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COMPACTIFICATION OF MINIMAL SUBMANIFOLDS OF HYPERBOLIC SPACE

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In this paper we study the geometry of complete minimal submanifolds of hyperbolic space $\mathbb{H}^n$. Specifically, we are interested in m-dimensional submanifolds whose second fundamental form $\mathcal{A}$ satisfies $\int_M |\mathcal{A}|^m < \infty$ where $|\mathcal{A}|$ is the norm of $\mathcal{A}$.

To motivate this hypothesis we briefly outline the main results when the ambient space is $\mathbb{R}^n$. Osserman [15] and Chern-Osserman [3], showed that for a complete minimal immersion (cmi for short) $M^2 \to \mathbb{R}^n$, with finite total curvature, it is possible to compactify M by the Gauss map $g\colon M \to G_{n,2}$ which maps $p \in M$ to the 2-plane $T_p(M)$. By the Weierstrass representation g is a holomorphic curve in $G_{n,2}$, viewed as the complex quadric $Q_{n-2} = \{z_1^2 + \cdots + z_n^2 = 0\}$ of the complex projective plane $\mathbb{C}P^{n-1}$. They showed that when the total curvature $C(M) = \int_M K$ is finite, M is of finite conformal type, i.e., M is conformally equivalent to a closed surface $\overline{M}$ with a finite number of points removed, and that g extends holomorphically to $\overline{M}$. In particular this implies that the total curvature is quantified by $C(M) = 2\pi k$, k an integer, and that M is properly immersed.

For a cmi $M^m \hookrightarrow \mathbb{R}^n$, $m \geq 2$, Anderson [2] has obtained a generalization of the Chern-Osserman result. He proved that $|\mathcal{A}|(p)$ goes to 0, as the distance $d(p, p_0)$ of p to a fixed point p_0 goes to infinity. Using the fact that the class of minimal submanifolds is invariant by the homotheties of $\mathbb{R}^n$, he proved that $|\mathcal{A}|(p) = \mu(p)/d^2(p,p_0)$, where $\mu(p) \to 0$ as $d(p,p_0) \to \infty$. Analysing the distance function of $\mathbb{R}^n$ restricted to M he concludes that M is properly immersed and that, outside a compact set, M is transversal to the spheres S_r

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of $\mathbb{R}^n$, with radius r and centered in p_0. In particular M is of finite topological type. Also, the flatness of $\mathbb{R}^n$ allows him to conclude the conformal type of M is finite. We state the result of [2] which will be our main concern.

Theorem 0.1 (Anderson). *Let $M^m \hookrightarrow \mathbb{R}^n$ be a cmi and suppose that $\int_M |\mathcal{A}|^m < \infty$. Then M is C^∞-diffeomorphic to a closed manifold $\overline{M}$ with a finite numbers of points removed. Also the Gauss map $g\colon M \to G_{n,m}$ extends to a C^{n-2} map $\bar{g}\colon \overline{M} \to G_{n,m}$ and the metric on M extends conformally to a metric of class C^{n-2} of $\overline{M}$.*

Thus each end of M^m is diffeomorphic to $S^{m-1} \times [0, \infty)$. Furthermore, Anderson proves also that in the case $m \geq 3$ all ends are embedded.

It is natural to consider the above problem when the ambient space is $\mathbb{H}^n$. We make use of the Sobolev inequalities [12] and of Simons equation [17] for the Laplacian of $\mathcal{A}$ on M to show that $|\mathcal{A}|(p)$ goes to zero as $\mathrm{dist}_M(p, p_0) \to \infty$, p_0 a fixed point of M. We do not have an estimate for the decreasing rate of $|\mathcal{A}|$ as good as in the Euclidean case, but the properties of the distance function of $\mathbb{H}^n$ restricted to M will allow us to bypass the absence of homotheties in $\mathbb{H}^n$ to conclude that M is properly immersed and meets transversally the geodesic spheres S_r of $\mathbb{H}^n$, at least outside some compact set of M.

For the special case of a cmi $M^2 \hookrightarrow \mathbb{H}^n$, we prove that M cannot have finite conformal type. Also we prove that the index of the operator $\mathbf{L} = -\Delta + 2 - |\mathcal{A}|^2$ is finite. When $n = 3$ this is just the stability operator. This extends in one direction a result of Fisher-Colbrie [6], namely, finite total "extrinsic" curvature $\int_M |\mathcal{A}|^2 < \infty$ implies the index of M is finite (the reciprocal assertion fails in the hyperbolic case). Here are the main results we will prove in this paper.

Theorem A. *Let $\varphi\colon M^m \hookrightarrow \mathbb{H}^n$ be a complete minimal immersion of a connected m-dimensional manifold M. Suppose that $\int_M |\mathcal{A}|^m < \infty$. Then M is properly immersed and is diffeomorphic to the interior of a compact manifold $\overline{M}$ with boundary. Furthermore φ extends to a continous map $\bar{\varphi}\colon \overline{M} \hookrightarrow \overline{\mathbb{H}}^n$, $\overline{\mathbb{H}}^n$ the compactified of $\mathbb{H}^n$.*

In the case of a minimal surface M we have information about the conformal type and the asymptotic behavior of M.

Theorem B. *Let $M^2 \hookrightarrow \mathbb{H}^n$ be a complete connected minimal surface with $\int_M |\mathcal{A}|^2 < \infty$. Then M is conformally equivalent to a compact surface $\overline{M}$ with a finite number of disks removed and the index of the operator $\mathbf{L} = -\Delta + 2 - |\mathcal{A}|^2$ is finite. Furthermore the asymptotic boundary $\partial_\infty M$ is a Lipschitz curve.*

We remark that the asymptotic behaviour of an immersion $M^m \hookrightarrow \mathbb{H}^n$ as above is very different from the situation in $\mathbb{R}^n$. In fact, any compact closed submanifold $V^{n-2} \subset \mathbb{H}^n$ of class $C^{2+\alpha}$, $\alpha > 0$, can be realized as the asymptotic boundary of a minimizing rectifiable current T^{n-1} of $\mathbb{H}^n$ [1]. The regularity result of Hardt-Lin [11] states that such a current is of class $C^{2+\beta}$, $\beta > 0$, in a neighbourhood of the sphere at infinity $\partial_\infty \mathbb{H}^n$. When $n \leq 7$, T^{n-1} is a smooth submanifold of $\mathbb{H}^n$. A direct calculation shows us that for a cmi $M^m \hookrightarrow \mathbb{H}^n$, which extends to a C^2-submanifold of $\overline{\mathbb{H}}^n$, we always have $\int_M |\mathcal{A}|^m < \infty$. This provides us with a lot of hypersurfaces satisfying the hypotheses of theorem A and having arbitrary topological type at infinity, as long as $n \leq 7$.

In view of theorem B a natural question arises : how regular at infinity is a surface satisfying the hypotheses of theorem B? It seems to the author that a C^1 regularity up to the boudary is necessary.

In section 1 we establish some notations and we prove a result about the essential spectrum of the Schrödinger operator over a complete Riemannian manifold. The index of this operator is also defined. In section 2 we develop the basic properties of the distance function of $\mathbb{H}^n$ when restricted to a submanifold. We prove a compactification theorem for submanifolds whose second fundamental form is small outside some compact set. In section 3 we prove the analytical part of theorem A and B and we make use of the results in section 2 to conclude the topological type is finite and that the immersion extends continuously to the compactified of M. The assertion about the conformal type in theorem B is proved in section 4.

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1. Notation and Known Results

1.1. Minimal submanifolds. Let $M^m \hookrightarrow N^n$ be a immersion of a m-dimensional manifold M into a n-dimensional Riemannian manifold N. Consider M with the metric induced by this immersion and denote by $\tilde{\nabla}$ and ∇ the Levi-Civita connexions of N and M respectively. For $p \in M$, the tangent space T_pN of N at p splits as an orthogonal direct sum $T_PN = T_pM \oplus \mathcal{N}_p(M)$, where $\mathcal{N}_p(M)$ is the normal fiber to M at p. The second fundamental form of the immersion is the symmetric bilinear form over T_pM defined by

$$\mathcal{A}(X_p, Y_p) = (\tilde{\nabla}_X Y)^{\perp}(p) \quad ; \quad X_p, Y_p \in T_pM$$

where X, Y are extensions of X_p and Y_p which are tangent to M.

Let us consider $\mathcal{A}$ as an element of $\mathrm{Hom}(\mathcal{S}_p(M), \mathcal{N}_p)$ where $\mathcal{S}_p(M)$ is the space of symmetric linear endomorphisms of T_pM. For the natural internal product of $\mathcal{S}_p(M)$ and $\mathcal{N}_p(M)$, let $\mathcal{A}_p^t \in \mathrm{Hom}(\mathcal{N}_p, \mathcal{S}_p(M))$ be the transpose of $\mathcal{A}$ and set $\mathcal{B}_p = \mathcal{A}_p \circ \mathcal{A}_p^t$. The norm of this application is by definition the norm $|\mathcal{A}|$ of $\mathcal{A}$. If $\{e_i\}_{i=1,\dots,m}$ is an orthonormal frame of $T_p(M)$ then

$$|\mathcal{A}|^2(p) = \sum_{i,j=1}^{m} |\mathcal{A}(e_i, e_j)|^2.$$

The trace of $\mathcal{A}$ is called the mean curvature H of M. With respect to the frame $\{e_i\}_{i=1,\dots,m}$ we have

$$H = \frac{1}{m} \sum_{i=1}^{m} \mathcal{A}(e_i, e_i).$$

The immersion $M^m \hookrightarrow N^n$ is called minimal if $H \equiv 0$. This is equivalent to saying that the immersion is a critical point for the volume functional, i.e., for any compact $K \subset M$ with piecewise smooth boundary, and for any piecewise smooth variation $F \colon I \times M \to N$ of ϕ, which leaves the exterior of K unchanged, we have $\frac{dV}{dt}(0) = 0$, where $V(t)$ is the volume of the submanifold $F(t, K)$.

If $M^m \hookrightarrow N^n$ is minimal, a domain $U \subset M$ is called stable if for any variation F as above whose variation vector field $E = F_* \frac{\partial}{\partial t}\big|_{t=0}$ is normal to M and compactly supported in U we have $\frac{d^2V}{dt^2}(0) \geq 0$.

