MAT 638, FALL 2020, Stony Brook University

TOPICS IN REAL ANALYSIS

OPEN PROBLEMS

Thursday, Dec 3, 2020 (last class)

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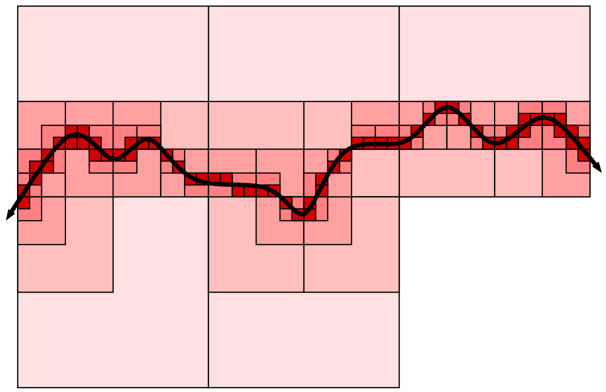
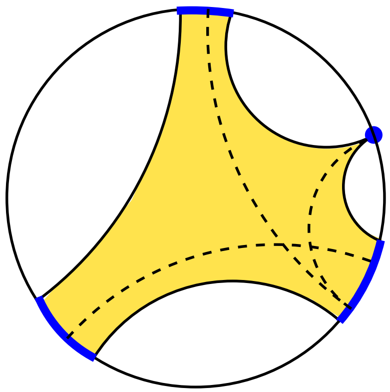
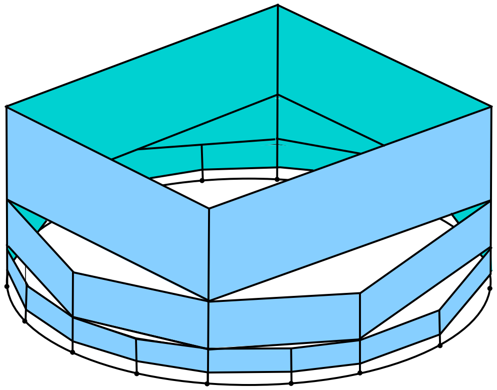






• **Hyperbolic conditions on codimension 1 surfaces:** Can our definitions be generalized to codimension 1 surfaces in $\mathbb{R}^n$ (or to general codimension)?

For example, we could take Γ to be an image of $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ under a self-homeomorphism of $\mathbb{R}^n$ (we will call this a Jordan surface) and consider the corresponding hyperbolic convex hull and minimal sub-manifolds in $(n+1)$-dimensional hyperbolic space $\mathbb{H}^{n+1}$.

To begin with, what is the correct generalization of the β^2-sum condition? It is easy to check that $\sum_Q \beta^2(Q)$ diverges for the $(n-1)$-sphere in $\mathbb{R}^n$, but both

$$\sum_Q \beta^2_{\Gamma,n-1}(Q)\mathrm{diam}(Q)^{n-1} < \infty,$$

$$\sum_Q \beta^{n-1}_{\Gamma,n-1}(Q) < \infty,$$

hold. The second implies the first, is scale invariant, and does not allow "corners", so is probably the more natural generalization of the Weil-Petersson condition to surfaces. Is it equivalent to natural generalization of our other conditions in term so Sobolev parameterizations, inscribed polyhedron, hyperbolic minimal surfaces and renormalized volumes?

What should renormalized volume mean? The hyperbolic volume of the truncated hyper-surface S_t may involve several negative powers of t.

• **Comparing different quantities:** The work of Takhtajan and Teo [30], Rohde and Wang [26] and Viklund and Wang [31] includes many explicit formulas relating the Dirichlet norm of $\log f'$ to the Kahler potential of the Weil-Petersson metric on universal Teichmuller space and the Loewner energy of the curve Γ. Are there similar formulas that relate these quantities to quantities discussed in this paper, e.g., Möbius energy, β^2-sums, Menger integrals, the curvature integral of a minimal surface associated to Γ or the renormalized area of this curve? If there is more than one such minimal surface, which surface?

