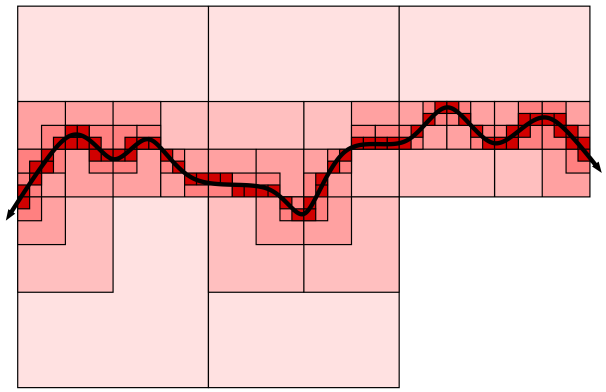
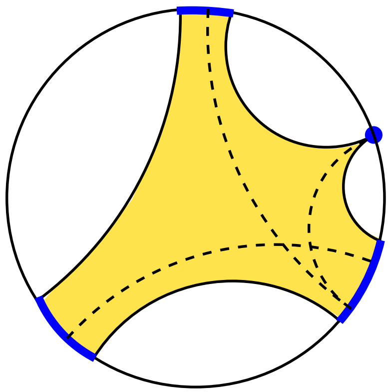
MAT 638, FALL 2020, Stony Brook University

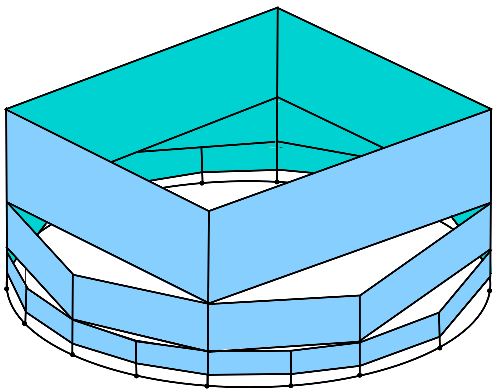
TOPICS IN REAL ANALYSIS

WEIL-PETERSSON CURVES, TRAVELING SALESMAN THEOREMS AND MINIMAL SURFACES

Christopher Bishop, Stony Brook







1. Introduction and overview

Weil-Petersson curves are rectifiable quasicircles that are related to many ideas:

- Teichmüller theory
- Sobolev spaces
- Geometric measure theory (Jones's β-numbers)
- Polygonal approximations
- Conformal mappings
- Quasiconformal maps
- Minimal surfaces
- Isoperimetric inequalities
- Renormalized area
- Integral Geometry
- Schramm-Loewner Evolutions (SLE)
- Brownian loop soup

The same curves were studied by Guo [33] and Cui [18] using the terms

“integrable Teichmüller space of degree 2”

“integrably asymptotic affine maps” respectively.

The name “Weil-Petersson class” is more common in recent papers and comes from work of Takhtajan and Teo [65] defining a Weil-Petersson metric on universal Teichmüller space.

Motivated by string theory, they wanted to put a Riemannian metric on space of smooth closed curves, i.e., the space of diffeomorphisms of the unit circle $\mathbb{T}$ into the complex numbers $\mathbb{C}$.

This means thinking of the space of closed curves as an infinite dimensional manifold and putting a Hilbert space norm on the tangent space at each point. This is not my point of view and I won’t discuss it in great detail.

Their metric is defined on all quasicircles, including non-smooth fractals. Weil-Petersson class is the closure of smooth curves in their metric.

Their metric space is disconnected. WP-class is connected component containing circle (and all smooth curves).

Need for metric on smooth closed curves also arises in computer vision and pattern recognition. The Weil-Petersson metric is used in David Mumford's approach to computer vision: how similar or dissimilar are two shapes? How do we morph one shape to another most efficiently? Use geodesics of Weil-Petersson metric.

For example, see the papers of Sharon and Mumford [62], Feiszli, Kushnarev and Leonard [23], and Feiszli and Narayan [24].

Some personal history.

In December of 2017, Mumford asked me to try to geometrically characterize elements of Weil-Petersson class.

This problem also stated in book of Takhtajan and Teo. I looked at the book, but did not see a way into the problem.

At the time I was busy with other things, but one year later (January of 2019) I attended a workshop on the geometry of random sets at IPAM (UCLA). Yilin Wang gave two talks: some known characterizations of WP class were listed at end of first talk. Material from the Takhtajan-Teo book that I had missed, but were similar to some of my own earlier work in 1990's with Peter Jones.

Yilin Wang's IPAM talk

Wang's talk gave me a way into the problem. That week I formulated a conjecture in terms of Peter Jones's β-numbers and proved it within a few weeks. Found several more conditions, some involving hyperbolic geometry and minimal surfaces.

During corona virus lockdown in Spring 2020, I had time to think about WP curves and was able to answer some more questions. I also realized that many of my characterizations make sense for curves in higher dimensions and are still equivalent.

Email conversations with David Mumford were very helpful: he observed connection to Sobolev smoothness of parameterizations.

David Mumford = 1974 Fields medalist in algebraic geometry. Later moved into in computer vision and pattern recognition using tools from conformal analysis, computational geometry and Riemmanian manifolds.

He has many interesting ideas, e.g., use information theory and entropy to measure the complexity of a curve.

Wikipedia page for David Mumford

Are there more characterizations of WP curves to be found?

Are WP curves in higher dimensions interesting, i.e., related to other known mathematical objects or ideas?

What about analogs of Weil-Petersson curves in Hilbert space? Other metric spaces?

What is a Weil-Petersson surface? Interesting or not?

Definition of WP class was motivated by string theory. We shall see WP curves are also related to idea of renormalized area, a concept coming from physics. Are these two things related?

Plan for lectures:

- Quick overview of results.
- Basic definitions: quasicircles, Dirichlet class, ...
- Known characterizations of WP class using conformal and QC maps
- How this relates to earlier work from 1990's
- Discuss some new characterizations, some easy proofs.
- Harder proofs. Start with β-numbers and traveling salesman theorem.

Definition	Description
1	$\log f'$ in Dirichlet class
2	Schwarzian derivative
3	QC dilatation in L^2
4	conformal welding midpoints
5	$\exp(i \log f')$ in $H^{1/2}$
6	arclength parameterization in $H^{3/2}$
7	tangents in $H^{1/2}$
8	finite Möbius energy
9	Jones conjecture
10	good polygonal approximations
11	β^2-sum is finite
12	Menger curvature
13	biLipschitz involutions
14	between disjoint disks
15	thickness of convex hull
16	finite total curvature surface
17	minimal surface of finite curvature
18	additive isoperimetric bound
19	finite renormalized area
20	dyadic cylinder
21	closure of smooth curves in $T_0(1)$
22	P_φ^- is Hilbert-Schmidt
23	double hits by random lines
24	finite Loewner energy
25	large deviations of SLE(0^+)
26	Brownian loop measure

The names of 26 characterizations of Weil-Peterson curves

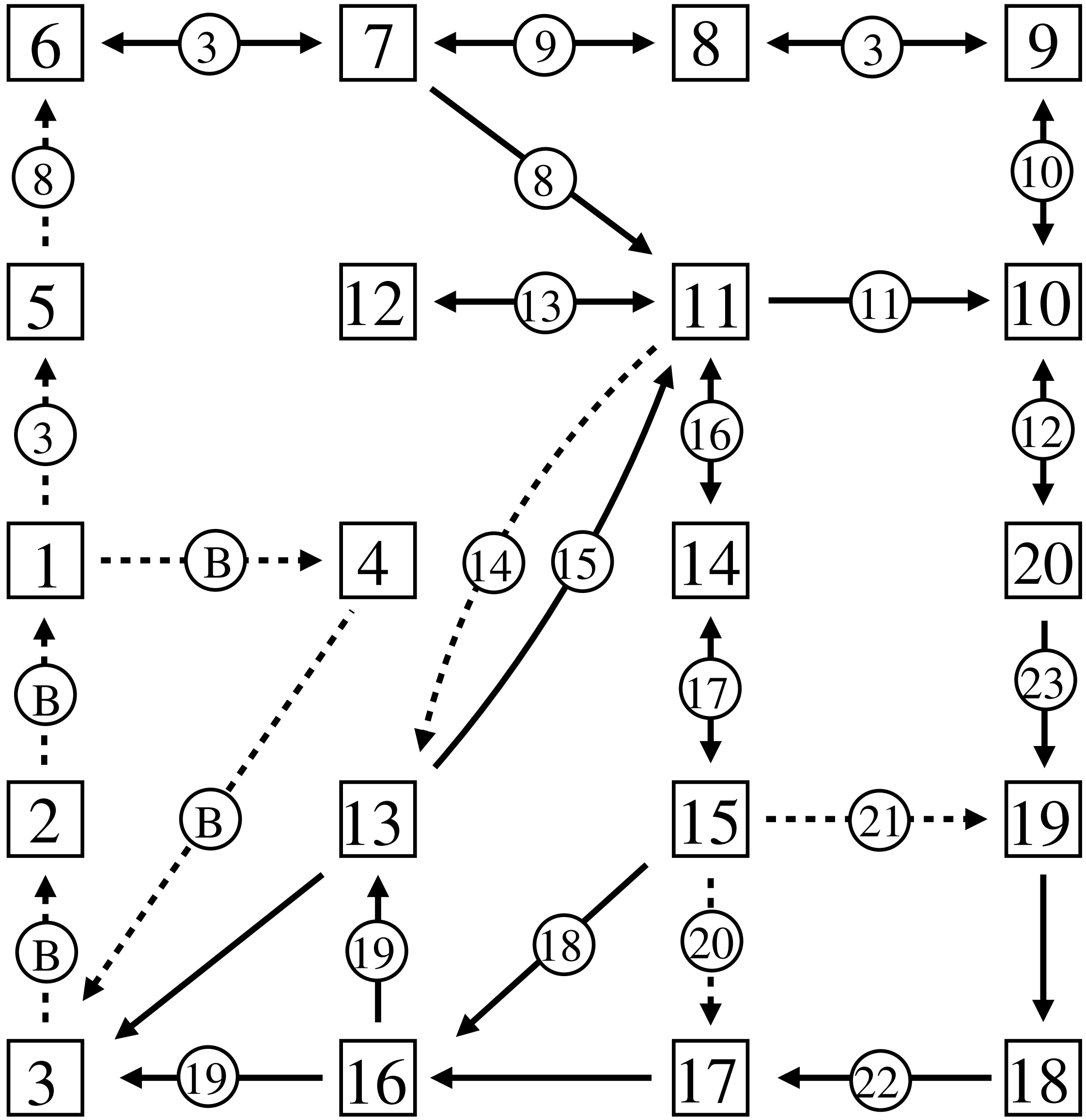
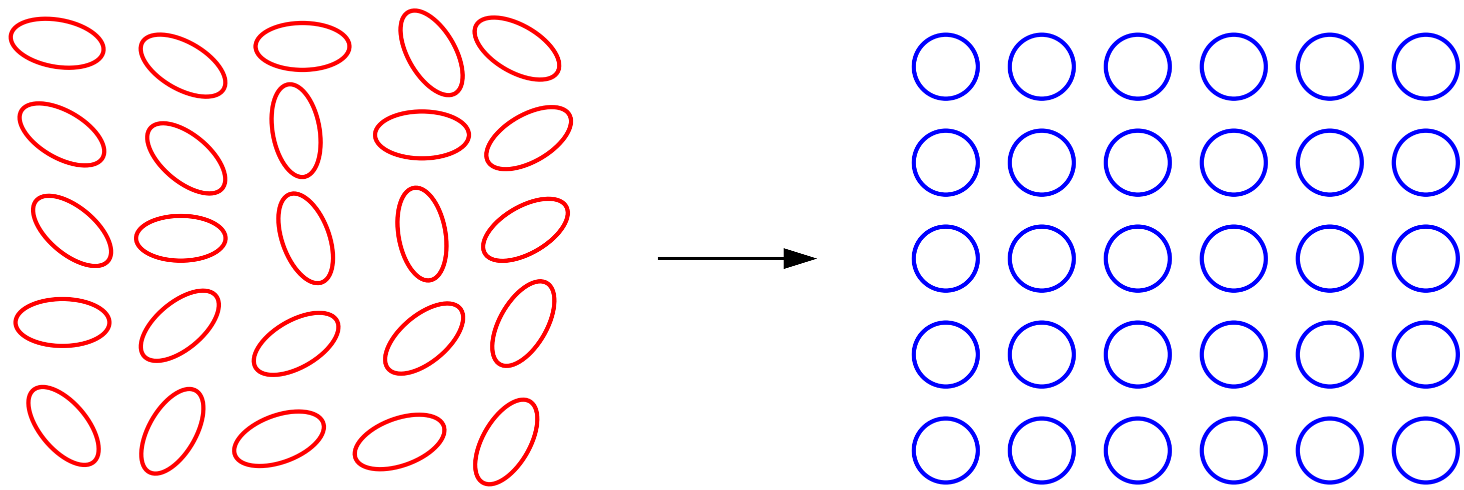


Diagram of implications between previous definitions.

Edge labels refer to sections of my preprint..

Thursday, August 27, 2020

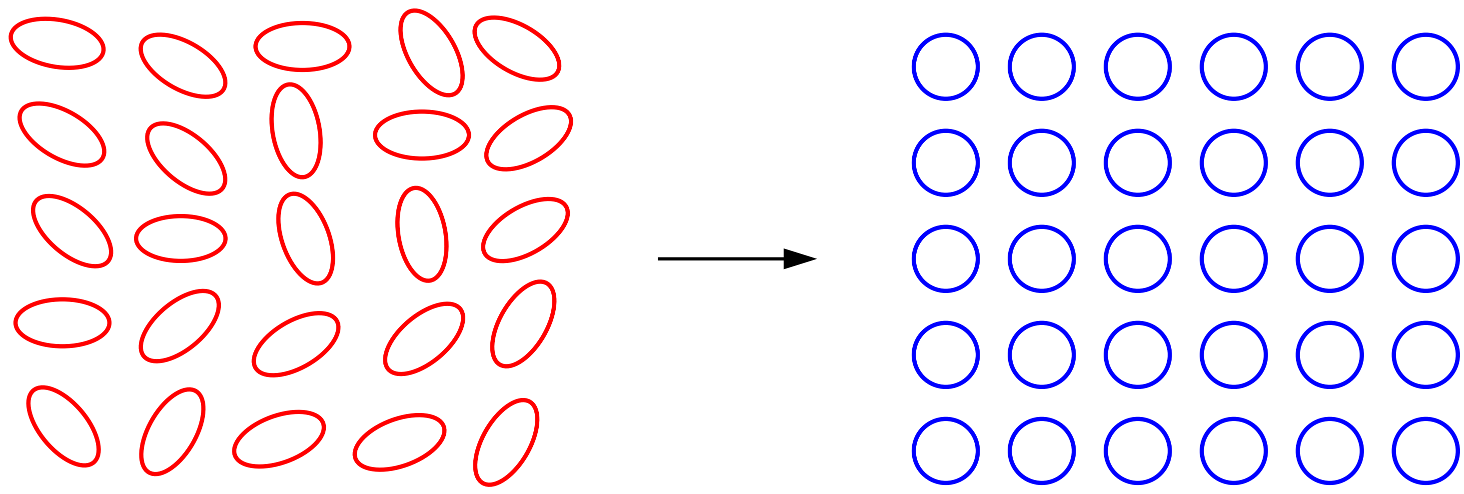
Quasiconformal (QC) maps send infinitesimal ellipses to circles.



Eccentricity = ratio of major to minor axis of ellipse.

For K-QC maps, ellipses have eccentricity $\leq K$

Quasiconformal (QC) maps send infinitesimal ellipses to circles.



Eccentricity = ratio of major to minor axis of ellipse.

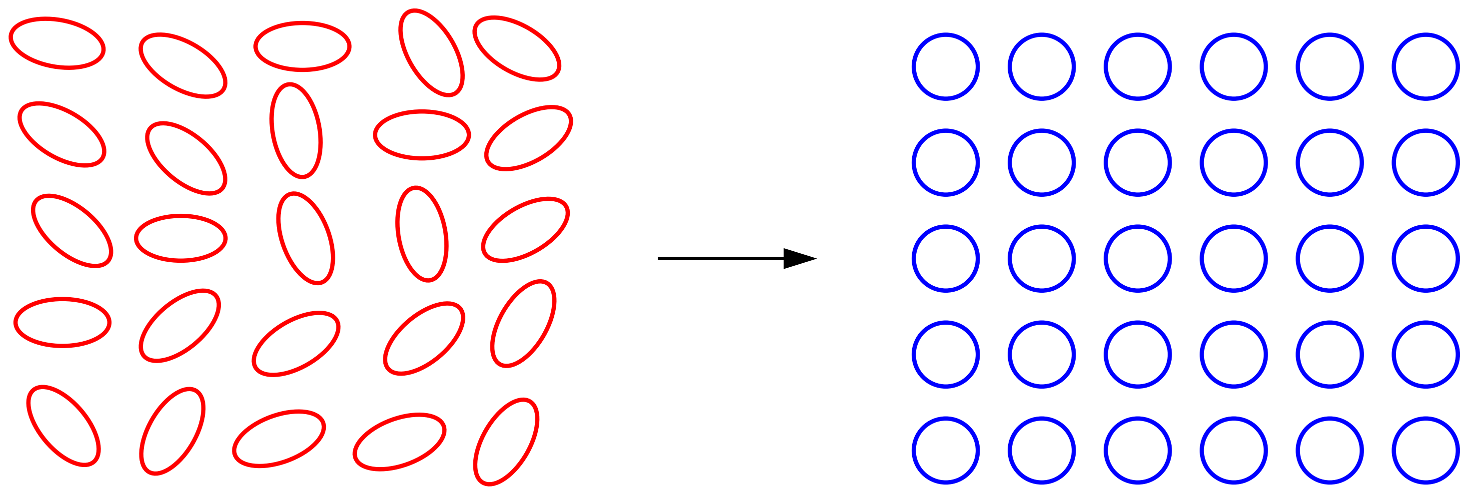
For K-QC maps, ellipses have eccentricity $\leq K$

Ellipses determined a.e. by measurable dilatation $\mu = f_{\bar{z}}/f_z$ with

$$|\mu| \leq \frac{K-1}{K+1} < 1.$$

Conversely, ...

Quasiconformal (QC) maps send infinitesimal ellipses to circles.



Mapping theorem: any such μ comes from some QC map f.

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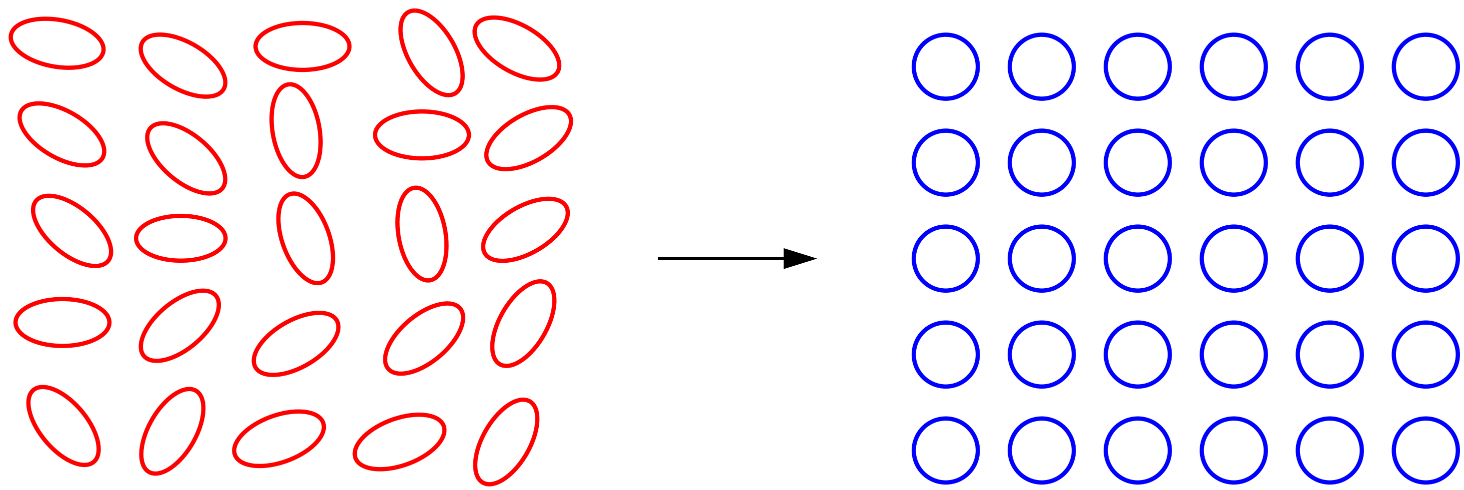


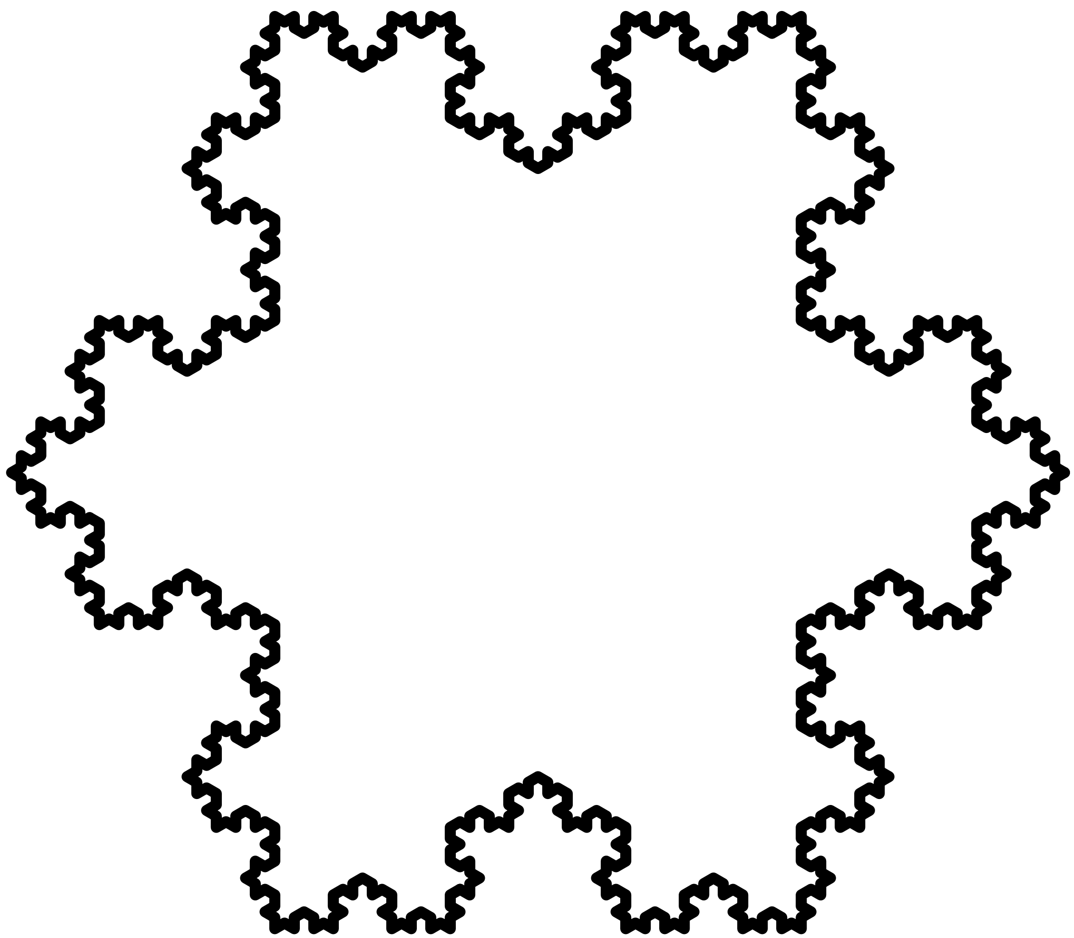
Mapping theorem: any such μ comes from some QC map f.

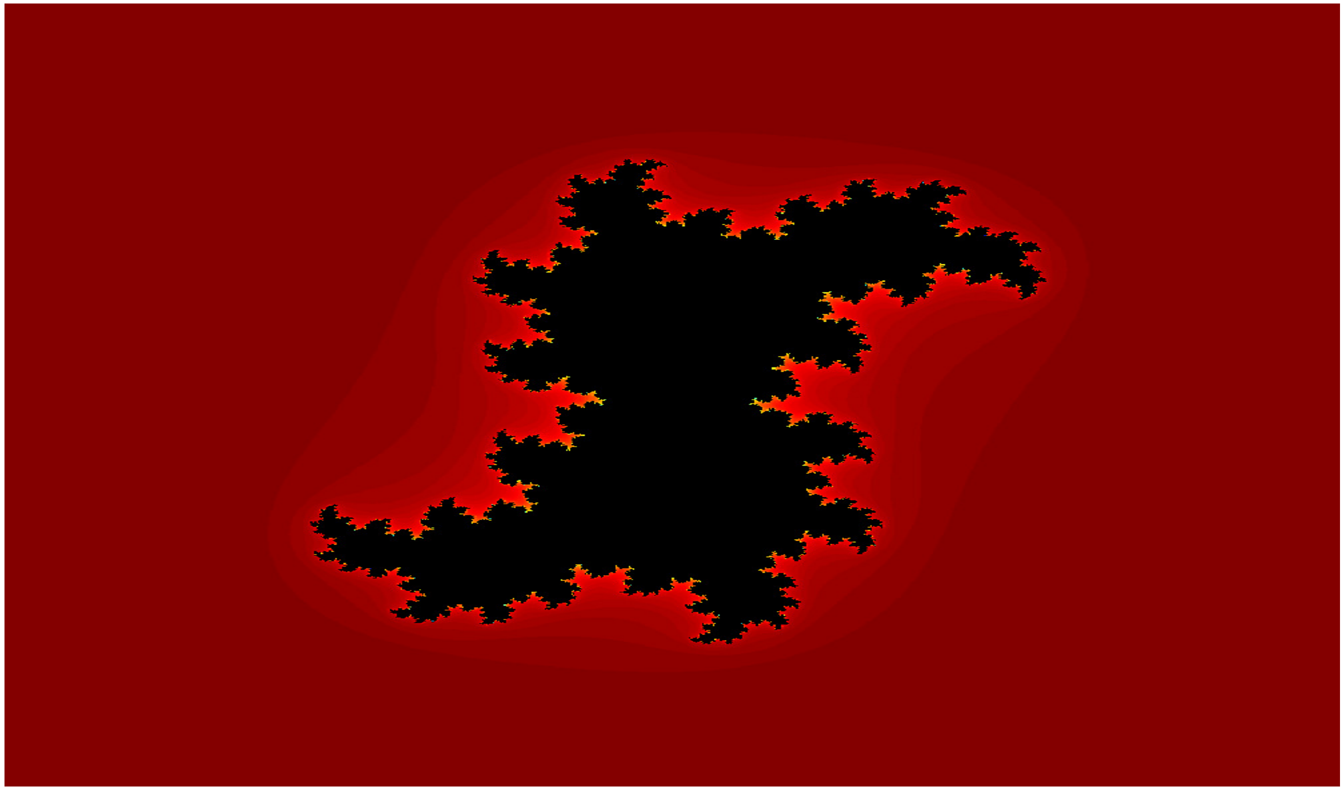
Cor: If f is holomorphic and ψ is QC, then there is a QC map φ so that $g = \psi \circ f \circ \varphi$ is also holomorphic.

A quasicircle is the image of the unit circle $\mathbb{T}$ under a quasiconformal mapping f of the plane, e.g., a homeomorphism of the plane that is conformal outside the unit disk $\mathbb{D}$, and whose dilatation $\mu = f_{\bar{z}}/f_z$ belongs to $\mathbb{B}_1^\infty$, the open unit ball in $L^\infty(\mathbb{D})$.

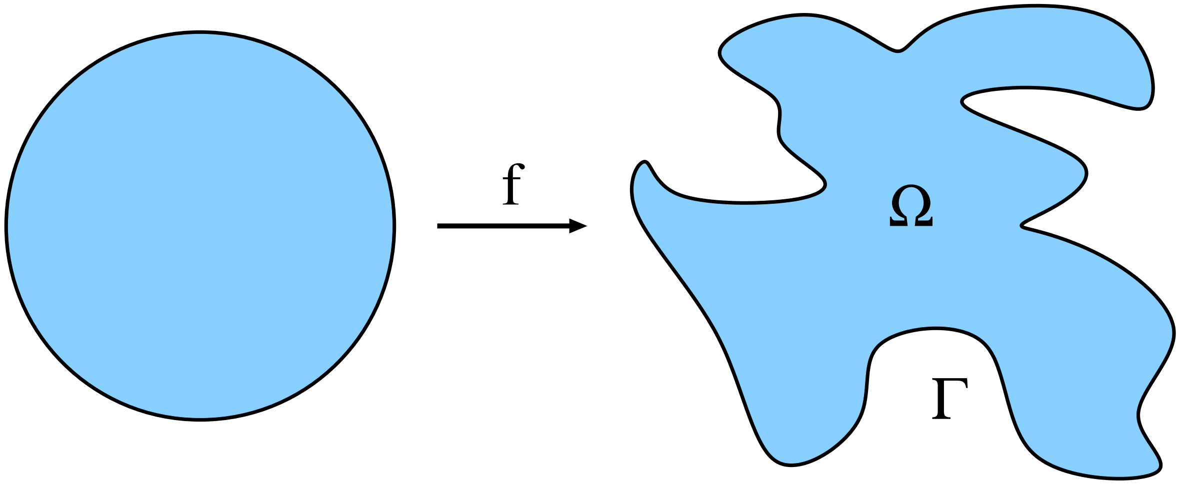
The collection of planar quasicircles corresponds to universal Teichmüller space $T(1)$ and the usual metric is defined in terms of $\|\mu\|_\infty$.

Any smooth curve is a quasicircle (diffeomorphism in QC on compact set). Also many non-smooth examples





Quadratic Julia set that is a quasicircle.

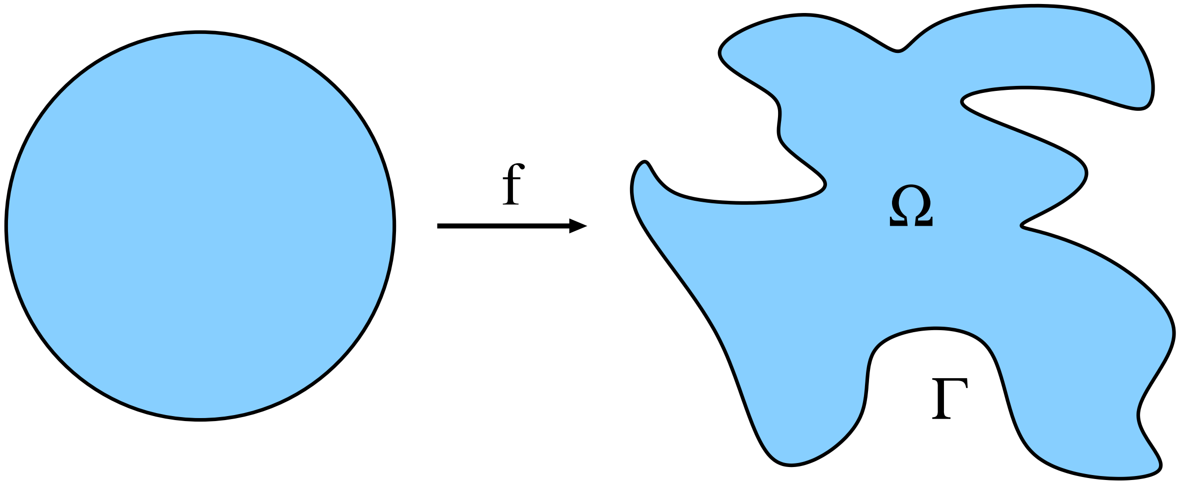


Suppose $\Gamma = \partial\Omega$ is Jordan curve, $f : \mathbb{D} \to \Omega$ is conformal.

Basic problem: how is geometry of Γ related to properties of f?

If f is quasiconformal, is geometry of Γ related to properties of $\mu_f = f_{\overline{z}}/f_z$?

If Γ is a quasicircle, f has a quasiconformal extension to plane, with dilation μ defined on $\mathbb{D}^* = \{|z| > 1\}$.



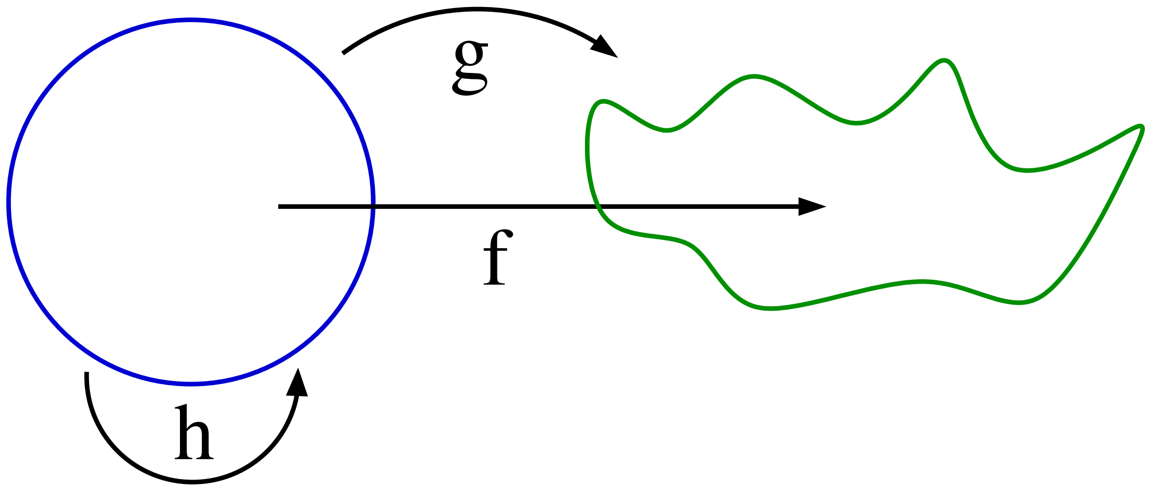
Also interested in Schwarzian derivative: $S(f) = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2.$

This is sort of second derivative that measures rate of change of best approximating Möbius transformation.

(Usual F'' measures rate of change of best approximating linear map.)

$|S(f)(z)|(1-|z|^2)^2 < 2$ implies f is conformal.
f conformal implies $|S(f)(z)|(1-|z|^2)^2 < 6$.

- $\Gamma = f(\mathbb{T})$ is associated to a conformal welding $h = g^{-1} \circ f : \mathbb{T} \to \mathbb{T}$:

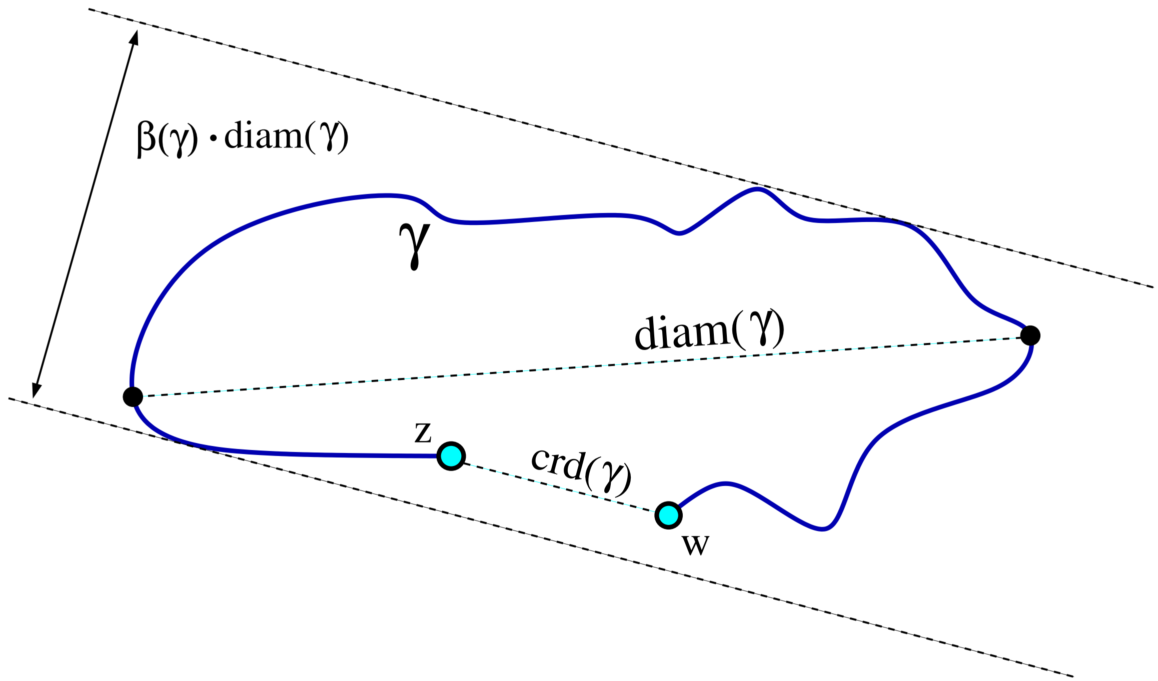


Two similar curves have same welding.

Möbius images have same welding.

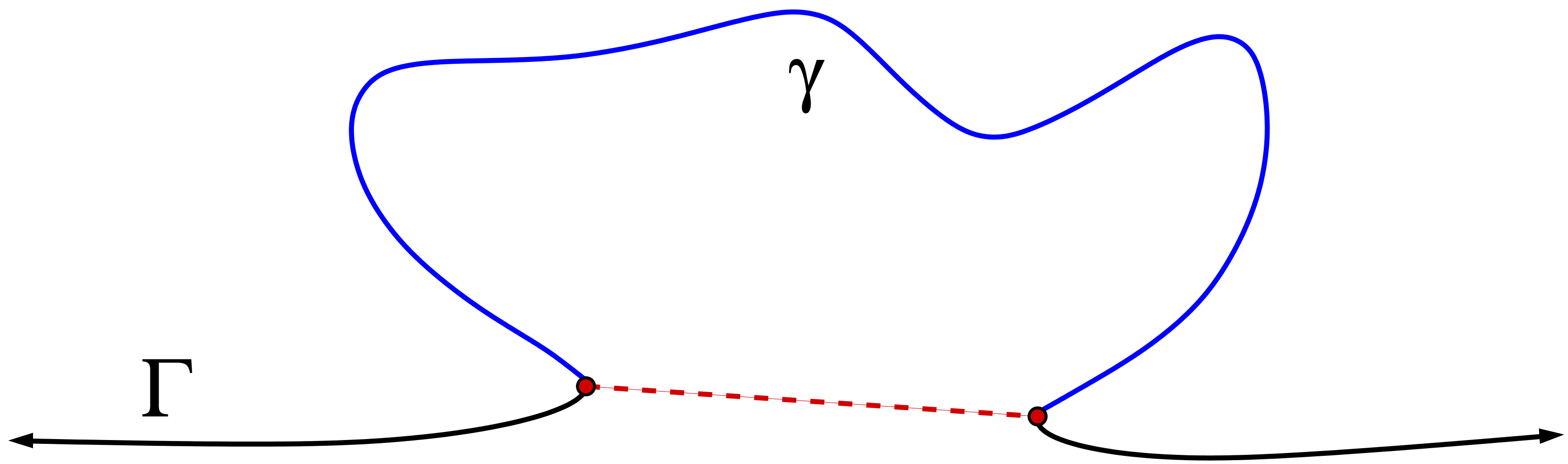
For quasicircles welding determines Γ up to Möbius image.

Not true in general (e.g., curves of positive area).



If γ is a planar Jordan arc with endpoints z, w, we set:

- $\mathrm{diam}(\gamma) =$ diameter of γ
- $\mathrm{crd}(\gamma) = z - w =$ chord length of γ
- $\ell(\gamma) =$ length of γ,
- $\Delta(\gamma) = \ell(\gamma) - \mathrm{crd}(\gamma) =$ excess length
- $\beta(\gamma) = \sup\{z \in \gamma : \mathrm{dist}(z, L)/\mathrm{diam}(\gamma)\}$, $L =$ line through z, w



Γ is a **quasicircle** iff $\mathrm{diam}(\gamma) = O(\mathrm{crd}(\gamma))$ for $\gamma \subset \Gamma$.

(Called Ahlfors 3-point condition.)

Γ is **chord-arc** iff $\ell(\gamma) = O(\mathrm{crd}(\gamma))$ for $\gamma \subset \Gamma$.

Space of quasicircles (modulo certain identifications) is called universal Teichmüller space.

Motivated by problems arising in string theory (e.g. [13], [14]), Takhtajan and Teo [65] defined a Weil-Petersson metric on universal Teichmüler space $T(1)$ that makes it into a Hilbert manifold. This structure on $T(1)$ is related to the Weil-Petersson metric on finite dimensional Teichmüller spaces.

This topology on $T(1)$ has uncountably many connected components, but one of these components, denoted $T_0(1)$, is exactly the closure of the smooth curves; this is the Weil-Petersson class. $T_0(1)$ is naturally a topological group: identfy elements bu conformal weldings and then compose these circle homomorphisms,

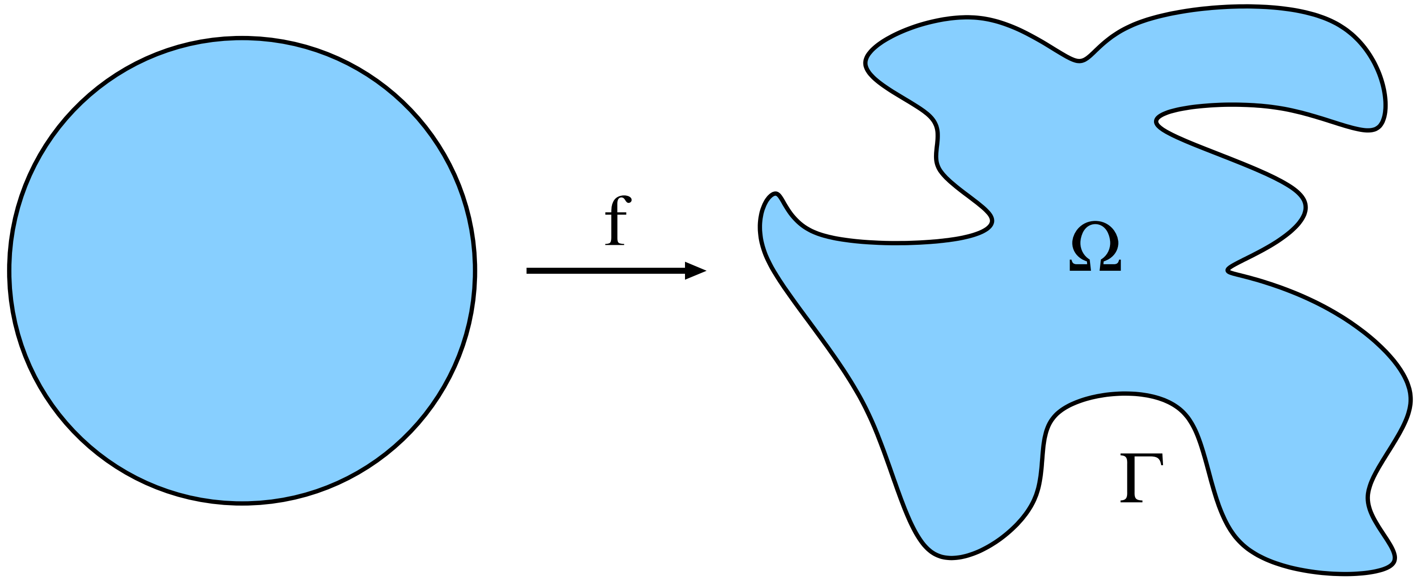
$T(1)$ is group but not a topological group; the group operation is not continuous in general.

These curves are precisely the images of $\mathbb{T}$ under quasiconformal maps with dilatation $\mu \in L^2(d\mathrm{A}_\rho) \cap \mathbb{B}_1^\infty$, where A_ρ is hyperbolic area on $\mathbb{D}$. Thus, roughly speaking, Weil-Petersson curves are to L^2 as quasicircles are to L^∞.

I won't give the original definition of WP class from Takhtajan and Teo paper now. Instead we will work from equivalent definitions (also in their paper) in terms of the conformal map f from the unit disk, $\mathbb{D}$, to the domain Ω bounded by Γ.

There are several results in geometric function theory that say "Γ has geometric property X iff $\log f'$ is in function space Y".

I will mention a few examples to give the flavor. We won't use these results.



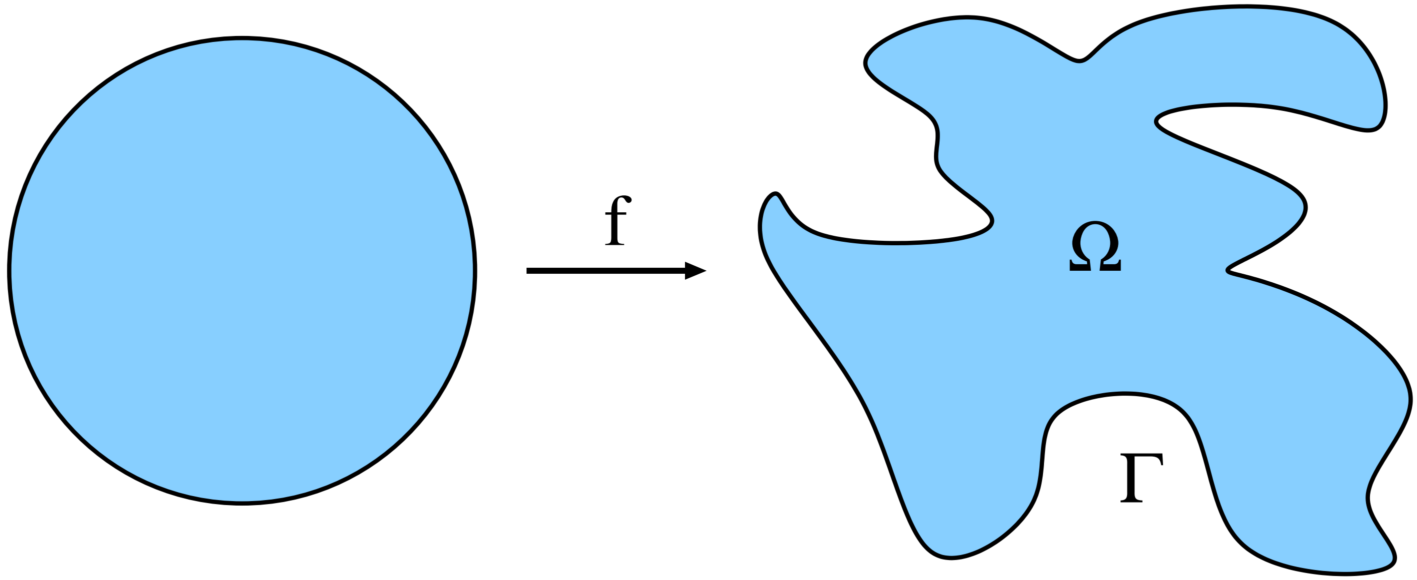
Theorem (Pommerenke, 1978): Γ is asymptotically conformal, i.e.,

$$\beta(\gamma) \to 0, \text{ as } \mathrm{diam}(\gamma) \to 0,$$

iff $\log f'$ is in little Bloch class

$$\mathcal{B}_0 = \left\{ g \text{ holomorphic on } \mathbb{D} : |g'(z)| = o\left(\frac{1}{1-|z|}\right) \right\}.$$

Bloch space $= \mathcal{B} = \left\{ g \text{ holomorphic on } \mathbb{D} : |g'(z)| = O\left(\frac{1}{1-|z|}\right) \right\}$.