Let $\widetilde{\mathcal{R}}$ denote the curvature tensor of N and for $v \in \mathcal{N}_p(M)$ define $\mathcal{R}(v)$ by $\mathcal{R}(v) = \sum_{i=1}^m (\widetilde{\mathcal{R}}_{e_i,v} e_i)^{\perp}$. Note that for v unitary $\langle \mathcal{R}(v), v\rangle$ is just the Ricci curvature $Ric(v)$ of N in the direction of v. If F is a normal variation of a minimal surface as above we have [14],

$$\frac{d^2V}{dt^2}(0) = -\int_U \langle \Delta E + \mathcal{R}(E) + \mathcal{B}(E), E\rangle.$$

In the case of a minimal oriented surface $M^2 \hookrightarrow H^3$ the variation vector field is just $E = \xi\nu$, where ν is the normal vector of the immersion and ξ is a compactly supported function on M. So M is stable if

$$Q(\xi,\xi) = \int_M (|\nabla\xi|^2 + 2\xi^2 - |\mathcal{A}|^2\xi^2) \geq 0 \tag{1.1}$$

for all $\xi \in C_0^\infty(M)$.

1.2. Compactification of Hyperbolic Space. Two oriented rays $\gamma_1(s)$ and $\gamma_2(s)$ of $\mathbb{H}^n$ are said to be equivalent if there exists a real number c such that $d(\gamma_1(s), \gamma_2(s)) \leq c$ for all $s \geq 0$, where $d(p,q)$ denotes the hyperbolic distance between the points p and q. The sphere at infinity $\partial_\infty\mathbb{H}^n$ is defined as the space of equivalent classes of oriented rays. For a fixed point $O \in \mathbb{H}^n$, identify $\partial_\infty\mathbb{H}^n$ with the unit sphere $U_1 \subset T_O\mathbb{H}^n$ in the following way: for a unit vector $v \in U_1$ associate the equivalent class of the ray $\exp_O sv$, $s \geq 0$. This provides $\partial_\infty\mathbb{H}^n$ with a conformal sructure which is independent of the chosen point O. With this structure any isometry of $\mathbb{H}^n$ extends conformally to $\overline{\mathbb{H}}^n = \mathbb{H}^n \cup \partial_\infty\mathbb{H}^n$.

For $p \in \mathbb{H}^n \setminus \{O\}$ we define a "projection" $P\colon \mathbb{H}^n \setminus \{O\} \to U_1$ by $P(p) = \exp_O^{-1}(p)/|\exp_O^{-1}(p)|$. Let S_r be the geodesic sphere of $\mathbb{H}^n$ of radius r and centered at O. If $v_p \in T_p(S_r)$, a comparison between the Jacobi fields along geodesics gives

$$|d\exp_O^{-1}(v_p)| = \frac{r|v_p|}{\sinh r}$$

where in the left term the norm is the Euclidean norm of $T_O\mathbb{H}^n$ and in the right $|v_p|$ is the norm of $v_p \in T_p\mathbb{H}^n$. Thus, for a vector $v_p \in T_p(S_r)$ we get

$$|(dP)(v_p)| = \frac{|v_p|}{\sinh r}. \tag{1.2}$$

1.3. The spectrum of the Schrödinger operator. Let M be a Riemannian manifold and let q be a real smooth function. The operator $\mathbf{L} = -\Delta + q$ is formally self-adjoint over $C_c^\infty(M)$, where Δ is the Laplacian on M. When q is bounded below by a real constant and $M = \mathbb{R}^n$, Glazman [9] proved that $\mathbf{L}$ admits a unique self-adjoint extension to an unbounded operator on $L^2(M)$. The theorem of Dodziuk [5] stated below allows us to follow the steps of the Glazman's proof in the case of an arbitrary manifold M. For the sake of completeness we prove this generalization of Glazman's result and we also prove a theorem about the essential spectrum of $\mathbf{L}$.

Theorem 1.1 (Dodziuk). *Let M be a complete Riemannian manifold and let $q \in C^\infty(M)$ be a real function bounded below by a constant. Suppose $\phi \in C^\infty(M) \cap L^2(M)$ and $\mathbf{L}\phi \in L^2(M)$. Then $\nabla\phi \in L^2(M)$ and the functions $\phi\overline{\Delta\phi}$, $q|\phi|^2$ belong to $L^1(M)$. Also*

$$(-\Delta\phi, \phi) = (\nabla\phi, \nabla\phi) \quad \textit{and} \quad (\mathbf{L}\phi, \phi) = (\nabla\phi, \nabla\phi) + (q\phi, \phi)$$

where $(.,.)$ is the product of $L^2(M)$.

Theorem 1.2. *Let M be a complete Riemannian manifold and let $q \in C^\infty(M)$ be a real function bounded below by a constant. Then the operator $\mathbf{L} = -\Delta + q$ admits a unique self-adjoint extension to an unbounded operator on $L^2(M)$.*

Proof. It suffices to prove that the spaces $\mathcal{K}_\pm = \text{Image}\,(\mathbf{L} \pm iI)^\perp$ are trivial. Take $\phi \in \mathcal{K}_+$. As a distribution, ϕ satisfies the equation $\mathbf{L}\phi = i\phi$. For any relatively compact domain $\Omega \subset M$, the operator $\mathbf{L}$ is strictily elliptic. By the Friedrichs's regularity result [7] we have $\phi \in C^\infty(M) \cap L^2(M)$. Therefore, by the Dodziuk's theorem stated above we obtain

$$(\mathbf{L}\phi, \phi) = |\phi|^2 + (q\phi, \phi) = i|\phi|^2$$

But q is a real function, so $\phi \equiv 0$ and $\mathcal{K}_+ = \{0\}$. Analogously we have $\mathcal{K}_- = \{0\}$. □

Recall that for a self-adjoint operator $\mathbf{L}$ on a Hilbert space, the essential spectrum *ess*($\mathbf{L}$) is the set of points $\lambda \in \mathbb{R}$ such that there exists a bounded non-compact sequence $\{u_n\}_{n\in\mathbb{N}}$, $u_n \in$ Domain $(\mathbf{L})$, satisfying

$$\lim_{n\to\infty} \|(\mathbf{L}-\lambda I)u_n\| = 0.$$

A sub-sequence of $\{u_n\}_{n\in\mathbb{N}}$ for which there is no convergent sub-sequence is called characteristic for $(\lambda, \mathbf{L})$.

Now let $\mathbf{L} = -\Delta + q$ be as in theorem 1.2 and let N be a domain of M which is relatively compact and has C^∞ boundary. Consider the operator $l_N = -\Delta + q$ defined on $C^\infty(M \setminus N)$. The quadratic form $Q(\phi) = (l_N\phi, \phi)$ defined on $C_0^\infty(M \setminus N)$ is bounded below, so it admits a closed extension. We define $\mathbf{L}_N$ to be the Friedrichs's extension of l_N, determined by the closed extension of Q. We prove now the generalization of Glazman's theorem [9], p. 68. When $q \equiv 0$ this result was obtained by Donnelly [4].

Theorem 1.3. *ess*$(\mathbf{L}) \subset$ *ess*$(\mathbf{L}_N)$.

Proof. Suppose the sequence $\{u_n\}_{n\in\mathbb{N}}$ is characteristic for $(\lambda, \mathbf{L})$. Without loss of generality we can suppose it is an orthonormal characteristic sequence for $(\lambda, \mathbf{L})$. By virtue of theorem 1.2 (the operator is essentially self-adjoint) there exists $\{\phi_n\}_{n\in\mathbb{N}}$, $\phi_n \in C_0^\infty(M)$, such that, for $n \in \mathbb{N}$,

$$\|\phi_n - u_n\| \leq \frac{1}{n} \quad \text{and} \quad \|\mathbf{L}\phi_n - \mathbf{L}u_n\| \leq \frac{1}{n}.$$

This implies $\{\phi_n\}_{n\in\mathbb{N}}$ is also characteristic for $(\lambda, \mathbf{L})$ and in particular we have

$$(\mathbf{L}\phi_n - \lambda\phi_n, \phi_n) \to 0 \quad \text{as } n \to \infty.$$

So for n large enough we get

$$\|\nabla\phi_n\|^2 + (q\phi_n, \phi_n) - \lambda\|\phi_n\|^2 \leq 1.$$

Let $-q_0$ be a lower bound for q. We obtain, for n large,

$$\|\nabla\phi_n\|^2 \leq q_0(\phi_n, \phi_n) + \lambda\|\phi_n\|^2 + 1.$$

Thus $\{\phi_n\}_{n\in\mathbb{N}}$ is bounded in $W^{1,2}(M)$, the space of functions f with f and ∇f belonging to $L^2(M)$. Let Ω' be a relatively compact neighbourhood of N. The embedding $W^{1,2}(\Omega') \hookrightarrow L^2(\Omega')$ is compact, so there exists a sub-sequence $\{\phi'_n\}_{n\in\mathbb{N}}$ such that $\phi'_n/_{\Omega'}$ converges in $L^2(\Omega')$. Set $\omega_n = \phi'_{2n+1} - \phi'_{2n}$ and remark that ω_n is still characteristic for $(\lambda, \mathbf{L})$ and that $\omega_n \to 0$ in $L^2(\Omega')$. Let

Ω be a neighbourhood of N such that $N \subset \overline{\Omega} \subset \Omega'$ and let $\xi \in C_0^\infty(\Omega')$ be such that $\xi = 1$ on Ω. We have

$$(\mathbf{L}\omega_n - \lambda\omega_n, \xi^2\omega_n) \to 0 \quad \text{as } n \to \infty$$

and by Dodziuk's theorem

$$\|\xi\nabla\omega_n\|^2 + (\xi\omega_n, 2\omega_n\nabla\xi) + (q\xi\omega_n, \xi\omega_n) - \lambda(\xi\omega_n, \xi\omega_n) \to 0 \quad \text{as } n \to \infty.$$

Since $\|\omega_n\|_{L^2(\Omega')} \to 0$ and support$(\xi) \subset \Omega'$ we get $\|\nabla\omega_n\|_{L^2(\Omega)} \to 0$ as $n \to \infty$.