Examples in the preprint indicate there may not be any simple relation between these different quantities, but the estimates in this paper should prove that $\mathcal{LE}(\Gamma)$, $\text{Möb}(\Gamma)$, $|\mathcal{RA}(\Gamma)|$ and $\sum_Q \beta_\Gamma^2(Q)$ should all be comparable for quasi-circles with small constant. Example 2 in Section 23 shows that $\text{Möb}(\Gamma)$ need not be comparable in size to the other three in general, but are the other three always comparable to each other (at least when the values are large)?

• **Other knot energies:** There are a variety of other knot energies besides Möbius energies. For example,

$$E^{j,p}(\Gamma) = \int_\Gamma \int_\Gamma \left(\frac{1}{|x-y|^j} - \frac{1}{\ell(x,y)^j} \right)^p dxdy,$$

blows up for self-intersections if $jp \geq 2$ and is finite for smooth curves if $jp \leq 2p+1$. Sobolev smoothness properties for curves with finite $E^{j,p}$ energy are studied by Blatt in [8] (but there is a typo in Theorem 1.1, s should be $s = (jp-1)/(2p)$).

Another class of knot energies considered in [28] are the Menger energies

$$\mathcal{M}_p(\Gamma) = \int_\Gamma \int_\Gamma \int_\Gamma c^p(x,y,z)|dx||dy||dz|,$$

with $\mathcal{M}_2(\Gamma)$ being the usual condition that is equivalent to rectifiability. They show that for $p \geq 3$, finite energy curves are Jordan curves and for $p > 3$ then are even $C^{1,\alpha}$ and establish bounds on the β-numbers. The endpoint case $p = 3$ seems the most interesting, as this is the only scale-invariant Menger energy. Since $c(x,y,z) \lesssim 1/\ell(x,y,z)$, $\mathcal{M}_3(\Gamma)$ is less restrictive than the Weil-Petersson condition. What are the corresponding geometric characterizations of these curves?

- In the definition of renormalized area, we truncate a minimal surface in $\mathbb{R}^3_+$ with asymptotic boundary Γ at height t above the boundary. This gives a new curve Γ_t. Does this have lower energy than Γ (either Loewner or Möbius)? Is energy monotone in t? Are these truncations close to a geodesic in the Weil-Petersson metric? When the topology of Γ_t changes corners form, so the energies must blow up, so we may want to consider only simply connected surfaces.

• **Gradient flow on energy:** The idea of knot energies is that if we start with a knot in $\mathbb{R}^3$, and flow "downward", e.g., follow the gradient of energy, then the curve never crosses itself and we end at a "nice" representative of the same knot. This flow has been studied by various investigators; see [19], [24]. For Weil-Petersson curves in the plane does this flow always lead to a circle? Is it related to the Weil-Petersson metric or geodesics for this metric?

- **Wildness of fixed points of involutions:** Fixed point sets of C^1 involutions are locally flat by a result of Bochner [9], but this can fail for involutions in the Sobolev space $W^{1,p}$, $1 \leq p < 2$, see [22]. Can the fixed point sets of $W^{1,2}$, quasiconformal and biLipschitz involutions be wild curves?

• **Length convergence on minimal surfaces:** The estimates in this paper prove that if Γ is Weil-Petersson, S is a minimal surface with asymptotic boundary Γ and Γ_t is the curve on S at height t above the boundary, then

$$\int_0^1 |\ell(\Gamma_t) - \ell(\Gamma)| \frac{dt}{t^2} < \infty.$$

Does the converse hold? The direction stated above follows by writing

$$|\ell(\Gamma_t) - \ell(\Gamma)| \leq |\ell(\Gamma_t) - \ell(\Gamma_n)| + |\ell(\Gamma_n) - \ell(\Gamma)|,$$

where Γ_n is the usual dyadically inscribed polygon with $2^{-n-1} < t \leq 2^{-n}$. The second term on the right is integrable by Theorem 1.3 of the preprint, and is controlled using the β-numbers at scales smaller than t. The first term is controlled by Seppi's estimate and the ε-numbers at scale t; these, in turn, are controlled by sums of β-numbers over scales larger than t. Thus the question is whether $\ell(\Gamma_t)$ can be a much better approximation to $\ell(\Gamma)$ than $\ell(\Gamma_n)$ for some non-Weil-Petersson curves?

