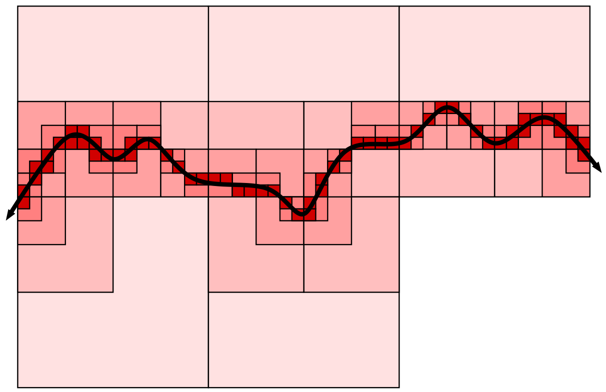
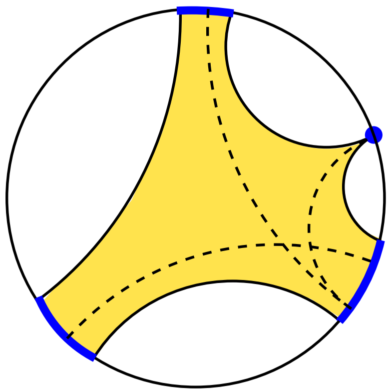
MAT 638, FALL 2020, Stony Brook University

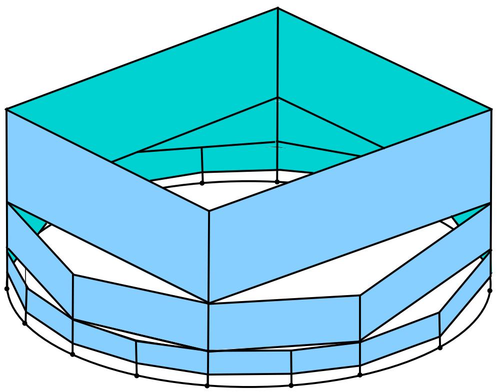
TOPICS IN REAL ANALYSIS

WEIL-PETERSSON CURVES, TRAVELING SALESMAN THEOREMS AND MINIMAL SURFACES

Christopher Bishop, Stony Brook







1. Thursday, November 5, 2020

2. Hyperbolic conditions in 3 dimensions

These slides continue earlier ones on the characterizations of Weil-Petersson curves in terms of Euclidean geometric conditions and a set of slides on Peter Jones's traveling salesman theorem.

These slides describe characterizations in terms of hyperbolic gometry, the curvature of mininal surfaces in hyperbolic space with asymptotic boundary Γ, and the renormalized area of these surfaces.

Definition	Description
1	$\log f'$ in Dirichlet class
2	Schwarzian derivative
3	QC dilatation in L^2
4	conformal welding midpoints
5	$\exp(i \log f')$ in $H^{1/2}$
6	arclength parameterization in $H^{3/2}$
7	tangents in $H^{1/2}$
8	finite Möbius energy
9	Jones conjecture
10	good polygonal approximations
11	β^2-sum is finite
12	Menger curvature
13	biLipschitz involutions

Definition	Description
14	between disjoint disks
15	thickness of convex hull
16	finite total curvature surface
17	minimal surface of finite curvature
18	additive isoperimetric bound
19	finite renormalized area
20	dyadic cylinder
21	closure of smooth curves in $T_0(1)$
22	P_φ^- is Hilbert-Schmidt
23	double hits by random lines
24	finite Loewner energy
25	large deviations of SLE(0^+)
26	Brownian loop measure

The names of 26 characterizations of Weil-Peterson curves

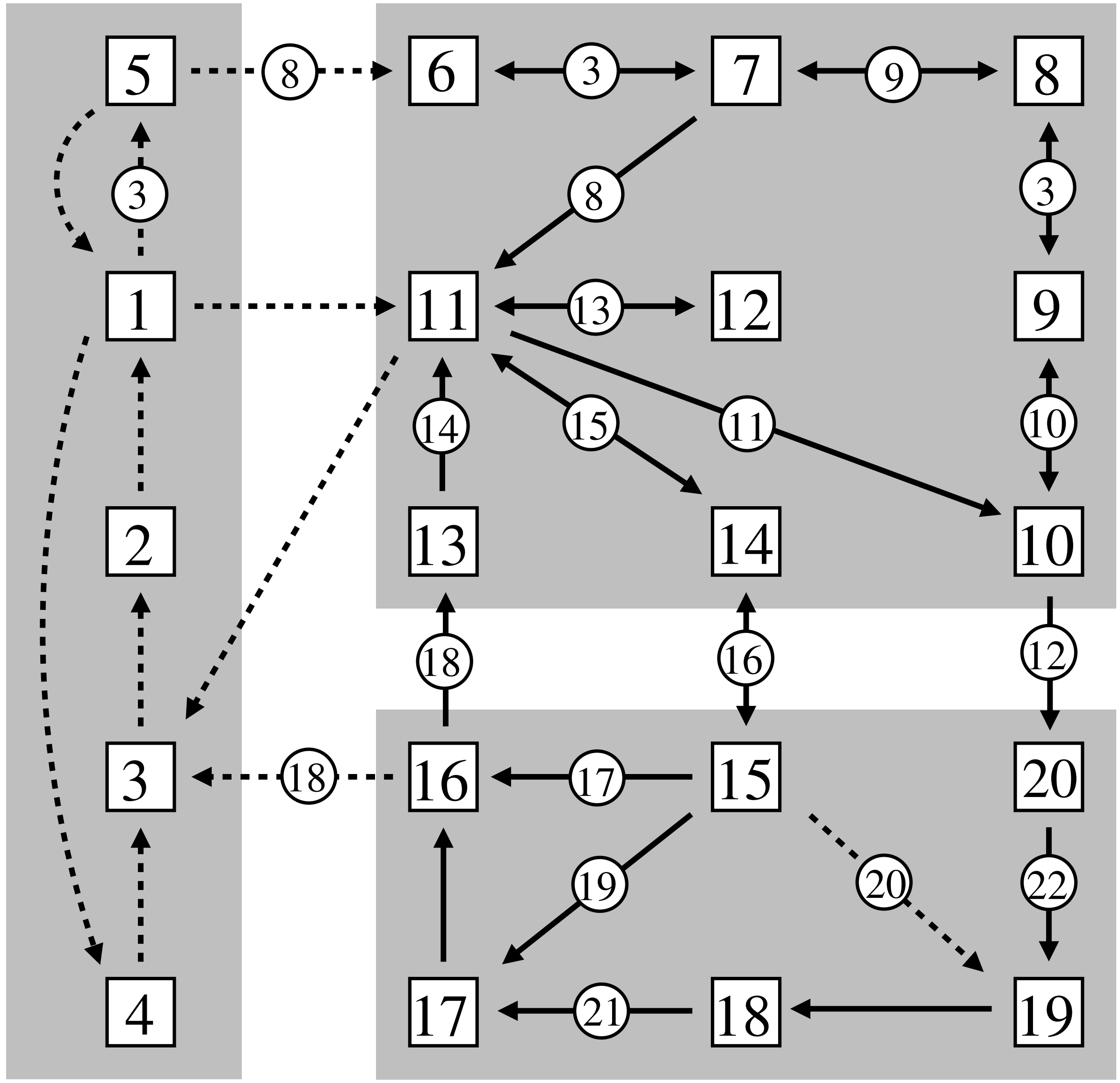


Diagram of implications between previous definitions.

Edge labels refer to sections of my preprint.

Many of our previous conditions involve sums or integrals over points $x \in \Gamma$ and scales $0 < t \leq \mathrm{diam}(\Gamma)$.

Thinking of t as a height instead of scale, we could interpret these as integrals over the cylinder $\Gamma \times (0, \mathrm{diam}(\Gamma)] \subset \mathbb{H}^{n+1} = \{(x,t) : x \in \mathbb{R}^n, t > 0\}$. Many of the conditions in this section will be integrals over various other surfaces in the upper half-space whose boundary on $\mathbb{R}^n$ is Γ.

The hyperbolic length of a (Euclidean) rectifiable curve in the unit disk $\mathbb{D}$ or in the n-dimensional ball $\mathbb{B}^n$ is given by integrating

$$\frac{ds}{1-|z|^2},$$

along the curve. In the upper half-space $\mathbb{H}^n$ we integrate $ds/2t$.

Note that this definition differs by a factor of 2 from that given in some sources; we have made our choice so that hyperbolic space has Gauss curvature -1.

The hyperbolic distance between two points is given by taking the infimum of all hyperbolic lengths of paths connecting the points. In the ball, hyperbolic geodesics are either diameters or subarcs of circles perpendicular to the boundary. In half-space model $\mathbb{H}^{n+1}$, hyperbolic geodesics are either vertical rays or semi-circles centered on the boundary.

The hyperbolic metric $\rho = \rho_\Omega$ on a simply connected planar domain Ω is defined by transferring the hyperbolic metric on $\mathbb{D}$ by a conformal map (the choice of the map makes no difference). The quasi-hyperbolic metric on Ω is defined by integrating

$$d\widetilde{\rho} = \frac{ds}{\operatorname{dist}(z, \partial\Omega)}.$$

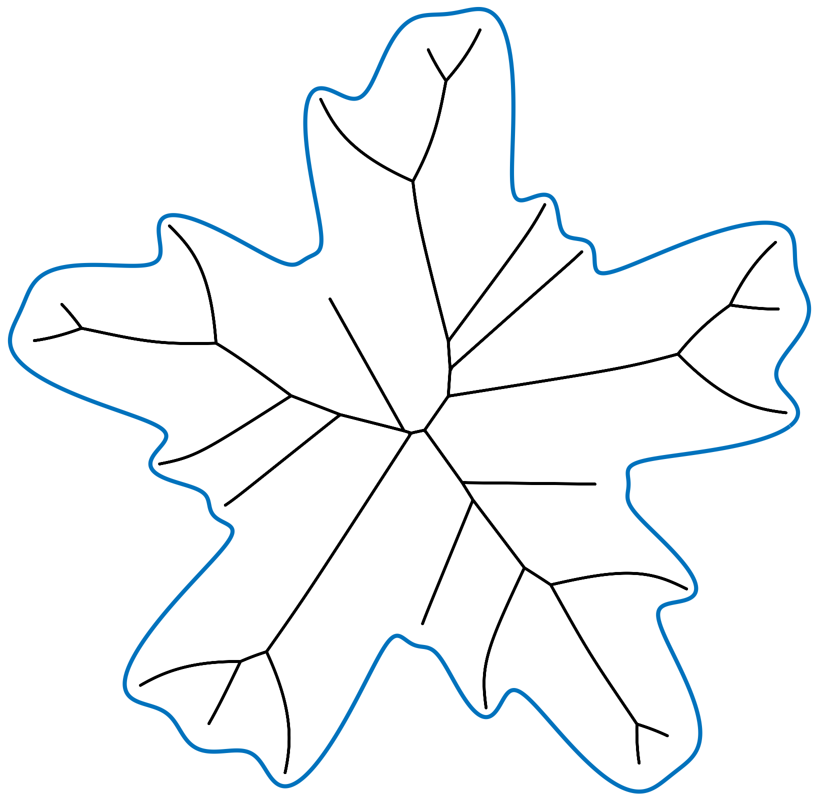
and Koebe's estimate for conformal maps implies

$$\rho_\Omega(z, w) \le \widetilde{\rho}_\Omega(z, w) \le 4 \cdot \rho_\Omega(z, w).$$

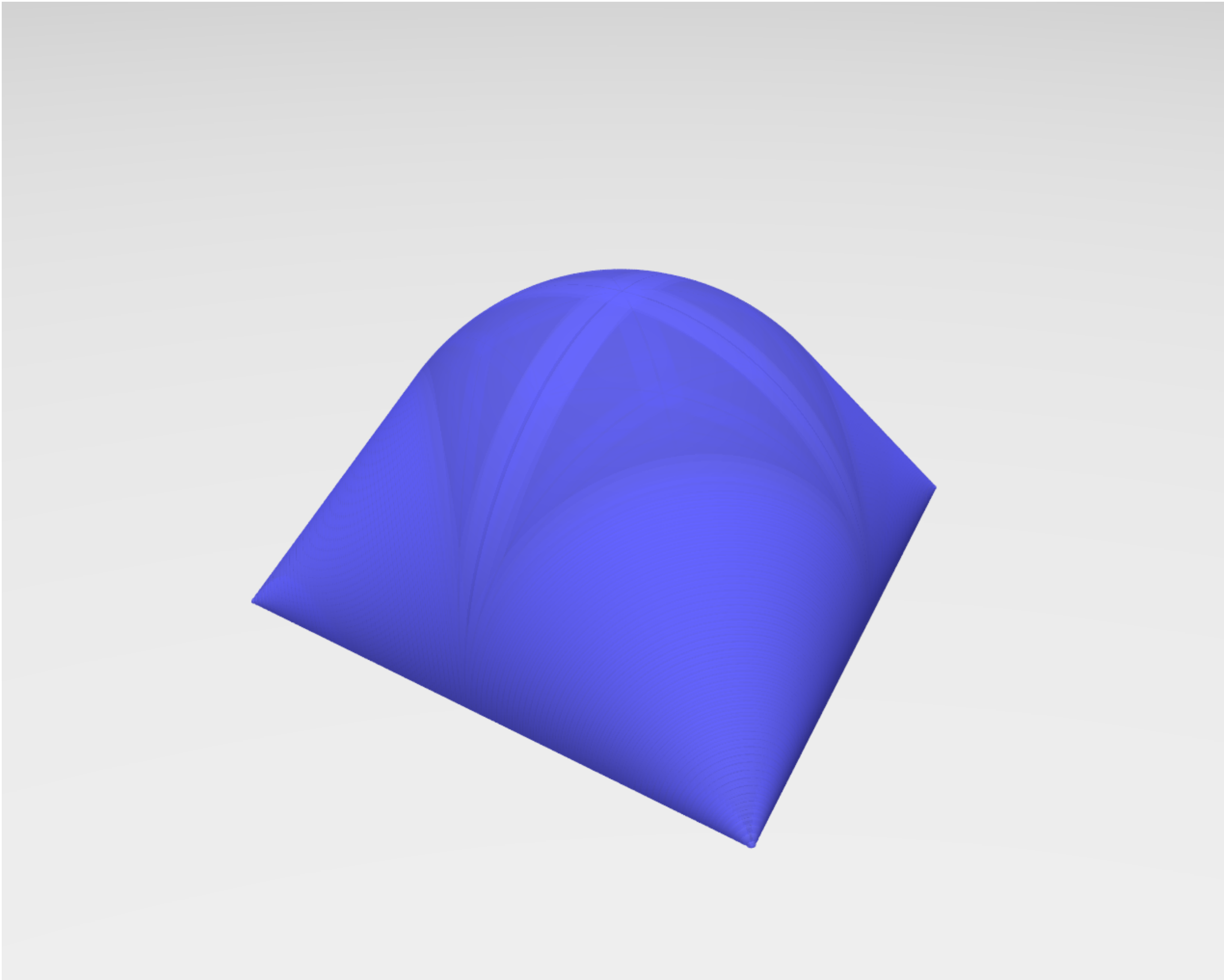
Given a closed curve Γ (or more generally, a compact set E) the hyperbolic convex hull, denoted $\mathrm{CH}(\Gamma)$, is the convex hull in $\mathbb{H}^{n+1}$ of all infinite geodesics that have both endpoints in Γ.