This allows us to construct a characteristic sequence for $(\lambda, \mathbf{L})$ which vanishes on a neighbourhood of N. As a matter of fact, let U be a neighbourhood of N such that $\overline{U} \subset \text{int}\,(\Omega)$, and let θ be a smooth function which satisfies $0 \leq \theta \leq 1$, $\theta = 0$ in U and $\theta = 1$ in $M \setminus \Omega$. Set $v_n = \theta\omega_n$, $n \in \mathbb{N}$, and remark that $v_n \in C_0^\infty(M \setminus N)$. Also the sequence $\{v_n\}_{n\in\mathbb{N}}$ is bounded and non-compact and

$$\begin{aligned}\|\mathbf{L}v_n - \lambda v_n\| \leq \|\mathbf{L}\omega_n - \lambda\omega_n\| &+ (\sup_M |\Delta\theta|)\|\omega_n\|_{L^2(\Omega)} \\ &+ 2(\sup_M |\nabla\theta|)\|\nabla\omega_n\|_{L^2(\Omega)}\,.\end{aligned}$$

Hence $\|\mathbf{L}v_n - \lambda v_n\| \to 0$ as $n \to \infty$ and the sequence $\{v_n\}_{n\in\mathbb{N}}$ is characteristic for $(\lambda, \mathbf{L}_N)$. $\square$

For a Riemannian manifold M and an operator $\mathbf{L}$ as in theorem 1.3 we define the index of $\mathbf{L}$ in the following way: Let Ω be a relatively compact domain of M with piecewise smooth boundary. The number of negative engeinvalues for the Dirichlet problem

$$\mathbf{L}u = \lambda u \quad ; \quad u_{/\partial\Omega} = 0$$

is finite. Let $ind_\Omega(\mathbf{L})$ be this number. Consider an exhaustion $\{\Omega_n\}_{n\in\mathbb{N}}$ of M by relatively compact domains with piecewise smooth boundary. The index $ind_M(\mathbf{L})$ of $\mathbf{L}$ is defined by

$$ind_M(\mathbf{L}) = \lim_{n\to\infty} ind_{\Omega_n}(\mathbf{L})$$

This limit does not depend on the chosen exhaustion [6], so the $ind_M(\mathbf{L})$ is well defined.

2. Submanifolds of Hyperbolic Space

In this section we develop some properties of the distance function of $\mathbb{H}^n$ restricted to a submanifold. In particular we will prove the following result:

Theorem 2.1. *Let $M^m \hookrightarrow \mathbb{H}^n$ be a complete immersion of a connected manifold M. Suppose there exists $\epsilon < 1$ and a compact set $C \subset M$ such that $|\mathcal{A}|(p) \leq \epsilon$ for $p \in M \setminus C$. Then the immersion is proper and for r large enough M is transversal to the geodesic spheres S_r of $\mathbb{H}^n$. In particular M is diffeomorphic to the interior of a compact manifold with boundary $\overline{M}$. Furthermore the immersion ϕ extends to a continuous map $\bar{\phi}: \overline{M} \hookrightarrow \overline{\mathbb{H}}^n$.*

2.1. The distance function of $\mathbb{H}^n$ restricted to submanifolds. Let $O \in \mathbb{H}^n$ be a fixed point and let $M^m \hookrightarrow \mathbb{H}^n$ be a isometric immersion. Let $d(q)$ be the distance of $q \in \mathbb{H}^n$ to O and let r be the restriction of d to M. Denote by $\tilde{\nabla}$ and ∇ the Levi-Civita connexions of $\mathbb{H}^n$ an M respectively. Let $\frac{\partial}{\partial d} = \tilde{\nabla} d$ denote the unitary radial vector field centered at O and defined on $\mathbb{H}^n \setminus \{O\}$.

For $p \in M$ let $\{E_i\}_{i=1,\ldots,m}$ be a frame tangent to M, defined in a neighbourhood of $p \in M$, orthonormal at p and satisfying $\nabla_{E_i} E_j(p) = 0$, for $i, j = 1, \ldots, m$. For $j = 1, \ldots, m$ we have $E_j r = \langle \frac{\partial}{\partial d}, E_j \rangle$, so

$$E_i E_j r = \langle \tilde{\nabla}_{E_i} \frac{\partial}{\partial d}, E_j \rangle + \langle \frac{\partial}{\partial d}, \tilde{\nabla}_{E_i} E_j \rangle \quad ; \quad i, j = 1, \ldots, m$$

where $\langle\ ,\ \rangle$ is the metric of $\mathbb{H}^n$. Recall that for a vector $v \in T_p S_r$ we have, $\tilde{\nabla}_v \frac{\partial}{\partial d} = \coth(r) v$. As $\tilde{\nabla}_{\frac{\partial}{\partial d}} \frac{\partial}{\partial d} = 0$, we obtain, for $i = 1, \ldots, m$

$$(\tilde{\nabla}_{E_i} \frac{\partial}{\partial d})(p) = \left(E_i - \langle E_i, \frac{\partial}{\partial d} \rangle \frac{\partial}{\partial d} \right) \coth r(p) \tag{2.1}$$

and for $i, j = 1, \ldots, m$ we get at p

$$E_i E_j r = \left(\delta_{ij} - \langle E_i, \frac{\partial}{\partial d} \rangle \langle E_j, \frac{\partial}{\partial d} \rangle \right) \coth r + \langle \frac{\partial}{\partial d}, \tilde{\nabla}_{E_i} E_j \rangle. \tag{2.2}$$

If the frame $\{E_i\}_{i=1,\ldots,m}$ is orthonormal, the Laplacian of r at p is given by $\Delta r(p) = \sum_{i=1}^m (E_i E_j r)(p)$. Hence we obtain, at all $p \in M$, $p \neq O$,

$$\Delta r = (m - |\nabla r|^2) \coth r + m \langle \frac{\partial}{\partial d}, H \rangle. \tag{2.3}$$

In the special case where M is a curve $\gamma \subset \mathbb{H}^n$ parametrized by the arc length s we get

$$r''(s) = (1-(r'(s))^2)\coth r + \langle \frac{\partial}{\partial d}, \tilde{\nabla}_{\gamma'(s)}\gamma'(s)\rangle. \tag{2.4}$$

A straightforward consequence of the above equations are the following lemmas:

Lemma 2.2. *Let $M^m \hookrightarrow \mathbb{H}^n$ be an immersion and let $p \in int(M)$ be a critical point of r. Suppose $|\mathcal{A}|(p) \leq 1$. Then p is a point of strict minimum for r.*

Proof. Let $\{e_i\}_{i=1,\dots,m}$ be a orthonormal frame of T_pM and let x be the normal coordinate system adapted to $\{e_i\}_{i=1,\dots,m}$, i.e., $x = \chi \circ (\exp_p)^{-1}$, where $\chi\colon T_pM \to \mathbb{R}^n$ is given by $\chi(\sum_{i=1}^m y^i e_i) = (y^1,\dots,y^m)$. Since $\nabla r(p) = 0$ we can choose the frame $\{e_i\}_{i=1,\dots,m}$ such that $\frac{\partial^2 r}{\partial x_i \partial x_j}(p) = 0$, for $i \neq j$. Setting $E_i = \frac{\partial}{\partial x_i}$, $i = 1,\dots,m$, by equation (2.2) we get

$$E_iE_ir = \coth r + \langle \frac{\partial}{\partial d}, \tilde{\nabla}_{E_i}E_i\rangle.$$

As $\nabla_{E_i}E_i(p) = 0$ we have $|\tilde{\nabla}_{E_i}E_i(p)| = |(\tilde{\nabla}_{E_i}E_i(p))^{\perp}| \leq |\mathcal{A}|(p) \leq 1$. So $E_iE_ir(p) > 0$, for $i = 1,\dots,m$, which implies, by the Taylor series expansion of r, that p is a point of strict minimum. □

Lemma 2.3. *Let $\gamma\colon [0,l) \subset \mathbb{H}^n$, $0 < l \leq \infty$, be a curve parametrized by the arc length s. Suppose the geodesic curvature of γ satisfies $|\tilde{\nabla}_{\gamma'(s)}\gamma'(s)| \leq \epsilon$, for some $\epsilon < 1$. Then $d(\gamma(0),\gamma(s)) \geq \sqrt{1-\epsilon}\, s$, $0 \leq s < l$.*

Proof. First observe that the curve γ is necessarily embedded ; otherwise, taking the intersection point as the origin of $\mathbb{H}^n$, the distance function $r(s)$ defined over γ would have a interior maximum, which contradicts lemma 2.2. Now taking the origin to be the point $\gamma(0)$, equation (2.4) says that

$$r''(s) = (1-(r'(s))^2)\coth r(s) + \langle \frac{\partial}{\partial d}, \tilde{\nabla}_{\gamma'(s)}\gamma'(s)\rangle$$

for all $0 < s < l$. Also $r(0) = 0$ and $\lim_{s\to 0} r'(s) = 1$. It suffices now to prove that $r'(s) \geq \sqrt{1-\epsilon}$, for all $0 < s < l$. Suppose this is not the case and let s_1 be the smallest positive real number for which $r'(s_1) = \sqrt{1-\epsilon}$. Then

$$r''(s_1) = \epsilon \coth r(s_1) + \langle \frac{\partial}{\partial d}, \tilde{\nabla}_{\gamma'(s_1)}\gamma'(s)\rangle$$

which, under the hypotheses of the lemma, implies $r''(s_1) > 0$. But this implies the existence of s_0, $0 < s_0 < s_1$, with $r'(s_0) < r'(s_1)$, violating the choice of s_1. □

2.2. Proof of theorem 2.1. First we prove that the immersion is proper and tranversal to the geodesic spheres S_r, for r large.

