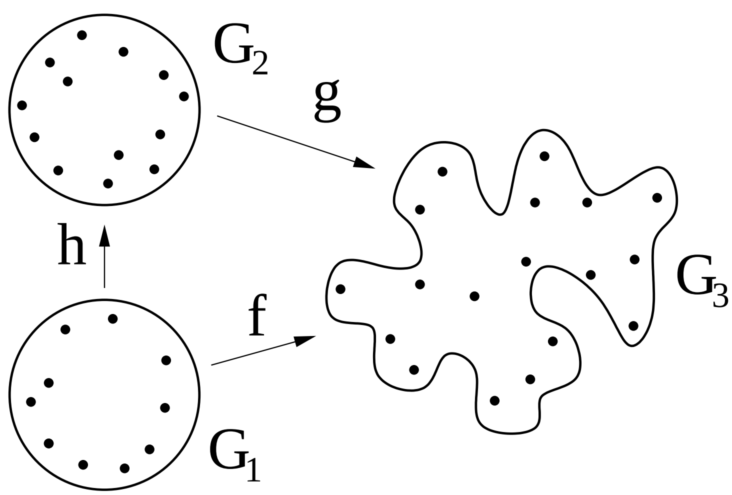
Finite area case by Sullivan (also see Bridgeman-Taylor, Bishop-Jones). Astala and Zinsmeister showed this is false if R has a Green's function. Is it true for other surfaces?



Theorem: If Ω/G has no Green's function then either $\partial\Omega =$ circle or $\dim(\partial\Omega) > 1$.

Want $s > 1$ s.t. $\sum_{g \in G} \mathrm{dist}(g(z_0), \partial\Omega)^s = \infty$. Hard part is to show for $s = 1$.

- No Green's function $\Rightarrow$ G_1-sum $= \infty$
- h quasiconformal $\Rightarrow$ G_2-sum $= \infty$
- $|g'|$ bounded below $\Rightarrow$ G-sum $= \infty$



$f =$ conformal, $g =$ expanding, $h =$ QC

I wouldn't even think of playing music if I was born in these times... I'd probably turn to something like mathematics. That would interest me.

Bob Dylan, 2005

"Ah!" replied Pooh. He'd found that pretending a thing was understood was sometimes very close to actually understanding it. Then it could easily be forgotton with no one the wiser...

Winnie-the-Pooh