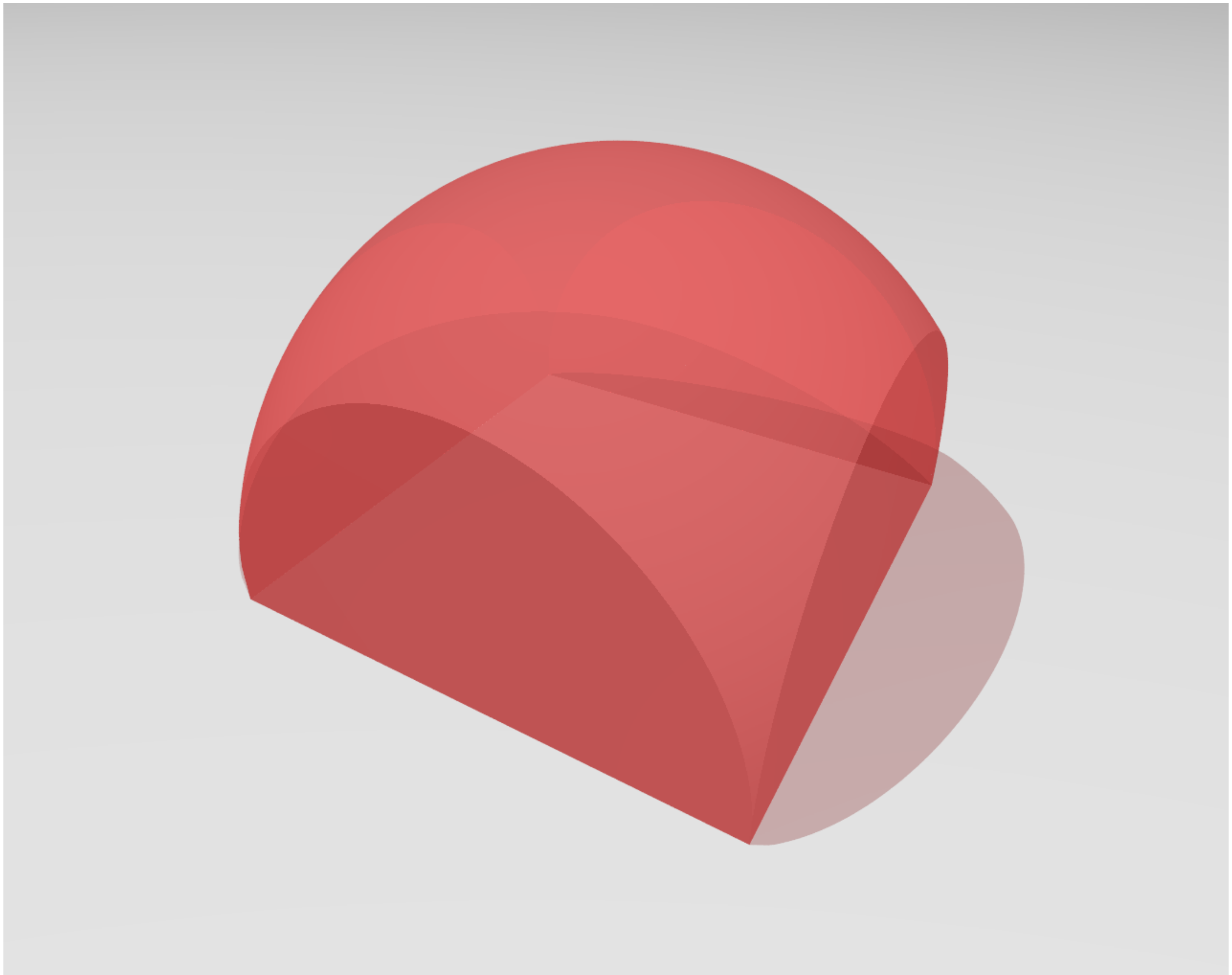
The complement of the convex hull is a union of hyperbolic half-spaces. Each such half-space intersects $\mathbb{R}^n$ in am open Euclidean ball (or half-space or exterior of a ball) that does not hit Γ. Conversely, each ball in $\mathbb{R}^n$ that does not intersect Γ corresponds to a hyperbolic half-space in the complement of $\mathrm{CH}(\Gamma)$.

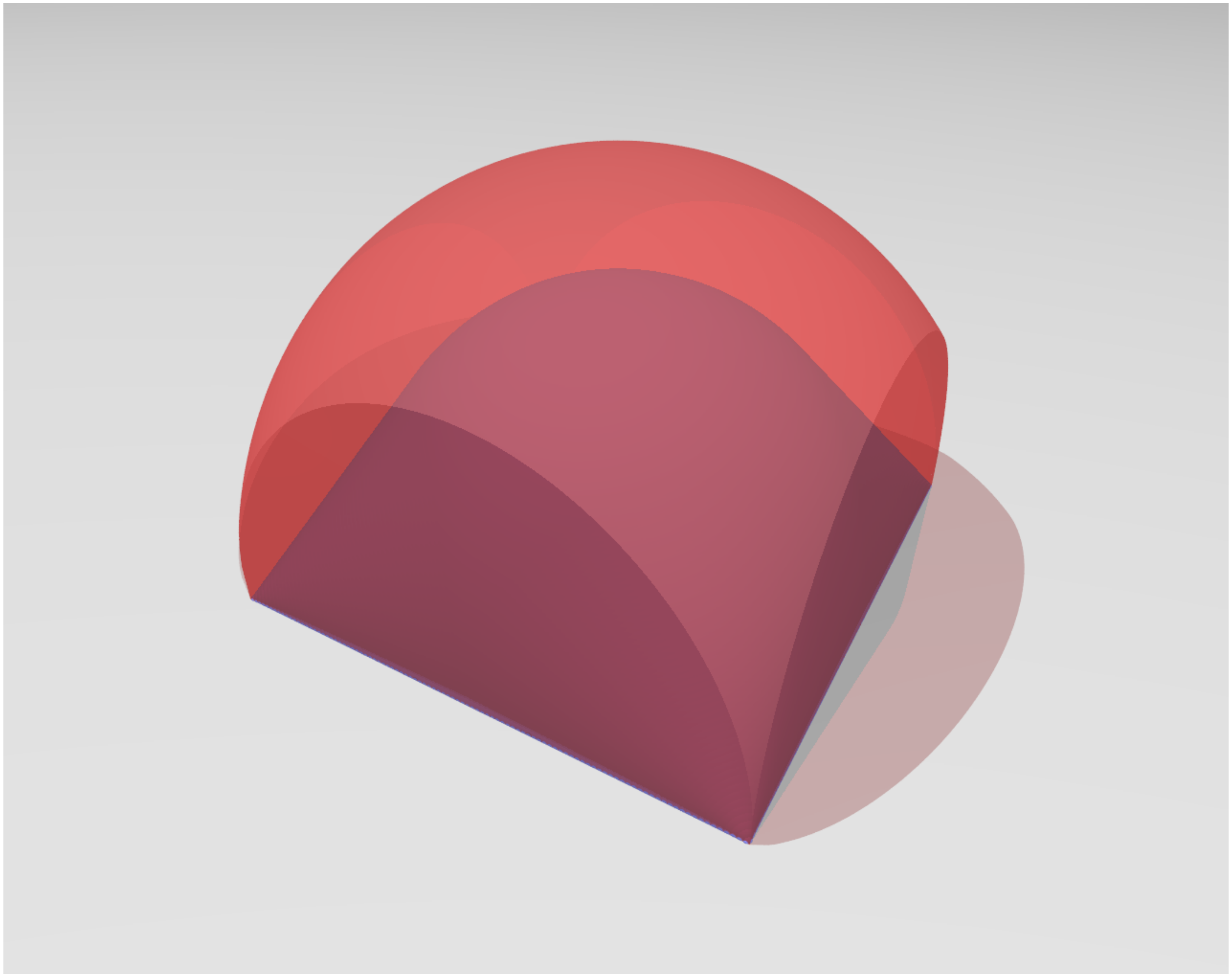
In the planar case it is only necessary to consider medial axis disks, i.e., those that hit the boundary in at least two points.

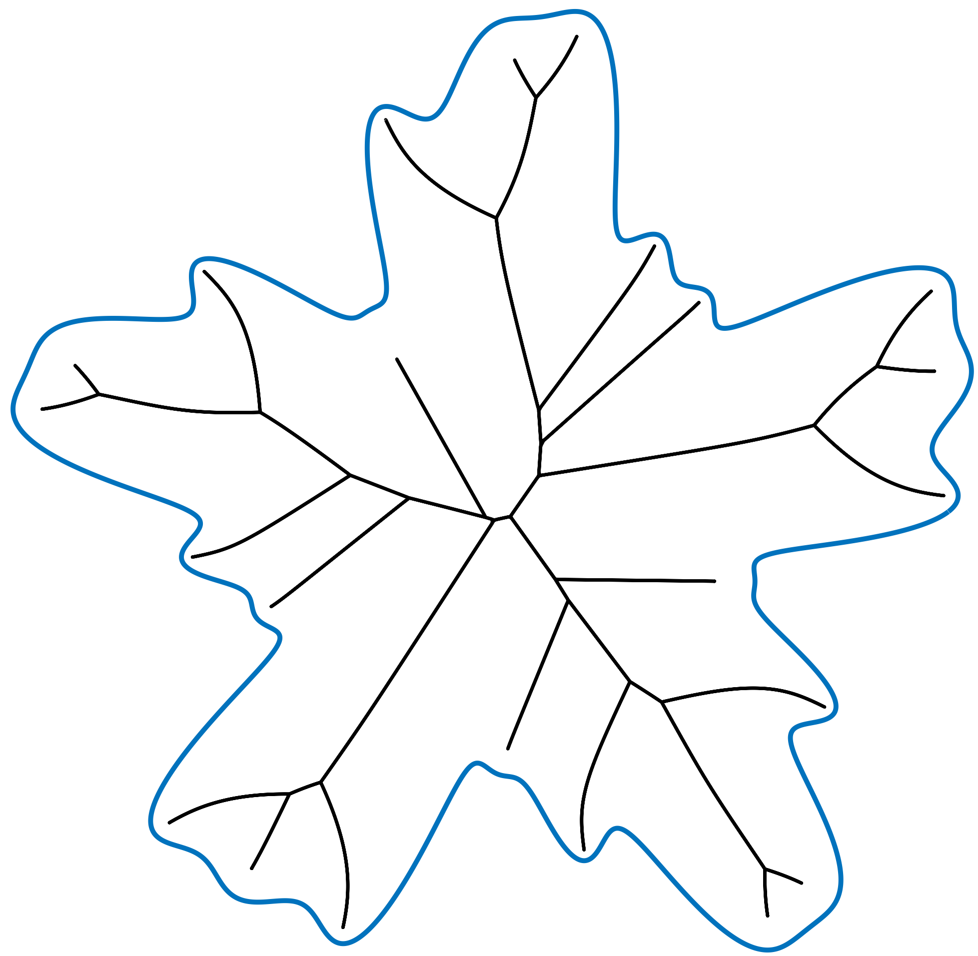


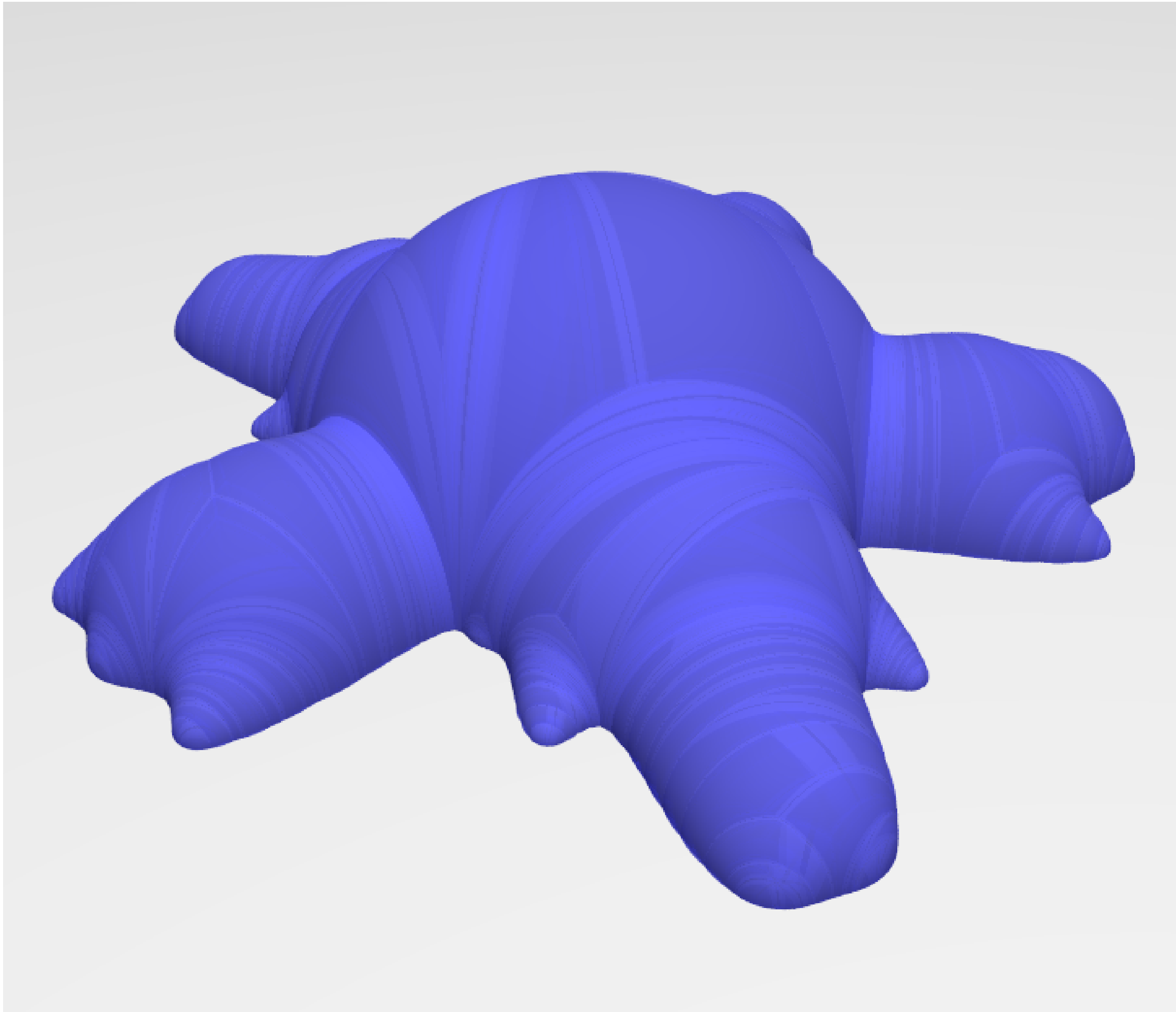
A planar curve Γ divides $\mathbb{R}^2$ into two components and the boundary of $\mathrm{CH}(\Gamma)$ has two corresponding connected components (unless Γ is a circle) called the domes of the two sides of Γ. In higher dimensions, the complement of a Jordan curve is connected and $\partial\mathrm{CH}(\Gamma)$ has a single component.