2.3. The immersion is proper.

Proof. Let $\bar{r} = \sup_{q \in C} r(q)$. For $p \in M \setminus C$ let γ be a geodesic of M, parametrized by arc length, which realises the distance between C and p. Say $\gamma(0) \in \partial C$ and $\gamma(l) = p$, where l is the length of γ. Of course $\gamma(s) \subset M \setminus C$, for all $s \in (0, l]$. As γ is a geodesic of M, we have $|\widetilde{\nabla}_{\gamma'(s)}\gamma'(s)| \leq |\mathcal{A}|(\gamma(s))$, $s \in [0, l]$. From lemma 2.3 we obtain

$$\begin{aligned} r(p) = d(O,p) &\geq d(\gamma(0), \gamma(l)) - d(O, \gamma(0)) \\ &\geq \sqrt{1-\epsilon}\, l - \bar{r} \ . \end{aligned}$$

Thus when $d(p, C)$ goes to infinity we get $r(p) \to \infty$, which means the immersion is proper. □

2.4. M is transversal to $\mathbf{S}_r$ for r large.

Proof. Let $\Omega_1 = M \cap B_{\bar{r}}$, where B_r denotes the closed geodesic ball of $\mathbb{H}^n$ of radius r. As the immersion is proper, Ω_1 is a compact set of M and by definition of $\bar{r}$, $|\mathcal{A}| \leq \epsilon$ in $M \setminus \Omega_1$. Suppose that there exists a critical point p of r in $M \setminus \Omega_1$. By lemma 2.2 p is a strict minimum for r. If γ is any geodesic joining p to $\partial\Omega_1$ then the maximum of the function $r(s)$ over $\gamma(s)$ is greater then $\max(\bar{r}, r(p))$. So this maximum is attained at a point $\bar{p} \in M \setminus \Omega_1$, which is impossible by lemma 2.2 applied to the geodesic γ. This contradiction implies that r has no critical points in $M \setminus \Omega_1$, i.e., M is transversal to S_r for $r \geq \bar{r}$. □

We have therefore a complete proper immersion $M^m \hookrightarrow \mathbb{H}^n$ such that the function r has no critical points in $M \setminus \Omega_1$ where $\Omega_1 = M \cap B_{\bar{r}}$, for some $\bar{r} > 0$. Furthermore $|\mathcal{A}| \leq \epsilon < 1$ in $M \setminus \Omega_1$. Let $\Sigma(r) = M \cap S_r$, so for $r \geq \bar{r}$, $\Sigma(r)$ is a compact $m-1$ dimensional submanifold of M. On $M \setminus \Omega_1$ define the fields

$\xi = \nabla r/|\nabla r|$ and $Y = \nabla r/|\nabla r|^2$. Let Ψ_t be the flow of Y. Thus Ψ_t maps $\Sigma(\bar{r})$ diffeomorphically into $\Sigma(\bar{r}+t)$, for $t \geq 0$. For a point p in $\Sigma(\bar{r})$ and $t \geq 0$, define $\alpha(p,t) = \sqrt{1-|\nabla r|^2}\,(p_t)$, where $p_t = \Psi_t(p)$. For $p \in \Sigma(\bar{r})$ this function satisfies

$$\frac{1}{2}\frac{\partial}{\partial t}\alpha^2 = -\langle \mathcal{A}(\xi,\xi), \frac{\partial}{\partial d}\rangle - \alpha^2 \coth(\bar{r}+t). \tag{2.5}$$

Moreover if $\eta \in T_p\Sigma(r)$, for $r \geq \bar{r}$, we have

$$\frac{1}{2}\eta(\alpha^2) = -\sqrt{1-\alpha^2}\,\langle \mathcal{A}(\eta,\xi), \frac{\partial}{\partial d}\rangle. \tag{2.6}$$

To see this let $\{N_i\}$, $i = 1,\ldots,k$, be a normal frame to M in a neighbourhood of p_t, where $k = n-m$ is the codimension of M. Write $\nabla r = \frac{\partial}{\partial d} - \sum_{i=1}^{k} \langle N_i, \frac{\partial}{\partial d}\rangle N_i$. For a vector $E \in T_{p_t}M$ we have

$$\nabla_E \nabla r = (\tilde{\nabla}_E \frac{\partial}{\partial d} - \sum_{i=1}^{k} \langle N_i, \frac{\partial}{\partial d}\rangle \tilde{\nabla}_E N_i)^T$$

so from equation (2.1) we obtain

$$\nabla_E \nabla r = (E - \langle E, \nabla r\rangle \nabla r)\coth r - \sum_{i=1}^{k} \langle N_i, \frac{\partial}{\partial d}\rangle (\tilde{\nabla}_E N_i)^T.$$

From $\frac{1}{2}\nabla_E |\nabla r|^2 = \langle \nabla_E \nabla r, \nabla r\rangle$ we get

$$\frac{1}{2}\nabla_E|\nabla r|^2 = \langle E, \nabla r\rangle(1-|\nabla r|^2)\coth r + |\nabla r|\langle \frac{\partial}{\partial d}, \mathcal{A}(E,\xi)\rangle \tag{2.7}$$

where we made use of the fact that $\langle \tilde{\nabla}_E N_i, \xi\rangle = -\langle N_i, \tilde{\nabla}_E \xi\rangle$ and that $\nabla r = |\nabla r|\xi$. Taking $E = \xi$ and remarking that $\frac{\partial}{\partial t} = \xi/|\nabla r|$ we obtain

$$\frac{1}{2}\frac{\partial}{\partial t}|\nabla r|^2 = (1-|\nabla r|^2)\coth r + \langle \frac{\partial}{\partial d}, \mathcal{A}(E,\xi)\rangle$$

which, after replacing $|\nabla r|^2$ by $1-\alpha^2$, is equation (2.5). In the same way, equation (2.6) is obtained from (2.7) with $E = \eta$.

Now we get the asymptotic behaviour of $|\nabla r|$.

Lemma 2.4. *On $M \setminus \Omega_1$ the function $|\nabla r|$ satisfies*

$$|\nabla r|^2(p_t) \geq (1-\epsilon)(1-e^{-2t}) \quad ; \quad \forall p \in \Sigma(\bar{r}).$$

Proof. By equation (2.5), for $p \in \Sigma(\bar{r})$ the function $\alpha(t) = \alpha(p,t)$ satisfies, for $t \geq 0$,

$$\frac{1}{2}(\alpha^2)'(t) \leq \epsilon - \alpha^2(t)$$

Let $f(t) = \epsilon + (1-\epsilon)\mathrm{e}^{-2t}$ be the solution of $\frac{1}{2}f'(t) + f(t) = \epsilon$, with $f(0) = 1$. The function $h(t) = f(t) - \alpha^2(t)$ satisfies $\frac{1}{2}h'(t) + h(t) \geq 0$ for $t \geq 0$ and $h(0) \geq 0$. This implies that $h(t) \geq 0$, for all $t \geq 0$. Thus for all $p \in \Sigma(\bar{r})$ and $t \geq 0$ we have, at p_t,

$$1 - |\nabla r|^2 \leq \epsilon + (1-\epsilon)\,\mathrm{e}^{-2t}. \quad \square$$

We are now able to finish the proof of theorem 2.1.

2.5. Asymptotic behavior.

Proof. If M is non orientable we replace M by the orientable double cover of M and remark that the hypotheses $|\mathcal{A}| \leq \epsilon$ outside a compact set is still satisfied. Let $P\colon \mathbb{H}^n \setminus \{O\} \to U_1$ be the projection on the unit sphere of $T_O\mathbb{H}^n$ as described in section 1. Denote by $\chi\colon \Sigma(\bar{r}) \times [0,\infty) \to U_1$ the map $\chi(p,t) = P \circ \Psi_t(p)$. We must prove that the 1-parameter family of immersions $\{\chi_t\}$, given by $\chi_t(p) = \chi(p,t)$ converges uniformly in $p \in \Sigma(\bar{r})$ as $t \to \infty$.

Observe that $|\frac{\partial\chi}{\partial t}(p,t)| = |dP(\gamma'(t))|$ where $\gamma(t) = \Psi_t(p)$ is the integral curve of Y with $\gamma(0) = p$. From equation (1.2) we have, after projection of the vector Y on $T_{p_t}S_{\bar{r}+t}$,

$$\left|\frac{\partial\chi}{\partial t}(p,t)\right| = \frac{\sqrt{1-|\nabla r|^2}}{|\nabla r|\sinh(\bar{r}+t)}.$$

By lemma 2.4, given δ, with $\epsilon < \delta < 1$, there exists t_1 such that for $t \geq t_1$, we have $|\nabla r|(p_t) \geq \sqrt{1-\delta}$. From the above equation we get, for $t \geq t_1$,

$$\left|\frac{\partial\chi}{\partial t}(p,t)\right| \leq \frac{1}{\sqrt{1-\delta}\sinh(\bar{r}+t)}$$

and this inequality implies

$$\int_0^\infty \left|\frac{\partial\chi}{\partial t}(p,t)\right| dt \to 0 \quad \text{as } t \to \infty$$

uniformly in $p \in \Sigma(\bar{r})$, so χ_t converges uniformly to a continous map $\chi_\infty\colon \Sigma(\bar{r}) \to \partial_\infty\mathbb{H}^n$. $\square$

2.6. Immersions transversal to geodesic spheres. Here we consider proper immersions $M^m \hookrightarrow \mathbb{H}^n$ which are transversal to the geodesic spheres S_r of $\mathbb{H}^n$ for $r \geq \bar{r}$. We are interested in the volume growth of $\Sigma(r) = M \cap S_r$. We suppose M to be orientable and denote by ω the volume form of M. On $M \setminus B(\bar{r})$ define $\sigma = \xi \rfloor \omega$, where $\xi = \nabla r/|\nabla r|$. If $r \geq \bar{r}$ and $\imath_r \colon \Sigma(r) \to M$ is the inclusion, let $\sigma_r = \imath_r^* \sigma$ be the volume form of $\Sigma(r)$. Up to sign, $\omega = \xi^b \wedge \sigma$, where ξ^b is the 1-form dual to the field ξ. For $p \in \Sigma(\bar{r})$ and $t \geq 0$ define $f(p,t)$ to be the positive function such that $f(p,t)\sigma_{\bar{r}} = \Psi_t^* \sigma_{\bar{r}+t}$. By definition of the function f we have, for $s, t \geq 0$ and $p \in \Sigma(\bar{r})$,

$$\frac{f(p,s+t)}{f(p,t)}\sigma(p_t) = (\Psi_s^*\sigma)(p_t).$$

Also, as the field $Y = \nabla r/|\nabla r|^2$ is invariant by the flow Ψ_t we get

$$(\Psi_t^*\xi^b)(p_t) = \frac{|\nabla r(p_{s+t})|}{|\nabla r(p_t)|}\xi^b(p_t).$$

It follows that the Lie derivative of ω in the direction Y is given by

$$L_Y\omega(p_t) = \frac{1}{f(p,t)}\left(\frac{\partial}{\partial t}f(p,t) - \frac{f(p,t)}{|\nabla r(p_t)|}\frac{\partial}{\partial t}(|\nabla r(p_t)|)\right).$$

But $L_Y\omega = \operatorname{div} Y\omega$ and $\frac{\partial}{\partial t}(|\nabla r(p_t)|) = \frac{1}{|\nabla r(p_t)|}\xi(|\nabla r|)(p_t)$. Also

$$\operatorname{div} Y = \frac{1}{|\nabla r|^2}\Delta r - \frac{2}{|\nabla r|^2}\xi(|\nabla r|)$$

because $\langle \nabla(|\nabla r|), \nabla r\rangle = |\nabla r|\xi(|\nabla r|)$. From these equations it follows that

$$f\Delta r = |\nabla r|^2\frac{\partial}{\partial t}f + f\xi(|\nabla r|). \tag{2.8}$$

Let $\gamma(s)$ be a integral curve of ξ such that $\gamma(0) = p_t$. Let $r(s) = d(\gamma(s))$ and recall that, since ξ is unitary $\langle \frac{\partial}{\partial d}, \widetilde{\nabla}_{\gamma'(0)}\gamma'(s)\rangle = \langle \frac{\partial}{\partial d}, \mathcal{A}(\xi,\xi)\rangle$. From equation (2.4) we have

$$r''(s) = (1 - (r'(s))^2)\coth r(s) + \langle \frac{\partial}{\partial d}, \mathcal{A}(\xi,\xi)\rangle$$

and using the fact that $r'(s) = |\nabla r|$ we get

$$\xi(|\nabla r|) = (1 - |\nabla r|^2)\coth r + \langle \frac{\partial}{\partial d}, \mathcal{A}(\xi,\xi)\rangle.$$

From equations (2.3) and (2.8) we obtain, with $\alpha^2 = 1 - |\nabla r|^2$, the desired equation for f

$$\frac{1}{f}(1-\alpha^2)\frac{\partial}{\partial t}f = (m-1)\coth r - \langle\frac{\partial}{\partial d}, \mathcal{A}(\xi,\xi)\rangle + m\langle H, \frac{\partial}{\partial d}\rangle. \tag{2.9}$$

3. Minimal immersions

In this section we prove theorem A and B. The statement about the conformal type of M in theorem B will be proved in the next section. First we state some basic inequalities.

