BRIAN WHITE - MINIMAL SURFACES (MATH 258) LECTURE NOTES

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Contents

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We would like to thank Brian White for an excellent class, as well as the audience for stimulating discussion. Additionally, we are grateful to David Hoffman for pointing out several typos as well as providing various pictures of minimal surfaces. Please be aware that the notes are a work in progress; it is likely that we have introduced numerous typos in our compilation process, and would appreciate it if these are brought to our attention.

1. Higher Dimensional Mapping Problem

The following section is somewhat out of place with respect to the rest of the notes, because some of the first year graduate students were in the middle of taking their qualifying exams, so this was a special lecture.

We'll briefly describe the *Plateau Problem* as follows: given a closed $(n-1)$-submanifold $\Gamma \subset \mathbb{R}^N$, can we "find" a least area manifold $M^n \subset \mathbb{R}^N$ so that $\partial M = \Gamma$. Of course, to make this precise we would need to define "∂" and "area." One of the first answers to this problem (for $n = 2$) was given by Douglas and Rado independently (cf. [Dou31, Rad30])

Theorem 1.1 (Douglas-Rado solution to the Plateau Problem)**.** *Given $\varphi : \partial B^2 \to \mathbb{R}^N$ a smooth embedding, there exists a smooth mapping $F : B^2 \to \mathbb{R}^N$ so that $F|_{\partial B^2} = \varphi$ and so that F minimizes the area:*

$$\operatorname{area}(F) = \int_{B^2} \sqrt{\left|\frac{\partial F}{\partial y}\right|^2 \left|\frac{\partial F}{\partial x}\right|^2 - \left(\frac{\partial F}{\partial x} \cdot \frac{\partial F}{\partial y}\right)^2}\, dxdy,$$

among all competing maps. Furthermore, F is an immersion away from isolated points, known as branch points.

We emphasize that Douglas and Rado only showed that F was smooth in the interior, so we have been somewhat imprecise in our statement of the above theorem.

Remark 1.2. When $N = 3$, Osserman has shown that there can be no interior branch points (see [Gul73]), while for $N > 3$, branch points do occur.

Furthermore, Douglas extended this in [Dou39b] to a solution for to Plateau's problem for surfaces with higher genus

Theorem 1.3. *Fix $\Gamma \subset \mathbb{R}^N$, a simple closed curve and $g \geq 0$. Among all oriented surfaces M with $\partial M = \Gamma$ and* $\operatorname{genus} M \leq g$*, there exists a smooth branched immersed surface in this class achieving the minimal area.*

As in the genus zero case, there cannot be interior branch points. We do note that it remains a major open problem to prove non-existence of boundary branch points in general.

Remark 1.4. We remark that it is impossible to minimize among surfaces of a fixed genus $g > 0$, as is easily seen by considering a planar curve in $\mathbb{R}^2 \subset \mathbb{R}^2$.

Thirty years later, a completely different solution to the Plateau Problem was found by Federer-Fleming in their paper "Normal and Integral Currents" [FF60], proving

Theorem 1.5 (Federer-Fleming)**.** *Given Γ a $(n-1)$-dimensional submanifold of $\mathbb{R}^N$, among all n-dimensional surfaces M with $\partial M = \Gamma$, there is a (not necessarily unique) M of minimal area (however, no regularity was established).*

Later, the regularity of the minimizer was drastically improved by Almgren and Hardt–Simon, cf. [Alm68, HS79]. In particular for minimal hypersurfaces, we have

Theorem 1.6. *For $n < 7$, if $\Gamma \subset \mathbb{R}^n$ is a $(n-2)$-dimensional submanifold, then a Federer-Fleming minimizer M^{n-1} is a smooth embedded submanifold. In general, the minimizer is not necessarily smooth, but could have a singular set of codimension 7 or higher.*

One of the particular features of this theory is that there is *no* control on the topology of the minimal surface M as there is in the Douglas result above. As such, one might consider the "higher dimensional mapping problem"

Question 1.7. Given $\varphi : \partial B^n \to \mathbb{R}^N$ a smooth embedding, among all maps $F : B^n \to \mathbb{R}^N$ with $F|_{\partial B^n} = \varphi$, does there exist a map of least area? As before, we define area by the formula

$$\operatorname{area}(F) = \int_{B^n} |\operatorname{Jac} F| \, dx.$$

In other words, this is the question whether or not the Douglas–Rado theory can be extended to higher dimensions. We have the following answer:

Theorem 1.8 (White [Whi83])**.** *For $\varphi : \partial B^n \to \mathbb{R}^{n+1}$, with $2 < n \leq 6$, there is a smooth least area mapping $F : B^n \to \mathbb{R}^{n+1}$ whose image is the Federer–Flemming least area surface along with lower dimensional pieces.*

It is not hard to see that there are examples of φ which must have F necessarily mapping an open subset of B^n to a set of n-dimensional measure zero. Furthermore, we may show

Theorem 1.9 (White [Whi83])**.** *For $n > 2$, in any codimension, the area of the Federer–Fleming solution is the same as the infimum over smooth maps F as described above.*

Proof of Theorem 1.8. We remark that a map $F : B^n \to \mathbb{R}^{n+1}$ with $F|_{\partial B} = \varphi$ is equivalent to a homotopy from φ to a constant map. Choose M to be a Federer–Fleming least area surface. We choose a triangulation of M, X. We will denote the k-skeleton of X by $X^{(k)}$. We'll first homotope $\partial M = \varphi(\partial B)$ into the $(n-1)$-skeleton $X^{(n-1)}$ to define F on $B_1 \backslash B_{1/2}$. To do so, for each n-cell in X which intersects ∂M, we homotope the region touching ∂M to the other part of the boundary of the cell. We may then repeat this until we have homotoped ∂M into $X^{(n-1)}$, i.e. $F(\partial B_{1/2}) \subset X^{(n-1)}$. Let $\hat{\varphi} := F|_{B_{1/2}}$ and note that $\hat{\varphi}$ is homogically trivial inside $X^{(n-1)}$. This follows from the construction of $\hat{\varphi}$ and the obvious fact that ∂M is homologically trivial inside of M (in the language of currents, it is not hard to see that the image of $\hat{\varphi}$ is the zero current). See Figure 1 for an illustration of this homotopy.

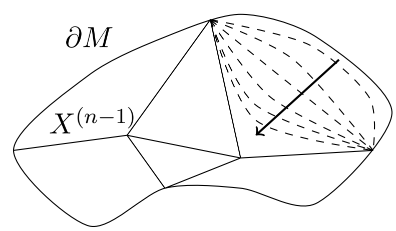

FIGURE 1. The homotopy from ∂M to $X^{(n-1)}$ to define $\hat{\varphi}$ in the proof of Theorem 1.8.

Now, we need to fill in $B_{1/2}$. Notice that there is no reason which the image of $\hat{\varphi}$ should be homotopically trivial inside of $X^{(n-1)}$. We let $Y := X^{(n-1)} \cup (0 \divideontimes X^{(n-2)})$, where the second term is the cone of the $(n-2)$ skeleton of X with respect to the origin 0. Because $\dim(Y) = n-1$, we see that $H^n(Y) = 0$. We claim that the image of $\hat{\varphi}$ is homotopically trivial inside of Y. This will conclude the proof, because we have already covered M by the image of $B_1 \backslash B_{1/2}$, so we just need to fill in the rest of $B_{1/2}$ without spending any n-area. In order to show that the image is homotopically trivial inside of Y, we recall

Theorem 1.10 (Hurewicz Theorem)**.** *For $m > 1$, if $\pi_1(Y) = \pi_2(Y) = \dots \pi_{m-1}(Y) = 0$, then the obvious map $\pi_m(Y) \to H^m(Y)$ is an isomorphism.*

This implies that for the Y constructed above, we have that $\pi_{n-1}(Y) \xrightarrow{\simeq} H_{n-1}(Y)$. As such, because the image of $\hat{\varphi}$ is homologically trivial in Y, it must also be homotopically trivial as well. This allows us to conclude the theorem, as remarked above. □

We also remark that the phenomena of the minimal disk approximating the topology of the Federer–Fleming solution also occurs in two dimension in a certain sense, when finding minimal annuli. For example, given two parallel circles, if they are close enough together, there is a minimal annulus, called the catenoid. On the other hand, if they are far enough apart, then there cannot exist an embedded minimal annulus between the two circles. So, the only minimal surface bounded by these disks is the one illustrated in Figure 2.

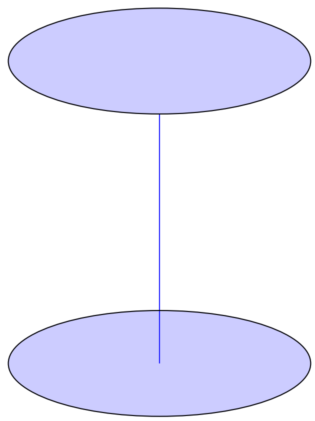

FIGURE 2. The only minimal annulus in the Douglas–Rado sense, which bounds two disks which are far apart is the two flat disks connected by a line.

2. Rado's Existence Theorem

In the present lecture we will concern ourselves with the existence aspect of the Plateau problem as studied by Rado in [Rad30]. As a reminder, in this version of the Plateau problem we are given a smooth simple closed curve $\Gamma \subset \mathbb{R}^N$, and we seek to find a surface M of least area for which $\partial M = \Gamma$, minimizing among the M with are smooth images of the open unit ball $D = B^2$. Naturally we wonder: does such an M exist? Is it smooth?

Theorem 2.1 (Rado)**.** *Let Γ be a simple closed curve in $\mathbb{R}^N$ and consider the class $\mathscr{C}_\Gamma$ of maps $f : \overline{D} \to \mathbb{R}^N$ that are continuous on $\overline{D}$, smooth (or $\mathcal{C}^1$ or locally Lipschitz) on D, and which map ∂D onto Γ monotonically. For $f \in \mathscr{C}_\Gamma$ we define:*

$$\operatorname{area}(f) = \int_D |\operatorname{Jac} f| = \int_D \sqrt{\left|\frac{\partial F}{\partial x}\right|^2 \left|\frac{\partial F}{\partial y}\right|^2 - \left(\frac{\partial F}{\partial x} \cdot \frac{\partial F}{\partial y}\right)^2}$$

In this context, there exists a map $F \in \mathscr{C}_\Gamma$ such that

(1) $\operatorname{area}(F)$ *is the infimum among* $\operatorname{area}(G)$, $G \in \mathscr{C}_\Gamma$,
(2) F *is harmonic,*
(3) F *is almost conformal, i.e.* $\left|\frac{\partial F}{\partial x}\right| = \left|\frac{\partial F}{\partial y}\right|$ *and* $\frac{\partial f}{\partial x} \cdot \frac{\partial f}{\partial y} = 0$, *and finally*
(4) *the singular points $p \in D$ for which $DF(p) = 0$ are isolated.*

Remark 2.2. The class $\mathscr{C}_\Gamma$ is a natural one in which to study the Plateau problem. Observe that $f \in \mathscr{C}_\Gamma$ mapping ∂D monotonically onto Γ is a slightly weaker hypothesis than the mapping being a homeomorphism. In this context, Rado's theorem guarantees not only that a minimizing F exists within this class, but in fact that a "nice" such map exists: one that is additionally harmonic and almost conformal, with isolated singular points.

Remark 2.3. Why is existence hard? The natural method to approach such existence problems is the *direct method.* Let α be the infimum of areas attained by surfaces M with $\partial M = \Gamma$, and suppose that $M_1, M_2, \ldots$ is a minimizing sequence for the area. We hope that a subsequence will converge to the surface M of least area. However, there exist very "bad" minimizing sequences as in Figure 3.

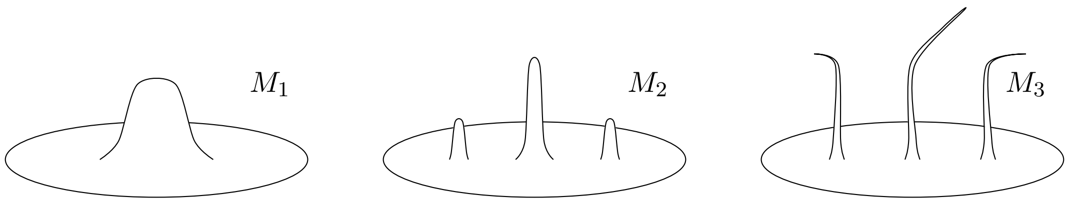

FIGURE 3. Example of a "bad" minimizing sequence.

Our need to avoid such "bad" sequences motivates the use of an *energy functional* in addition to the area functional we have already defined.

Definition 2.4. We define the energy of the map $f : D \to \mathbb{R}^N$ to be

$$\mathrm{E}(f) = \frac{1}{2}\int_D |Df|^2 = \frac{1}{2}\int_D \left|\frac{\partial f}{\partial x}\right|^2 + \left|\frac{\partial f}{\partial y}\right|^2$$

We are going to need three basic facts about the area and the energy functionals.

Fact 2.5 (Energy dominates area)**.** *For every map $f : D \to \mathbb{R}^N$ we have that* $\mathrm{area}(f) \leq \mathrm{E}(f)$*, with equality if and only if f is almost conformal.*

Fact 2.6 (Harmonic maps minimize energy)**.** *If $f, h : \overline{D} \to \mathbb{R}^N$ are smooth, $f \equiv h$ on ∂D and h is harmonic then* $\mathrm{E}(h) \leq \mathrm{E}(f)$*.*

Proof. The proof of this fact is based upon the observation that we may write $f = h + \varphi$ with $\varphi \equiv 0$ on ∂D, in which case by expanding the square and employing the divergence theorem we see that

$$\mathrm{E}(f) = \mathrm{E}(h+\varphi) = \mathrm{E}(h) + \mathrm{E}(\varphi) + \int_D Dh \cdot D\varphi = \mathrm{E}(h) + \mathrm{E}(\varphi) + \int_{\partial D} \varphi \cdot \frac{\partial h}{\partial \nu} - \int_D \varphi \cdot \Delta h$$

The last two terms vanish in view of h being harmonic and φ being identically zero on ∂D, and in particular we conclude that $\mathrm{E}(f) = \mathrm{E}(h) + \mathrm{E}(\varphi) \geq \mathrm{E}(h)$. $\square$

The final fact that we are going to need, whose proof is a simple exercise left to the reader, concerns itself with the conservation of areas and energy under certain compositions of functions:

Fact 2.7 (Conservation of area and energy)**.** *If $f : D \to \mathbb{R}^N$ is given, then* $\mathrm{area}(f \circ u) = \mathrm{area}(f)$ *for all diffeomorphisms $u : D \to D$ and* $\mathrm{E}(f \circ u) = \mathrm{E}(f)$ *when u is additionally conformal.*

Finally we return to Rado's theorem.

Proof of Theorem 2.1. We break the proof down into steps.

Step 1. Observe that the infimum of areas is the same if we only consider maps smooth up to the boundary. The idea is to take $f \in \mathscr{C}_\Gamma$ and approximate f by a map $\tilde{f} \in \mathscr{C}_\Gamma$ that is smooth up to the boundary and such that $\mathrm{area}(\tilde{f}) \leq \mathrm{area}(f) + \varepsilon$.

Step 2. Given an $f \in \mathscr{C}_\Gamma$ smooth up to the boundary and $\varepsilon > 0$, we claim that there exists a harmonic function $h \in \mathscr{C}_\Gamma$ such that $\mathrm{E}(h) \leq \mathrm{area}(f) + \varepsilon$. For $\delta > 0$ define $F_\delta : \overline{D} \to \mathbb{R}^{N+2}$ given by

$F_\delta(x,y) = (f(x,y), \delta x, \delta y)$. This is a smooth embedding into $\mathbb{R}^{N+2}$. By Korn–Lichtenstein there exists a conformal diffeomorphism $G_\delta : \overline{D} \to F_\delta(\overline{D})$. If π denotes the projection of $\mathbb{R}^{N+2}$ onto $\mathbb{R}^N$ and we set $\widehat{G_\delta} = \pi \circ G_\delta$ and take $h_\delta : \overline{D} \to \mathbb{R}^N$ with prescribed values $\widehat{G_\delta}$ at the boundary ∂D then

$$\mathrm{E}(h_\delta) \leq \mathrm{E}(\widehat{G_\delta}) \leq \mathrm{E}(G_\delta) = A(G_\delta) = A(F_\delta)$$

because harmonic maps minimize energy, projections decrease energy, conformal maps have equal energy and area, and area is independent of parametrization. In particular area$(F_\delta) \to$ area(f) as $\delta \searrow 0$, so in particular we may pick our harmonic function to be h_δ for $\delta > 0$ small enough.

Remark 2.8. So far it has been important that we haven't specified a fixed parametrization of the boundary but we have been free to choose any one of them.

Remark 2.9. If A_Γ denotes the infimum of attainable areas among $f \in \mathscr{C}_\Gamma$ and E_Γ denotes the infimum of attainable energies from within the same class, then Step 2 guarantees that $A_\Gamma = E_\Gamma$. In particular, we may focus on minimizing energy from this point on.