Theorem (Pommerenke, 1978): Γ is asymptotically smooth, i.e.,

$$\frac{\Delta(\gamma)}{\mathrm{crd}(\gamma)} = \frac{\ell(\gamma) - \mathrm{crd}(\gamma)}{\mathrm{crd}(\gamma)} \to 0, \text{ as } \mathrm{diam}(\gamma) \to 0,$$

iff $\log f' \in \mathrm{VMOA}$.

Bounded mean oscillation (BMO) is the space of functions so that

$$m_I(f - m_I(f)) = O(1),$$

where $m_I(f)$ is the mean value of f over I, i.e.,

$$m_I(f) = \frac{1}{|I|} \int_I f dx.$$

Here $|I|$ is Lebesgue measure of I. In other words, f is in BMO if

$$\|f\|_{\mathrm{BMO}} = \sup_I \frac{1}{|I|} \int_I |f - m_I(f)| dx < \infty.$$

$L^\infty \subset \mathrm{BMO}$, but $\log|x| \in \mathrm{BMO}$.

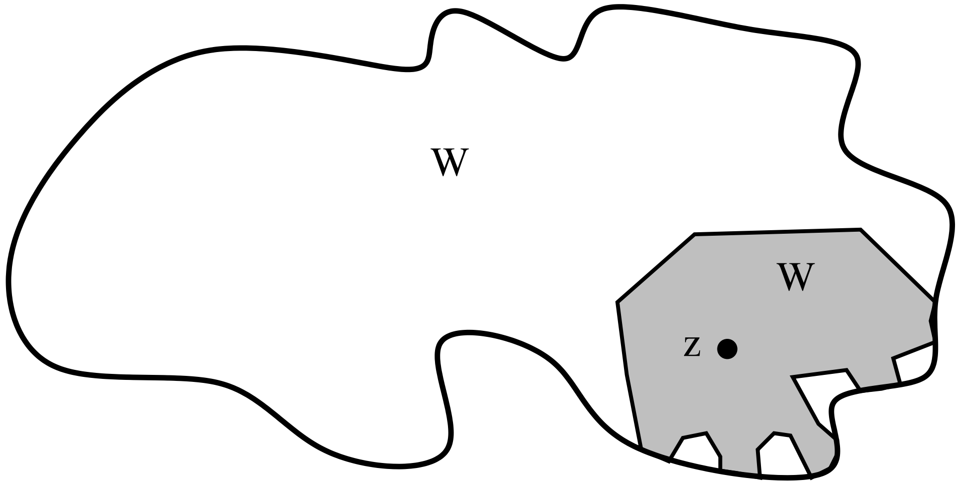
Vanishing Mean Oscillation (VMO)

$$\lim_{|I| \to 0} \frac{1}{|I|} \int_I |f - m_I(f)| dx \to 0.$$

A holomorphic function on $\mathbb{D}$ is in BMOA (VMOA) if it is the harmonic extension of a BMO (VMO) function on the circle.

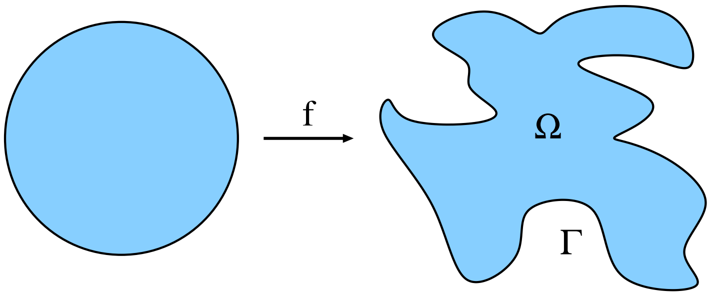
Theorem (B.-Jones): $\log f' \in \mathrm{BMOA}$ iff for every $z \in \Omega$ there is a chord-arc subdomain $z \in W \subset \Omega$ with

$$\mathrm{diam}(W) \simeq \ell(\partial W) \simeq \ell(\partial W \cap \Gamma) \simeq \mathrm{dist}(z, \Gamma).$$



Expands on closely related work of Kari Astala and Michel Zinsmeister who gave characterization in terms of Schwarzian derivatives. They developed a whole "BMO-Teichmüller" theory parallel to stadard "Bloch" version.

Consider the conformal mapping $f : \mathbb{D} \to \Omega$, the domain bounded by Γ.



Dirichlet space = holomorphic F on $\mathbb{D}$ with $F' \in L^2(dxdy)$.

Then Γ is Weil-Petersson if and only if $\log f'$ is in the Dirichlet space, i.e.,

$$(\log f')' = f''/f' \in L^2(\mathbb{D}, dxdy),$$

or

$$\int_{\mathbb{D}} |(\log f')'|^2 dxdy < \infty.$$

Tuesday, September 1, 2020

Takhtajan and Teo [65] showed this condition is the same as

$$\frac{1}{\pi}\iint_{\mathbb{D}}\left|\frac{f''(z)}{f'(z)}\right|^2 dxdy+\frac{1}{\pi}\iint_{\mathbb{D}^*}\left|\frac{g''(z)}{g'(z)}\right|^2 dxdy+4\log\frac{|f'(0)|}{|g'(\infty)|}<\infty.$$

where g is a conformal map from $\mathbb{D}^*=\{|z|>1\}$ to $\mathbb{C}\setminus\overline{\Omega}$ so that $g(\infty)=\infty$.

They called this quantity the universal Liouville action, and showed that it is Möbius invariant.

More recently, Yilin Wang [66] proved it equals the Loewner energy of Γ, as defined by her and Steffen Rohde in [59]; we will denote it by $\mathcal{LE}(\Gamma)$. This provides a connection to SLE (Schramm-Loewner Evolutions).

The definition of Weil-Petersson in terms of the Dirichlet space implies:

Theorem 1.1. *Γ is Weil-Petersson iff it is chord-arc and the arclength parameterization is in the Sobolev space $H^{3/2}(\mathbb{T})$.*

By definition $f \in H^{3/2}$ if f is absolutely continuous and $f' \in H^{1/2}$.

$H^{1/2}(\mathbb{T})$ has several equivalent definitions.

Lemma 1.2. *Suppose $f \in L^2(\mathbb{T})$. Then the following are equivalent.*

(1) $f(z) = \sum_{-\infty}^{\infty} a_n z^n$ *where* $\sum_{-\infty}^{\infty} n|a_n|^2 < \infty$ *(Fourier coefficients).*
(2) $\int_{\mathbb{D}} |\nabla Pf(z)|^2 dxdy < \infty$, *where Pf denotes the Poisson extension of f to $\mathbb{D}$. (Dirichlet integral)*
(3) *f is the a.e. radial limits of a function* $u \in W^{1,2}(\mathbb{D}) = \{u : \int_{\mathbb{D}} |u|^2 + |\nabla u|^2 dxdy < \infty$. *(Trace theorem for Sobolev spaces)*
(4) $\int_{\mathbb{T}} \int_{\mathbb{T}} \left| \frac{f(x)-f(y)}{x-y} \right|^2 dxdy < \infty$. *(Douglas formula)*

It was previously known that the tangent space at a point of $T_0(1)$ is naturally identified with $H^{3/2}$ (see [65]), but it was not previously known how to identity $T_0(1)$ itself with a subset of $H^{3/2}(\mathbb{T})$.

H^s diffeomorphisms of circle, $s > 3/2$, is a topological group under composition ([15], [36]). $H^{3/2}$ is not.

Theorem 1.1 makes $H^{3/2}$ into topological group via identification with $T_0(1)$.

Shen proved conformal weldings h of WP curves are characterized by $\log h' \in H^{1/2}$.

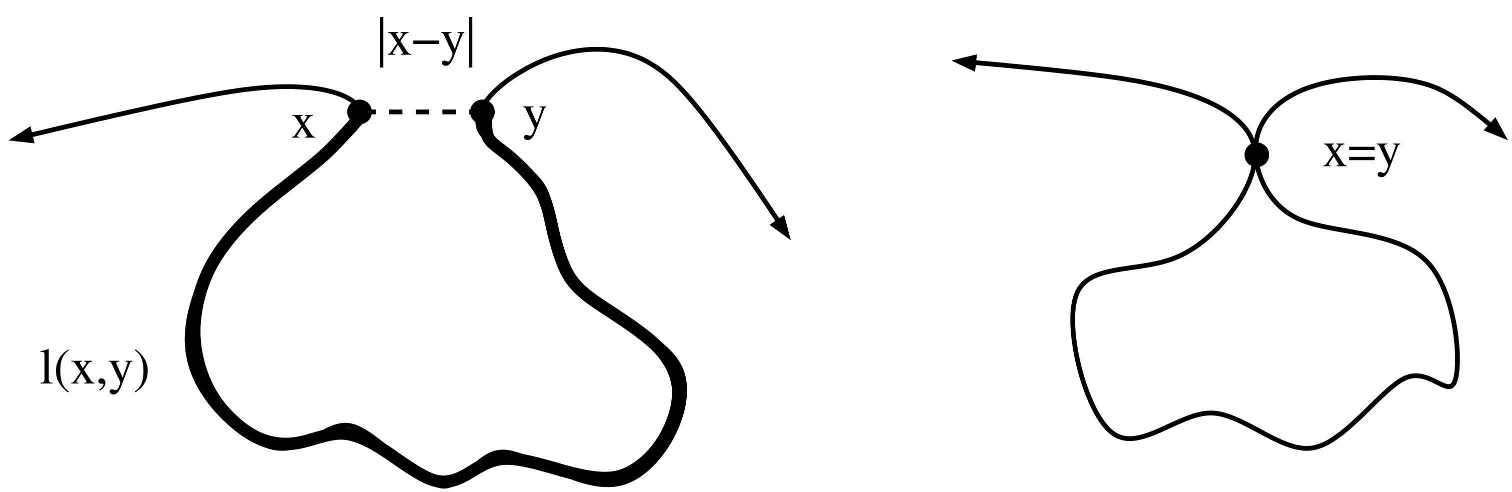
Theorem 1.1 was one of the last characterizations that I discovered. David Mumford noticed that it was implied by another characterization from an earlier draft.

Current draft does not use my original approach. Instead it uses an idea from knot theory.

Theorem 1.3. Γ *is Weil-Petersson iff it has finite Möbius energy, i.e.,*

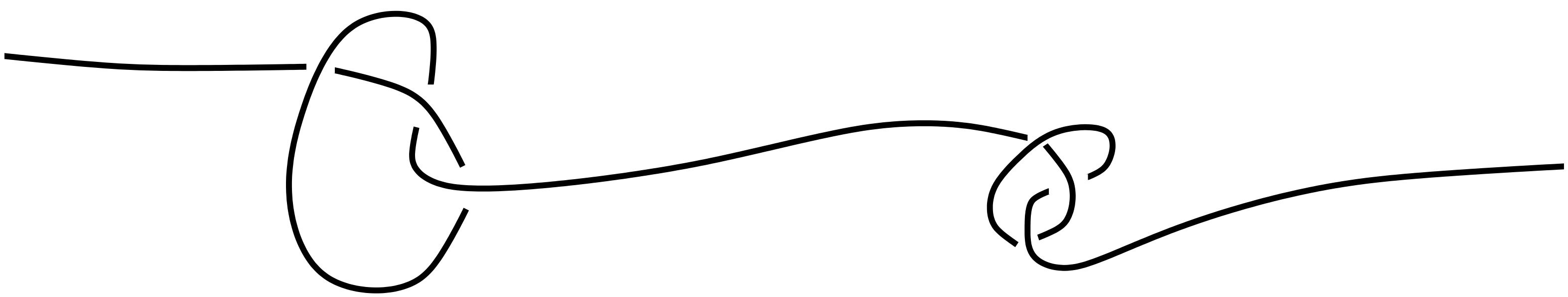
$$\text{Möb}(\Gamma) = \int_\Gamma \int_\Gamma \left(\frac{1}{|x-y|^2} - \frac{1}{\ell(x,y)^2} \right) dxdy < \infty. \tag{1.1}$$

Möbius energy is one of several "knot energies" introduced by O'Hara [49], [51], [50]. It blows up when the curve is close to self-intersecting, so continuously deforming a curve in $\mathbb{R}^3$ to minimize the Möbius energy should lead to a canonical "nice" representative of each knot type.



This was proven for irreducible knots by Freedman, He and Wang [34], who also showed that $\text{Möb}(\Gamma)$ is Möbius invariant (hence the name), that $\text{Möb}(\Gamma)$ attains its minimal value 4 only for circles, that finite energy curves are chord-arc, and in $\mathbb{R}^3$ they are topologically tame (there is an ambient isotopy to a smooth embedding).

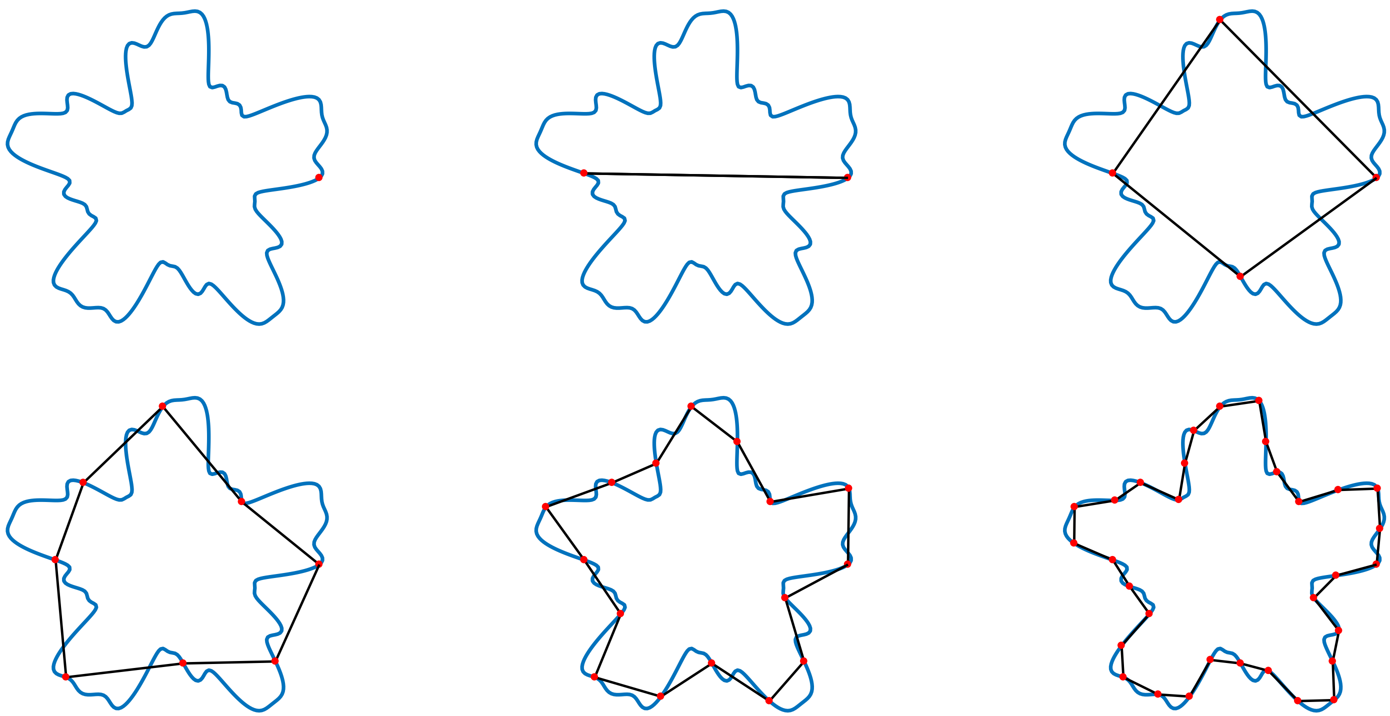
Theorem 1.3 follows from Theorem 1.1 by a result of Blatt [11] (proof later).



Pulling knot tight can change topology. Must renormalize by Möbius transformations. Possible for irreducible knots.

Define dyadic decomposition. If a closed Jordan curve Γ has finite length $\ell(\Gamma)$, choose a base point $z_1^0 \in \Gamma$ and for each $n \geq 1$, let $\{z_j^n\}$, $j = 1, \dots, 2^n$ be the unique set of ordered points with $z_1^n = z_1^0$ that divides Γ into 2^n equal length intervals (called the nth generation dyadic subintervals of Γ).

Let Γ_n be the inscribed 2^n-gon with these vertices. Clearly $\ell(\Gamma_n) \nearrow \ell(\Gamma)$.



Theorem 1.4. *With notation as above, a curve Γ is Weil-Petersson if and only if*

$$\sum_{n=1}^{\infty} 2^n \left[\ell(\Gamma) - \ell(\Gamma_n)\right] < \infty \tag{1.2}$$

with a bound that is independent of the choice of the base point.

We need to say the bound is independent of the base point.

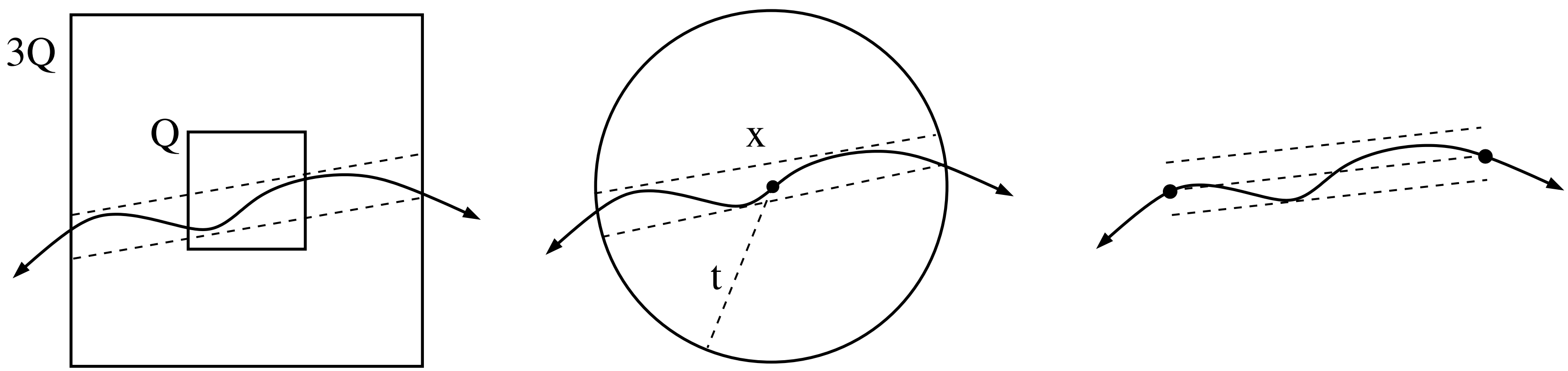
A square is not WP, but $\ell(\Gamma) = \ell(\Gamma_2)$ if the base point is a corner.

Definition of β-numbers

Given a curve $\Gamma \subset \mathbb{R}^2$, $x \in \mathbb{R}^2$ and $t > 0$, define

$$\beta_\Gamma(x,t) = \inf_L \sup_{z \in D(x,t)} \frac{\operatorname{dist}(z,L)}{t},$$

where the infimum is over all lines hitting $D(x,t)$. $\widetilde{\beta}$ uses lines through x.



There are equivalent versions using dyadic cubes or using subarcs: $\beta_\Gamma(Q)$, $\beta(\Gamma_j)$.

Lemma 1.5. *Suppose* $-1 < s < 2$ *and* $\Gamma \subset \mathbb{R}^n$ *is Jordan curve (either closed or an arc). Then the following are equivalent:*

$$\sum_{Q \in \mathcal{D}} \beta_\Gamma^2(Q) \mathrm{diam}(Q)^s < \infty, \tag{1.3}$$

$$\int_0^\infty \iint_{\mathbb{R}^n} \beta^2(x,t) \frac{dxdt}{t^{n+1-s}} < \infty, \tag{1.4}$$

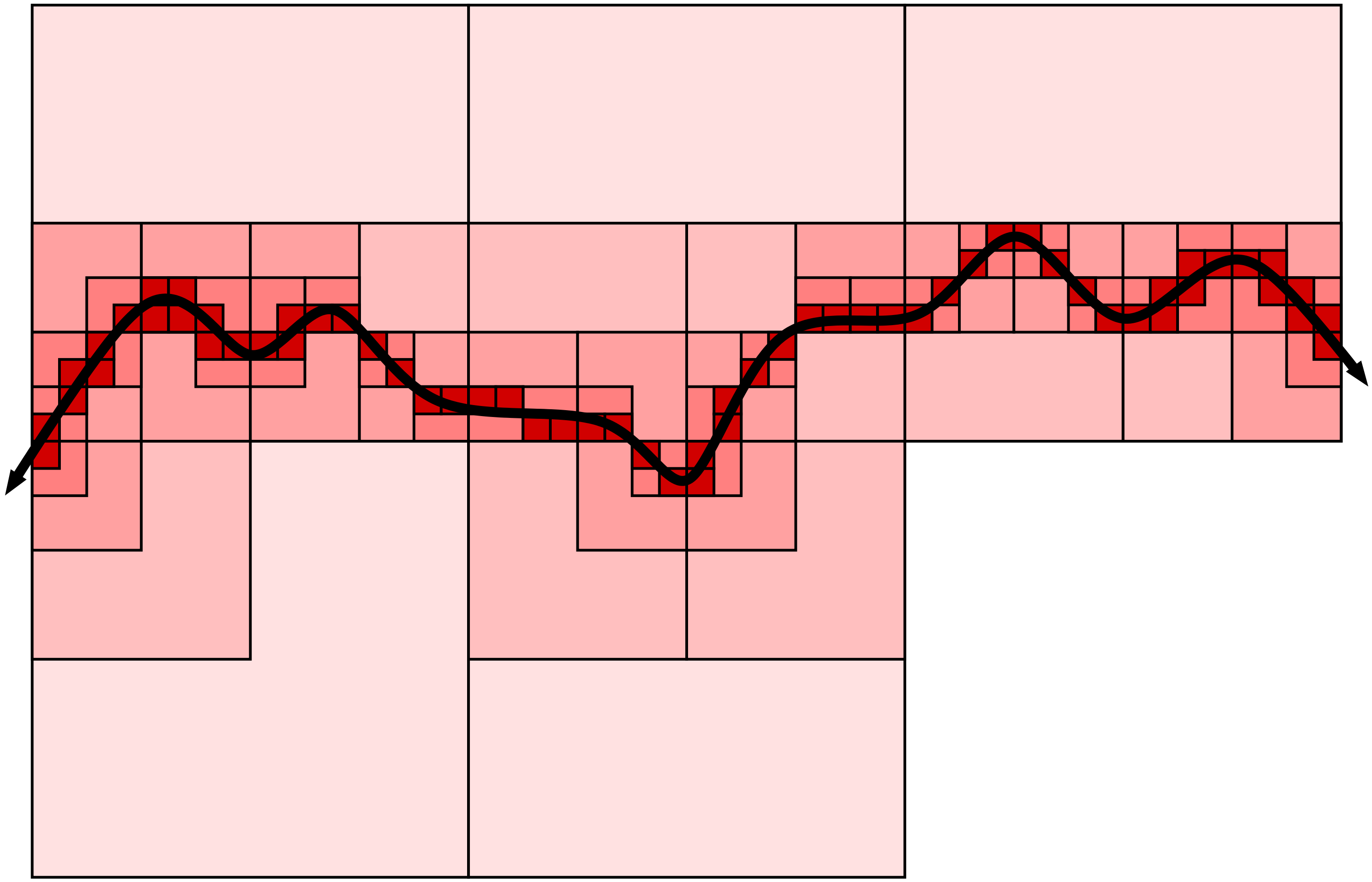
$$\int_0^\infty \int_\Gamma \widetilde{\beta}^2(x,t) \frac{dsdt}{t^{2-s}} < \infty, \tag{1.5}$$

$$\sum_j \beta^2(\Gamma_j) \mathrm{diam}(Q)^s < \infty, \tag{1.6}$$

where dx *is volume measure on* $\mathbb{R}^n$, ds *is arclength measure on* Γ, *and the sum in (6.7) is over a multi-resolution family* $\{\Gamma_j\}$ *for* Γ. *All four quantities are comparable with constants that depend only on* n.

Proved in

The travelings salesman theorem for Jordan curves



$$\ell(\Gamma) \simeq \mathrm{diam}(Q) + \sum_{Q} \beta_{\gamma}^{2}(Q)\mathrm{diam}(Q)$$

If both A and B depend on some parameter,

$$A \lesssim B$$

means that $A \leq C \cdot B$ for some $C < \infty$ independent of that parameter.

Is the same as $A = O(B)$.

$A = O(1)$ means A is bounded, independent of the parameter.

$A \gtrsim B$ iff $B \lesssim A$.

$A \simeq B$ means both $A \lesssim B$ and $A \gtrsim B$ hold. In this case we say A and B are "comparable".

$A = o(B)$ means $A/B \to 0$ as the parameter tends to infinity.

$A = o(1)$ means $A \to 0$ as the parameter tends to infinity.

A multi-resolution family in a metric space X is a collection of sets $\{X_j\}$ in X such that there is are $N, M < \infty$ so that

(1) For each $r > 0$, the sets with diameter between r and Mr cover X,

(2) each bounded subset of X hits at most N of the sets X_k with $\mathrm{diam}(X)/M \leq \mathrm{diam}(X_k) \leq M\mathrm{diam}(X)$.

(3) any subset of X with positive, finite diameter is contained in at least one X_j with $\mathrm{diam}(X_j) \leq M\mathrm{diam}(X)$.

Dyadic intervals are not a multi-resolution family, e.g., $X = [-1, 1] \subset \mathbb{R}$ is not contained in any dyadic interval, violating (3).

However, the family of triples of all dyadic intervals (or cubes) do form a multi-resolution family. Similarly, if we add all translates of dyadic intervals by $\pm 1/3$, we get a multi-resolution family (this is sometimes called the "$\frac{1}{3}$-trick", [52]).

The analogous construction for dyadic squares in $\mathbb{R}^n$ is to take all translates by elements of $\{-\frac{1}{3}, 0, \frac{1}{3}\}^n$.

Peter Jones's traveling salesman theorem in [38] says that the shortest curve Γ containing $E \subset \mathbb{R}^2$ has length

$$\begin{aligned} \ell(\Gamma) &\simeq \operatorname{diam}(E) + \sum_Q \beta_E^2(Q)\operatorname{diam}(Q) \\ &\simeq \operatorname{diam}(E) + \iint \beta_E^2(x,t)\frac{dxdt}{t} \end{aligned}$$

Jones's TST was extended to higher (finite) dimensions by Kate Okikiolu [52], but with constants that grow exponentially with the dimension, and later Raanan Schul [61] proved a version that holds for sets in Hilbert space, and thus in $\mathbb{R}^n$ with constants that are independent of n. This is one of only a handful of problems in Euclidean analysis where dimension independent bounds are known. Extensions to curves in other metric spaces are given in [19], [25], [42], [43].

Theorem 1.6. *A Jordan curve* Γ *is Weil-Petersson if and only if*

$$\sum_{Q} \beta_\Gamma^2(Q) < \infty. \tag{1.7}$$

Equivalent:

$$\int_\Gamma \int_0^\infty \beta_\Gamma^2(x,t) \frac{dt dx}{t^2} < \infty.$$

β-numbers measure curvature of Γ at a particular point and scale.

This says that Γ is Weil-Petersson iff curvature is square integrable over all locations and scales".

We will have many variations of this were curvature is measured in different ways: Schwarzian derivatives, Menger curvature, the Gauss curvatures of minimal surfaces.

Rectifiable: $\sum_Q \beta_\Gamma^2(Q)\,\mathrm{diam}(Q) < \infty$.

Weil-Petersson: $\sum_Q \beta_\Gamma^2(Q) < \infty$.

Thus WP is stronger version of rectifiability. (Even stronger than chord-arc).

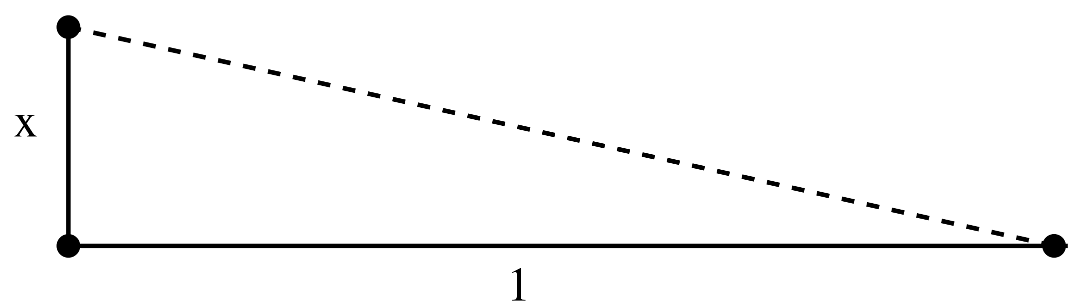
The fact that $\sum \beta_\Gamma^2(Q) < \infty$ holds for $H^{3/2}$ curves is fairly straightforward, but the reverse implication seems less so. For curves in $\mathbb{R}^2$, we can prove this direction though a chain of function theoretic characterizations that eventually lead back to the $H^{3/2}$ condition. For curves in $\mathbb{R}^n$, $n \geq 3$, we will need an improvement of Jones's original TST.

Original version:

$$\ell(\Gamma) \simeq \mathrm{diam}(\Gamma) + \sum_Q \beta^2(Q)\mathrm{diam}(Q).$$

$$\ell(\Gamma) \leq (1+\delta)\mathrm{diam}(\Gamma) + C(\delta)\sum_Q \beta^2(Q)\mathrm{diam}(Q).$$

Can't take $\delta = 0$. Can choose 3 points so optimal length is $1 + x$, diameter is $1 + O(x^2)$ and β^2-sum is $O(x^2)$:



But we can take $\delta = 0$ if $E = \Gamma$ is a Jordan curve.

Theorem 1.7. *If $\Gamma \subset \mathbb{R}^n$ is a Jordan arc, then*

$$\begin{aligned}
\ell(\Gamma) &= \operatorname{diam}(\Gamma) + O\left(\sum_Q \beta^2(Q)\operatorname{diam}(Q)\right) \\
&= \operatorname{crd}(\Gamma) + O\left(\sum_Q \beta^2(Q)\operatorname{diam}(Q)\right) \\
&= \operatorname{crd}(\Gamma) + O\left(\int_{\mathbb{R}^n}\int_0^\infty \beta_\Gamma^2(x,t)\right)
\end{aligned}$$

Here $\operatorname{crd}(\Gamma) = |z - w|$ denotes the distance between the endpoints z, w of Γ.

The point of Theorem 1.7 is that the $\operatorname{diam}(\Gamma)$ term in TST can be replaced by the smaller value $\operatorname{crd}(\Gamma)$, and that this term is only multiplied by "1" in the estimate (1.8).

I plan to prove this (and the general TST for $\mathbb{R}^n$) in this course.

Thursday, September 3, 2020

The Weil-Petersson class is Möbius invariant, so we should seek a Möbius invariant version of the β-numbers as well.

The β-numbers locally trap Γ between two lines. We will introduce ϵ-numbers that globally trap Γ between two disjoint disks. We then extend these disks to hemispheres in the upper half-space $\mathbb{R}^3_+ = \mathbb{R}^3_+$ and measure the distance between these hemispheres in the hyperbolic metric on $\mathbb{R}^3_+$.

The hyperbolic upper half-space is defined as

$$\mathbb{R}^3_+ = \mathbb{R}^3_+ = \{(x,t) : x \in \mathbb{R}^2, t > 0\},$$

with the hyperbolic metric $d\rho = ds/2t$ is chosen so that $\mathbb{R}^3_+$ has constant Gauss curvature -1.

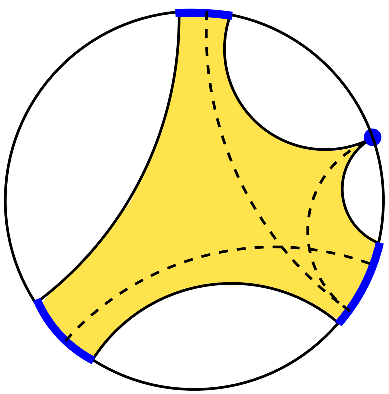
The ball model uses

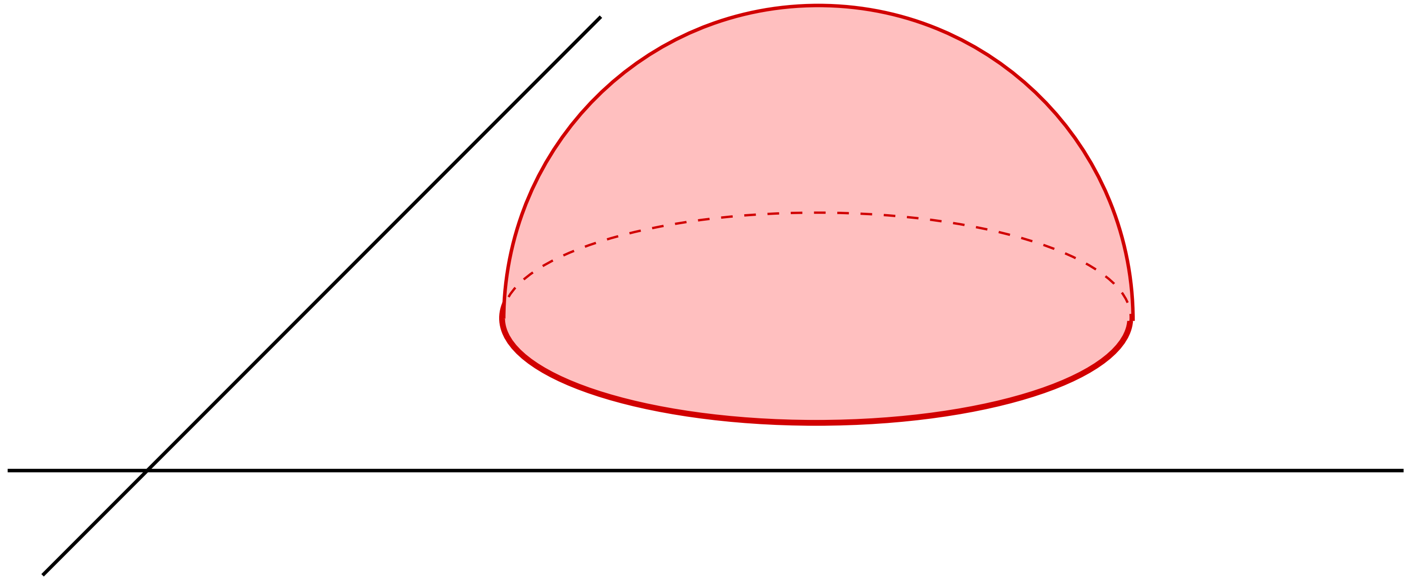
$$\mathbb{B} = \{x \in \mathbb{R}^3 : |z| < 1\},$$

with the metric $d\rho = ds/(1 - |z|^2)$.

Geodesics are circles (or lines) perpendicular to boundary.

The hyperbolic convex hull of $\Gamma \subset \mathbb{R}^2$, denoted $\mathrm{CH}(\Gamma)$, is the smallest convex set in $\mathbb{R}^3_+$ that contains all (infinite) hyperbolic geodesics with both endpoints in Γ.





For a circle in plane, hyperbolic convex hull is a hemisphere.

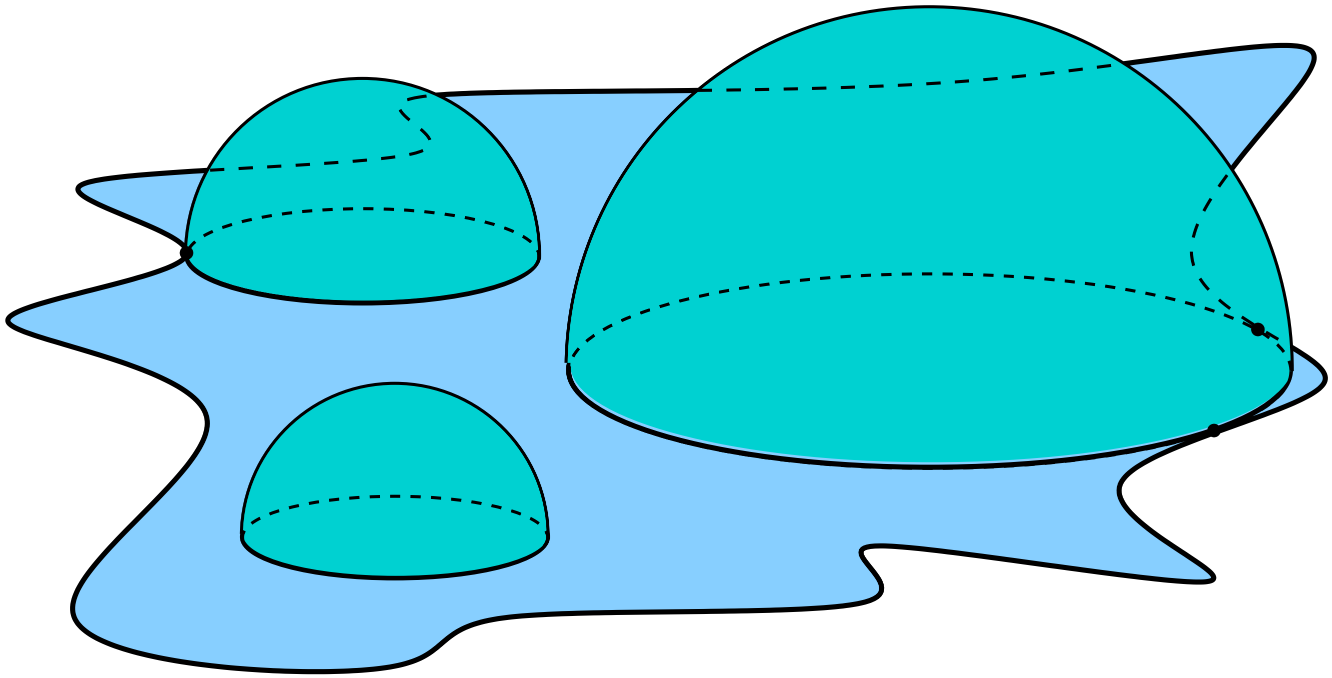
In general, $\mathrm{CH}(\Gamma)$ has non-empty interior.

There are two boundary surfaces, each asymptotic to Γ.

For experts: if Γ is the limit set of a quasi-Fuchsian group G, then $\mathrm{CH}(\Gamma)$ corresponds to convex core of the corresponding 3-manifold $M = \mathbb{R}^3_+/G$.

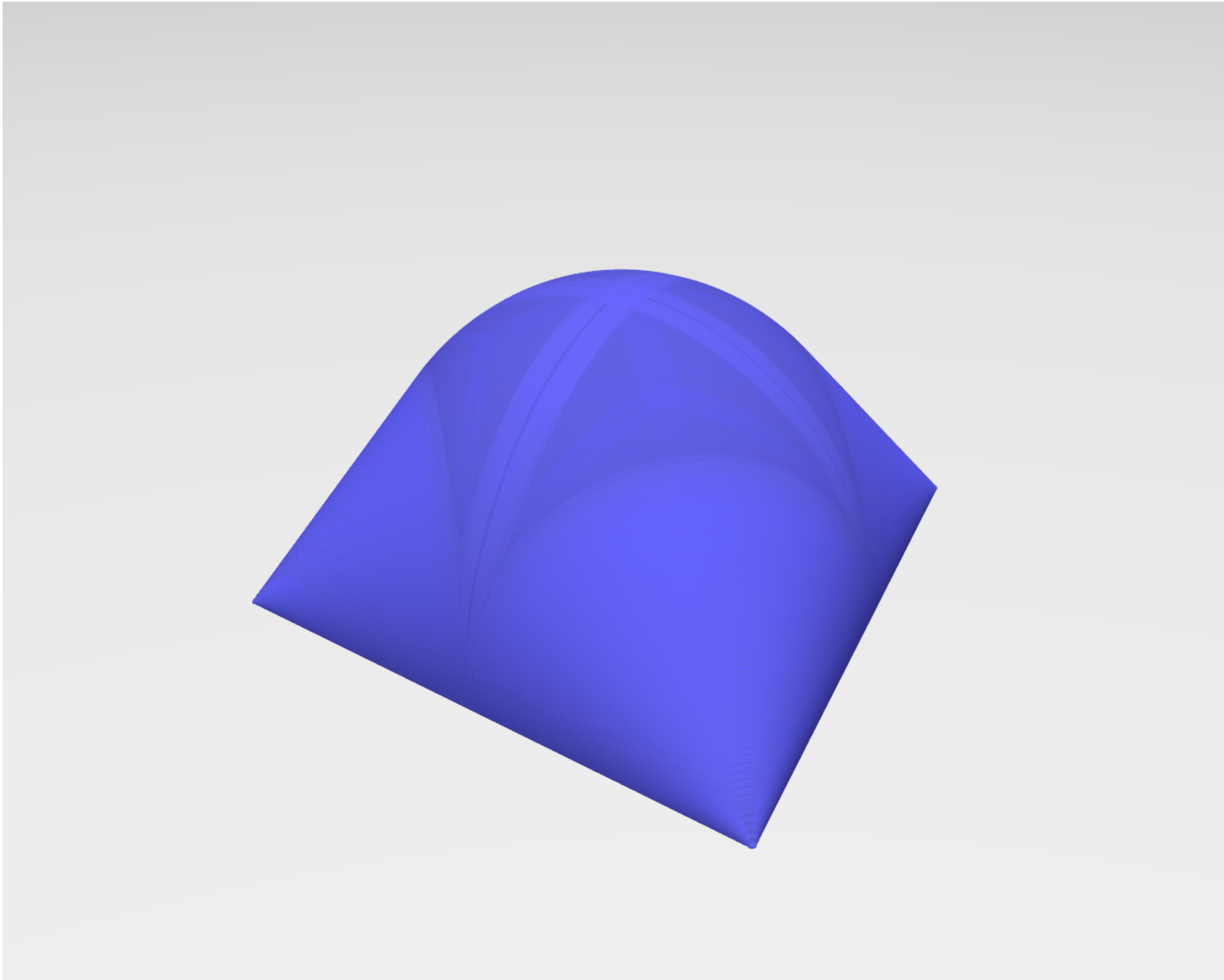
Suppose Ω is Jordan domain with boundary Γ.

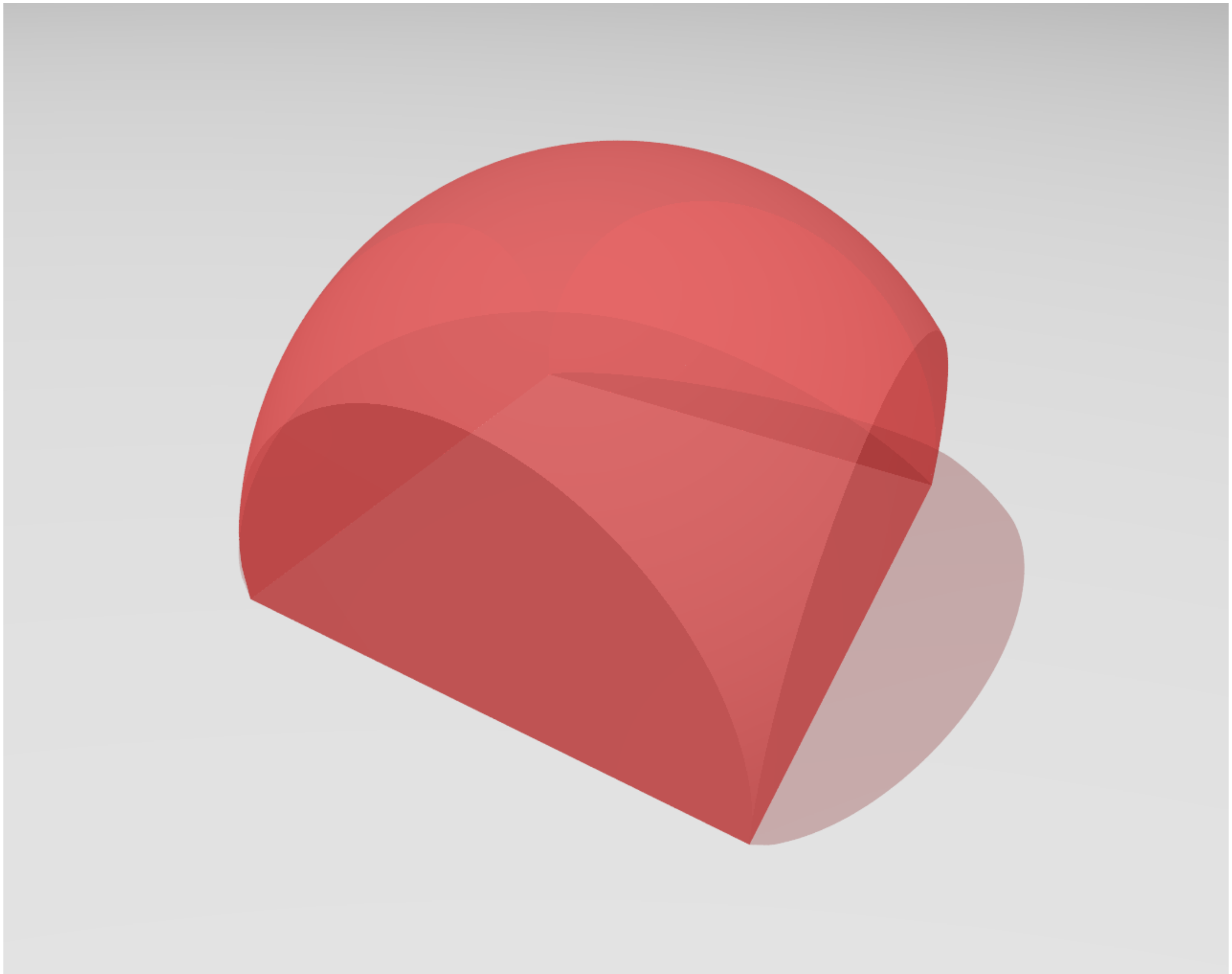
The dome of Ω is upper envelope of all hemispheres with base disk in Ω.