3.1. Simons and Sobolev inequalities. Let $\varphi \colon M^m \hookrightarrow \mathbb{H}^n$ be a cmi and denote $u = |\mathcal{A}|$. Simons'equation [17], applied to minimal submanifolds of $\mathbb{H}^n$, tell us that u satisfies

$$\Delta u + mu + mu^3 \geq 0 \tag{3.1}$$

in the distribution sense. Let ξ be a compactly supported smooth function on M and let $q \geq 1$ be a real number. Multiplying (3.1) by $\xi^2 u^{2q-1}$, integrating by parts, rearranging terms and taking square roots we obtain

$$\|\nabla \xi u^q\|_2 \leq c_1\sqrt{q}(\|\xi u^q\|_2 + \|\xi u^{q+1}\|_2 + \|u^q \nabla\xi\|_2) \tag{3.2}$$

for a constant c_1 which depends only on m.

From Sobolev inequality [12], for any smooth function h compactly supported in M we have

$$\|h\|_{\frac{m}{m-1}} \leq c_2\|\nabla h\|_1 \tag{3.3}$$

where c_2 does not depends on h. From (3.3) and the Holder inequality we have, for $1 \leq r < m$,

$$\|h\|_{\frac{mr}{m-r}} \leq c_2\frac{r(m-1)}{m-r}\|\nabla h\|_r. \tag{3.4}$$

These inequalities are valid in case h is a bounded compactly supported function and $h \in W^{1,r}$, the space of functions in $L^r(M)$, whose gradient ∇h also belongs to $L^r(M)$. We remark that this is the case when $h = \xi u^q$, ξ a smooth function compactly supported on M.

3.2. Proof of theorem A. We first prove an analytical lemma.

Lemma 3.1. *Given $m \geq 2$, there exists universal constants $\epsilon > 0$ and $c > 0$, depending only on m, with the folowing property : If $\varphi\colon M^m \hookrightarrow \mathbb{H}^n$ is a minimal immersion of an open manifold M, and $x_0 \in M$ is such that the closed geodesic ball $B(1)$ of radius 1 centered at x_0 is compact in M, then $\left(\int_{B(1)} |\mathcal{A}|^m\right)^{\frac{1}{m}} \leq \epsilon$ implies*

$$|\mathcal{A}|(x_0) \leq c\Big(\int_{B(1)} |\mathcal{A}|^m\Big)^{\frac{1}{m}}.$$

Proof. We deal separetely the cases $m \geq 3$ and $m = 2$.

case $m \geq 3$. For $\xi \in C_c^\infty(B(1))$ denote by χ the characteristic function of the support of ξ. If $s > 2$ the Holder's inequality gives us

$$\int_M \xi^2|\mathcal{A}|^2u^{2q} \leq \|\chi|\mathcal{A}|^2\|_{\frac{s}{2}}\|\xi^2u^{2q}\|_{\frac{s}{s-2}} = \|\chi|\mathcal{A}|^2\|_{\frac{s}{2}}\|\xi u^q\|^2_{\frac{2s}{s-2}}.$$

We take $r = 2$ in the Sobolev inequality (3.4), apply (3.2) and the above inequality to obtain

$$\|\xi u^q\|_{\frac{2m}{m-2}} \leq c_4\sqrt{q}\left(\|u^q|\nabla\xi|\|_2 + \|\xi u^q\|_2 + \|\chi|\mathcal{A}|^2\|^{\frac{1}{2}}_{\frac{s}{2}}\|\xi u^q\|_{\frac{2s}{s-2}}\right) \tag{3.5}$$

where c_4 depends only on m. Suppose that

$$c_4\sqrt{\frac{m}{2}}\Big(\int_{B(1)}|\mathcal{A}|^m\Big)^{\frac{1}{m}} \leq \frac{1}{2}. \tag{3.6}$$

With this assumption, from (3.5) with $s = m$ and $q = \frac{m}{2}$ we have

$$\|\xi u^{\frac{m}{2}}\|_{\frac{2m}{m-2}} \leq c_5\,[\|u^{\frac{m}{2}}|\nabla\xi|\|_2 + \|\xi u^{\frac{m}{2}}\|_2]$$

and

$$\|\xi u^{\frac{m}{2}}\|_{\frac{2m}{m-2}} \leq c_5(\sup_{B(1)}|\nabla\xi| + \sup_{B(1)}|\xi|)\Big(\int_{B(1)}|\mathcal{A}|^m\Big)^{\frac{1}{2}}$$

where $c_5 = c_4\sqrt{2m}$. Taking ξ such that $0 \leq \xi \leq 1$, $\xi = 1$ on $B(\frac{3}{4})$, $\xi = 0$ on the exterior of $B(1)$ and such that $|\nabla\xi| \leq 8$, we have by the above inequality

$$\||\mathcal{A}|^{\frac{m}{2}}\|_{\frac{2m}{m-2},B(\frac{3}{4})} \leq 10c_5\Big(\int_{B(1)}|\mathcal{A}|^m\Big)^{\frac{1}{2}} \tag{3.7}$$

where the norm in the left side is taken over the ball $B(\frac{3}{4})$. We want to use (3.7) to get control of the $L_{\frac{2m}{m-2}}$ norm of ξu^q in terms of its L_2 norm.

Let ϵ be the greatest positive real number such that if $\int_{B(1)} |\mathcal{A}|^m \leq \epsilon^m$ then inequality (3.6) and

$$10c_5\Big(\int_{B(1)}|\mathcal{A}|^m\Big)^{\frac{1}{2}} \leq 1 \tag{3.8}$$

are satisfied. The constant ϵ depends only on m. Remark that for $s = \frac{m^2}{m-2}$ we have

$$\||\mathcal{A}|^2\|_{\frac{s}{2},B(\frac{3}{4})} = \||\mathcal{A}|^{\frac{m}{2}}\|^{\frac{2}{m}}_{\frac{2m}{m-2},B(\frac{3}{4})}.$$

Therefore, assuming $\int_{B(1)} |\mathcal{A}|^m \leq \epsilon^m$, from (3.5) and (3.7) we get , for $s = \frac{m^2}{m-2}$,

$$\|\xi u^q\|_{\frac{2m}{m-2}} \leq 2c_4\sqrt{q}(\|u^q|\nabla\xi|\|_2 + \|\xi u^q\|_2 + \|\xi u^q\|_{\frac{2s}{s-2}}) \tag{3.9}$$

for all smooth ξ with support in the ball $B(\frac{3}{4})$.

On the other hand, for $s = \frac{m^2}{m-2}$, and any $\delta > 0$ we have the interpolation formula

$$\|\xi u^q\|_{\frac{2s}{s-2}} \leq \delta\|\xi u^q\|_{\frac{2m}{m-2}} + \delta^{-\sigma}\|\xi u^q\|_2. \tag{3.10}$$

where $\sigma = \frac{m}{m-2}$. Given $q \geq 1$ we chose δ such that $c_4\delta\sqrt{q} = \frac{1}{4}$. Thus $\delta^{-\sigma} = (4c_4)^\sigma q^{\frac{\sigma}{2}}$ and from (3.9) and (3.10) we get, for any $\xi \in C_c^\infty(B(\frac{3}{4}))$

$$\|\xi u^q\|_{\frac{2m}{m-2}} \leq c_6\sqrt{q}(\|u^q|\nabla\xi|\|_2 + (1 + q^{\frac{\sigma}{2}})\|\xi u^q\|_2) \tag{3.11}$$

for some constant c_6 which depends only on m.

Now we iterate to obtain a bound for $|\mathcal{A}|$ over $B(\frac{1}{4})$. For $i \in \mathbb{N}$, let $B_i = B(\frac{1}{4} + \frac{1}{2^{i+1}})$. Let ξ_i, $0 \leq \xi_i \leq 1$, be a Lipschitz function which satisfies

$$\xi_i = 1 \quad \text{on } B_{i+1} \quad ; \quad \xi_i = 0 \quad \text{on } M \setminus B_i$$

and such that $|\nabla\xi_i| \leq 2^{i+2}$. From (3.11) with $\xi = \xi_i$ we get

$$\|\xi_i u^q\|_{2\sigma} \leq c_6\sqrt{q}\,(2^{i+3} + q^{\frac{\sigma}{2}})\,\|\chi_i u^q\|_2$$

where χ_i is the characteristic function of support(ξ_i). Squaring the above inequality we obtain

$$\Big(\int_{B_{i+1}}|\mathcal{A}|^{2q\sigma}\Big)^{\frac{1}{\sigma}} \leq c_6^2 q\,(2^{i+3} + q^{\frac{\sigma}{2}})^2\int_{B_i}|\mathcal{A}|^{2q}. \tag{3.12}$$

Let $2q = m\sigma^i$ and observe that, for this choice of q we have $c_6^2 q(2^{i+3}+q^{\frac{i\sigma}{2}}) \leq c_7^i$ for some constant c_7 depending only on m. Hence from (3.12) we obtain

$$\left(\int_{B_{i+1}} |\mathcal{A}|^{m\sigma^{i+1}}\right)^{\frac{1}{\sigma}} \leq c_7^i \int_{B_i} |\mathcal{A}|^{m\sigma^i}. \tag{3.13}$$

Define $I_i = \left(\int_{B_i} |\mathcal{A}|^{m\sigma^i}\right)^{\frac{1}{\sigma^i}}$. From (3.13) we get $I_{i+1} \leq c_7^{\frac{i}{\sigma^i}} I_i$. Since $\sigma > 1$ the series $\sum_{i=1}^{\infty} \frac{i}{\sigma^i}$ converges. Thus there exist a real number c depending only m such that

$$I_{i+1} \leq c^m I_0.$$

This implies the norm L^∞ of $|\mathcal{A}|^m$ over the ball $B(\frac{1}{4})$ is bounded by $c^m \int_{B(1)} |\mathcal{A}|^m$ which is the conclusion of the lemma for $m \geq 3$. □

case $m = 2$. We prove first there exist δ such that if $\left(\int_{B(1)} |\mathcal{A}|^2\right)^{\frac{1}{2}} \leq \delta$ then the operator $\mathbf{L} = -\Delta + 2 - |\mathcal{A}|^2$ is positive defined on the ball $B(\frac{1}{2})$. In the case $n = 3$ this means that the ball $B(\frac{1}{2})$ is stable.