• **Harmonic measure on curves in $\mathbb{R}^3$:** Harmonic measure on a plane curve can be defined as the hitting distribution of a Brownian motion. In higher dimensions, a Brownian motion almost surely never hits a fixed smooth curve, so this does not make sense. An alternative possibility is to think harmonic measure as the equilibrium probability measure that minimizes energy with respect to the Newton kernel ($\log \frac{1}{z}$ in the plane, $|x|^{n-2}$ in $\mathbb{R}^n$). For curves in $\mathbb{R}^n$ no measure supported on a smooth curve gives finite energy, but we there are finite energy probability measures on an ϵ-thickening of the curve.

In particular there is an energy minimizing one (the equilibrium distribution of a unit charge constrained to stay on the ϵ-tube). What measure do these converge to as $\epsilon \nearrow 0$? Is it absolutely continuous with respect to arc-length? Are $H^{3/2}$ curves in $\mathbb{R}^4$ characterized by some property of this measure, e.g., being comparable to arclength measure? Perhaps there some connection to recent work of David, Engelstein, Feneuil, Mayboroda defining harmonic measure on subsets of $\mathbb{R}^n$ with codimension greater than 1. See [14], [13].

• **Möbius energy and SLE:** Yilin Wang showed Weil-Petersson curves are related to the large deviations theory of Schramm-Loewner evolutions (SLE) as the parameter κ tends to zero. It is intriguing that they are also characterized in terms of the rate of blow-up of a self-repulsive energy that prevents self-intersections. Is there some more direct connection between these two ideas?

A SLE(κ) curve has Hausdorff dimension $1 + \kappa/8$ for $0 < \kappa \leq 8$ and we expect the ϵ-truncation of the energy integral for an α-dimensional measure and kernel $|x|^{2-d}$ to grow like $\epsilon^{2-d+\alpha}$. Do SLE paths have energy that grows like $\epsilon^{-1+\kappa/8}$, or are they, in some sense, optimal among such curves?

Is there something interesting to say regarding hyperbolic convex hulls and minimal surfaces of an SLE path when $\kappa > 0$, e.g. can we compute an "expected curvature" for the corresponding minimal surface? When $\kappa \geq 8$ the paths become plane filling, but do the corresponding minimal surfaces still make sense and if so, can we characterize their properties (e.g., growth rate of renormalized area) in terms of κ?

• **Brylinski's beta function:** Given a rectifiable curve $\gamma \subset \mathbb{R}^3$, in [12] Jean-Luc Brylinski defines a function $B_\gamma(s) = \int_\gamma \int_\gamma |z-w|^s dzdw$, where dz, dw denote arclength measure. This integral converges and defines a holomorphic function for $\mathbb{R}(s) > 0$. For example, if γ is a circle, then he shows

$$B_\gamma(s) = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{s+1}{2})}{2\Gamma(\frac{s}{2}+1)},$$

where Γ is the usual Gamma function, the analytic extension of $\int_0^\infty x^{s-1}e^{-x}dx$.

More generally, he shows that if γ is smooth, then $B_\gamma(s)$ extends to be meromorphic on the whole plane with poles only possible at the negative odd integers. He also shows that for smooth curves $B_\gamma(-2) = \text{Möb}(\gamma) - 4$ and

$$\text{Möb}(\gamma, s) = \int_\gamma \int_\gamma (\ell(z,w)^s - |z-w|^s) dzdw,$$

extends to be holomorphic in $\mathbb{R}(s) > -3$. Is this extension (or something like it) true for all Weil-Petersson curves, not just the smooth ones?

• **Renormalized volume of hyperbolic 3-manifolds:** Let G be a quasi-Fuchsian group, M its hyperbolic quotient 3-manifold, R_1, R_2 the two Riemann surfaces comprising the boundary at ∞ of M, and Γ its limit set. There are a variety of papers that relate the volume $\mathrm{CH}(\Gamma)$, the renormalized volume of M, and the Weil-Petersson distance between R_1 and R_2. For example, see [10], [11], [21], [27].

