MAT 638, FALL 2020, Stony Brook University

TOPICS IN REAL ANALYSIS

WEIL-PETERSSON CURVES, TRAVELING SALESMAN THEOREMS AND MINIMAL SURFACES

QUESTIONS

(1) Can we use Ahlfors' Schwarzians for curves in Euclidean spaces to describe Weil-Petersson curves in $\mathbb{R}^n$? (Martin Chuaqui Farrú, 2-26-20)

(2) Geometrically chacterize quasicircles $\Gamma = f(\mathbb{T})$ so that μ_f is in L^p for hyperbolic area, $p \neq 2$ ($p = 2$ is the Weil-Peterson class). Perhaps in terms of the β-numbers or arclength parameterization. (Martin Chuaqui Farrúm, 8-26-20)

(3) Is Brennan's conjecture true for domains bounded by a Weil-Petersson curve. Brennan's conjecture is that for any simply connected domain Ω in the plane any conformal map $f : \Omega \to \mathbb{D}$ satisfies $f' \in L^p(\Omega, dxdy)$ for all $p < 4$. This is known false for $p = 4$ ($\Omega = \mathbb{R} \setminus [0, \infty)$) and is known for $4/3 < p < 3.399$ (maybe a better estimate is known now). Counterexamples to Brennan's conjecture are expected to fractalsm which WP curves can't be. This problem may be just an exercise using the definitions of the Dirichlet space and L^p. (Mayank Goswami, 8-27-20)

(4) Are WP curves dense in all quasicircles? It depends on what dense means. WP curves include all smooth curves, which are dense in all closed curves for the Hausdorff metric, WP curves are definitely not dense for the Weil-Peterson metric (they are a closed set in this metric, by definition). There is not a $(1+\epsilon)$-QC map from the von Koch snowflake to any WP curve (the Hausdorff dimension can't decrease from log4/log 3 to 1 so fast), so they are not dense for the metric coming from $\|\mu\|_\infty$ (the Teichmüller metric). Is there a natural metric were WP curves are dense in all quasicircles but the smooth curves are not? (Mayank Goswami, 8-27-20)

(5) How can we tell if two quasicircles are in the same or different connected componentsfor the Weil-Petersson metric on universal Teichmuller space? Is it true that two curves are in the same component iff one is the image of the other under a QC map whose dilatation is in L^2 with respect to the hyperbolic area measure on the complement of the domain curve? I asked Leon Takhtajan and he responded " Yes, I think it is correct since right translations act transitively on the set of components." The paper of Takhtajan and Teo.) Are two polygons in the same component iff they have

the same angles for some cyclic ordering of each? Is the Hausdoff dimension constant on each connected component? (Martin Chuaqui Farrúm, 8-29-20)

(6) If Γ is a quasicircle and f is a QC map whose diatation is in L^2 for hyperbolic area on the complement of Γ does $f(\Gamma)$ have the same Hausdorff dimension as Γ? Is it a bi-Lpschitz image of Γ. If this is true for quasicircles, is it also true for general closed Jordan curves? General closed sets?

(7) Does the Weil-Petersson class for an infinite dimensional Lie group? It is a topological group with the structure of an infinite dimemnsional manifold, but is something else needed to make it a Lie group? I am not sure if there is a standard way to define an infinfinite dimensional Lie group. I asked Leon Takhtajan and he responded " I do not know whether it is a Lie group. Probably not, since its holomorphic tangent space is a Hilbert space $A_2(D)$ and I do not know how to endow it with the Lie bracket. However, the Diff(S^1) is a Lie group with the Lie algebra being smooth vector fields on S^1. For points in $T_0(1)$ corresponding conformal welding can be characterized by Theorem 1.12 in Chapter 2, so maybe one can still

define a Lie bracket, though it may not map $H^{3/2}$ into itself." Also see Wikipedia Infinite Dimensional Lie groups.