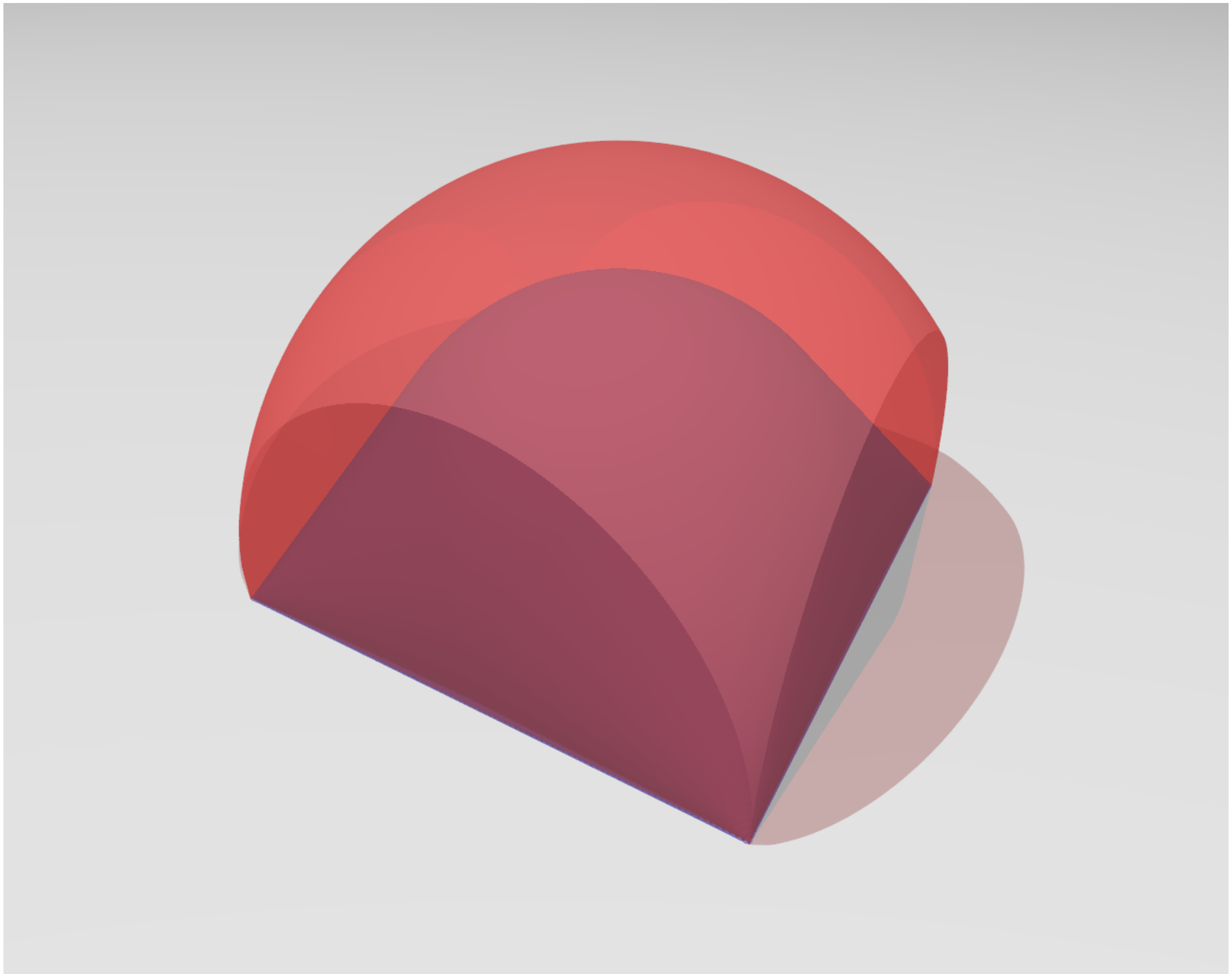


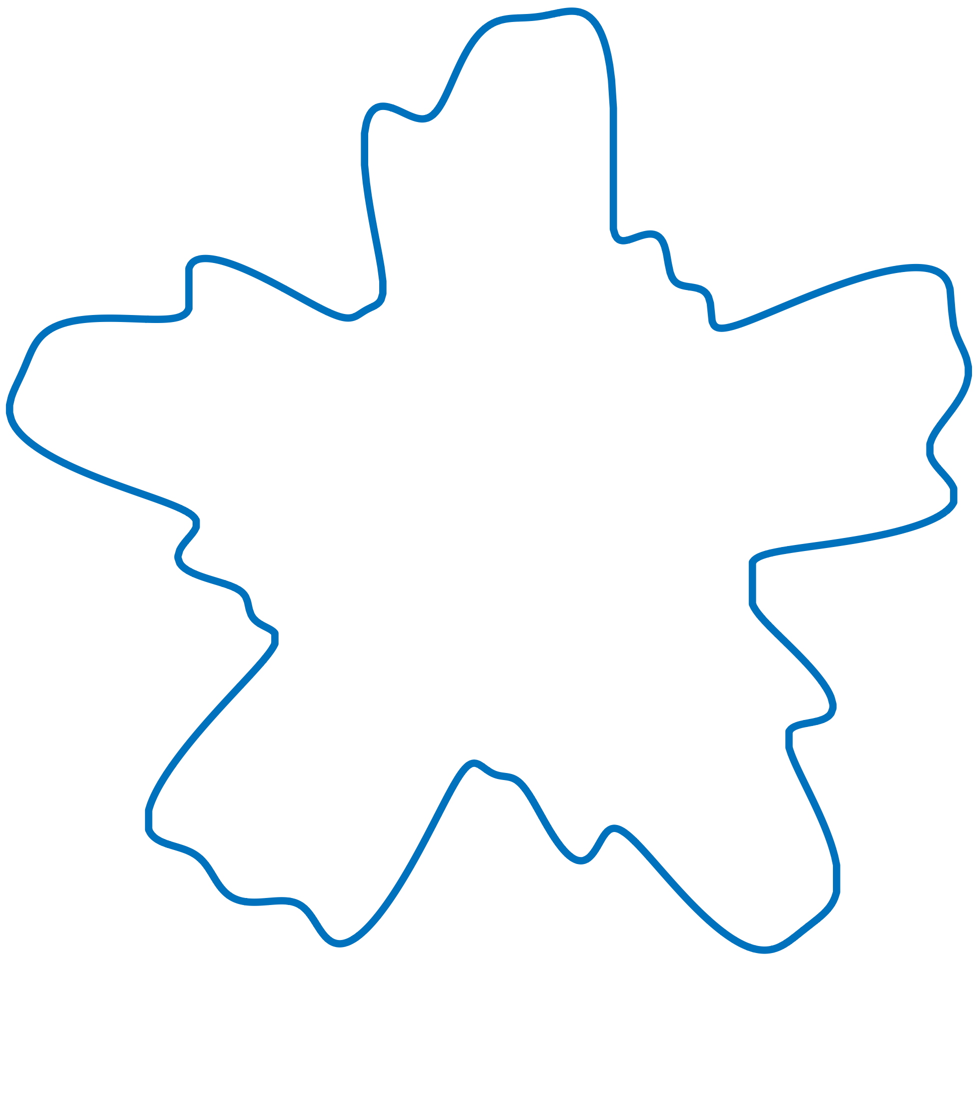
Region above dome is intersection of half-spaces, hence convex.

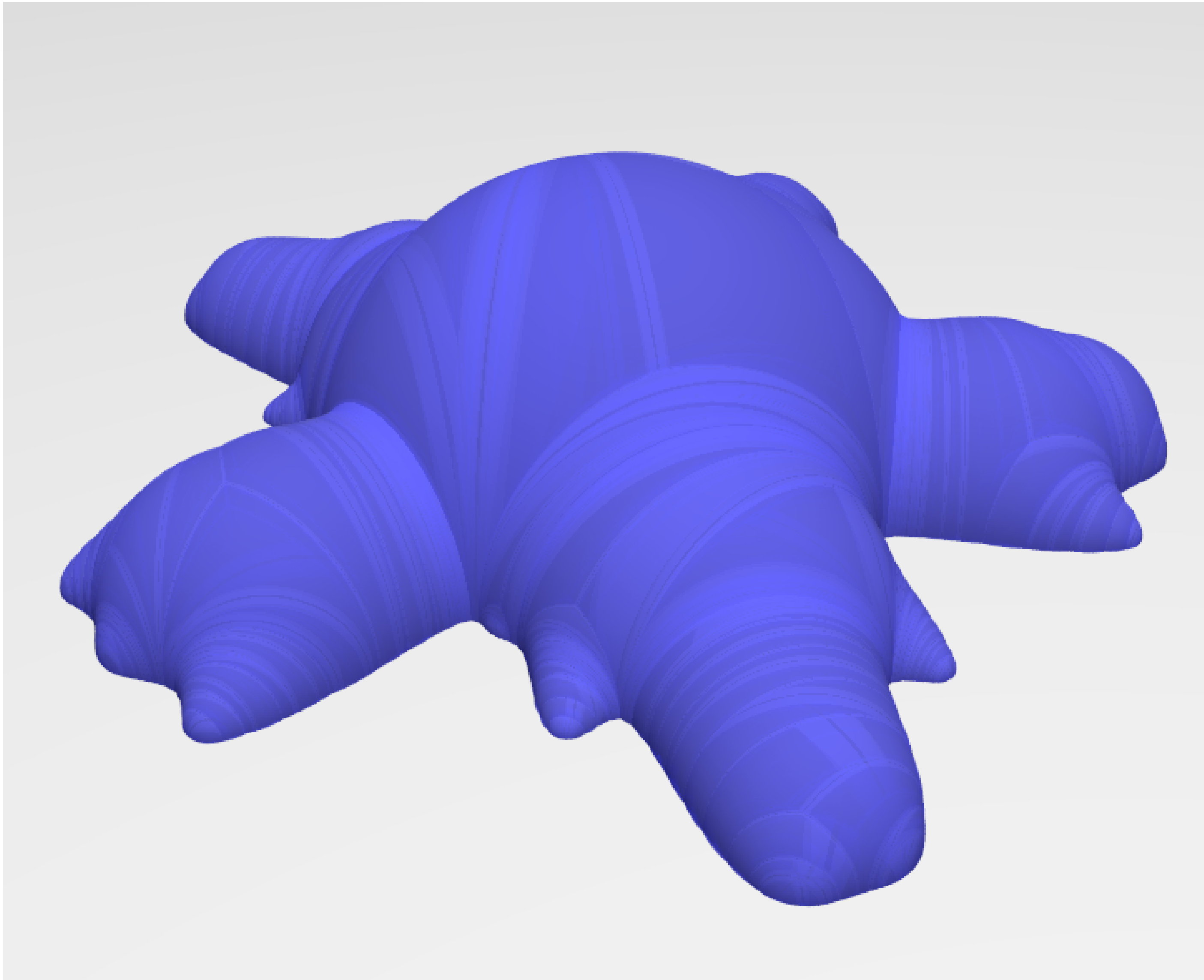
CH(Γ) is region between domes of "inside" and "outside" of Γ.

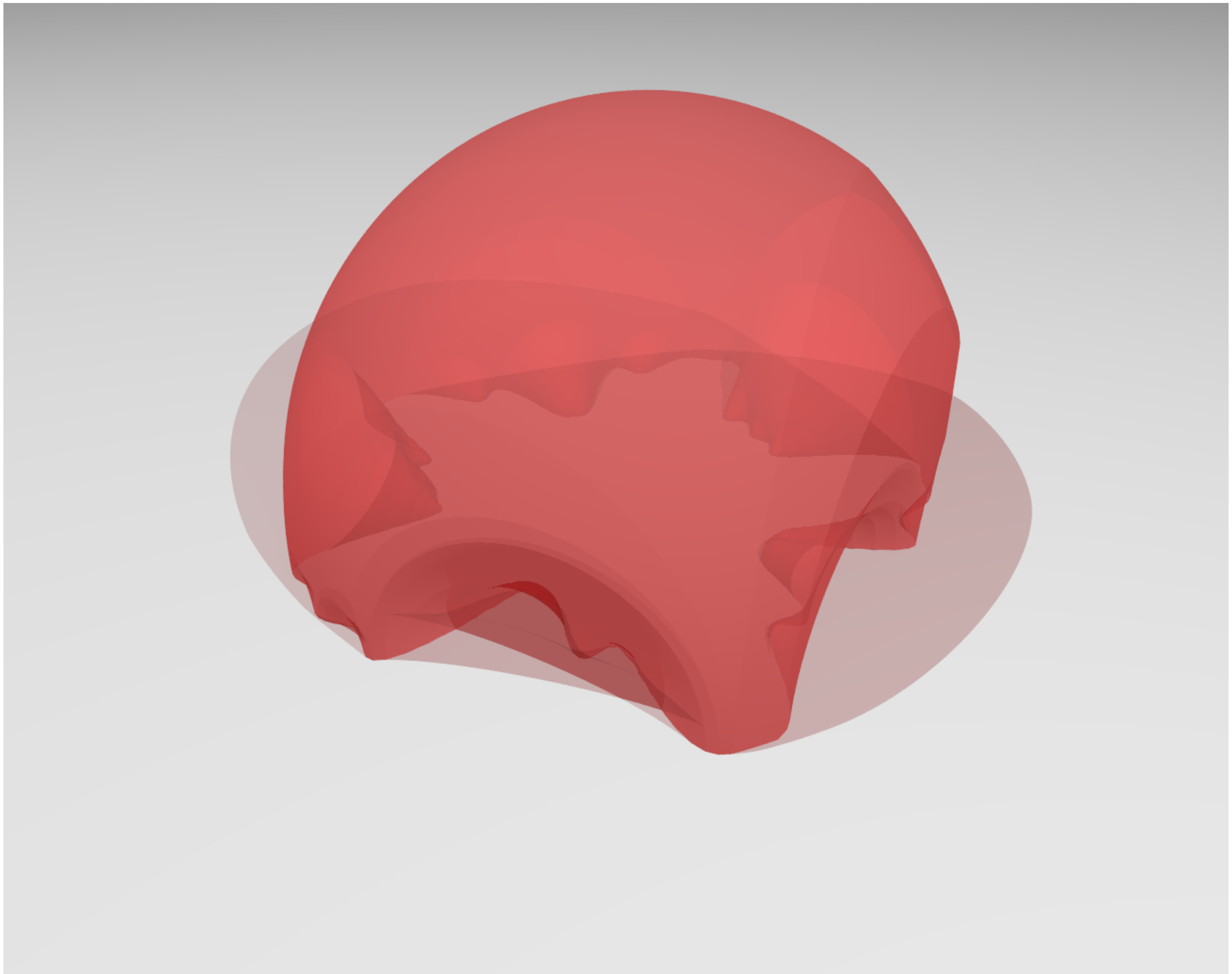


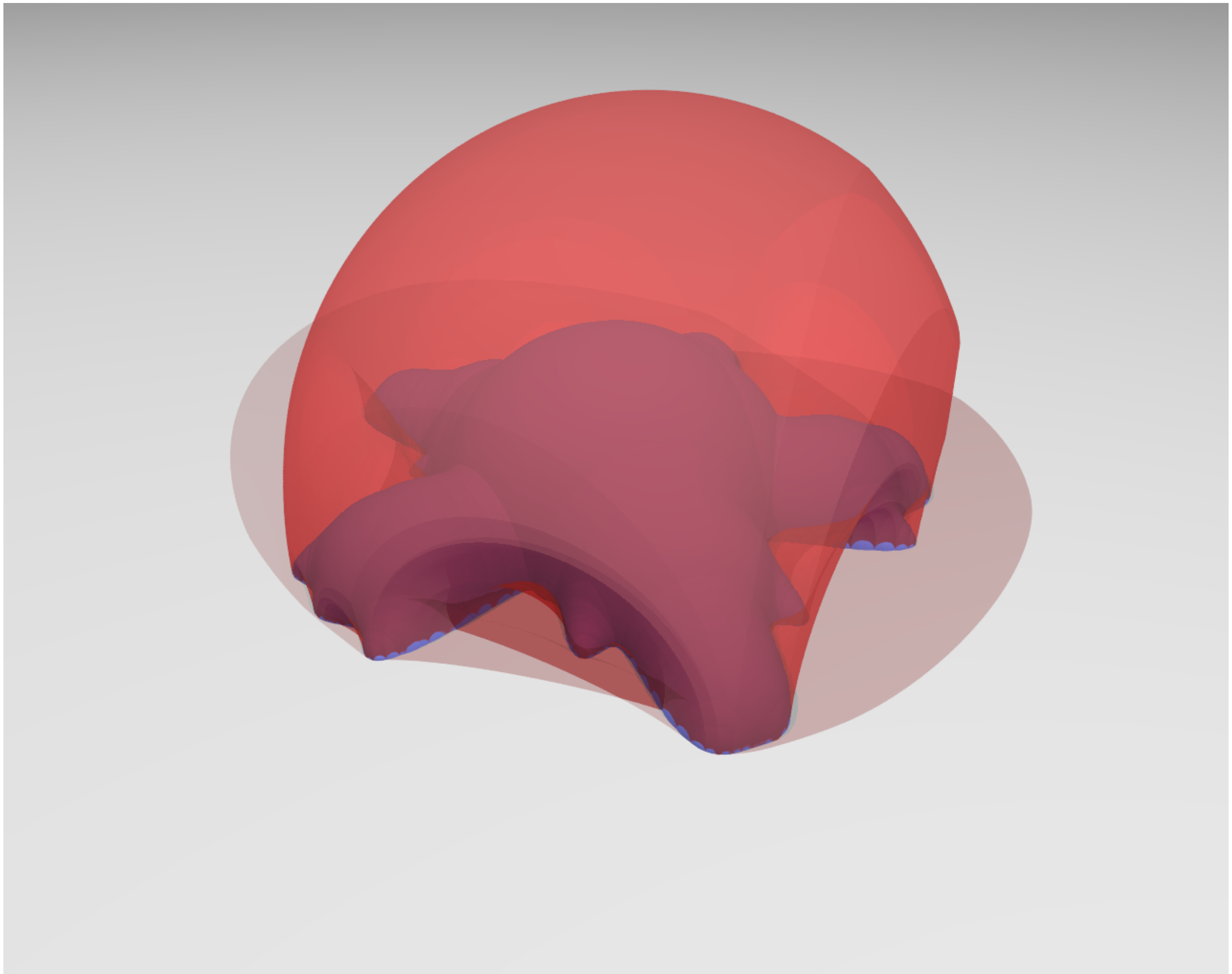


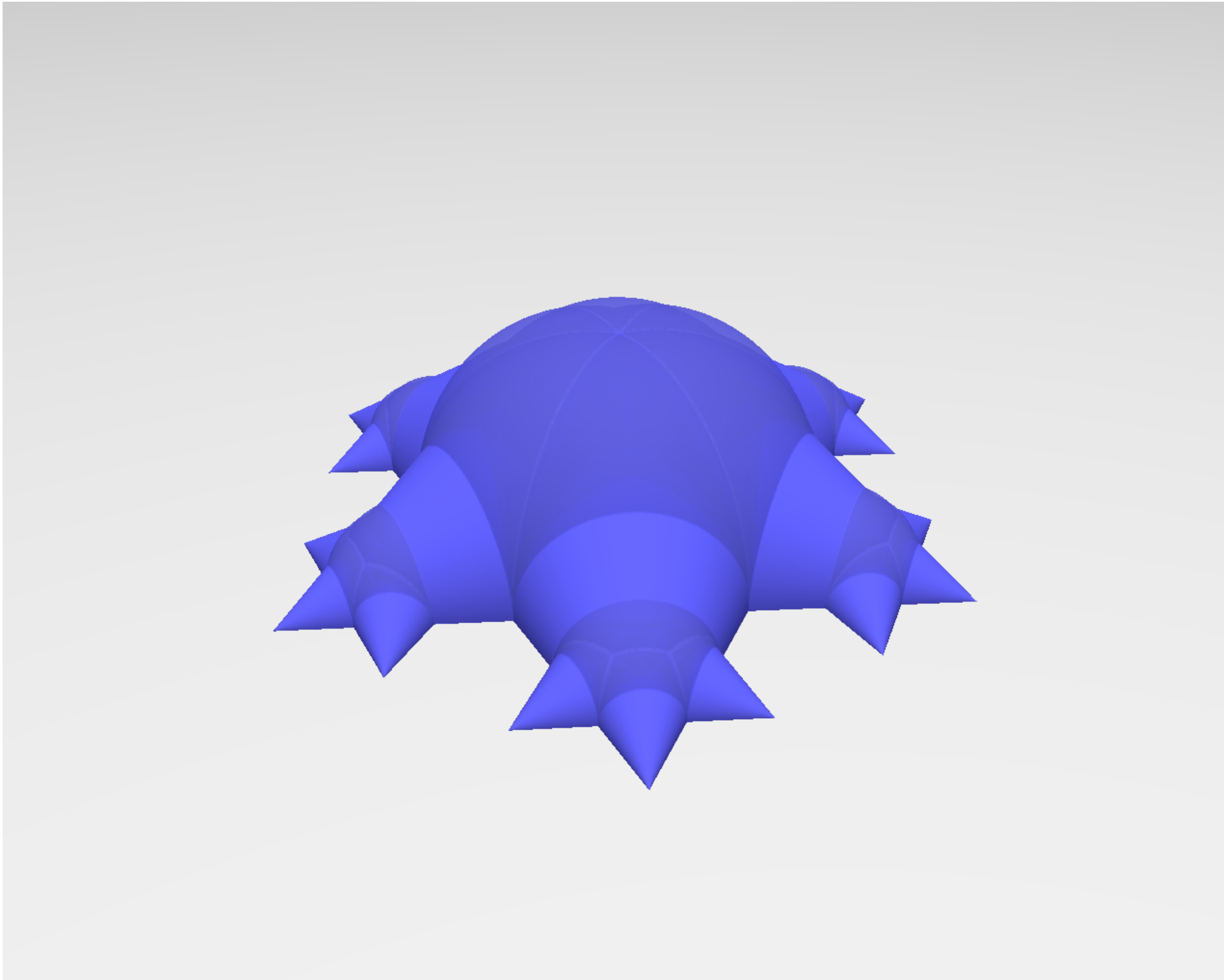


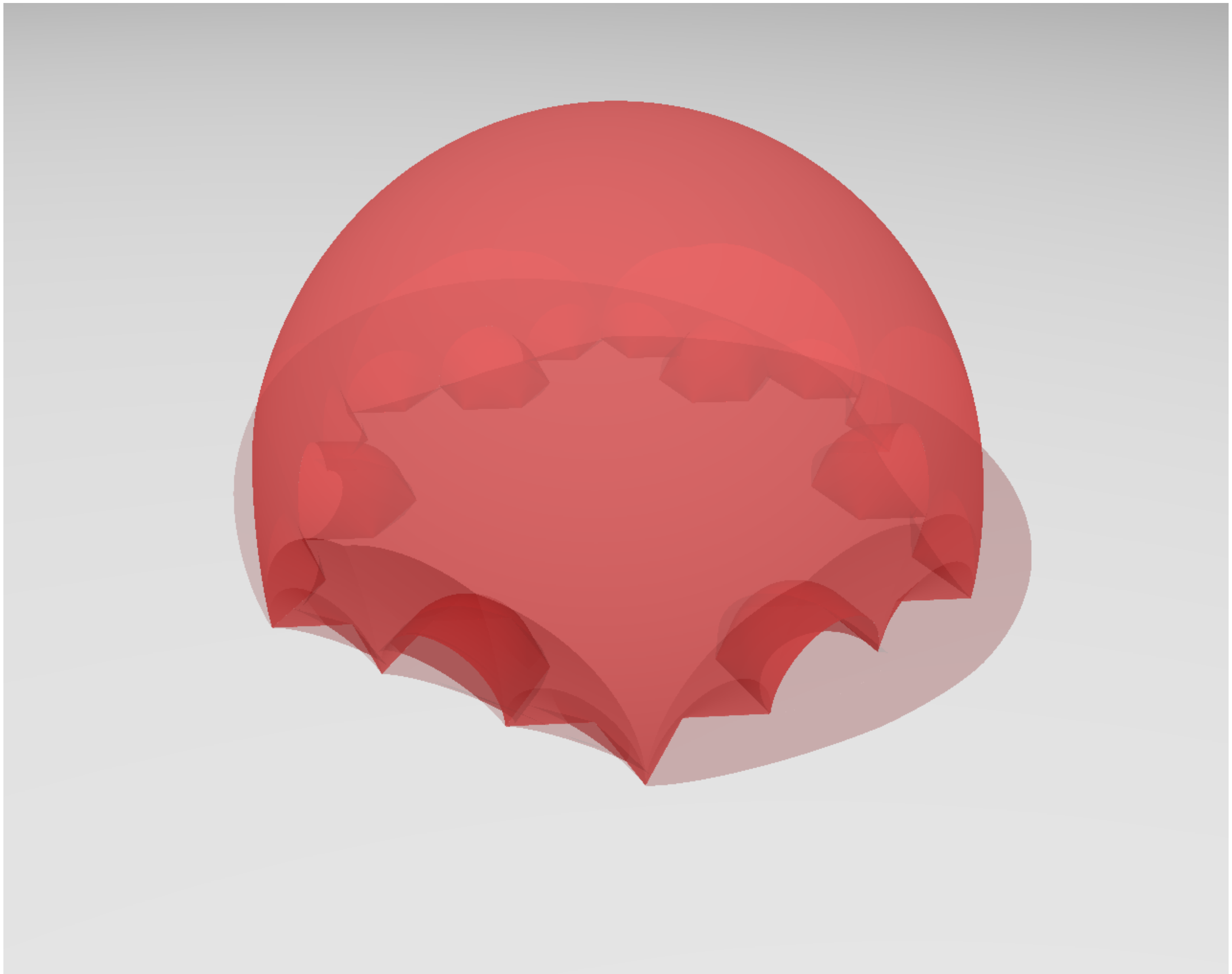


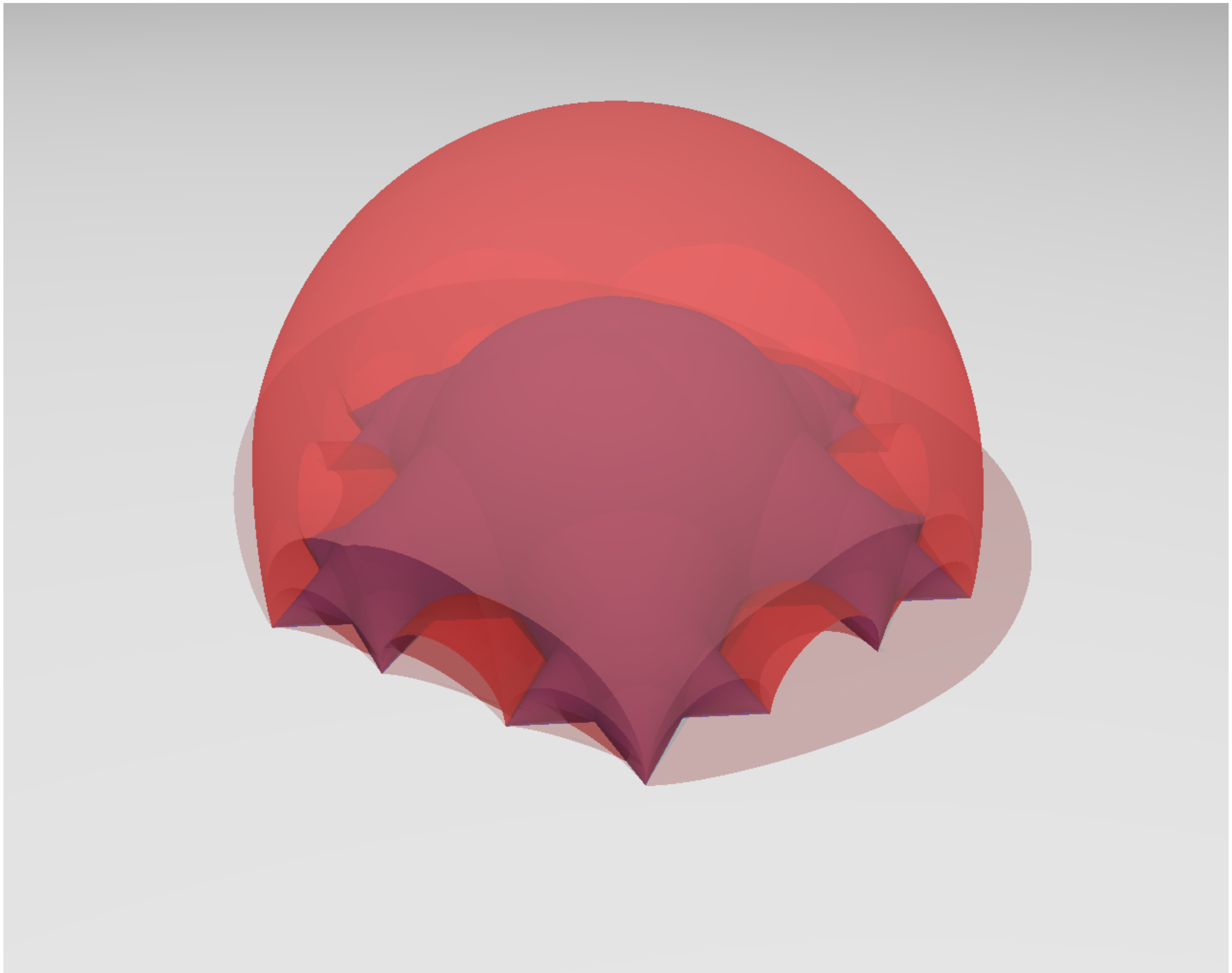








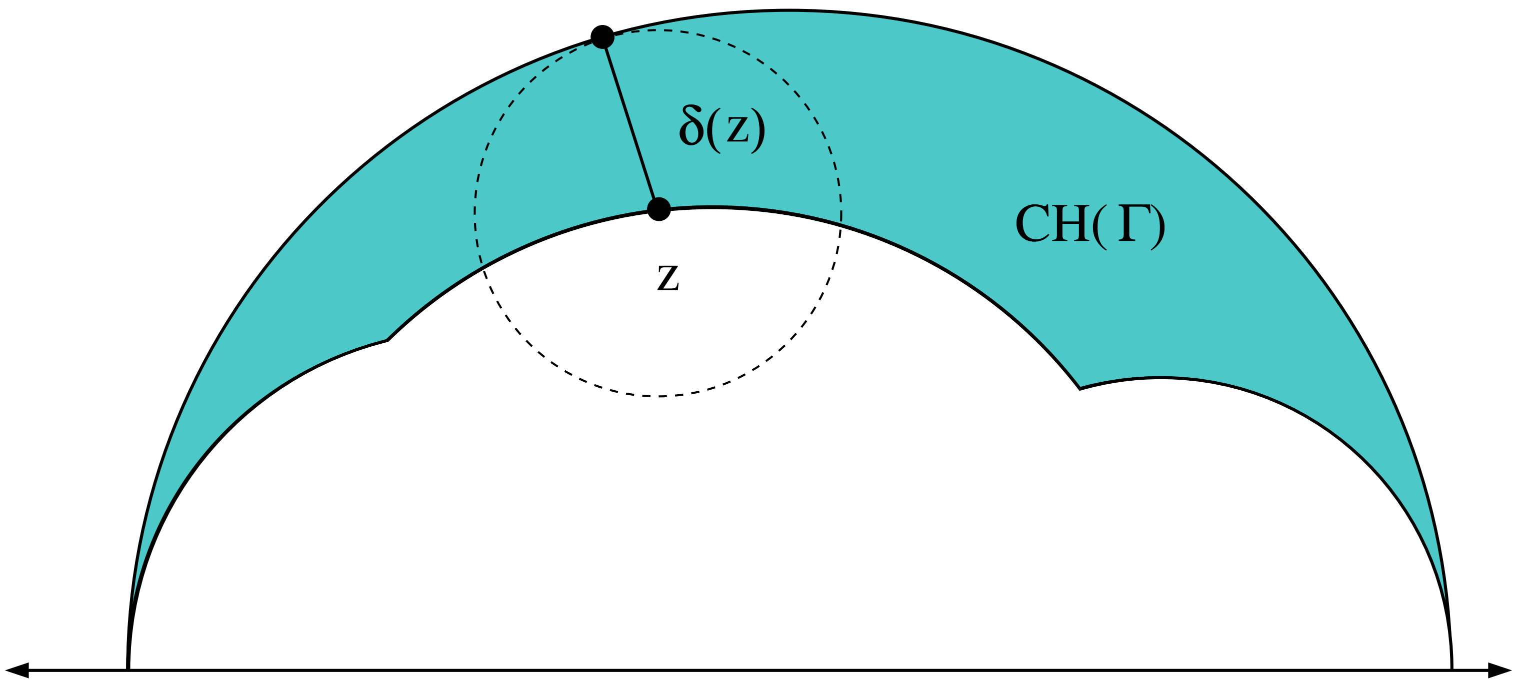




Except when Γ is a circle, $\mathrm{CH}(\Gamma)$ has non-empty interior and two boundary surfaces (both with asymptotic boundary Γ). We define $\delta(z)$ to be the maximum of the hyperbolic distances from z to the two boundary components of $\mathrm{CH}(\Gamma)$. This function serves as our Möbius invariant version of the β-numbers.

Instead of integrating over all points x in the plane and all scales $t > 0$, our hyperbolic Weil-Petersson criteria will involve integrating over points (x, t) on some surface $S \subset \mathbb{R}^3_+$ that has $\Gamma \subset \mathbb{R}^2$ as its asymptotic boundary; usually S will be one of the two connected components of $\partial\mathrm{CH}(\Gamma)$, the cylinder $\Gamma \times (0, 1]$, or a minimal surface contained in $\mathrm{CH}(\Gamma)$.

Let $\delta(z)$ be the hyperbolic distance to farther boundary component.



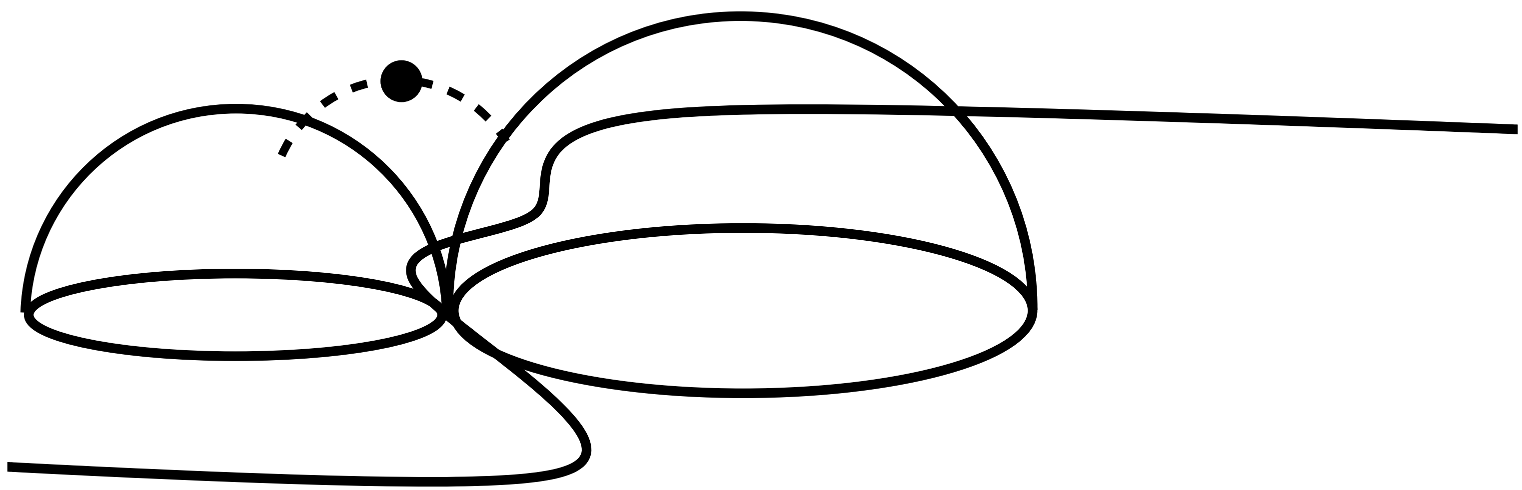
For $z \in \mathrm{CH}(\Gamma)$, $\delta(z)$ measures "width" of convex hull near z.

$\delta(z) = 0$ iff Γ is a circle (hull has no interior).

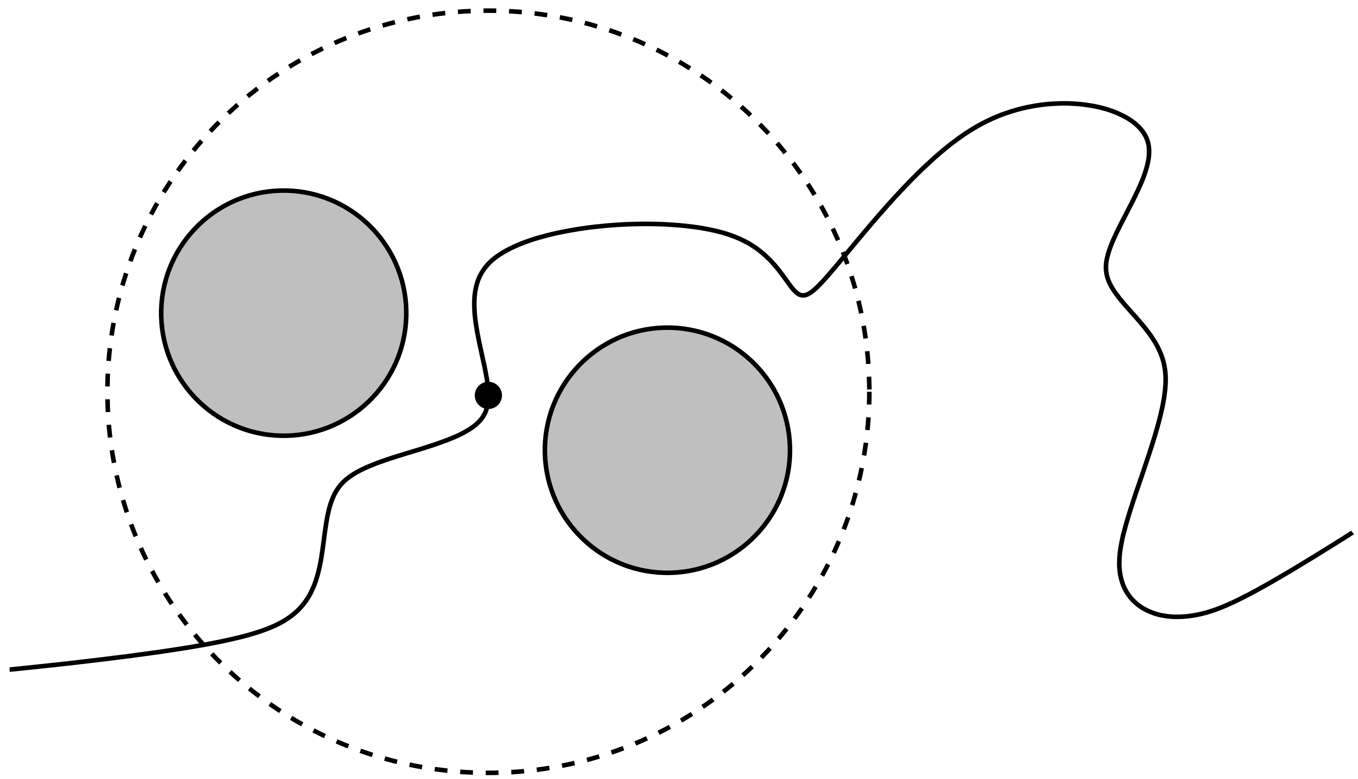
If $\delta(z) = \epsilon$ then $\delta = O(\epsilon)$ on a unit neighborhood of z (proof later).

Given a point $z$in the convex hull of Γ there is a geodesic segment through z this is perpendicular to half-spaces that have their boundaries on either side of Γ.

The hyperbolic length of this is approximately δ and is conformally invariant.



For quasicircles, $\delta(z) \in L^\infty$ (not conversely).



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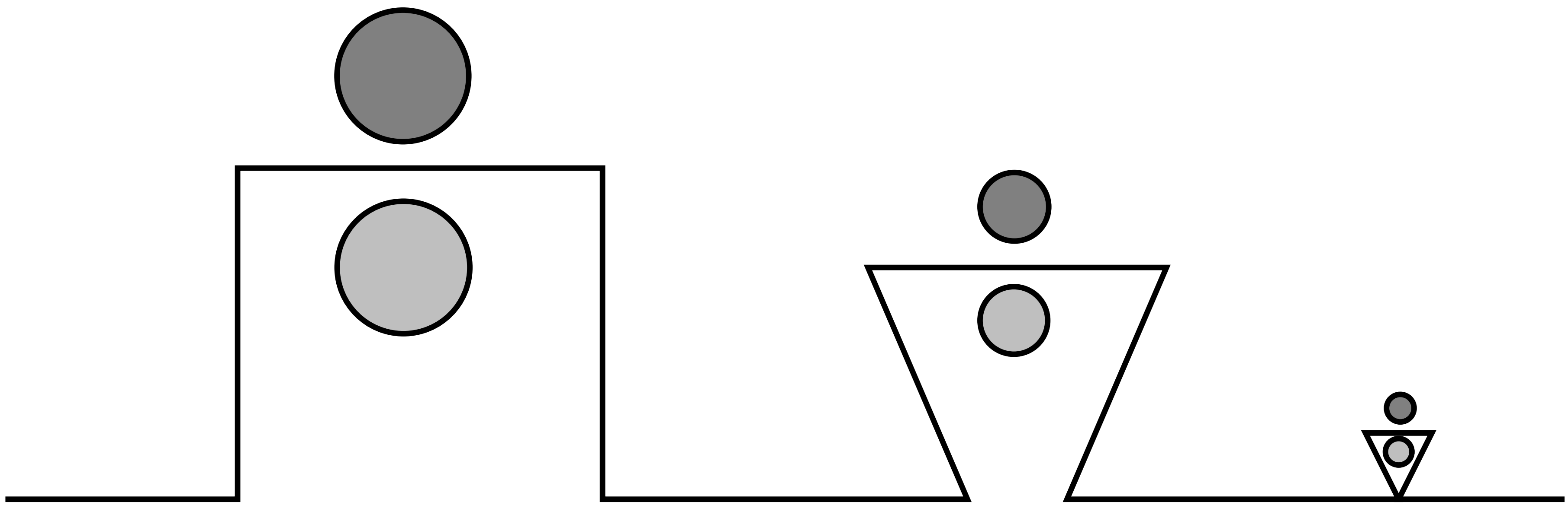


Suppose $S \subset \mathbb{R}^3_+$ is a 2-dimensional, properly embedded sub-manifold that has an asymptotic boundary that is a closed Jordan curve in $\mathbb{R}^2$. The Euler characteristic of S will be denoted $\chi(S)$, and equals $2-2g-h$ if S is a surface of genus g with h holes.

We let $K(z)$ denote the Gauss curvature of S at z. The hyperbolic metric $d\rho = ds/2t$ is chosen so that $\mathbb{R}^3_+$ has constant Gauss curvature -1. If the principle curvatures of S at z are $\kappa_1(z), \kappa_2(z)$, then $K(z) = -1 + \kappa_1(z)\kappa_2(z)$ (this is the Gauss equation). The norm of the second fundamental form is given by $|\mathcal{K}(z)|^2 = \kappa_1(z)^2 + \kappa_2(z)^2$.

The surface S is called a minimal surface if $\kappa_1 = -\kappa_2$ (the mean curvature is zero). In this case we will write $\kappa = |\kappa_j|$, $j = 1, 2$ and so $K(z) = -1 - \kappa^2(z)$.

Surface looks like a "saddle" with curvature κ in one direction and $-\kappa$ in perpendicular direction.

The surface S is called area minimizing if any compact Jordan region $\Omega \subset S$ has minimal area among all compact surfaces in $\mathbb{R}^3_+$ with the same boundary. All such surfaces are minimal, but not conversely.

Michael Anderson [6] has shown that every closed Jordan curve on $\mathbb{R}^2$ bounds a simply connected minimal surface in $\mathbb{R}^3_+$, but there may be other minimal surfaces with boundary Γ that are not disks (example later).

Any minimal surface S with boundary Γ is contained in $\mathrm{CH}(\Gamma)$ and the principle curvatures of S at a point z can be controlled by the function $\delta(z)$ introduced above. Let A_ρ denote hyperbolic area and ℓ_ρ hyperbolic length.

Minimal surface trapped between two parallel planes must itself be flat. Minimal surfaces analogous to harmonic functions: supremum controls gradient.

Theorem 1.8. *For a closed curve $\Gamma \subset \mathbb{R}^2$, the following are equivalent:*

(1) $\Gamma \subset \mathbb{R}^2$ *is a Weil-Petersson curve.*

(2) Γ *asymptotically bounds a surface* $S \subset \mathbb{R}^3_+$ *so that*

$$\int_S |\delta(z)|^2 d\mathrm{A}_\rho(z) < \infty.$$

(3) Γ *asymptotically bounds a surface* $S \subset \mathbb{R}^3_+$ *so that* $|\mathcal{K}(z)| \to 0$ *as* $z \to \mathbb{R}^2 = \partial\mathbb{R}^3_+$ *and*

$$\int_S |\mathcal{K}(z)|^2 d\mathrm{A}_\rho(z) < \infty.$$

(4) *Every minimal surface S asymptotically bounded by Γ has finite Euler characteristic and finite total curvature, i.e.,*

$$\int_S |\kappa(z)|^2 d\mathrm{A}_\rho(z) = \int_S |K(z) + 1| d\mathrm{A}_\rho(z) < \infty.$$

(5) *There is some minimal surface S with finite Euler characteristic and asymptotic boundary Γ so that S is the union of a nested sequence of compact Jordan subdomains $\Omega_1 \subset \Omega_2 \subset \dots$ with*

$$\limsup_{n\to\infty} \left[\ell_\rho(\partial\Omega_n) - \mathrm{A}_\rho(\Omega_n)\right] < \infty.$$

Usual Euclidean isoperimetric inequality

$$L^2 \geq 4\pi A,$$

where A is area, L is boundary length.