The ideas in these papers seem very similar to our results characterizing Weil-Petersson curves Γ in terms of the "thickness" of the hyperbolic convex hull of Γ and the renormalized area of a surface with boundary Γ. Is there a precise connection between the results of this paper and the papers mentioned above? In [30], Takhtajan and Teo show that the usual Weil-Petersson metric for compact surfaces can be recovered from their Weil-Petersson metric on the universal Teichmüller space. Is this helpful in making the connection suggested above?

• **Computing minimal surfaces:** Given a planar closed curve Γ can we efficiently compute a minimal surface $S \subset \mathbb{R}^3_+$ that has Γ as its asymptotic boundary? Can we compute all such surfaces? Efficiently compute the number of minimal surfaces with boundary Γ?

• **How do minimal surfaces lie in the convex hull?:** Sullivan's convex hull theorem, e.g., [15], [29], [6], says that there is a K-quasiconformal map from a simply connected domain Ω to its dome that is the identity on $\Gamma = \partial\Omega$ and K is bounded, independent of Ω. If Γ is a Jordan curve, then Sullivan's theorem also applies to the other complementary component Ω^*. In general, any L-quasiconformal reflection across Γ might have L much larger than K, and hence any QC map between the domes of Ω and Ω^* fixing Γ also has large constant.

Suppose S is a simply connected minimal surface with asymptotic boundary Γ. Is S about half-way between the two domes quasiconformally, i.e., there are $O(\sqrt{L})$ quasiconformal maps from S to each of the convex hull boundaries? Any bound in terms of L? Can S remain $O(1)$-quasiconformally close to one of the domes as $L \nearrow \infty$? What if Γ is not a quasicircle; can S be quasiconformally equivalent to either dome by a map fixing Γ?

• **Detecting Weil-Petersson components of $T(1)$:** The Hilbert manifold topology of Takhtajan and Teo divides the universal Teichmüller space into uncountable many connected components. Can we geometrically characterize when two curves belong to the same component? The current paper has done this for the component containing the unit circle. Perhaps some condition can be given saying that the convex hulls are quasi-isometric with constants that tend to 1 in a square integrable sense near the boundary of hyperbolic space. Are Γ_1, Γ_2 in the same component iff $\Gamma_2 = f(\Gamma_1)$ for some planar QC map f whose dilatation is in L^2 for hyperbolic area on the complement of Γ_1? This may be known.

- A closely related problem is to construct a natural section for universal Teichmüller space, i.e., a natural choice of one quasicircle from each connected component. A good starting point might be Rohde's paper [25] that gives such a choice for quasicircles modulo biLipschitz images.

- Alexakis and Mazzeo also prove that if a minimal surface minimizes renormalized area, then it is area minimizing, i.e., if S_1, S_2 are embedded minimal surfaces with the same asymptotic boundary Γ and $S_1$is area minimizing in $\mathbb{R}^3_+$ then $\mathcal{RA}(S_1) \leq \mathcal{RA}(S_2)$, and equality holds if and only if S_2 is also an area minimizer. Can this be extended to non-Weil-Petersson curves by replacing the inequality between renormalized area by a comparison of their growth near the boundary?

In [2] Alexakis and Mazzeo impose an L^2 curvarture condition but integrate against a weight that blows up logarithmically near the boundary, in order to obtain C^1 control of the boundary curve Γ. They note that this by itself is not enough to allow such control: the current paper shows precisely what boundary curves are possible under this weaker hypothesis.

In terms of β-numbers, their stronger condition roughly says that

$$\sum_Q \beta_\Gamma^2(Q)|\log \operatorname{diam}(Q)|^{2p} < \infty,$$

for some $p > 1$. It would be interesting to explore the consequences of this assumption in terms of the conditions given in this paper, and to explore the possible analogs of this paper's results in the higher dimensional settings considered in [1].

