Remark on "Locally univalent functions, VMOA and the Dirichlet space" by Gallardo-Gutiérrez, González, Pérez-González, Pommerenke, and Rättyä.

This note describes what I believe is a small error in the proof of Theorem 4 of [1]:
Theorem 4: *If f is conformal on the unit disk $\mathbb{D}$, $\log f'$ is in the Dirichlet class and $\Gamma = f(\mathbb{T})$ then*

$$\int_\Gamma \int_\Gamma \frac{\ell(w_1, w_2) - |w_1 - w_2|}{|w_1 - w_2|^3} |dw_1||dw_2| < \infty.$$

I believe the result itself is true (I have a proposed alternate proof that uses a refinement of Peter Jones' traveling salesman theorem). The proof in [1] starts with Lemma 10 on page 581. This lemma correctly states that for h in the Hardy space H^1, we have

$$\frac{1}{|I_a|}\int_{I_a} |h(t)||dt| - \frac{1}{|I_a|}\left|\int_{I_a} h(t)|dt|\right| \le 20 \left(\int_{\mathbb{T}} |h(t)| P_a(t)|dt| - |h(a)|\right),$$

where $I_a \subset \mathbb{T}$ is the arc centered at $a/|a|$ of length $2\pi(1-|a|)$. However, it is possible for the left side to be much smaller that the right side. This happens, for example, when $h(z) = z$. The left side above then has size $\simeq (1-|a|)^2$ but the right side is $\simeq 1 - |a|$. On page 582, Lemma 10 is applied to g', where $w = g(z)$ is a conformal map, to deduce

$$\begin{aligned}
I(g) &\equiv \int_\Gamma \int_\Gamma \frac{\ell(w_1, w_2) - |w_1 - w_2|}{|w_1 - w_2|^3} |dw_1||dw_2| \\
&\lesssim \int_\Gamma \int_\Gamma \frac{|I_a|}{|w_1 - w_2|^3} \left(\int_{\mathbb{T}} |g'(t)| P_a(t)|dt| - |g'(a)|\right) |dw_1||dw_2| \\
&\simeq \int_\Gamma \int_\Gamma \frac{|I_a|}{|w_1 - w_2|^3} \left(\int_{\mathbb{D}} \frac{|g''(z)|^2}{|g'(z)|}(1 - |\varphi_a(z)|^2) dA_\rho(z)\right) |dw_1||dw_2|
\end{aligned}$$

where $\varphi_a(z) = (z-a)/(1-\overline{a}z)$ and the second inequality follows from a standard application of Green's theorem. However, the last line above can be infinite when $I(g)$ is finite. For example, if g is conformal on a neighborhood of $\overline{\mathbb{D}}$ and $|g''/g'| \simeq 1$ on $\mathbb{D}$. For example, consider the conformal map onto a domain with analytic boundary; then g' is non-zero on a neighborhood of the boundary and we can replace $g(z)$ by $g(rz)$, to get an example where g'' is non-zero on the boundary too. Given this, then $\log g'$ is certainly in the Dirichlet class and $|w_1 - w_2| = |g(z_1) - g(z_2)| \simeq |z_1 - z_2|$.

The latter estimate implies

$$\begin{aligned}\int_\Gamma\int_\Gamma \frac{|I_a|}{|w_1-w_2|^3}\left(\int_{\mathbb{D}} \frac{|g''(z)|^2}{|g'(z)|}(1-|\varphi_a(z)|^2)d\mathrm{A}_\rho(z)\right)|dw_1||dw_2| \\ \simeq \int_\Gamma\int_\Gamma \frac{|z_1-z_2|}{|z_1-z_2|^3}\left(\int_{\mathbb{D}}(1-|\varphi_a(z)|^2)d\mathrm{A}_\rho(z)\right)|dz_1||dz_2| \\ \simeq \int_\Gamma\int_\Gamma \frac{|z_1-z_2|}{|z_1-z_2|^3}|z_1-z_2||dz_1||dz_2| \\ \simeq \int_\Gamma\int_\Gamma \frac{|dz_1||dz_2|}{|z_1-z_2|}=\infty.\end{aligned}$$

However, for smooth curves $\ell(w_1,w_2)-|w_1-w_2| \lesssim |w_1-w_2|^3$, so $I(g)$ is finite. Thus even in smooth cases, the proof is bounding the finite value $I(g)$ by an infinite quantity. This possibly infinite quantity is mistakenly proven to be finite due to a slight error on the fourth line of page 584. Lemma 11 gives a correct estimate

$$\iint_{F(z)} \frac{|d\zeta_1||d\zeta_2|}{|\zeta_1-\zeta_2|^3} \simeq \frac{1}{1-|z|},$$

where $F(z)\subset \mathbb{T}\times\mathbb{T}$ is the set of pairs $\{\zeta_1,\zeta_2\}$ so that $z\in Q(\zeta_1,\zeta_2)$ whose base interval has endpoints $\{\zeta_1,\zeta_2\}$. However, on the fourth line of page 584, it is mistakenly stated that this lemma implies that

$$\iint_{F_k(z)} \frac{|d\zeta_1||d\zeta_2|}{|\zeta_1-\zeta_2|^3} \simeq \frac{2^k}{1-|z|},$$

where $F_k(z)$ is the set of pairs so that $z\in 2^kQ(\zeta_1,\zeta_2)$; the correct bound is

$$\iint_{F_k(z)} \frac{|d\zeta_1||d\zeta_2|}{|\zeta_1-\zeta_2|^3} \simeq \frac{4^k}{1-|z|},$$

but this causes the factor $\sum_{k=0}^\infty 2^{-k}$ in line 12 of page 584 to become $\sum_{k=0}^\infty 1$, which obviously diverges. To see that 4^k is correct (at least as a lower bound), note that the arc I_z contains 2^k pairs of intervals of length $\simeq 2^{-k}(1-|z|)$, separated by $\simeq 2^{-k}(1-|z|)$ and so that $z\in 2^kQ(\zeta_1,\zeta_2)$, where one of ζ_1,ζ_2 is chosen from each interval of a pair. Thus each such pair contributes $2^k/(1-|z|)$ to the integral, and hence the 2^k pairs together contribute $4^k/(1-|z|)$.

References

[1] Eva A. Gallardo-Gutiérrez, María J. González, Fernando Pérez-González, Christian Pommerenke, and Jouni Rättyä. Locally univalent functions, VMOA and the Dirichlet space. *Proc. Lond. Math. Soc. (3)*, 106(3):565–588, 2013.