In a space of constant negative curvature

$$L^2 \geq 4\pi A\chi + A^2,$$

where $\chi = \chi(\Omega)$ is the Euler characteristic of Ω. Hence

$$L - A \geq \frac{4\pi A\chi}{L + A}$$

For topological disk, $\chi = 1$, so

$$L - A \geq \frac{4\pi A}{L + A} > 0$$

A simply connected minimal surface corresponds to WP curve iff $L - A$ is bounded above for some exhaustion of surface by compact Jordan subdomains.

For surface in upper half-space with boundary on $\mathbb{R}^2$, we can form sub-domains by cutting at a certain height.

Truncate $S \subset \mathbb{R}^3_+$ at a fixed height above the boundary, i.e.,

$$S_t = S \cap \{(x, y, s) \in \mathbb{R}^3_+ : s > t\}, \quad \partial S_t = S \cap \{(x, y, s) \in \mathbb{R}^3_+ : s = t\}$$

and define the renormalized area of S to be

$$\mathcal{RA}(S) = \lim_{t \searrow 0} \left[\mathrm{A}_\rho(S_t) - \ell_\rho(\partial S_t) \right]$$

when this limit exists and is finite.

This was introduced by Graham and Witten [32]. Related to quantum entanglement, black holes,...

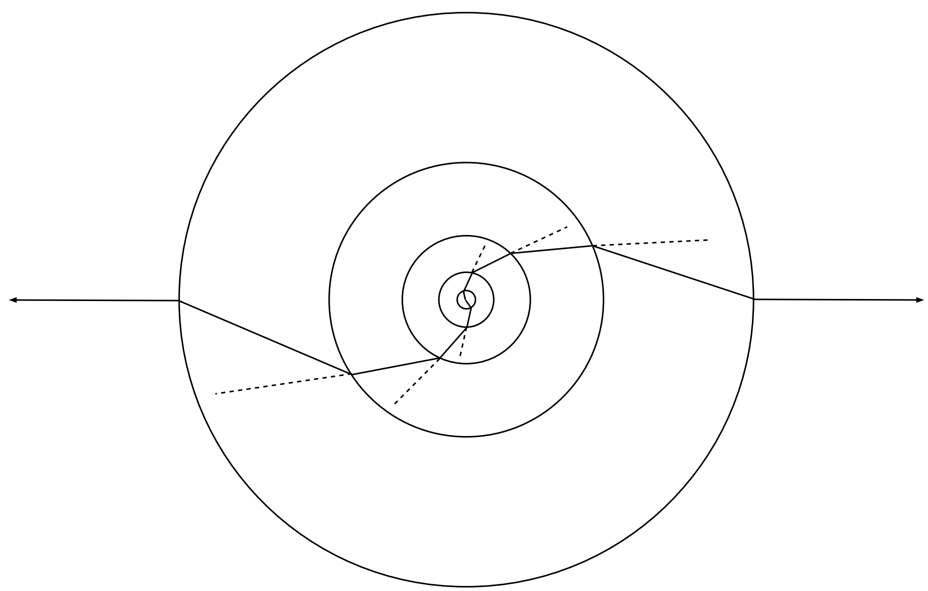
Corollary 1.9. *For any closed curve $\Gamma \subset \mathbb{R}^2$ and for any minimal surface $S \subset \mathbb{R}^3_+$ with finite Euler characteristic and asymptotic boundary Γ,*

$$\mathcal{RA}(S) = -2\pi\chi(S) - \int_S \kappa^2(z) d\mathrm{A}_\rho, \tag{1.8}$$

In other words, either Γ is Weil-Petersson and both sides are finite and equal, or Γ is not Weil-Petersson and both sides are $-\infty$.

Proposition 3.1 of Alexakis and Mazzeo's paper [5] gives a version of (1.8) for surfaces in the setting of n-dimensional Poincaré-Einstein manifolds (that formula also contains a term involving the Weyl curvature), but they use the additional assumption that Γ is $C^{3,\alpha}$.

However, Weil-Petersson curves need not be even C^1, in general. We can build examples with infinite spirals. Need angles to satisfy $\sum \theta_n = \infty$, but $\sum \theta_n^2 < \infty$.



Corollary 1.9 shows that the Alexakis-Mazzeo result holds without any conditions on Γ, at least in the case of $\mathbb{R}^3_+$. Their proof uses power series expansions near boundary; need to control several terms.

Our proof of Corollary 1.9 will show that the exact method of truncation in the definition of renormalized area is not important, and that it can be defined intrinsically on S, without explicit reference to the boundary:

Corollary 1.10. *Suppose $S \cup_n K_n \subset \mathbb{R}^3_+$ is a minimal surface where $K_1 \subset K_2 \subset \dots$ are nested compact sets such that $S \setminus K_n$ is a topological annulus for all n. Then*

$$-2\pi\chi(S) - \int_S \kappa^2(z) d\mathrm{A}_\rho, = \mathcal{RA}(S) = \lim_{n\to\infty} \sup_{\Omega \supset K_n} [\mathrm{A}_\rho(\Omega) - \ell_\rho(\partial\Omega)]$$

where the supremum is over compact domains $K_n \subset \Omega \subset S$ bounded by a single Jordan curve. All terms are finite and equal, or all are $-\infty$.

Möbius energy is also an example of renormalization, namely the Hadamard regularization of a divergent integral.

$$\mathrm{M\ddot{o}b}(\Gamma) = \int_\Gamma \int_\Gamma \left(\frac{1}{|x-y|^2} - \frac{1}{\ell(x,y)^2} \right) dx dy < \infty.$$

Given the divergent integral of a function that blows up on a set E, this is defined by integrating the function outside a t-neighborhood of E, writing the result as power series in t, and taking the constant term of this series as the renormalized value of the integral (of course, this depends on exactly how we choose the neighborhoods).

To apply Hadamard renormalization to Möbius energy, note that the integral of the first term in (1.1) is infinite, but for smooth curves the truncated version equals

$$\iint_{\ell(x,y)>t} \frac{dxdy}{|x-y|^2} = \frac{2\ell(\Gamma)}{t} + C + O(t). \tag{1.9}$$

Regularizing the other term in (1.1) (e.g., Lemma 2.3 of [34]) gives

$$\iint_{\ell(x,y)>t} \frac{dxdy}{\ell(x,y)^2} = \frac{2\ell(\Gamma)}{t} - 4, \tag{1.10}$$

so that $\text{Möb}(\Gamma) = 4 + C$.

The divergent integral in (1.9) is the energy of arclength measure with respect to a inverse cube law, e.g., electrostatics in four dimensions. It is infinite because Brownian motion in $\mathbb{R}^4$ almost surely misses any rectifiable curve, but Weil-Petersson curves are exactly those for which the electrostatic energy of arclength measure blows up as slowly as possible (up to an additive constant).

Incidentally, the Loewner energy of Γ can also be written as type of renormalization involving the Lawler-Werner Brownian loop measure of random closed curves hitting both sides of a neighborhood of Γ. This measure tends to infinity as the neighborhood shrinks, but subtracting the corresponding quantity for a circle gives a multiple of $\mathcal{LE}(\Gamma)$ in the limit.

Thursday, Sept 10 , 2020

Class canceled

Tuesday Sept 15, 2020

Since Loewner energy, Möbius energy and renormalized area are all Möbius invariant quantities that characterize Weil-Petersson curves, it seems natural to ask if they are essentially the same quantity, or at least comparable in size.

There are examples showing that for any large M we can have

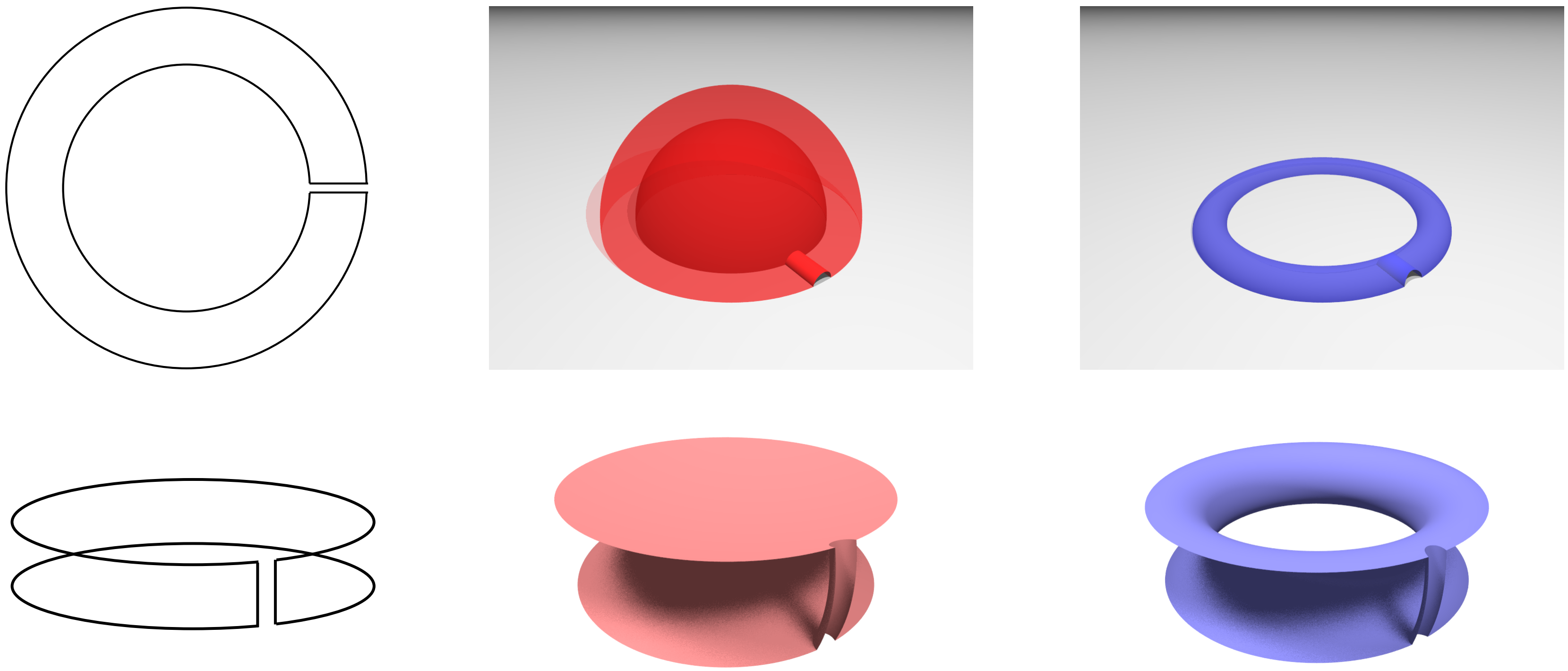
(1) $\mathcal{LE}(\Gamma_1) \simeq \text{Möb}(\Gamma_1) \simeq \mathcal{RA}(\Gamma_1) \simeq M$.

(2) $\mathcal{LE}(\Gamma_2) \simeq \mathcal{RA}(\Gamma_2) \simeq M$ but $\text{Möb}(\Gamma_2) \simeq M \log M$.

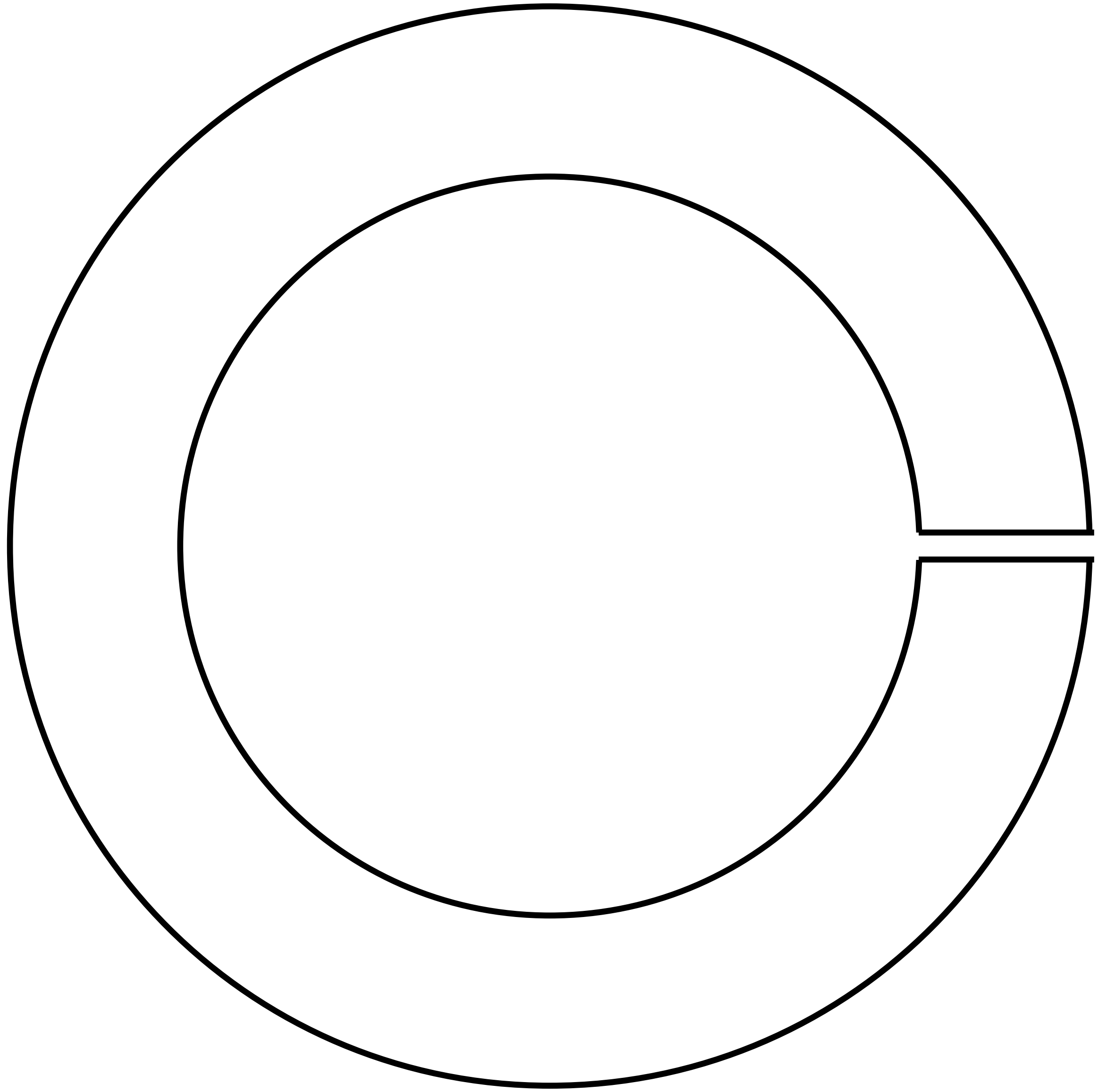
(3) $|\mathcal{LE}(\Gamma_3) - \mathcal{LE}(\Gamma_4)| \simeq M \simeq |\mathcal{RA}(\Gamma_3)|$ but $|\mathcal{RA}(\Gamma_3) - \mathcal{RA}(\Gamma_4)| \simeq 1$.

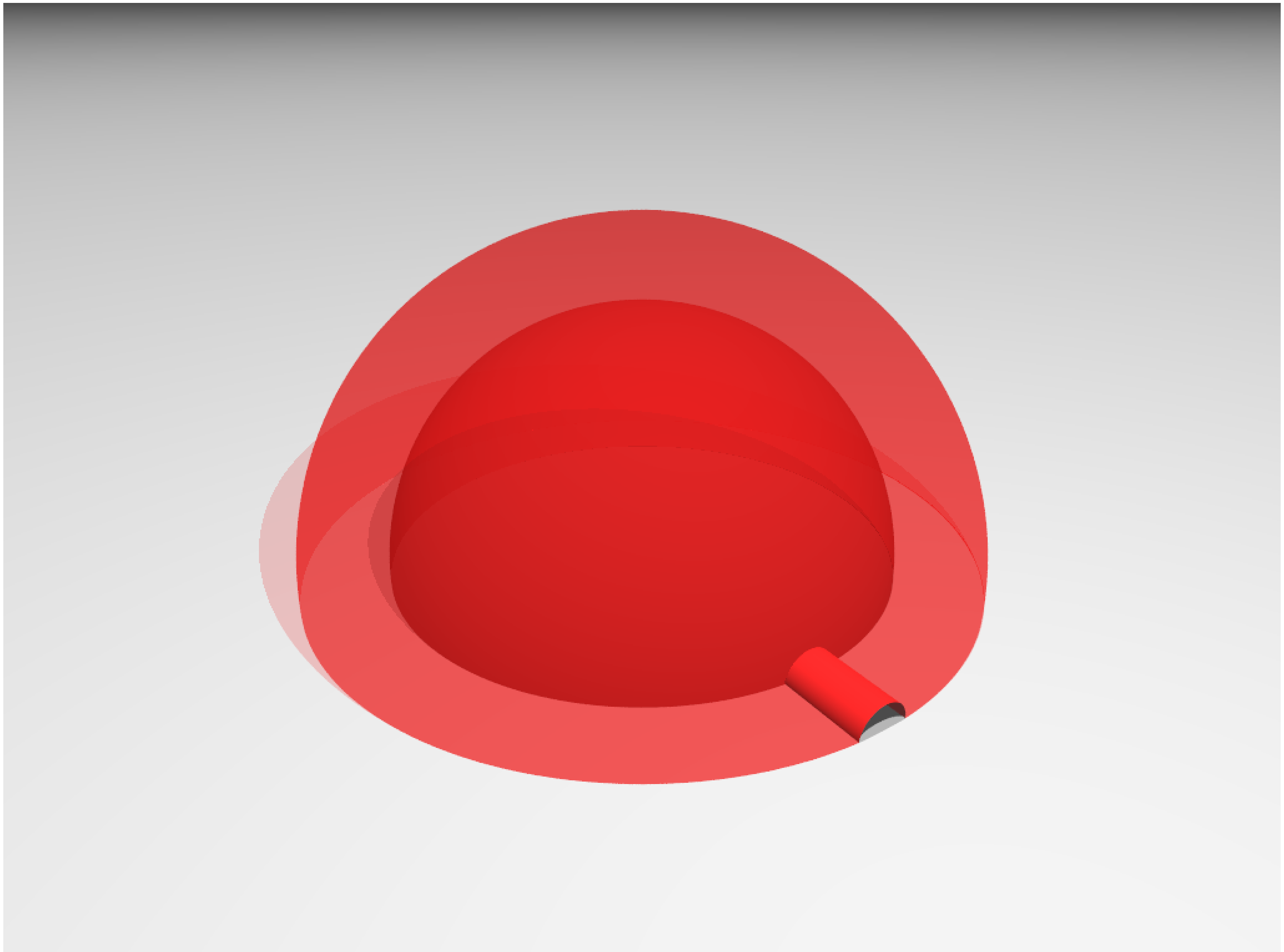
Thus it is unclear whether there is any simple relation among these quantities.

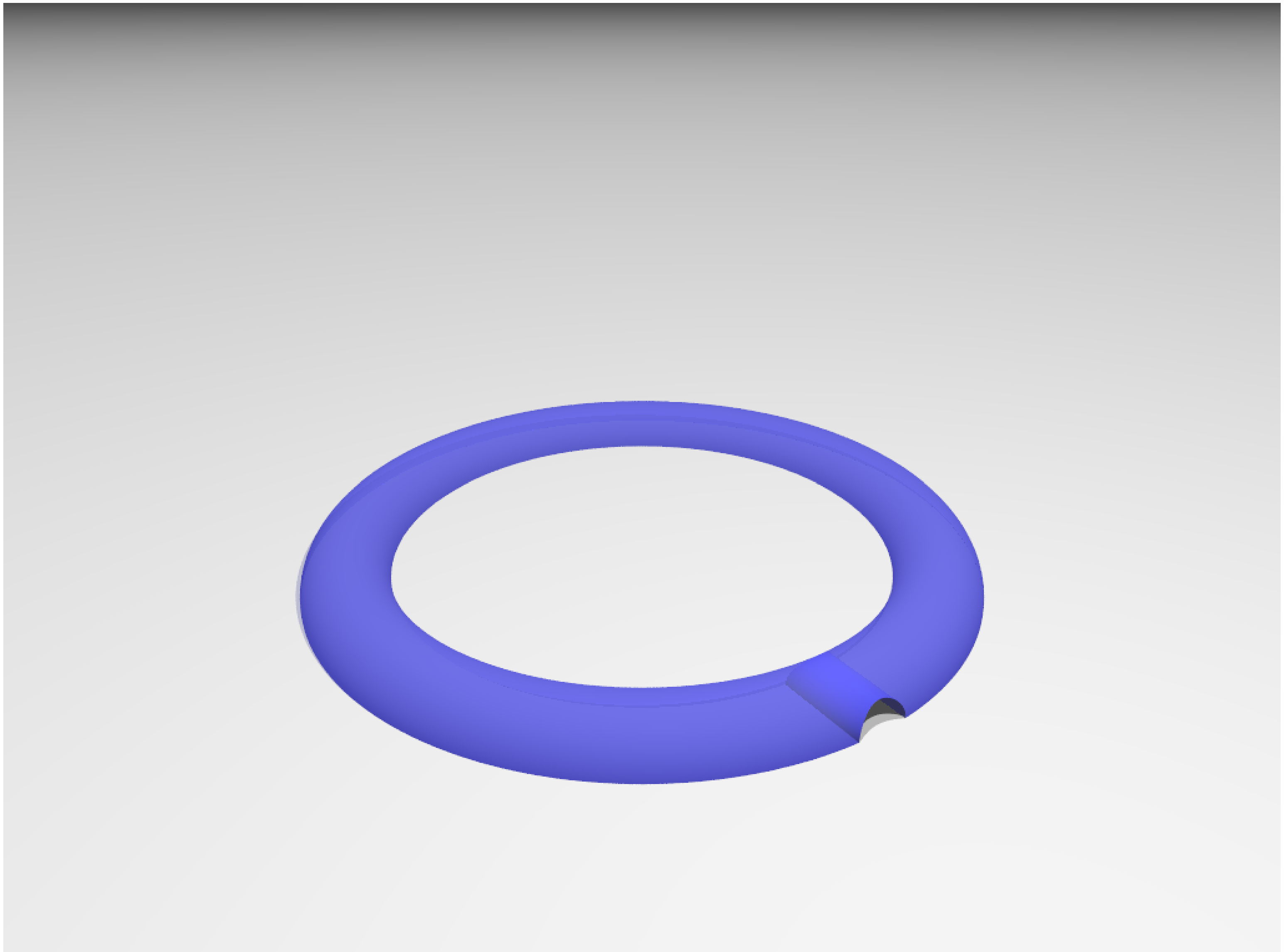
In these estimates, $\mathcal{RA}(\Gamma)$ denotes $\mathcal{RA}(S)$ for some choice of minimal surface S with asymptotic boundary Γ. This surface need not be unique and a different choice can lead to very different values of the renormalized area.

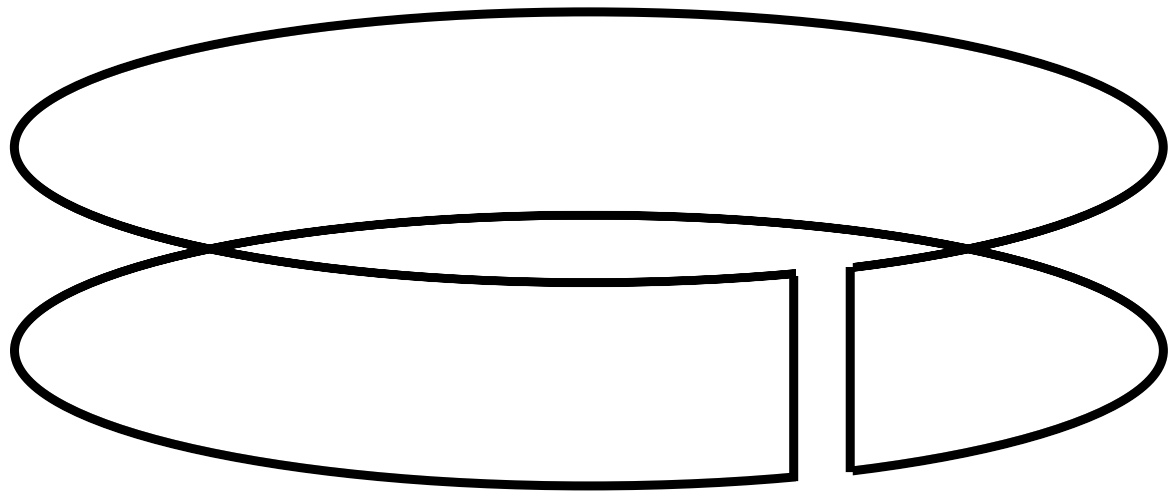


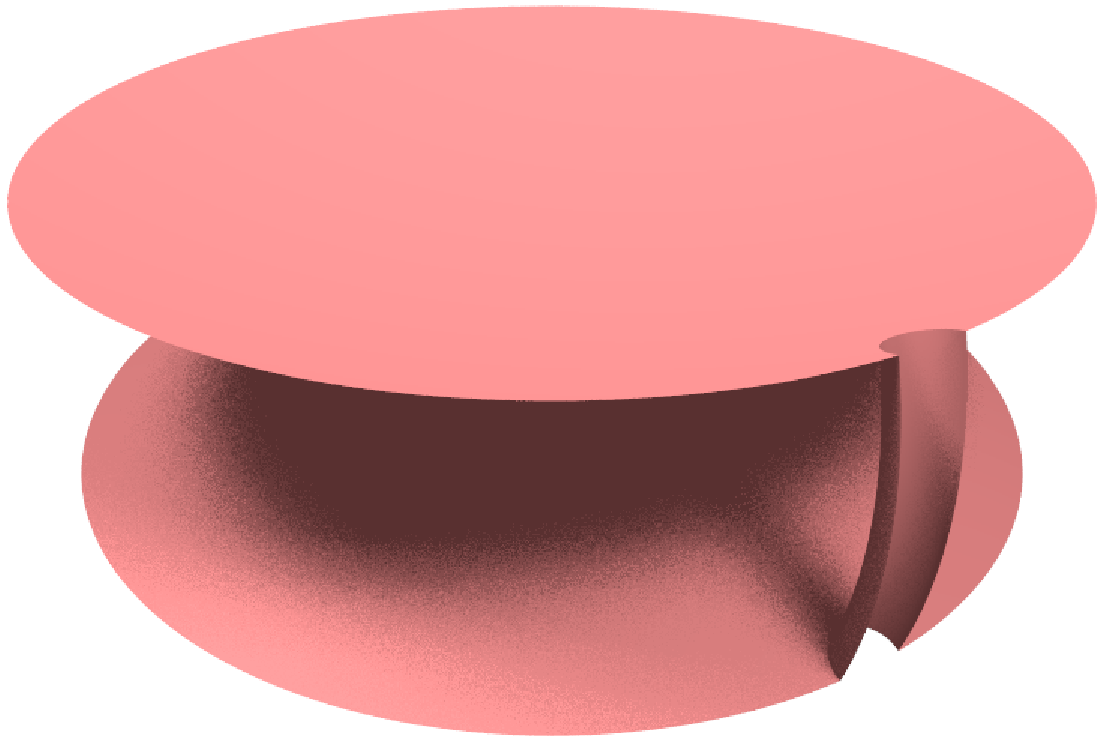
The example from Anderson's paper [6] where the absolute area minimizer is not simply connected; it can have very different renormalized area from the simply connected minimal surface. Top row shows the upper half-space model and the lower row is in the ball model.

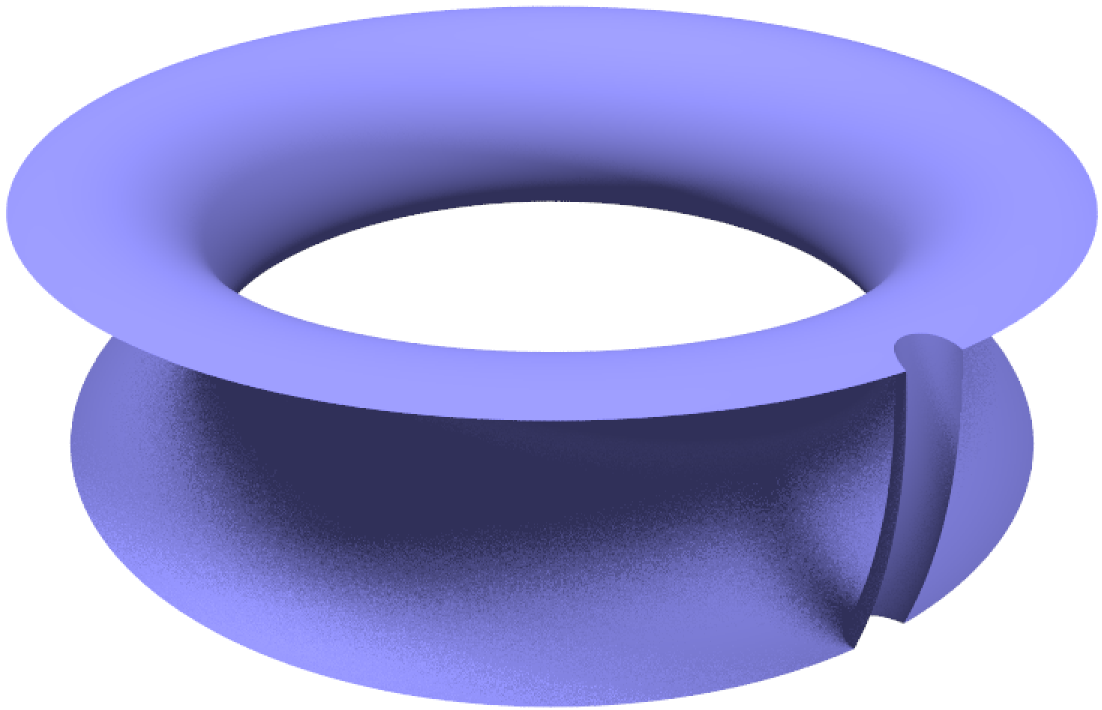












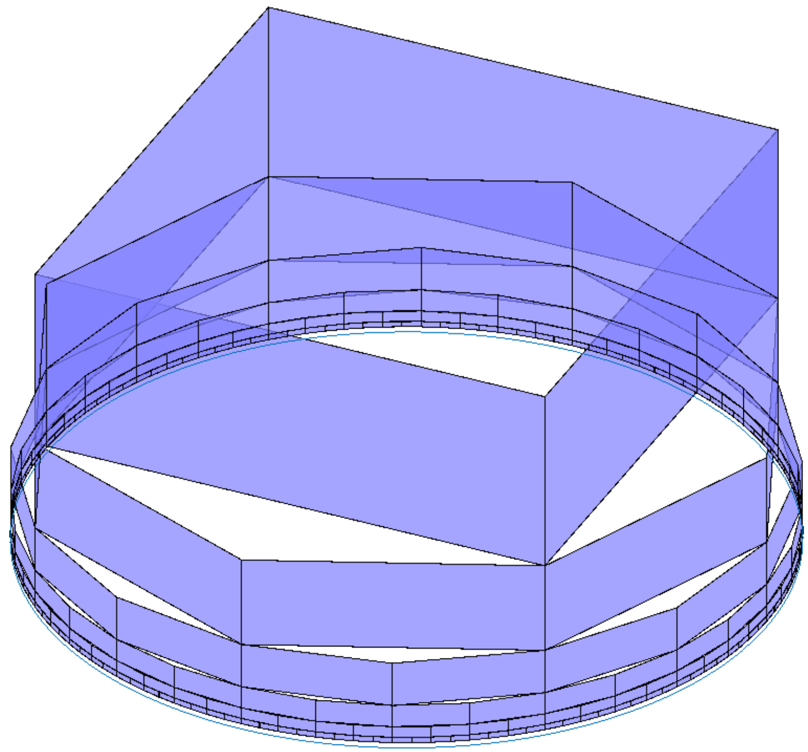
The dyadic dome

There is a discrete version of this that might be relevant to computational questions, and that illustrates the connection between our Euclidean and hyperbolic conditions.

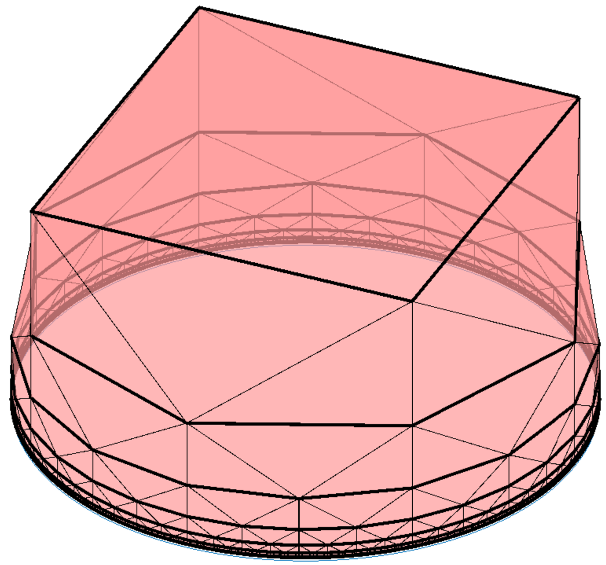
Define the "dyadic cylinder"

$$X = \bigcup_{n=1}^{\infty} \Gamma_n \times [2^{-n}, 2^{-n+1}),$$

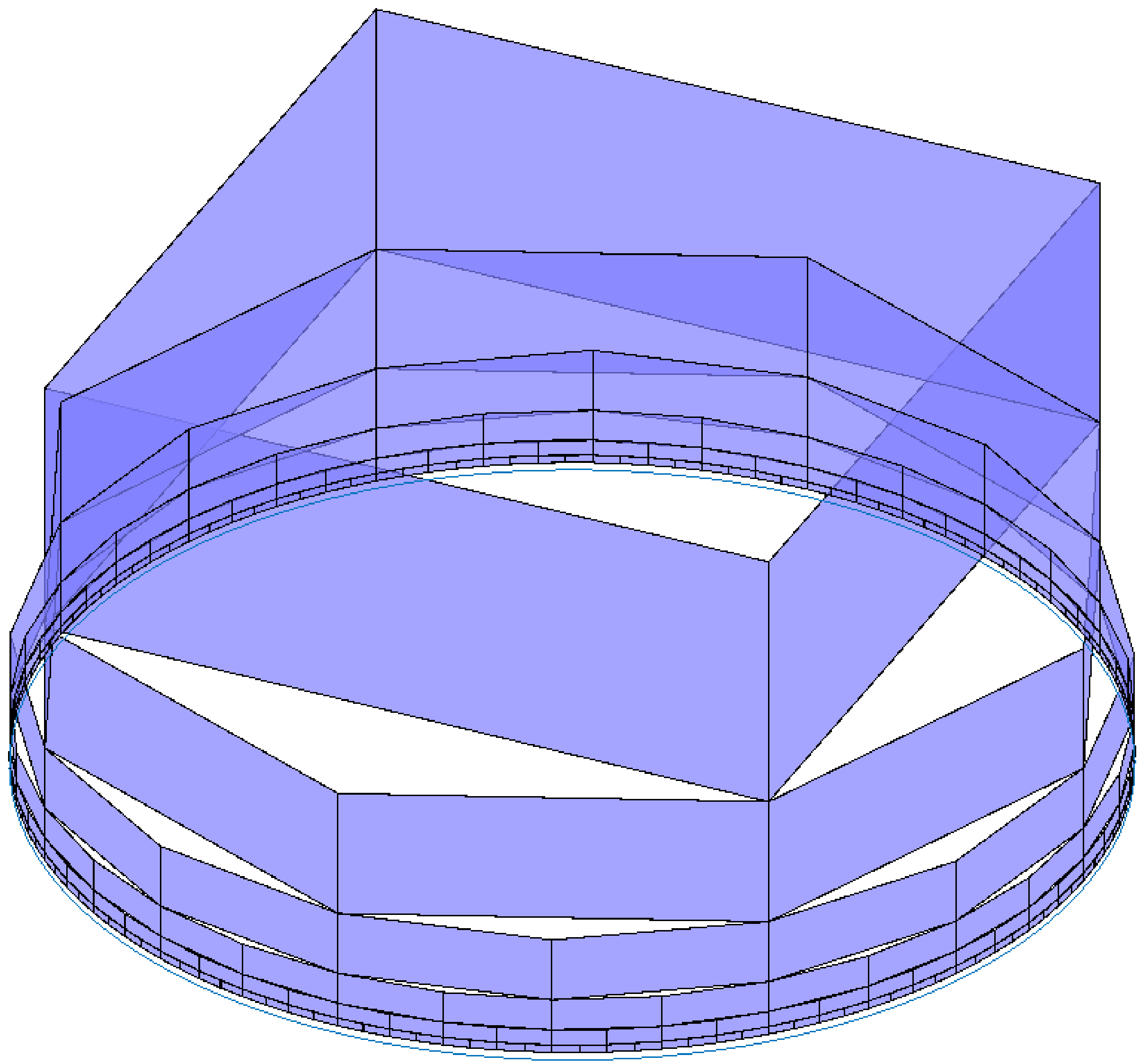
where $\{\Gamma_n\}$ are the dyadic polygonal approximations to Γ, as in Theorem 1.4.

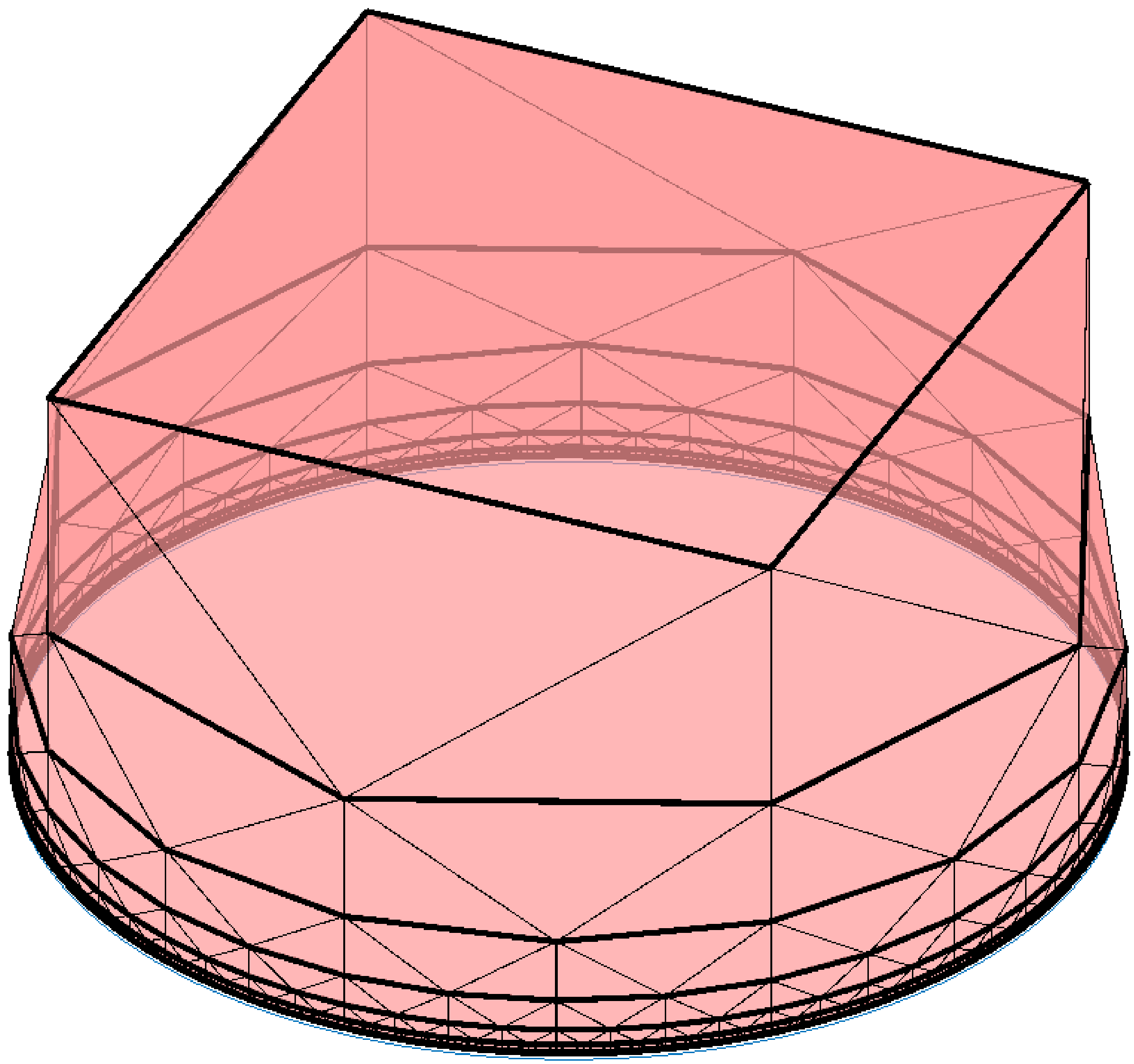


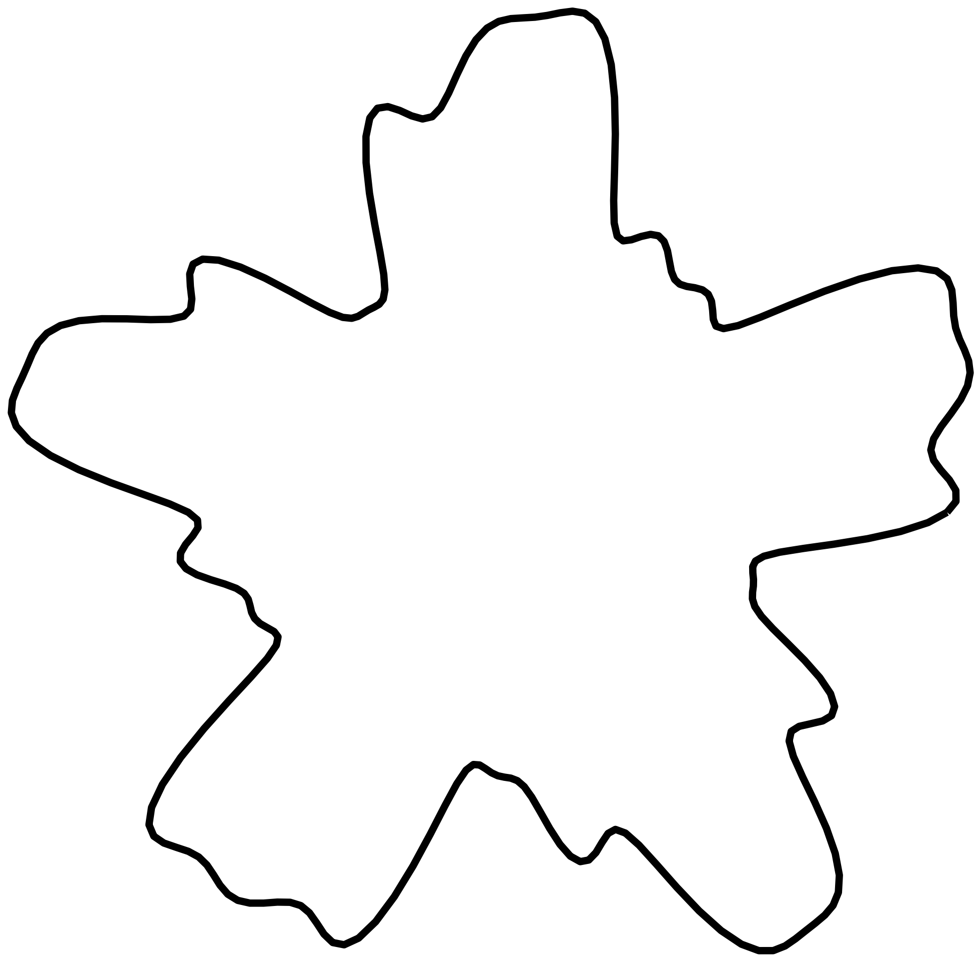
X is a union of vertical rectangular panels that do not quite meet up, so it has holes, but we can replace the vertical rectangles by tilted triangles that form a simply connected surface, the "dyadic dome",