• **Characterizing subsets of Weil-Petersson curves:** Peter Jones's traveling salesman theorem characterizes the subsets of the plane that lie on some rectifiable curve by $\sum_Q \beta_E^2(Q)\mathrm{diam}(Q) < \infty$. Does the analogous sum $\sum_W \beta_E^2(Q) < \infty$ characterize subsets of Weil-Petersson curves? This condition is obviously necessary since the β^2-sum for any such curve would dominate the sum for the set. More generally, if we define a collection of sets by the convergence of some series involving β-numbers, is every set in that collection always contained in a curve from that collection?

• **Curves with smoothness between Weil-Petersson and rectifiable:** What can we say about a curve if e.g.,

$$\sum_Q \beta_\Gamma^2(Q)\mathrm{diam}(Q)^s < \infty,$$

a condition interpolating between rectifiability ($s = 1$) and the Weil-Petersson class ($s = 0$)? Are these $H^{(3-s)/2}$-curves? See Corollary 2 of [16], but β means something different there and is not directly comparable to our β-numbers. Similar sums occur in [4] and [5] related to Hölder parameterizations of curves. In [18], Silvia Ghinassi considers curves for which

$$\int_0^1 \beta_\Gamma^2(x,t)t^{-2\alpha}dt < M < \infty,$$

and shows they have parameterizations that are $C^{1,\alpha}$, i.e., f' is α-Hölder. The β-sum definition implies the Weil-Petersson class forms a subset of the $\alpha = 1/2$ case.

• **Sharper Schwarzian estimate:** The refined version of the traveling salesman theorem should have an analog for the Schwarzian derivative. In Lemma 3.9 of [7], Peter Jones and I proved that for a conformal map $f : \mathbb{D} \to \Omega$, where $\Gamma = \partial\Omega$ is a quasicircle, we have

$$\ell(\Gamma) \lesssim \operatorname{diam}(\Gamma) + \iint_{\mathbb{D}} |f'(z)||S(f)(z)|^2(1-|z|^2)^3 dxdy.$$

Recalling that $S(f) \equiv 0$ implies Γ is a circle, can we improve this to

$$\ell(\Gamma) - \pi \cdot \operatorname{diam}(\Gamma) \simeq \iint_{\mathbb{D}} |f'(z)||S(f)(z)|^2(1-|z|^2)^3 dxdy?$$

Is there a similar estimate for estimating $\Delta(\gamma) = \ell(\gamma) - \operatorname{crd}(\gamma)$ for sub-arcs?

A first guess for a function theoretic analog might be

$$\Delta(\gamma) \simeq \iint_Q |f'(z)||S(f)(z)|^2(1-|z|^2)^3 dxdy,$$

where $\gamma = f(I)$ and Q is the Carleson square with base I, but this fails for any Möbius transformation taking $\mathbb{D}$ to a disk (the right side is zero and the left is not), so some modification is needed. Finding the correct version of this, or its analogs in terms of $\log f'$ or μ, should give new ways to deduce geometric conditions from function theoretic ones.

• **Angles of inscribed dyadic polygons:** Suppose $\{z_j^n\}$ are a choice of dyadic points in Γ, and

$$\theta(n,k) = \arg\left(\frac{z_{j+1}^n - z_j^n}{z_j^n - z_{j-1}^n}\right),$$

be the angles between adjacent nth generation segments. It is not hard to show that if Γ is Weil-Petersson, then