Let ξ be a smooth compact supported function on $B(1)$ and χ the characteristic function of the support of ξ. From the Sobolev inequality (3.3) we have

$$\|\xi u^2\|_2 \leq 2c_2(\|\xi u \nabla u\|_1 + \|u^2 \nabla \xi\|_1)$$

and by the Cauchy-Schwartz inequality we get

$$\|\xi u^2\|_2 \leq 2c_2 \|\chi u\|_2 (\|\xi \nabla u\|_2 + \|u \nabla \xi\|_2)$$

Taking $q = 1$ in (3.2) and rearranging terms we obtain

$$\|\xi \nabla u\|_2 \leq 2c_1(\|\xi u\|_2 + \|\xi u^2\|_2 + \|u \nabla \xi\|_2) \tag{3.14}$$

and from these last two inequalities we get

$$\|\xi \nabla u\|_2 \leq c_8(\|\xi u\|_2 + \|\chi u\|_2(\|\xi \nabla u\|_2 + \|u \nabla \xi\|_2) + \|u \nabla \xi\|_2) \tag{3.15}$$

for some constant c_8 which does not depends on ξ. Let $\delta_1 = \min(\frac{1}{2c_8}, 1)$ and suppose

$$\left(\int_{B(1)} |\mathcal{A}|^2\right)^{\frac{1}{2}} \leq \delta_1. \tag{3.16}$$

From (3.15) we get

$$\|\xi\nabla u\|_2 \le 2c_8\left(\|\xi u\|_2 + 2\|u\nabla\xi\|_2\right) \tag{3.17}$$

for all ξ compactly supported in $B(1)$.

Now take ξ with compact support in $B(1)$ and satisfying

$$\xi = \begin{cases} 1 & ; \quad \text{on } B(\frac{3}{4}) \\ 0 & ; \quad \text{on } B(1)\setminus B(\frac{3}{4}) \end{cases}$$
$$0 \le \xi \le 1 \quad ; \quad |\nabla\xi| \le 8.$$

For any such ξ we have, from (3.17)

$$\|\xi\nabla u\|_2 \le 20c_8\|\chi u\|_2 \tag{3.18}$$

Let ϕ be a smooth compactly supported function on $B(\frac{1}{2})$ and let ξ be as above, so that (3.18) is verified. From Sobolev inequality (3.3) with ξ replaced by $\xi\phi$ and from Schwartz inequality we have

$$\frac{1}{2c_2}\|\mathcal{A}\phi\xi\|_2 \le \|\xi\nabla|\mathcal{A}|\|_2\|\phi\|_2 + \|\xi\mathcal{A}\|_2\|\nabla\phi\|_2 + \|\phi\|_2\|\mathcal{A}|\nabla\xi|\|_2. \tag{3.19}$$

By (3.18) there exist $\delta < \delta_1$ such that if

$$\left(\int_{B(1)}|\mathcal{A}|^2\right)^{\frac{1}{2}} \le \delta$$

then

$$\|\xi\nabla|\mathcal{A}|\|_2 < \frac{1}{4c_2} \quad ; \quad \|\xi\mathcal{A}\|_2 < \frac{1}{4c_2} \quad ; \quad \|\mathcal{A}|\nabla\xi|\|_2 < \frac{1}{4c_2}.$$

Therefore, if $\left(\int_{B(1)}|\mathcal{A}|^2\right)^{\frac{1}{2}} \le \delta$ we have, from (3.19),

$$\int_M|\mathcal{A}|^2\phi^2 \le 2\int_M\phi^2 + \int_M|\nabla\phi|^2 \tag{3.20}$$

for all $\phi \in C_c^\infty(B(\frac{1}{2}))$.

Since the immersion is minimal we have by the Gauss equation $K = -1 - \frac{1}{2}|\mathcal{A}|^2$. Also our surface satisfies the "stability" equation (3.20) on the compact ball of radius $\frac{1}{2}$, the Sobolev inequality (3.3) and Simon's inequality (3.1). So we have all the requirements to apply Schoen's stability result [16]: there exists constants c_9, c_{10} and $0 < \mu \leq \frac{1}{2}$, not depending on the immersion such that

$$\begin{cases} \int_{B(\mu)}(1+|\mathcal{A}|^2) & \leq c_9 \\ \sup_{B(\mu)}|\mathcal{A}| & \leq c_{10} \end{cases} \tag{3.21}$$

This enable us to find a bound for $|\mathcal{A}|$ on $B(\frac{\mu}{4})$ in terms of the L_2 norm of $|\mathcal{A}|$ on $B(\mu)$. From (3.4) we have, with $r = \frac{4}{3}$,

$$\|\xi|\mathcal{A}|^q\|_4 \leq c_{11}\|\nabla\xi|\mathcal{A}|^q\|_{\frac{4}{3}}$$

for some constant c_{11} and for any function ξ compactly supported in $B(\mu)$. From Holder inequality and the estimate on the area given by (3.21) we get

$$\|\xi|\mathcal{A}|^q\|_4 \leq c_{12}\|\nabla\xi|\mathcal{A}|^q\|_2$$

for some constant c_{12} which does not depends on the immersion. By (3.2) and the bound of $|\mathcal{A}|$ on $B(\mu)$ we obtain, for some constant c_{13},

$$\|\xi|\mathcal{A}|^q\|_4 \leq c_{13}\sqrt{q}\left(\sup_{B(\mu)}|\xi| + \sup_{B(\mu)}|\nabla\xi|\right)\|\chi|\mathcal{A}|^q\|_2.$$

where χ is the characteristic function of $B(\mu)$.

An iteration method analogous to that used in the proof of case $m \geq 3$ gives that there exists constants c_{14} and $\epsilon < \delta$ such that if $\left(\int_{B(1)}|\mathcal{A}|^2\right)^{\frac{1}{2}} \leq \epsilon$ then

$$\sup_{B(\frac{\mu}{4})}|\mathcal{A}| \leq c_{14}\left(\int_{B(\epsilon)}|\mathcal{A}|^2\right)^{\frac{1}{2}}$$

and this finishes the proof of the lemma. □

Now theorem A is an easy consequence of lemma 3.1. In fact, for a cmi $M^m \hookrightarrow \mathbb{H}^n$ satisfying $\int_M |\mathcal{A}|^m < \infty$, there exists $r_0 > 0$ such that

$$\left(\int_{M\setminus B(r_0)}|\mathcal{A}|^m\right)^{\frac{1}{m}} \leq \epsilon,$$

ϵ as in lemma 3.1. Thus we have

$$\sup_{M\setminus B(r+1)}|\mathcal{A}| \leq c\left(\int_{M\setminus B(r)}|\mathcal{A}|^m\right)^{\frac{1}{m}}$$

for $r \geq r_0$. Therefore $|\mathcal{A}|(p)$ goes uniformly to 0 as $p \to \infty$. The result now follows from theorem 2.1 of section 2.

Another consequence of lemma 3.1 is a "topological gap phenomenom"

Corollary 3.2. *Let $M^m \hookrightarrow \mathbb{H}^n$ be a connected complete minimal immersion. Then there exists $\bar{\epsilon}$ such that if $\int_M |\mathcal{A}|^m \leq \bar{\epsilon}$ then M is simply connected.*

Proof. It suffices to take $\bar{\epsilon} = \epsilon/c$ for ϵ and c as in lemma 3.1. Thus $|\mathcal{A}| \leq 1$ on M. If $\pi_1(M)$ is non trivial then there exists a geodesic γ of M with coincidents ending points. As M is minimal $|\widetilde{\nabla}_{\gamma'}\gamma'| \leq 1$. The contradiction follows from lemma 2.3. Thus $\pi_1(M)$ is trivial. □

3.3. Proof of theorem B. The assertion about the conformal type will be proved in the next section.

3.4. $\partial_\infty(M)$ is a Lipschitz.