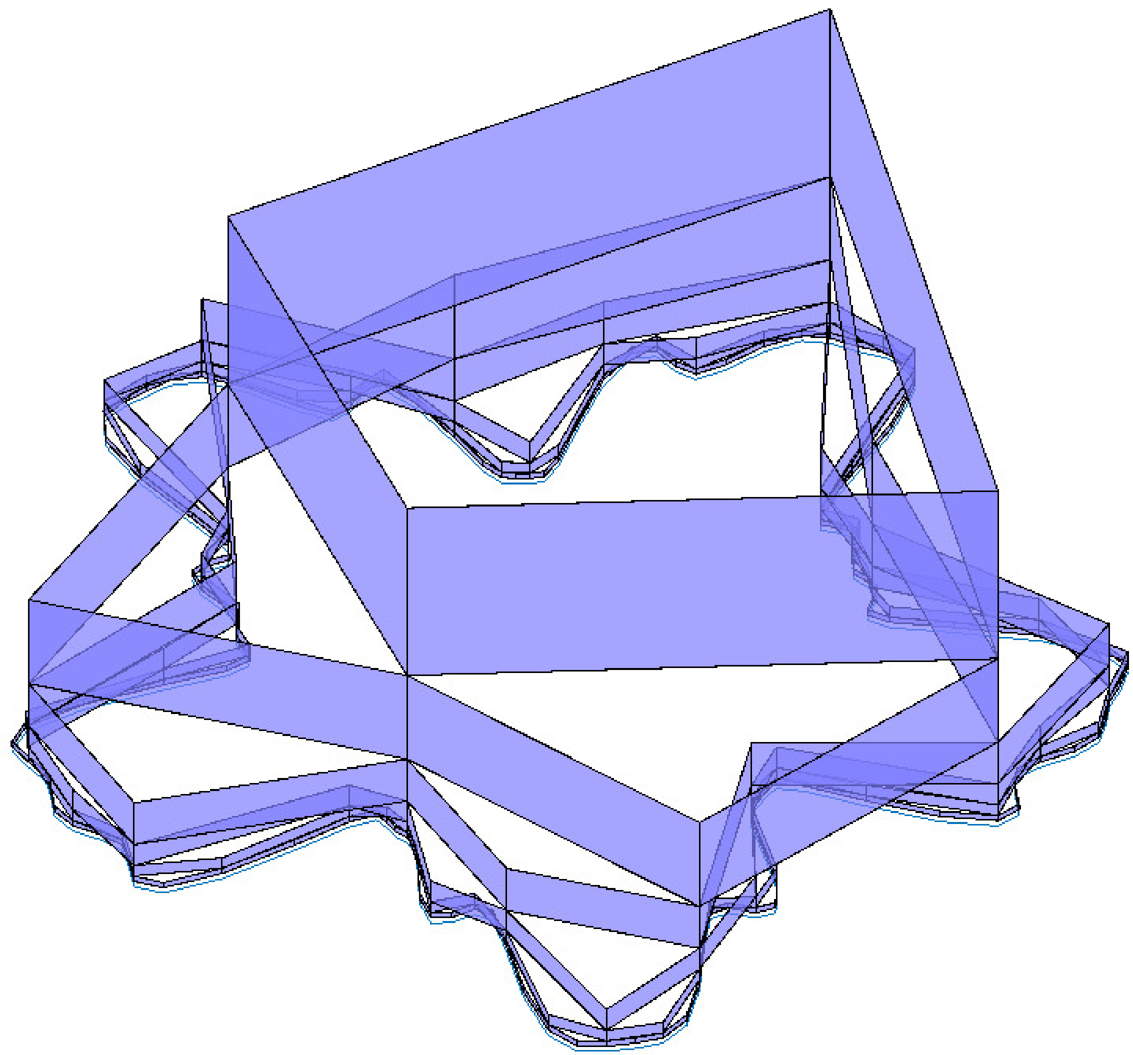


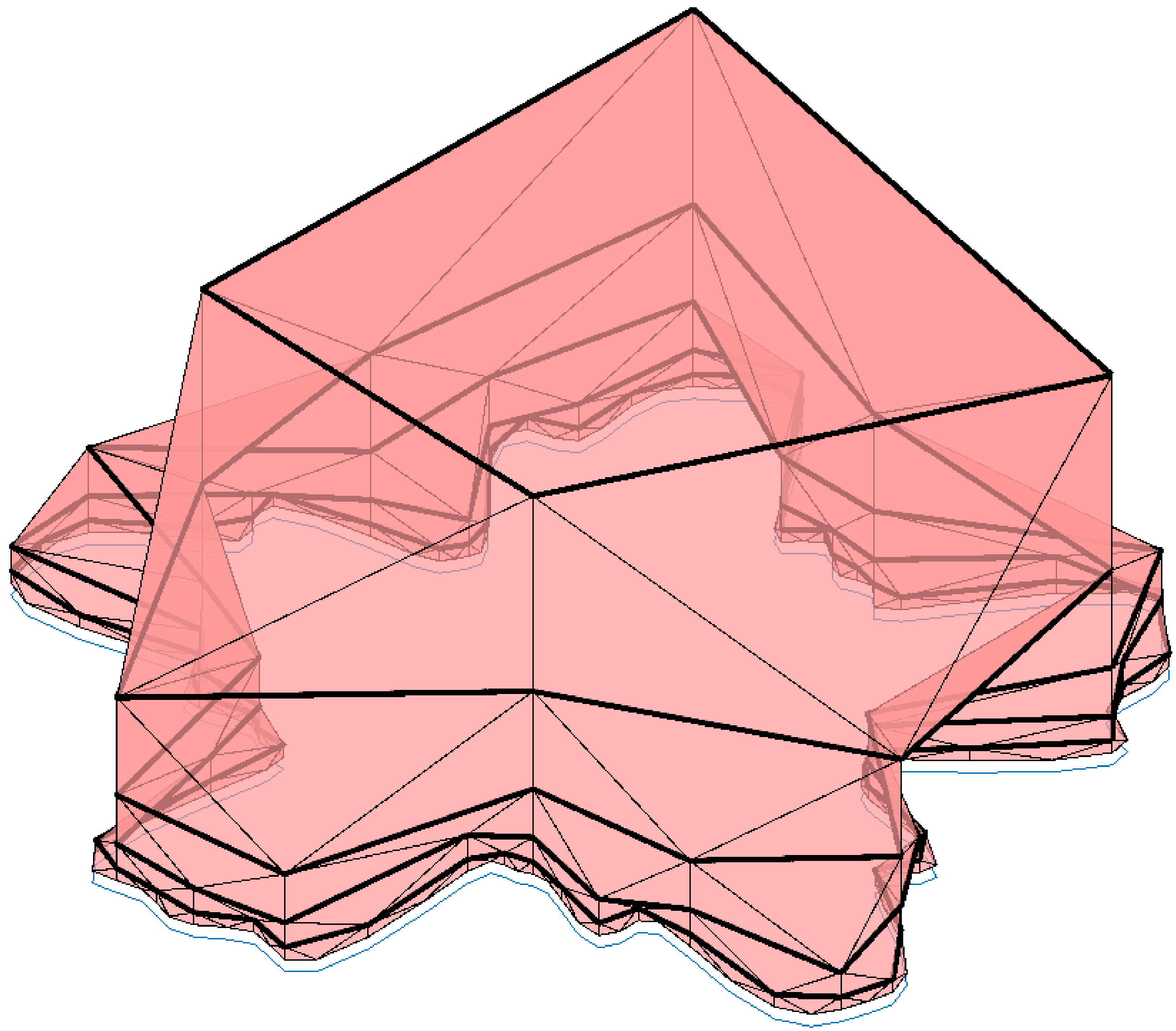
Theorem 1.11. Γ *is Weil-Petersson iff the corresponding dyadic dome has finite renormalized area.*

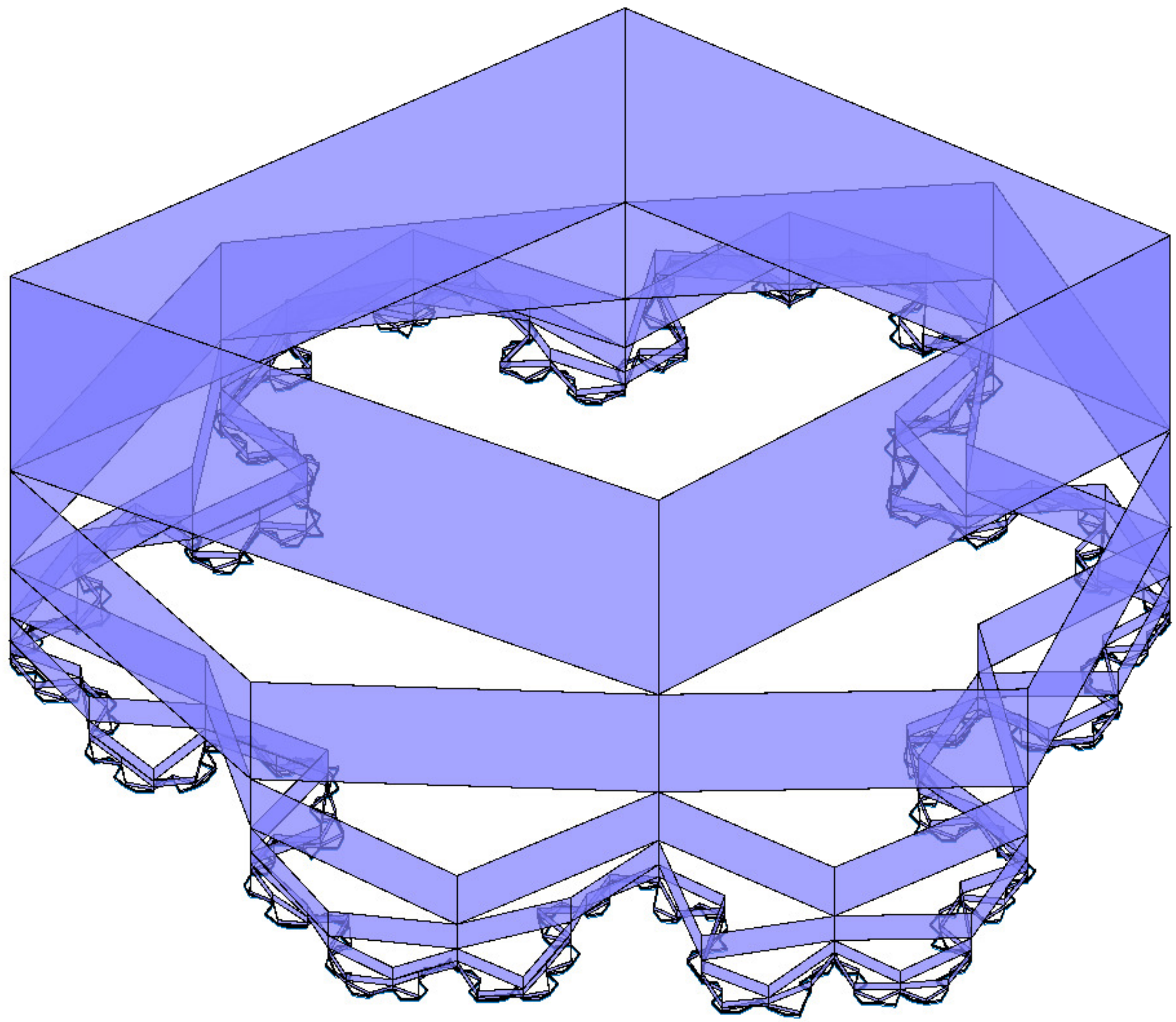


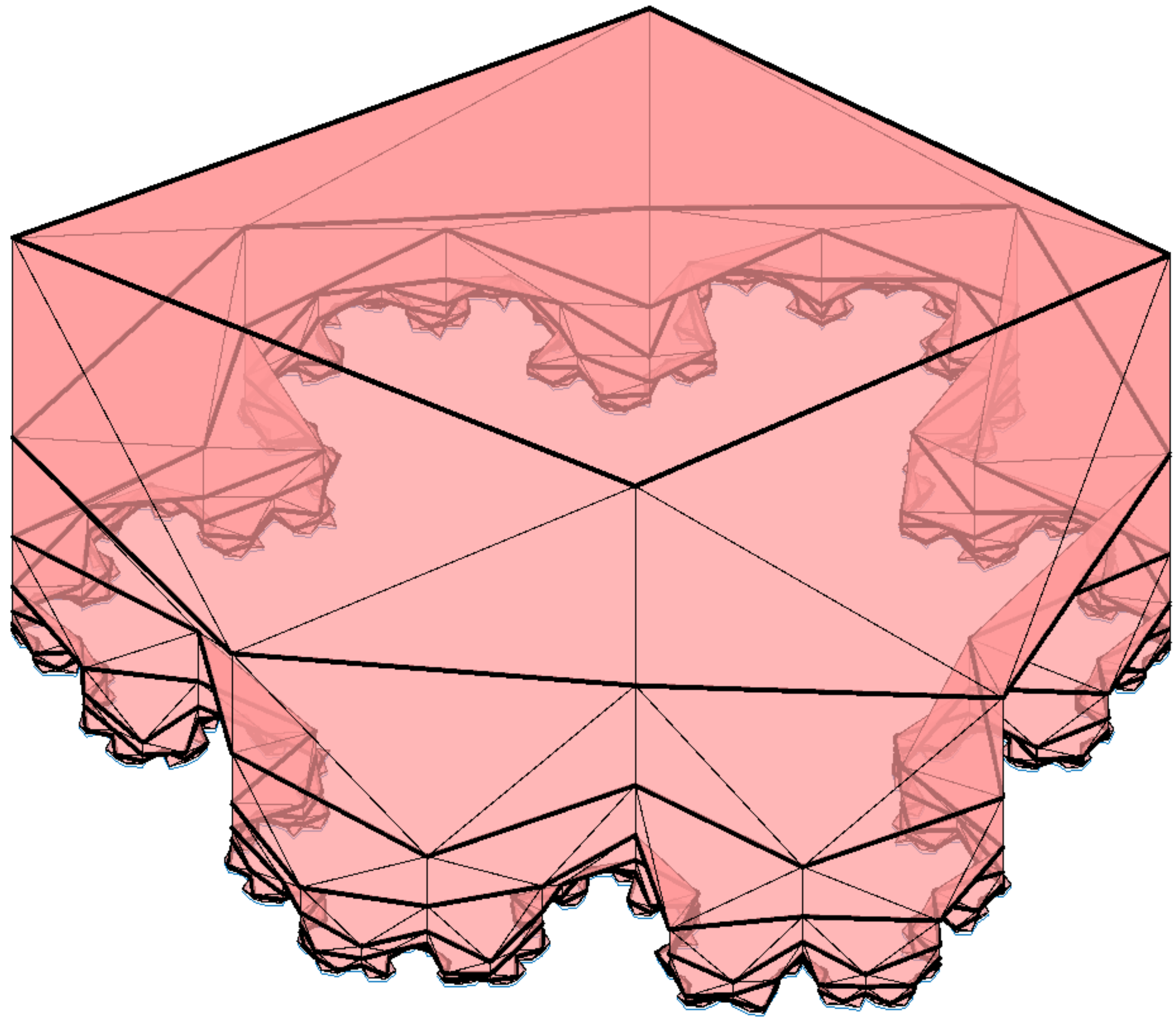










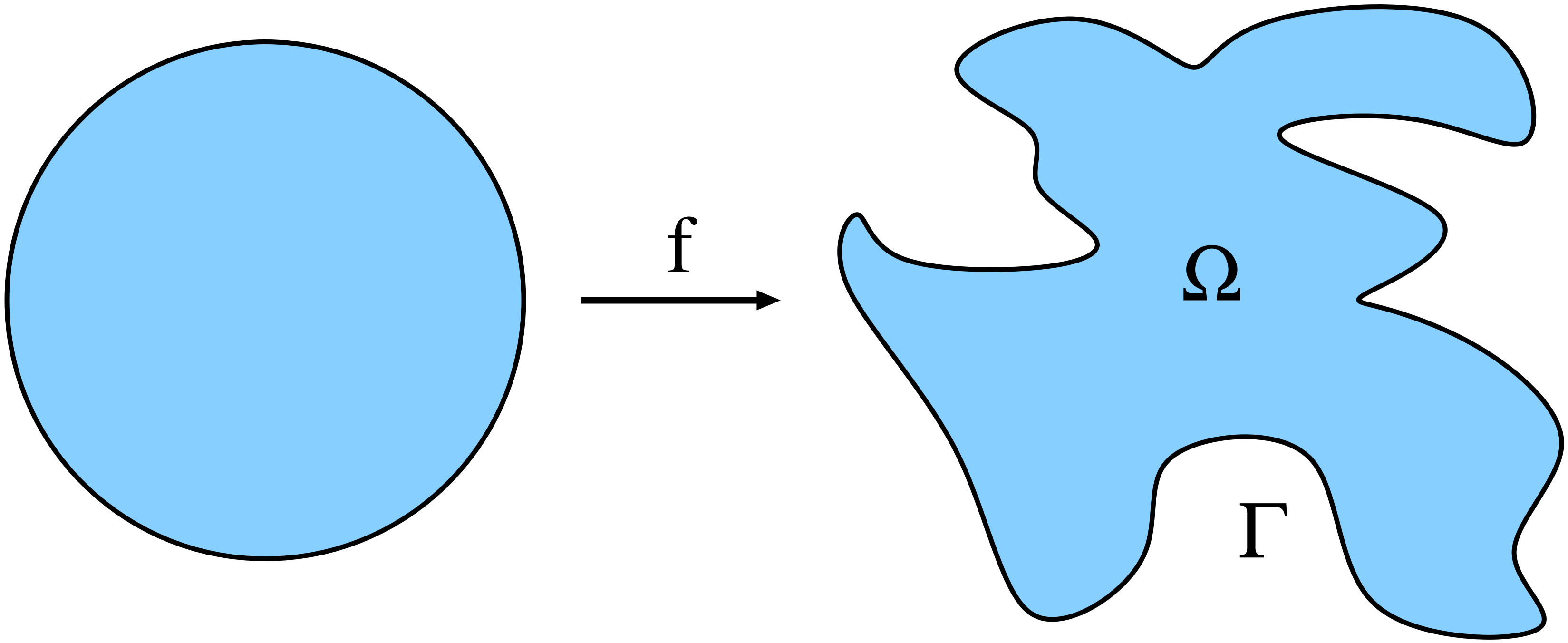


2. Function theoretic definitions of the Weil-Petersson class

A quasiconformal (QC) map h of a planar domain Ω is a homeomorphism of Ω to another planar domain Ω' that is absolutely continuous on almost all lines and whose dilatation $\mu = h_{\overline{z}}/h_z$ is satisfies $\|\mu\|_\infty \leq k < 1$. See [3] or [40] for the basic properties of such maps.

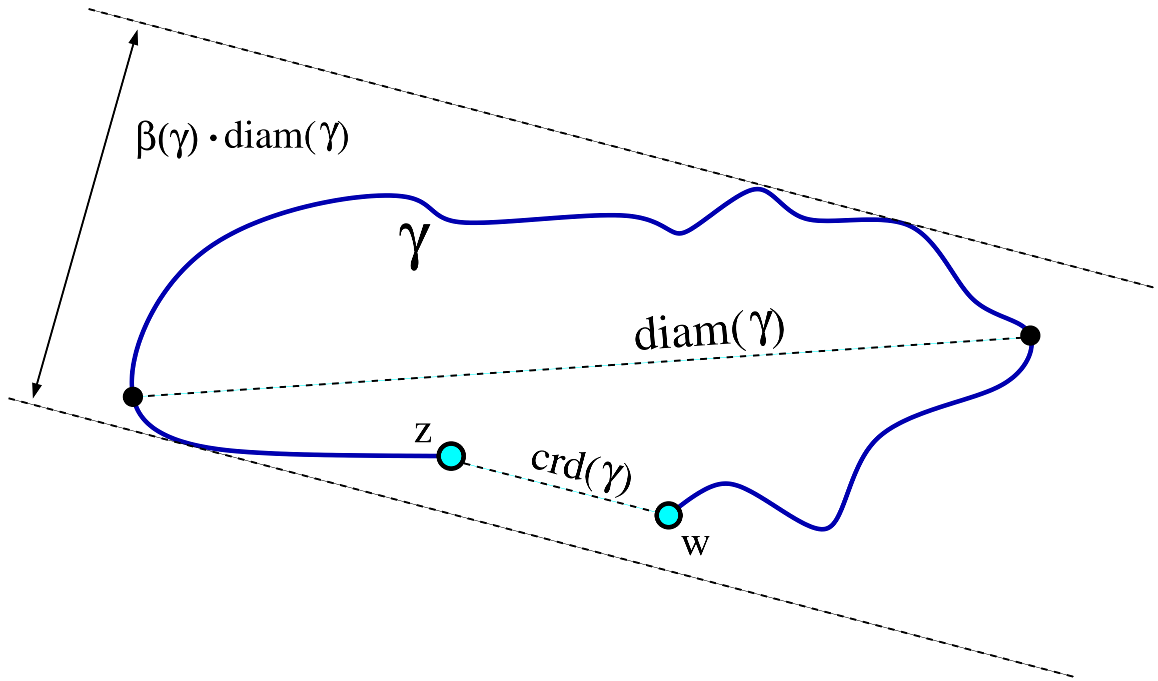
We say the h is a planar quasiconformal map if $\Omega = \Omega' = \mathbb{R}^2$. The measurable Riemann mapping theorem says that given such a μ, there is a planar quasiconformal map h with this dilatation. If μ is supported on the unit disk, $\mathbb{D}$, then there is a quasiconformal $h : \mathbb{D} \to \mathbb{D}$ with this dilatation.

A quasiconformal map h is called K-quasiconformal if its dilatation satisfies $\|\mu\|_\infty \leq k = (K-1)/(K+1)$. More geometrically, at almost every point h is differentiable and its derivative (which is a real linear map) send circles to ellipses of eccentricity at most K (the eccentricity of an ellipse is the ratio of the major to minor axis).



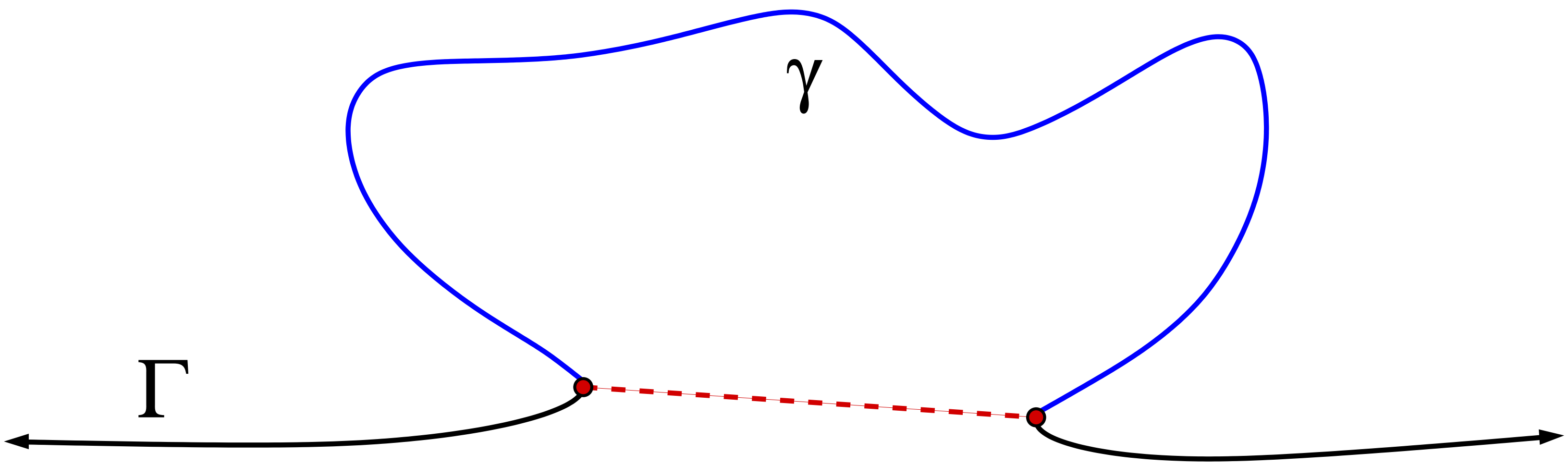
Suppose $\Gamma = \partial\Omega$ is Jordan curve, $f : \mathbb{D} \to \Omega$ is conformal.

Basic problem: how is geometry of Γ related to properties of f?



If γ is a planar Jordan arc with endpoints z, w, we set:

- $\mathrm{diam}(\gamma)$ = diameter of γ
- $\mathrm{crd}(\gamma) = z - w$ = chord length of γ
- $\ell(\gamma)$ = length of γ,
- $\Delta(\gamma) = \ell(\gamma) - \mathrm{crd}(\gamma)$ = excess length
- $\beta(\gamma) = \sup\{z \in \gamma : \mathrm{dist}(z, L)/\mathrm{diam}(\gamma)\}$, L = line through z, w



Γ is a **quasicircle** iff $\mathrm{diam}(\gamma) = O(\mathrm{crd}(\gamma))$ for $\gamma \subset \Gamma$.

Γ is **chord-arc** iff $\ell(\gamma) = O(\mathrm{crd}(\gamma))$ for $\gamma \subset \Gamma$.

If f is conformal on $\mathbb{D}$, then f' is never zero, so $\Phi = \log f'$ is a well defined holomorphic function on $\mathbb{D}$. Recall that the Dirichlet class is the Hilbert space of holomorphic functions F on the unit disk such that $|F(0)|^2 + \int_{\mathbb{D}} |F'(z)|^2 dxdy < \infty$. In other words, the Dirichlet space consists of the holomorphic functions in the Sobolev space $W^{1,2}(\mathbb{D})$ (functions with one derivative in $L^2(dxdy)$).

Definition 1. Γ *is a quasicircle and* $\Gamma = f(\mathbb{T})$, *where* f *is conformal on* $\mathbb{D}$ *and* $\log f'$ *is in the Dirichlet class.*

This definition immediately provides some geometric information about the curve Γ. For a Jordan arc γ, let $\ell(\gamma)$ denote its arclength and let $\mathrm{crd}(\gamma) = |z-w|$ where z, w are the endpoints of γ. If $\log f'$ is in the Dirichlet class, then $\log f' \in \mathrm{VMOA}$ (vanishing mean oscillation; see Chapter VI of [28]). The John-Nirenberg theorem (e.g., Theorem VI.2.1 of [28]) then implies f' is in the Hardy space $H^1(\mathbb{D})$, so Γ is rectifiable.

Even stronger, a theorem of Pommerenke [56] implies that Γ is asymptotically smooth, i.e., $\ell(\gamma)/\mathrm{crd}(\gamma) \to 1$ as $\ell(\gamma) \to 0$, i.e., a Weil-Petersson curve has "no corners".

Asymptotic smoothness implies Γ is chord-arc; a fact observed in [26] (see also Theorem 2.8 of [57], but there is a gap due to the non-standard definition of "quasicircle" in a result quoted from [22].)

Bounded mean oscillation (BMO) is the space of functions so that

$$m_I(f - m_I(f)) = O(1),$$

where $m_I(f)$ is the mean value of f over I, i.e.,

$$m_I(f) = \frac{1}{|I|} \int_I f dx.$$

Here $|I|$ is Lebesgue measure of I. In other words, f is in BMO if

$$\|f\|_{\mathrm{BMO}} \sup_I \frac{1}{|I|} \int_I |f - m_I(f)| dx < \infty.$$

$m_I(f)$ can be replaced by any constant c_I.

Equivalent definition

$$\sup_I \frac{1}{|I|} \int_I |f - m_I(f)|^2 dx < \infty.$$

$L^\infty \subset \mathrm{BMO}$, but $\log |x| \in \mathrm{BMO}$.

Vanishing Mean Oscillation (VMO)

$$\lim_{|I|\to 0} \frac{1}{|I|} \int_I |f - m_I(f)|^2 dx \to 0.$$

The John-Nirenberg Theorem says that if f in in BMO, then

$$|\{x \in I : |f(x) - m_I(f)| > \lambda\}| \leq C \exp(-\lambda/C\|f\|_{\mathrm{BMO}}).$$

In particular, if f is in BMO then $\int e^{cf} < \infty$ for some $c > 0$ depending on the BMO norm of f. See Theorem VI.2.1 of Garnett's *Bounded Analytic Functions*.

If f is in VMO, then we can write f as the sum of a continuous function and function with small BMO norm (Theorem VI.5.1 of *BAF*). This implies that if f is in VMO then $\exp(cf)$ is integrable for any c (not just small c).

A positive measure on the unit disk is a **Carleson measure** if for every disk $D = D(x, r)$ centered on the unit circle,

$$\mu(D) = O(r).$$

If u is the harmonic extension of g from unit circle to disk, then g is in BMO iff

$$\mu = |\nabla u(z)|^2 \log \frac{1}{|z|} dxdy$$

is a Carleson measure. See Theorem VI.3.4 of Garnett's *Bounded Analytic Functions*.

g is in VMO iff this μ is a vanishing Carleson measure. This means $\mu(D) = o(r)$.

If f is a conformal map and $g = \log f'$ is in the Dirichlet space, then it it also in VMO:
$$\int_{D(x,r)\cap\mathbb{D}} |g'|^2 \log\frac{1}{|z|} dxdy \lesssim r \int_{D(x,r)\cap\mathbb{D}} |g'|^2 dxdy = ro(1),$$
where we have used the dominated convergence theorem.

Thus $\log f'$ in Dirichlet space implies $\log f'$ is in VMO, hence $|f'| = \exp(\log f')$ is integrable. Hence f' is in the Hardy space $H^1(\mathbb{T})$ and $\Gamma = f(\mathbb{T})$ is rectifiable.

An estimate of Beurling [9] (simplified and extended by Chang and Marshall in [16] and [44]) says that $\log|f'|$ in the Dirichlet class implies $\int \exp(\alpha \log^2 |f'|^2) ds < \infty$ for all $\alpha \le 1$. So being in Dirichlet class is stronger than VMO.

It is easy to see that a function F is in the Dirichlet class if and only if $F(\mathbb{D})$ has finite area, when counted with multiplicity. It is also easy to check that if the inradius of a simply connected domain Ω is small, then the conformal map $g : \mathbb{D} \to \Omega$ is in the Bloch space with small norm, i.e.,

$$\|g\|_{\mathcal{B}} = \sup_{z \in \mathbb{D}} |g'(z)|(1 - |z|^2),$$

is small. A standard result (e.g., Theorem VII.2.1 of [29]) then says that $f = \int_0^z e^g dz$ is a conformal map onto a Jordan domain.

Thus choosing Ω to have finite area, small in-radius but containing unbounded rays gives a conformal map f with $g = \log f'$ in the Dirichlet class such that $f(\mathbb{T})$ contains infinite spirals. Hence Weil-Petersson curves need not be C^1 (there are even examples where the spirals are dense on the curve).

We can also build examples of spirals by hand and verify the β^2-sum characterization of Weil-Petersson curves.

Set $F = \log f'$. We claim

$$\iint_{\mathbb{D}} |F'(z)|^2 dxdy < \infty$$

is equivalent to

$$I := \iint_{\mathbb{D}} |F''(z)|^2 (1 - |z|^2)^2 dxdy < \infty.$$

Proof. Assume F has the power series expansion $F(z) = \sum_{n=0}^{\infty} b_n z^n$, and then a simple computation in polar coordinates leads to

$$\iint_{\mathbb{D}} |F'(z)|^2 dxdy = 2\pi \sum_{n=1}^{\infty} n^2 |b_n|^2 \int_0^1 r^{2n-1} dr = \sum_{n=1}^{\infty} (\pi n)|b_n|^2,$$

and hence

$$\begin{aligned} I &= \iint_{\mathbb{D}} |F''(z)|^2 (1-|z|^2)^2 dxdy \\ &= 2\pi \sum_{n=1}^{\infty} n^2 (n-1)^2 |b_n|^2 \int_0^1 r^{2n-4}(1-2r^2+r^4) r dr \\ &= 2\pi \sum_{n=1}^{\infty} n^2 (n-1)^2 |b_n|^2 \left(\frac{1}{2n-2} - \frac{2}{2n} + \frac{1}{2n+2}\right) \\ &= \sum_{n=1}^{\infty} 2\pi \frac{n(n-1)}{n+1} |b_n|^2 \simeq \sum_{n=2}^{\infty} \pi n |b_n|^2 \end{aligned}$$

Thus both infinite series (and hence both integrals) diverge or converge together.

$\square$

If we expand out $F' = (\log f')'$ and F'' we see that

$$\int_{\mathbb{D}} |F'|^2 dxdy = \int_{\mathbb{D}} |(\log f')'|^2 dxdy = \int_{\mathbb{D}} \left|\frac{f''}{f'}\right|^2 dxdy < \infty \tag{2.1}$$

can be replaced by the condition

$$\int_{\mathbb{D}} |F''|^2 (1 - |z|^2)^2 dxdy \tag{2.2}$$

$$= \int_{\mathbb{D}} \left| \left(\frac{f'''}{f'}\right) - \left(\frac{f''}{f'}\right)^2 \right|^2 (1 - |z|^2)^2 dxdy < \infty. \tag{2.3}$$

This integrand is reminiscent of the Schwarzian derivative of f given by

$$S(f) = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2. \tag{2.4}$$

The quantities in (2.2) and (2.4) are very similar, except that a factor of 1 has been changed to 3/2. However, this represents a non-linear change, and it is difficult to compare the two quantities directly, e.g., for a Möbius transformation sending $\mathbb{D}$ to a half-plane, the Schwarzian is constant zero, but the expression in (2.2) blows up to infinity at a boundary point.

Nevertheless, for conformal maps into bounded quasidisks, the integrals of these two quantities are simultaneously finite or infinite:

Definition 2. Γ *is quasicircle and* $\Gamma = f(\mathbb{T})$*, where* f *is conformal on* $\mathbb{D}$ *and satisfies*

$$\int_{\mathbb{D}} |S(f)(z)|^2(1 - |z|^2)^2 dxdy < \infty. \tag{2.5}$$

Proposition 1 of Cui's paper [18] says that Definitions 2 and 1 are equivalent to each other.

See also Theorem II.1.12 of Takhtajan and Teo's book [65] and Theorem 1 of [55] by Pérez-González and Rättyä.

I can give a proof in class later.

If f is univalent on $\mathbb{D}$ then

$$\sup_{z\in\mathbb{D}} |S(f)(z)|(1-|z|^2)^2 \leq 6. \tag{2.6}$$

See Chapter II of [41] for this and other properties of the Schwarzian.

If f is holomorphic on the disk and satisfies (2.6) with 6 replaced by 2, then f is injective, i.e., a conformal map. If 2 is replaced by a value $t < 2$, then f also has a K-quasiconformal extension to the plane, where K depends only on t. This is due to Ahlfors and Weill [2], who gave a formula for the extension and its dilatation

$$f(w) = f(z) + \frac{(1-|z|^2)f'(z)}{\overline{z} - \frac{1}{2}(1-|z|^2)(f''(z)/f'(z))} \tag{2.7}$$

$$\mu(w) = -\frac{1}{2}(1-|z|^2)^2 S(f)(z) \tag{2.8}$$

where $w \in \mathbb{D}^*$ and $z = 1/\overline{w} \in \mathbb{D}$.

See Section 4 of [17] for a lucid discussion of the Ahlfors-Weill extension and a proof that when $t = 2$, this extension gives a homeomorphism of the sphere. See also Formula (3.33) of [53] and or Equation (9) of [58].

The Alhfors-Weill extension shows that Definition 2 implies:

Definition 3. $\Gamma = f(\mathbb{T})$ *where* f *is a quasiconformal map of the plane that is conformal on* $\mathbb{D}^*$ *and whose dilatation* μ *on* $\mathbb{D}$ *satisfies satisfies*

$$\int_{\mathbb{D}} \frac{|\mu(z)|^2}{(1-|z|^2)^2} dxdy < \infty. \tag{2.9}$$

This was shown to be equivalent to Definition 2 by Guizhen Cui; see Theorem 2 of [18].

The integral in (2.9) is the same as

$$\int_{\mathbb{D}} |\mu(z)|^2 d\mathrm{A}_\rho < \infty, \tag{2.10}$$

where $d\mathrm{A}_\rho$ denotes integration against hyperbolic area. Thus Γ is Weil-Petersson iff if is image of a QC map whose dilatation in in L^2 for hyperbolic area on the disk.

Another variation on this theme is to consider the map $R(z) = f(1/\overline{f^{-1}(z)})$. This is an orientation reversing quasiconformal map of the sphere to itself that fixes Γ pointwise, exchanges the two complementary components of Γ and whose dilatation satisfies

$$\int_{\Omega\cup\Omega^*} |\mu(z)|^2 dA_\rho(z) < \infty, \tag{2.11}$$

where dA_ρ is hyperbolic area on each of the domains Ω, Ω^*.

This version is sometimes easier to check, and we will use it interchangeably with Definition 3. The map R is called a quasiconformal reflection across Γ (and constructing R is often the easiest way to check Γ is WP).

f is K-biLipschitz if

$$\frac{1}{K} \leq \frac{|f(x) - f(y)|}{|x - y|} \leq K,$$

for all x, y.

BiLipschitz implies quasiconformal.

Quasicircles in plane always have biLipschitz reflections.

Later we will formulate a biLipschitz variation of Definition 3, that we shall discuss later and is easier to extend to higher dimensions: Γ is Weil-Petersson iff it is the fixed point set of a orientation reversing biLipschitz mapping of $\mathbb{R}^2$ so that "the local biLipschitz constant $1 + \rho(x)$ is in L^2 on the complement of the curve". We will have to make this last part more precise later.

Thursday, Sept 17 , 2020

A circle homeomorphism $\varphi : \mathbb{T} \to \mathbb{T}$ is called a conformal welding if $\varphi = f^{-1} \circ g$ where f, g are conformal maps from the two sides of the unit circle to the two sides of a closed Jordan curve Γ.

There are many weldings associated to each Γ, but they all differ from each other by compositions with Möbius transformations of $\mathbb{T}$. Not every circle homeomorphism is a conformal welding, but weldings are dense in the homeomorphisms in various senses; see [27].

A circle homeomorphism is called M-quasisymmetric if it maps adjacent arcs in $\mathbb{T}$ of the same length to arcs whose length differ my a factor of at most M; we call φ quasisymmetric if it is M-quasisymmetric for some M.

The quasisymmetric maps are exactly the circle homeomorphisms that can be continuously extended to quasiconformal self-maps of the disk, and are also exactly the conformal weldings of quasicircles. See [3].

The QS maps form a meager set in the space of all circle homeomorphism, but the set of conformal weldings is residual (it contains a G_δ). Thus most conformal weldings are not QS.

F_σ = countable union of closed sets
G_δ = countable intersection of open sets
residual = contains a dense G_δ = topologically generic
meager = contained in nowhere dense F_σ = topologically rare

See my recent preprint
Conformal removability is hard

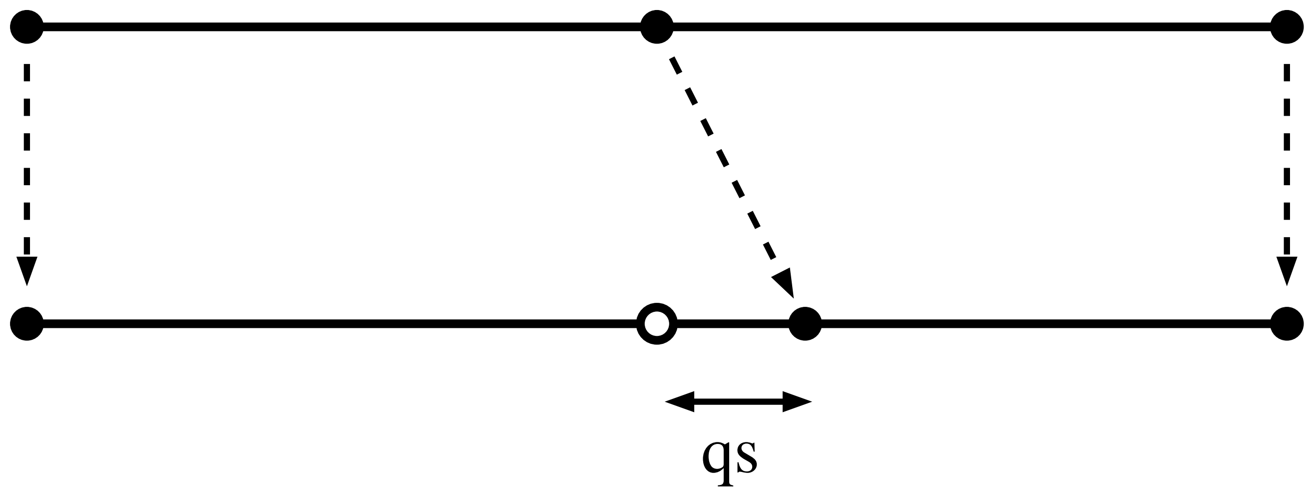
A quasisymmetric homeomorphism is called symmetric if the constant M tends to 1 on small scales (Pommerenke [56] proved such weldings characterize curves where $\log f'$ is in the little Bloch space; see also [27] by Gardiner and Sullivan and [64] by Strebel).

These papers were part of the origin of Takhtajan and Teo's book, and hence this class.

Weil-Petersson class corresponds to replacing "tends to zero" with "is square summable".

More precisely, if $I \subset \mathbb{T}$ is an arc, let $m(I)$ denote its midpoint. For a homeomorphism $\varphi : \mathbb{T} \to \mathbb{T}$ define

$$\mathrm{qs}(\varphi, I) = \frac{|\varphi(m(I)) - m(\varphi(I))|}{\ell(\varphi(I))}.$$



Definition 4. Γ *is closed Jordan curve whose welding map* φ *satisfies*

$$\sum_I \mathrm{qs}^2(\varphi, I) < \infty, \tag{2.12}$$

where the sum is over some dyadic decomposition of $\mathbb{T}$*.*

This makes sense, because we expect $\mathrm{qs}(\varphi, I)$ to control size of dilatation μ of the QC extension of φ to the unit disk.

Weil-Petersson weldings were first characterized by Yuliang Shen [63] in terms of the Sobolev space $H^{1/2}$. We will describe his result a little later.

A **Whitney decomposition** of an open set $\Omega \subset \mathbb{R}^n$ is a countable collection of closed sets $\{Q_j\}$ inside Ω so that

(1) $\cup_j Q_j = \Omega$,

(2) the Q_j have disjoint interiors

(3) $\mathrm{diam}(Q_j) \simeq \mathrm{dist}(Q_j, \partial\Omega)$

(4) Q_j contains a ball of radius comparable to its diameter.

Lemma 2.1. *Whitney decompositions always exist if $\partial\Omega \neq \emptyset$.*

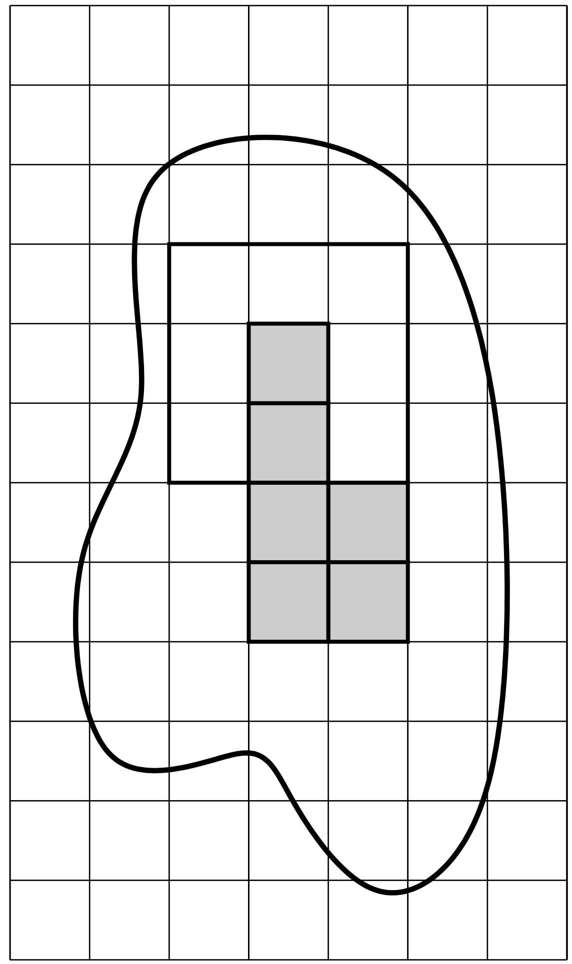
Proof. For each $x \in \Omega$ take the largest dyadic cube Q containing x so that $3Q \subset \Omega$. (A largest one clearly exists.)

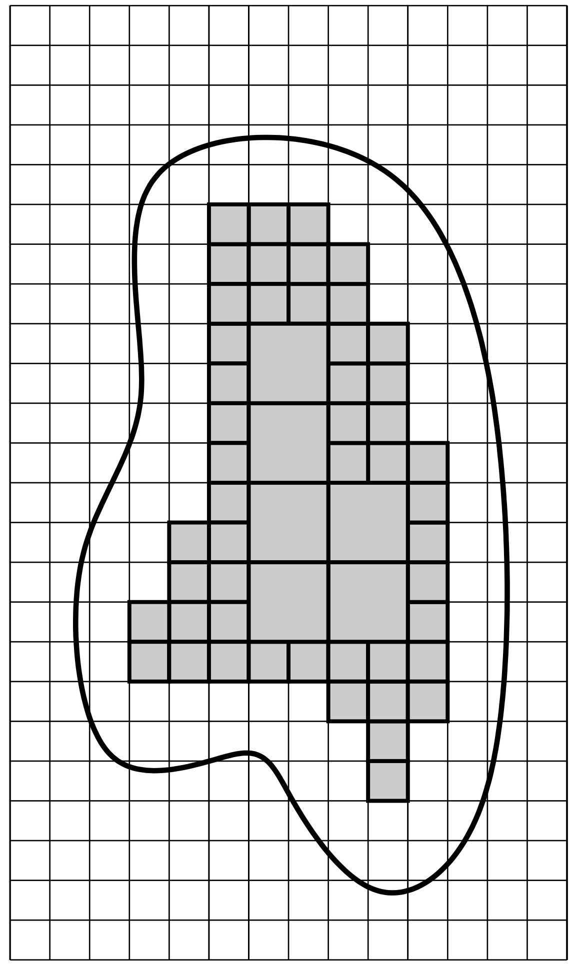
Clearly these cover Ω since every x is in such a square.

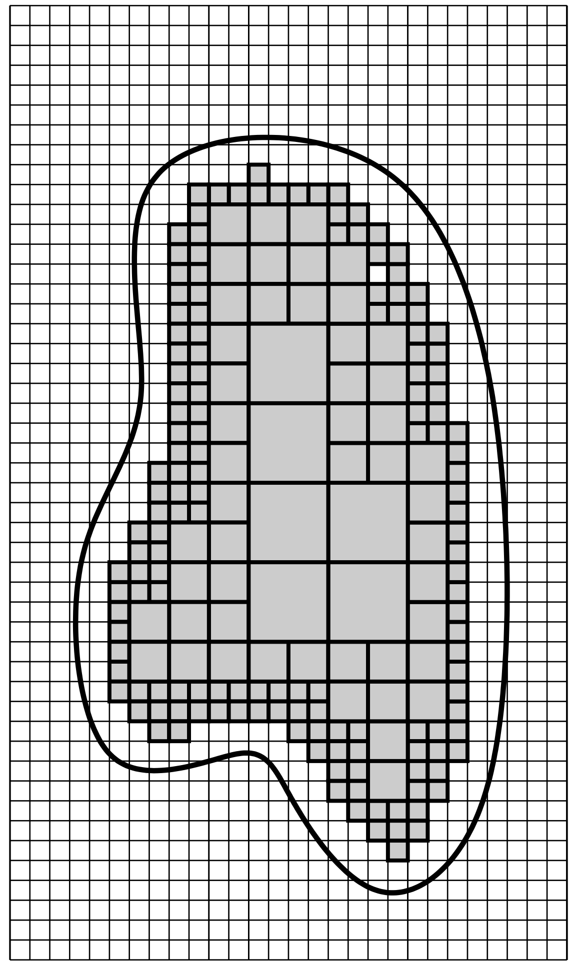
The nested property of dyadic square implies disjoint interiors.