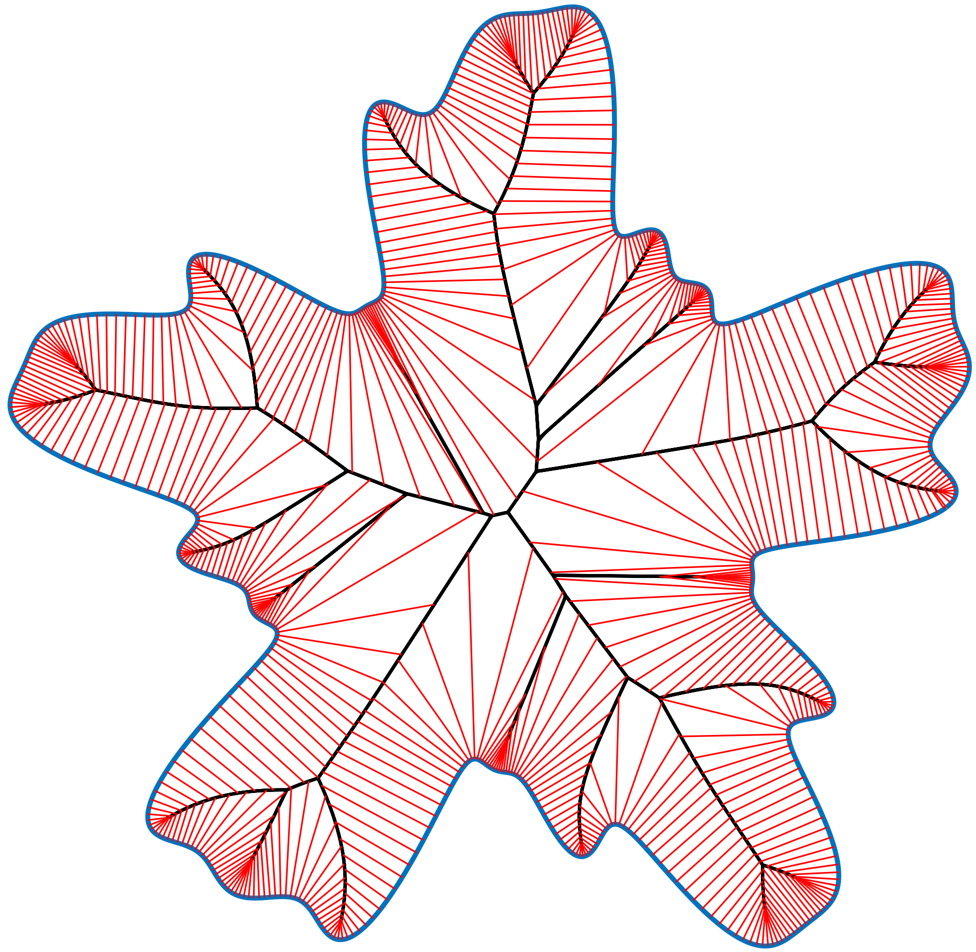
$$\sum_{n=1}^{\infty}\sum_{k=1}^{2^n}\theta^2(n,k) < \infty,$$