Step 3. Suppose that $h_1, h_2, \ldots$ is a sequence of harmonic maps $h_n : \overline{D} \to \mathbb{R}^N$ in our class $\mathscr{C}_\Gamma$ such that $\mathrm{E}(h_n) \to A_\Gamma$. Since we are imposing the uniform boundary condition $h_n(\partial D) = \Gamma$, by the maximum principle we conclude that the sequence of h_n is uniformly bounded and thus normal. By Montel's theorem in complex analysis, our sequence h_n converges smoothly on compact subsets of D to a harmonic function h, and by Fatou's lemma $E(h) \leq \liminf_n E(h_n) = A_\Gamma$. The main question is whether or not the limiting function h is in our class $\mathscr{C}_\Gamma$. In particular since Montel's theorem tells us nothing about behavior at the boundary, we need to worry about that. See Figure 4 for an example of what could go wrong.

Remark 2.10. Consider the sequence of harmonic functions $h_n = r^n \sin n\theta$ on $\overline{D}$. Then certainly $h_n \to 0$ smoothly on compact subsets of D, but on the other hand $\mathrm{E}(h_n) \to \infty$. This motivates that there is something to worry about in the passage to the limit, as in this example we seem to have lost something in doing so.

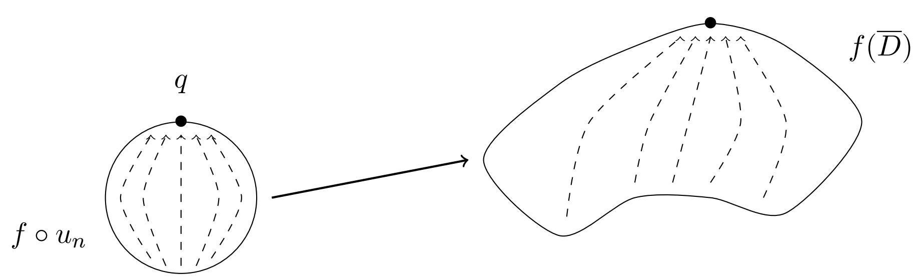

FIGURE 4. Equicontinuity at the boundary is not automatic. We could have a sequence of conformal maps $u_n : D \to D$ such that $u_n(p) \to q$ (fixed) for all $p \in D$.

Step 4. We need to show that h_n is equicontinuous on the boundary. This is actually not automatic, so instead we impose a three point condition! Fixing $a, b, c \in \partial D$ and $A, B, C \in \Gamma$ with the same orientation we require that $a \mapsto A$, $b \mapsto B$, and $c \mapsto C$ at all previous steps of the proof and invoke the Courant–Lebesgue lemma.

Lemma 2.11 (Courant–Lebesgue). *Suppose $\Omega \subset \mathbb{R}^2$ open, $p \in \mathbb{R}^2$, $u : \Omega \to \mathbb{R}^N$, $\mathrm{E}(u) \leq E < \infty$, and $R \in (0,1)$ are given. Then there exists a radius r between R and $\sqrt{R}$ such that*

$$\left(\text{length}\, u|_{\Omega \cap \partial B_r(p)}\right)^2 \leq \frac{8\pi E}{\log 1/R}$$

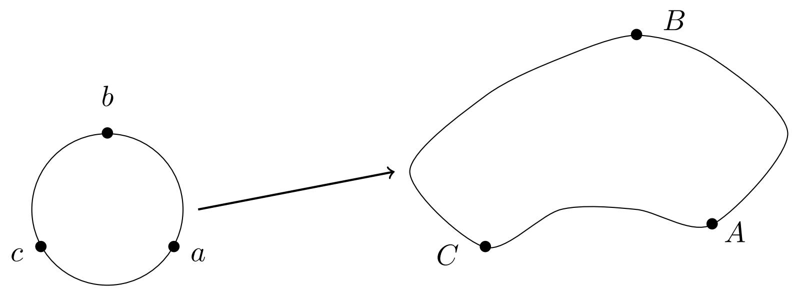

FIGURE 5. Three point condition, $a \mapsto A$, $b \mapsto B$, $c \mapsto C$.

Proof. We may assume that $p = 0$. If $L(r) = \operatorname{length} u|_{\Omega\cap\partial B_r}$ and $L = \min_{R\le r\le\sqrt{R}} L(r)$ then

$$L^2 \le L(r)^2 = \left(\int_{\Omega\cap\partial B_r}\left|\frac{\partial u}{\partial\theta}\right| d\theta\right)^2 \le 2\pi\int_{\Omega\cap\partial B_r}\left|\frac{\partial u}{\partial\theta}\right|^2 d\theta = 2\pi r^2\int_{\Omega\cap\partial B_r}\left|\frac{1}{r}\frac{\partial u}{\partial\theta}\right|^2 d\theta$$

On the other hand

$$|Du|^2 = \left|\frac{\partial u}{\partial x}\right|^2 + \left|\frac{\partial u}{\partial y}\right|^2 = \left|\frac{1}{r}\frac{\partial u}{\partial\theta}\right|^2 + \left|\frac{\partial u}{\partial r}\right|^2$$

and therefore

$$L^2 \le 2\pi r^2\int_{\Omega\cap\partial B_r}|Du|^2 d\theta$$

Rearranging and integrating over $R \le r \le \sqrt{R}$ gives

$$L^2\log\frac{\sqrt{R}}{R} = \int_R^{\sqrt{R}}\frac{L^2}{r}dr \le 2\pi\int_R^{\sqrt{R}}\int_{\Omega\cap\partial B_r}|Du|^2 r\,d\theta\,dr \le 4\pi\,\mathrm{E}(u)$$

and the result follows upon rearranging. □

Now we return to Step 4 of the proof. Recall that we have chosen $h_n : \overline{D} \to \mathbb{R}^N$ a sequence of harmonic maps in our class $\mathscr{C}_\Gamma$ with $\mathrm{E}(h_n) \to A_\Gamma$ and which satisfy the three point condition $a \mapsto A$, $b \mapsto B$, $c \mapsto C$. We claim that now we can prove equicontinuity, i.e. that for $p_n, q_n \in \partial D$ and $|p_n - q_n| \to 0$ we get $|F_n(p_n) - F_n(q_n)| \to 0$. Write $R_n = |p_n - q_n| \ll 1$. By Courant–Lebesgue there exists a $r_n \in (R_n, \sqrt{R_n})$ such that

$$(\operatorname{length} h_n|_{C_n})^2 \le \frac{8\pi}{\log 1/R_n} \to 0$$

where $C_n = D \cap \partial B_{r_n}(p_n)$. Then $|F_n(p_n') - F_n(q_n')| \le \operatorname{length} h_n|_{C_n} \to 0$ for p_n', q_n' the intersection points $C_n \cap \partial D$. If S_n denotes the little arc of ∂D with endpoints p_n', q_n' ($p_n, q_n \in S_n$) we claim that $F_n(S_n)$ is the short arc of Γ with endpoints $F_n(p_n')$, $F_n(q_n')$, not the long one. Indeed, since S_n is short it contains at most one of $\{a, b, c\}$ so $F_n(S_n)$ contains at most one of $\{A, B, C\}$, making it the short arc and thus $|F_n(p_n) - F_n(q_n)| \to 0$, proving equicontinuity.

Step 5. We check that the limiting function along some subsequence is in the correct class $\mathscr{C}_\Gamma$, harmonic and almost conformal. By passing to a subsequence we can assume that $h_n|_{\partial D}$ converges uniformly to a continuous function on the boundary, i.e. it is Cauchy. The function $h_n - h_m$ is harmonic so by the maximum principle for harmonic functions

$$\max_{\overline{D}}|h_n - h_m| = \max_{\partial D}|h_n - h_m| \to 0$$

Therefore $h_n \to \varphi$ uniformly on $\overline{D}$ and smoothly in the interior (Montel's theorem), with φ mapping the boundary monotonically. Then $\varphi \in \mathscr{C}_\Gamma$, $\mathrm{E}(\varphi) = A_\Gamma$, $\Delta\varphi = 0$, and $A_\Gamma \le \operatorname{area}(\varphi) \le \mathrm{E}(\varphi) = A_\Gamma$ which forces φ to be almost conformal.

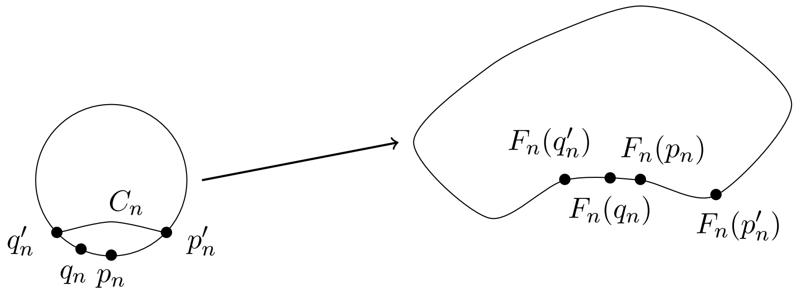

FIGURE 6. Establishing equicontinuity at the boundary by the Courant-Lebesgue lemma.

Step 6. Interior branch points are isolated. This comes from the fact that the minimizer φ is a harmonic function on D and therefore for $z = x + iy$ and the differentials

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \text{ and } \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

we see that

$$\Delta\varphi = 0 \Leftrightarrow \frac{\partial}{\partial \overline{z}}\frac{\partial \varphi}{\partial z} = 0 \Leftrightarrow \frac{\partial \varphi}{\partial z} \text{ is holomorphic}$$

Recall that zeros of holomorphic functions are isolated and the result follows. □

With a slightly more refined argument the precise version of the theorem we can get is:

Theorem 2.12. *Suppose $F : \overline{D} \to \mathbb{R}^N$ represents a classical minimal disk: $F : \overline{D} \to \mathbb{R}^N$ is continuous, $F : \partial D \to \Gamma$ is continuous and monotone, $\Delta F = 0$, F is almost conformal, and* $\operatorname{area}(F)\,(= \operatorname{E}(F)) < \infty$*. If $k \geq 1$ and Γ is $\mathcal{C}^{k,\alpha}$ then $F|_{\overline{D}}$ is $\mathcal{C}^{k,\alpha}$ too. In particular if Γ is smooth then $F : \overline{D} \to \mathbb{R}^N$ is smooth as well. Also $\{p \in \overline{D} : DF(p)\}$ is a finite set, i.e. branch points are isolated.*

Remark 2.13. As a reminder, in the case $N = 3$ it has been shown [Gul73] that there can be no interior branch points. The existence of boundary branch points in $\mathbb{R}^3$ is an open problem.

3. Douglas Theorem for Surfaces of Higher Genus

In this lecture we seek to extend the Douglas-Rado theorem to surfaces whose topology is more complicated than that of a disk. To this end there is the following theorem by Douglas [Dou39a].

Theorem 3.1 (Douglas Theorem for Annuli)**.** *Suppose there exists an annulus with boundary $\Gamma_1 \cup \Gamma_2$, where Γ_1, Γ_2 are disjoint oriented simple closed curves in $\mathbb{R}^N$, whose area is smaller than that of any pair of disks ("Douglas condition"). Then there exists a radius $R \in (1, \infty)$ and a continuous map $F : A(1, R) \to \mathbb{R}^N$, $A(1, R) = \{p \in \mathbb{R}^2 : 1 \leq |p| \leq R\}$, such that $\Delta F = 0$, F is almost conformal, $F : \partial A(1, R) \to \Gamma_1 \cup \Gamma_2$ monotonically such that* $\operatorname{area}(F)\,(= \operatorname{E}(F))$ *is the least possible area among annuli.*

Proof. The proof is more or less the same as before, the only catch being that we have no control over the optimal radius R because different annuli $A(1, R)$ are not conformally equivalent. Pick a minimizing sequence $F_i : A(1, R_i) \to \mathbb{R}^N$ with $\operatorname{E}(F_i)$ decreasing to the infimum of areas. Without loss of generality F_i is harmonic. In order to pass to a subsequence and argue as before, we need to check that R_i is bounded away from 1 and ∞.

Step 1. R_i stays away from 1. If we were to have $R_i \to 1$ along some subsequence then we could argue that $\operatorname{E}(F_i) \to \infty$ in a way similar to before. This gives a contradiction.

Step 2. R_i stays away from ∞. Recall that we have a uniform upper bound $\operatorname{E}(F_i) \leq E < \infty$. Then by Courant-Lebesgue there exist $r_i \in (1, R_i)$ such that $\operatorname{length} F_i|_{\partial B_{r_i}} \to 0$. We can perform

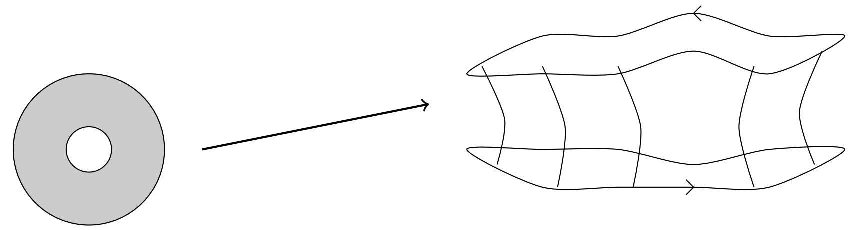

FIGURE 7. Plateau problem for annuli

FIGURE 8. Split the narrow neck into two disjoint disks.

surgery and split up the narrow neck into two small disjoint disks that are filled up. Then we get a pair of disk solutions to the Plateau problem whose area converges to the infimum, which contradicts the Douglas condition. □

Remark 3.2. As the narrow neck picture above suggests, one way to get the Douglas condition is for Γ_1, Γ_2 to be "close together" with the "same orientation."

The proof similarly extends to mappings from surfaces of higher genus. The statement is:

Theorem 3.3 (Douglas Theorem for Higher Genus, [Dou39a])**.** *Suppose Γ is a smooth embedded closed curve (not necessarily connected). For each $g < \infty$ there exists a least area genus $\leq g$ surface that solves the Plateau problem for Γ.*

Theorem 3.4 (Alternative Formulation of the Douglas Theorem)**.** *Let Γ be as above. If A_k denotes the infimum of areas among surfaces of genus k with boundary Γ then trivially $A_0 \geq A_1 \geq \ldots$ and if $A_g < A_{g-1}$ ("Douglas condition") then there exists a least area genus g surface.*

4. Intersections of Minimal Surfaces, Meeks-Yau Theorem

In this lecture we studied (self-)intersections of minimal surfaces in $\mathbb{R}^3$ (or more generally in 3-manifolds) and the question of whether or not the solution to the Plateau problem is embedded. We start off with an application of Douglas's higher genus theorem.

Theorem 4.1. *Suppose that in the Douglas theorem we have $A_{g-1} > A_g = A_{g+1}$ in $\mathbb{R}^3$ and that M is a least area genus g surface with $\partial M = \Gamma$. Then M does not intersect itself transversely in its interior.*

Proof. Suppose that M intersected itself transversely on at least one point. If $p \in M$ is the point where two sheets of M meet transversely, we can perform surgery around p and add a handle between the different sheets and smoothing as in Figure 9. This gives a genus $g + 1$ surface with smaller area, a contradiction. □

Remark 4.2. This gives a sufficient hypothesis for the "Douglas condition." If we have a genus g minimizer which crosses itself then we can construct a genus $g + 1$ surface with strictly smaller area thus guaranteeing $A_g > A_{g+1}$.

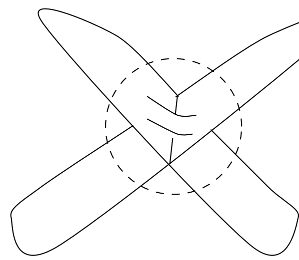

FIGURE 9. Constructing a higher genus surface with smaller area due to transverse intersection.

One reason regularity of solutions to the Plateau problem in $\mathbb{R}^3$ is simpler than regularity in $\mathbb{R}^N$ in general is because minimal surfaces intersect in predictable ways in $\mathbb{R}^3$ and 3-manifolds in general. The following theorem helps quantify this statement.

Theorem 4.3. *Suppose M, M' are smooth embedded minimal surfaces in $\mathbb{R}^3$ (or a 3-manifold). Suppose M, M' are tangent at a point p. Then for some neighborhood $\mathcal{U}$ of p either (1) the surfaces overlap, i.e. $M \cap \mathcal{U} = M' \cap \mathcal{U}$, or (2) $(M \cap M') \cap \mathcal{U}$ consists of k curves that meet at k equal angles at p ($k-1$ being the order of contact) and the point p of tangency is isolated.*

Proof. (In the special case of M' being the horizontal plane in $\mathbb{R}^3$.) Suppose our surface M is (locally) parametrized by $F : D \to \mathbb{R}^3$ so that F is harmonic and conformal. If f is the third coordinate of F then f is harmonic too and $M \cap M' = F(\{f = 0\})$, $f(0) = 0$, $Df(0) = 0$. Then either $f \equiv 0$, which is case (1), or $f = \operatorname{Re}\left(a_0 z^Q + a_1 z^{Q+1} + \ldots\right)$ for $Q \geq 2$. We may change coordinates conformally and assume $f = \operatorname{Re}\left(z^Q(1 + \widehat{a}_1 z + \ldots)\right)$. For z near 0 we can perform yet another conformal change of coordinates $w = z(1 + \widehat{a}_1 z + \ldots)^{1/Q}$ by fixing a choice of Q-th roots, in which case $M \cap M'$ locally consists of points $\operatorname{Re}\left(w^Q\right) = r^Q \sin Q\theta$, which is case (2). $\square$

Corollary 4.4. *If M, M' are two smooth embedded minimal surfaces in a 3-manifold N^3, and $X = \{p : M \cap \mathcal{U} = M' \cap \mathcal{U}$ for some neighborhood $\mathcal{U}$ of $p\}$, then X is simultaneously both closed and open in $(M \cup M') \setminus (\partial M \cup \partial M')$.*

Proof. Clearly X is open by definition, so it suffices to show its complement is open too. If q is in the complement then it is either not in M or not in M', which is an open condition, or at worse it is an isolated tangency point with curves of transverse intersection leading into it by the previous theorem, which is also an open condition. $\square$

This corollary allows us to prove that area minimizing disks are disjoint provided they merely stay away from each others' boundary.