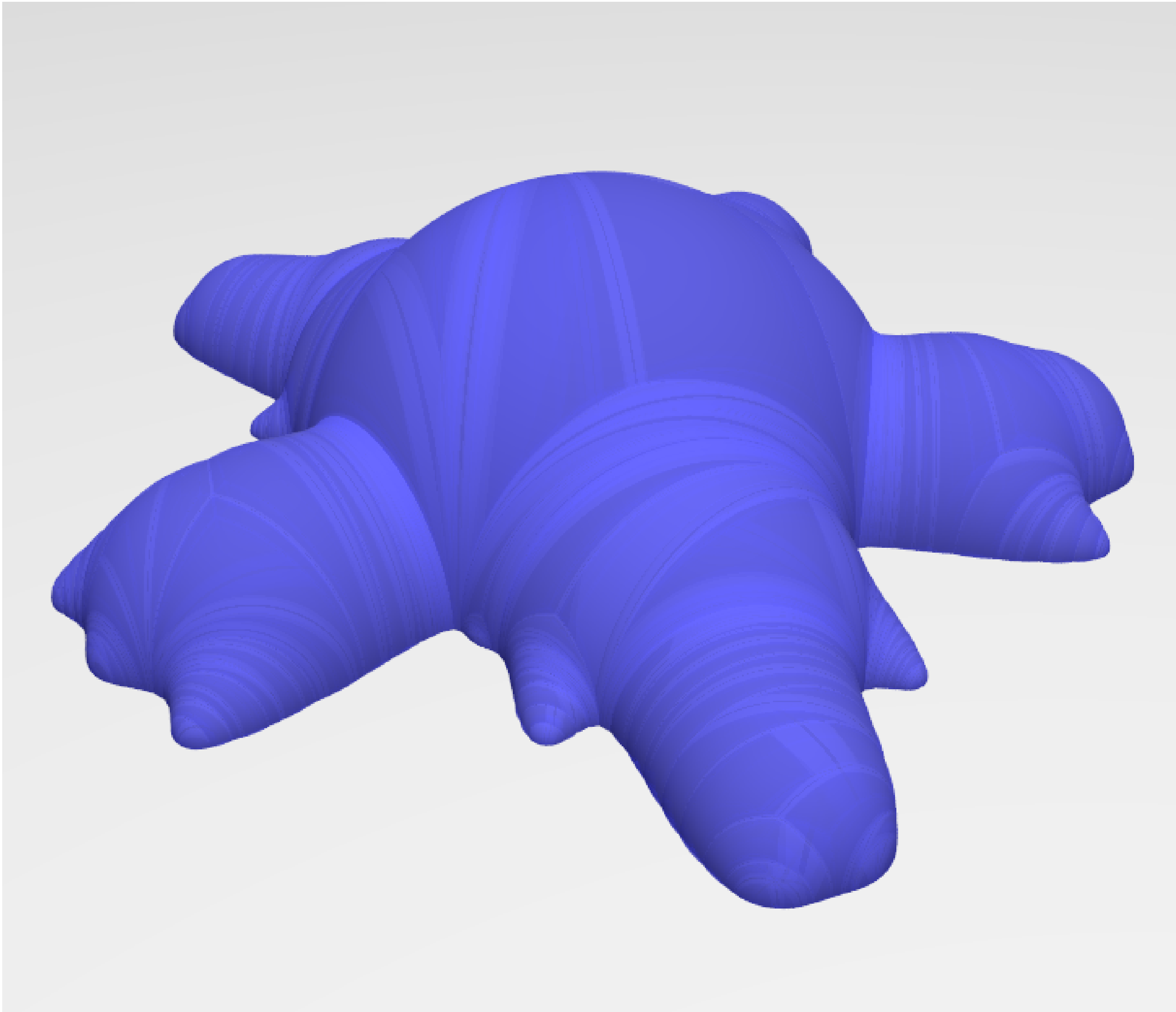


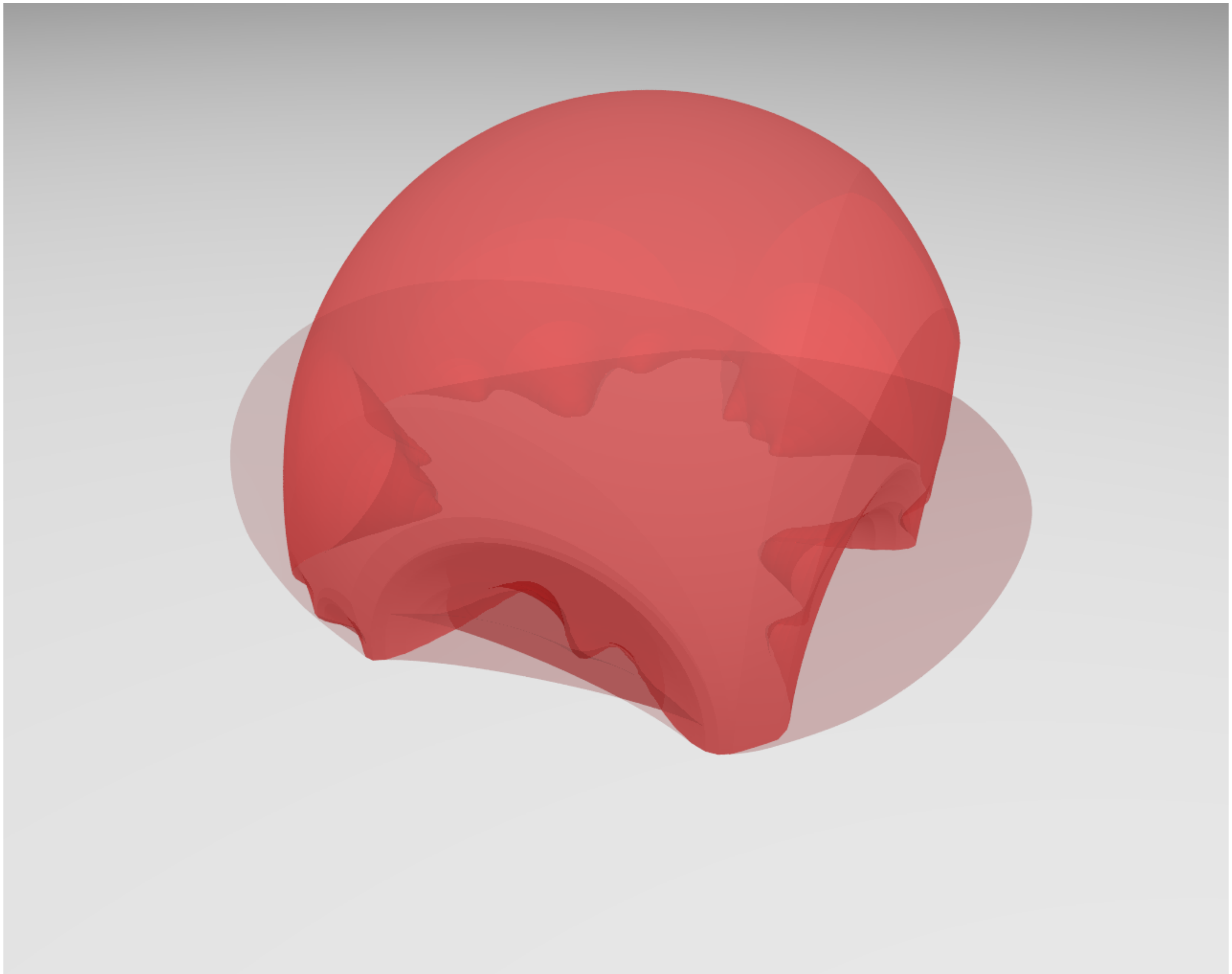


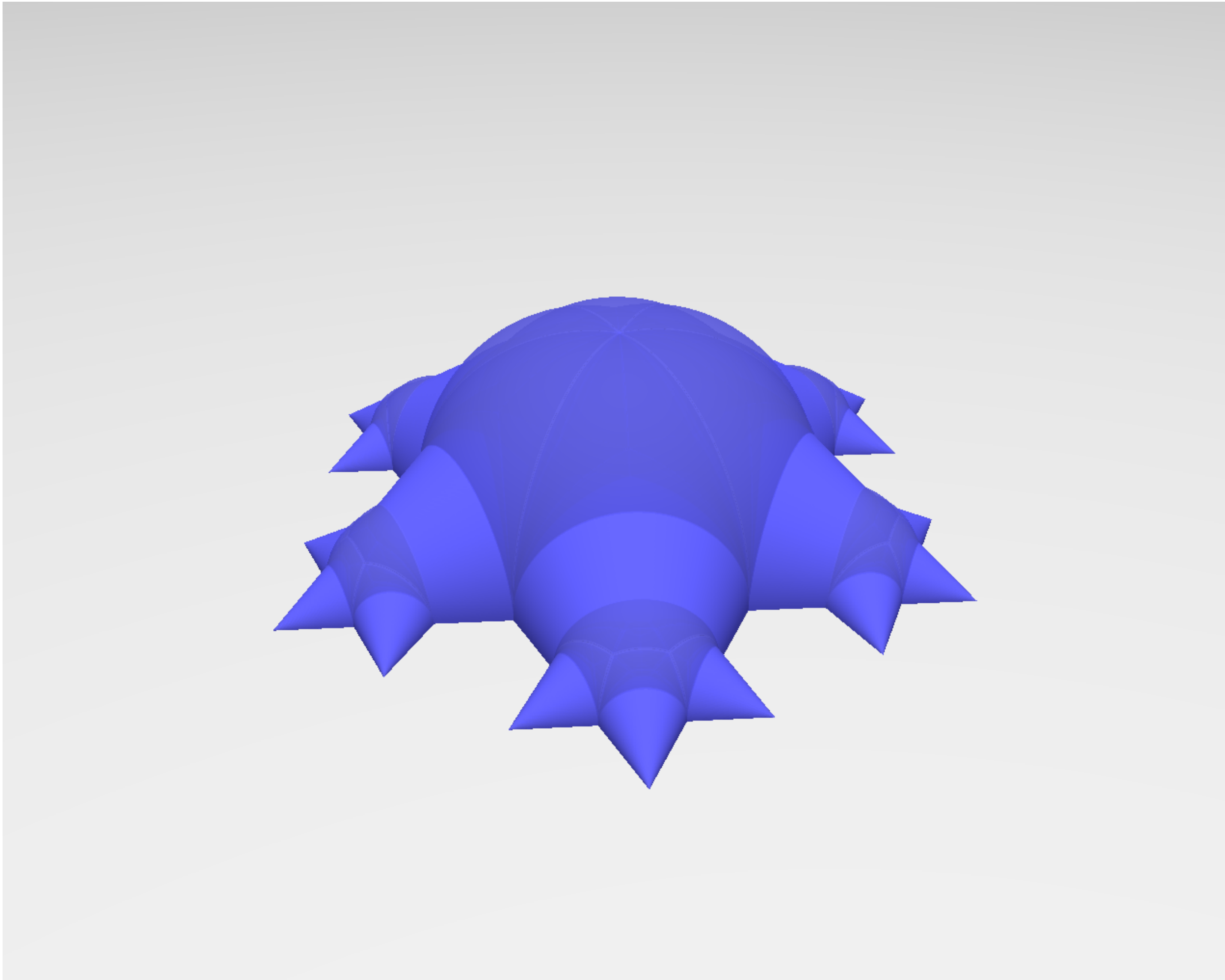


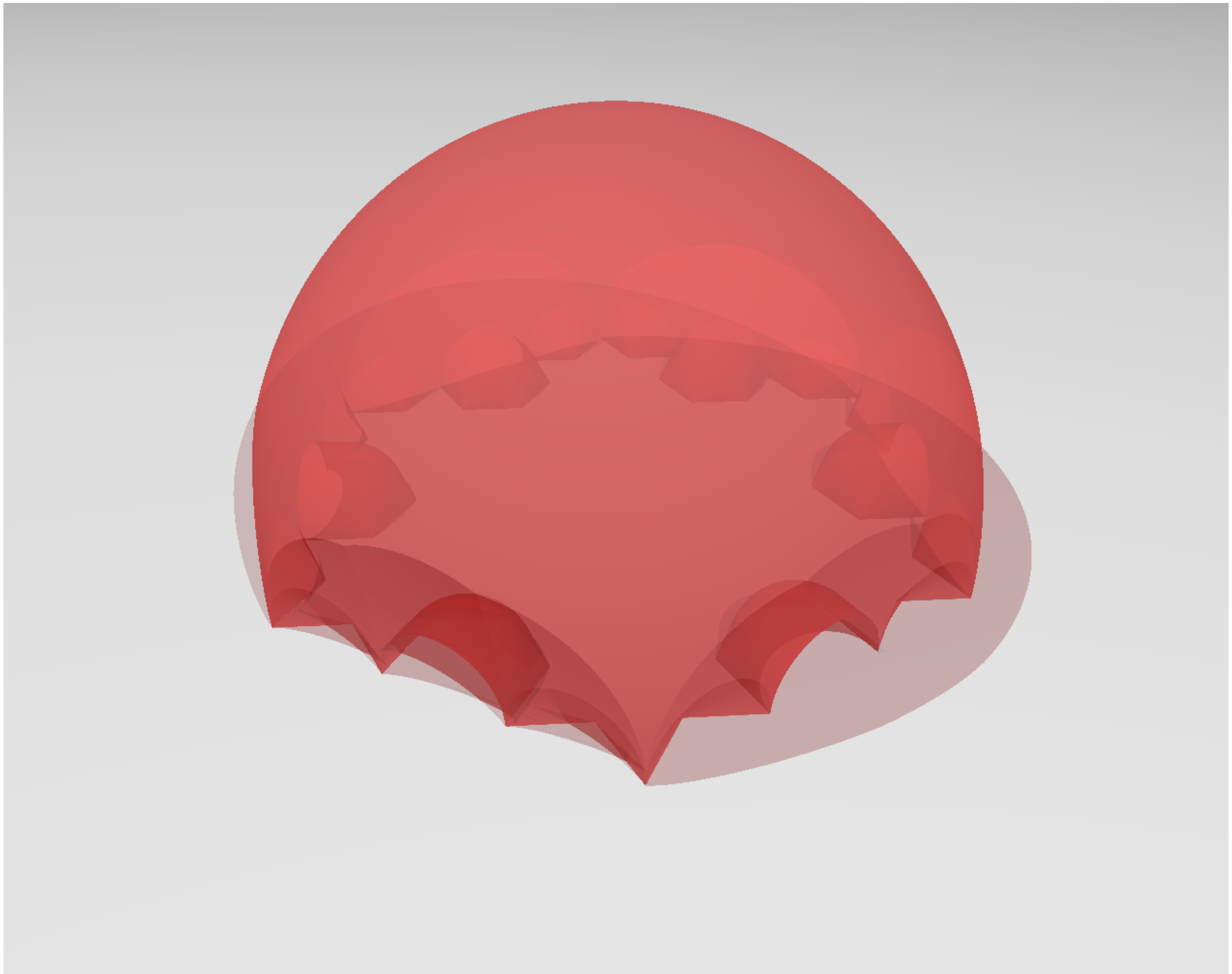


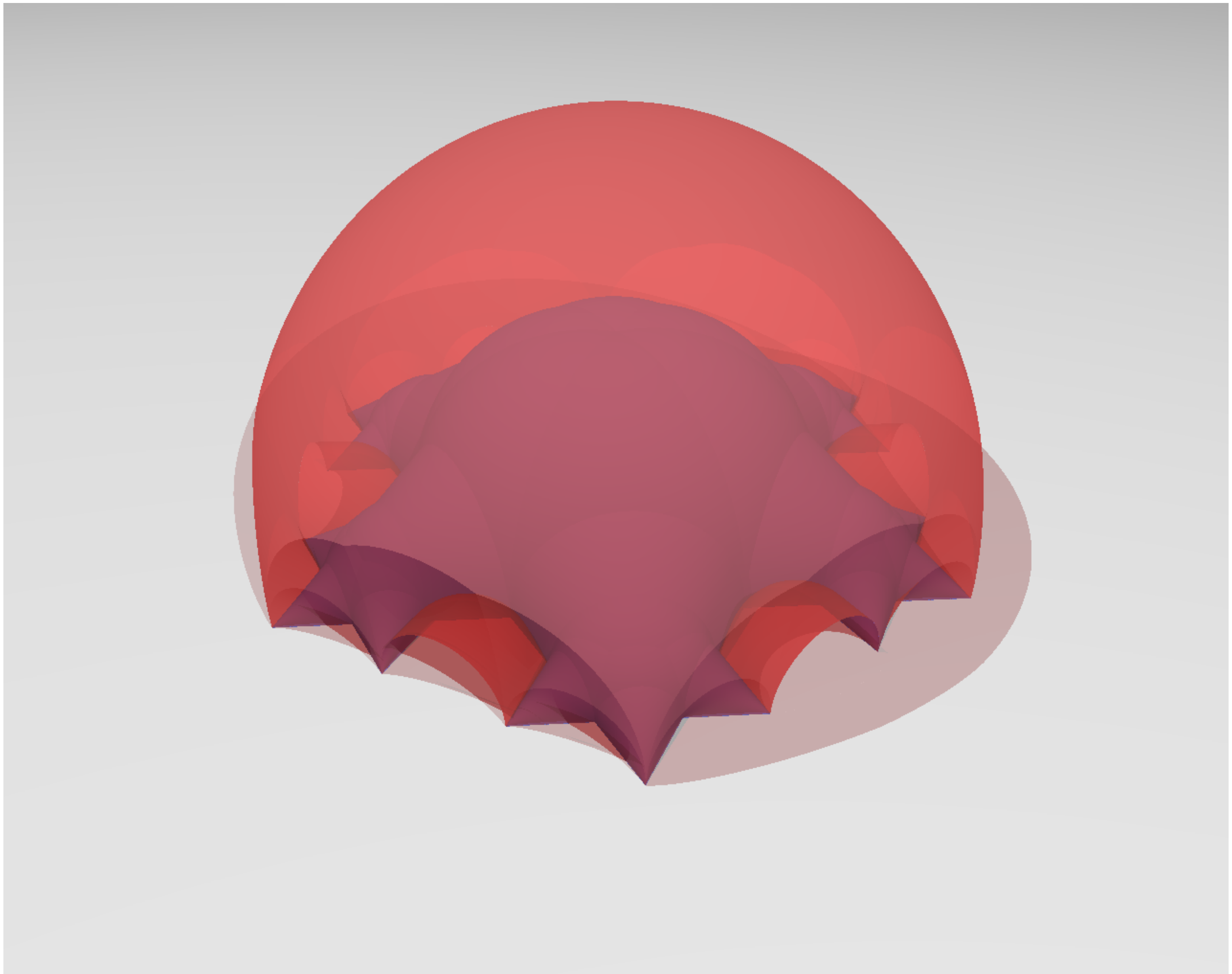












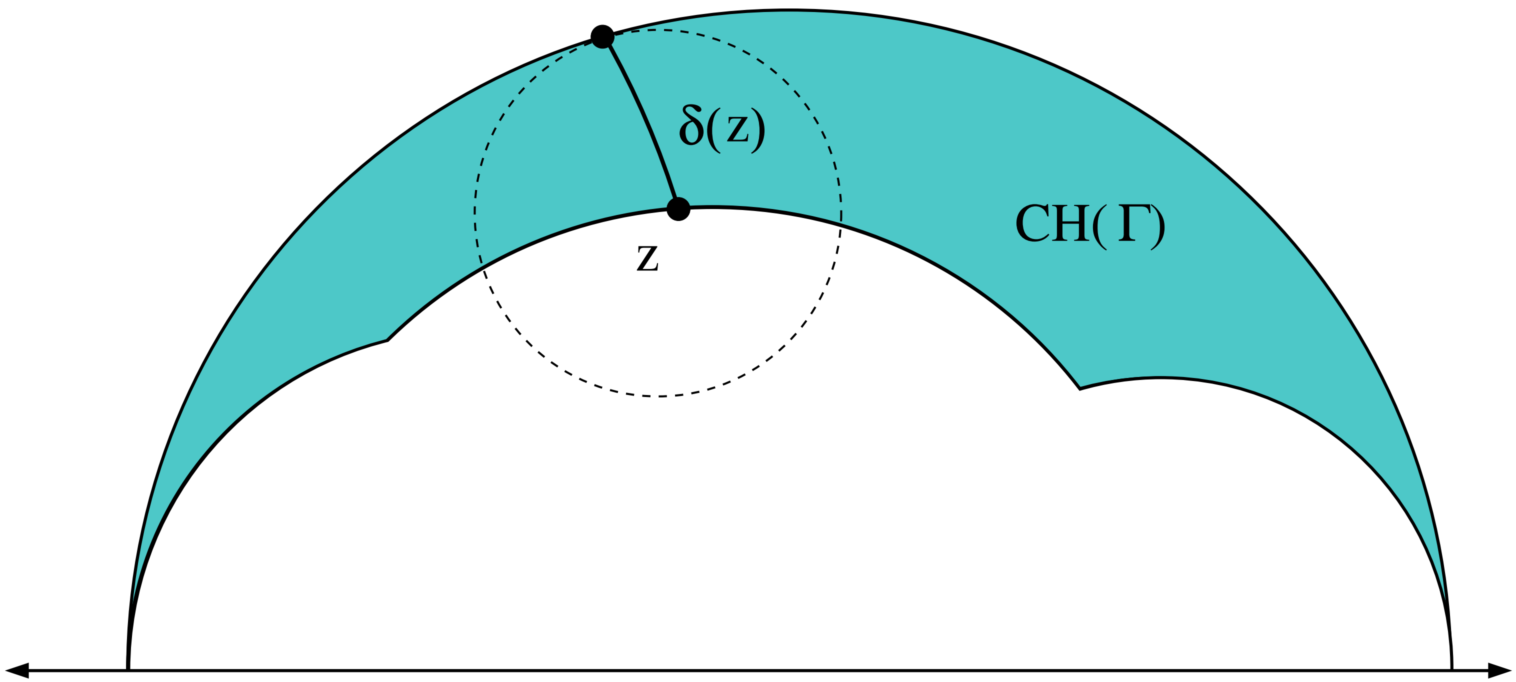
As noted above, the boundary $\mathrm{CH}(\Gamma) \subset \mathbb{R}^3_+$ has two connected components, S_1, S_2. Each of these surfaces meets $\mathbb{R}^2$ exactly along Γ and each is isomorphic to the hyperbolic unit disk when given its hyperbolic path metric.

Each of these surfaces is also a pleated surface. This means that it is a disjoint union of non-intersecting infinite geodesics for $\mathbb{B}^3$ (possibly uncountably many) and at most countably many regions lying on hyperbolic planes, each region bounded by disjoint hyperbolic geodesics. Roughly speaking, each surface is a copy of the hyperbolic disk that has been "bent" along a collection of disjoint geodesics, and there is an associated bending measure that gives the amount of bending on each geodesic.

The bending measure actually measures arcs that are transverse to the bending geodesics and in general it may have both atoms and continuous parts. However, for convex hulls of Weil-Petersson curves, no atoms will occur. For more about

convex hulls and pleated surfaces, see [5] by Epstein and Marden (or the revised version [6]). For an overview of domes and convex hulls see Marden's paper [10]; also his book [9] for a discussion related to hyperbolic 3-manifolds.

For a point $z \in \mathrm{CH}(\Gamma)$ we define $\delta(z) = \max(\mathrm{dist}_\rho(z, S_1), \mathrm{dist}_\rho(z, S_2))$, i.e., $\delta(z)$ is the hyperbolic distance to the farther of the two boundary components of $\mathrm{CH}(\Gamma)$. For z inside the convex hull, $\delta(z)$ measures the "thickness" of the convex hull of Γ near z.



For $z \in \mathrm{CH}(\Gamma)$, $\delta(z)$ measures "width" of convex hull near z.

$\delta(z) = 0$ iff Γ is a circle (hull has no interior).

Definition 15.

$$\int_{\partial \mathrm{CH}(\Gamma)} \delta^2(z) d\mathrm{A}_\rho(z) < \infty, \tag{2.1}$$

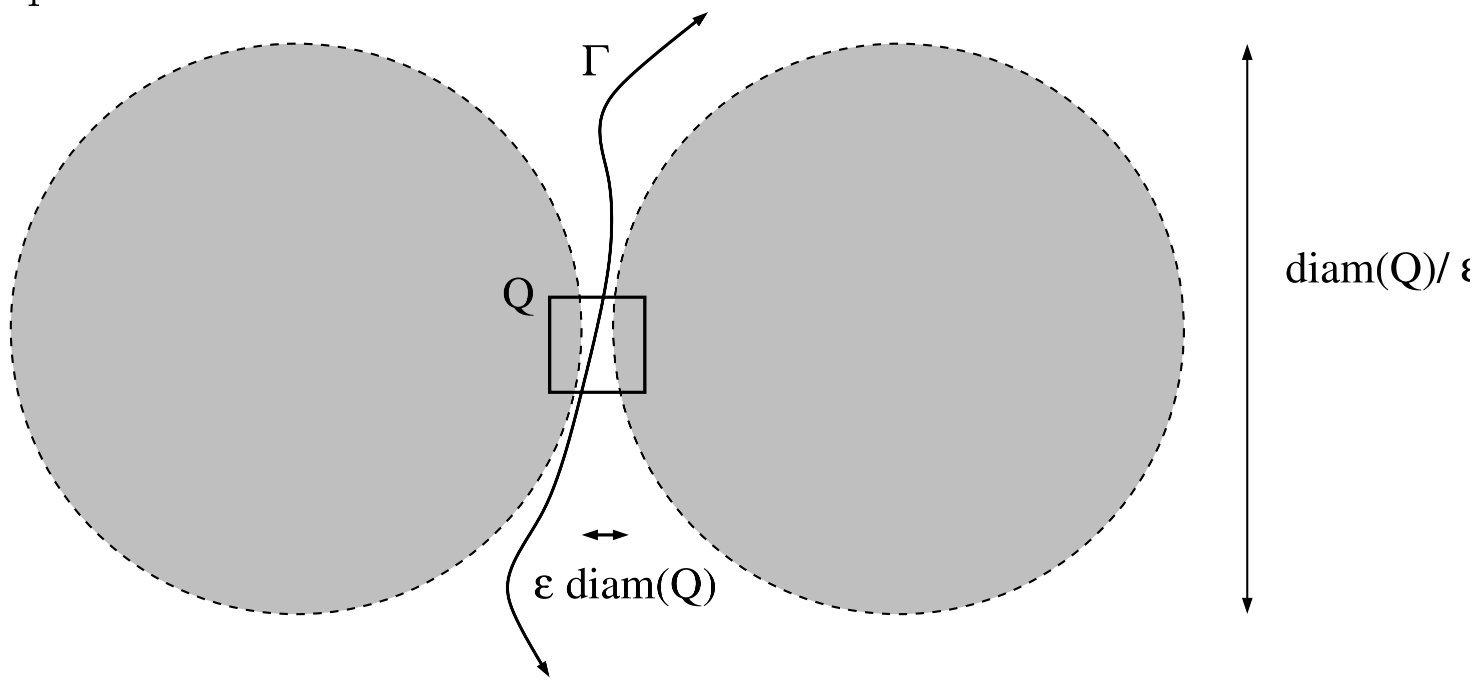
where $d\mathrm{A}_\rho$ *denotes hyperbolic surface area on* $\partial\mathrm{CH}(\Gamma)$.

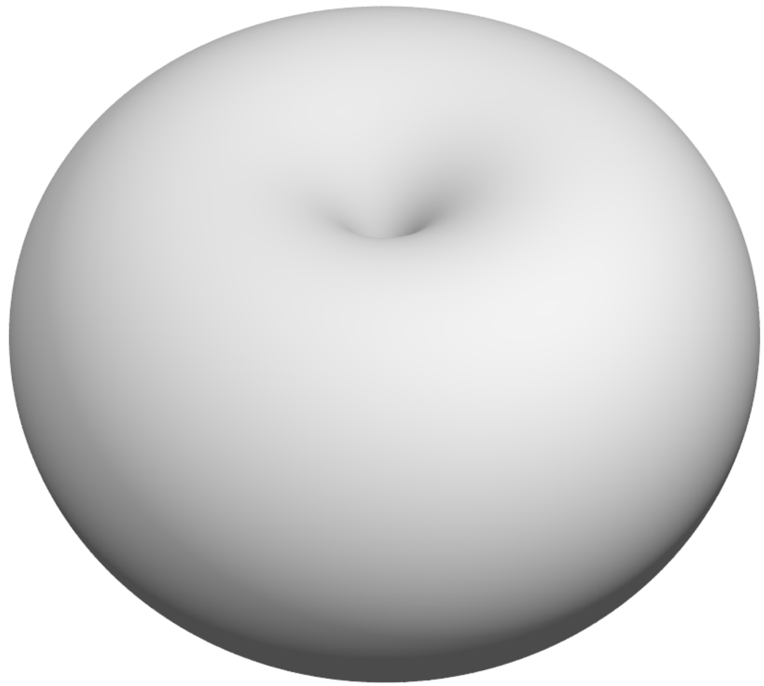
We have integrated over all of $\partial\mathrm{CH}(\Gamma)$, but the proof will show that if the integral over one component is finite, then so is the integral over the other one.

We want to show this follows from Definition 14 on ε-numbers.

Recall from before:

Given a dyadic square Q let $\varepsilon_\Gamma(Q)$ be the infimum of the $\epsilon \in (0,1]$ so that $3Q$ hits a line L, a point z and a disk D so that D has radius $\ell(Q)/\epsilon$, z is the closest point of D to L and neither D nor its reflection across L hits Γ.





In higher dimensions the disk D is replaced by a ball B of radius $\mathrm{diam}(Q)/\epsilon$ that attains its distance ϵ from L at $z \in Q$, and that the full rotation of B around L does not intersect Γ. Thus Γ is surround by a "fat torus". The centers of the balls form a $(n-2)$-sphere that lies in a $(n-1)$-hyperplane perpendicular to L.

If no such line, point and disk exist, we set $\varepsilon_\Gamma(Q) = 1$.

Definition 14. Γ *is chord-arc and satisfies*

$$\sum_Q \varepsilon_\Gamma^2(Q) < \infty \tag{2.2}$$

where the sum is over dyadic squares hitting Γ *with* $\operatorname{diam}(Q) \le \operatorname{diam}(\Gamma)$*.*

Next we show ε controls δ.

Recall that each ball $B \subset \mathbb{R}^n$ is the boundary of a hyperbolic half-space H in $\mathbb{H}^{n+1}$, and two balls are disjoint iff the corresponding half-spaces are disjoint.