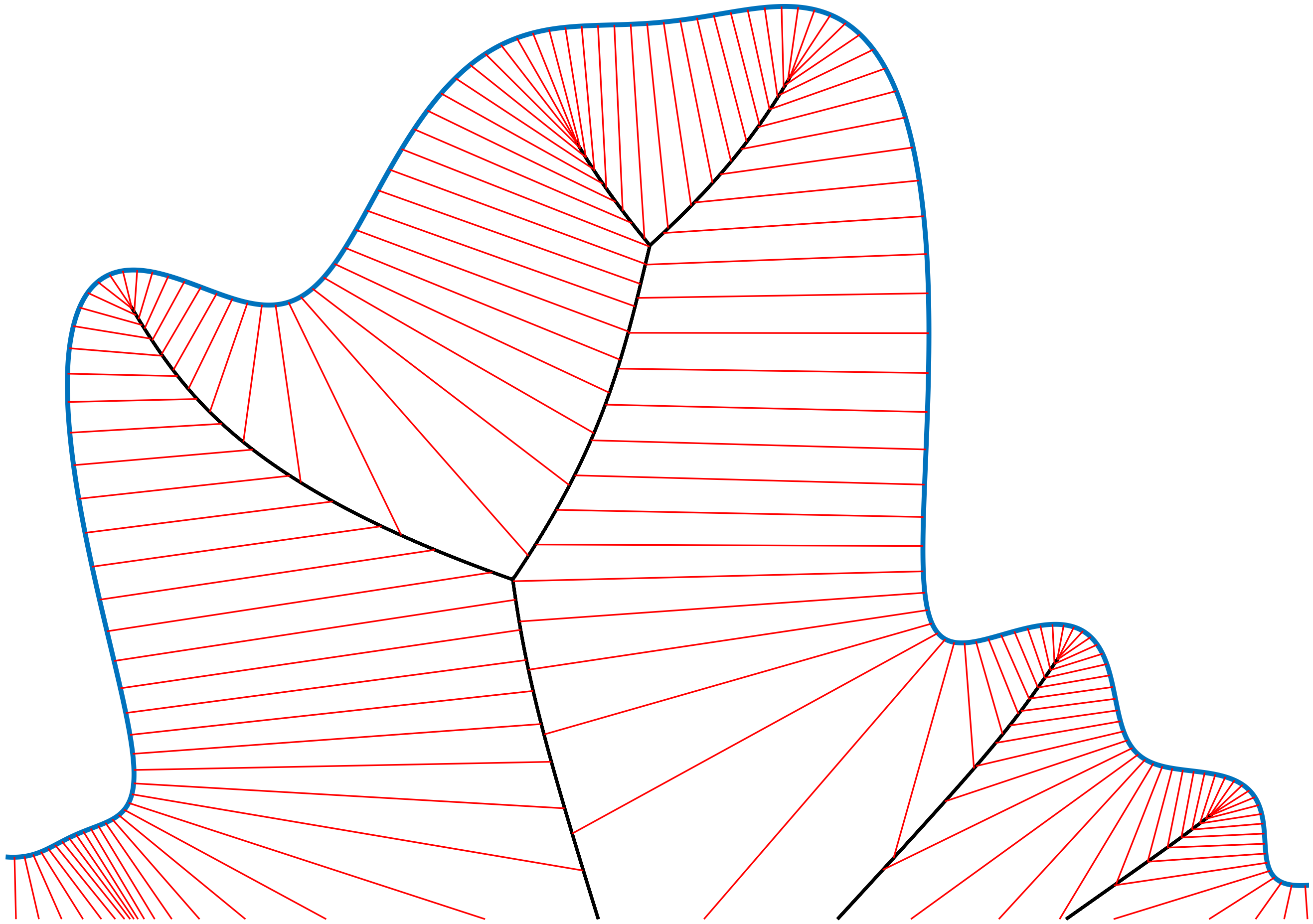
with a uniform bound independent of the choice of dyadic base point. Is the converse true? What if we also assume Γ is chord-arc?

In general, θ can be zero at a point, even if β is large, e.g., at the center of a spiral. This is reminiscent of the longstanding ϵ^2-conjecture of Carleson, recently proved by Jaye, Tolsa and Villa [20].

• **The medial axis:** The medial axis $\mathrm{MA}(\Omega)$ of a domain Ω is the set of centers of disks $D(x,r) \subset \Omega$ so that $\mathrm{dist}(x,y) = r$ for at least two points $y \in \partial\Omega$. See [17] for its basic properties (it is called the skeleton of Ω there). David Mumford has asked if Weil-Petersson curves can be characterized in terms of the medial axis of their complementary domains. This means we know both the set and the distance function to the boundary, (a line segment, with different distance functions, can be the medial axis of both WP and non-WP curves).

The cleanest statement I am aware of is the following. The region $\Omega \setminus \mathrm{MA}(\Omega)$ is foliated by directed line segments that connect each point to its unique nearest point on $\partial\Omega$. For each hyperbolic unit ball $B_\rho(w, 1)$ in Ω we assign the supremum of the difference between directions for the segments hitting B. Then Γ is Weil-Petersson iff Γ is chord-arc and this function is in $L^2(\Omega, d\mathrm{A}_\rho)$. This says Γ is Weil-Petersson iff the nearest point foliation is orthogonal to the boundary with an L^2 error. Is there a "nice" characterization in terms of the medial axis itself?





• **New characterizations of old curve families:** The function theoretic characterizations of the Weil-Petersson class are exactly analogous to known characterizations of other classes, e.g., when $\log f'$ is in VMO [23] or BMO [3], [7]. Do these classes have other characterizations analogous to the ones discussed in this paper? For example, Michel Zinsmeister has asked if anything interesting can be said about the domes and minimal surfaces associated to boundaries of BMO domains.

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