Theorem 4.5. *Suppose M, M' are least area embedded disks in N^3. Suppose, further, that they do not intersect at boundary points in the sense that $\partial M \cap M' = M \cap \partial M' = \emptyset$. Then M and M' are disjoint.*

Proof. First, M and M' cannot overlap anywhere because by the corollary the overlap would extend to the boundary. Consequently $M \cap M'$ consists of finitely many points that are joined by smooth curves. By elementary graph theory, $M \cap M'$ contains a closed curve. We claim this contradicts the area minimizing nature of M, M'. Indeed suppose Σ, Σ' are the regions of M, M' that are bounded by the common closed curve. Then $\operatorname{area}(\Sigma) = \operatorname{area}(\Sigma')$, otherwise one could reduce the area of either M or M' by replacing the region of larger area by the region of smaller area. But now $(M \setminus \Sigma) \cup \Sigma'$ is an area minimizing disk which is not smooth, a contradiction. $\square$

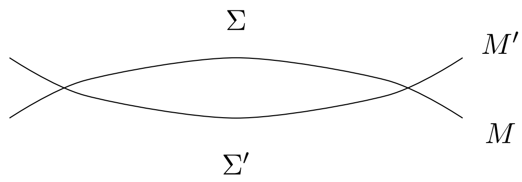

Figure 10. Intersection of two embedded minimal surfaces in a 3-manifold.

Remark 4.6. In reality we have only proved in previous lectures that the solution *we constructed* is smooth, not that any area minimizing solution is smooth. But this does hold, for example in this case by Osserman's theorem.

There are certain situations in which we can guarantee that our surfaces stay away from each others' boundary. For example we may make use of the following lemma:

Lemma 4.7 (Convex Hull Property)**.** *If $F : \overline{D} \to \mathbb{R}^N$ is continuous and $\Delta F = 0$, then $F(D)$ is contained in the convex hull of $F(\partial D)$. Furthermore, interior points get mapped to points in the interior of the convex hull*

Proof. The convex hull is the intersection among all half-spaces containing $F(\partial D)$. If ν is the normal to the half-space, the maximum of $F \cdot \nu$ is attained on ∂D so F never leaves any one of those half-spaces. The interior point claim follows from the strong maximum principle. □

Corollary 4.8. *If M, M' are minimal surfaces in the unit ball $B_1 \subset \mathbb{R}^3$ such that ∂M and $\partial M' \subset \partial B_1$, and also $\partial M \cap \partial M' = \emptyset$, then $\partial M \cap M' = M \cap \partial M' = \emptyset$.*

The next result we will state but not prove generalizes Dehn's lemma which was stated in 1910 and proved in 1957 in [Pap57a], [Pap57b]. This result is due to Meeks and Yau and (using the lemma above) proves in full generality a conjecture due to Osserman, which we state as its corollary.

Theorem 4.9 (Meeks-Yau Theorem, [MY80])**.** *Suppose $F : \overline{D} \to N^3$ is a least area disk with $F|_{\partial D}$ an embedding, $F(\partial D) = \Gamma$ a simple closed curve. Suppose $F^{-1}(\Gamma) = \partial D$, i.e. no interior point gets mapped to Γ. Then F is a smooth embedding.*

Corollary 4.10 (Osserman's conjecture)**.** *Let Γ be a simple closed curve on the boundary of the unit ball in $\mathbb{R}^3$, i.e. $\Gamma \subset \partial B_1 \subset \mathbb{R}^3$. Let $F : \overline{D} \to \mathbb{R}^3$ be the least area disk mapping ∂D to Γ. Then F is a smooth embedding.*

5. Branch Points

Recall that a branch point is locally modeled by $F : D \subset \mathbb{R}^2 \simeq \mathbb{C} \to \mathbb{C} \times \mathbb{R}^{N-2} \simeq \mathbb{R}^N$ where $F(0) = 0$ and $DF(0) = 0$. If F is minimal, we can expand it in a power series, and write $F(z) = (z^Q, 0) + o(z^Q)$. By a non-conformal change of coordinates, we may assume that $F(z) = (z^Q, f(z))$ for some function f. Hence, we have written this as a "multivalued graph," i.e. if we let $w = z^Q$, we're considering the "graph" of $f(w^{1/Q})$ (we won't take this point of view, but it's useful to keep in mind). At this point we can distinguish between "false" and "true" branch points.

False branch points. It is entirely possible that $f(z) = \psi(z^Q)$ for some ψ. In this case, the graph of F is the same as the graph of ψ and the surface was actually smooth, only the parametrization was bad.

True branch points. These are simply all the branch points that are not false. Relevant to these is

Theorem 5.1 (Osserman [Oss70])**.** *If $F : M^2 \to N^n$ is almost conformal and harmonic (i.e. minimal) and if F has a true interior brach point at $p \in M$, then F is not area minimizing.*

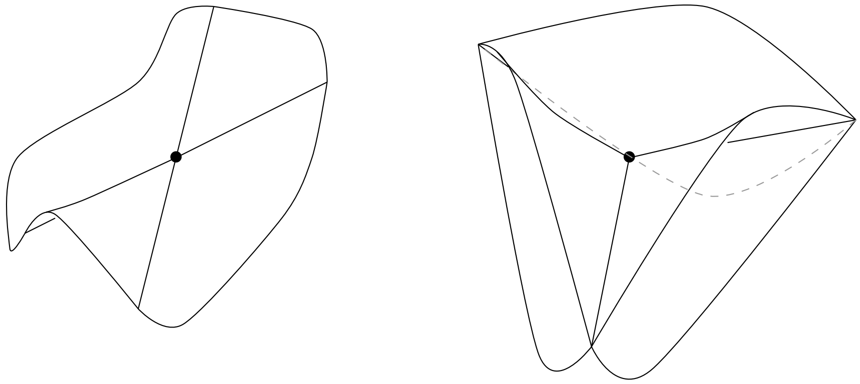

Figure 11. Examples of false branch points (left) and true branch points (right).

Proof. There exists a curve of transverse self intersection going to a branch point as in Figure 12. In particular, we could attempt to apply the same handle attaching argument as in Figure 9, which

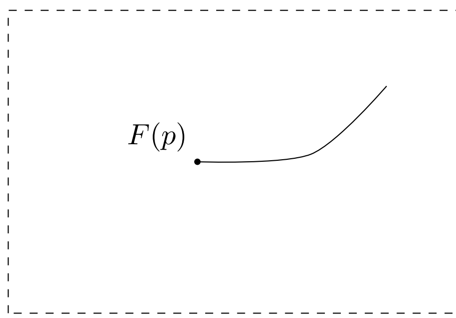

Figure 12. A curve of self-intersection near a branch point.

would in fact decrease the area. However, we cannot use this exact argument, as it could potentially raise the genus of the surface, which would not prove anything. Osserman's key observation was that in the branch point case, we can decrease the area without changing the topology. Osserman's

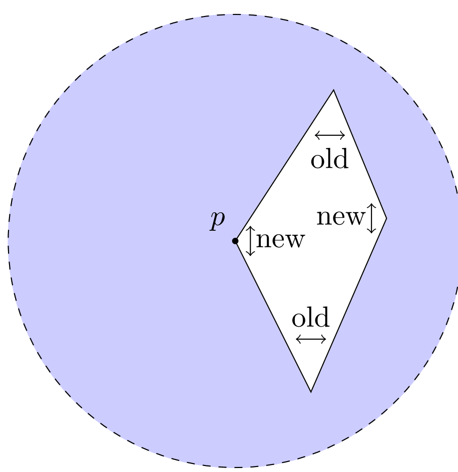

Figure 13. Osserman's area decreasing modification of a surface with a branch point.

gluing argument is illustrated in Figure 13, but essentially the idea is that because p is a branch point, there are at least two preimages of the curve of self intersection found above Then, by a clever gluing argument, one may show that by gluing the surface along these curves, it is possible to decrease the area without changing the topology. □

6. First Variation and Monotonicity Formulae

Suppose that $M^m \subset N^n$ is a submanifold. We'll consider variations of M, i.e. a family $f_t : M^m \to N^n$ so that $f_0(x) \equiv x$ along M. We'd like to compute $\frac{d}{dt}\big|_{t=0} \text{area}(f_t(M))$. First, recall that

$$\text{area}(f_t(M)) = \int_M |\operatorname{Jac}(f_t)| dx = \int_M \sqrt{\det(Df_t)^T Df_t} dx,$$

and assuming everything is smooth enough, we thus have that (using the fact that f_0 is an isometry, so $Df_0 = \text{Id}$)

$$\begin{aligned}
\frac{d}{dt}\Big|_{t=0} \text{area}(f_t(M)) &= \int_M \frac{d}{dt}\Big|_{t=0} \sqrt{\det(Df_t)^T Df_t} dx \\
&= \int_M \frac{1}{2\sqrt{\det\left((Df_t)^T Df_t\right)|_{t=0}}} \frac{d}{dt}\Big|_{t=0} \det\left((Df_t)^T Df_t\right) \\
&= \frac{1}{2}\int_M \operatorname{tr}\left(\frac{d}{dt}\Big|_{t=0} (Df_t)^T Df_t\right) dx
\end{aligned}$$

To calculate this, we first define a vector field by $X(x) := \frac{d}{dt}\big|_{t=0} f_t(x)$. In normal coordinates p around a point, we know that $Df_t = \left(\frac{\partial (f_t)^j}{\partial x^i}\right)_{i=1,\dots,m, j=1,\dots n}$, so

$$\begin{aligned}
\operatorname{tr}\left(\frac{d}{dt}\Big|_{t=0} \left((Df_t)^T Df_t\right)\right) &= \sum_{i=1}^m \sum_{k=1}^n \frac{d}{dt}\Big|_{t=0} \left(\frac{\partial (f_t)^k}{\partial x^i}\right)^2 \\
&= 2\sum_{i=1}^m \sum_{k=1}^n \frac{\partial (f_0)^k}{\partial x^i}\frac{\partial X^k}{\partial x^i} \\
&= 2\sum_{i=1}^m e_i \cdot \nabla_{e_i} X.
\end{aligned}$$

In particular, this shows that

$$\frac{d}{dt}\Big|_{t=0} \text{area}(f_t(M)) = \int_M \operatorname{div}_M X \; dx.$$

This is one version of the formula for the *first variation of area*. We will now derive several related versions which are also of much importance. By splitting up X into its component which is tangential to M, denoted X^T and component which is normal to M, denoted $X^\perp$, we have that

$$\int_M \operatorname{div}_M X = \int_M \operatorname{div}_M X^\perp + \int_M \operatorname{div}_M X^T$$

First, by Stoke's theorem, we have that

$$\int_M \operatorname{div}_M X^T = \int_{\partial M} X^T \cdot \nu = \int_{\partial M} X \cdot \nu$$

where ν is the (outward) pointing unit normal of ∂M. We have also used the fact that $\nu \in \Gamma(\partial M, TM)$, so $\nu \cdot X^\perp = 0$. On the other hand, we have that

$$\begin{aligned}\operatorname{div}_M X^\perp &= \sum_{i=1}^m e_i \cdot \nabla_{e_i} X^\perp \\ &= \sum_{i=1}^m \left(\nabla_{e_i} (\underbrace{e_i \cdot X^\perp}_{=0}) - (\nabla_{e_i} e_i) \cdot X^\perp \right) \\ &= -\sum_{i=1}^m \mathrm{II}(e_i, e_i) \cdot X^\perp \\ &= -H \cdot X^\perp \\ &= -H \cdot X,\end{aligned}$$

where we have defined the mean curvature H to be the trace of the second fundamental form, as above. As such, we may rewrite the first variation formula as

$$\frac{d}{dt}\Big|_{t=0} \operatorname{area}(f_t(M)) = \int_M \operatorname{div}_M X \, dx = -\int_M H \cdot X + \int_{\partial M} X \cdot \nu.$$

In particular, if $H \neq 0$ at some point in the interior of M, we may find a vector field so that $X \cdot H > 0$ near the point and $X \cdot H = 0$ away from the point, so $\frac{d}{dt}\big|_{t=0} \operatorname{area}(f_t(M)) < 0$. Thus, if M is area minimizing, then necessarily $H = 0$. In particular, we define

Definition 6.1. A submanifold M is minimal if and only if $\frac{d}{dt}\big|_{t=0} \operatorname{area}(f_t(M)) = 0$ for all compactly supported variations $X \in \Gamma(M \backslash \partial M, TN)$. Equivalently, this holds if and only if $H \equiv 0$.

Now we want to establish the *monotonicity formula*. Suppose that M is a compact minimal submanifold of $\mathbb{R}^N$. Let $X(x) = \vec{x}$. It is not hard to check that $\operatorname{div}_M X = m$. In particular, the first variation formula yields

$$m \operatorname{area}(M) = \int_{\partial M} X \cdot \nu_M \, ds.$$

On the other hand, let $C = 0 \mathbin{⋇} \partial M$ (the cone over ∂M with respect to the origin) as illustrated

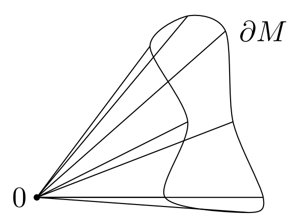

FIGURE 14. The cone over ∂M.

in Figure 14. The first variation formula also yields

$$m \operatorname{area}(C) = -\int_C H_C \cdot X + \int_{\partial M} X \cdot \nu_{cone}.$$

However, because X is tangent to C, the first term vanishes. Furthermore, we claim that

$$X \cdot \nu_M \leq X \cdot \nu_{cone}.$$

To see this, for $p \in \partial M$ consider $W \subset T_p\mathbb{R}^N$ the two dimensional subset which is normal to $T_p\partial M$. We claim that among all unit vectors $\vec{n} \in W$, $X \cdot \vec{n}$ is maximized at ν_{cone}. To see this, note that

$$\|X\|^2 \geq \| \operatorname{proj}_{T_p\partial M \oplus \langle \vec{n} \rangle} X\|^2 = \| \operatorname{proj}_{T_p\partial M} X\|^2 + \| \operatorname{proj}_{\langle \vec{n} \rangle} X\|^2 = \| \operatorname{proj}_{T_p\partial M} X\|^2 + | \langle \vec{n}, X \rangle |,$$

However, because $T_p\partial M \oplus \langle \nu_{cone} \rangle = T_pC$, it is clear that for $\vec{n} = \nu_{cone}$, we have equality in the first inequality above. This proves the claim.

As such, we see that

$$\text{area}(M) \leq \text{area}(C).$$

This allows us to derive the monotonicity formula as follows. Let $A(r) = \text{area}(M \cap B_r(p))$. For $0 < r < \text{dist}(p, \partial M)$, we have that

$$A'(r) \geq L(r) = L(\partial(B_r(p) \cap M)) = L(M \cap \partial B_r(p)).$$

The first inequality follows from the co-area formula. On the other hand, above formula implies that

$$A(r) \leq \text{area}(0 \times\!\!\!\!\times (M \cap \partial B_r(p))) = \frac{r}{m} L(r).$$

Thus $A'(r) \geq \frac{m}{r} A$, and integrating this we have

Theorem 6.2 (Monotonicity Formula)**.** *For M a minimal surface, the quantity*

$$\frac{\text{area}(M \cap B_r(p))}{r^m}$$

is an increasing function of r for $0 < r < \text{dist}(p, \partial M)$. Furthermore, the quantity is strictly increasing unless M is a cone with vertex p.

We may define

$$\Theta(M, p, r) := \frac{\text{area}(M \cap B_r(p))}{\omega_m r^m}$$

where ω_m is the area of the unit m-ball. The monotonicity formula is equivalent to the claim that $\Omega(M, p, r)$ is an increasing function of r for $r < \text{dist}(p, \partial M)$.

In fact, letting $E_p = \{\lambda(x - p) + p : x \in \partial M, \lambda \geq 1\}$ denote the *exterior cone* over ∂M with respect to p, the same proof as above shows

Theorem 6.3 (Extended Monotonicity Formula, Ekholm–White–Wienholtz [EWW02])**.** *The density $\Theta(M \cup E_p, p, r)$ is increasing for $r \in (0, \infty)$.*

This allows us to discuss the "density at infinity" for minimal submanifolds of $\mathbb{R}^N$.