Lemma 2.1. *Suppose $B_1, B_2 \subset \mathbb{R}^n$ are disjoint balls of radius r that are distance ϵ apart. Then the hyperbolic distance between the corresponding half-spaces is $\simeq \sqrt{\epsilon/r}$.*

Proof. The nearest points on the half-spaces will occur over the line connecting the centers of B_1 and B_2, so it suffices to do this calculation in the copy of the hyperbolic upper half-plane lying above this line; this is a simple calculus exercise.

We can normalize so the balls both have have radius 1 and the distance between them is $\eta = \epsilon/r$. The intersection of the hemispheres with this plane are two half-circles. At height t above $\mathbb{R}^2$, these circles are Euclidean distance $\eta + O(t^2)$ apart, hence hyperbolic distance $\simeq t + \eta/t$ apart. This is minimized when $t = \sqrt{\eta} = \sqrt{\epsilon/r}$. $\square$

Lemma 2.2. *If* $z \in \mathrm{CH}(\Gamma)$*, then* $\delta(w) = O(\delta(z))$ *for all* $w \in \mathrm{CH}(\Gamma) \cap \mathrm{B}_\rho(\mathrm{z}, 1)$.

Proof. The point is that if H_1, H_2 are two disjoint hyperbolic half-planes that both within distance δ of a point z, then their boundaries remain within distance $O(\delta)$ of each other inside $B_\rho(z, 1)$ (imagine $z = 0$ in the ball model). $\square$

Lemma 2.3. *Definition 14 implies Definition 15.*

Proof. Lemmas 2.1 and 2.2 imply that if $\varepsilon_\Gamma(Q)$ is small (say less than $1/100$), then $\delta(z) \lesssim \varepsilon_\Gamma(Q)$ for every point $z \in T(Q) = Q \times [\ell(Q)/2, \ell(Q)] \subset \mathbb{H}^{n+1}$. Thus $\sum_Q \delta^2(Q)$ is bounded by a uniform multiple of $\varepsilon_\Gamma^2(Q)$. $\square$

It is fairly easy to show Definition 15 implies Definition **??** directly using Lemma 3.3 from [1]. Next we recall Lemma 3.3 from [1]

Lemma 2.4. *Suppose* Ω *is simply connected and*

$$\{|z| < 1 - \delta\} \subset \Omega \subset \mathbb{D}.$$

If $F : \mathbb{D} \to \Omega$ *is conformal map such that* $F(0) = 0$ *and* $F'(0) > 0$*, then*

$$|F(z) - z| = O(\delta),$$

for all $|z| \leq 1/2$ *and*

$$\max(|F'(0) - 1|, |F''(0)|, |F'''(0)|) = O(\delta).$$

In particular, $|S(F)(0)| = O(\delta)$*.*

Combined with our earlier discussion, and the fact that the Schwarzian is invariant under post-composition by Möbius transformations, this gives

Lemma 2.5. *Suppose* Γ *is closed curve on the sphere with complementary domains* Ω_1, Ω_2*, whose domes are denoted* S_1, S_2*. If* $f : \mathbb{D} \to \Omega_1$ *is conformal, then*

$$|S(f)(z)|(1 - |z|^2)^2 = O(\delta(N(f(z))),$$

where and N *is the nearest projection map from* Ω_1 *to* S_1*.*

If Γ is a quasicircle, then each point z of one boundary component is within a uniformly bounded hyperbolic distance $\delta(z)$ of the other boundary component, i.e., if Γ is a quasicircle then $\delta(z) \in L^\infty(\partial \mathrm{CH}(\Gamma), d\mathrm{A}_\rho)$.

This holds because both complementary components of a quasicircle are uniform domains [11], and thus for every $x \in \Gamma$ and $0 < r \leq \mathrm{diam}(\Gamma)$, both complementary components contain disks of diameter $\simeq r$ inside $D(x, r)$.

The converse is not true, since non-quasicircles may also have $\delta(z) \in L^\infty$.

Definition 15 says that the Weil-Petersson class corresponds to

$$\delta(z) \in L^2(\partial \mathrm{CH}(\Gamma), d\mathrm{A}_\rho)$$

. The condition $\delta(z) \in L^1(\partial \mathrm{CH}(\Gamma), d\mathrm{A}_\rho)$ is equivalent to $\mathrm{CH}(\Gamma)$ having finite hyperbolic volume.

However, for a closed curve, this is always either zero (for lines and circles) or infinite (everything else); we leave this as an exercise.