Proof. We assume M is orientable. From theorem A we know that a cmi $\varphi\colon M^2 \hookrightarrow \mathbb{H}^n$ satisfying $\int_M |\mathcal{A}|^2 < \infty$ is properlly immersed and $|\mathcal{A}|(p) \to 0$ as $p \to \infty$. In particular M meets transversally the geodesic spheres S_r centered at a fixed point $O \in \mathbb{H}^n$, for $r > \bar{r}$, $\bar{r}$ large enough. Let $\Sigma(r) = M \cap S_r$ as in section 2. Recall that for $p \in \Sigma(\bar{r})$ and $t \geq 0$, $f(p,t)$ is the norm of $d\Psi_t(\eta(p))$ where Ψ_t is the flow of $Y = \nabla r/|\nabla r|^2$ and η is the unitary vector field defined on $M \setminus B(\bar{r})$ and orthogonal to $\xi = \nabla r/|\nabla r|$. Let l be the length of $\Sigma(\bar{r})$ and let $\gamma\colon [0,l] \to \Sigma(\bar{r})$ be a parametrization of $\Sigma(\bar{r})$ by arc length. Define

$$x\colon [0,l) \times [0,\infty) \mapsto M \setminus B(\bar{r})$$

$$x(\theta,t) = \Psi_t(\gamma(\theta))$$

and remark that $\frac{\partial x}{\partial \theta}(\theta,t) = f(\theta,t)\eta(\theta,t)$ where $f(\theta,t) = f(\gamma(\theta),t)$ and $\eta(\theta,t) = \eta(x(\theta,t))$. Also $\frac{\partial x}{\partial t}(\theta,t) = Y(x(\theta,t))$. In the coordinate system given by x the area element is $dS = (f/|\nabla r|)d\theta dt$. Set $\alpha(\theta,t) = \alpha(x(\theta,t))$, where $\alpha = \sqrt{1-|\nabla r|^2}$. When $H=0$, equations (2.5), (2.6) and (2.9) give

$$\frac{1}{2}\frac{\partial}{\partial t}\alpha^2 = -\langle \mathcal{A}(\xi,\xi), \frac{\partial}{\partial d}\rangle - \alpha^2 \coth(\bar{r}+t) \tag{3.22}$$

$$\frac{1}{2}\frac{\partial \alpha^2}{\partial \theta} = -f\langle \mathcal{A}(\eta,\xi), \frac{\partial}{\partial d}\rangle\sqrt{1-\alpha^2} \tag{3.23}$$

$$\frac{1}{f}(1-\alpha^2)\frac{\partial}{\partial t}f = \coth(\bar{r}+t) - \langle \frac{\partial}{\partial d}, \mathcal{A}(\xi,\xi)\rangle. \tag{3.24}$$

From (3.22) and (3.24) we obtain

$$\frac{1}{f}\frac{\partial}{\partial t}f = \coth(\bar{r}+t) + \frac{2\alpha^2}{1-\alpha^2}\coth(\bar{r}+t) + \frac{1}{2}\frac{\partial}{\partial t}\ln(1-\alpha^2). \tag{3.25}$$

Assume for the moment that there exists a positive real number C such that

$$\int_0^\infty \alpha^2(\theta,t)\,dt \le C \quad ; \quad \forall\theta\in[0,l]. \tag{3.26}$$

Since $|\mathcal{A}|(p)\to 0$ as $p\to\infty$ we take $\bar{r}$ large enough to have $|\mathcal{A}|\le\frac{1}{2}$ and $\alpha^2\le\frac{1}{4}$ on $M\setminus B(\bar{r})$. This is possible by lemma 2.4. Integrating both sides of (3.25) and using (3.26) we obtain

$$f(p,t) = e^{t+h(p,t)} \tag{3.27}$$

for some bounded function $h(p,t)$ defined on $M\setminus B(\bar{r})$.

Let $\chi\colon \Sigma(\bar{r})\times[0,\infty)\to U_1$ be as in the proof of theorem 2.1, U_1 the unit sphere of $T_O\mathbb{H}^n$, so that $\chi_t(p)=\chi(p,t)$ is just the projection of the curve $\Sigma(\bar{r}+t)$ in the sphere at infinity $\partial_\infty\mathbb{H}^n\cong U_1$. From equation (1.2) we have that

$$|(d\chi_t)(\eta(p))| = \frac{f(p,t)}{\sinh r}$$

and hence, by (3.27), we get a bound for the length of the immersions $\chi_t(\Sigma(\bar{r}))$. But a sequence of uniformly convergent curves of U_1 whose lengths are uniformly bounded converges to a Lipschitz curve of U_1.

To prove (3.26) we first prove that $\alpha\in L_2(M)$.

As $\alpha^2\le\frac{1}{4}$ on $M\setminus B(\bar{r})$ we have $|\nabla r|\ge\frac{3}{4}$ and therefore $fd\theta dt\le dA\le\frac{4}{3}fd\theta dt$. Let $D_t=\{\Psi_s(p)\mid p\in\Sigma(\bar{r})\,;\,0\le s\le t\}$ be the annuli of M bounded by $\Sigma(\bar{r}+t)$ and $\Sigma(\bar{r})$. Since $\langle\mathcal{A}(\xi,\xi),\frac{\partial}{\partial d}\rangle=\langle(\tilde{\nabla}_\xi\xi)^\perp,(\frac{\partial}{\partial d})^\perp\rangle$, by the Cauchy-Schwartz inequality we have

$$|\langle\mathcal{A}(\xi,\xi),\frac{\partial}{\partial d}\rangle|\le|\mathcal{A}|\sqrt{1-|\nabla r|^2}=|\mathcal{A}|\alpha. \tag{3.28}$$

From (3.22) and (3.24) we get

$$\frac{\partial}{\partial t}(\alpha^2 f)+\frac{1-2\alpha^2}{1-\alpha^2}f\alpha^2\coth(\bar{r}+t)=-\left(\frac{3-\alpha^2}{1-\alpha^2}\right)f\langle\mathcal{A}(\xi,\xi),\frac{\partial}{\partial d}\rangle$$

and, making use of (3.28), we have

$$\frac{\partial}{\partial t}(\alpha^2 f) + \frac{1}{2}\frac{f\alpha^2}{|\nabla r|} \le \frac{4}{3}\frac{f|\mathcal{A}|\alpha}{|\nabla r|}.$$

Integrating this inequality over $[0,l] \times [0,t]$, using Holder inequality in the right term and remembering that $f(\theta,0) = 1$, $\forall \theta \in [0,l]$, we obtain

$$\int_0^l \alpha^2 f(\theta,t)\, d\theta - \int_0^l \alpha^2(\theta,0)\, d\theta + \frac{1}{2}\int_{D_t}\alpha^2 \le \frac{4}{3}\left(\int_{D_t}|\mathcal{A}|^2\right)^{\frac{1}{2}}\left(\int_{D_t}\alpha^2\right)^{\frac{1}{2}}.$$

As $f > 0$ this implies that $\alpha \in L_2(M)$. By equation (3.24) and the fact that $|\mathcal{A}| < 1$ on $M \setminus B(\bar{r})$ we have $\frac{\partial f}{\partial t} > 0$ for $t \ge 0$. Thus $f \ge 1$ on $M \setminus B(\bar{r})$. As $\alpha \in L_2(M)$, this implies that the integral $\int_0^l\int_0^\infty \alpha^2\, d\theta dt$ is finite. Hence for almost all $\theta \in [0,l]$ the integral $\int_0^\infty \alpha^2(\theta,t)dt$ is finite. Changing the parametrization of $\Sigma(\bar{r})$ if necessary we can assume that

$$\int_0^\infty \alpha^2(0,t)dt < \infty.$$

Define $I(\theta,t) = \int_0^t \alpha^2(\theta,s)ds$. From (3.23) we have for $\theta_0 \in [0,l]$,

$$I(\theta_0,t) - I(0,t) = \int_0^{\theta_0} \frac{\partial I}{\partial \theta}(\theta,t)\, d\theta \le 2\int_0^{\theta_0}\int_0^t f|\mathcal{A}|\alpha\, dt d\theta.$$

Thus, since $dA \ge f d\theta dt$, we have by the Cauchy-Schwartz inequality

$$I(\theta_0,t) \le I(0,t) + 2\left(\int_M |\mathcal{A}|^2\right)^{\frac{1}{2}}\left(\int_M \alpha^2\right)^{\frac{1}{2}}. \quad \square$$

3.5. The operator $\mathbf{L} = -\Delta + 2 - |\mathcal{A}|^2$ has finite index.

Proof. From theorem A we know that $|\mathcal{A}|(p) \to 0$ as $p \to \infty$, so for any $\epsilon \in (0,2)$ there exists a compact set $N_\epsilon \subset M$ such that

$$(\mathbf{L}\phi,\phi) \ge (-\Delta\phi + \epsilon\phi,\phi) \quad \phi \in C_c^\infty(M \setminus N_\epsilon).$$

Therefore the spectrum of the restriction $\mathbf{L}_N$ of $\mathbf{L}$ to the exterior of N_ϵ is contained in the interval $[\epsilon,\infty)$. By theorem 1.3 we have that the essential spectrum of $\mathbf{L}$ is contained in $[\epsilon,\infty)$. Thus for any $\delta < \epsilon$, the number of eigenvalues of $\mathbf{L}$ smaller than δ is finite. In particular the index of $\mathbf{L}$ is finite. $\square$

4. Conformal Type of Minimal Surfaces

It's well known that there exists no complete conformal metric $ds^2 = e^{2u}|dz|$ on the complex plane $\mathbb{C}$, whose Gauss curvature satisfies $K \le -1$. In this section we prove that any conformal metric on the complex plane, whose Gauss curvature is sufficiently negative outside a compact set must necessarily have non-negative total curvature. This will enable us to prove that the punctured disk $D^* = \{0 < |z| < 1\}$ can't be conformally immersed in $\mathbb{H}^n$ in such a manner that the immersion is a complete (at the origin) minimal surface.

Lemma 4.1. *Let $ds^2 = e^{2u}|dz|$ be a conformal metric defined on $\mathbb{C}$. Suppose that the Gauss curvature satisfies $K(z) \le -1/|z|^2$ oustide a compact set $\Omega \subset \mathbb{C}$. Then $\int_{\mathbb{C}} KdA \ge 0$.*

Proof. Let (ρ, θ) be the polar coordinates of $\mathbb{C}$ and let $dA = e^{2u}\rho d\rho d\theta$ be the area element for the metric ds^2. For $r > 0$ we integrate both sides of the Gauss equation $\Delta u = -Ke^{2u}$ over the disc $\{|z| \le r\}$ to obtain

$$\int_{|z|\le r} \Delta u dx dy = -\int_0^r \int_0^{2\pi} e^{2u} K \rho d\rho d\theta = -\int_{|z|\le r} KdA.$$

Let $I(r) = \frac{1}{2\pi}\int_0^{2\pi} u(r,\theta)d\theta$ and denote by $I'(r)$ the derivative of $I(r)$. By the Green's formula and the above equation we get

$$rI'(r) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial u}{\partial r}(r,\theta) r d\theta = -\frac{1}{2\pi}\int_0^r \int_0^{2\pi} e^{2u} K \rho d\rho d\theta. \tag{4.1}$$

Taking derivatives with respect to r gives

$$\frac{1}{r}[rI'(r)]' = -\frac{1}{2\pi}\int_0^{2\pi} e^{2u} K d\theta.$$

Let $r_0 > 0$ be such that $\Omega \subset \{|z| < r_0\}$. Then, for $|z| \ge r_0$ we have $K(z) \le -1/|z|^2$ and by Jensen's inequality we get

$$r[rI'(r)]' \ge \frac{1}{2\pi}\int_0^{2\pi} e^{2u(r,\theta)}\, d\theta \ge e^{\frac{1}{2\pi}\int_0^{2\pi} 2u(r,\theta)\, d\theta} = e^{2I(r)}. \tag{4.2}$$