Theorem 6.4. *For M a minimal submanifold of $\mathbb{R}^N$ with $\partial M = \emptyset$ or ∂M compact and M unbounded, the density at infinity, defined by*

$$\Theta(M) := \lim_{r \to \infty} \Theta(M, p, r)$$

is well defined independently of the choice of p and furthermore $\Theta \geq 1$.

Proof. We first note that we may replace M by $M \backslash B_R(p)$ for R so that $\partial M \subset B_R(p)$ and $\partial B_R(p)$ intersects M transversely (as it does for a.e. R) and as such, may assume that ∂M is smooth.

Let E_p be the exterior cone of ∂M with respect to p. By the extended monotonicity formula, we have that $\Theta(M \cup E_p, p, r)$ is an increasing function of $r \in (0, \infty)$. Thus, we have that

$$\lim_{r \to \infty} \Theta(M \cup E_p, p, r)$$

exists. Furthermore, for large r, letting $C = 0 \times\!\!\!\!\times \partial M$,

$$\Theta(M \cup E_p, p, r) = \Theta(M, p, r) + \Theta(E_p, p, r) = \Theta(M, p, r) + \Theta(C, p, r) + \Theta(E_p \backslash C, p, r).$$

It is clear that $\Theta(C, p, r)$ is independent of r and $\Theta(E_p \backslash C, p, r) \to 0$ as $r \to \infty$, as it has finite total area. This shows that the limit defining $\Theta(M)$ exists. To see that it is independent of the choice of point p, notice that

$$M \cap B_r(p) \subseteq M \cap B_{r+|p-q|}(q),$$

so from this we see that
$$\frac{\operatorname{area}(M\cap B_r(p))}{\omega_m r^m} \leq \frac{\operatorname{area}(M\cap B_{r+|p-q|}(q)}{\omega_m r^m} = \frac{\operatorname{area}(M\cap B_{r+|p-q|}(q)}{\omega_m (r+|p-q|)^m}\left(\frac{r+|p-q|}{r}\right)^m.$$
Taking the limit as $r\to\infty$, we see that
$$\Theta(M,p,\infty)\leq\Theta(M,q,\infty)$$
but we could clearly reverse the role of p and q. This shows that $\Theta(M)$ does not depend on the choice of point p.

Finally, it remains to show that $\Theta\geq 1$. If $\partial M=\emptyset$, this follows easily from taking $p\in M$ and using the monotonicity formula (it is not hard to check that $\lim_{r\to 0}\Theta(M,p,r)\geq 1$). On the other hand, if $\partial M\neq\emptyset$, by assumption there exists $q\in M$ which is a distance $R\gg 1$ away from ∂M. The extended monotonicity formula implies that
$$\Theta(M\cup E_q,q,r)\geq\lim_{t\to 0}\Theta(M\cup E_q,q,t)\geq 1.$$
On the other hand, taking $r\to\infty$, we see that
$$\Theta(M)+\Theta(E_q)\geq 1.$$
By the same argument as above, we see that $\Theta(E_p)=\Theta(C)$. However, as $R\to\infty$ the cone angle tends to 0, and as such $\Theta(C)\to 0$. This shows that $\Theta(M)\geq 1$. □

7. Limits of Minimal Surfaces

Suppose that $M_i\subset\Omega$ is a sequence of minimal submanifolds of some open set Ω with $\partial M_i\subset\partial\Omega$. Can we take the limit of the M_i? Can singularities arise in the limit? We address these questions here, proving a sequence of theorems.

Theorem 7.1. *If the M_i are minimal surfaces and each $M_i=\operatorname{graph}(f_i:\mathbb{B}^m\to\mathbb{R}^{N-m})$ with $\|f_i\|_{C^2}\leq C<\infty$ then, up to extracting a subsequence, the f_i converge smoothly on compact subsets of $\mathbb{B}^n$ to f so that $M=\operatorname{graph}(f)$ is a minimal surface.*

Proof. By the Arzelà-Ascoli theorem, we may extract a subsequence so that $f_i\to f$ in $C^{1,\alpha}$ on compact sets. Now, an easy PDE argument (using Schauder estimates for linear elliptic equations) shows that we may in fact upgrade this convergence to smooth convergence on compact sets. It is easy to see that the limiting function has its graph a minimal surface. □

For the next convergence theorem, for a minimal surface M and point $p\in M$, we define $|A|(M,p)$ to be the norm of the second fundamental form of M at p. We note that from now on, one could assume that Ω is a domain in a Riemannian manifold as the above theorem will still hold locally near a point (the manifold is approximately flat on small scales). However, we will not worry too much about this remark.

Theorem 7.2. *For $M_i\subset\Omega$ a sequence of minimal submanifolds with $\partial M_i\subset\partial\Omega$, if $|A|(M_i,\cdot)$ is uniformly bounded on compact subsets of Ω and we have the area bound for $K\Subset\Omega$, $\operatorname{area}(M\cap K)\leq A_K<\infty$, then up to a subsequence, the M_i converge smoothly to some minimal surface M.*

Proof. We'd like to apply the previous theorem. Choose $p_i\in M_i\cap K$. Extracting a subsequence, the p_i converge to some p. We may further assume that $\operatorname{Tan}(M_i,p_i)\to T$ for some plane T. Then, we may express the surfaces M_i locally as graphs (a bounded number, by the area bounds) over T (by the bound on the second fundamental form) and use the above theorem to extract a subsequence. Then, a diagonal argument allows us to find a sequence converging smoothly on all of Ω. □

However, what if we don't have second fundamental form bounds? Then, we should expect the possibility of singularities forming. The following theorem helps us have a better understanding of how this happens

Theorem 7.3. *If $M_i \subset \Omega$ are minimal with $\partial M_i \subset \partial\Omega$, suppose that $|A|(M_i, \cdot)$ has a local maximum at a sequence of points p_i, converging to some $p \in M$. Suppose further that $|A|(M_i, p_i) \to \infty$. Then, letting $\lambda_i := |A|(M_i, p_i)$ the blown up surfaces $\lambda_i(M_i - p_i)$ converge smoothly to a minimal surface $M \subset \mathbb{R}^N$ with $\partial M = \emptyset$. Furthermore, $|A|(M, \cdot) \leq |A|(M, 0) = 1$.*

Proof. Letting $M_i' := \lambda_i(M_i - p_i)$, it is not hard to check by scaling that $|A|(M_i', \cdot) \leq 1$. Then, the previous theorem guarantees the desired convergence. $\square$

An example of this is given by the catenoid, as in Figure 15. If we scale the catenoid down by

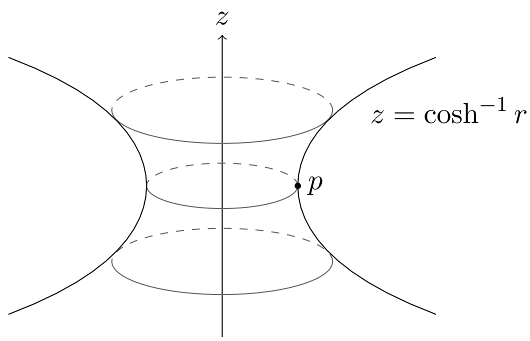

FIGURE 15. The catenoid. The point p is a local maximum for $|A|(\cdot)$.

$\frac{1}{n}$, then it converges to two planes (away from the origin). However, the curvature is blowing up at the points $\frac{1}{n}p$ which are converging to the origin. Performing the rescaling above will just lead to the catenoid again, which is a simple example of what is possible to get as a rescaling limit in the above theorem.

Finally, one might wonder what happens if there is no local maxima of $|A|(\cdot)$? It turns out that by using a point-picking argument, we can still attain a good blowup limit

Theorem 7.4. *For $M_i \subset \Omega$ minimal with $\partial M_i \subset \partial\Omega$, assume that $p_i \in M_i$ with $p_i \to p \in \Omega$ have $|A|(M_i, p_i) \to \infty$. Then, there exists (up to a subsequence) new points $q_i \in M_i$ with $q_i \to p$ so that for $\lambda_i := |A|(M_i, q_i)$, $\lambda_i(M_i - q_1)$ converges smoothly to M a smooth minimal submanifold of $\mathbb{R}^N$ with no boundary and $|A|(M, \cdot) \leq |A|(M, 0) = 1$.*

Proof. Let $r_i < \operatorname{dist}(p, \partial\Omega)$ tend to zero slowly enough so that still $r_i|A|(M_i, p_i) \to \infty$ (e.g. one could take $r_i := |A|(M_i, p_i)^{-1/2}$). We'll choose $q_i \in M_i \cap B_{r_i}(p_i)$ to maximize

$$|A|(M_i, \cdot) \operatorname{dist}(\cdot, \partial B_{r_i}(p_i)).$$

Letting $R_i := \operatorname{dist}(q_i, \partial_{r_i}(p_i))$, notice that q_i *also* automatically maximizes

$$|A|(M_i, \cdot) \operatorname{dist}(\cdot, \partial B_{R_i}(q_i))$$

on $M_i \cap B_{R_i}(q_i)$. This is because $\operatorname{dist}(\cdot, \partial B_{R_i}(q_i)) \leq \operatorname{dist}(\cdot, \partial B_{r_i}(p_i))$ here. See Figure 16. Now, we see that $|A|(M_i, q_i)R_i \geq |A|(M, p_i)r_i \to \infty$, and $R_i \leq r_i \to 0$. In particular, we may define $\lambda_i := |A|(M_i, q_i)$ and $M_i' = \lambda_i(M_i - q_i)$. Because $|A|^2 * \operatorname{dist}$ is a scale invariant quantity, we see that

$$|A|(M_i', x) \operatorname{dist}(x, \partial B_{\lambda_i R_i}(0)) \leq \lambda_i R_i,$$

by the above choices. As such,

$$|A|(M_i, x) \leq \frac{\lambda_i R_i}{\lambda_i R_i - |x|} \to 1$$

as $i \to \infty$. Thus, the same argument as in the previous theorem yields the desired conclusion. $\square$

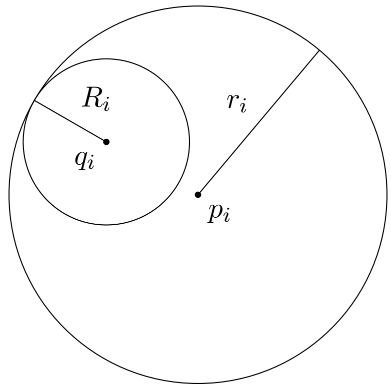

FIGURE 16. The point picking argument.

By the above theorem, if $M_i \subset \Omega$ are a sequence of minimal surfaces with $\partial M_i \subset \partial\Omega$, and if the curvature of the M_i is blowing up somewhere, we may find such a rescaling converging to some minimal surface M' in $\mathbb{R}^N$ with no boundary. For each M_i in the sequence under consideration, we define a Radon measure $|M_i|$ by

$$|M_i|(U) := \text{area}(M_i \cap U).$$

It is not hard to see that up to a subsequence, the $|M_i|$ converge as Radon measures to some Radon measure μ. If the metric in Ω is the flat metric, then because $\Theta(M_i, x, r)$ is non-decreasing in r (for $r < \text{dist}(x, \partial\Omega)$), it is not hard to see that so is $\Theta(\mu, x, r)$. So $\Theta(\mu, x) := \lim_{r\searrow 0} \Theta(\mu, x, r)$ exists.

Claim 7.5. *For $y_i \to y \in \Omega$ and M_i as above, and $r_i \to 0$, we have that*

$$\limsup_{i\to\infty} \Theta(M_i, y_i, r_i) \leq \Theta(\mu, y).$$

Proof. Fixing some r, for $r_i < r$, we have that

$$\Theta(M_i, y_i, r_i) \leq \Theta(M_i, y_i, r)$$

by monotonicity. As such

$$\limsup_{i\to\infty} \Theta(M_i, y_i, r_i) \leq \limsup_{i\to\infty} \Theta(M_i, y_i, r) \leq \Theta(\mu, x, r)$$

where the second inequality is from the convergence of the measures. Letting $r \to 0$ finishes the proof. ☐

In fact, we see that the local density of μ at y is related to the density at infinity of the rescaled limit M'

Theorem 7.6. *With M_i and M' as above $\Theta(M') \leq \Theta(\mu, x)$.*

Proof. We have that for a.e. R

$$\Theta(M', 0, R) = \lim_{i\to\infty} \Theta(M_i', 0, R) = \lim_{i\to\infty} \Theta(M_i, x_i, R/\lambda_i) \leq \Theta(\mu, x).$$

The last inequality is by the previous claim. ☐

Corollary 7.7. *With M' as above, we have that $\Theta(M') < \infty$.*

We remark that if $\dim M' = 2$, this condition is often referred to as *quadratic area growth.* We now prove

Theorem 7.8 (Easy Version of Allard's Regularity Theorem [Whi05])**.** *Under the above hypothesis, if the curvature of the M_i are blowing up at $x \in \Omega$, then $\Theta(\mu, x) > 1$.*

Proof. By the above theorem, we know that $\Theta(M') \le \Theta(\mu, x)$. On the other hand,

$$1 \le \Theta(M', 0) \le \Theta(M')$$

by monotonicity, with equality in the second inequality if and only if M' is a cone centered at 0. Because M' is smooth, this can only be true if it is a union of planes through the origin, which cannot be true as M' is not flat, by construction. □

In fact, we can improve this slightly

Theorem 7.9 ([Whi05]). *There is $\epsilon = \epsilon(n, N)$ so that if M' is a complete, minimal n-dimensional surface in $\mathbb{R}^N$ with $|A|(M', \cdot) \le |A|(M', 0) = 1$, then $\Theta(M') \ge 1 + \epsilon$.*

Proof. Let $\alpha := \inf\{\Theta(M') : M' \text{ as in theorem}\}$. We may assume $\alpha < \infty$. As such, we choose a minimizing sequence M'_i with $\Theta(M'_i) \to \alpha$. Passing to a subsequence, the curvature bounds guarantee smooth convergence to some M' (with maximal curvature 1 at the origin). As such, for a.e. R, we have that

$$\Theta(M', 0, R) = \lim_{i\to\infty} \Theta(M'_i, 0, R) \le \lim_{i\to\infty} \Theta(M'_i) = \alpha.$$

Because M' is non-flat, we thus see that $1 < \Theta(M') = \alpha$, as desired. □

One might wonder if there is always equality in Theorem 7.6. An example of strict inequality is provided by the Costa–Hoffman–Meeks surfaces, cf. [HM85].

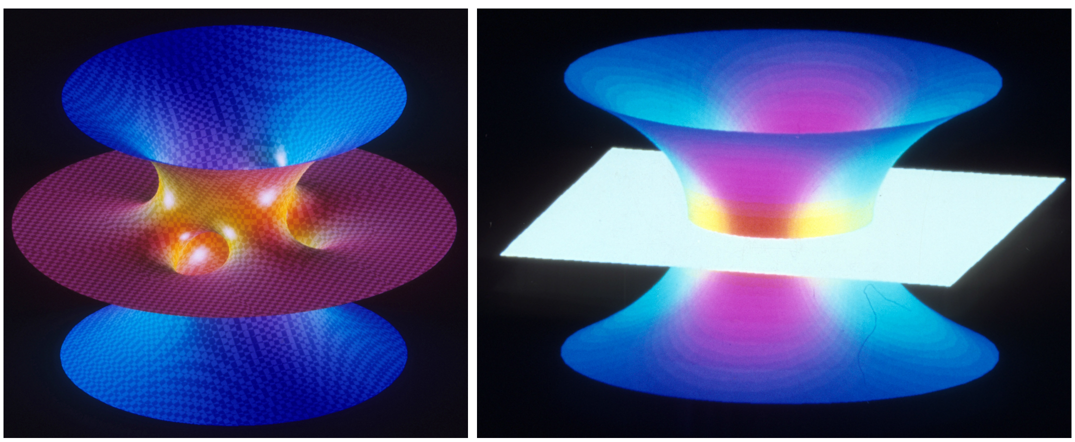

FIGURE 17. A Costa–Hoffman–Meeks surface along with the limiting catenoid and plane

These are a sequence of embedded minimal surfaces M_g which are of genus g and are asymptotic to a catenoid and a plane. In particular, as the genus tends to ∞, the curvature is blowing up on a circle. The surfaces along with their limit as the genus tends to infinity are illustrated[1] in Figure 17. On the other hand, the rescaled surfaces $\frac{1}{g}M_g$ converge to a multiplicity three plane. As such, writing μ for the limit, $\Theta(\mu, 0) = 3$. However, rescaling to obtain a smooth limit, one finds Scherk's singly periodic surface, which looks like the union of two planes, and as such has density at infinity 2.

The following exercises are "true," by GMT methods, but it is not clear how to solve them using classical minimal surface methods

Exercise 1. If M' is a minimal surface with $\dim M' = 2$, if $\Theta(M') < \infty$, show that $\Theta(M') \in \mathbb{N}$. A special case is to show that $\alpha := \min\{\Theta(M') : M' \text{ is not flat}\} = 2$.

[1]Thanks to David Hoffman for providing these pictures.

A roughly equivalent problem is

Exercise 2. Suppose that M_i are minimal surfaces (two dimensional) and $|M_i| \to \mu$ as Radon measures. Suppose further that there is a 2-plane P so that $\mu = \theta|P|$. Show that $\theta \in \mathbb{N}$.