The part of the convex hull within unit distance of z has hyperbolic volume bounded by $O(\delta(0))$. Therefore Definition 15 can be restated as:

$$\iint_{\mathbb{B}} \operatorname{dist}(z, \mathbb{B} \setminus \mathrm{CH}(\Gamma)) dV(z) < \infty, \tag{2.3}$$

here dV denotes hyperbolic volume.

As mentioned above, each boundary component of $\mathrm{CH}(\Gamma)$ is is 'pleated" surfaces. This means that the foliation supports a transverse measure: this assigns mass to any curve that crosses the foliation transversely.

In the case of a region Ω that is a finite union of disks, the boundary component $\mathrm{CH}(\partial\Omega)$ that faces Ω lies on a finite union of hemispheres and the transverse measure is atomic: is assigns a mass to each bending geodesic equal to the angle formed by the two hemisphere that meet along that geodesic.

Every simply connected planar domain Ω can be exhausted by finite unions of disks in such a way that the domes converge to the dome of Ω and the atomic bending measures converge to the bending measure for the dome of Ω. See [5].

For $z \in \partial \mathrm{CH}(\Gamma)$, let Bnd(z) denote the amount of bending in a unit neighborhood of z on the component of $\partial \mathrm{CH}(\Gamma)$ containing z.

Proposition 2.6. Γ *is WP iff it satisfies*

$$\int_{\partial \mathrm{CH}(\Gamma)} \mathrm{Bnd(z)}^2 \mathrm{dA}_\rho(\mathrm{z}) < \infty, \tag{2.4}$$

where dA_ρ *denotes hyperbolic surface area.*

As in Definition 15 if is equivalent to just integrate over one of the connected components of $\partial \mathrm{CH}(\Gamma)$.

3. Tuesday, November 10, 2020

If we use the ball model of hyperbolic space, assume $0 \in S$, then we can rotate so that the xy-plane is the tangent plane to S at 0 and S looks like the graph of $\kappa_1 x^2 + \kappa_2 y^2$ (plus higher order terms), where κ_1, κ_2 are the principle curvatures of S at 0.

The mean curvature is $H = (\kappa_1 + \kappa_2)/2$, and the norm-squared of the second fundamental form is $|K|^2 = \kappa_1^2 + \kappa_2^2$, the norm of the 2×2 diagonal matrix K with entries κ_1, κ_2.

The mean curvature of a minimal surface is zero.

Definition 16. $\Gamma \subset \mathbb{R}^2$ *is the boundary of a smooth surface* $S \subset \mathbb{R}^3_+$ *such that* $\mathcal{K}(z) \to 0$ *as* z *tends to the boundary of hyperbolic space and*

$$\int_S |\mathcal{K}(z)|^2 d\mathrm{A}_\rho(z) < \infty, \tag{3.1}$$

where $\mathcal{K}$ *is the second fundamental form of* S*.*

Lemma 3.1. *Definition 15 implies Definition 16.*

Proof. In both $n = 2$ and higher dimensions we create a triangulated surface where adjacent triangles are very close to parallel, and smooth this surface to obtain a surface with small principle curvatures.

In dimensions ≥ 2, the discrete surface can be the dyadic dome, discussed later, and the principle curvatures are controlled by the β-numbers.

In the special case $n = 2$, we can also use a discretization of the usual hyperbolic dome of one side of Γ.

Suppose S is one component of $\partial \mathrm{CH}(\Gamma)$. It is known that S, with its hyperbolic path metric, is isomorphic to the hyperbolic disk (e.g., [6], [10], [9]). The hyperbolic unit disk can be triangulated by geodesic triangles with hyperbolic diameters $\simeq 1$ and angles bounded strictly between 0 and π, e.g., take the tesselation corresponding to a Fuchsian triangle group, or obtain a triangulation by connecting the center of each Whitney box for $\mathbb{D}$ to the box's vertices.

Fix such a triangulation of $\mathbb{D}$ and map the vertices to S via the isometry. Each triple of image vertices corresponding to a triangle on $\mathbb{D}$ lies on a hyperbolic plane and determines a triangle on this plane. Create a new surface S_1 by gluing these triangles together along their edges.

Because the vertices lie in $\mathrm{CH}(\Gamma)$, convexity implies each triangle, and hence all of S_1, also lie in $\mathrm{CH}(\Gamma)$.

Consider two triangles T_1, T_2 in S_1 that meet along a common edge e. Normalize so that one endpoint of e is the origin in the ball model of hyperbolic 3-space, e lies along the x axis and T_1 lies in the xy-plane. Then T_2 lies in Euclidean plane that makes some angle θ with the xy-plane, and by our assumptions, it contains a point p (e.g., the vertex of T_2 not on e) that is hyperbolic distance $\simeq 1$ from 0 and Euclidean distance $\simeq 1$ from the x-axis.

Then p is Euclidean distance $\simeq \theta$ from the xy-plane. Because both triangles lie inside $\mathrm{CH}(\Gamma)$ and $\mathrm{CH}(\Gamma)$ is trapped between two hyperbolic half-planes that each come within hyperbolic distance $\delta(0)$ of the origin, we must have $\theta \lesssim \delta(0)$ (we are using Lemma 2.2).

If T is component triangle of S_1, let $\theta(T)$ be the maximum angle T makes with any of its neighboring triangles, and think of $\theta(z)$ as a function on S_1 that is constant on triangles. Since $\theta(z)$ can be bounded by a uniform multiple of $\delta(w)$ for a point w that is a uniform hyperbolic distance away, we get

$$\int_{S_1} \theta^2(z) dA_\rho(z) \lesssim \int_{S_1} \delta^2(z) dA_\rho(z) < \infty.$$

The principle curvatures of S_1 are zero inside each triangle and a measure along the edges. However, by smoothing S_2 we can obtain a surface S_2 so that the principle curvatures tend to zero as we approach infinity and are bounded by $O(\max_{T^*} \theta(z))$, where T^* denotes the union of all component triangles that touch T (including those that only touch at a vertex). Then

$$\int_{S_2} |K|^2(z) dA_\rho(z) \lesssim \int_{S_1} \delta^2(z) dA_\rho(z) < \infty.$$

For $n \geq 2$ essentially the same proof works if we take the dyadic dome for our triangulated surface with asymptotic boundary Γ. The angles between adjacent faces are easily bounded by the β-numbers of the corresponding arcs of Γ, which, after smoothing, proves that Definition 15 implies Definition 16. $\square$

Lemma 3.2. *For $n = 2$, Definition 16 implies Definition 3.*

Proof. This implication is due to Charles Epstein [4]. He proves that for a surface $S \subset \mathbb{R}^3_+$ whose principle curvatures $|\kappa_1(p)|, |\kappa_2(p)|$ are bounded strictly below 1, the Gauss map from the surface to the plane at infinity is quasiconformal. Recall that the Gauss map sends a point p on S to the endpoint on $\mathbb{R}^2$ of the hyperbolic geodesic ray starting at p that is normal to S. There are actually two Gauss maps from S to $\mathbb{R}^2$ depending on which "side" of S the geodesic ray is in.

In the case when the surface has asymptotic limit Γ, a curve on $\mathbb{R}^2$, the composition of one of these maps with the inverse of the other defines a quasiconformal reflection across Γ.

By Proposition 5.1 of [4], the dilatation of the composed Gauss maps is

$$\begin{aligned} D(z) &= \max\left(\left|\frac{1+\kappa_1(p)}{1-\kappa_1(p)}\cdot\frac{1-\kappa_2(p)}{1+\kappa_2(p)}\right|^{1/2}, \left|\frac{1-\kappa_1(p)}{1+\kappa_1(p)}\cdot\frac{1+\kappa_2(p)}{1-\kappa_2(p)}\right|^{1/2}\right) \\ &= 1+O(|\kappa_1(p)|+|\kappa_2(p)|), \end{aligned}$$

where $p \in S$ is the point corresponding to $z \in \mathbb{R}^2$.

Therefore the dilatation satisfies

$$|\mu(z)| = O(|\kappa_1(p)| + |\kappa_2(p)|).$$

Moreover, on page 121 of [4], Epstein shows that the Jacobian J of this map satisfies

$$C_1|(1 \mp \kappa_1)(1 \mp \kappa_2)| \leq J \leq C_2|(1 \pm \kappa_1)(1 \pm \kappa_2)|.$$

In particular, $J \simeq 1$ if $|\kappa_1|, |\kappa_2|$ are both uniformly bounded below 1.

Definition 16 implies that κ_1, κ_2 are both small outside some compact ball B around the origin. Thus the Gauss map for S defines a quasiconformal reflection in some neighborhood U of Γ and inside this neighborhood

$$\int_U |\mu(z)|^2 d\mathrm{A}_\rho(z) \lesssim \int_{S\setminus B} |\mathcal{K}_0(z)|^2 d\mathrm{A}_\rho(z),$$

where $d\mathrm{A}_\rho$ is the hyperbolic area measure on $\mathbb{R}^2 \setminus \Gamma$ and S respectively and $\mathcal{K}_0$ is the trace-free second fundamental form of S.

Extend this reflection to the rest of $\mathbb{R}^2$ by some diffeomorphism of one component of $\mathbb{R}^2 \setminus U$ to the other that agrees with the reflection given by the Gauss map on ∂U. This gives a global quasiconformal reflection across Γ that satisfies (**??**), as desired. $\square$

Recall that a Whitney decomposition of an open set $W \subset \mathbb{R}^n$ is a collection of dyadic cubes Q with disjoint interiors, whose closures cover W and which satisfy

$$\mathrm{diam}(Q) \simeq \mathrm{dist}(Q, \partial W).$$

The existence of such decompositions is a standard fact (e.g., for each $z \in W$, take the maximal dyadic cube Q so that $z \in Q \subset 3Q \subset W$, see Section I.4 of [8]).

Definition of $\rho(Q)$:

Suppose U is a neighborhood of $\Gamma \subset \mathbb{R}^n$ and $R : U \to U' \subset \mathbb{R}^n$ is a homeomorphism fixing each point of Γ. For each Whitney cube Q for $W = \mathbb{R}^n \setminus \Gamma$, with $Q \subset U$, define $\rho(Q)$ to be the infimum of values $\rho > 0$ so that R is $(1+\rho)$-biLipschitz on Q and $\mathrm{dist}(\frac{z+R(z)}{2}, \Gamma) \leq \rho \cdot \mathrm{diam}(Q)$ for $z \in Q$ (the latter condition ensures $R(z)$ is on the "opposite" side of Γ from z). R is called an involution if $R(R(z)) = z$.

Definition 13. *There is homeomorphic involution R defined on a neighborhood of Γ that fixes Γ pointwise, and so that*

$$\sum_{Q} \rho^2(Q) < \infty. \tag{3.2}$$

The sum is over all Whitney cubes for $\mathbb{R}^n \setminus \Gamma$ that lie inside U.

Lemma 3.3. *For $n \geq 2$, Definition 16 implies Definition 13.*

Proof. We consider only points z on S that are at height $\leq t_0$ above $\mathbb{R}^n$ where t_0 is chosen so small that that if $z = (x, t) \in \mathbb{R} \times (0, t_0)$, then the principle curvatures at z are all very small, say $\leq 1/100$.

There is an $(n-1)$-sphere of directions in the tangent space of $\mathbb{H}^{n+1}$ at z that are perpendicular to S. These directions define a tangent $(n-1)$-dimensional hyperbolic hyper-plane H_z that passes through z, and the boundary of H_z on $\mathbb{R}^n$ is a Euclidean $(n-2)$-sphere S_z whose center is within $O(t \cdot \sup_w \max_j |\kappa_j(w)|)$ of x. We define R on this sphere by taking the antipodal map.

We claim that such circles foliate a neighborhood U of Γ and that R is Lipschitz. If so, then R is a biLipschitz involution that fixes Γ. Let $K_r = K_r(z)$ be an upper bound for $\max |\kappa_j||$ in a hyperbolic r-ball around z.

Given $z, w \in S$ that are $t < r$ apart in the hyperbolic metric, let γ be the geodesic segment in $\mathbb{H}^{n+1}$ connecting them. The perpendicular hyperplanes H_z, H_w are both within $O(K_r)$ of orthogonal to γ and hence the corresponding spheres S_z, S_w are within $O(K_r \cdot t)$ of each other, but are also at least distance $\gtrsim K \cdot t$ apart (this is easiest to see in the ball model of hyperbolic space, setting $z = 0 \in \mathbb{B}^{n+1}$).

Thus the antipodal maps preserve distance between points on the same sphere and increase the distance between points on different spheres by at most $O(K \cdot t)$. Thus R is Lipschitz, as desired.

Moreover, if two such spheres intersect the same Whitney cube Q of $\mathbb{R}^n \setminus \Gamma$, then then both have radii $\simeq \ell(Q)$ and centers that are within $O(\ell(Q))$ of each other. Thus the corresponding points on S are within hyperbolic distance $O(1)$ of each other.

Therefore the argument above implies that $\rho(Q) = O(K_r(z))$ for some point $z \in S$ and $\sum_Q \rho^2(Q)$ is finite if $\leq \int_S |K_r(z)|^2$ is. Hence Definition 16 implies Definition 13. $\square$

Definition	Description
1	$\log f'$ in Dirichlet class
2	Schwarzian derivative
3	QC dilatation in L^2
4	conformal welding midpoints
5	$\exp(i \log f')$ in $H^{1/2}$
6	arclength parameterization in $H^{3/2}$
7	tangents in $H^{1/2}$
8	finite Möbius energy
9	Jones conjecture
10	good polygonal approximations
11	β^2-sum is finite
12	Menger curvature
13	biLipschitz involutions

14	between disjoint disks
15	thickness of convex hull
16	finite total curvature surface
17	minimal surface of finite curvature
18	additive isoperimetric bound
19	finite renormalized area
20	dyadic cylinder
21	closure of smooth curves in $T_0(1)$
22	P_φ^- is Hilbert-Schmidt
23	double hits by random lines
24	finite Loewner energy
25	large deviations of SLE(0^+)
26	Brownian loop measure

The names of 26 characterizations of Weil-Peterson curves

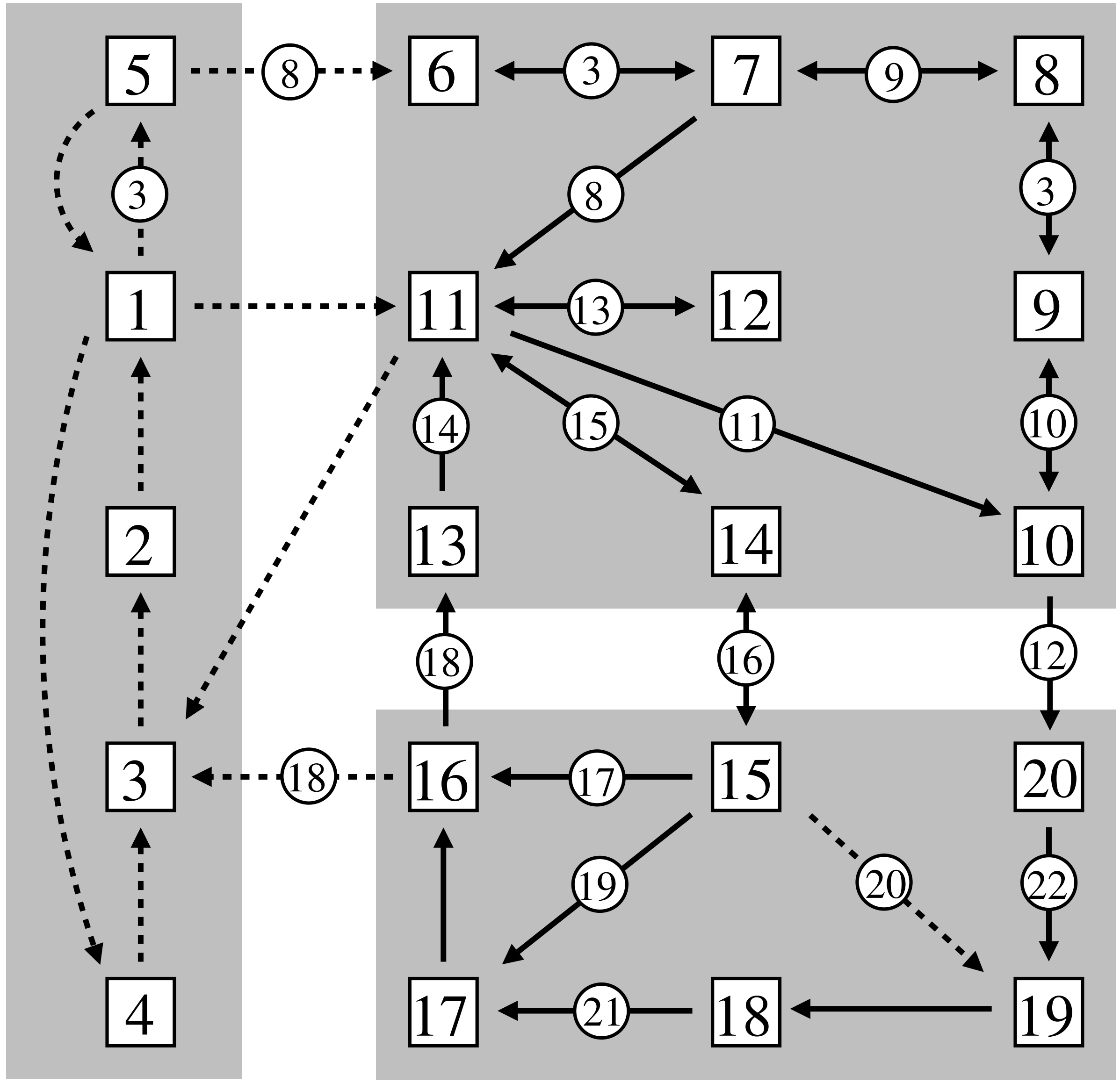


Diagram of implications between previous definitions.