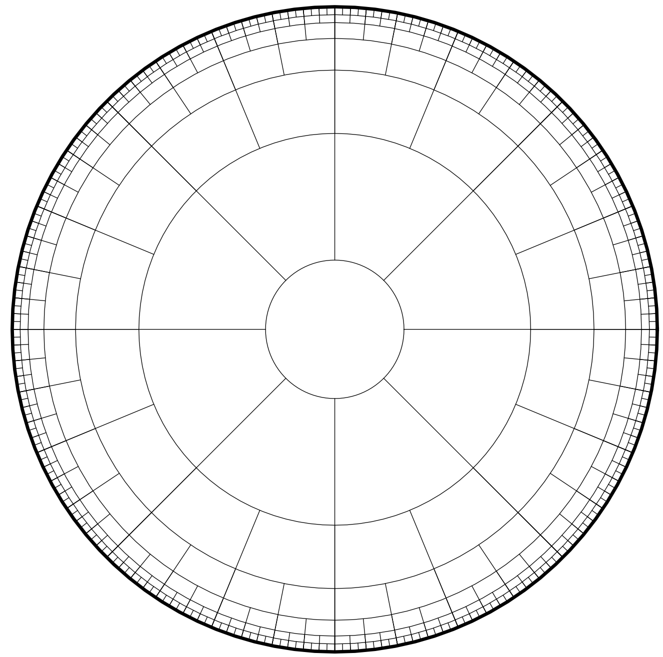
Upper diameter bound clear since $3Q \subset \Omega$.

Maximality implies the lower diameter bound (otherwise we would have chosen a bigger cube). $\square$









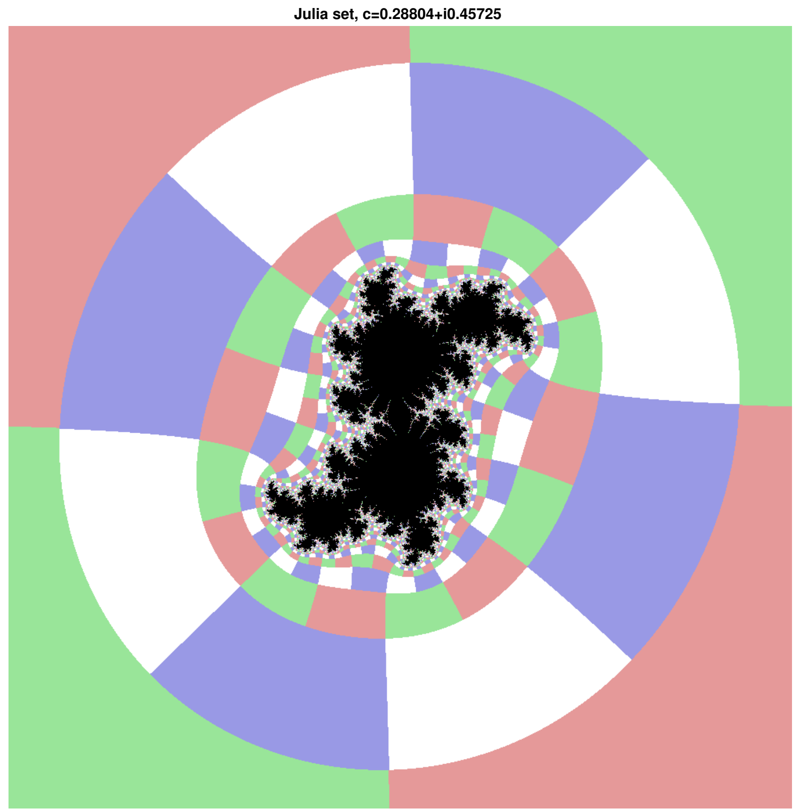
In disk we define a **Carleson square** associate to an arc $I \subset \mathbb{T}$ as

$$Q_I = \{z \in \mathbb{D} : z/|z| \in I, 0 < 1 - |z| \leq |I|\}.$$

and its "top half" as

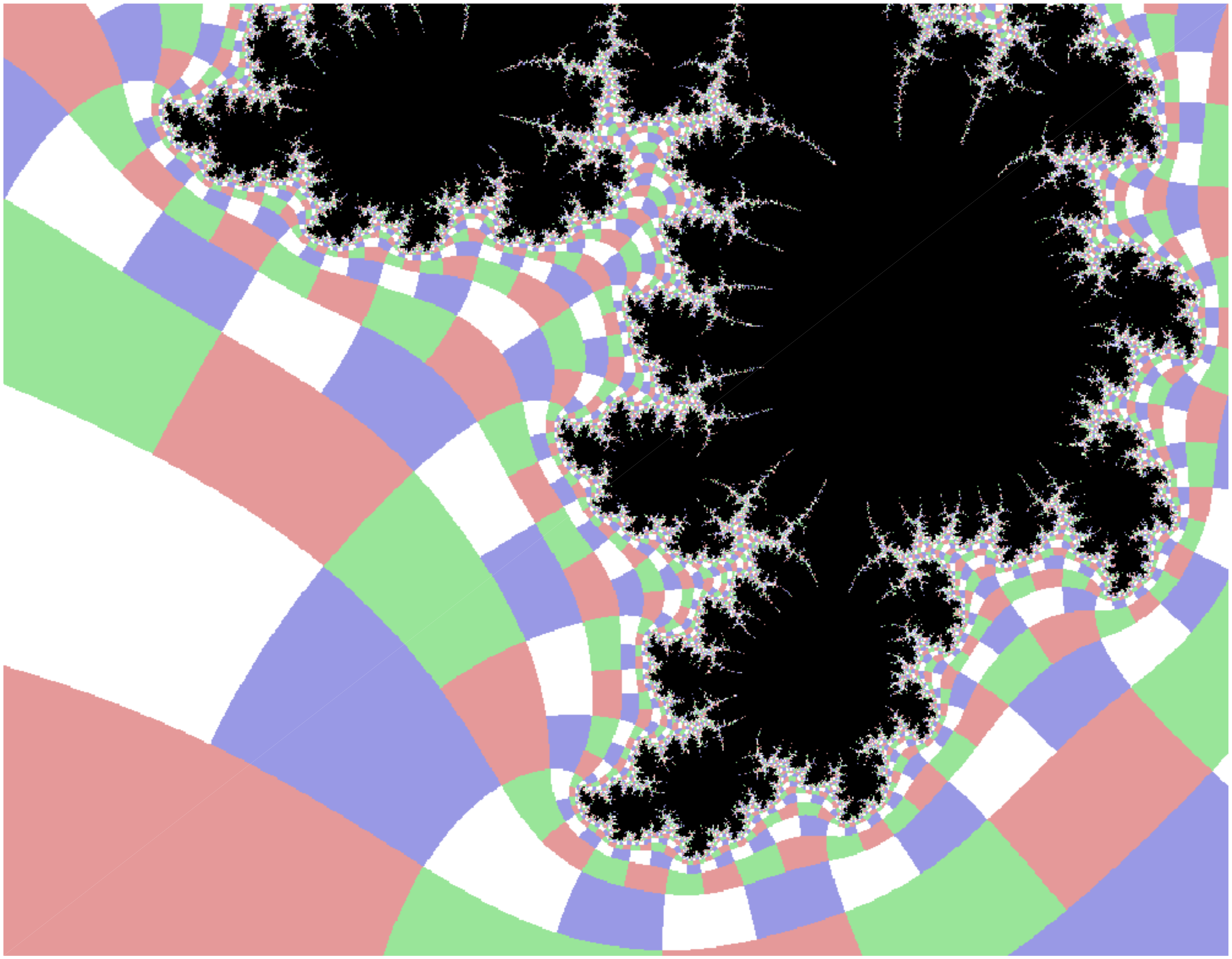
$$T_I = \{z \in \mathbb{D} : z/|z| \in I, |I|/2 \leq 1 - |z| \leq |I|\}.$$

As I ranges over the dyadic intervals of $\mathbb{T}$, this gives a Whitney decomposition (we also use a disk near the origin.)



A Whitney decomposition for the complement of the Julia set of a quadratic polynomial. Here the elements of the decomposition are chosen to respect the dynamics: near the Julia set each Whitney box has two preimages that are also Whitney boxes.

Julia set, c=0.28804+i0.45725



$\log f'$	$\mathcal{B}_0$	BMOA	VMOA	Dirichlet
$(\log f')'(1-\lvert z\rvert^2)$	$C_0(\mathbb{D})$	$\mathrm{CM}(\mathbb{D})$	$\mathrm{CM}_0(\mathbb{D})$	$L^2(dA_\rho)$
$S(z)(1-\lvert z\rvert^2)^2$	$C_0(\mathbb{D})$	$\mathrm{CM}(\mathbb{D})$	$\mathrm{CM}_0(\mathbb{D})$	$L^2(dA_\rho)$
μ	$C_0(\mathbb{D})$	$\mathrm{CM}(\mathbb{D})$	$\mathrm{CM}_0(\mathbb{D})$	$L^2(dA_\rho)$
$h = g^{-1} \circ f$	symmetric	strongly quasisymmetric	$\log h' \in \mathrm{VMO}$	$\log h' \in H^{1/2}$
$\Gamma = f(\mathbb{T})$	asymptotically conformal	Bishop-Jones condition	asymptotically smooth	

Each of four columns is a theorem giving 5 equivalent conditions.

Conditions become more restrictive moving left to right.

CM = Carleson measure, CM_0 = vanishing Carleson measure,

C_0 = continuous on disk, vanishing on boundary

Strongly quasisymemtric = h is abssolutely continuous and h' is an A^∞ weight.

$\log f'$	$\mathcal{B}_0$	BMOA	VMOA	Dirichlet
$(\log f')'(1-\|z\|^2)$	$C_0(\mathbb{D})$	$\mathrm{CM}(\mathbb{D})$	$\mathrm{CM}_0(\mathbb{D})$	$L^2(dA_\rho)$
$S(z)(1-\|z\|^2)^2$	$C_0(\mathbb{D})$	$\mathrm{CM}(\mathbb{D})$	$\mathrm{CM}_0(\mathbb{D})$	$L^2(dA_\rho)$
μ	$C_0(\mathbb{D})$	$\mathrm{CM}(\mathbb{D})$	$\mathrm{CM}_0(\mathbb{D})$	$L^2(dA_\rho)$
$h = g^{-1} \circ f$	symmetric	strongly quasisymmetric	$\log h' \in \mathrm{VMO}$	$\log h' \in H^{1/2}$
$\Gamma = f(\mathbb{T})$	asymptotically conformal	Bishop-Jones condition	asymptotically smooth	

Theorem (Pommerenke, 1978): Γ is asymptotically conformal, i.e.,

$$\beta(\gamma) \to 0, \text{ as } \operatorname{diam}(\gamma) \to 0,$$

iff $\log f'$ is in little Bloch class

$$\mathcal{B}_0 = \left\{ g \text{ holomorphic on } \mathbb{D} : |g'(z)| = o\left(\frac{1}{1-|z|}\right)\right\}.$$

Bloch space $= \mathcal{B} = \left\{ g \text{ holomorphic on } \mathbb{D} : |g'(z)| = O\left(\frac{1}{1-|z|}\right)\right\}.$

$\log f'$	$\mathcal{B}_0$	BMOA	VMOA	Dirichlet
$(\log f')'(1-\|z\|^2)$	$C_0(\mathbb{D})$	$\mathrm{CM}(\mathbb{D})$	$\mathrm{CM}_0(\mathbb{D})$	$L^2(dA_\rho)$
$S(z)(1-\|z\|^2)^2$	$C_0(\mathbb{D})$	$\mathrm{CM}(\mathbb{D})$	$\mathrm{CM}_0(\mathbb{D})$	$L^2(dA_\rho)$
μ	$C_0(\mathbb{D})$	$\mathrm{CM}(\mathbb{D})$	$\mathrm{CM}_0(\mathbb{D})$	$L^2(dA_\rho)$
$h = g^{-1} \circ f$	symmetric	strongly quasisymmetric	$\log h' \in \mathrm{VMO}$	$\log h' \in H^{1/2}$
$\Gamma = f(\mathbb{T})$	asymptotically conformal	Bishop-Jones condition	asymptotically smooth	

Theorem (Pommerenke, 1978): Γ is asymptotically smooth, i.e.,

$$\frac{\Delta(\gamma)}{\mathrm{crd}(\gamma)} = \frac{\ell(\gamma) - \mathrm{crd}(\gamma)}{\mathrm{crd}(\gamma)} \to 0, \text{ as } \mathrm{diam}(\gamma) \to 0,$$

iff $\log f' \in \mathrm{VMOA}$.

$\log f'$	$\mathcal{B}_0$	BMOA	VMOA	Dirichlet
$(\log f')'(1-\|z\|^2)$	$C_0(\mathbb{D})$	$\mathrm{CM}(\mathbb{D})$	$\mathrm{CM}_0(\mathbb{D})$	$L^2(dA_\rho)$
$S(z)(1-\|z\|^2)^2$	$C_0(\mathbb{D})$	$\mathrm{CM}(\mathbb{D})$	$\mathrm{CM}_0(\mathbb{D})$	$L^2(dA_\rho)$
μ	$C_0(\mathbb{D})$	$\mathrm{CM}(\mathbb{D})$	$\mathrm{CM}_0(\mathbb{D})$	$L^2(dA_\rho)$
$h = g^{-1} \circ f$	symmetric	strongly quasisymmetric	$\log h' \in \mathrm{VMO}$	$\log h' \in H^{1/2}$
$\Gamma = f(\mathbb{T})$	asymptotically conformal	Bishop-Jones condition	asymptotically smooth	

Astala-Zinsmeister theorem:

$$\log f' \in \mathrm{BMO} \Leftrightarrow |S(f)|^2(1-|z|^2)^3 dzxdy \text{ is Carleson.}$$

μ is a Carleson measure if $\mu(D(x,r)) = O(r)$.

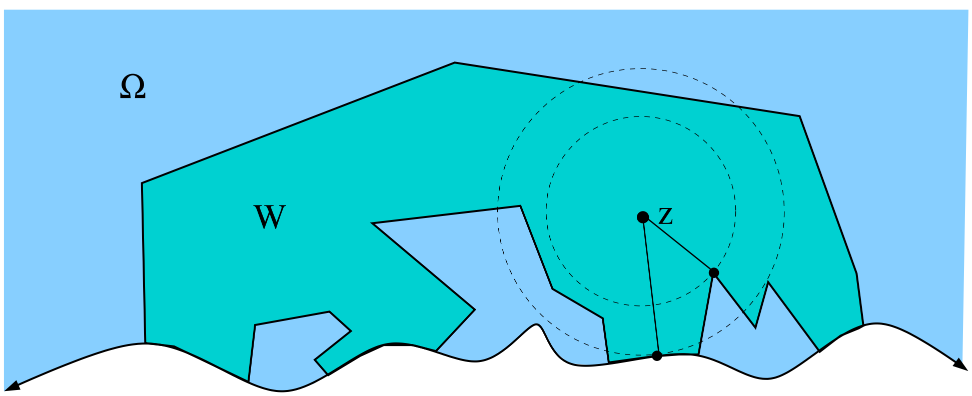
BMOA are homomorphic functions such that

$$|g'(z)|^2(1-|z|^2)dxdy \text{ is Carleson}$$

$\log f'$	$\mathcal{B}_0$	BMOA	VMOA	Dirichlet
$(\log f')'(1-\|z\|^2)$	$C_0(\mathbb{D})$	$\mathrm{CM}(\mathbb{D})$	$\mathrm{CM}_0(\mathbb{D})$	$L^2(dA_\rho)$
$S(z)(1-\|z\|^2)^2$	$C_0(\mathbb{D})$	$\mathrm{CM}(\mathbb{D})$	$\mathrm{CM}_0(\mathbb{D})$	$L^2(dA_\rho)$
μ	$C_0(\mathbb{D})$	$\mathrm{CM}(\mathbb{D})$	$\mathrm{CM}_0(\mathbb{D})$	$L^2(dA_\rho)$
$h = g^{-1} \circ f$	symmetric	strongly quasisymmetric	$\log h' \in \mathrm{VMO}$	$\log h' \in H^{1/2}$
$\Gamma = f(\mathbb{T})$	asymptotically conformal	Bishop-Jones condition	asymptotically smooth	

Bishop-Jones: for all $z \in \Omega$ there is chord-arc $W \subset \Omega$ with $z \in W$ and

$$\ell(\partial W) \simeq \ell(\partial W \cap \partial\Omega) \simeq \mathrm{dist}(z, \partial\Omega) \simeq \mathrm{dist}(z, \partial W)$$



Definition	Description
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The names of 26 characterizations of Weil-Peterson curves

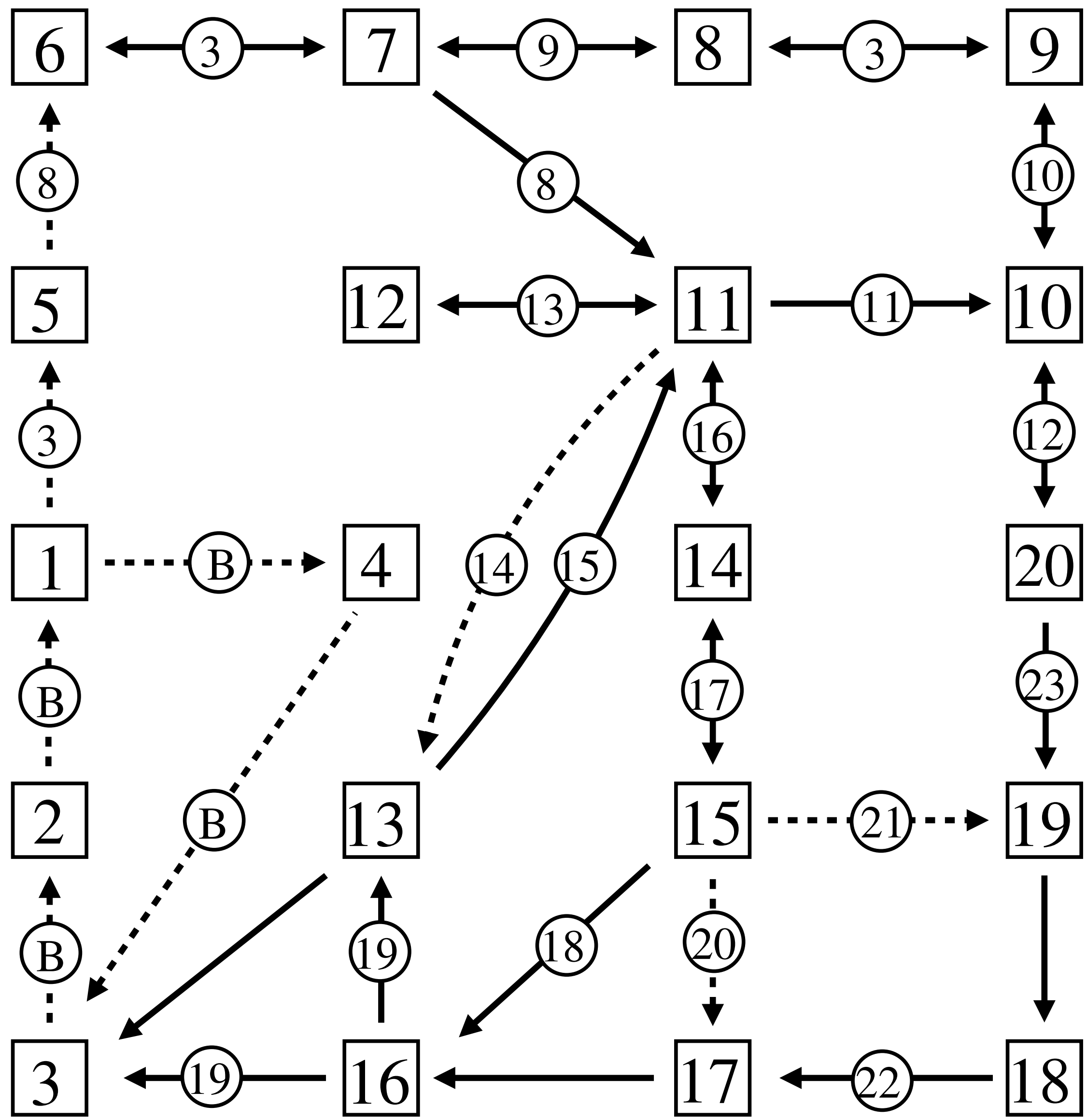

Diagram of implications between previous definitions.

Edge labels refer to sections of my preprint..

3. Sobolev type definitions of the Weil-Petersson class

Definition 1 can be interpreted in terms of Sobolev spaces. The space $H^{1/2}(\mathbb{T}) \subset L^2(\mathbb{T})$ is defined by the finiteness of the seminorm

$$\begin{aligned} D(f) &= \iint_{\mathbb{D}} |\nabla u(z)|^2 dxdy \\ &= \frac{1}{8\pi}\int_0^{2\pi}\int_0^{2\pi}\left|\frac{f(e^{is})-f(e^{it})}{\sin\frac{1}{2}(s-t)}\right|^2 dsdt \simeq \int_{\mathbb{T}}\int_{\mathbb{T}} \frac{|f(z)-f(w)|^2}{|z-w|^2}|dz||dw|. \end{aligned}$$

where u is the harmonic extension of f to $\mathbb{D}$.

The equality of the first and second integrals is called the Douglas formula, after Jesse Douglas who introduced it in his solution of the Plateau problem [21]. See also Theorem 2.5 of [4] (for a proof of the Douglas formula) and [60] (for more information about the Dirichlet space).

For $s \in (0,1)$ we define the space $H^s(\mathbb{T})$ using

$$\int_{\mathbb{T}} \int_{\mathbb{T}} \frac{|f(z) - f(w)|^2}{|z-w|^{1+2s}} |dz||dw| < \infty.$$

See [1] and [20] for additional background on fractional Sobolev spaces.

See also [48] by Nag and Sullivan; in the authors' words its "purpose is to survey from various different aspects the elegant role of $H^{1/2}$ in universal Techmüller theory" (a role we seek to explore in this paper too).

Shen [63] proved Γ is Weil-Petersson iff its welding map satisfies $\log \varphi' \in H^{1/2}$.

To see necessity, observe that $\log f'$ is in the Dirichlet class on $\mathbb{D}$ if and only if its radial boundary values satisfy $\log f' \in H^{1/2}(\mathbb{T})$. Thus Definition 1 implies $\log f', \log g' \in H^{1/2}(\mathbb{T})$ and a simple computation shows $\log \varphi'(x) = -\log f'(\varphi(x)) + \log g'(x)$. Beurling and Ahlfors [8] proved $H^{1/2}(\mathbb{T})$ is invariant under pre-compositions with quasisymmetric circle homeomorphisms, so $\log \varphi' \in H^{1/2}(\mathbb{T})$.

At present I don't plan to prove the converse here.

Tuesday, September 22, 2020

As noted above, $\log f'(z)$ is in the Dirichlet class on $\mathbb{D}$ if and only if the radial limits $\log|f'|$ and $\arg(f')$ are in $H^{1/2}(\mathbb{T})$.

Since $\arg(f')$ can be unbounded, it is, perhaps, surprising that this is equivalent to $f'/|f'| \in H^{1/2}$:

Definition 5. $\Gamma = f(\mathbb{T})$ *is chord-arc and* $\exp(i \arg f') = f'/|f'| \in H^{1/2}(\mathbb{T})$.

In other words, Γ is WP iff the unit tangent direction at $f(z)$ defines an $H^{1/2}$ function on the circle.

One direction is easy:

Definition 1 implies $\log f' = \log|f| + i \arg f'$ is in the Dirichlet class, so $\arg f' \in H^{1/2}(\mathbb{T})$. Using $|e^{ix} - e^{iy}| \le |x-y|$ and the Douglas formula we get

$$\int_{\mathbb{T}}\int_{\mathbb{T}} \left| \frac{e^{i\arg f'(x)} - e^{i\arg f'(y)}}{x-y} \right|^2 dxdy \le \int_{\mathbb{T}}\int_{\mathbb{T}} \left| \frac{\arg f'(x) - \arg f'(y)}{x-y} \right|^2 dxdy < \infty.$$

Thus $\exp(i \arg f') \in H^{1/2}(\mathbb{T})$.

The converse direction seems harder. We shall give two proofs of it: one by following a chain of geometric characterizations of the Weil-Petersson class, and a direct function theoretic proof.

Let $a : \mathbb{T} \to \Gamma$ be an orientation preserving arclength parameterization (i.e., a multiplies the arclength of every set by $\ell(\Gamma)/2\pi$). For $z \in \Gamma$, let $\tau(z)$ be the unit tangent direction to Γ with its usual counterclockwise orientation.

Then $\tau(a(x)) = a'(x)2\pi/\ell(\Gamma)$, where $a' = \frac{da}{d\theta}$ on $\mathbb{T}$. Thus $a' = \exp(i \arg f') \circ \varphi$ where $\varphi = a^{-1} \circ f$ is a circle homeomorphism.

We shall prove that this map φ is quasisymmetric (and hence so is its inverse). It is a result of Beurling and Ahlfors [8] that pre-composing with such maps preserves $H^{1/2}(\mathbb{T})$, so Definition 5 is equivalent to saying $a' \in H^{1/2}(\mathbb{T})$.

Every arclength parameterization is Lipschitz hence absolutely continuous, and therefore the distributional derivative of a equals its pointwise derivative a'. Thus, for arclength parameterizations, $a' \in H^{1/2}(\mathbb{T})$ is the same as $a \in H^{3/2}(\mathbb{T})$. Therefore Definition 5 is equivalent to

Definition 6. Γ *is chord-arc and the arclength parameterization* $a : \mathbb{T} \to \Gamma$ *is in the Sobolev space* $H^{3/2}(\mathbb{T})$*.*

Proving this is equivalent to Definition 1 gives Theorem 1.1.

Lemma 3.1. *Definition 5 implies Definition 6.*

Proof. Suppose f is a conformal map from $\mathbb{D}$ to the bounded complementary component of Γ. Let $a : \mathbb{T} \to \Gamma$ be an orientation preserving arclength parameterization and let $\varphi = a^{-1} \circ f : \mathbb{T} \to \mathbb{T}$. We claim this circle homeomorphism is quasisymmetric.

To prove this, consider to adjacent arcs I, J of the same length. Since Definition 1 is known to be equivalent to Definition 3, f has a quasiconformal extension to the whole plane, hence it is also a quasisymmetric map and this implies that $f(I)$ and $f(J)$ have comparable diameters. See [31] or Section 4 of [35].

Since we also know that Γ is chord-arc, this implies that $f(I)$ and $f(J)$ have comparable lengths, hence $\varphi(I)$ and $\varphi(J)$ also have comparable lengths, since a preserves arclength. This is the definition of quasisymmetry for φ.

Note that $a' = \exp(i \arg f') \circ \varphi$. Beurling and Ahlfors proved in [8] that $H^{1/2}$ is invariant under composition with a quasisymmetric homeomorphism of $\mathbb{T}$. Thus $a' \in H^{1/2}$ iff $\exp(i \arg f) \in H^{1/2}$. Since a is Lipschitz, it is also absolutely continuous, so its weak derivative agrees with its pointwise derivative a'. Hence $a \in H^{3/2}(\mathbb{T})$. $\square$

Previous to Shen's result described earlier, Gay-Balmaz and Ratiu [30] had proved that if Γ is Weil-Petersson, then $\varphi \in H^s(\mathbb{T})$ for all $s < 3/2$, but Shen [63] gave examples not in $H^{3/2}(\mathbb{T})$ or Lipschitz.

Thus Theorem 1.1 implies that having an $H^{3/2}$ arclength parameterization is not equivalent to having an $H^{3/2}$ conformal welding. These are equivalent conditions for $s > 3/2$: for such weldings the Sobolev embedding theorem implies that φ' is Hölder continuous, which implies that the conformal mappings f, g have non-vanishing, Hölder continuous derivatives (e.g.,[39]), and therefore φ is biLipschitz.

This implies Γ has an H^s arclength parameterization (copy the argument following Definition 5, using the fact that biLipschitz circle homeomorphisms preserve $H^s(\mathbb{T})$ for $1/2 < s < 1$, e.g., [12]).

When identified with quasisymmetric circle homeomorphisms, elements of the universal Teichmüller space $T(1)$ form a group under composition. It is not a topological group under the usual topology because left multiplication is not continuous (e.g., Theorem 3.3 in [41] or Remark 6.9 in [37]).

However, the subgroup $T_0(1)$ is a topological group with its Weil-Petersson topology. Circle diffeomorphisms in $H^s(\mathbb{T})$ with $s > 3/2$ also form a group, e.g., [36], [63], and by the previous paragraph this means H^s curves are identified with a topological group via conformal welding. Even though $H^{3/2}$-diffeomorphisms of the circle are not a group, Theorem 1.1 shows the set of $H^{3/2}$ curves can also be identified with a group via conformal welding, namely $T_0(1)$.

See also [7], [30], [45], [46] for relevant discussions of groups, weldings, Sobolev embeddings and immersions.

Assuming Γ is chord-arc,

$$\frac{1}{C} \leq \frac{|a(x)-a(y)|}{|x-y|} \leq 1, \quad x, y \in \mathbb{T},$$

so setting $z = a(x), w = a(y)$, we have

$$\begin{aligned}
\int_\Gamma \int_\Gamma \left|\frac{\tau(z)-\tau(w)}{z-w}\right|^2 |dz||dw| &= \int_{\mathbb{T}} \int_{\mathbb{T}} \left|\frac{a'(x)-a'(y)}{a(x)-a(y)}\right|^2 dxdy \\
&= \int_{\mathbb{T}} \int_{\mathbb{T}} \left|\frac{a'(x)-a'(y)}{x-y} \cdot \frac{x-y}{a(x)-a(y)}\right|^2 dxdy \\
&\simeq \int_{\mathbb{T}} \int_{\mathbb{T}} \left|\frac{a'(x)-a'(y)}{x-y}\right|^2 dxdy
\end{aligned}$$

Thus Definition 6 is equivalent to:

Definition 7. Γ *is chord-arc and*
$$\int_\Gamma \int_\Gamma \left| \frac{\tau(z) - \tau(w)}{z - w} \right|^2 |dz||dw| < \infty.$$

This is very similar to saying $f'/|f'| \in H^{1/2}(\mathbb{T})$, but the under lying measure is wrong. The condition on $f'/|f'|$ transported to Γ would involve integrating against harmomic measure, the image of Lebesgue measure under the conformal map. The definition above is in terms of arclength measure.

Thus in this sense, harmonic measure and arclength are "essentially the same" on Weil-Petersson curves.

We will prove this is equivalent to:

Definition 8. Γ *has finite Möbius energy, i.e,*

$$\mathrm{M\ddot{o}b}(\Gamma) = \int_\Gamma \int_\Gamma \left(\frac{1}{|z-w|^2} - \frac{1}{\ell(z,w)^2} \right) dz dw < \infty.$$

Blatt [11] proved directly that Definition 6 is equivalent to Definition 8 (but there is a typo in Theorem 1.1 of [11]: it is stated that $s = (jp-2)/(2p)$, but this should be $s = (jp-1)/(2p)$, as given in the proof).

A Jordan curve with a $H^{3/2}$ arclength parameterization is chord-arc (Lemma 2.1 of [11], because this assumption prevents bending on small scales, but there is no quantitative bound on the chord-arc constant. However, such a bound is possible in terms of $\text{Möb}(\Gamma)$. This is Lemma 1.2 of [34], but for the reader's convenience, we sketch a proof here.

Lemma 3.2. *Finite Möbius energy implies chord-ard.*

Proof. If $|z - w| \leq \epsilon$, but $\ell(z, w) \geq M\epsilon$, let $\sigma_k, \sigma_k' \subset \gamma(z, w)$ be arcs of length $2^k\epsilon$ that are path distance (on Γ) $2^k\epsilon$ from z and w respectively, for $k = 1, \ldots, K = \lfloor \log_2(M) \rfloor - 4$. Then $\sigma_k \cup \sigma_k'$ has diameter at most $\epsilon(1 + 2^{k+1})$ in $\mathbb{R}^n$, but these two arcs are at least distance $(M - 2^{k+2})\epsilon \geq M\epsilon/2$ apart on Γ.

Thus

$$\begin{aligned}
\int_{\sigma_k}\int_{\sigma_k'}\left(\frac{1}{|z-w|^2}-\frac{1}{\ell(v,w)^2}\right)dzdw \;&\geq\; \left[\frac{1}{(2^{k+2}\epsilon)^2}-\frac{1}{(M/2)^2}\right](2^k\epsilon)(2^k\epsilon)\\
&\geq\; \frac{1}{16}-\frac{2^{2K+2}}{M^2}\\
&\geq\; \frac{1}{16}-2^{-6}>\frac{1}{32}
\end{aligned}$$

Summing over k shows $\text{Möb}(\Gamma)\geq K/32 \gtrsim \log M$, so $\text{Möb}(\Gamma)<\infty$ implies Γ is chord-arc. $\square$

Using the fact that Γ is chord-arc, we get

$$\begin{aligned}
\text{Möb}(\Gamma) &= \int_\Gamma \int_\Gamma \frac{\ell(z,w)^2 - |z-w|^2}{|z-w|^2 \ell(z,w)^2} dz dw \\
&= \int_\Gamma \int_\Gamma \frac{(\ell(z,w) - |z-w|)(\ell(z,w) + |z-w|)}{|z-w|^2 \ell(z,w)^2} dz dw \\
&\simeq \int_\Gamma \int_\Gamma \frac{\ell(z,w) - |z-w|}{|z-w|^3}.
\end{aligned}$$

Thus Definition 8 holds iff

Definition 9. Γ *is chord-arc and satisfies*

$$\int_\Gamma \int_\Gamma \frac{\ell(z,w) - |z-w|}{|z-w|^3} |dz||dw| < \infty. \tag{3.1}$$

In [26], Gallardo-Gutiérrez, González, Pérez-González, Pommerenke and Rättyä claim that (3.1) follows from Definition 1, but their proof contains a small error. They state the converse as a conjecture of Peter Jones; our results prove both directions.

This definition does not immediately look like a "curvature is square integrable" criterion, but it can easily be put in this form. Set

$$\kappa(z,w) = \sqrt{24} \cdot \sqrt{\frac{\ell(z,w) - |z-w|}{|z-w|^3}}.$$

If Γ is smooth, then it is easy to check that $\kappa(x) = \lim_{y\to x} \kappa(x,y)$, is the usual Euclidean curvature of Γ at x. Thus (3.1) can be rewritten as

$$\int_\Gamma \int_\Gamma \kappa^2(z,w)|dz||dw| < \infty, \tag{3.2}$$

and this has much more of a "L^2-curvature" flavor.

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The names of 26 characterizations of Weil-Peterson curves

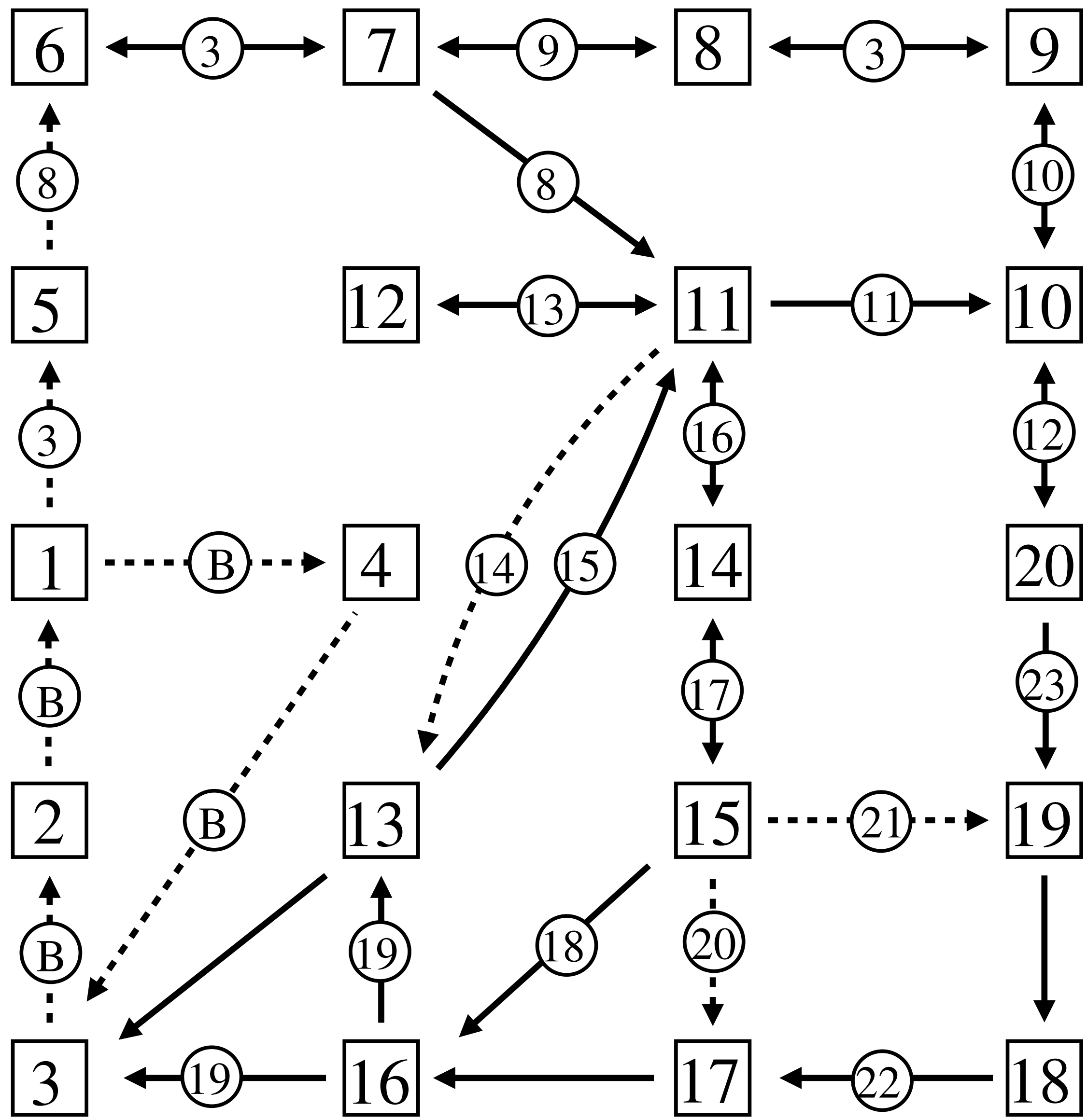


Diagram of implications between previous definitions.

Edge labels refer to sections of my preprint.

4. Thursday, Sept 24, 2020

5. β-NUMBERS

A dyadic interval I in $\mathbb{R}$ is one of the form $(2^{-n}j, 2^{-n}(j+1)]$. A dyadic cube in $\mathbb{R}^n$ is the product of n dyadic intervals of the same length. This length is called the side length of Q and is denoted $\ell(Q)$. Note that $\mathrm{diam}(Q) = \sqrt{n}\ell(Q)$.

For a positive number $\lambda > 0$, we let λQ denote the cube concentric with Q but with diameter $\lambda \mathrm{diam}(Q)$, e.g., $3Q$ is the "triple" of Q, a union of Q and $3^n - 1$ adjacent copies of itself. We let $Q^{\uparrow}$ denote the parent of Q; the unique dyadic cube containing Q and having twice the side length. Q is one of the 2^n children of $Q^{\uparrow}$.

A multi-resolution family in a metric space X is a collection of sets $\{X_j\}$ in X such that there is are $N, M < \infty$ so that

(1) For each $r > 0$, the sets with diameter between r and Mr cover X,

(2) each bounded subset of X hits at most N of the sets X_k with $\mathrm{diam}(X)/M \leq \mathrm{diam}(X_k) \leq M\mathrm{diam}(X)$.

(3) any subset of X with positive, finite diameter is contained in at least one X_j with $\mathrm{diam}(X_j) \leq M\mathrm{diam}(X)$.

Dyadic intervals are not multi-resolution family, e.g., $X = [-1, 1] \subset \mathbb{R}$ is not contained in any dyadic interval, violating (3). However, the family of triples of all dyadic intervals (or cubes) do form a multi-resolution family. Similarly, if we add all translates of dyadic intervals by $\pm 1/3$, we get a multi-resolution family (this is sometimes called the "$\frac{1}{3}$-trick", [52]).

The analogous construction for dyadic squares in $\mathbb{R}^n$ is to take all translates by elements of $\{-\frac{1}{3}, 0, \frac{1}{3}\}^n$.

We often deal with functions α that map a collection of sets into the non-negative reals, and will wish to decide if the sum $\sum_j \alpha(X_j)$ over some multi-resolution family converges or diverges. We will frequently use the following observation to switch between various multi-resolution families without comment.

Lemma 5.1. *Suppose $\{X_j\}$, $\{Y_k\}$ are two multi-resolution families on a space X and that α is a function mapping subsets of X to $[0,\infty)$ that satisfies $\alpha(E) \lesssim \alpha(F)$, whenever $E \subset F$ and $\operatorname{diam}(F) \lesssim \operatorname{diam}(E)$. Then*

$$\sum_j \alpha(X_j) \simeq \sum_k \alpha(Y_k).$$

Proof. By Condition (3) above, each X_j is contained in some set $Y_{k(j)}$ of comparable diameter. Hence $\alpha(X_j) \lesssim \alpha(Y_{k(j)})$ by assumption. Each Y_k is contained in a comparably sized X_m, and X_m can contain at most a bounded number of comparably sized subsets X_j. Thus each Y_k is only chosen boundedly often as a $Y_{k(j)}$. Thus $\sum_j \alpha(X_j) \lesssim \sum_k \alpha(Y_k)$. The opposite direction follows by reversing the roles of the two families. $\square$

For a Jordan arc γ with endpoints z, w recall $\mathrm{crd}(\gamma) = |z - w|$ and define $\Delta(\gamma) = \ell(\gamma) - \mathrm{crd}(\gamma)$. We will prove Definition 9 is equivalent to:

Definition 10. Γ *is chord-arc and*

$$\sum_j \frac{\Delta(\Gamma_j)}{\ell(\Gamma_j)} < \infty \tag{5.1}$$

for some multi-resolution family $\{\Gamma_j\}$ *of arcs on* Γ.

Condition (5.1) is just a reformulation of (1.2), since if $\{\Gamma_j\}$ corresponds to a dyadic decomposition of Γ we have

$$\sum_n 2^n[\ell(\Gamma) - \ell(\Gamma_n)] = \sum_j \Delta(\gamma_j)/\ell(\gamma_j). \tag{5.2}$$

Thus proving that Definition 10 is equivalent to being Weil-Petersson essentially proves Theorem 1.4.

There is a slight gap here because Definition 10 uses a sum over a multi-resolution family and Theorem 1.4 is in terms of dyadic intervals.

However, the theorem assumes a bound that is uniform over all dyadic decompositions, and this includes the $\frac{1}{3}$-translates of a single dyadic family, and these form another multi-resolution family (recall the "$\frac{1}{3}$-trick" from above). Conversely, we will show that $\Delta(\gamma) \leq \Delta(3\gamma)$, so the dyadic sum can be bounded by the sum over dyadic triples, a multi-resolution family. Thus (5.1) for any multi-resolution family is equivalent to (5.2) with a uniform bound over all dyadic decompositions of Γ.