8. Genus and Total Curvature of Minimal Surfaces

We define the genus of a surface (possibly non-compact) as follows: first we use the classical definition for closed surfaces. Secondly, we demand that the genus is additive in the usual way, and finally we define the genus for a compact surface with boundary to be the genus of the surface formed by capping off each boundary component.

This definition is quite well behaved with respect to the discussion of the previous section. In particular, if $\dim M_i = 2$, $M_i \subset \Omega$ and $\partial M_i \subset \partial\Omega$ are a sequence of minimal surfaces whose curvatures are blowing up, then if $\text{genus}(M_i) \leq g$, this implies that $\text{genus}(M') \leq g$ (where M' is the surface obtained as in the previous section by blowing up the parts with concentrating curvature to obtain a smooth limit). In fact, the same thing holds even if we only require $\text{genus}(M_i \cap B_\epsilon(x)) \leq g$, where x is the point of curvature blowup.

As such, we're naturally led to the study of blowups M' with finite genus. Our main results are

Theorem 8.1. *Suppose that M is a two dimensional complete, properly immersed (without boundary) minimal surface with quadratic area growth and finite genus. Then*

$$\int_M |K|\, dA < \infty,$$

i.e. M has finite total curvature.

Remark 8.2. Suppose we have a surface $M^2 \subset \mathbb{R}^3$ with principal curvatures κ_1, κ_2. Then the mean curvature is $H = \kappa_1 + \kappa_2$ and the Gauss curvature $K = \kappa_1\kappa_2$. In particular

$$K = \frac{H^2}{2} - \frac{|A|^2}{2}$$

If M is minimal then $K = -\frac{1}{2}|A|^2$ and $|K| = \frac{1}{2}|A|^2$. In particular, the theorem guarantees we have global L^2 finiteness of the curvature norm.

In the converse direction we have the following theorem due to Osserman (in $\mathbb{R}^3$) and Chern-Osserman (in $\mathbb{R}^N$, $N \geq 4$):

Theorem 8.3 (Osserman [Oss64], Osserman-Chern [CO67])**.** *Suppose $M^2 \subset \mathbb{R}^N$ is a complete orientable (without boundary) minimal surface with finite total curvature; i.e.*

$$\int_M |K|\, dA < \infty$$

Then

(1) M is conformally equivalent to a closed surface Σ with finitely many points removed (and hence has finite genus),

(2) the Gauss map $p \mapsto T_pM$ extends continuously to Σ,

(3) The Gauss curvature integrates to an integer multiple of 4π (in $\mathbb{R}^3$) or 2π (for $N \geq 4$), i.e.

$$-\int_M K\, dA = \int_M |K|\, dA = \begin{cases} 4\pi j & \text{for } N = 3 \\ 2\pi j & \text{for } N \geq 4 \end{cases}$$

(4) M is proper in $\mathbb{R}^N$ and exhibits quadratic area growth.

There are various things we can say before turning to the proof of either theorem. For starters, note that surfaces with quadratic area growth and which are properly immersed form a natural class of surfaces to study, because all rescaling limit surfaces M' turn out to be such. We already know they exhibit quadratic area growth from Corollary 7.7, but why are they properly immersed? It turns out to be a consequence of area growth and our pointwise curvature bounds:

Lemma 8.4. *Suppose $M^2 \subset \mathbb{R}^N$ is a minimal surface (without boundary) such that $\Theta(M) < \infty$ and whose pointwise curvature is uniformly bounded. Then M is properly immersed.*

Proof. Suppose $p_i \in M$ is a sequence of points that escapes to infinity intrinsically but not extrinsically. In view of our uniform curvature bounds around each p_i there exists a ball of fixed radius in which M looks like a graph. After passing to a subsequence we may assume that the p_i accumulate to some $q \in \mathbb{R}^n$ and that on slightly smaller balls the graphs converge smoothly to a common graph. This contradicts finiteness of area in a neighborhood of q. $\square$

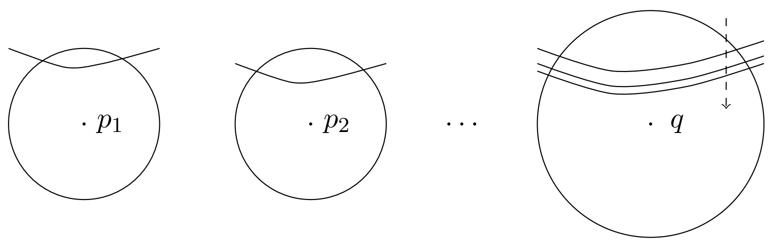

FIGURE 18. Illustration of the contradiction obtained by the existence of accumulation points.

Before proceeding to the proof of the first main theorem, we present an application of the monotonicity formula which forces a bound on just how spread out the boundary of a minimal surface can be in terms of its length.

Lemma 8.5. *Let $M^2 \subset \mathbb{R}^N$ be a compact minimal surface with boundary ∂M. Then*

$$\max_{p \in M} \operatorname{dist}_{\mathbb{R}^N}(p; \partial M) \leq \frac{\operatorname{length}(\partial M)}{2\pi}$$

Equality is attained if and only if M is a planar disk.

Proof. Let $p \in M \setminus \partial M$, E_p denote the exterior of the cone $p \divideontimes \partial M$, and C denote the full cone. By the extended monotonicity formula

$$1 \leq \Theta(M, p) \leq \Theta(M \cup E_p, p, \infty) = \Theta(C, p, \infty)$$

If $r = \operatorname{dist}(p; \partial M)$ and $\pi : \mathbb{R}^N \setminus \{p\} \to \partial B_1(p)$ denotes the standard projection function then invariance of C under dilations gives

$$\Theta(C, p, \infty) = \frac{\operatorname{length}(\pi(\partial M))}{2\pi} \leq \frac{\operatorname{length}(\partial M)}{2\pi r}$$

and the result follows by rearranging r and taking the supremum over all $p \in M$. $\square$

Remark 8.6. There exists an analogous statement for higher dimensional minimal surfaces M^m in $\mathbb{R}^N$. It gives

$$\max_{p \in M} \operatorname{dist}_{\mathbb{R}^N}(p; \partial M) \leq \frac{\left(\mathscr{H}^{m-1}(\partial M)\right)^{\frac{1}{m-1}}}{m\omega_m}$$

Here $\mathscr{H}^{m-1}$ denotes $(m-1)$-dimensional Hausdorff measure and ω_m the total volume of the the unit ball in $\mathbb{R}^m$.

Corollary 8.7. *Suppose that $M^2 \subset \mathbb{R}^N$ is as above, connected, and such that $\partial M = \Gamma_1 \cup \Gamma_2$, for a pair of curves Γ_1, Γ_2. Then*

$$\operatorname{dist}_{\mathbb{R}^N}(\Gamma_1, \Gamma_2) \leq \frac{\operatorname{length}(\partial M)}{\pi}$$

Proof. Let $d = \operatorname{dist}_{\mathbb{R}^N}(\Gamma_1, \Gamma_2)$, pick a point $p \in M$ that is at least $d/2$-apart from both Γ_1, Γ_2, and apply the lemma. □

With these tools at hand we can return to the proof the first main theorem.

Proof of Theorem 8.1. Let us first do an easy special case to get some intuition; suppose that M is simply connected. Pick $p \in M$ and let

$$\begin{aligned} M(r) &= \{x \in M : \operatorname{dist}_M(x, p) \leq r\} \subset B_r(p) \\ A(r) &= \operatorname{area}(M(r)) \\ L(r) &= \operatorname{length}(\partial M(r)) \end{aligned}$$

Since $K < 0$, $\partial M(r)$ is a smooth curve for all r, and $A'(r) = L(r)$. Differentiating again and using the first variation formula and Gauss-Bonet

$$A''(r) = L'(r) = \int_{\partial M(r)} k\, ds = 2\pi - \int_{M(r)} K\, dA = 2\pi + \int_{M(r)} |K|\, dA$$

Quadratic area growth (on the ambient space) forces quadratic area growth intrinsically. Since A'' above is monotone, it must be uniformly bounded and letting $r \to \infty$ we get an L^1 bound on $|K|$.

Now let's do the general case, with M not necessarily a disk. It is clear that there are finitely many ends because each end must contribute at least 1 to the density at infinity by Theorem 6.4. From topology we know that such 2-manifolds with finitely many ends and finite genus are homeomorphic to closed surfaces with finitely many points removed. Thus, each end is annular.

For E an annular end and homotopic curves $\Gamma \sim \Gamma_i$ in E with $\Gamma_i \to \infty$, the corollary shows that $\operatorname{length}(\Gamma_i) \to \infty$. That is, ends grow wider as they escape to infinity. At this point we may replace Γ by the shortest curve in M homotopic to Γ because the lengths diverge as we escape to infinity. This choice of Γ is geodesic in M.

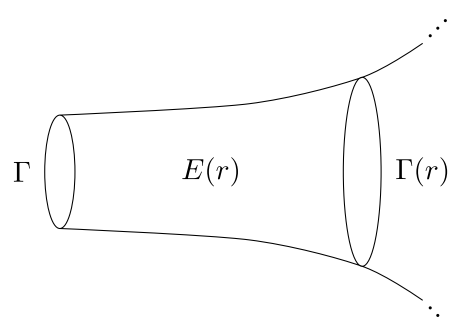

FIGURE 19. Intrinsic area growth on an annular end E.

As before let's define

$$\begin{aligned} E(r) &= \{x \in E : \operatorname{dist}_M(x, \Gamma) \leq r\} \\ \Gamma(r) &= \partial E(r) \setminus \Gamma \\ A(r) &= \operatorname{area}(E(r)) \\ L(r) &= \operatorname{length}(\partial E(r)) \end{aligned}$$

In view of $K < 0$, $\Gamma(r)$ is smooth still, and by the same computation as above $A'(r) = L(r)$ and

$$A''(r) = L'(r) = \int_{\Gamma(r)} k\, ds + \int_{\Gamma} k\, ds$$

The curve Γ is geodesic so $k \equiv 0$. By Gauss-Bonet and the vanishing characteristic of $E(r)$:

$$A''(r) = -\int_{E(r)} K\, dA = \int_{E(r)} |K|\, dA$$

on each end. Our quadratic area growth assumption in turn forces an upper bound on $A''(r)$ and thus finiteness on total curvature in E, as before. The result follows since we have finitely many ends and the remainder of M is compact. □

Proof of Theorem 8.3. The first part of the theorem is true in a very general setting; indeed Huber [Hub57] showed that any complete surface $M^2 \subset \mathbb{R}^N$ such that

$$\int_M |K_-|\, dA < \infty$$

(where K_- denotes the negative part of curvature) is conformally equivalent to a closed surface Σ with finitely many points removed.

The rest of the proof we present in the convenient special case $N = 3$. Equivalent to $p \mapsto T_pM$ in this case is the map $\vec{n} : M \to \mathbb{S}^2$, $\vec{n}(p)$ being a unit normal to M at p. It is standard that $K = \operatorname{Jac} \vec{n}$ and therefore by the area formula

$$\operatorname{area}(\vec{n}(\mathcal{U})) = \int_{\mathcal{U}} |K|\, dA$$

provided the area on the left accounts for multiplicity. Let p be one of the punctures. Choose E to be a neighborhood with small total curvature, say

$$\int_E |K|\, dA < 2\pi$$

Then $\operatorname{area}(\vec{n}(E)) < 2\pi < \operatorname{area}(\mathbb{S}^2)$, so in particular $\vec{n}(E)$ misses at least three distinct points $q_1, q_2, q_3 \in \mathbb{S}^2$. On the other hand $\vec{n} : E \to \mathbb{S}^2 \cong \mathbb{C} \cup \{\infty\}$ is conformal and without loss of generality q_3 corresponds to ∞ on the Riemann sphere, so by Picard's theorem $\vec{n} : E \to \mathbb{S}^2$ is meromorphic and therefore extends to a holomorphic function $\vec{n} : E \cup \{p\} \to \mathbb{S}^2$.

Remark 8.8. The precise nature of $\vec{n}$ on minimal surfaces is that of a conformal, orientation-reversing map (provided we endow $\mathbb{S}^2$ with the natural orientation). Indeed fix a base point $q \in M$ and label the principal directions e_1, $e_2 \in T_qM$ with corresponding principal curvatures κ_1, κ_2. Since our surface is minimal, $\kappa_1 = -\kappa_2 = \kappa$ for some κ and therefore

$$D\vec{n}(q) = \begin{pmatrix} \kappa_1 & \\ & \kappa_2 \end{pmatrix} = \begin{pmatrix} \kappa & \\ & -\kappa \end{pmatrix}$$

If we reverse the orientation on $\mathbb{S}^2$ then $\vec{n}$ is conformal and orientation-preserving.

Returning to the proof of the theorem, recall that M is conformally equivalent to $\Sigma \setminus \{p_1, \ldots, p_k\}$ and that $\vec{n}$ extends to the closed surface Σ. Therefore

$$\int_M K\, dA = \int_M \operatorname{Jac} \vec{n}\, dA = \operatorname{area}(\vec{n}(\Sigma)) = \deg \vec{n} \cdot \operatorname{area}(\mathbb{S}^2)$$

is an integer multiple of 4π. Finally, it remains to prove properness and quadratic area growth. Fix an annular end E, with ∂E some compact curve. We may assume that T_pE is nearly horizontal, say with slope ≤ 1, uniformly for $p \in E$ by going far out enough on E. In view of the uniform bound on slopes, we may re-metrize E by the flat metric of its projection on the horizontal plane Π and this metric is equivalent to the original one. Therefore for some uniform constant $C > 0$

$$C \operatorname{dist}_M(p; \partial E) \leq \operatorname{dist}_\Pi(\pi(p); \pi(\partial E)) \leq \operatorname{dist}_{\mathbb{R}^N}(p; \partial E)$$

In particular when $\operatorname{dist}_M(p; \partial E) \to \infty$ we also have $\operatorname{dist}_{\mathbb{R}^N}(p; \partial E) \to \infty$ which gives properness and similarly (by measuring the lengths of boundary curves by these two equivalent metrics) we can deduce as before quadratic area growth as in the previous theorem. □

Now we will analyze the structure of the annular ends of a properly immersed minimal surface with finite genus and quadratic area growth. Fix such an end E. By rotating E, we may assume that $\operatorname{slope}(\operatorname{Tan}(E, p)) \leq \epsilon$ for all $p \in E$.

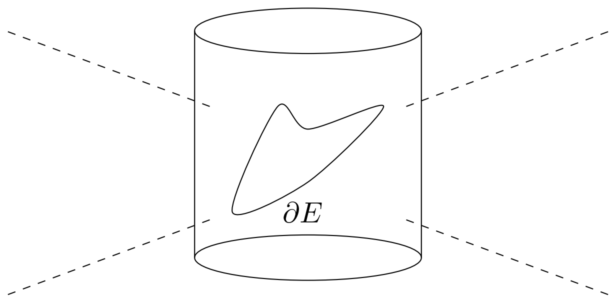

FIGURE 20. An illustration of an annular end which must be a finite sheeted graph outside of a cylinder. By the slope assumption made below, E must stay within the dashed lines.

Claim 8.9. *There is some cylinder $C = B^{\mathbb{R}^2}(0, R) \times \mathbb{R}^{N-2}$ so that $E \backslash C$ is a k-sheeted multi-graph for some finite k.*

Proof. We first observe that by the assumption that the tangent planes have small slope, $\pi : E \backslash C \to \mathbb{R}^2 \backslash B^{\mathbb{R}^2}(0, R)$ is a covering map. It cannot be infinitely sheeted because it is proper, and must stay within the dashed region in Figure 20 by the slope assumption. □

In particular, we have that

Theorem 8.10. *If $E \backslash C$ is embedded in $\mathbb{R}^3$ then it is 1-sheeted, i.e. it is a graph over $\mathbb{R}^2 \backslash B^{\mathbb{R}^2}(0, R)$.*

Proof. First of all, notice that because E is *one* end, $E \backslash C$ must be connected (we've removed a compact set), except for possibly some compact regions. These compact regions cannot occur here, because otherwise $E \backslash C$ could not be a k-sheeted graph, with k being fixed independent of the point. Now, because we've assumed $E \backslash C$ is embedded, we may consider the "top most sheet" of the graph. Clearly this is an open and closed set, so because $E \backslash C$ is connected, the "top most sheet" must be equal to $E \backslash C$. This clearly implies that $k = 1$. □

We remark that Enneper's surface, as in[2] Figure 21 shows that the previous theorem fails without the embeddedness assumption.

Theorem 8.11 ([Whi87]). *Let M^2 be an orientable minimal surface in $\mathbb{R}^3$ with*

$$\frac{1}{2} \int_M |A|^2 = \int_M |K| \leq 4\pi - \epsilon.$$

Then, there is C_ϵ so that $\beta(M, p) \operatorname{dist}_M(p, \partial M) \leq C_\epsilon < \infty$.

[2]Thanks to David Hoffman for providing this figure.