Edge labels refer to sections of my preprint.

The following is due to Andrea Seppi [13]:

Lemma 3.4. *Suppose S is an embedded minimal disk in $\mathbb{B}^3$ that has an asymptotic bounding quasicircle $\Gamma \subset \mathbb{S}^2$. Suppose $0 \in S$ and that S lies between two disjoint hyperbolic planes that both at most distance ϵ from 0, one on either side of the xy-plane. Then the tangent plane of S at 0 makes angle at most $O(\epsilon)$ with the xy-plane and the absolute values of the principle curvatures of S at 0 are both bounded by $O(\epsilon)$.*

This is essentially Propositions 4.14 and 4.15 of [13]; see Equation (32) in particular.

Given a minimal surface S that is trapped between two hyperbolic planes P_-, P_+, Seppi considers the function $u(z) = \sinh(\text{dist}(z, P_-))$ for $z \in S$ and uses the fact that this satisfies the equation $\Delta_S u - 2u = 0$, where Δ_S is the Laplace-Beltrami operator for the surface S. The Schauder estimates for this equation imply that

$$\|u\|_{C^2(B(x,r/2))} \leq C\|u\|_{C^0(B(x,r))}.$$

In order to get a uniform bound for C, we must bound the curvature of S, and Seppi gives an argument for this assuming the boundary of S is a quasicircle (this covers our application, since Weil-Petersson curves are quasicircles). Finally, the sup norm of u is bounded in terms of the distance between P_- and P_+ near z, and that we have shown is $O(\delta(z))$, e.g. Lemma 2.2.

One small technical point is that Seppi requires the point z to be on a geodesic segment that meets both P_- and P_+ orthogonally. However, it is very simple to see that if z is between two disjoint hyperbolic planes that each come within ϵ of z, then there are also two disjoint planes that come within $O(\epsilon)$ and satisfy the orthogonality condition for z.

Seppi's estimate implies that near the boundary of hyperbolic space we have

$$\int_S |\mathcal{K}|^2 d\mathrm{A}_\rho \lesssim \int_{\partial \mathrm{CH}(\Gamma)} \delta^2(z) d\mathrm{A}_\rho < \infty,$$

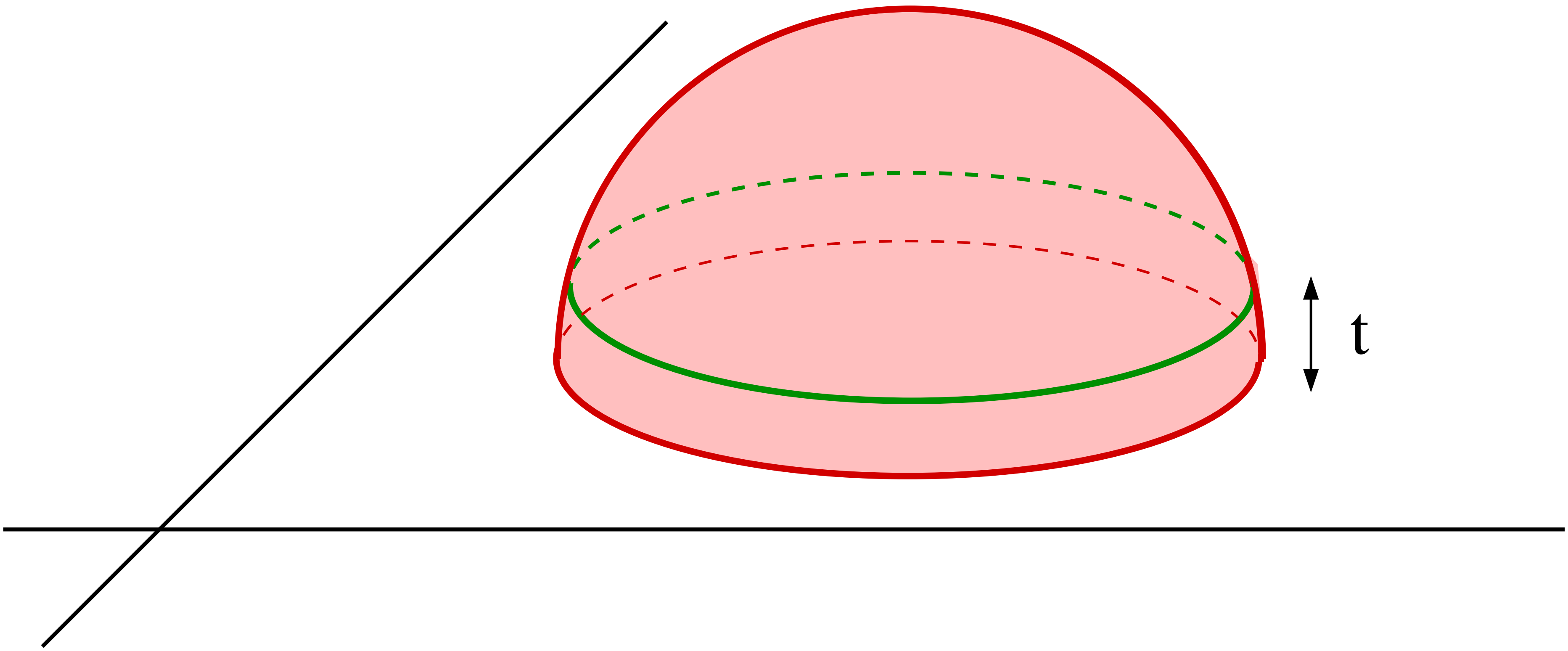
when Γ is Weil-Petersson. Thus, for $n = 2$ Definition 15 implies Definition 17.

As we discussed in the introduction to the course, a 2-surface $S \subset \mathbb{H}^{n+1}$ with boundary curve $\Gamma \subset \mathbb{R}^n$ is said to have finite renormalized area if

$$\mathcal{RA}(S) = \lim_{t \searrow 0} \left[\mathrm{A}_\rho(S_t) - L_\rho(\partial S_t) \right]$$

exists and is finite, where

$$S_t = \{(x, y, s) \in S : s \geq t\}, \quad \partial S_t = \{(x, y, s) \in S : s = t\}.$$



Lemma 3.5. *For $n = 2$, Definition 15 implies 19.*

Proof. Using the Gauss-Bonnet theorem

$$\begin{aligned}
\mathrm{A}_\rho(S_t) - L_\rho(\partial S_t) &= \int_{S_t} 1 d\mathrm{A}_\rho - \int_{\partial S_t} 1 dL_\rho \\
&= \int_{S_t} (1+\kappa^2) d\mathrm{A}_\rho - \int_{S_t} \kappa^2 d\mathrm{A}_\rho - \int_{\partial S_t} 1 dL_\rho \\
&= -\int_{S_t} K d\mathrm{A}_\rho - \int_{S_t} \kappa^2 d\mathrm{A}_\rho - \int_{\partial S_t} 1 dL_\rho \\
&= -2\pi\chi(S_t) + \int_{\partial S_t} \kappa_g dL_\rho - \int_{S_t} \kappa^2 d\mathrm{A}_\rho - \int_{\partial S_t} 1 dL_\rho \\
&= -2\pi\chi(S_t) - \int_{S_t} \kappa^2 d\mathrm{A}_\rho + \int_{\partial S_t} (\kappa_g - 1) dL_\rho
\end{aligned}$$

where κ_t is the geodesic curvature of ∂S_t in S_t.

Since we are assuming Definition 15 holds, we know from earlier results that the β's tend to zero and this implies that near the boundary, any minimal surface is nearly vertical (trapped between nearly touching hyperbolic planes) and therefore it has finite Euler characteristic.

Seppi's estimate implies
$$\int_{S_t} \kappa^2 d\mathrm{A}_\rho = O(\int_{S_t} \delta^2 d\mathrm{A}_\rho).$$
Since Γ is Weil-Petersson, our earlier results imply this integral converges to a finite limit as $t \searrow 0$.

The geodesic curvature κ_g of the boundary curve comes from two components. There is a vertical component of size 1 due to the curve lying on the horizontal plane. There is a horizontal component due to the curvature of ∂S_t in this plane. This component has size bounded by the principle curvatures of the surface, that by Seppi's estimate are bounded by $O(\delta(z))$.

The geodesic curvature κ_g is given by projecting this vector onto the tangent space of S_t, that our previous estimates show makes an angle at most $O(\delta)$ with the vertical. Thus $|\kappa_g| = 1 + O(\delta^2)$. Hence

$$\int_{\partial S_t} (\kappa_g - 1) ds = O\left(\int_{\partial S_t} \delta^2(z) ds \right) \tag{3.3}$$

Note that since δ^2 has finite integral over the whole surface its integral over the annulus $A_t = S_t \setminus S_{t+1}$ tends to zero with t. Moreover, Lemma 2.2 implies the integral of $\delta^2(z)$ over ∂S_t is dominated by a multiple of the area integral over A_t and hence the boundary integral in (3.3) must tend to zero. This proves the lemma. $\square$

The estimate $|\kappa_g| = 1 + O(\delta^2)$ also follows from from Equation (2.4) of [3]:

$$\kappa_g = \frac{1}{\nabla r}(\coth r + \langle \mathcal{K}(e, e), \nabla \perp r\rangle),$$

where r is the hyperbolic distance to some fixed point (say the origin in the ball model), Dr is the gradient of r in $\mathbb{H}^{n+1}$, ∇r is the projection of Dr onto the tangent space of S, $\nabla^{\perp} r$ is the projection of Dr onto the normal space of S, and $\mathcal{K}$ is the second fundamental form of S.

Corollary 3.6. *Suppose $S \cup_n K_n \subset \mathbb{R}^3_+$ is a minimal surface where $K_1 \subset K_2 \subset \ldots$ are nested compact sets such that $S \setminus K_n$ is a topological annulus for all n. Then*

$$-2\pi\chi(S) - \int_S \kappa^2(z) d\mathrm{A}_\rho, = \mathcal{RA}(S) = \lim_{n\to\infty} \sup_{\Omega\supset K_n} [\mathrm{A}_\rho(\Omega) - L_\rho(\partial\Omega)]$$

where the supremum is over compact domains $K_n \subset \Omega \subset S$ bounded by a single Jordan curve. As above, either all terms are finite and equal, or all are $-\infty$.

Proof of Corollary 3.6. The inequality

$$\mathcal{RA}(S) \leq \sup\{\mathrm{A}_\rho(\Omega) - L_\rho(\partial\Omega)\}$$

is obvious since the truncated surfaces in the definition of $\mathcal{RA}(S)$ are among the domains used in the supremum on the right.

To prove the other direction note that if $D(z, R) \subset \Omega$, then $\chi(S) = \chi(\Omega) = \chi(\Omega_t)$ for all $0 \leq t \leq T/2$ if R is large enough. Then by Lemma 3.7

$$\mathrm{A}_\rho(\Omega) - L_\rho(\partial\Omega) \leq -(2\pi \mp \epsilon)\chi(D(z, R/2)) - (1-\epsilon)\int_{D(z,R/2)} \kappa^2 d\mathrm{A}_\rho.$$

Taking $R \nearrow \infty$, and applying the Monotone Convergence Theorem, we get

$$\mathrm{A}_\rho(\Omega) - L_\rho(\partial\Omega) \leq -(2\pi \mp \epsilon)\chi(\Omega) - (1-\epsilon)\int_S \kappa^2 d\mathrm{A}_\rho.$$

Then taking $\epsilon \searrow 0$ gives

$$\limsup_{R\nearrow\infty} \sup_{\Omega:\Omega\supset D(z,R)} \mathrm{A}_\rho(\Omega) - L_\rho(\partial\Omega) \leq -2\pi\chi(\Omega) - \int_S \kappa^2 d\mathrm{A}_\rho. \quad \square$$

Suppose that $S \subset \mathbb{H}^{n+1}$ is a minimal surface with asymptotic boundary $\Gamma \subset \mathbb{R}^n$. As before, for $t > 0$ let $S_t = S \cap \{(x, s) \in \mathbb{R}^n \times (t, \infty)\}$ be the part of S above height t and let $S_t^* = S \setminus S_t$ be the part below height t.

We assume that for t small enough, S_t^* is real analytic and a topological annulus. Suppose $\Omega \subset S_t^*$ is a compact sub-annulus with one boundary component equal to $\Gamma_t = S \cap \mathbb{R}^n \times \{t\}$, and the other boundary component a smooth curve Γ.

Let $T = T(\Omega)$ be the distance in S between Γ and Γ_t. For $0 \le s \le T$, let

$$\Omega(s) = \{z \in \Omega : d_S(z, \Gamma) > s\}, \quad \Gamma(s) = \{z \in \Omega : d_S(z, \Gamma) = s\}.$$

Here d_S refers to distance on the surface S. Note that $\Gamma(0) = \partial\Omega$ and $\Omega(0) = \Omega$. Also note that $\chi(\Omega) = 0$ (it is an annulus) and $\chi(\Omega(s)) \ge 0$ since $\Omega(s)$ is the union of a topological annulus and possibly some disks.

Let $A(s)$ be the hyperbolic area of $\Omega(s)$ and $L(s)$ the hyperbolic length of $\Gamma(s) = \partial\Omega(s) \setminus \Gamma_t$. In particular, $A(0) = \mathrm{A}_\rho(\Omega)$ and $L(0) = L_\rho(\Gamma)$.

The Gauss-Bonnet theorem says that

$$\int_{\Omega(s)} K d\mathrm{A}_\rho + \int_{\partial\Omega(s)} \kappa_g dL_\rho = 2\pi\chi(\Omega(s))$$

where κ_g is the geodesic curvature of $\partial\Omega$ in Ω. For points in $\Gamma_t \subset \partial\Omega$, this is the negative of κ_g^S, the geodesic curvature of Γ_t in S_t. Since $\partial\Omega(s) = \Gamma_t \cup \Gamma(s)$ and $\chi(\Omega(s)) \geq 0$, we get

$$-\int_{\Gamma(s)} \kappa_g dL_\rho = \int_{\Gamma_t} \kappa_g dL_\rho + \int_{\Omega(s)} K d\mathrm{A}_\rho - 2\pi\chi(\Omega(s)) \leq \int_{\Gamma_t} \kappa_g dL_\rho + \int_{\Omega(s)} K d\mathrm{A}_\rho$$

Lemma 3.7. *Suppose $\frac{2}{T} < \epsilon \leq 1$. With notation as above,*

$$L_\rho(\partial\Omega) - \mathrm{A}_\rho(\Omega) \geq -C(S,t) + (1-\epsilon)\int_{\Omega(1/\epsilon)} \kappa^2 d\mathrm{A}_\rho,$$

where

$$C(S,t) = \max\left(\int_{\Gamma_t} \kappa_g^S dL_\rho, L_\rho(\Gamma_t)\right).$$

Proof. This follows from known facts about the isoperimetric inequality on negatively curved surfaces. Our presentation follows that of Chavel and Feldman [2], although they attribute the basic facts to Faila [7].