Lemma 5.2. *If $\gamma, \gamma' \subset \Gamma$ are adjacent, then $\Delta(\gamma) + \Delta(\gamma') \leq \Delta(\gamma \cup \gamma')$.*

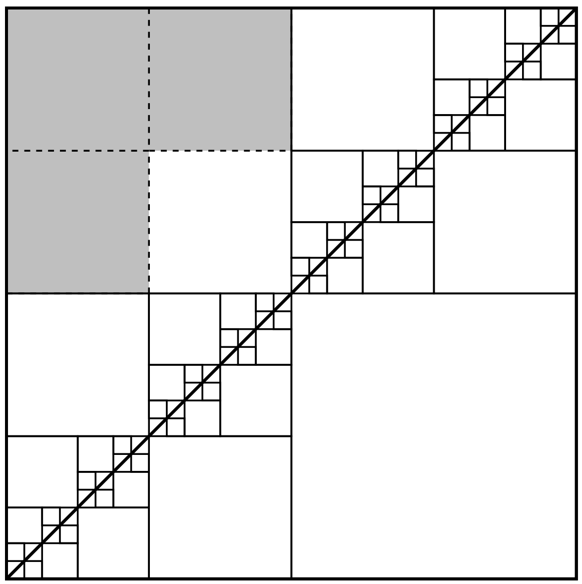
Proof. Note that $\ell(\gamma \cup \gamma') = \ell(\gamma) + \ell(\gamma')$, and $\operatorname{crd}(\gamma \cup \gamma') \leq \operatorname{crd}(\gamma) + \operatorname{crd}(\gamma')$, so

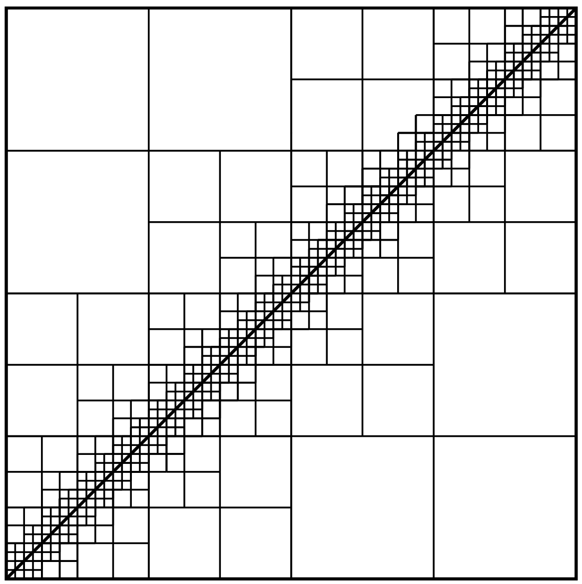
$$\begin{aligned} \Delta(\gamma \cup \gamma') &= \ell(\gamma \cup \gamma') - \operatorname{crd}(\gamma \cup \gamma') \\ &\geq \ell(\gamma) + \ell(\gamma') - \operatorname{crd}(\gamma) - \operatorname{crd}(\gamma') = \Delta(\gamma) + \Delta(\gamma'). \quad \square \end{aligned}$$

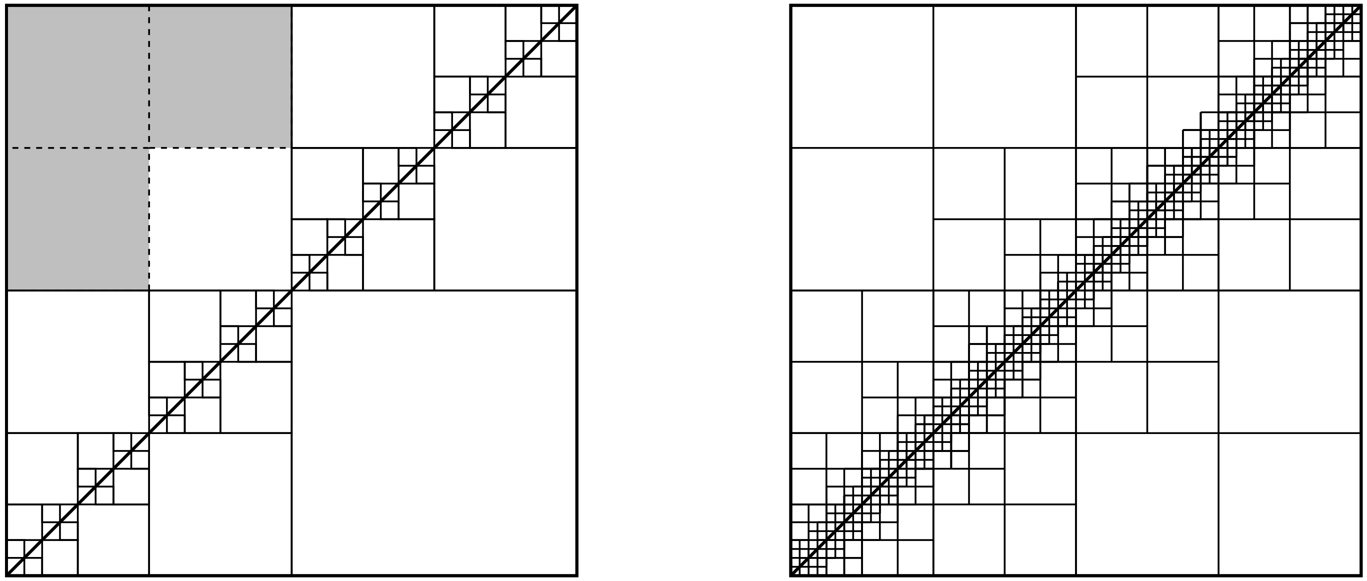
Corollary 5.3. *If $\gamma \subset \gamma'$ then $\Delta(\gamma) \leq \Delta(\gamma')$.*

Lemma 5.4. *Definition 9 is equivalent to Definition 10.*

Proof. Without loss of generality we may rescale Γ so that is has length 1. We identify $\Gamma \times \Gamma$ with the torus $\mathbb{T}^2 = [0,1]^2$, let U be the torus minus the diagonal, and take a Whitney decomposition of U by dyadic squares $\{Q_j\}$.

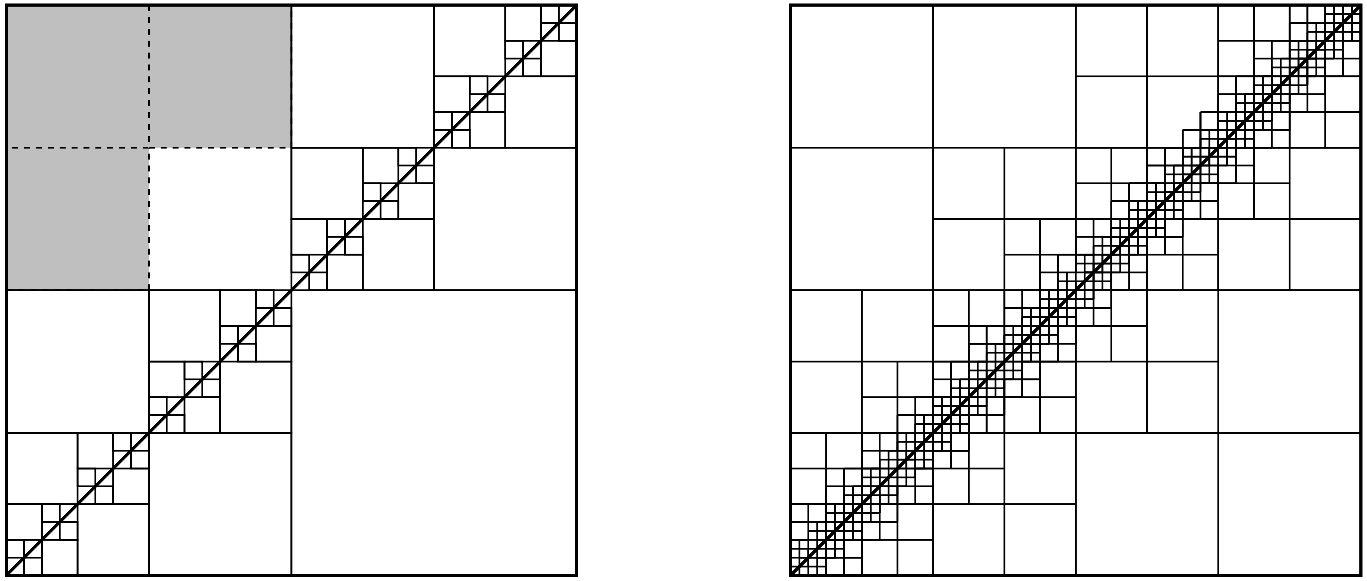




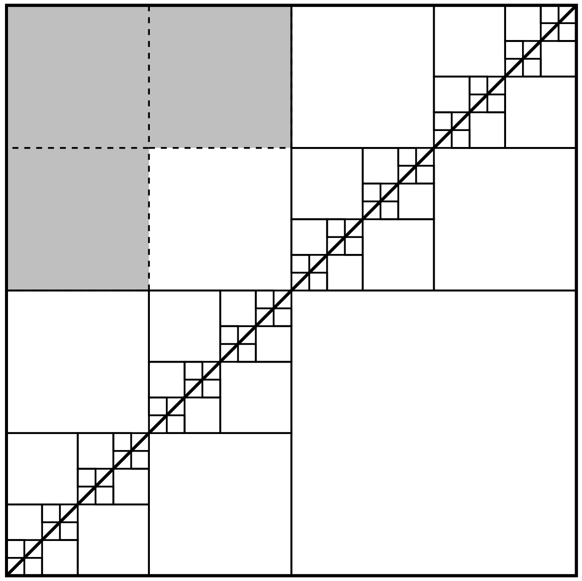
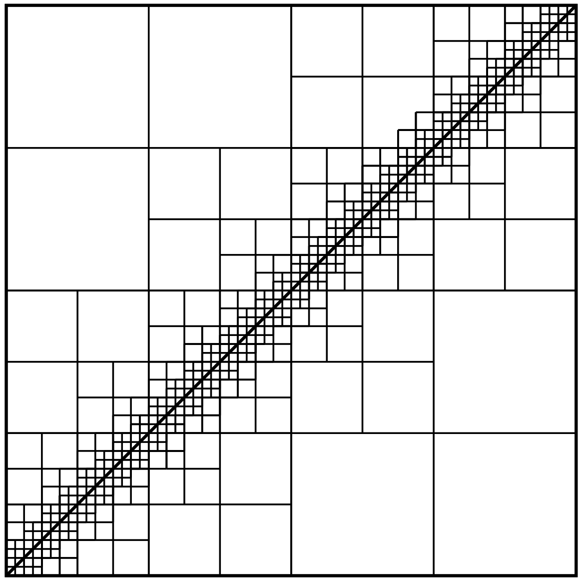


Elements of the decomposition are denoted $\{W_j\}$, and each is a product of dyadic arcs $W_j = \gamma_j \times \gamma_j'$. For each W_j, we can write $\gamma_j \cup \gamma_j' = \Gamma_j \setminus \Gamma_j'$ for arcs Γ_j, Γ_j' so that all four arcs have comparable lengths.

Recall that $\mathrm{crd}(\gamma) = |z-w|$ where z, w are the endpoints of γ and that $\Delta(\gamma) \equiv \ell(\gamma) - \mathrm{crd}(\gamma)$. We sometimes write $\Delta(z, w)$ for $\Delta(\gamma)$ when γ has endpoints z, w, and it is clear from context which arc connecting these points we mean. We say two subarcs of Γ are adjacent if they have disjoint interiors, but share a common endpoint.



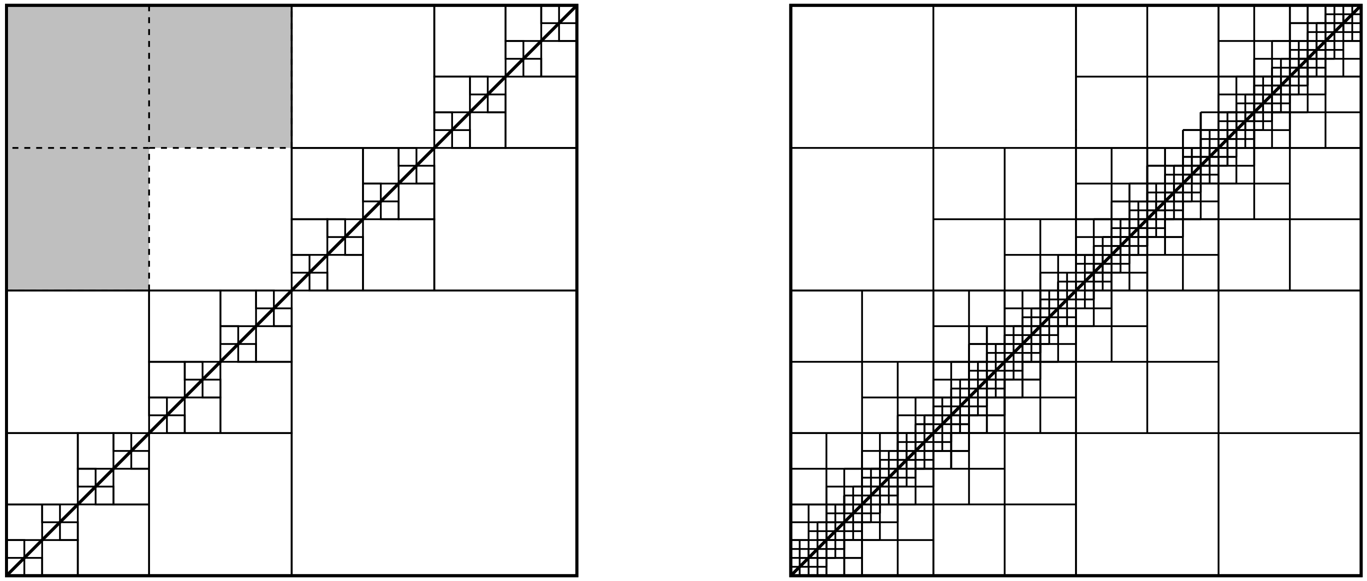
Now, fix j and consider the Whitney box $W_j = \gamma_j \times \gamma_j'$. If $\gamma \subset \Gamma_j$ is any arc with one endpoint in γ_j and the other in γ_j' then $\Gamma_j' \subset \gamma \subset \Gamma_j$, and hence $\Delta(\Gamma_j') \leq \Delta(\gamma) \leq \Delta(\Gamma_j)$. Because Γ is chord-arc, if $z \in \gamma_j'$ and $w \in \gamma_j$, then $|z - w| \gtrsim \ell(\Gamma_j') \simeq \ell(\Gamma_j)$.

We can therefore write the integral from Definition 9 as

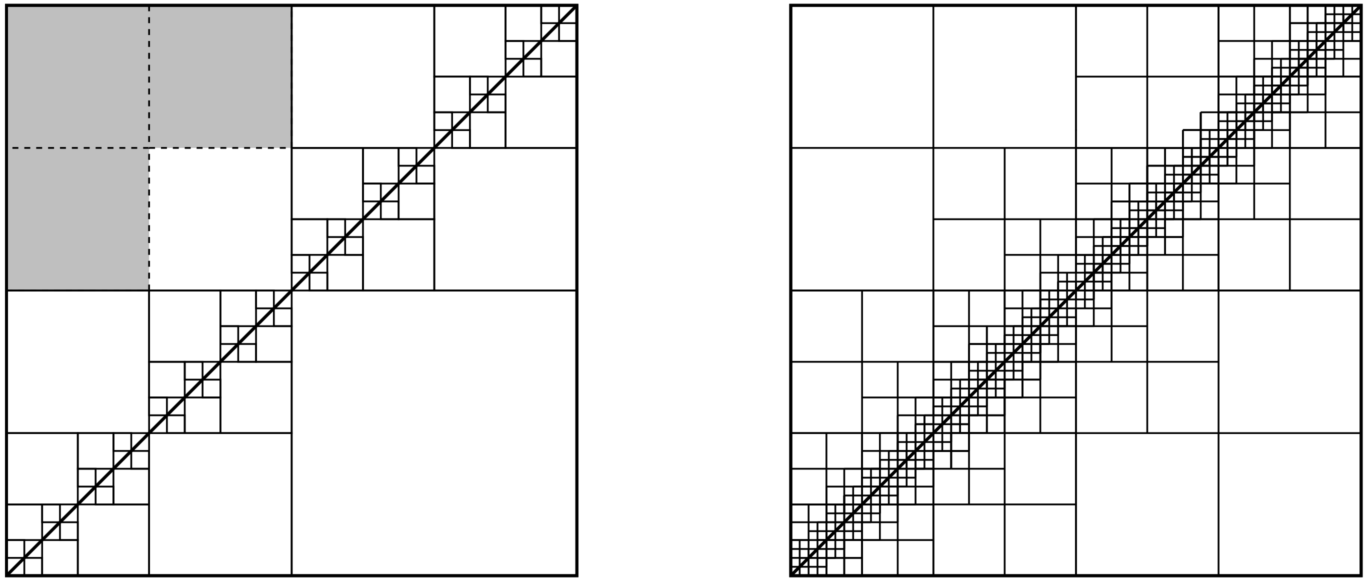
$$\int_\Gamma \int_\Gamma \frac{\ell(z,w) - |z-w|}{|z-w|^3} |dz||dw| = \sum_j \int_{W_j} \frac{\Delta(z,w)}{|z-w|^3} |dz||dw|$$

$$\lesssim \sum_j \frac{\Delta(\Gamma_j)}{\ell(\Gamma_j)^3} \ell(\Gamma_j)^2 = \sum_j \frac{\Delta(\Gamma_j)}{\ell(\Gamma_j)}.$$



Reversing the argument, now assume Γ'_j is some dyadic subinterval of Γ and let γ_j, γ'_j be the equal length dyadic arcs adjacent to Γ'_j.

$$\int_{\gamma_j}\int_{\gamma'_j} \frac{\ell(z,w) - |z-w|}{|z-w|^3}|dz||dw| \gtrsim \frac{\Delta(\Gamma'_j)}{\ell(\Gamma'_j)}.$$



The squares $W_j = \gamma_j \times \gamma_j'$ arising in this way have bounded overlap, so

$$\int_\Gamma \int_\Gamma \frac{\ell(z,w) - |z-w|}{|z-w|^3} |dz||dw| \gtrsim \sum_j \frac{\Delta(\Gamma_j')}{\ell(\Gamma_j')},$$

where the sum is over all dyadic subintervals of Γ. This works for any dyadic decomposition $\{\Gamma_j\}$ of Γ, and hence for a multi-resolution family. This gives the equivalence of Definitions 9 and 10. $\square$

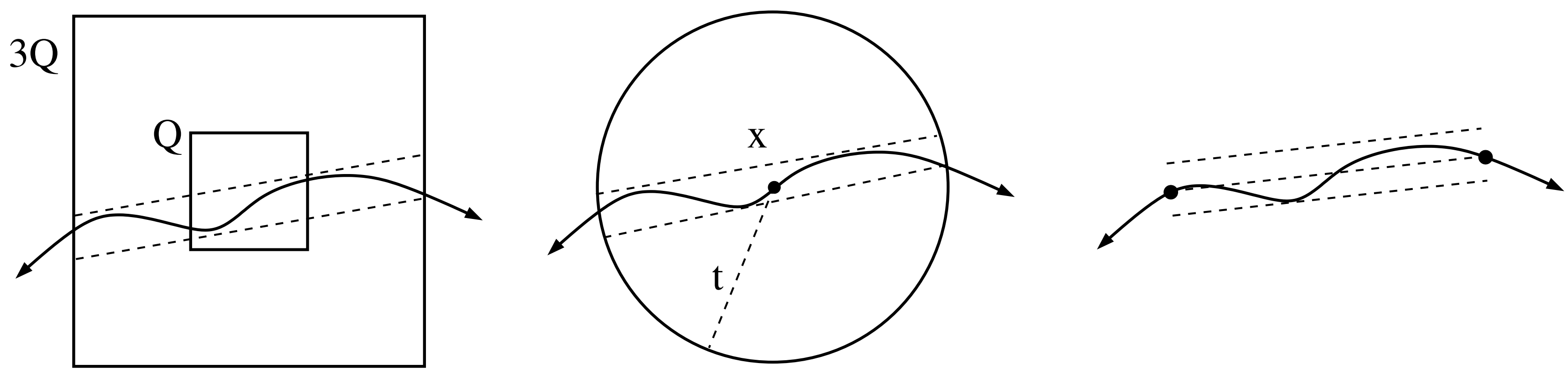
6. Tuesday, Sept 29, 2020

The Beta-numbers:

Given a set $E \subset \mathbb{R}^n$ and a dyadic cube Q, define Peter Jones's β-number as

$$\beta(Q) = \beta_E(Q) = \frac{1}{\operatorname{diam}(Q)} \inf_L \sup\{\operatorname{dist}(z, L) : z \in 3Q \cap E\},$$

where the infimum is over all lines L that hit $3Q$.



Peter Jones invented the β-numbers as part of his traveling salesman theorem [38]. One consequence of his theorem is that for a Jordan curve Γ,

$$\ell(\Gamma) \simeq \mathrm{diam}(\Gamma) + \sum_Q \beta_\Gamma(Q)^2 \mathrm{diam}(Q), \tag{6.1}$$

where the sum is over all dyadic cubes Q in $\mathbb{R}^n$. Our main geometric characterization of Weil-Petersson curves is to simply the "diam(Q)" terms from (6.1).

Definition 11. Γ *is a closed Jordan curve that satisfies*

$$\sum_Q \beta_\Gamma(Q)^2 < \infty, \tag{6.2}$$

where the sum is over all dyadic cubes.

This is not terribly surprising (in retrospect). Peter Jones and I proved (Lemma 3.9 of [10], or Theorem X.6.2 of [29]) that if Γ is a M-quasicircle, then

$$\ell(\Gamma) \simeq \operatorname{diam}(\Gamma) + \iint |f'(z)||S(f)(z)|^2(1-|z|^2)^3 dxdy \tag{6.3}$$

with constants depending only on M.

By Koebe's distortion theorem

$$|f'(z)|(1-|z|^2) \simeq \operatorname{dist}(f(z), \partial\Omega),$$

and thus the factor on the left is analogous to the diam(Q) in Jones's β^2-sum. Dropping this term from (6.3) gives exactly the integral in Definition 2:

$$\iint |S(f)(z)|^2(1-|z|^2)^2 dxdy \tag{6.4}$$

Thus Definition 11 in the plane is a direct geometric analog of this.

It will be convenient to consider several equivalent formulations of condition (6.2). For $x \in \mathbb{R}^2$ and $t > 0$, define

$$\beta_\Gamma(x,t) = \frac{1}{t} \inf_L \max\{\mathrm{dist}(z,L) : z \in \Gamma, |x-z| \le t\},$$

where the infimum is over all lines hitting the disk $D = D(x,t)$ and let $\widetilde{\beta}_\Gamma(x,t)$ be the same, but where the infimum is only taken over lines L hitting x.

Since this is a smaller collection, clearly $\beta(x,t) \le \widetilde{\beta}(x,t)$ and it is not hard to prove that $\widetilde{\beta}(x,t) \le 2\beta(x,t)$ if $x \in \Gamma$.

Given a Jordan arc γ with endpoints z, w we let

$$\beta(\gamma) = \frac{\max\{\mathrm{dist}(z,L) : z \in \gamma\}}{|z-w|},$$

where L is the line passing through z and w.

Lemma 6.1. *If Γ is a closed Jordan curve or a Jordan arc in $\mathbb{R}^n$ such that (6.2) holds, then Γ is a chord-arc curve. Moreover, (6.2) holds if and only if any of the following conditions holds:*

$$\int_0^\infty \iint_{\mathbb{R}^n} \beta^2(x,t) \frac{dxdt}{t^{n+1}} < \infty, \tag{6.5}$$

$$\int_0^\infty \int_\Gamma \widetilde{\beta}^2(x,t) \frac{dsdt}{t^2} < \infty, \tag{6.6}$$

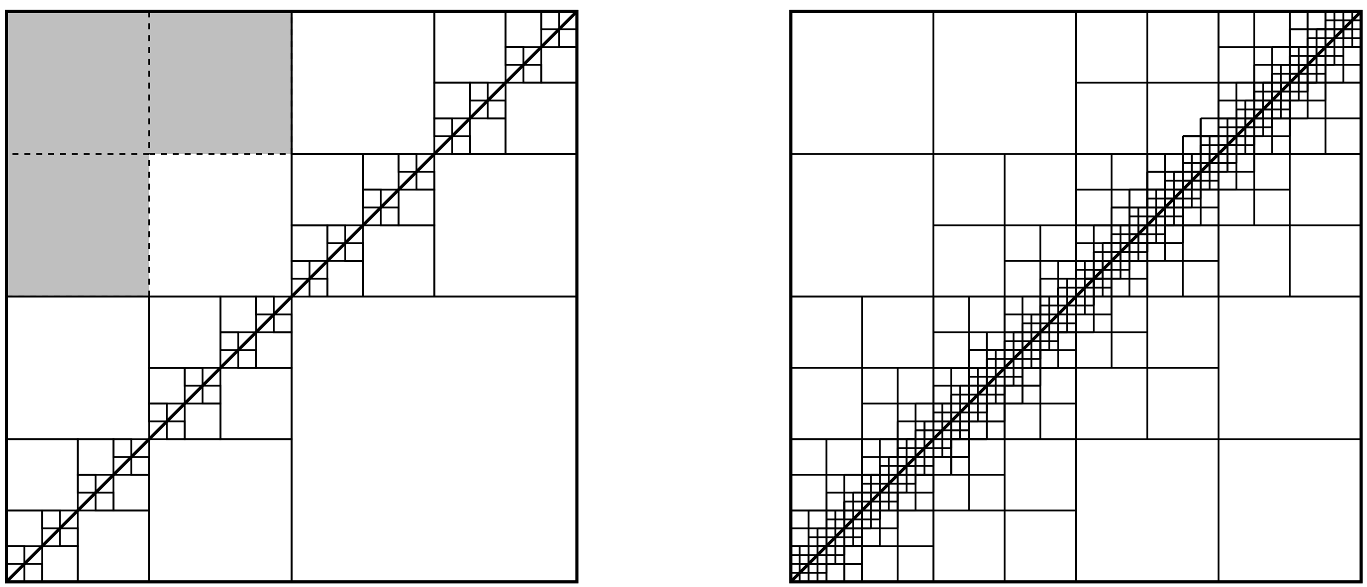
$$\sum_j \beta^2(\Gamma_j) < \infty, \tag{6.7}$$

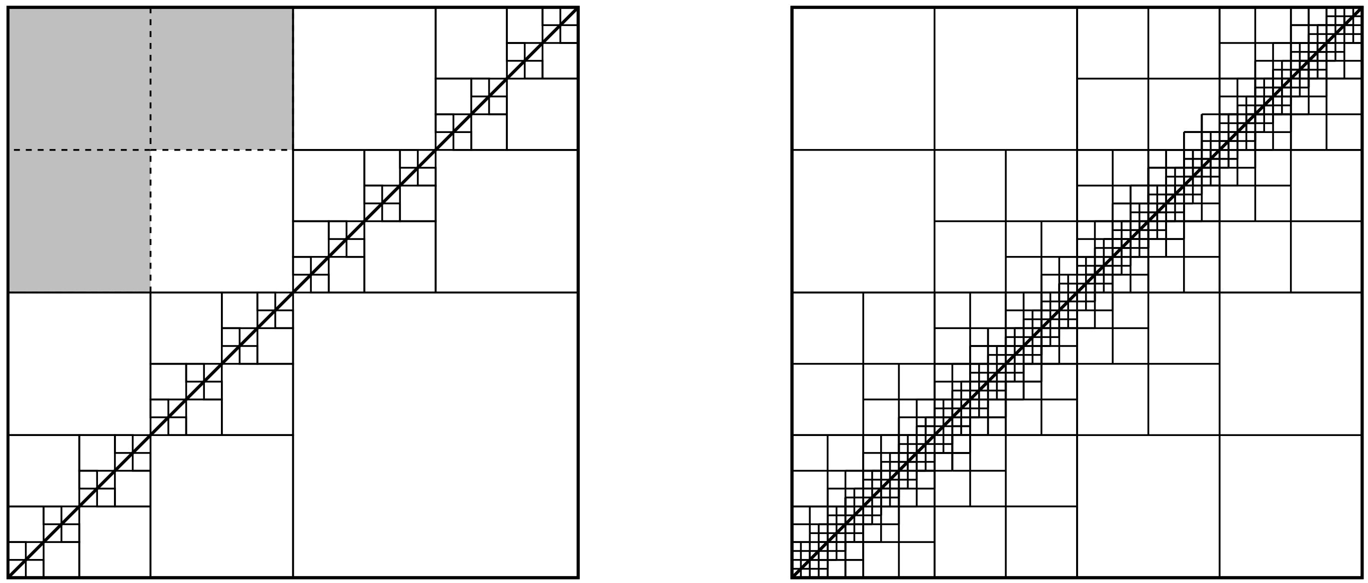
where dx is volume measure on $\mathbb{R}^n$, ds is arclength measure on Γ, and the sum in (6.7) is over a multi-resolution family $\{\Gamma_j\}$ for Γ. Convergence or divergence in (6.5) and (6.6) is not changed if $\int_0^\infty$ is replaced by $\int_0^M$ for any $M > 0$.

$H^{3/2}$ implies Beta-numbers:

Lemma 6.2. *Definition 7 implies Definition 11.*

Proof. Let U be the torus $\mathbb{T} \times \mathbb{T}$ minus the diagonal. Take a Whitney decomposition of U, i.e., a covering of U by squares Q with disjoint interiors and the property that $\mathrm{diam}(Q) \simeq \mathrm{dist}(Q, \partial U)$. We will think of $\mathbb{T}$ as $[0,1]$ with its endpoints identified, and use dyadic squares in $[0,1]^2$ as elements of our decomposition.





Each element W_j of the decomposition can be written as $W_j = \gamma_j \times \gamma_j'$ where $\gamma_j \cup \gamma_j' = \Gamma_j \setminus \Gamma_j'$ and all these arcs have comparable lengths (in fact, γ_j and γ_j' have the same length).

For each Whitney piece $W_j = \gamma_j \times \gamma_j'$, choose a $w_0 \in \gamma_j'$ so that

$$\ell(\gamma_j') \int_{\gamma_j} |\tau(z) - \tau(w_0)|^2 |dz| \leq 2 \int_{\gamma_j'} \int_{\gamma_j} |\tau(z) - \tau(w)|^2 |dz||dw|.$$

(We can do this because a positive measurable function must take a value that is less than or equal to twice its average.)

Let L be the line through one endpoint of γ_j' in direction $\tau(w)$. Then the maximum distance D that γ_j can attain from L satisfies

$$d \lesssim \int_{\gamma_j} |\tau(z) - \tau(w_0))||dz| \leq \left(\int_{\gamma_j} |\tau(z) - \tau(w_0)|^2 |dz| \right)^{1/2} \ell(\gamma_j)^{1/2}.$$

Therefore (using the fact that γ is chord-arc),

$$\begin{aligned}
\beta^2(\gamma_j) \simeq d^2/\mathrm{diam}(\gamma_j) &\lesssim \frac{1}{\ell(\gamma_j)} \int_{\gamma_j} |\tau(z) - \tau(w_0)|^2 |dz| \\
&\leq \frac{2}{\ell(\gamma_j)^2} \int_{\gamma_j} \int_{\gamma_j'} |\tau(z) - \tau(w)|^2 |dz||dw| \\
&\lesssim \int_{\gamma_j} \int_{\gamma_j'} \left| \frac{\tau(z) - \tau(w)}{z - w} \right|^2 |dz||dw|.
\end{aligned}$$

Summing over all Whitney pieces proves that the β^2-sum is finite when taken over all arcs of the form $\{\gamma_j\}$. By construction every dyadic interval in $[0,1]$ (except for $[0,\frac{1}{2}]$, $[\frac{1}{2},1]$ and $[0,1]$) occurs as a γ_j at least once and at most three times, so this bounds the sum of $\beta^2(\gamma)$ over all dyadic subintervals of Γ for a fixed base point, with an estimate independent of the basepoint.

Thus it holds for some multi-resolution family of arcs (recall the $\frac{1}{3}$-trick for making such a family from three translates of the dyadic family). Because of Lemma 6.1, this proves the lemma. $\square$

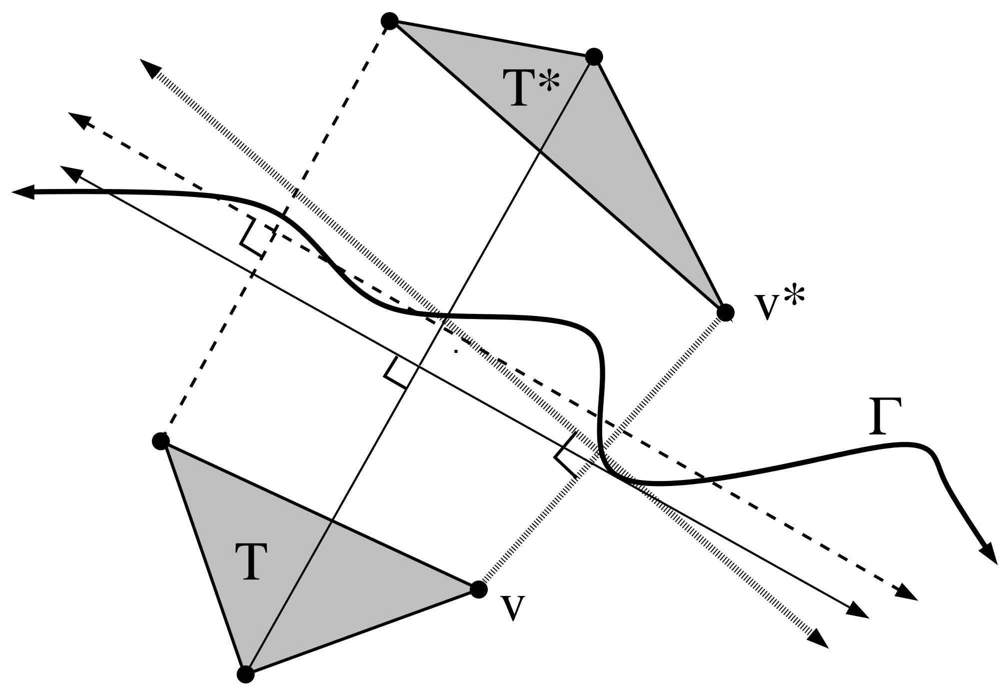
Beta-numbers imply BiLipschitz reflection:

Lemma 6.3. *Definition 11 implies Definition 13 for $n = 2$.*

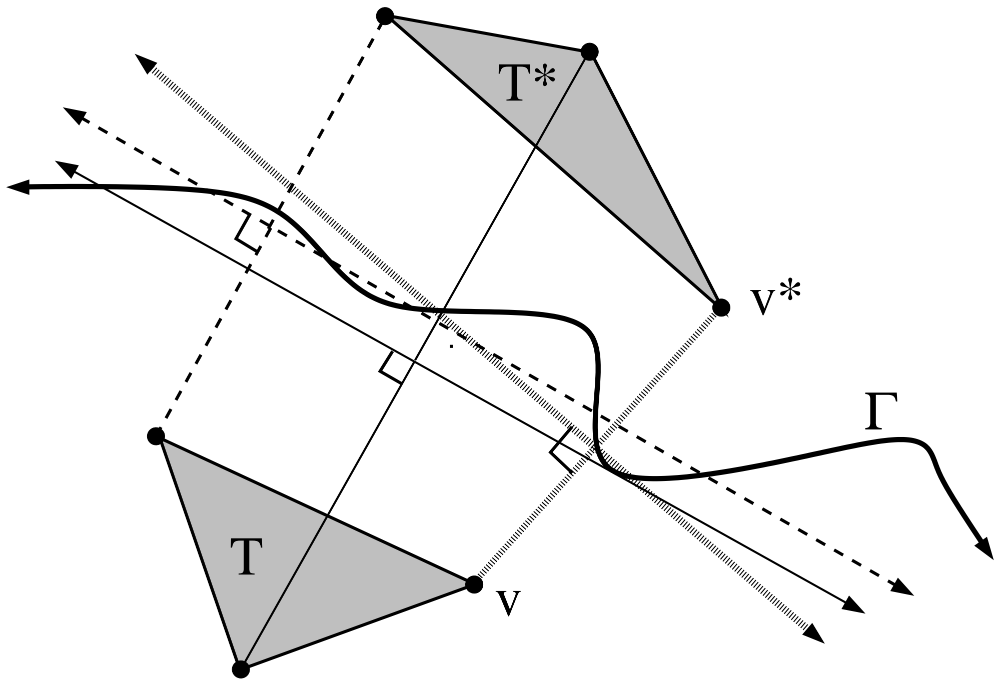
Proof. Since $\sum_Q \beta_\Gamma^2(Q) < \infty$, only finitely many of the β's can be larger than $1/1000$. Let $U(\epsilon)$ denote the ϵ-neighborhood of Γ, and choose ϵ_0 so small that $U(\epsilon_0)$ only contains dyadic squares Q with $\beta_\Gamma(Q) < 1/1000$. Let Ω be the bounded complementary component of Γ and consider a Whitney decomposition for Ω using dyadic squares.

Form a triangulation of Ω by connecting the center of each square to the vertices on its boundary. Note that neighboring triangles have comparable diameters and that all angles are bounded uniformly above 0 and below π.

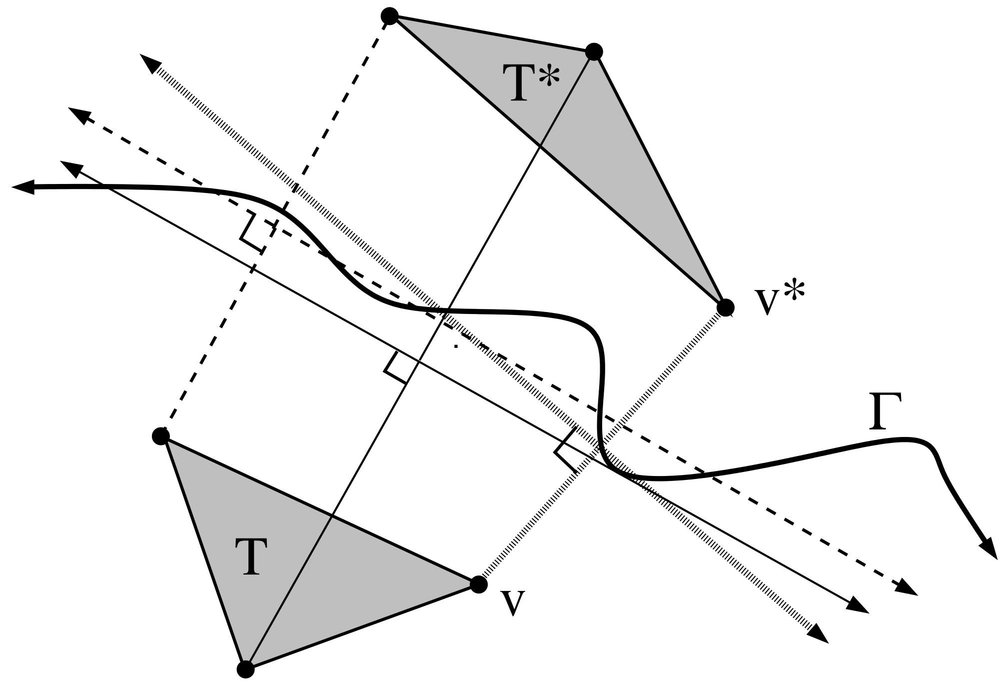
We will define a reflection across Γ that is defined on a neighborhood of Γ and is piecewise affine on the above triangles. Let $\mathcal{S}_k$ be the collection of squares Q in the Whitney decomposition so that $\ell(Q) = 2^{-k}$ and let $\mathcal{S} = \cup_{k>k_0}\mathcal{S}_k$ where k_0 is chosen so that the elements of $\mathcal{S}$ are all contained in $U(\epsilon/100)$. Order the elements of $\{Q_j\}_1^\infty = \mathcal{S}$ so that side lengths are non-decreasing.



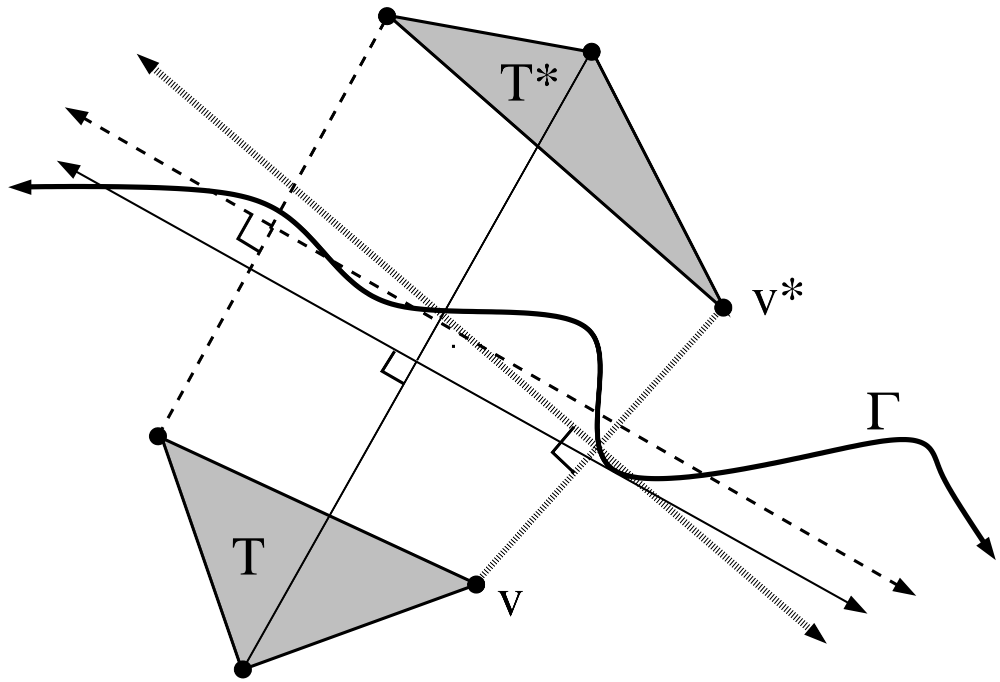
For each Q_j choose a dyadic square Q_j' of comparable size that hits Γ and so that $3Q_j'$ contains Q_j. Note that $Q_j' \subset U$, so $\beta_\Gamma(Q_j')$ is small. To begin, choose a line L_j that minimizes the definition of $\beta_\Gamma(Q_j')$. Reflect all four vertices of Q_1 across L_1. In general, reflect each vertex v of Q_j across L_j to a point v^* in Ω^*, if it was not already reflected by belonging to some Q_k with $k < j$.



The main point is that each vertex v belongs a uniformly bounded number of Q_J's and the different possible reflections v^* of v corresponding to these different squares all lie within distance $\beta_j \cdot \mathrm{dist}(v, \Gamma)$ of each other, where Q_j is any of the Whitney squares having v as a corner and $\beta_j = \beta_\Gamma(Q'_j)$.



This occurs because all the lines we might use have directions that differ by at most $O(\beta_j)$, and they all pass within $O(\beta_j \ell(Q'_j))$ of some point in Q'_j. We now define affine maps on each element of our triangulation that lies inside $U(\epsilon_0/1000)$ by sending each vertex to its reflection v^*. Suppose T is a triangle associated to Q_j. Then $\mathrm{diam}(T) \simeq \mathrm{dist}(T, \Gamma)$. The reflected vertices of T form a triangle T^* that is within $O(\beta_j)$ of being congruent to T.



Extending the map between vertices linearly, we get an affine map from T to T^* that is biLipschitz with constant $1 + O(\beta(Q'_j))$. Moreover, for each $z \in T$ we have $\mathrm{dist}(\frac{1}{2}(z + R(z)), \Gamma) = O(\beta_j \mathrm{diam}(Q'_j))$. Thus we have $\rho(Q_j) = O(\beta(Q'_j))$, and so the ρ^2-sum is finite if the β^2-sum is finite. $\square$

Beta-numbers implies $\mu \in L^2$:

Corollary 6.4. *For $n = 2$, Definition 13 implies Definition 3*

Proof. The homeomorphism R constructed above on a neighborhood of U of $\Gamma \subset \mathbb{R}^2$ is clearly quasiconformal on $U \setminus \Gamma$. Since Γ is a quasicircle, it is removable for quasiconformal homeomorphisms and hence our map is quasiconformal on all of U, i.e., we have defined a quasiconformal reflection across Γ on a neighborhood U of Γ. Each triangle T has hyperbolic area $\simeq 1$, so

$$\int_T |\mu(z)|^2 d\mathrm{A}_\rho(z) = O(\beta_\Gamma^2(Q)),$$

for some dyadic square Q with $\mathrm{diam}(Q) \simeq \mathrm{dist}(Q, T) \simeq \mathrm{diam}(T)$. Therefore

$$\int_U |\mu(z)|^2 d\mathrm{A}_\rho(z) = O\left(\sum_Q \beta_\Gamma^2(Q)\right)$$

since each Q occurs for only boundedly many T. Extend this map quasiconformally to the rest of Ω to get a reflection satisfying Definition 3. $\square$

7. Thursday, Oct 1, 2020

Menger curvature

The Menger curvature of three points $x, y, z \in \mathbb{R}^n$ is $c(x, y, z) = 1/R$ where R is the radius of the circle passing through these points.

Equivalently,

$$c(x,y,z) = \frac{2\text{dist}(x, L_{yz})}{|x-y||x-z|}, \tag{7.1}$$

where L_{yz} is the line passing through y and z, or

$$c(x,y,z) = 2\frac{\sin\theta}{|x-y|}, \tag{7.2}$$

where θ is the angle opposite $[x, y]$ in the triangle with vertices $\{x, y, z\}$. The perimeter of this triangle is denoted by $\ell(x, y, x) = |x - y| + |y - z| + |z - x|$.

Definition 12. Γ *is chord-arc and satisfies*

$$\int_\Gamma\int_\Gamma\int_\Gamma \frac{c(x,y,z)^2}{\ell(x,y,z)}|dx||dy||dz| < \infty. \tag{7.3}$$

It is known that the conditions

$$\int_\Gamma\int_\Gamma\int_\Gamma c(x,y,z)^2|dx||dy||dz| < \infty. \tag{7.4}$$

$$\sum_Q \beta_\Gamma^2(Q)\ell(Q) < \infty, \tag{7.5}$$

are equivalent, and the analog of dropping the length term from (7.5), would be to divide by a term that scales like length in (7.4), which gives (7.3).

I will not prove this in class.

The preprint merely indicates how to modify certain lines of the proof of the equivalence of (7.4) and (7.5) in Pajot's book [54].

Recall that a Whitney decomposition of an open set $W \subset \mathbb{R}^n$ is a collection of dyadic cubes Q with disjoint interiors, whose closures cover W and which satisfy

$$\operatorname{diam}(Q) \simeq \operatorname{dist}(Q, \partial W).$$

The existence of such decompositions is a standard fact (e.g., for each $z \in W$, take the maximal dyadic cube Q so that $z \in Q \subset 3Q \subset W$, see Section I.4 of [29]).

Definition of $\rho(Q)$:

Suppose U is a neighborhood of $\Gamma \subset \mathbb{R}^n$ and $R : U \to U' \subset \mathbb{R}^n$ is a homeomorphism fixing each point of Γ. For each Whitney cube Q for $W = \mathbb{R}^n \setminus \Gamma$, with $Q \subset U$, define $\rho(Q)$ to be the infimum of values $\rho > 0$ so that R is $(1+\rho)$-biLipschitz on Q and $\mathrm{dist}(\frac{z+R(z)}{2}, \Gamma) \leq \rho \cdot \mathrm{diam}(Q)$ for $z \in Q$ (the latter condition ensures $R(z)$ is on the "opposite" side of Γ from z). R is called an involution if $R(R(z)) = z$.

Definition 13. *There is homeomorphic involution R defined on a neighborhood of Γ that fixes Γ pointwise, and so that*

$$\sum_{Q} \rho^2(Q) < \infty. \tag{7.6}$$

The sum is over all Whitney cubes for $\mathbb{R}^n \setminus \Gamma$ that lie inside U.

Bi-Lipschitz reflections control beta-numbers:

We need some preliminary results. For a dyadic square Q' define

$$P(Q') = \left(\frac{1}{\text{diam}(Q')} \sum_{Q \subset 3Q'} \rho^2(Q)\text{diam}(Q) \right)^{1/2},$$

where we sum over Whitney cubes Q inside $3Q'$.

Lemma 7.1. *With notation as above,*

$$\sum_{Q'} P^2(Q') \lesssim \sum_{Q} \rho^2(Q),$$

where the first sum is over all dyadic cubes in U $\mathbb{R}^n$ and the second is over all Whitney cubes for $U \setminus \Gamma$. In particular, $P(Q)$ is bounded if Definition 13 holds.