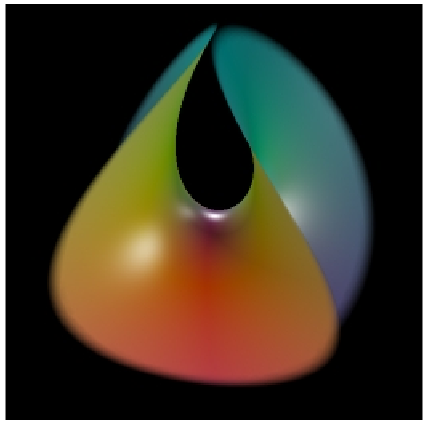

FIGURE 21. Enneper's surface is not a single sheeted graph outside of some cylinder (of course it is not embedded, so this shows that the embedded assumption may not be dropped in Theorem 8.10).

Proof. Suppose that there is a sequence of counterexamples, i.e. $M_i \ni p_i$ with

$$\frac{1}{2}\int_{M_i} |A|^2 \leq 4\pi - \epsilon,$$

but $|A|(M_i, p_i)\,\mathrm{dist}(p_i, \partial M_i) \to \infty$. We may assume that the M_i are smooth compact manifolds with boundary. Furthermore, by choosing a different p_i if necessary, we may assume that p_i maximizes $|A|(M_i, \cdot)\,\mathrm{dist}(\cdot, \partial M)$. By translating and scaling, we may arrange that $p_i = 0$ and $|A|(M_i, 0) = 1$. As such, for $p \in M_i$

$$|A|(M_i, p) \leq \frac{\mathrm{dist}_{M_i}(0, \partial M)}{\mathrm{dist}_{M_i}(p, \partial M_i)} \leq \frac{\mathrm{dist}_{M_i}(0, \partial M_i)}{\mathrm{dist}_{M_i}(0, \partial M_i) - \mathrm{dist}_{M_i}(0, p)}.$$

As long as the intrinsic distance from p to 0 is bounded, the right hand side converges to 1. Thus, we have smooth convergence of the M_i to some nonflat M. However, the integral bound contradicts Theorem 8.3.[3] □

Of course, from the proof it is obvious that we could prove similar statements corresponding to higher codimension or non-orientability in Theorem 8.3. In fact, a similar statement holds in a Riemannian manifold, i.e.

Theorem 8.12. *Suppose that $M_i \subset \Omega \subset (N^n, g)$, where Ω is a bounded domain in a Riemannian manifold N. Suppose further that*

$$\frac{1}{2}\int_M |A|^2 \leq 2\pi - \epsilon$$

Then, for $\lambda > 0$, there is $C = C(\lambda) > 0$ so that if $\mathrm{dist}_M(p, \partial M) \leq \lambda$, then $|A|(M, p)\,\mathrm{dist}_M(p, \partial M) \leq C$.

The added requirement that the point p is close enough to the boundary comes from the fact that in order to obtain a blowup limit in $\mathbb{R}^n$, we need that the second fundamental form term is blowing up.

As a consequence of the above theorems, we have that

[3]We remark that because we have curvature bounds on intrinsic balls, not extrinsic balls, we do not know that the limit will be properly immersed. However, we may still apply Theorem 8.3 to obtain a contradiction, because properness was *not* assumed there.

Theorem 8.13. *Suppose that $M_i \subset \Omega \subset \mathbb{R}^N$ (or a Riemannian manifold) is a sequence of minimal surfaces, with $\partial M_i \subset \partial\Omega$ and*

$$\frac{1}{2}\int_{M_i} |A|^2 \le S < \infty$$

for some S. After extracting a subsequence, there is a finite set of points X so that $\beta(M_i, \cdot)$ is uniformly bounded on compact subsets of $\Omega\backslash X$.

Proof. Define Radon measures β_i by

$$\beta_i(U) := \frac{1}{2}\int_{M_i \cap U} |A|(M_i, \cdot)^2.$$

After extracting a subsequence, $\beta_i \to \beta$ weakly. Let $X := \{p \in \Omega : \beta(p) \ge 2\pi\}$. Clearly X has at most $\frac{\beta(\Omega)}{2\pi} \le \frac{S}{2\pi}$ points. For $p \in \Omega\backslash X$, we have that $\beta(p) < 2\pi - \epsilon$ for some $\epsilon > 0$. Therefore, there is some $r > 0$ so that $\beta(\overline{B}_r(p)) < 2\pi - \epsilon$, and thus for i large enough $\beta_i(\overline{B}_r(p)) < 2\pi - \epsilon$. As such, we have uniform second fundamental form bounds on $M_i \cap B_{\frac{r}{2}}(p)$ (because this is clearly contained in the intrinsic ball of the same radius), as desired. □

Theorem 8.14. *Suppose that $M \subset \mathbb{R}^3$ is a properly embedded, simply connected minimal surface of quadratic area growth. Then M is a flat plane.*

More generally, we could replace "simply connected" with "finite genus and one end."

Proof. After rotating the coordinates, by Theorem 8.10, for $\epsilon > 0$, we may find R large enough so that letting $C_R := \mathbb{R}^2\backslash B^{\mathbb{R}^2}(0, R)$, then $M\backslash C_R$ is a graph with slope $\le \epsilon$. This implies that $\Theta(M) = 1$, and thus it is a plane by monotonicity. □

As above, we may turn this uniqueness statement into a curvature estimate.

Theorem 8.15. *Suppose that $M_i \subset \Omega \subset \mathbb{R}^3$ are minimal disks with $\partial M_i \subset \partial\Omega$. Then, given area bounds on compact sets, there is some subsequence which converges smoothly to M, which is a smooth embedded minimal disk possibly with multiplicity.*

To prove this, we first need the following lemma

Lemma 8.16. *Suppose that M is a minimal disk in $\mathbb{R}^n$ and B is some ball. If $B \cap \partial M = \emptyset$, then $M \cap B$ is a union of disks.*

Proof. Let $C \subset M \cap B$ be a simple closed curve. Thus $C = \partial D$, for $D \subset M$ a disk. Thus, D is a minimal disk, and so $D \subset B$ by the convex hull property. □

Using this, we have

Proof of Theorem 8.15. Suppose that the curvature of the sequence M_i is blowing up at some point p. Then, by the usual argument we may rescale to obtain M' a nonflat smooth minimal surface in $\mathbb{R}^3$ with quadratic area growth and no boundary. To see that M' is simply connected, if $C \subset M'$ is a simple closed curve, by smooth convergence, there are $C_i \in M_i'$ (the rescaled M_i) so that $C_i \to C$. Choose a ball B containing C, $B \supset C$. Then, by smooth convergence, for i large enough $B \supset C_i$. As in the above lemma, C_i bounds a disk $D_i \subset M_i'$ inside of B. Thus, the limit of the disks D_i must be some disk $D \subset M' \cap B$. This proves that M' is simply connected (alternatively, one may use the fact that the genus may only decrease under a smooth limit of minimal surfaces). Furthermore M' is embedded (possibly with multiplicity). This is because it is the smooth limit of embedded surfaces, it cannot have a transverse self-intersection. Thus, the only possibility is self-tangency, which implies that it is embedded with multiplicity by the maximum principle. □

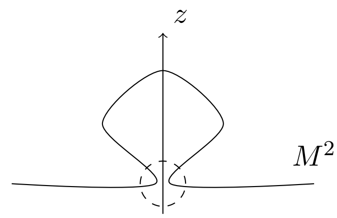

FIGURE 22. There is a metric g on $\mathbb{R}^3$ so that the surface of revolution formed by the illustrated curve is a minimal surface. However, the dashed ball intersects M in an annulus, not a disk.

We emphasize that this is false in a general 3-manifold! The thing that could potentially go wrong is that the above lemma could fail because the convex hull property will not work in the same manner. This is illustrated in Figure 22. See also [Whi89] for similar examples.

On the other hand, if Ω is a geodesic ball of radius R in some M^3 and if all geodesic balls of radius $\leq R$ inside of Ω have smooth convex boundary (alternatively we could assume that $\Omega = B^{\mathbb{R}^3}(0,1)$ with some metric so that all balls $B^{\mathbb{R}^3}(p,r)$ have convex boundary), then the above proof works. For example, it holds for convex domains in hyperbolic space.

9. Removable Singularities

Assume that $M^2 \subset B_1(0)\backslash\{0\} \subset \mathbb{R}^n$ is a proper, branched, minimal immersion with $\partial M \subset \partial B_1(0)$ and $0 \in \overline{M}$.

Proposition 9.1. *For M as above, we have that* $\operatorname{area}(M) < \infty$.

Proof. Let $M(r,R) := \{x \in M : r \leq |x| \leq R\}$ and $X(x) := \vec{x}$. As in the proof of the monotonicity formula, we have that

$$\begin{aligned} 2 = \operatorname{div}_M X &= \operatorname{div}_M X^\perp + \operatorname{div}_M X^\parallel \\ &= -X \cdot H + \operatorname{div}_M X^\parallel \\ &= 0 + \operatorname{div}_M X^\parallel. \end{aligned}$$

As such,

$$\begin{aligned} 2\operatorname{area}(M(r,R)) &= \int_{M(r,R)} \operatorname{div}_M X^\parallel \\ &= \int_{\partial M(r,R)} X \cdot \nu \\ &\leq \int_{\Gamma(R)} X \cdot \nu \\ &\leq R \operatorname{length}(\Gamma(R)) \end{aligned}$$

where $\Gamma(R) = M \cap \partial B_R(0)$. The inner boundary term has been thrown away because the normal must necessarily have a negative dot product with X. As such, because the above bound is independent of R, we see that

$$\operatorname{area}(M \cap B_R) \leq \frac{R}{2} \operatorname{length}(\Gamma(R)).$$

□

We remark that this final inequality is exactly what we used to prove monotonicity, i.e.

Corollary 9.2. *The density $\Theta(M,0,R)$ is non-decreasing for $R \in (0,1)$.*

Now we claim

Proposition 9.3. *M has finitely many ends at 0.*

Proof. Fix $x \in M$ a distance r away from 0. Then, we have that $|M \cap B_r(x)| \geq \pi r^2$. By inclusion, this clearly implies that $|M \cap B_{2r}(0)| \geq \pi r^2$, so $\Theta(M, 0, 2r) \geq \frac{1}{4}$. Letting $r \to 0$, we have that $\Theta(M, 0) \geq \frac{1}{4}$. However, this clearly works for each component of M, and thus establishes the proposition. □

We may now prove

Theorem 9.4. *Suppose that M has finite genus. Then $M \cup \{0\}$ is a branched minimal surface.*

Proof. Because we have shown that M has a finite number of ends E we have that M is topologically a compact manifold with boundary (corresponding to $\partial M \subset \partial B_1(0)$) with a finite number of punctures. As such, each end is topologically a punctured disk. There are thus two cases: either the end is conformal to an annulus or to a punctured disk. In the second case, we have that the end may be parametrized near 0 by $F : \mathbb{D}^2 \backslash \{0\} \to \mathbb{R}^3$, a bounded harmonic map. Then, the removable singularities theorem for harmonic maps guarantees that F extends to the origin.

Finally, we claim that the case of an annulus does not happen. Without loss of generality, we may assume that the annulus is $\{1 \leq |x| \leq \rho\}$. By Schwartz reflection, we may extend $F : \{1 \leq |x| \leq \rho\} \to \mathbb{R}^3$ to an annulus with smaller inner radius, $F : \{1/\rho \leq |x| \leq \rho\} \to \mathbb{R}^3$ by $F(z) := -F\left(\frac{1}{\bar{z}}\right)$. This extended map is harmonic and almost conformal. However, any such map may only have *isolated* points where $DF = 0$. On the other hand, it is clear that DF would vanish on $\{|x| = 1\}$, by construction. This is a contradiction, so the annular case does not occur. □

Open Question 9.5. Is the assumption of finite genus necessary?

In fact, this is still open, even if we assume that M is a smooth, embedded minimal surface in $M \backslash \{0\}$. What could we say about the tangent cone to the origin for a counterexample? One possibility which we do not know how to rule out is a union of planes (with multiplicity).

10. Gauss Bonnet and Branch Points

Suppose that M is a branched surface with smooth boundary (for simplicity we will assume that there are no branch points on ∂M, but we could also deal with this case if necessary). Observe that

$$-2\pi \sum_{\text{branch pt. } p} \operatorname{ord} p + \int_M K dA + \int_{\partial M} k ds = 2\pi \chi(M).$$

Here, $\operatorname{ord} p$ is the integer m so that locally the branch point looks like the graph of $z \mapsto z^{m+1}$. To prove this, just cut out a little ball around each branch point and apply Gauss–Bonnet. It is not hard to determine the boundary terms coming from the small balls near the branch points. Given this, we may prove

Theorem 10.1. *Suppose that we are given $M_i \subset \Omega \subset M^n$, branched minimal surfaces with $\partial M_i \subset \partial \Omega$, with area uniformly bounded on compact sets. Suppose that one of the following two criteria are satisfied:*

(1) *For all $K \Subset \Omega$, we have uniform total curvature bounds on K, i.e.*

$$\sup_i \int_{M_i \cap K} |A|^2(M_i, \cdot) < \infty$$

(2) *For open sets U with $\overline{U} \Subset \Omega$, we have a uniform genus bound, $\sup_i \operatorname{genus}(M_i \cap U) < \infty$.*

Then, after passing to a subsequence, we have smooth convergence $M_i \to M$ where M is a branched minimal surface, possibly with multiplicity, and the convergence is smooth away from isolated points.

Proof. If condition (1) holds, we've already done all of the work to prove this: Theorem 8.13 guarantees convergence away from a finite set of points, and then the removable singularities result, Theorem 9.4 shows that the limiting surface is in fact a smooth branched minimal immersion.

Thus, we will show that (2) $\Rightarrow$ (1). Because the result is local, we may assume that $\Omega = B_5(0) \subset \mathbb{R}^n$ with some Riemannian metric. We may further assume that $\operatorname{genus} M_i \leq g < \infty$, $\operatorname{area}(M_i) \leq A < \infty$ and that all (Euclidean) balls in Ω are convex with respect to the metric. Recall that

$$\chi(M) = 2\#(\text{components}) - 2\operatorname{genus} - 2\#(\text{boundary components})$$

We'd like to uniformly bound the Euler characteristic for any M_i (which we will just denote M for simplicity). For $r \geq 1$, we denote by $M(r)$ the union of all components of $M \cap B_r(0)$ that intersect $B_1(0)$. Without loss of generality we may assume that $M = M(5)$. By monotonicity, for any minimal surface in Ω, say Σ, and each $B_r(x) \subset \Omega$ with $x \in \Sigma$ and $B_r(x) \cap \partial\Sigma = \emptyset$, we have that

$$\operatorname{area}(\Sigma \cap B_r(x)) \geq \alpha r^2$$

(here α is some uniform constant, which would be 1 in flat space, but now possibly has some dependence on the metric). Now, we have

Claim 10.2. *For $2 \leq r \leq 3$, we have that*

$$\#\{\partial M(r)\} \leq \frac{2A}{\alpha} + g$$

and

$$\#\{M(r)\} \leq \frac{A}{\alpha}.$$

Here, A is the assumed uniform area bound.

Proof. Each component, C of $M(r)$ has a point $x \in B(0,1) \cap C$, by definition of $M(r)$ and the choice $r \geq 2$. Thus,

$$\operatorname{area}(C) \geq \operatorname{area}(C \cap B_1(1)) \geq \alpha.$$

This proves the second claim. Also, each component of $M \backslash M(r)$ has boundary in $\partial B_r(0)$ *and* $\partial B_5(0)$. The first statement is because by definition, C touches $B_1(0)$, and the second follows from the assumption that balls are convex, and the maximum principle. In particular, this implies that there is $x \in C \cap \partial B_4(0)$. As such,

$$\operatorname{area}(C) \geq \operatorname{area}(C \cap B_1(x)) \geq \alpha$$

As above, this implies that

$$\#\{M \backslash M(r)\} \leq \frac{A}{\alpha}.$$

Now, if we imagine taking M and removing each component of $M \cap \partial B_r(0)$, each removal either: increases the number of components *or* decreases the genus (this fact follows easily from topology). As such, we have the bound

$$\#\{\partial M(r)\} \leq \#\{M(r)\} + \#\{M \backslash M(r)\} + g,$$

which proves the first claim, after being combined with the above bounds. $\square$

The crucial observation now is that this claim allows us to uniformly bound the Euler characteristic of $M \cap B_r(0)$. As such, if we could uniformly bound

$$\int_{\partial M(r)} k ds,$$

then we would have total curvature bounds on $M \cap B_1(0)$, and we'd be done.[4] FIrst, we note that we can choose r so that we may bound the length of $\partial M(r)$ as follows:

$$A \geq \text{area}(M) \geq \int_0^5 \text{length}(M \cap \partial B) \geq \int_2^3 \text{length}(M \cap \partial B).$$

Thus, we may find some $r \in (2,3)$ so that $|\partial M(r)| \leq \frac{A}{3-2} = A$. Now, we claim that we may find a curve in $B_r(0) \cap M$ which is obtained by pushing the original boundary inwards slightly so that we can control the curvature integral on this curve. To do this, take P, a finite set of points on $\partial M(r)$ (at least two on each component) so that they are separated by geodesic distance inside of $M \cap B_r(0)$ at most 1. In particular, we may do this so that

$$|P| \leq 2(\#\{\partial M(r)\}) + L.$$

Now, we replace each arc between the points with a shortest curve in $M(r)$. Where the new arcs stick to the boundary, the curvature integral has a good sign, and where they are in the interior, they must be geodesics. Furthermore, they cannot intersect $B_1(0)$, because $r \geq 2$ and the new acs are of length at most 1. There are also contributions to the curvature integral from the points in P, because the new boundary curve is not necessarily smooth there. However, the contributions of a single point is at most π, so we have

$$\int_{\partial\Sigma} k \leq \pi |P|$$

where Σ is the interior of the part of M bounded by this new polygonal arc. As such, we have bounds on the total curvature, so we've reduced the theorem to assumption (1), which we have discussed above. $\square$

11. Nonlinear and Linear PDE's

Here we will discuss how to apply linear PDE techniques to nonlinear PDE's (most importantly to us, the minimal surface equation). In particular, this will allow us to understand how minimal surfaces intersect in a better way. We'll first begin with a toy model theorem

Theorem 11.1. *Suppose that $F : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth function. Suppose further that there are $x, y \in \mathbb{R}^n$ so that $F(x) = F(y) = 0$. Then, we can find an "interesting" linear map M so that $M(x-y) = 0$.*

Of course, $M = 0$ would work, but this is not "interesting." What we mean by this will be clear from the proof.