As shown in [7], the function $A(s)$ is continuously differentiable and decreasing on $[0,T]$, and $A'(s) = -L(t)$ (Theorem 5 of [7]). Similarly, by Theorem 3 of [7], $L(s)$ is continuous on $[0,T]$, analytic except for finitely many points, and (except for these points)

$$L'(s) \leq -\int_{\Gamma(s)} \kappa_g dL_\rho.$$

Using the remarks about Gauss-Bonnet before the lemma, we get

$$L'(s) \leq \int_{\Gamma_t} \kappa_g dL_\rho + \int_{\Omega(s)} K d\mathrm{A}_\rho. \tag{3.4}$$

Thus

$$\begin{aligned} L'(s) - A'(s) \ &\leq \ \int_{\Gamma_t} \kappa_g dL_\rho + \int_{\Omega(s)} K d\mathrm{A}_\rho + L(s) \\ &= \ \int_{\Gamma_t} \kappa_g dL_\rho - \int_{\Omega(s)} (1+\kappa^2) d\mathrm{A}_\rho + L(s), \end{aligned}$$

which implies

$$L'(s) - A'(s) \ \leq \ L(s) - A(s) + \int_{\Gamma_t} \kappa_g dL_\rho - \int_{\Omega_s} \kappa^2 d\mathrm{A}_\rho. \tag{3.5}$$

By the isoperimetric inequality for surfaces with $K \leq -1$ (e.g., Equation (4.30) of [12]), we have

$$L_\rho(\partial\Omega(s))^2 = (L(s) + L_\rho(\Gamma_t))^2 \geq 4\pi\chi(\Omega_s)A(s) + A(s)^2,$$

and this implies

$$L(s) - A(s) \geq \frac{4\pi\chi(\Omega_s)A(s)}{L(s) + L_\rho(\Gamma_t) + A(s)} - L_\rho(\Gamma_t) \geq -L_\rho(\Gamma_t) \tag{3.6}$$

since $\chi(\Omega_s) \geq 0$.

Assume for the moment that

$$L(0) - A(0) \leq -L_\rho(\Gamma_t) + \int_{\Omega(1/\epsilon)} \kappa^2 d\mathrm{A}_\rho. \tag{3.7}$$

Then we claim there must be a $s \in [0, 1/\epsilon]$ so that

$$L'(s) - A'(s) \geq -\epsilon \int_{\Omega_{1/\epsilon}} \kappa^2 d\mathrm{A}_\rho. \tag{3.8}$$

If not, then by integrating and using (3.7) we get

$$\begin{aligned} L(\frac{1}{\epsilon}) - A(\frac{1}{\epsilon}) &= L(0) - A(0) + \int_0^{1/\epsilon} L'(x) - A'(x) dx \\ &< -L_\rho(\Gamma_t) + \int_{\Omega(1/\epsilon)} \kappa^2 d\mathrm{A}_\rho + \frac{1}{\epsilon}\left[-\epsilon \int_{\Omega(1/\epsilon)} \kappa^2 d\mathrm{A}_\rho\right] \\ &= -L_\rho(\Gamma_t) \end{aligned}$$

which contradicts (3.6) for $s = 1/\epsilon$, proving there is at least one such point s.

Let a be the infimum of values s where (3.8) holds. Since we have assumed that κ is not constant zero, this bound is negative if ϵ is small enough (which forces T to be large). Thus $L(s) - A(s)$ has a negative derivative except for finitely many points in $[0, a]$ and therefore $L(a) - A(a) \leq L(0) - A(0)$. Using (3.8) and (3.5) with $s = a$,

$$\begin{aligned} -\epsilon \int_{\Omega(1/\epsilon)} \kappa^2 d\mathrm{A}_\rho &\leq L'(a) - A'(a) \\ &\leq L(a) - A(a) + \int_{\Gamma_t} \kappa_g dL_\rho - \int_{\Omega(a)} \kappa^2 d\mathrm{A}_\rho \\ &\leq L(0) - A(0) + \int_{\Gamma_t} \kappa_g dL_\rho - \int_{\Omega(a)} \kappa^2 d\mathrm{A}_\rho \end{aligned}$$

This implies

$$L(0) - A(0) \geq -\int_{\Gamma_t} \kappa_g dL_\rho + \int_{\Omega_t} \kappa^2 d\mathrm{A}_\rho - \epsilon \int_{\Omega_E} \kappa^2 d\mathrm{A}_\rho.$$

Now since $0 \leq a \leq 1/\epsilon$, we have $\Omega(1/\epsilon) \subset \Omega(a)$, so

$$\int_{\Omega(a)} \kappa^2 d\mathrm{A}_\rho - \epsilon \int_{\Omega(1/\epsilon)} \kappa^2 d\mathrm{A}_\rho \ \geq \ (1-\epsilon) \int_{\Omega(a)} \kappa^2 d\mathrm{A}_\rho \geq (1-\epsilon) \int_{\Omega(1/\epsilon)} \kappa^2 d\mathrm{A}_\rho$$

and hence

$$L(0) - A(0) \geq -\int_{\Gamma_t} \kappa_g dL_\rho + (1-\epsilon) \int_{\Omega(1/\epsilon)} \kappa^2 d\mathrm{A}_\rho. \tag{3.9}$$

Thus either (3.7) fails or (3.9) holds. In either case we have proven the lemma.

$\square$

Lemma 3.8. *Definition 18 implies 17.*

Proof. Fix a point $z \in S$ and a large disk $D = D(z, R)$ around z. For n large enough, Ω_n contains $D(z, 2R)$ and so $\Omega_n(R)$ contains $D(z, R)$. So if R is large enough, κ is as small as we wish in $\Omega_n^*(R) = \Omega_n \setminus \Omega_n(R)$. Lemma 3.7 with $\epsilon = 1/2$ then implies

$$\int_{D(z,R)} \kappa^2 d\mathrm{A}_\rho \leq 2C(S, t) + 2[L_\rho(\partial\Omega_n) - \mathrm{A}_\rho(\Omega_n)].$$

The first term on the right is independent of n, and Definition 18 says the second term is bounded independent of n. Therefore

$$\int_{D(z,R)} \kappa^2 d\mathrm{A}_\rho = O(1),$$

with a bound independent of R. Taking $R \nearrow \infty$ and applying the Monotone Convergence Theorem shows $\int_{S_t^*} \kappa^2 d\mathrm{A}_\rho < \infty$, as desired. $\square$

Definition	Description
1	$\log f'$ in Dirichlet class
2	Schwarzian derivative
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The names of 26 characterizations of Weil-Peterson curves

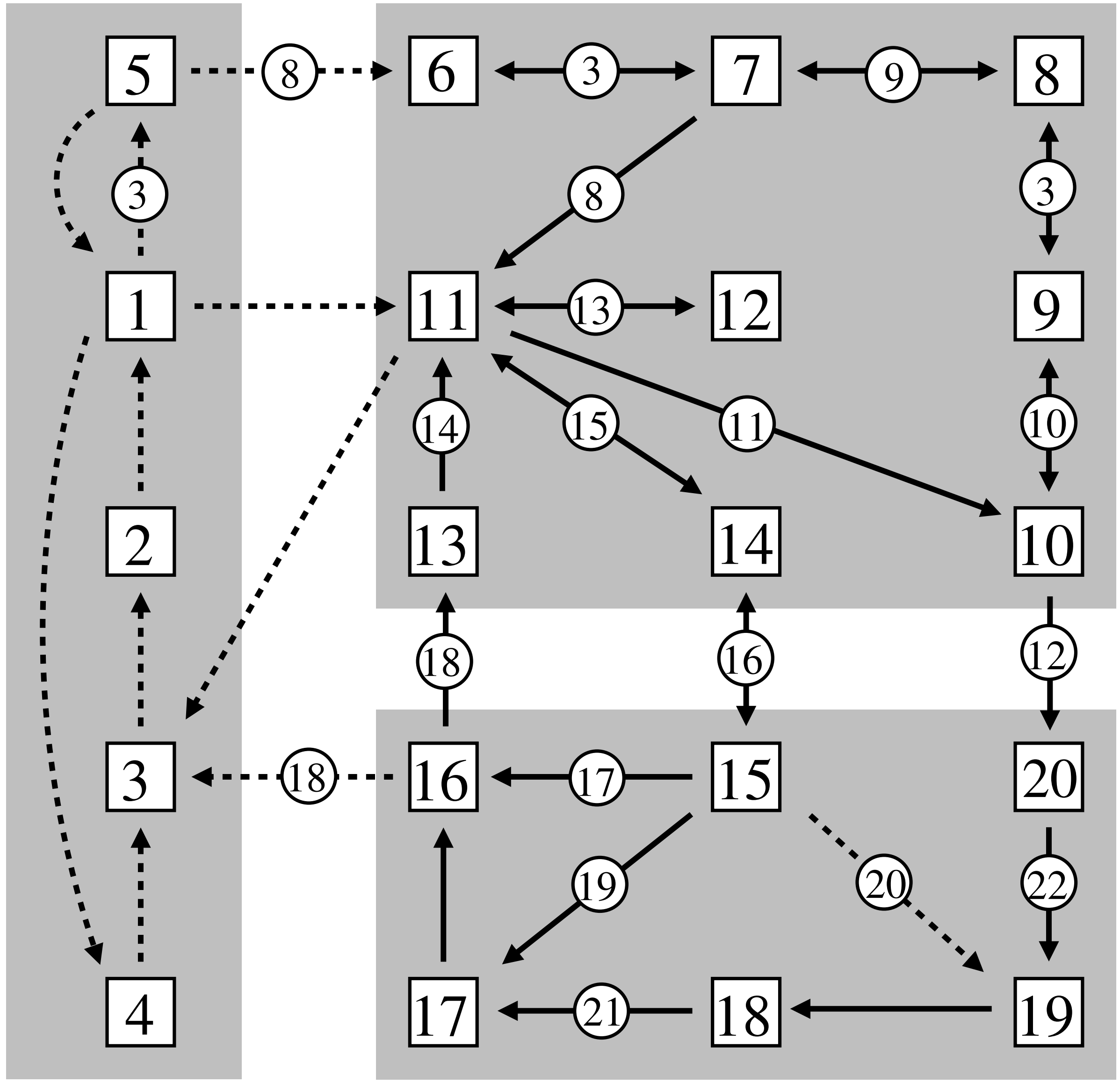


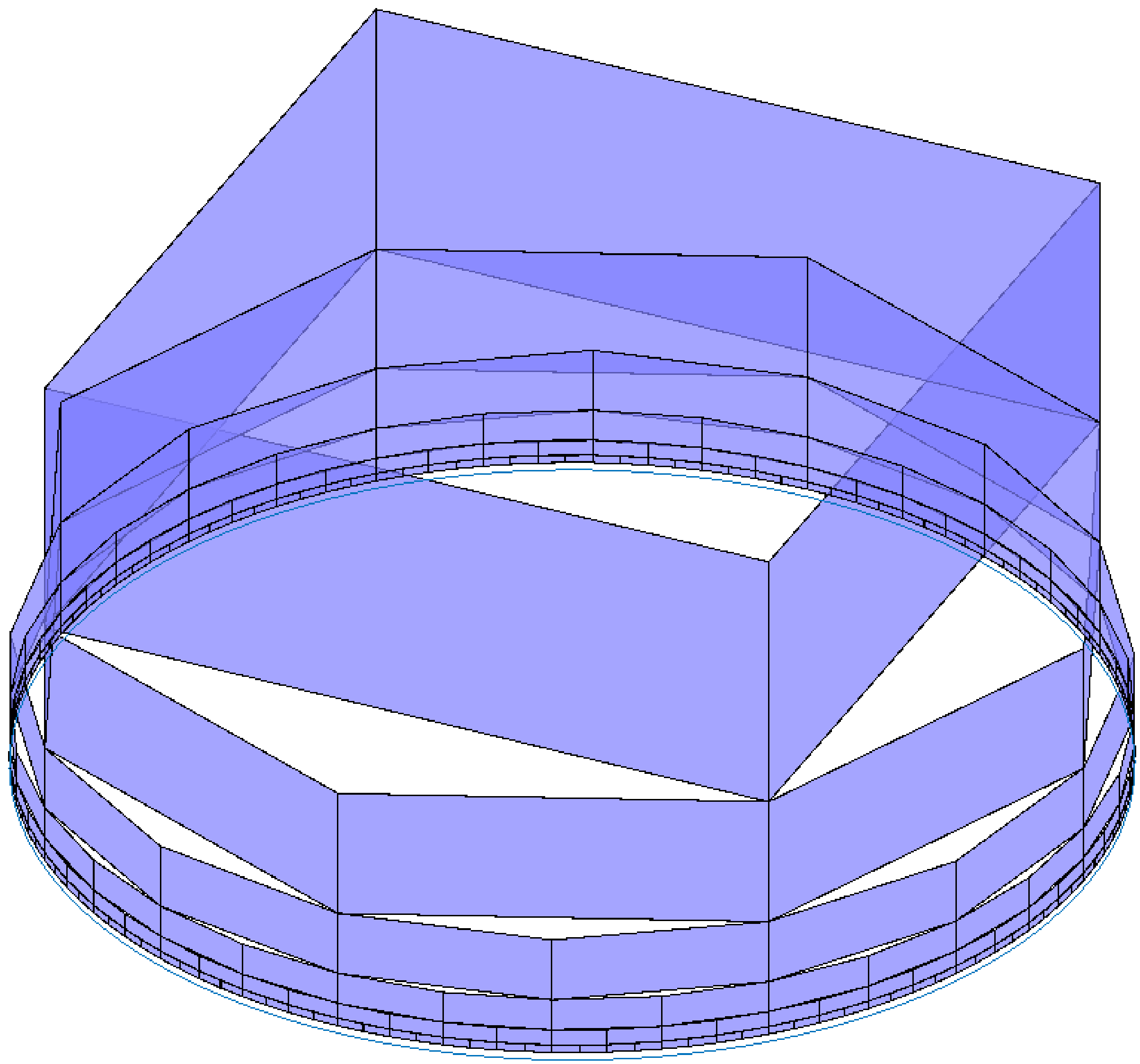
Diagram of implications between previous definitions.

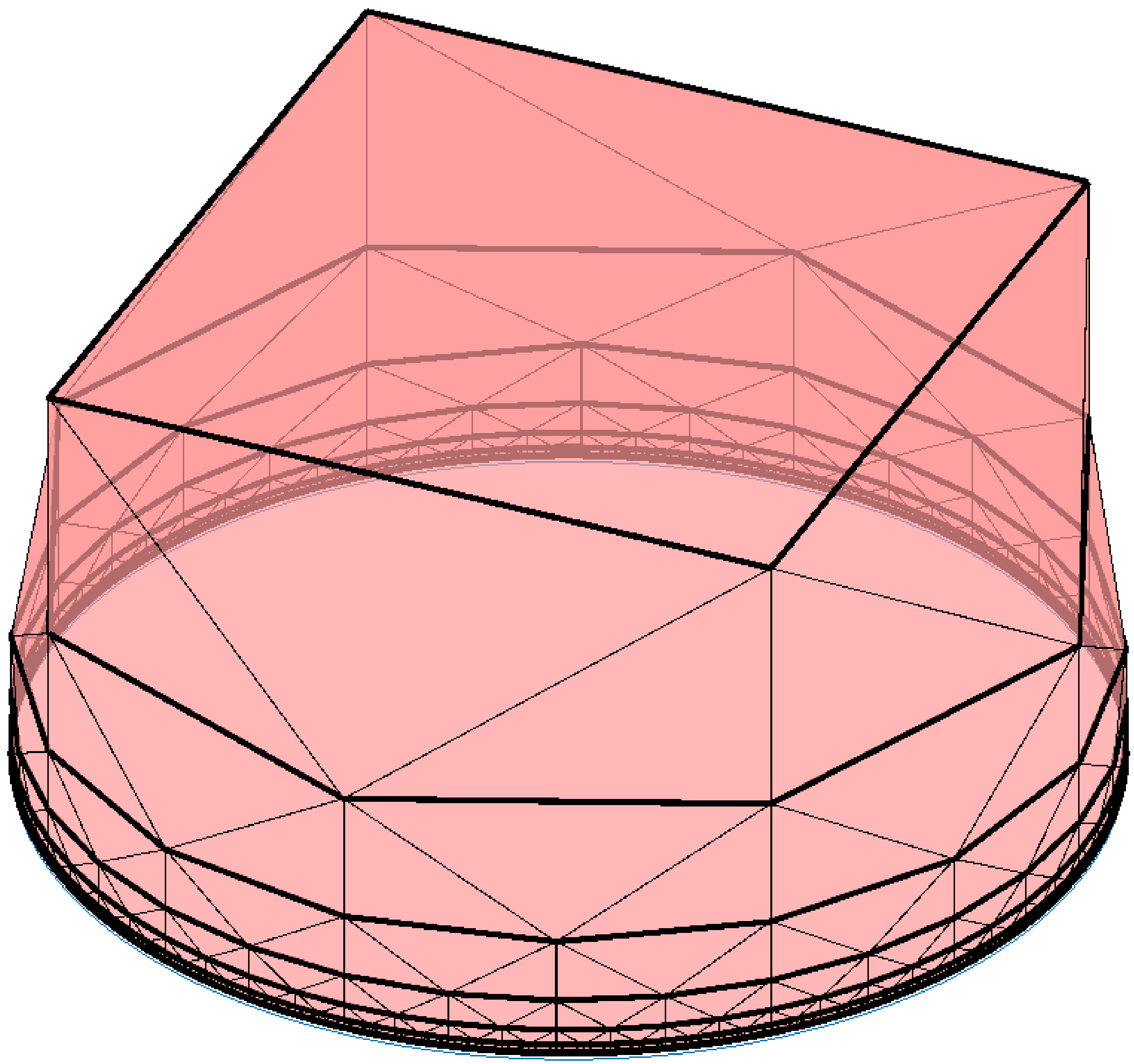
Edge labels refer to sections of my preprint.