Proof. We use the defintion and reverse the order of summation (Tonelli's thm):

$$\begin{aligned}\sum_{Q'} P^2(Q') &= \sum_{Q'} \sum_{Q\subset 3Q'} \rho^2(Q)\frac{\mathrm{diam}(Q)}{\mathrm{diam}(Q')} \\ &= \sum_{Q} \rho^2(Q) \sum_{Q':Q\subset 3Q'} \frac{\mathrm{diam}(Q)}{\mathrm{diam}(Q')} \\ &= \sum_{Q} \rho^2(Q) O\left(\sum_{n=1}^{\infty} 2^{-n}\right) \lesssim \sum_{Q} \rho^2(Q),\end{aligned}$$

since the sum over Q' only involves $O(1)$ cubes of each size $2^n\mathrm{diam}(Q)$. $\square$

Lemma 7.2. *A map $R : U \to U'$ satisfying Definition 13 is biLipschitz on U.*

Proof. Suppose $z, w \in U$, and and $|z - w| \leq 3 \max(\mathrm{dist}(z, \Gamma), \mathrm{dist}(w, \Gamma))$. (This is the "hard" case; when z, w are relatively far apart, the argument is easier and will be given below).

Without loss of generality we may assume $\mathrm{dist}(z, \Gamma) \geq \mathrm{dist}(w, \Gamma)$. Let S be the segment between z and w. Then $|R(z) - R(w)| \leq \ell(R(S))$.

The segment S may hit Γ, but R is the identity at such points, and $S \setminus \Gamma$ consists of at most countably many open subsegments, each covered by its intersection with Whitney cubes Q for $\mathbb{R}^n \setminus \Gamma$.

The length of each such intersection is increased by at most a factor of $\rho(Q)$. Therefore,

$$|R(z) - R(w)| - |z - w| \lesssim \sum_{Q \cap S \neq \emptyset} \rho(Q)\mathrm{diam}(Q)),$$

where the sum is over all Whitney cubes that hit S.

By the Cauchy-Schwarz inequality, the right side above is less than

$$\begin{aligned}
&\lesssim \left(\sum_{Q\cap S\neq\emptyset} \rho^2(Q)\mathrm{diam}(Q)\right)^{1/2} \left(\sum_{Q\cap S\neq\emptyset} \mathrm{diam}(Q)\right)^{1/2} \\
&\lesssim \left(\ell(S)\sum_{Q\cap S\neq\emptyset} \rho^2(Q)\mathrm{diam}(Q)\right)^{1/2} \\
&\lesssim \left(\ell(S)\sum_{Q\subset 3Q'} \rho^2(Q)\mathrm{diam}(Q)\right)^{1/2} \\
&\lesssim P(Q')\mathrm{diam}(Q'), \\
&\lesssim \mathrm{diam}(Q'), \\
&\lesssim |z-w|,
\end{aligned}$$

Thus $|R(z)-R(w)| = O(|z-w|)$, as desired.

When $|z - w| \geq 3\max(\text{dist}(z, \Gamma), \text{dist}(w, \Gamma)$ we can choose $z', w'\Gamma$, with $|z - z'| = \text{dist}(z, \Gamma)$ and similarly for w, w'. The previous case applies to each of these pairs.

Since z', w' are fixed by R we thus have

$$|R(z) - R(w)| \leq |R(z) - z'| + |z' - w'| + |w' - R(w)| \lesssim |z - w|.$$

Then R is Lipschitz. Since $R = R^{-1}$ is an involution, it is automatically biLipschitz. $\square$

8. Tuesday, Oct 6, 2020

biLipschitz involution implies β-numbers:

Lemma 8.1. *Definition 13 implies Definition 11.*

Proof. First note that

$$\begin{aligned}
\sum_{Q'} P^2(Q') &= \sum_{Q'} \sum_{Q \subset 3Q'} \rho^2(Q) \frac{\operatorname{diam}(Q)}{\operatorname{diam}(Q')} \\
&= \sum_{Q} \rho^2(Q) \sum_{Q' : Q \subset 3Q'} \frac{\operatorname{diam}(Q)}{\operatorname{diam}(Q')} \lesssim \sum_{Q} \rho^2(Q),
\end{aligned}$$

since the sum over Q' only involves $O(1)$ cubes of each size. Thus it suffices to show that $\beta(Q') = O(P(Q'))$.

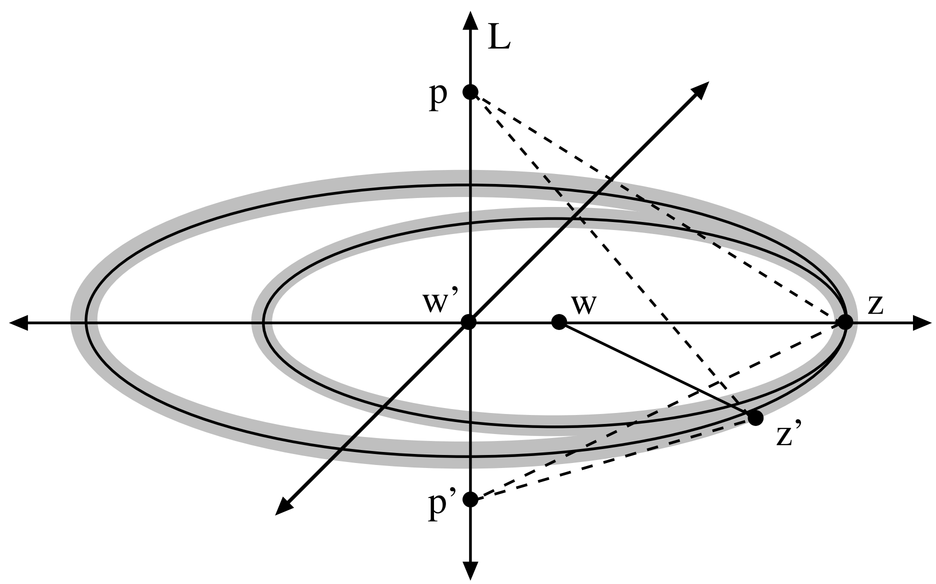
Normalize so $\ell(Q') = 1$. Choose two points $p, q \in \Gamma \cap 3Q'$ with $|p - q| \simeq 1$ and let L be the line through p and q. Choose $w \in \Gamma \cap 3Q'$

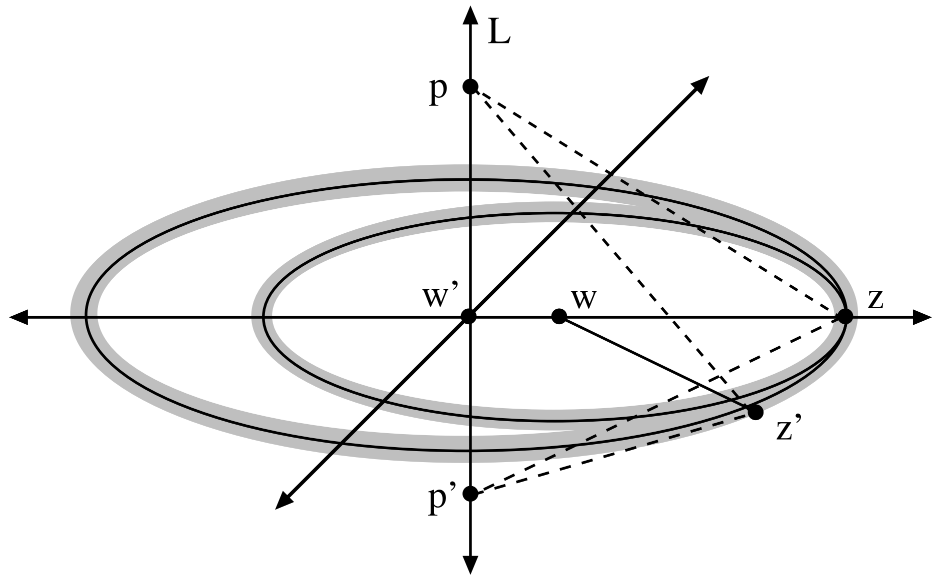
Now choose w to maximize the distance on $\Gamma \cap 3Q'$ from L. Let $\beta = \mathrm{dist}(w, L0$. It suffices to show that $\beta = O(P(Q'))$.

We may fix a large $M < \infty$ and assume that $P(Q') \leq 1/M^2$ and $MP(Q') \leq \beta \leq 1/M$, for otherwise there is nothing to do. We will show this gives a contradiction if M is large enough.

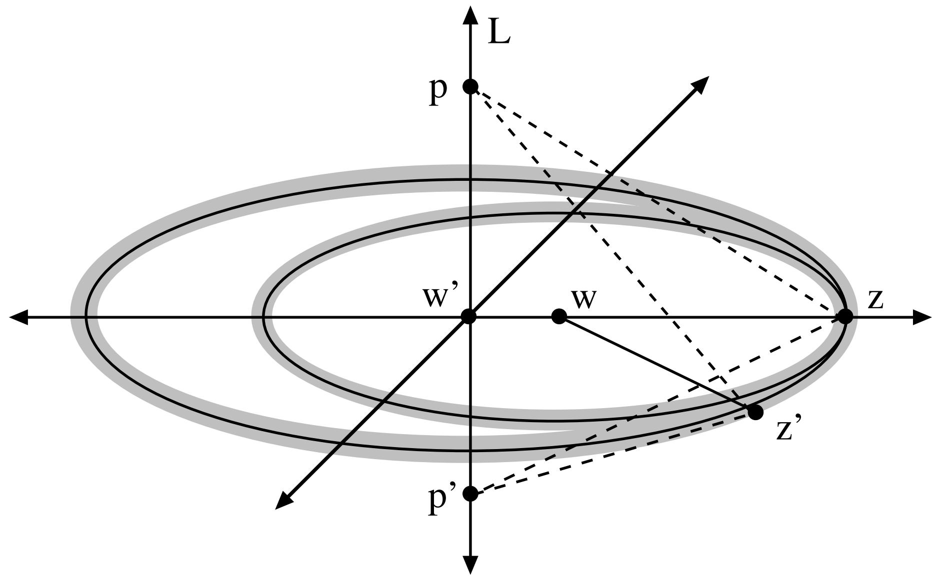
Let w' be the closest point on L to w and let z be the point on the ray from w' through w so that $\text{dist}(z, L) = \frac{1}{2}\ell(Q')$.

Let Q be the Whitney square for $\mathbb{R}^n \setminus \Gamma$ containing z and let $z' = R(z)$. Note that the p, q, w, w', z, z' all lie in a three dimensional sub-space, so, without loss of generality, we may assume L is the z-axis in $\mathbb{R}^3$, $w' = 0$, $w = (\beta, 0, 0)$, and $z = (1, 0, 0)$.





The points p, q satisfy $|p| \simeq |q| \simeq |p-q| \simeq 1$. Since z and z' are the same distance from each of these points, up to a factor of $O(P(Q'))$, we deduce z' lies inside a $O(P(Q'))$ neighborhood of the circle $x^2+y^2=1$ in the xy-plane.



Similarly, since z and z' are equidistant from w, up to a factor of $O(P(Q'))$, the points z' lies within a $O(P(Q'))$ neighborhood of the sphere of radius $1-\beta$ around z.

However, since $P(Q') \ll \beta \ll 1$, these two regions only intersect in the half-space $\{x > 0\}$ and thus z' also lies in this half-space. Thus $q = (z + z')/2$ has x-coordinate $\geq 1/2$ and, by the definition of ρ, it is within $\rho(Q)$ of a point $q' \in \Gamma$. But $\rho(Q) \lesssim P(Q') \ll 1$ (since it is one of the cubes in the sum defining $P(Q')$).

This implies there is a point q' of Γ that is about unit distance from L, contradicting the assumption that the maximum distance was $\beta \leq 1/M \ll 1$. Thus $\beta(Q') \leq M \cdot P(Q')$, as desired, and we have proven that Definition 13 implies Definition 11. $\square$

The Smith conjecture:

We can also extend R to be a biLipschitz involution on the sphere $\mathbb{S}^n$, except in the case when Γ is knotted in $\mathbb{R}^3$. This is imposible because the (positive) solution of the Smith conjecture implies the fixed set of an orientation preserving diffeomorphic involution of $\mathbb{S}^3$ is an unknotted closed curve. See [47].

Except for knotted curves in $\mathbb{R}^3$, we can say that Weil-Petersson curves are exactly the fixed point sets of biLipschitz involutions of $\mathbb{S}^n$ that satisfy (7.6).

Although the Smith conjecture was stated for diffeomorphisms, John Morgan explains on page 4 of [47] that its proof extends to homeomorphisms when the fixed point set is locally flat (locally ambiently homeomorphic to a segment). This holds in our case by Theorem 4.1 of [34] (finite Möbius energy implies tamely embedded), and the fact that Definition 13 implies Definition 8.

This completes the proof that (1) $\Rightarrow$ (11) $\Rightarrow$ (3) and hence the proof of Theorem 1.6 in the plane, since (3) $\Rightarrow$ (1) is already known.

Definition	Description
1	$\log f'$ in Dirichlet class
2	Schwarzian derivative
3	QC dilatation in L^2
4	conformal welding midpoints
5	$\exp(i \log f')$ in $H^{1/2}$
6	arclength parameterization in $H^{3/2}$
7	tangents in $H^{1/2}$
8	finite Möbius energy
9	Jones conjecture
10	good polygonal approximations
11	β^2-sum is finite
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13	biLipschitz involutions
14	between disjoint disks
15	thickness of convex hull
16	finite total curvature surface
17	minimal surface of finite curvature
18	additive isoperimetric bound
19	finite renormalized area
20	dyadic cylinder
21	closure of smooth curves in $T_0(1)$
22	P_φ^- is Hilbert-Schmidt
23	double hits by random lines
24	finite Loewner energy
25	large deviations of SLE(0^+)
26	Brownian loop measure

The names of 26 characterizations of Weil-Peterson curves

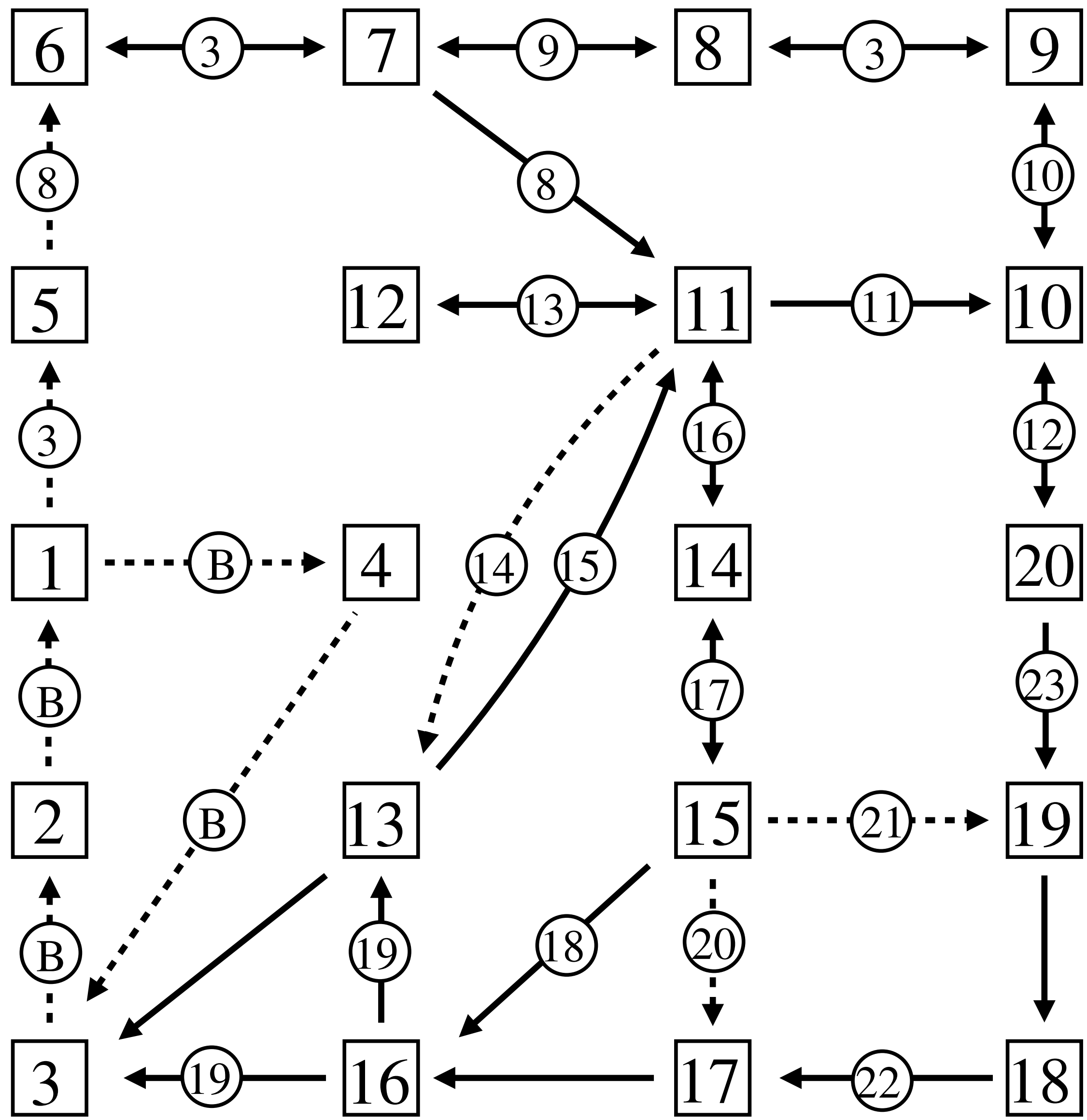


Diagram of implications between previous definitions.

Edge labels refer to sections of my preprint.

Beta-numbers imply (discrete) Jones conjecture for $n \geq 3$:

Lemma 8.2. *Definition 11 implies Definition 10.*

Proof. Let $\{\Gamma_j\}$ be a dyadic decomposition of Γ. For each j, choose a dyadic cube Q_j that hits Γ_j and has diameter between $\mathrm{diam}(\Gamma_j)$ and $2 \cdot \mathrm{diam}(\Gamma_j)$. Note that any such dyadic square can only be associated to a uniformly bounded number of arcs Γ_j in this way, because there are only a bounded number of arcs Γ_j that have the correct size and are close enough to Q_j; this uses the fact that Γ is chord-arc. Because Γ is chord-arc, $\mathrm{diam}(\Gamma_j) \simeq \ell(\Gamma_j) \simeq \mathrm{diam}(Q_j)$.

Recall the TST for Jordan arcs: If $\Gamma \subset \mathbb{R}^n$ is a Jordan arc, then

$$\Delta(\Gamma) = \ell(\Gamma) - \mathrm{crd}(\Gamma) = O\left(\int_{\mathbb{R}^n}\int_0^\infty \beta_\Gamma^2(x,t)\right)$$

Using this,

$$\Delta(\Gamma_j) \simeq \sum_{Q \subset 3Q_j} \beta_{\Gamma_j}^2(Q)\ell(Q).$$

Since $\beta_{\Gamma_j}(Q) \leq \beta_\Gamma(Q)$, we get

$$\begin{aligned}\sum_j \frac{\Delta_j}{\ell(\Gamma_j)} &\simeq \sum_j \sum_{Q \subset 3Q_j} \beta_{\Gamma_j}^2(Q)\frac{\ell(Q)}{\ell(Q_j)} \\ &\lesssim \sum_j \sum_{Q \subset 3Q_j} \beta_\Gamma^2(Q)\frac{\ell(Q)}{\ell(Q_j)} \simeq \sum_Q \beta_\Gamma^2(Q) \cdot \sum_{j : Q \subset 3Q_j} \frac{\ell(Q)}{\ell(Q_j)}.\end{aligned}$$

Note that for each Q with $\mathrm{diam}(Q) \le \mathrm{diam}(\Gamma)$ and $Q \cap \Gamma \neq \emptyset$, there is a cube of the form Q_j from above, that has diameter comparable to $\mathrm{diam}(Q)$ and such that $Q \subset 3Q_j$. Moreover, there there can only be a uniformly bounded number of dyadic squares Q_j of a given size so that $3Q_j$ contains Q, so each Q_j can only be chosen a bounded number of times.

Thus the sum over the j's in the last line above is bounded by a multiple of a geometric series and so is uniformly bounded. Thus

$$\sum_j \frac{\Delta(\Gamma_j)}{\ell(Q_j)} \lesssim \sum_Q \beta_\Gamma^2(Q). \quad \square \tag{8.1}$$

This proves (6)-(11) are equivalent in all finite dimensions, assuming the TST for Jordan curves in all finite dimensions.

What about infinite dimensions?

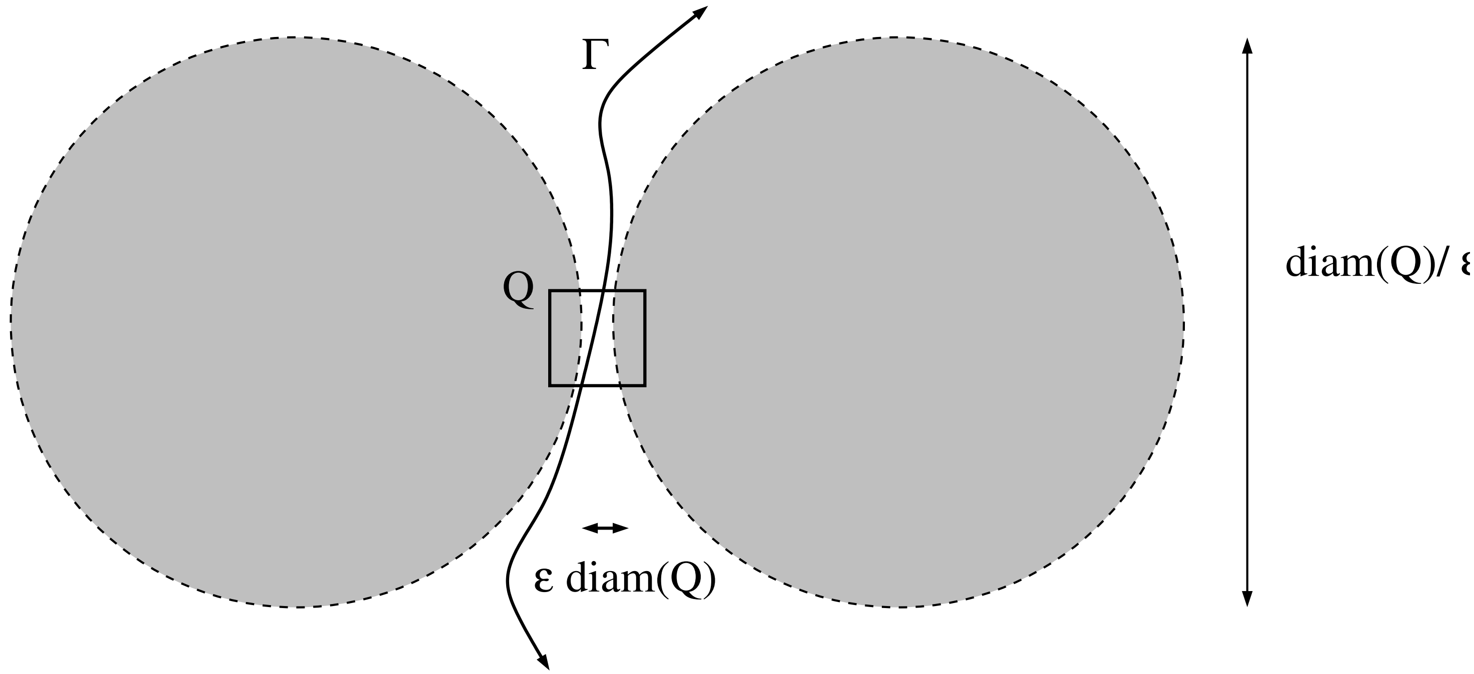
9. Thursday, Oct 8, 2020

ε-numbers versus β-numbers:

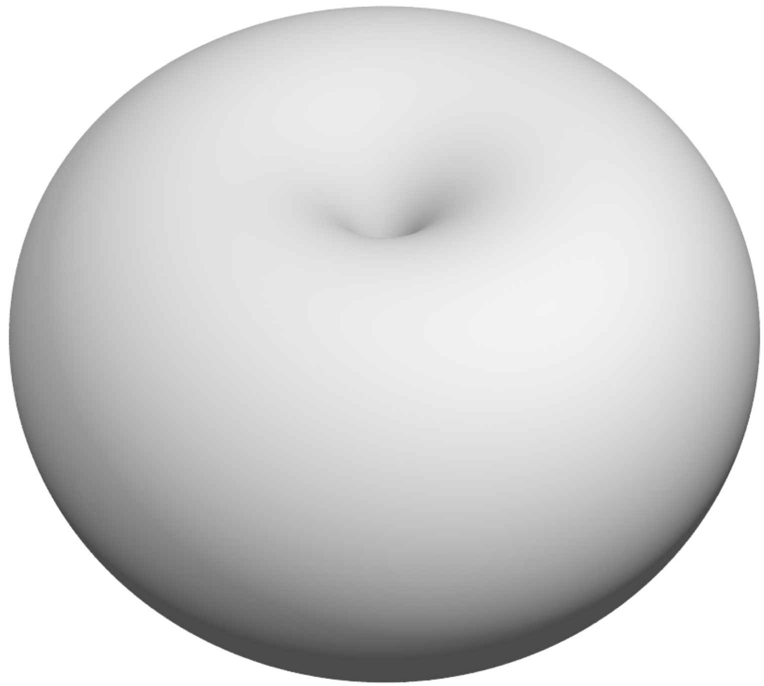
Towards a Möbius invariant version of β-numbers.

Next, we give a variation of the β-numbers that uses solid tori instead of cylinders and provides a stepping stone to the hyperbolic conditions discussed later.

We start with the definition in the plane. Given a dyadic square Q let $\varepsilon_\Gamma(Q)$ be the infimum of the $\epsilon \in (0,1]$ so that $3Q$ hits a line L, a point z and a disk D so that D has radius $\ell(Q)/\epsilon$, z is the closest point of D to L and neither D nor its reflection across L hits Γ.



Γ
diam(Q)/ ε
Q
ε diam(Q)



In higher dimensions the disk D is replaced by a ball B of radius $\mathrm{diam}(Q)/\epsilon$ that attains its distance ϵ from L at $z \in Q$, and that the full rotation of B around L does not intersect Γ. Thus Γ is surround by a "fat torus". The centers of the balls form a $(n-2)$-sphere that lies in a $(n-1)$-hyperplane perpendicular to L.

If no such line, point and disk exist, we set $\varepsilon_\Gamma(Q) = 1$.

It is easy to see that $\beta_\Gamma(Q) = O(\epsilon_\Gamma(Q))$, but the opposite direction can certainly fail for a single square Q. Nevertheless, we will see that that the corresponding sums over all dyadic squares are simultaneously convergent or divergent.

Definition 14. Γ *is chord-arc and satisfies*

$$\sum_Q \varepsilon_\Gamma^2(Q) < \infty \tag{9.1}$$

where the sum is over dyadic squares hitting Γ *with* $\mathrm{diam}(Q) \le \mathrm{diam}(\Gamma)$.

Lemma 9.1. *Definition 11 is equivalent to Definition 14.*

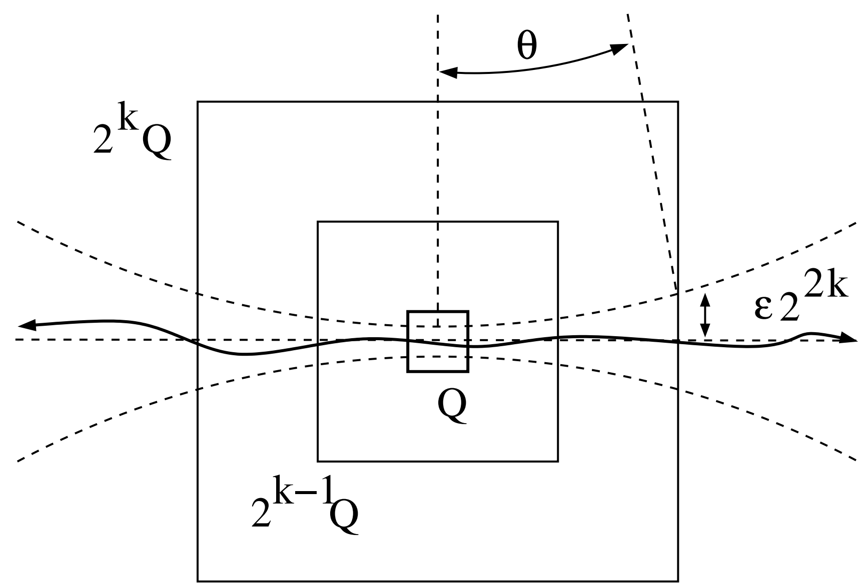
Proof. It is easy to see that $\beta_\Gamma(Q) \lesssim \varepsilon_\Gamma(Q)$, but the reverse direction can certainly fail for a single square Q. However, we shall prove that the sum of $\varepsilon^2_\Gamma(Q)$ over all dyadic squares is bounded iff the sum of $\beta^2_\Gamma(Q)$ is.

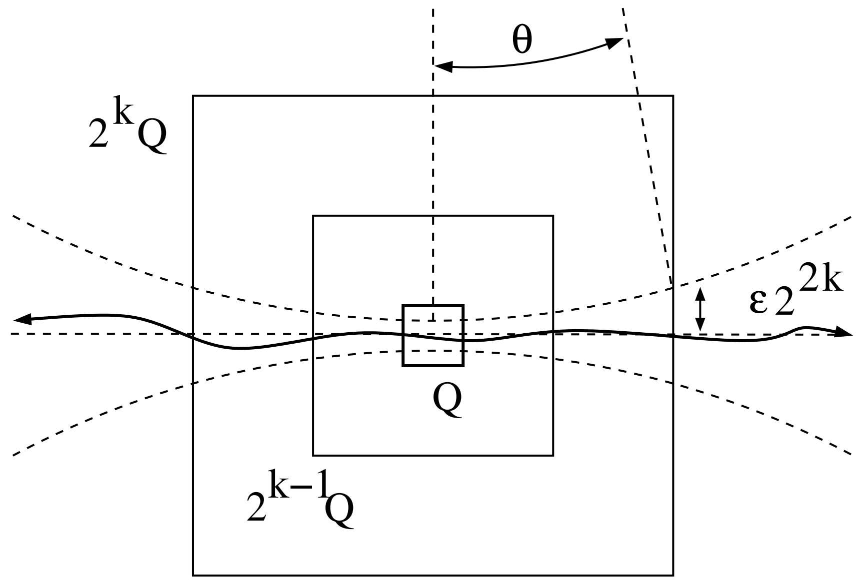
Fix $x \in \Gamma$ and a dyadic cube Q_0 containing x with $\mathrm{diam}(Q_0) \le \mathrm{diam}(\Gamma)$, for some $N \ge 10$. Renormalize so $\mathrm{diam}(Q_0) = 1$. For $k \ge 1$, let Q_k be the dyadic cube containing Q_0 and with diameter $\mathrm{diam}(Q_k) = 2^k \mathrm{diam}(Q_0)$. Let

$$\epsilon = 2A \sum_{k=1}^{\infty} 2^{-k} \beta_\Gamma(Q_k) = 2A \sum_{Q' : Q \subset Q'} \beta_\Gamma(Q_k) \frac{\mathrm{diam}(Q)}{\mathrm{diam}(Q')},$$

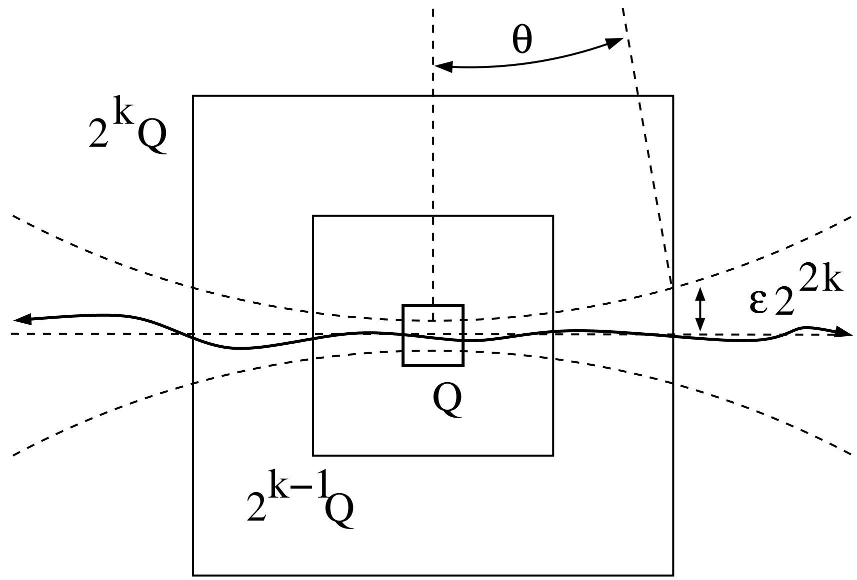
where the constant $0 < A < \infty$ will be chosen later. I claim that $\varepsilon_\Gamma(Q) \lesssim \epsilon$.

To prove this, we construct a large ball B whose rotations are disjoint from Γ. Let L be a line through x that minimizes in the definition of $\beta_\Gamma(Q_0)$. Let $L^\perp$ be the perpendicular hyperplane through x and let $z \in L^\perp$ be distance $1/\epsilon$ from x. Let $B = B(z, r)$ where $r = (1/\epsilon) - \epsilon$.



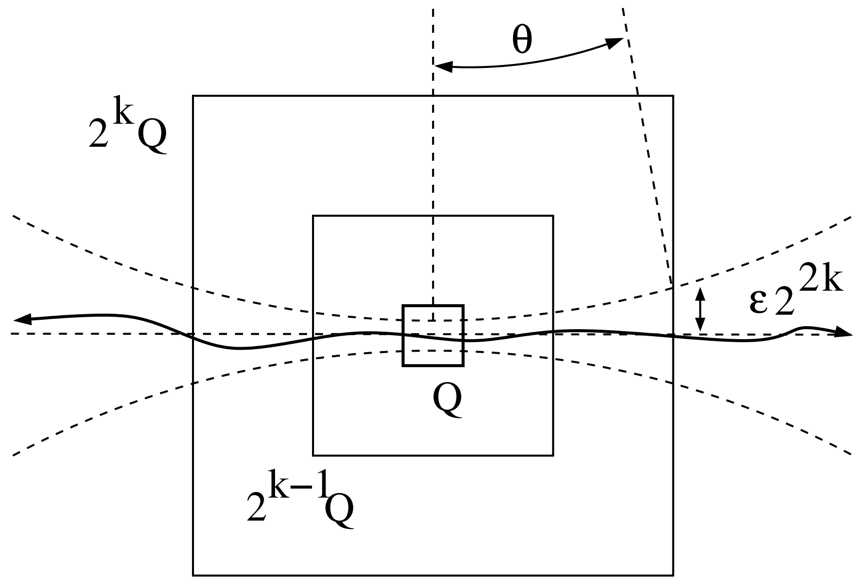


Then $\operatorname{dist}(B, L) = \epsilon$ and for $0 \leq n \leq N = \lfloor \log_2 \frac{1}{\epsilon} \rfloor$, simple trigonometry shows that $\operatorname{dist}(B \backslash 3Q_n, L) \geq C_1 \epsilon 2^{2n}$ (we can do the calculation in the plane generated by L and z.



On the other hand, the distance between $\Gamma \cap 3Q_n$ and L is $\leq C_2 \sum_{k=0}^{n} \beta_\Gamma(Q_k) 2^k$, because the angle between the best approximating lines for Q_k and Q_{k+1} is $O(\beta_\Gamma(Q_{k+1}))$. Therefore B and $\Gamma \cap 2Q_N$ will be disjoint, if for every $0 \leq n \leq N$ we have

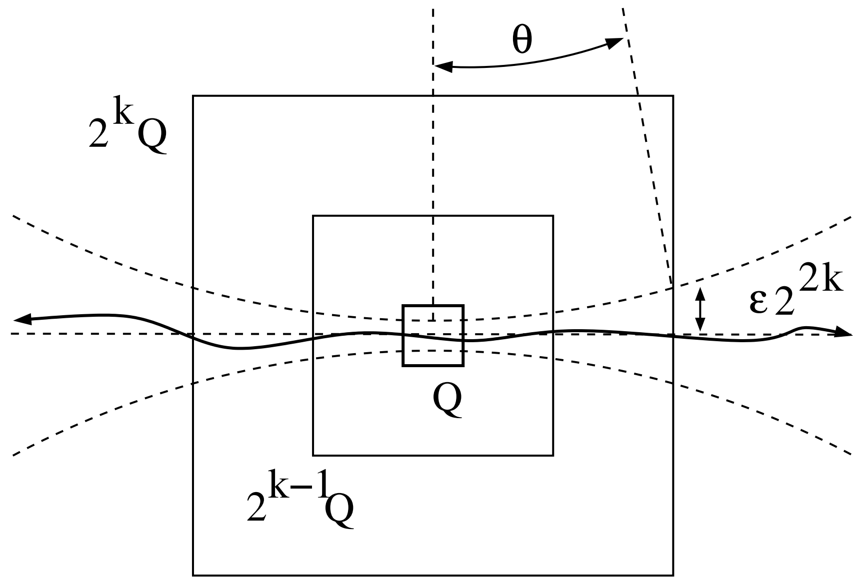
$$\sum_{k=0}^{n} \beta_\Gamma(Q_k) 2^k < (C_1/C_2)\epsilon 2^{2n}.$$



Note that

$$\max_{0\le n\le N} 2^{-2n}\sum_{k=0}^{n}\beta_\Gamma(Q_k)2^k \le \sum_{n=0}^{N} 2^{-2n}\sum_{k=0}^{n}\beta_\Gamma(Q_k)2^k$$
$$\le \sum_{k=0}^{N}\beta_\Gamma(Q_k)2^k\sum_{n=k}^{N}2^{-2n} \le \sum_{k=0}^{N}\beta_\Gamma(Q_k)2^{-k} = \epsilon/(2A) = (C_1/C_2)\epsilon,$$

if we take $A = \frac{1}{2}C_2/C_1$. This holds for every choice of z in $L^\perp$ that is distance $1/\epsilon$ from L, so we have proven that $\varepsilon_\Gamma(Q) \lesssim \epsilon$, as claimed.



The part of the ball of radius $\mathrm{diam}(Q)/\varepsilon(Q)$ that lies in $2^kQ \setminus 2^{k+1}Q$ makes angle $\theta \simeq \varepsilon 2^k$ with the perpendicular ray from L to z and hence (since we are assuming $\mathrm{diam}(Q) = 1$) is distance approximately $\varepsilon^{-1}(1 - \cos(\theta)) \simeq \varepsilon\theta^2 = \varepsilon 2^{2k}$ from the line L.

Summing over all dyadic cubes gives

$$\int_Q \varepsilon_\Gamma^2(Q) \lesssim \sum_Q \left[\sum_{Q':Q\subset Q'} \beta_\Gamma(Q')\frac{\operatorname{diam}(Q)}{\operatorname{diam}(Q')}\right]^2$$

$$\lesssim \sum_Q \left[\sum_{Q':Q\subset Q'} \beta_\Gamma(Q')\left(\frac{\operatorname{diam}(Q)}{\operatorname{diam}(Q')}\right)^{3/4}\left(\frac{\operatorname{diam}(Q)}{\operatorname{diam}(Q')}\right)^{1/4}\right]^2$$

and by Cauchy-Schwarz we get

$$\lesssim \sum_Q \left[\sum_{Q':Q\subset Q'} \beta_\Gamma^2(Q')\left(\frac{\operatorname{diam}(Q)}{\operatorname{diam}(Q')}\right)^{3/2}\right] \cdot \left[\sum_{Q':Q\subset Q'} \left(\frac{\operatorname{diam}(Q)}{\operatorname{diam}(Q')}\right)^{1/2}\right].$$

The second term is dominated by a geometric series, hence bounded. Thus

$$\int_Q \varepsilon_\Gamma^2(Q) \lesssim \sum_{Q'} \beta_\Gamma^2(Q') \sum_{Q:Q\subset Q'} \frac{\operatorname{diam}(Q)^{3/2}}{\operatorname{diam}(Q')^{3/2}}.$$

Since Definition 11 implies Γ is chord-arc, the number of dyadic cubes inside Q' of size $\operatorname{diam}(Q')2^{-k}$ and hitting Γ is at most $O(2^k)$.

Thus the right side is bounded by

$$\lesssim \sum_{Q'} \beta_\Gamma^2(Q') \sum_{k=0}^{\infty} O(2^k) 2^{-3k/2} \lesssim \sum_{Q'} \beta_\Gamma^2(Q') \sum_{k=0}^{\infty} 2^{-k/2} \lesssim \sum_{Q'} \beta_\Gamma^2(Q')$$

and so the ε^2-sum is finite if the β^2-sum is finite, as desired. $\square$

It is sometimes convenient to assume that the balls in the definition of ε_Γ are small compared to $\mathrm{diam}(\Gamma)$. This is easy to obtain if we replace $\varepsilon_\Gamma(Q)$ by

$$\widetilde{\varepsilon}_\Gamma(Q) = \max(\varepsilon_\Gamma, (\mathrm{diam}(Q)/\mathrm{diam}(\Gamma))^\alpha)$$

for some $1/2 < \alpha < 1$. Because $\alpha < 1$, we get

$$\frac{\mathrm{diam}(D)}{\mathrm{diam}(\Gamma)} = \frac{\mathrm{diam}(Q)/\widetilde{\varepsilon}_\Gamma(Q)}{\mathrm{diam}(\Gamma)} \leq \left(\frac{\mathrm{diam}(Q)}{\mathrm{diam}(\Gamma)}\right)^{1-\alpha} \to 0.$$

Clearly $\varepsilon_\Gamma \leq \widetilde{\varepsilon}_\Gamma$. Moreover,

$$\sum_{Q:Q\cap\Gamma\neq\emptyset} \widetilde{\varepsilon}^2_\Gamma(Q) \lesssim \sum_Q \varepsilon^2_\Gamma(Q) + \sum_Q \left(\frac{\mathrm{diam}(Q)}{\mathrm{diam}(\Gamma)}\right)^{2\alpha}$$

where the second sum is finite because $\alpha < 1/2$ and Γ being chord-arc implies the number of dyadic squares of size $\simeq 2^{-n}$ hitting Γ is $O(2^n)$.

For chord-arc curves either type of ε-number works and second one allows us to get more local estimates.

Definition	Description
1	$\log f'$ in Dirichlet class
2	Schwarzian derivative
3	QC dilatation in L^2
4	conformal welding midpoints
5	$\exp(i \log f')$ in $H^{1/2}$
6	arclength parameterization in $H^{3/2}$
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26	Brownian loop measure

The names of 26 characterizations of Weil-Peterson curves

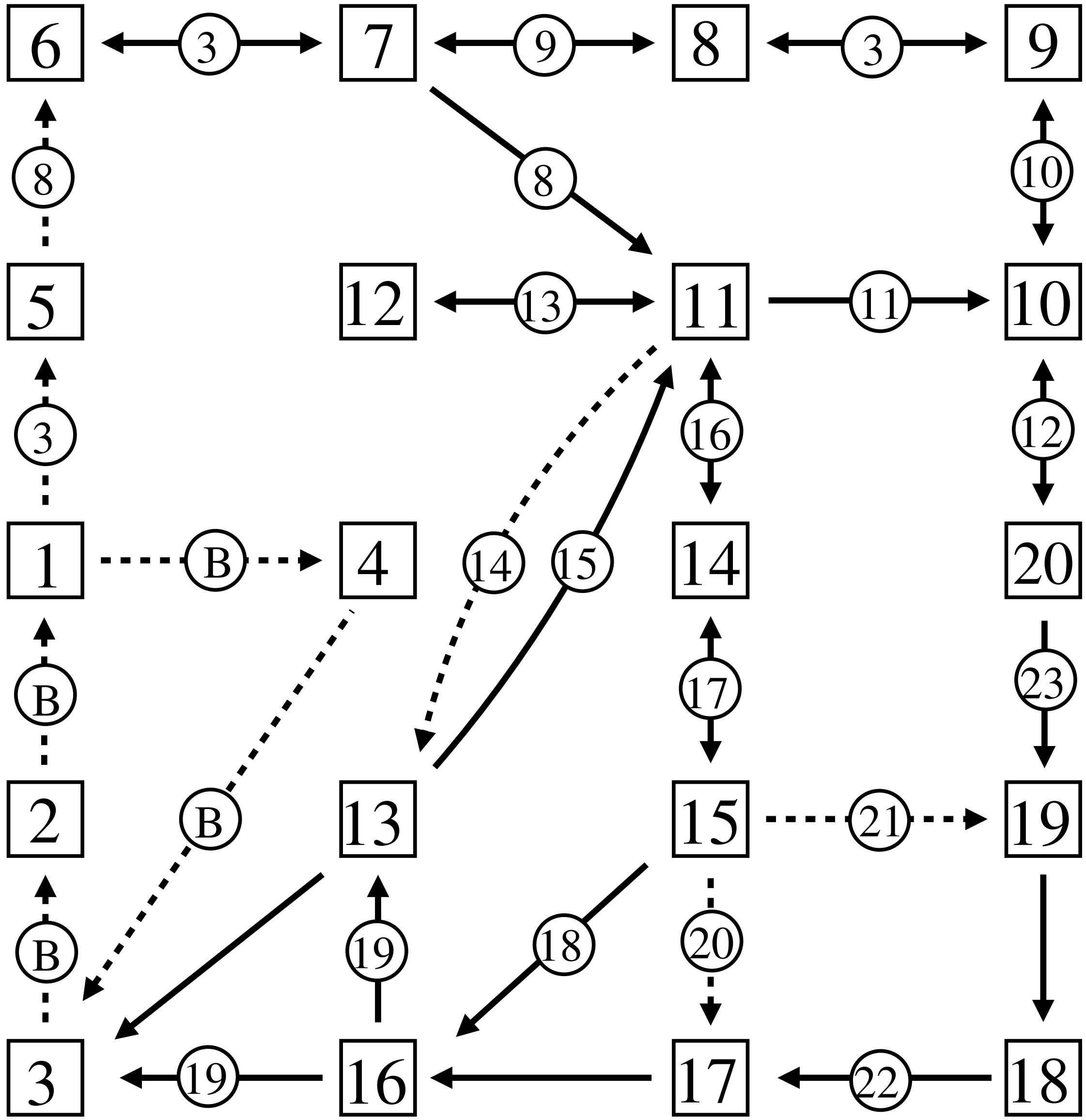

Diagram of implications between previous definitions.

Edge labels refer to sections of my preprint.

10. The traveling salemans theorem in all dimensions

These lectures use a different sets of slides based on my preprint "The traveling salesman theorem for Jordan curves".

11. Hyperbolic conditions in 3 dimensions

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