Proof. We use the Fundamental Theorem of Calculus and the chain rule to write

$$\begin{aligned} F(y) - F(x) &= \int_0^1 \frac{d}{dt}\left(F(x + t(y-x))\right) dt \\ &= \int_0^1 DF|_{x+t(y-x)}(y-x) dt \\ &= \underbrace{\left(\int_0^1 DF|_{x+t(y-x)} dt\right)}_{:=M}(y-x) \end{aligned}$$ $\square$

Now, we'll discuss the PDE version of this

[4]We briefly remark that because we are not in flat space, we do not have the identity (which is true for minimal surfaces in $\mathbb{R}^n$) $\int_\Sigma K dA = -\frac{1}{2}\int_\Sigma \beta^2$, but instead there is a correction term (from the Gauss equations) involving the sectional curvature of the ambient metric on $T_p\Sigma$. However, given area bounds, this term is uniformly bounded, so everything works like we expect.

Theorem 11.2. *Consider a (nonlinear) partial differential operator $L[\cdot]$, defined by*

$$L[u] = a_{ij}(x, u, Du)D_{ij}u + b_i(x, u)D_i u + c(x, u)u + f(x, u).$$

Then, for u and v, there is a linear PDE operator $\tilde{L}$ (which depends on u, v) so that

$$L[u] - L[v] = \tilde{L}[u - v].$$

Furthermore $\tilde{L}$ is "nice," e.g. if L is elliptic, then so is $\tilde{L}$.

As an example of this, consider $M^n \subset \mathbb{R}^{n+1}$ a minimal hypersurface which is a graph of $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$. This is the same as requiring that

$$L[u] = \operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) = \left(\frac{\delta_{ij}}{\sqrt{1 + |\nabla u|^2}} - \frac{D_i u D_j u}{\sqrt{1 + |\nabla u|^2}}\right) D_{ij}u = 0.$$

We remark that this is in a much simpler form as compared to the full generality of the above theorem, i.e.

$$L[u] = a_{ij}(Du)D_{ij}u.$$

We now consider u and v, two solutions to the MS. We compute

$$\begin{aligned} L[u] - L[v] &= a_{ij}(Du)D_{ij}u - a_{ij}(Dv)D_{ij}v \\ &= a_{ij}(Du)D_{ij}u - a_{ij}(Du)D_{ij}v + a_{ij}(Du)D_{ij}v - a_{ij}(Dv)D_{ij}v \\ &= a_{ij}(Du)D_{ij}(u - v) + \underbrace{(a_{ij}(Du) - a_{ij}(Dv))}_{:=\beta_{ij}} D_{ij}v. \end{aligned}$$

Now, we apply the fundamental theorem of calculus trick to β_{ij} as used in the finite dimensional theorem above

$$\beta_{ij} = \int_0^1 \frac{d}{dt}\left(a_{ij}(Du + t(Dv - Du))\right) dt = \left(\int_0^1 Da_{ij}|_{Du+t(Dv-Du)} dt\right)(Du - Dv).$$

In particular, writing

$$b_k := \left(\int_0^1 Da_{ij}|_{Du+t(Dv-Du)} dt\right)_k D_{ij}v,$$

we have that

$$\tilde{L}[u - v] = L[u] - L[v] = a_{ij}(Du)D_{ij}(u - v) + b_k D_k(u - v).$$

Suppose that u_α, v_α are two sequences of solutions to the MSE converging (say smoothly) to another solution w. Then, of course the operator we have just constructed, $\tilde{L}_\alpha[\cdot]$, depends on α. However, as $\alpha \to \infty$, it is not hard to check that $L_\alpha[\cdot]$ converges to the linearization of $L[\cdot]$ at w! We recall that the linearization of $L[\cdot]$ is defined by

$$\hat{L}[\varphi] := \frac{d}{dt}\Big|_{t=0} L[w + t\varphi].$$

This, of course, should not be too surprising. Recalling that in the finite dimensional theorem we found the linear operator

$$M = \int_0^1 DF_{x+t(y-x)} dt.$$

As $x_\alpha, y_\alpha \to z$, clearly M converges to $DF|_z$.

So far, it is not clear that our observations above are actually of any use. However, we will now see that they allow us to apply several theorems about linear PDE to nonlinear PDE (which in our case will be the minimal surface equation). For example

Theorem 11.3. *Suppose that $M, N \subset \Omega$ are two minimal connected hypersurfaces with $\partial M, \partial N \subset \partial\Omega$. We suppose further that M divides Ω into two components U and V. If $N \subset \overline{V}$, then if $M \cap N \neq \emptyset$ then $M = N$.*

Proof. To prove this, at $p \in M \cap N$, we may locally write M and N as a graph of u and v (respectively) over their (common) tangent plane. The above results allow us to find a linear (elliptic) PDE for $v - u$. Because $v - u \geq 0$ and $v - u = 0$ at p, we may then apply the strong maximum principle to conclude that $u \equiv v$ in the neighborhood of p. Thus, $M \cap N$ is open. It is obviously closed, and so thus $M = N$. □

Furthermore, we can prove a more refined version of this theorem in the case of an ambient 3-manifold

Theorem 11.4. *Suppose that M, N are a pair of immersed, connected minimal surfaces in a 3-dimensional manifold. We suppose that they are tangent at a point $p \in M \cap N$. Then, they intersect in a bouquet of rays leaving p at equal angles.*

Proof. We may locally write M and N as graphs over their tangent planes (in normal coordinates at p), i.e. $u : \mathbb{B}^2 \to \mathbb{R}$ so that $u(0) = v(0) = 0$ and $Du(0) = Dv(0) = 0$. Then $\varphi = u - v$ satisfies the PDE

$$a_{ij} D_{ij} \varphi + b_i D_i \varphi + c\varphi = 0.$$

It is not hard to check that $a_{ij}(0) = \delta_{ij}$. Now, supposing that everything (the metric and the minimal surfaces) are real analytic, i.e. φ has a power series

$$\varphi(x) = p(x) + \text{higher order terms},$$

where $p(x)$ is a homogeneous polynomial of degree $n \geq 2$. Now, considering the $(n-2)$-th order term in the above PDE, it is clear that this is

$$\delta_{ij} D_{ij} P = 0.$$

In other words, P is a harmonic polynomial (with respect to the flat metric). In particular, it is of the form $ar^n \sin\theta$. This has the property that its zeros are rays leaving the origin at equal angles. There are higher order terms, but it is not hard to see that these do not mess up the equal angle property (of course, the zero set will not be exactly straight lines, but it will approximately look this way near p).

If everything is only C^∞ (or even less regular), one might expect that the above argument would totally fail. Of course, there are widely known examples of C^∞ functions all of whose derivatives vanish at a point. For such functions, the above argument would have no hope of working. However, it is an amazing fact that such "vanishing to infinite order" cannot happen for C^∞ solutions to elliptic PDE! This is known as "unique continuation." In fact, the hypothesis one needs to expand φ in a partial power series are quite weak, for example, requiring that the background metric g is in C^2 is enough. This "partial power series" is then enough to run the above argument. For a proof of this statement, see [GL86, GL87]. □

12. Second Variation Formula, Stability

References for this section include [Sim83, Law75]. Given $M \subset N^n$ a minimal surface, we can embed M into a two parameter family $F_{x,y} : M \to N^n$, where $F_{0,0} = \text{id}$. For x, y small, we define $M_{x,y}$ and $A_{x,y} = \text{area}(M_{x,y})$. We'll furthermore assume that F is compactly supported in the interior of M (in particular F is supported away from ∂M, if it is non-empty). We may further assume that $\left.\frac{\partial F}{\partial x}\right|_{(0,0)}$ and $\left.\frac{\partial F}{\partial y}\right|_{(0,0)}$ point in the normal direction, as tangential variations do not change the area. We've already seen that

$$\frac{\partial A}{\partial y} = -\int_{M_{x,y}} H_{x,y} \cdot \frac{\partial F}{\partial y} \operatorname{Jac}(F_{x,y}) dA$$

and we'd like to compute $\frac{\partial^2 A}{\partial x \partial y}$. We won't actually do the computation here, but just observe that

$$\frac{\partial^2 A}{\partial x \partial y} = -\int_{M_{x,y}} \left(\left\langle DH, \frac{\partial F}{\partial x} \right\rangle \cdot \frac{\partial F}{\partial y} \ \mathrm{Jac}(F_{x,y}) + H \cdot \frac{\partial^2 F}{\partial x \partial y} \mathrm{Jac}(F_{x,y}) + H \cdot \frac{\partial F}{\partial y} \frac{\partial}{\partial x} (\mathrm{Jac}(F_{x,y})) \right) dA.$$

In particular, at $(x, y) = (0, 0)$ the second two terms vanish (because we've assumed that $M_{0,0}$ is minimal), and we see that

$$\frac{\partial^2 A}{\partial x \partial y} = -\int_{M_{x,y}} \left\langle DH, \frac{\partial F}{\partial x} \right\rangle \cdot \frac{\partial F}{\partial y} dA$$

In particular, we see that the second variation formula is roughly equivalent to the derivative of the mean curvature. We further observe that because second derivatives of A commute, DH is clearly a (second order) linear differential operator.

Assuming that M is a two-sided hypersurface, and writing $\frac{\partial F}{\partial x} = f\nu$ and $\frac{\partial F}{\partial y} = g\nu$, one may compute that

$$\frac{\partial^2 A}{\partial x \partial y} = -\int_M \left(\Delta f + \mathrm{Ric}(\nu, \nu) f + |A|^2 f \right) g dA.$$

It is convenient to define the *Jacobi operator* by

$$J(f) = -\Delta f - \mathrm{Ric}(\nu, \nu) f - |A|^2 f.$$

We note that in fact, $J(f) = 0$ is the linearization of the minimal surface equation on M!

Definition 12.1. A minimal surface M is *stable* if $\frac{\partial^2 A}{\partial x^2} \geq 0$ for all compactly supported variations as above. Equivalently (for M a two-sided hypersurface as above), it is stable exactly when $\int J(f) f \geq 0$ for all compactly supported f.

Theorem 12.2. *Suppose that M is a closed, two-sided stable minimal hypersurface in N^n where* $\mathrm{Ric}_N \geq 0$. *Then* $|A| \equiv 0$ *and* $\mathrm{Ric}(\nu, \nu) = 0$ *on* M.

Proof. Simply take $f \equiv 1$ to obtain

$$0 \leq -\int_M (\mathrm{Ric}(\nu, \nu) + |A|^2) dA.$$

The theorem easily follows. □

Corollary 12.3. *If N has positive Ricci curvature, then there are no stable, two-sided minimal hypersurfaces.*

We remark that two-sided is necessary as $\mathbb{R}P^1 \hookrightarrow \mathbb{R}P^2$ and $\mathbb{R}P^2 \hookrightarrow \mathbb{R}P^3$ are both area minimizing[5]

In three dimensions, we have much stronger results, in particular we mention the following result due to Fischer-Colbrie–Schoen and do Carmo–Peng

Theorem 12.4 ([FCS80, dCP79]). *If $M \subset \mathbb{R}^3$ is a complete orientable stable hypersurface, then it is a plane.*

As usual, a Bernstein theorem corresponds to curvature estimates, giving

Corollary 12.5. *For M a stable, orientable (but not necessarily complete) minimal surface in $\mathbb{R}^3$, we have*

$$\mathrm{dist}_M(x, \partial M) |A|(M, x) \leq C < \infty.$$

[5] Both facts can be seen as follows: (1) the first case is homotopically nontrivial and the second is homologically nontrivial, so we may minimize area in the respective classes to find a smooth area minimizing representative, and (2) lifting to the S^2 or S^3 double cover, we may classify geodesics in S^2 and minimal two spheres in S^3 to be exactly the equators. The first fact is obvious, and the second follows from a Hopf differential argument due to Almgren [Alm66].

There are similar statements in a 3-manifold, see [FCS80] where they are expressed very nicely in terms of geometric quantities, e.g. scalar curvature, of the ambient manifold.

Now, we discuss another result concerning stability which holds in any dimension

Theorem 12.6. *Suppose that M_i, M_i' are disjoint sequences of minimal surfaces which converge smoothly to some minimal hypersurface M which is two-sided. Then, M is stable.*

We'll give several proofs of this statement. The first one is rather analytic:

"Analytic" Proof. We may write M_i, M_i' as normal graphs over M of u_i and u_i'. Then, as discussed in Section 11, $L_i(u_i - u_i') = 0$ and L_i converges to J_M. Choose $p \in M$ and let $v_i = \frac{u_i - u_i'}{u_i(p) - u_i'(p)}$. In particular, $L_i v_i = 0$ and $v_i(p) = 1$. By the Harnack inequalty, v_i is then uniformly bounded away from 0 and ∞ on compact sets of M (the bounds are uniform in i, as the L_i's coefficients are smoothly converging to those of J_M as just remarked). Now, by Schauder estimates, we have $|v_i|_{C^{2,\alpha}(K)} \leq C(K, \tilde{K})|v_i|_{C^0(\tilde{K})}$ where $K \subset \tilde{K}$ are compact sets. Crucially, C is uniform in i, for the same reason as in the Harnack estimates. As such, we have a $C^{2,\alpha'}$ (and by higher regularity estimates, automatically C^∞) convergent subsequence $v_i \to v$. Clearly $v(p) = 1, v > 0$ and $J_M(v) = 0$.

We thus may conclude that M has a positive Jacobi field (it does not need to have compact support). Now, we claim the general fact that if M has a positive Jacobi field, then it is stable. To see this, suppose otherwise, i.e. that M is unstable. By a cutoff argument, it is not hard to see that this implies that there is $M' \subset M$ which is compact and unstable. Because $J_{M'}$ is a self adjoint, elliptic, second order linear differential operators, there is a full $L^2(M')$ basis of eigenfunctions $\varphi_1, \varphi_2, \ldots$ (with $\varphi_k|_{\partial M'} = 0$) and corresponding eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots$. Clearly M' is stable if and only if $\lambda_1 \geq 0$.

Because v is positive and φ_1 is positive in M' and zero outside of M' we may scale φ_1 so that $v - \varphi_1 \geq 0$ and $v(q) - \varphi_1(q) = 0$ for some $q \in M'$. On the other hand, we have that

$$J_{M'}(v - \varphi_1) = -\lambda_1 \varphi_1 > 0.$$

The last inequality follows from the assumption that M' is unstable, so $\lambda_1 < 0$. However, this is easily seen to violate the maximum principle. □

We now give a second proof, which is very simple if we are willing to take for granted several (difficult to prove) facts from GMT

"GMT" proof. We'll suppose that we are in a ball for simplicity. This proof will work in general if we are in a domain whose boundary has positive mean curvature. Choose a least area surface Σ_i with $\partial\Sigma_i = \partial M_i'$ and which is constrained to lie between M_i and M_i' (see Figure 23). By Allard's regularity theorem Σ_i is smooth and is furthermore converging smoothly to M. Furthermore, the Σ_i are locally area minimizing, they must be stable and it is not hard to see that the smooth limit of stable minimal surfaces is itself stable. □

Finally we give a geometric proof, at least in a special case that $M_i' = M$ for all i and that M_i lies on one side of M (in fact, this is often how this theorem is used, so this is not such an essential restriction).

"Geometric" Proof. Suppose that M' is a compact submanifold of M which is strictly unstable, as in the first proof. Then, $J_{M'}\varphi_1 = \lambda_1\varphi_1$ with $\lambda_1 < 0$. Let $M_t = \text{graph}(t\varphi\nu)$. The mean curvature is pointing outward for small t (by the second variation formula). However, the M_i act as barriers for this! See Figure 23. This cannot occur, so M must be stable.

□

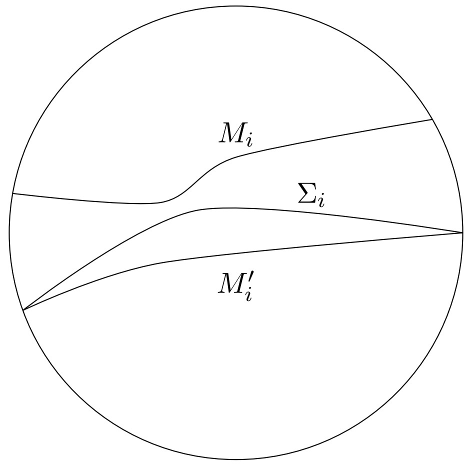

FIGURE 23. Minimizing between M_i and M_i'.