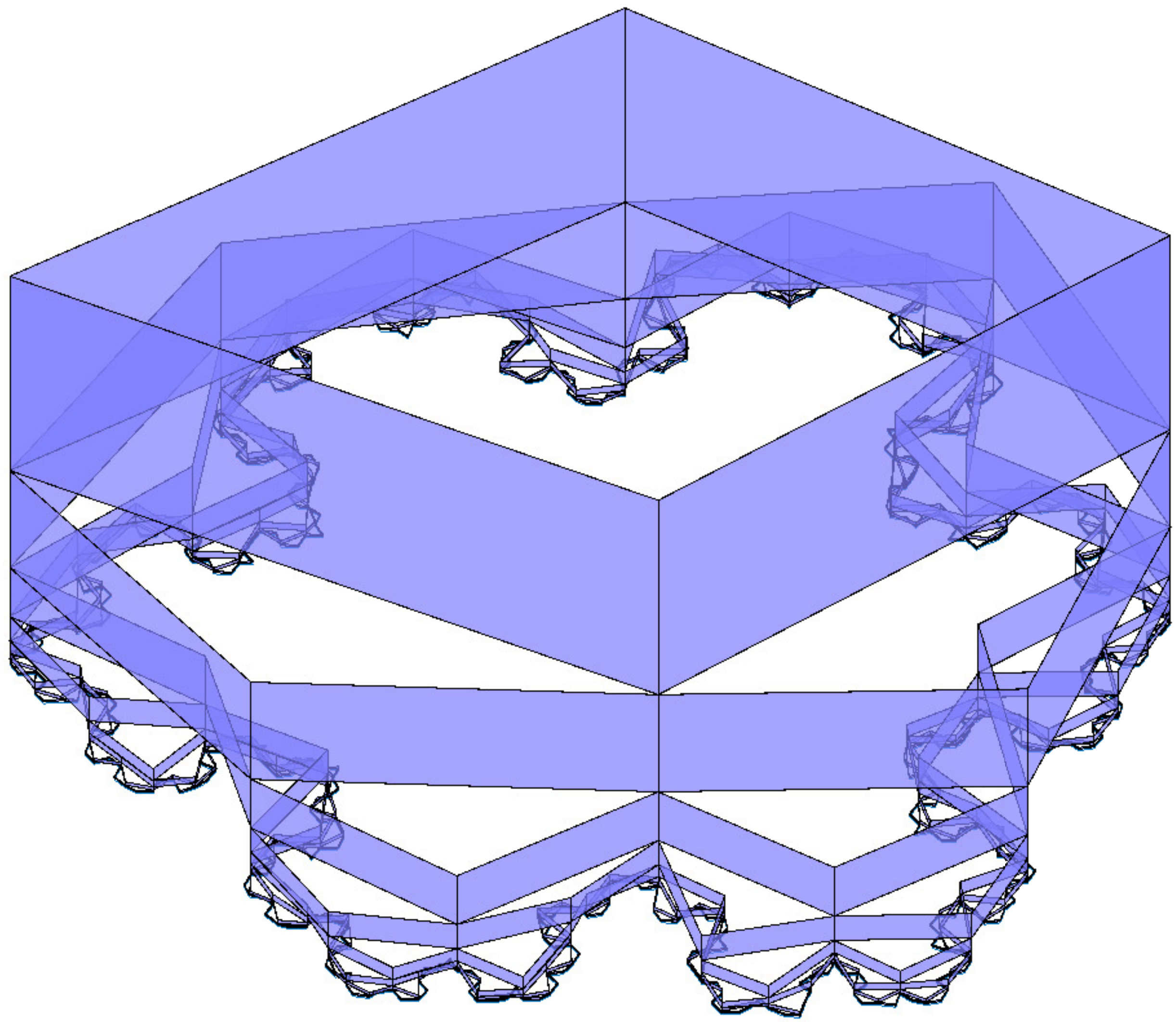
Next we discuss an idea that combines our Eulcidean and hyperbolic characterizations in one object.

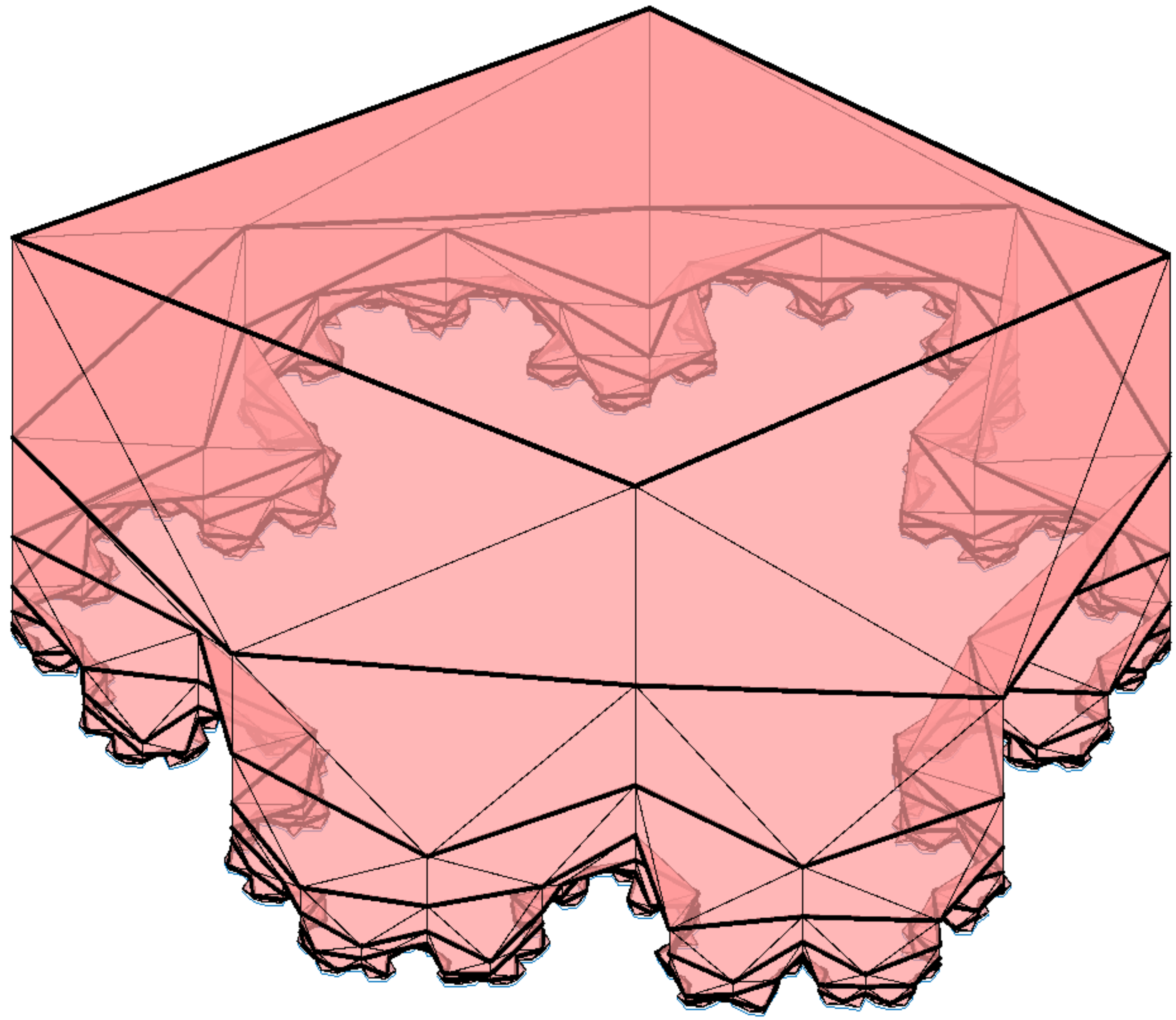
Define a "dyadic cylinder" associated to Γ by $X = \bigcup_{n=0}^{\infty} \Gamma_n \times (2^{-n-1}, 2^{-n}]$, where Γ_n is the 2_n-gon inscribed in Γ corresponding to a dyadic decomposition of Γ into subarcs of length $2^{-n}\ell(\Gamma)$.

Note that each "layer" of X between heights 2^{-n} and 2^{-n+1} consists of 2^n Euclidean rectangles (or "panels") in vertical planes that meet along vertical edges (called "hinges"). Alternate vertices of the top edge of one layer agree with the bottom vertices of the next layer up, but there are triangular horizontal "holes" between the layers.









Lemma 3.9. *If Γ a closed rectifiable Jordan curve, then Γ is Weil-Petersson if and only if every corresponding dyadic cylinder X has finite renormalized area.*

Proof. First we show that the Weil-Petersson condition implies finite renormalized area. A simple calculation as above shows that the part of X between heights 2^{-n} and 2^{-n+1} has hyperbolic area $2^{n-1}\ell(\Gamma_n)$.

Similarly, if $2^{-n-1} \leq t \leq 2^{-n}$, then

$$\mathrm{A}_\rho(X_t) = \sum_{k=0}^{n} 2^{k-1}\ell(\Gamma_k) + (\frac{1}{t} - 2^n)\ell(\Gamma_{n+1}).$$

and hence

$$\begin{aligned}
\mathrm{A}_\rho(X_t) - \frac{1}{t}\ell(\Gamma) &= \mathrm{A}_\rho(X_t) - (\frac{1}{t} - 2^n + 1 + \sum_{k=1}^{n} 2^{k-1})\ell(\Gamma) \\
&= -\ell(\Gamma) - \sum_{k=1}^{n} 2^k[\ell(\Gamma) - \ell(\Gamma_k)] + (\frac{1}{t} - 2^n)(\ell(\Gamma) - \ell(\Gamma_{n+1}) \\
&= -\ell(\Gamma) - \sum_{k=1}^{n} 2^k[\ell(\Gamma) - \ell(\Gamma_k)] + O(2^n[\ell(\Gamma) - \ell(\Gamma_{n+1})]) \\
&\to -\ell(\Gamma) - \sum_{k=1}^{\infty} 2^k[\ell(\Gamma) - \ell(\Gamma_k)]
\end{aligned}$$

since the infinite series is convergent when Γ is Weil-Petersson.

Finally, for $2^{-n-1} \leq t \leq 2^{-n}$, note that $\ell(\partial X_t) = \ell(\Gamma_{n+1})/t$, so

$$\frac{1}{t}[\ell(\partial X_t) - \ell(\Gamma)] \leq 2^{n+1}[\ell(\Gamma_{n+1}) - \ell(\Gamma)] \to 0,$$

since these are terms of a summable series. Thus $\mathrm{A}_\rho(X_t) - L_\rho(\partial X_t)$ has a finite limit and X has finite renormalized area.

Next we consider the converse: finite renormalized area implies Γ is Weil-Petersson. Suppose $\mathcal{RA}(X) < \infty$. First we deduce that Γ is rectifiable. If $t = 2^{-n}$, then

$$\mathrm{A}_\rho(X_t) - L_\rho(\partial X_t) = \left(\sum_{k=1}^{n} 2^{k-1}\ell(\Gamma_k)\right) - 2^n\ell(\Gamma_n) = O(1),$$

or equivalently,

$$\ell(\Gamma_n) = \frac{1}{2}\ell(\Gamma_n) + \frac{1}{4}\ell(\Gamma_{n-1}) + \cdots + 2^{-n}\ell(\Gamma_1) + O(2^{-n}),$$

and hence (since $\{\ell(\Gamma_n)\}$ is non-decreasing),

$$\begin{aligned}
\ell(\Gamma_n) &= \frac{1}{2}\ell(\Gamma_{n-1}) + \frac{1}{4}\ell(\Gamma_{n-2}) + \cdots + O(2^{-n}) \\
&\le \frac{1}{2}\ell(\Gamma_{n-1}) + \frac{1}{4}\ell(\Gamma_{n-1}) + \cdots + O(2^{-n}) \\
&\le \ell(\Gamma_{n-1}) + O(2^{-n})
\end{aligned}$$

which clearly implies $\ell(\Gamma) < \infty$.

To show that Γ is Weil-Petersson, note that

$$\begin{aligned}
\mathrm{A}_\rho(X_t) - L_\rho(\partial X_t) &= \left(\sum_{k=1}^{n} 2^{k-1}\ell(\Gamma_k)\right) - 2^n \ell(\Gamma_n) \\
&= \left(\sum_{k=1}^{n} 2^{k-1}\ell(\Gamma_k)\right) - (1+1+2+\ldots 2^{n-1})\ell(\Gamma_n) \\
&= -\frac{1}{2}\sum_{k=1}^{n} 2^k[\ell(\Gamma_n) - \ell(\Gamma_k)] - \ell(\Gamma_n).
\end{aligned}$$

By the Monotone Converge Theorem (for counting measure on $\mathbb{N}$), this tends to

$$-\tfrac{1}{2}\textstyle\sum_{k=1}^{\infty} 2^k[\ell(\Gamma) - \ell(\Gamma_k)] - \ell(\Gamma).$$

Thus if $\mathrm{A}_\rho(X_t) - L_\rho(\partial X_t)$ is bounded below, then

$$\sum_{k=1}^{\infty} 2^k [\ell(\Gamma) - \ell(\Gamma_k)] < \infty,$$

with a bound independent of the choice of the dyadic decomposition.

Hence finite renormalized area implies Γ is Weil-Petersson. $\square$

Next we want to show that finite renormalized area for the dyadic cylinder implies it for the minimal surface.

Lemma 3.10. *Let X denote the dyadic cylinder associated to Γ. If X has finite renormalized area, then $\mathrm{A}_\rho(S_t) - \mathrm{A}_\rho(X_t)$ has a finite limit as $t \searrow 0$*

Proof. If Definition 20 holds, so does Definition 14. For each vertical rectangle R making up a side (or a "panel") of X, we have a Lipschitz map from this panel to a portion of S that changes area by at most an additive factor of $O(\varepsilon_\Gamma^2(Q))$, where Q is the dyadic cube associated to the center of R.

Due the vertical "hinges" between adjacent panels, some points of S might be hit twice or not at all by the Lipschitz maps associated to those panels. However, the angles between these panels are bounded by $O(\varepsilon_\Gamma(Q))$ and hyperbolic distance between S and X is also bounded by $O(\varepsilon_\Gamma(Q))$. Thus the total error is at most $O(\varepsilon_\Gamma^2(Q))$, which is summable over all the panels of X. Thus the difference between the hyperbolic areas of S and X above height t has a finite limit $t \searrow 0$. □

Lemma 3.11. *With X as above, $L_\rho(S_t) - L_\rho(X_t)$ had a finite limit as $t \searrow 0$*

Proof. The same argument as in the previous lemma works again: the Lipschitz map from each panel of X to S, preserves length up to an additive factor of $O(\varepsilon_\Gamma^2(Q))$ and the errors caused by the corners are bounded by the same magnitude. $\square$

So if the dyadic cylinder has finite renormized area, then $\lim_{t \searrow 0} \mathrm{A}_\rho(X_t) - L_\rho(\partial X_t)$ exists and is finite. The preceding lemmas imply the same for S, so it also has finite renormalized area.

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