Set $t = \ln r$, and $m(t) = I(e^t)$. Denoting derivatives with respect to t by a dot we have

$$\dot{m}(t) = ((rI'(r))(e^t) \quad \text{and} \quad \ddot{m}(t) = (r[rI'(r)]')(e^t)$$

From the above we get, with $t_0 = \log r_0$,

$$\dot{m}(t) = -\frac{1}{2\pi}\int_{|z|\leq e^t} K dA$$
$$\ddot{m}(t) \geq e^{2m(t)} \quad \text{for } t \geq t_0.$$

Suppose now that the conclusion of the lemma is not verified. From the above equations and the fact that the integral of K over the disc $\{|z| \leq e^t\}$ is a decreasing function of t for t large, there exists real numbers $a > 0$ and $t_1 > t_0$ such that $\dot{m}(t) \geq a$ for all $t \geq t_1$. Thus

$$\frac{d}{dt}(\dot{m}(t))^2 = 2\dot{m}(t)\ddot{m}(t) \geq 2\dot{m}(t)e^{2m(t)} = \frac{d}{dt}e^{2m(t)} \quad ; \quad t \geq t_1.$$

Integrating both sides from t_1 to $t \geq t_1$ we get

$$\dot{m}^2(t) \geq e^{2m(t)}(1 + ce^{-2m(t)}) \quad ; \quad c = \dot{m}^2(t_1) - e^{2m(t_1)}.$$

Since $\dot{m}(t) \geq a$ for $t \geq t_1$ we have $m(t) \to \infty$ as $t \to \infty$. Take $t_2 > t_1$ such that for $t \geq t_2$ we have $ce^{-2m(t)} > -\frac{1}{4}$. Thus for $t \geq t_2$ we obtain

$$\dot{m}(t)e^{-m(t)} \geq \frac{1}{2}$$

and integrating both sides from t_2 to $t \geq t_2$ we get

$$-e^{-m(t)} + e^{-m(t_2)} \geq \frac{1}{2}(t - t_2).$$

But the left side of this inequality is bounded and the variable t is supposed to be defined all over the reals. This contradiction establishes that for t large enough we have $\int_{|z|\leq e^t} K dA > 0$, that proves the lemma. □

For the sake of completeness we prove the following known lemma.

Lemma 4.2. *Let $ds^2 = e^{2u}|dz|$ be a complete conformal metric defined on $\mathbb{C}$ such that the Gauss curvature K satisfies $K \leq -1$ outside some compact set. Let $d(z)$ be the distance from the origin with respect to the metric ds^2 and let $B(r) = \{z \mid d(z) \leq r\}$ be the geodesic ball of radius r. Let $L(r)$ denote the length of $\partial B(r)$. Then $L(r) \to 0$ as $r \to \infty$.*

Proof. By the precedent lemma $\int_{\mathbb{C}} KdA$ exists and is non-negative. As $K \leq -1$ outside a compact set, the total area of the complete surface $(\mathbb{C}, ds^2)$ is finite. A result of Huber [13, theorem 12] tells us that for a complete surface of finite total curvature and finite total area we have equality in the Cohn-Vossen inequality; thus

$$\int_{\mathbb{C}} K\,dA = 2\pi\chi(\mathbb{C}) = 2\pi$$

where $\chi(\mathbb{C})$ is the Euler characteristic of the plane. For almost all r the boundary $\partial B(r)$ is a finite union of piecewise differentiable Jordan curves and for those r the derivative $L'(r)$ exists and satisfies [18, theorem 1]

$$L'(r) \leq 2\pi(2 - 2h(r) - c(r)) - \int_{B(r)} K\,dA$$

where $c(r)$ = number of connected components of $\partial B(r)$ and $h(r)$ = number of handles inside $B(r)$. In our case $h(r) = 0$ and $c(r) \geq 1$, so

$$L'(r) \leq 2\pi - \int_{B(r)} K\,dA$$

Let r_0 be such that for $r \geq r_0$ we have $K \leq -1$ on $\mathbb{C}^* \setminus B(r_0)$. Thus, for $r \geq r_0$, $\int_{B(r)} K\,dA$ is a decreasing function of r which goes to 2π as $r \to \infty$. Hence $L'(r) < 0$ for $r \geq r_0$. By the co-area formula we have

$$\int_0^\infty L(r)\,dr \leq \int_0^\infty \left(\int_{\partial B(R)} |\nabla r|^{-1} ds \right) dr = \text{Area}(\mathbb{C}, ds^2).$$

Therefore $\int_0^\infty L(r)\,dr < \infty$ and $L'(r) < 0$ for almost all $r \geq r_0$ and this implies that $L(r) \to 0$ as $r \to \infty$. □

4.1. Proof of theorem B (conformal type). For $r > 0$, we denote by $D^*(r)$ the punctured disc $\{0 < |z| < r\}$ and we let D^* be the unit punctured disc. The assertion about the conformal type of the ends of a minimal surface in hyperbolic space is a consequence of the following

Lemma 4.3. *Let $x\colon D^* \hookrightarrow \mathbb{H}^n$ be a conformal minimal immersion. Then there exists a path $\gamma\colon [0,1) \to D^*$ converging to the origin 0 as $t \to 1$ and such that $\int_\gamma ds < \infty$, where ds^2 is the metric on D^* induced by the immersion x.*

Proof. We consider the Poincaré model of $\mathbb{H}^n$ so that $\mathbb{H}^n$ is the unit ball $\{|x| < 1\}$ of $\mathbb{R}^n$ endowed with the metric $d\eta^2 = 4|dx|^2/(1-|x|^2)^2$. The area element dA of the metric ds^2 is given by

$$(4.3) \qquad dA = \frac{1}{2}\|\nabla x\|^2 dudv$$

where $z = u + iv$ is a point of $\mathbb{C}$ and $\|\nabla x\|^2 = \frac{4}{1-|x|^2}(|x_u|^2 + |x_v|^2)$ is the hyperbolic norm of $\nabla x = (\nabla x^1, \dots, \nabla x^n)$.

As x is minimal we have by Gauss equation $K \leq -1$ on D^*. Extend the metric ds^2 to a smooth metric $d\tilde{s}^2$ on $\mathbb{C}^* \cup \{\infty\}$ such that inside $D^*(\frac{1}{2})$ it coincides with ds^2. Let us suppose that the conclusion of the lemma does not hold. This means the metric $d\tilde{s}^2$ is a complete conformal metric on $\mathbb{C}^* \cup \{\infty\}$ satisfying $K \leq -1$ outside some compact set. By lemma 4.1 the total curvature is finite and in particular the area $\int_{D^*(\frac{1}{2})} dA$ of $x(D^*(\frac{1}{2}))$ is finite. Therefore

$$(4.4) \qquad \int_{D(\frac{1}{2})} \|\nabla x\|^2 dudv < \infty.$$

Also, by the monotonicity theorem of Anderson [1], $x(D^*(\frac{1}{2}))$ is contained in a compact set of $\mathbb{H}^n$; otherwise $x(D^*(\frac{1}{2}))$ would have infinite area. This fact and (4.4) implies that the restriction of the immersion x to $D^*(\frac{1}{2})$ belongs to $H_2^1(D(\frac{1}{2}), \mathbb{H}^n)$, the space of maps $f\colon D(\frac{1}{2}) \mapsto \mathbb{H}^n$ such that f and $|\nabla f|$ belong to $L_2((D(\frac{1}{2}))$ (v. [10]).

On $D^*(\frac{1}{2})$ the conformal minimal immersion x satisfies the system of equations

$$(4.5) \qquad \Delta x^i = F^i(x, \nabla x) \quad ; \quad \text{for } i = 1, \dots, n$$

where

$$F^i(x, \nabla x) = \frac{2}{1-|x|^2}\Big(x^i|\nabla x|^2 - 2\langle x, x_u\rangle x_u^i - 2\langle x, x_v\rangle x_v^i\Big).$$

We assert that x is a weak solution of (4.5) on $D(\frac{1}{2})$. In fact if $\phi = (\phi^1, \dots, \phi^n)$ is a smooth map compactly supported in $D(\frac{1}{2})$ then the integrals

$$I_i = \int_{D(\frac{1}{2})} \Big(\langle \nabla x^i, \nabla \phi^i\rangle + F^i(x, \nabla x)\phi^i\Big)\, dudv \quad ; \quad i = 1, \dots, n$$

are well defined since x is bounded in $D(\frac{1}{2})$ and $x \in H_2^1(D(\frac{1}{2}), \mathbb{H}^n)$. Let $z_0 \in \mathbb{C}^*$ be a fixed point and let $B(r)$ be the geodesic ball for the metric $d\tilde{s}^2$, of radius r and centered in z_0. For $i = 1, \ldots, n$ the integrals I_i can be written as

$$I_i = \lim_{r \to \infty} \int_{D^*(\frac{1}{2}) \cap B(r)} \left(\langle \nabla x^i, \nabla \phi^i \rangle + F^i(x, \nabla x) \phi^i \right) dudv.$$

Observe that for r large enough the boundary $\partial B(r)$ is contained in $D(\frac{1}{2})$ and that, by lemma 4.2, the length of $\partial B(r)$ goes to 0 as $r \to \infty$. Let $\{r_k\}$, $k \in \mathbb{N}$, be a sequence with $r_k \to \infty$ as $k \to \infty$, and such that $\partial B(r_k)$ is a finite union of piecewise smooth curves. By equation (4.4) and Green's formula we get

$$I_i = \lim_{k \to \infty} \int_{\partial B(r_k)} \phi^i \frac{\partial x^i}{\partial \nu} |dz|$$

where ν is the interior normal to $\partial B(r_k)$, defined but for a finite number of points. Since $|\frac{\partial x^i}{\partial \nu}| \leq |\nabla x|$ we have that $|\frac{\partial x^i}{\partial \nu}||dz| \leq ds$ on $D^*(\frac{1}{2})$. Hence

$$|I_i| \leq \left(\max_{D(\frac{1}{2})} |\phi| \right) \int_{\partial B(r_k)} ds.$$

As the length of $\partial B(r_k)$ goes to 0 as $k \to \infty$ we have $I_i = 0$ for $i = 1, \ldots, n$, and therefore x is a weak solution of (4.5) on $D(\frac{1}{2})$. By the regularity result of Grüter [10, theorem 3.8] a minimal immersion x as above is of class $C^{1,\alpha}$ on $D(\frac{1}{2})$, for all $0 < \alpha < 1$. But this implies that any path γ converging to the origin and having finite Euclidean length has also finite length in the induced metric ds. This contradiction establishes the lemma. □

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