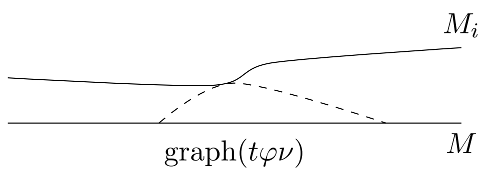

FIGURE 24. The "geometric" proof, using M_i as barriers for the second variation in the direction of an unstable Jacobi field.

Even though it's not immediately obvious, Theorem 12.6 can take us a long way in enhancing previous results of ours, at least in the case of two-sided minimal surfaces. We begin with a necessary remark.

Remark 12.7. Observe that a (two-sided) minimal surface M is stable if and only if $M \setminus Q$ is stable, for Q some discrete subset of M. More generally it is also true that $\lambda_k(M) = \lambda_k(M \setminus Q)$.

In the special case where M is two-sided, Q is a collection of points $\{p_1, \ldots, p_N\}$ and our ambient space is Euclidean space we can argue stability in the non-trivial direction ($\Leftarrow$) as follows. If M were unstable then there would exist u with

$$\int |\nabla u|^2 - |A|^2 u^2 < 0$$

Then for an appropriate choice of cut-off function $\varphi \in \mathcal{C}_c^\infty(M \setminus \{p_1, \ldots, p_n\})$ we're going to have

$$\int |\nabla(\varphi u)|^2 - |A|^2 |\varphi u|^2 < 0$$

as well, meaning that $M \setminus Q$ would be unstable too.

Theorem 12.8. *Let $M_i \subset \Omega$, $\partial M_i \subset \partial\Omega$ be oriented, embedded minimal surfaces with locally finite area and locally finite genus. We already know that (passing to a subsequence) $M_i \to M$, with M a smooth embedded surface with multiplicity, and the convergence is smooth except perhaps on a discrete set of points. The added observation is that any component Σ of M with multiplicity > 1 is necessarily stable.*

Proof. If Q denotes the discrete set of points away from which we have smooth convergence, then applying Theorem 12.6 to $\Sigma \setminus Q \subset \Omega \setminus Q$ (which is the limit of multiple sequences of sheets) tells us that $\Sigma \setminus Q$ is stable, and hence that Σ is stable too. $\square$

Corollary 12.9. *In the context of the theorem, if a component $\widehat{\Sigma}$ is unstable then $\widehat{\Sigma} \cap Q = \emptyset$.*

Proof. If $\widehat{\Sigma}$ is an unstable component then it necessarily has multiplicity 1, so by Allard's regularity theorem convergence is smooth *everywhere* on $\widehat{\Sigma}$. □

Another enhancement we get is the following:

Theorem 12.10. *If $M_i \subset \Omega$, $\partial M_i \subset \partial\Omega$ be as before with curvature blowing up at some $p \in \Omega$ then we already know that we can pick $p_i \to p$, $\lambda_i \to \infty$, such that $\lambda_i(M_i - p_i) \to M'$ smoothly where M' is a non-flat surface. The added observation is that M' has multiplicity* 1.

Proof. If M' had multiplicity > 1 then it would be stable; however by [dCP79] (or for a simpler proof [Pog81]) the only such is the plane which contradicts the non-flatness of M'. □

13. Getting Local Area Bounds

In this section we explore the process of obtaining local area bounds in a very general setting. We are going to need to introduce some new notation.

Definition 13.1. Suppose that $M_i \subset \Omega$ is a sequence of submanifolds (or even varifolds), not necessarily minimal. The *area blow-up set* is defined as

$$Z = \{x \in \Omega : \limsup_{i\to\infty} |M_i|(B_r(x)) = \infty \quad \forall r > 0\}$$

where we're using the standard GMT notation $|M_i|(B_r(x)) = \text{area}(M_i \cap B_r(x))$.

It is not hard to see that Z is a closed subset of Ω.

Definition 13.2. A closed subset $Z \subset \Omega$ is called an (m, h)*-set* provided the following condition holds. For every $f : \Omega \to \mathbb{R}$ smooth and every point $p \in Z$ that is a local maximum for the restriction $f|_Z$, we have: $\text{tr}_m D^2 f(p) \leq h|Df(p)|$. Here tr_m denotes the sum of the smallest m eigenvalues of the corresponding matrix.

Remark 13.3. The moral of the story in this section is that if the M_i are minimal then Z behaves "like a minimal surface *without boundary.*" These definitions allow us to quantify that. In particular the definition of an (m, h)-set, which resembles in spirit the maximum principle, (roughly) is a set-theoretic version to describe m-dimensional manifolds without boundary with mean curvature $|H| \leq h$. In particular, if $M^m \subset \Omega$, $\partial M \subset \partial\Omega$, then M is an (m, h)-set $\Leftrightarrow |H| \leq h$.

Theorem 13.4 (Main Theorem)**.** *Suppose $M_i \subset \Omega$ are m-dimensional submanifolds (or arbitrary varifolds, not necessarily rectifiable) such that $|H_{M_i}(\cdot)| \leq h$, $\sup_i |\partial M_i|(\mathcal{U}) < \infty$ for all $\mathcal{U} \subset\subset \Omega$. Then the area blow-up set Z is an (m, h)-set.*

The following facts are easy to check:

Fact 13.5. *Z is an (m, h)-set $\Leftrightarrow \mu Z$ is an $(m, h/\mu)$-set.*

Fact 13.6. *If $Z_i \to Z$ in the Hausdorff sense (e.g. $Z = \{$subsequential limits of sequences $p_i \in Z_i\}$) and each Z_i is an (m, h)-set, then Z is an (m, h)-set too.*

Fact 13.7. *Z is an (m, h)-set $\Rightarrow Z$ is an (m, h') set for each $h' > h$.*

Fact 13.8. *The set $\{h : Z$ is an (m, h)-set$\}$ is closed.*

The following theorem is particularly useful:

Theorem 13.9 (Constancy Theorem, [Whi12])**.** *Suppose Z is a subset of a connected m-manifold $M \subset \Omega$ and that Z is an (m, h)-subset of Ω. Then either $Z = \emptyset$ or $Z = M$.*

We will now give the proof of the main theorem about area blowup

Proof of Theorem 13.4. Suppose the theorem is false. Then, there is a smooth function $f : \Omega \to \mathbb{R}$ so that $f|_Z$ has a local maximum for $p \in Z$ but

$$\operatorname{tr}_m D^2 f(p) - h|Df(p)| > 0.$$

Without loss of generality, we may assume that $\Omega = B_1(0) \subset \mathbb{R}^n$, $p = 0$, $f|_Z$ attains its strict local maximum at 0 and $\{f \geq t\}$ is compact for all t. The first two assumptions are easy to satisfy, as this is a purely local result. For the second two, it is not hard to see that we may replace f by

$$f(x) - \frac{|x|^4}{1 - |x|^2}.$$

We may find a smaller ball $B \subset \Omega$ around 0 so that there is $\delta > 0$ with

$$\operatorname{tr}_m(D^2 f) - h|Df| \geq \delta > 0$$

on B. Furthermore, by adding a constant to f, we may assume that

$$\max_{Z \backslash B^\circ} f < 0 < f(p)$$

and $N := \{f \geq 0\}$ is compact. Additionally, we may assume that 0 is a regular value for f, so ∂N is smooth. Because $N \backslash B^\circ$ is compact and disjoint from Z, we have the uniform bound

$$|M_i|(N \backslash B^\circ) \leq A < \infty.$$

Furthermore, because N is compact, we have that

$$\max_N f|Df| \leq \Gamma < \infty$$

and

$$\min_N [f \operatorname{tr}_m D^2 f] \geq -\tau$$

for constants Γ, τ. Now, fix B^*, an even smaller ball inside of B, so that $f \geq \gamma > 0$ on B^*. These sets are illustrated in Figure 25.

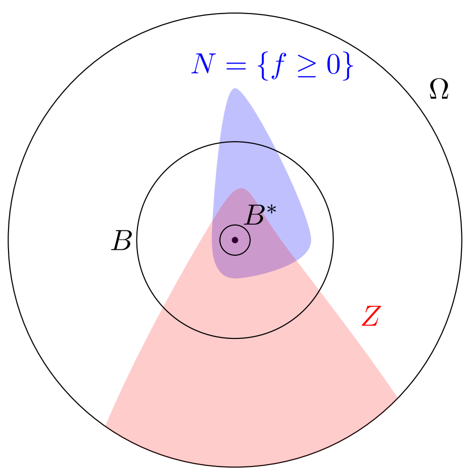

FIGURE 25. The various sets in the proof of Theorem 13.4.

Define a vector field X by

$$X := \nabla \left(\frac{1}{2} f^2 \right) = f \nabla f = f Df^T.$$

By assumption, $|X| = f|Df| \leq \Gamma$ on N. Furthermore

$$DX = f D^2 f + Df^T Df.$$

Notice that the second term has nonnegative eigenvalues. As such, on N (where $f \geq 0$)

$$\operatorname{tr}_m(DX) \geq \operatorname{tr}_m(fD^2 f) = f \operatorname{tr}_m(D^2 f).$$

As such, using X in the first variation formula for $M_i \cap N$, we have that

$$\begin{aligned}
\int_{M_i \cap N} \operatorname{div}_{M_i} X &= -\int_{M_i \cap N} H_{M_i} \cdot X + \int_{\partial(M_i \cap N)} \nu \cdot X \\
&\leq \int_{M_i \cap N} h|X| + \int_{\partial M_i \cap N} |X| + \int_{M_i \cap \partial N} |X| \\
&= \int_{M_i \cap N} h|X| + \int_{\partial M_i \cap N} |X|
\end{aligned}$$

The last line follows as clearly $X \equiv 0$ on ∂N, by definition of X and N. Furthermore, $|X| \leq \Gamma$ in N, so we thus have that

$$\int_{M_i \cap N} \operatorname{div}_{M_i} X \leq \int_{M_i \cap N} h|X| + \Gamma |\partial M_i|(N).$$

The final term is $O(1)$ by the assumption in the theorem. Now, we split the integrand into the pieces inside of B and outside of B, obtaining

$$\begin{aligned}
\int_{M_i \cap N \cap B} (\operatorname{tr}_m DX - h|X|) &\leq \int_{M_i \cap N \cap B} (\operatorname{div}_{M_i} X - h|X|) \\
&\leq \int_{(M_i \cap N) \setminus B} (h|X| - \operatorname{div}_{M_i} X) + O(1) \\
&\leq C|M_i|(N \setminus B) + O(1) \leq O(1).
\end{aligned}$$

Now, we will show that the left hand side goes to ∞. We have that

$$\begin{aligned}
\int_{M_i \cap N \cap B} (\operatorname{tr}_m DX - h|X|) &\geq \int_{M_i \cap N \cap B} f(\operatorname{tr}_m D^2 f - h|Df|) \\
&\geq \int_{M_i \cap N \cap B^*} f(\operatorname{tr}_m D^2 f - h|Df|) \\
&\geq \gamma\delta |M_i|(B^*).
\end{aligned}$$

Here, we have used the fact that f and $\operatorname{tr}_m D^2 f - h|Df|$ are both positive on $N \cap B$. However, by the assumption that $p \in Z$, the area of B^* inside of M_i must be blowing up (at least after passing to a subsequence), which is a contradiction! $\square$

This theorem allows us to give a computation free proof of a fact we mentioned above

Theorem 13.10. *Suppose $M \subset \Omega$ is an m-dimensional submanifold with no boundary in Ω. Suppose $|H_M| \leq h$. Then M is an (m, h)-set*

Proof. Let M_k be M with multiplicity k. Then, Z, the area blowup set is M. Thus, the above theorem shows that M is an (m, h) set. $\square$

Of course, one may also prove this (and the opposite direction claimed above) by direct computation. Now, we show that (m, h) sets act like minimal surfaces in a crucial way: they satisfy a barrier principle

Theorem 13.11 (Barrier Principle)**.** *Suppose that Z is an (m, h) set in Ω and Z lies inside a closed region N with smooth boundary. If $p \in Z \cap \partial N$ then*

$$\sum_{i=1}^{m} k_i(p) \leq h.$$

where $k_1(p) \leq k_2(p) \leq \dots k_{n-1}(p)$ are the principle curvatures of ∂N with respect to the inward unit normal.

Proof. Let $u : \Omega \to \mathbb{R}$ be the signed distance function to ∂N with $u > 0$ outside of N. Let, for $\alpha > 0$ to be chosen, $f = e^{\alpha u}$. We easily compute

$$Df = \alpha e^{\alpha u} Du \quad \text{and} \quad D^2 f = \alpha e^{\alpha u} D^2 u + \alpha^2 e^{\alpha u} Du^T Du.$$

In a basis of $T_p N$ made up of principal directions, it is not hard to check that

$$D^2 u = \begin{pmatrix} k_1 & & & & \\ & k_2 & & & \\ & & \ddots & & \\ & & & k_{n-1} & \\ & & & & 0 \end{pmatrix}.$$

From this, it is easy to see that

$$D^2 f(p) = \begin{pmatrix} \alpha k_1 & & & & \\ & \alpha k_2 & & & \\ & & \ddots & & \\ & & & \alpha k_{n-1} & \\ & & & & \alpha^2 \end{pmatrix}.$$

Choosing α large enough so that α^2 is larger than the other terms, it is clear that

$$\operatorname{tr}_m D^2 f = \alpha \sum_{i=1}^{m} k_i(p).$$

Thus, because Z is an (m, h) set, we have that

$$0 \geq \operatorname{tr}_m(D^2 f(p)) - h|Df(p)| = \alpha \left(\sum_{i=1}^{m} k_i(p) - h \right).$$

Rearranging this yields the desired result. □

We now give a proof of the constancy theorem (which is fundamental to applications of these ideas)

Proof of Theorem 13.9. Suppose not. Then, we may find $p \in M \backslash Z$ and $q \in Z$ which is the nearest point to p in Z. Dilating around q yields an $(m, 0)$-set Z_∞ in $\mathbb{R}^n$ which is contained in a half plane contained in $\mathbb{R}^m$ (it is not hard to see that we may take a subsequential limit in the Hausdorff topology on closed sets, and that rescaling Z by λ yields a $(m, h/\lambda)$-set, so the limit is an $(m, 0)$-set). We assume that Z_∞ is contained in the set $\{x_1 < 0, x_{m+1} = \dots, x_n = 0\}$. Then, the function

$$f(x) = x_1 + (x_1)^2 + \sum_{i>m} (x_i)^2$$

attains a local maximum at 0, when restricted to Z. However

$$D^2 f(0) = \begin{pmatrix} 2 & & & & & & \\ & 0 & & & & & \\ & & \ddots & & & & \\ & & & 0 & & & \\ & & & & 2 & & \\ & & & & & \ddots & \\ & & & & & & 2 \end{pmatrix},$$

so its easy to see that $\operatorname{tr}_m D^2 f(0) = 2 > 0$. □

Finally, to illustrate the power of these results, we give an example of this in Figure 26. Suppose that we have a sequence of minimal surfaces M_i who lie in the region outside of both a catenoid *and* a cylinder. Suppose further that ∂M_i lies on the boundary of the cylinder, and has controlled length (independent of i).

We recall that by a celebrated result of Hoffman–Meeks [HM90], it is not possible to have a proper, immersed minimal surface in $\mathbb{R}^3$ disjoint from a catenoid. However, examining the proof of this result, all that it uses is the barrier principle for minimal surfaces, which is also true for $(m, 0)$-sets. So, it is not hard to see that the same result holds for $(2, 0)$-sets in $\mathbb{R}^3$ (in particular, these sets are acting like minimal surfaces *without boundary* as mentioned before).

As such, the area blow-up set of M_i, Z must be either empty, or contain the catenoid entirely. However, we have also assumed that the M_i are exterior to the cylinder, so Z cannot reach the "neck" of the catenoid! Thus Z is empty.

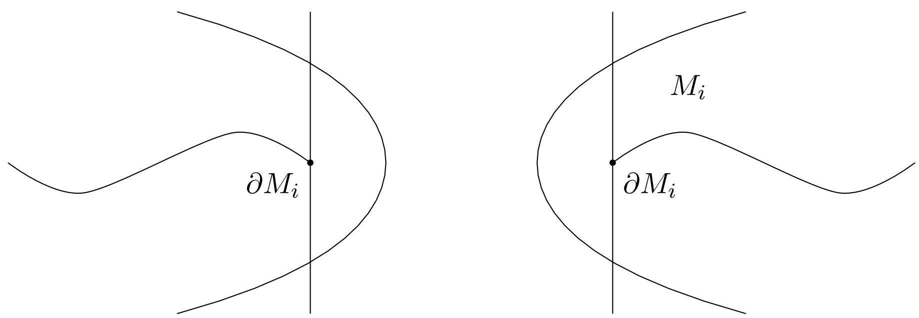

FIGURE 26. An example of the application of the constancy theorem.

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