BRIAN WHITE - TOPICS IN GMT (MATH 258) LECTURE NOTES

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These are lecture notes from Brian White's class on flat chains and geometric measure theory taught in the Spring quarter of 2012 at Stanford University. I have probably introduced errors in the complication process—if you spot any, please let me know at ochodosh@math.stanford.edu.

Much thanks to Brian White for a great class, as well as to the fellow audience members for exciting discussion. Additional thanks is due to Nick Alikakos for suggesting the inclusion of Examples 2.10, 2.11 and 2.12, as well as to Felix Schulze for some corrections.

1. INTRODUCTION

We are interested in the Plateau problem:

Problem 1.1 (Plateau Problem). *Given a closed $(m-1)$ dimensional surface $\Gamma \subset \mathbb{R}^N$, find an m-dimensional surface S of least area among surfaces with $\partial S = \Gamma$.*

We have certainly not yet made any decisions about the exact setup of the problem, but there are obvious questions of existence, uniqueness, and regularity once we do. To show existence, one might try the following method: take a minimizing sequence $S_1, S_2, \ldots$ such that $\partial S_i = \Gamma$ and

$$\text{area}(S_i) \to \alpha := \inf\{\text{area}(S) : \partial S = \Gamma\}.$$

We might hope that by extracting a subsequence, we could have $S_i \to S$ in some sense, for some surface S where S solves Plateau's problem. To do this, we need existence of finite area surfaces S with $\partial S = \Gamma$, compactness with respect to some topology, and furthermore, lower semicontinuity of area with respect to this topology along with continuity of the boundary operator (so that $S_i \to S$ implies that $\Gamma = \partial S_i \to \partial S$, so $\partial S = \Gamma$).

What can go wrong with this sort of plan? We give the following example:

Example 1.2. *Figure 1 shows a sequence of disks with small fingers. The disk spans a (flat) circle, so the unique minimizer is the disk. However, we can make the fingers as long as we desire while making the area closer and closer to that of the flat disk, by making the fingers very narrow. Certainly, this sequence of disks with fingers does not converge to the flat disk in a classical topology!*
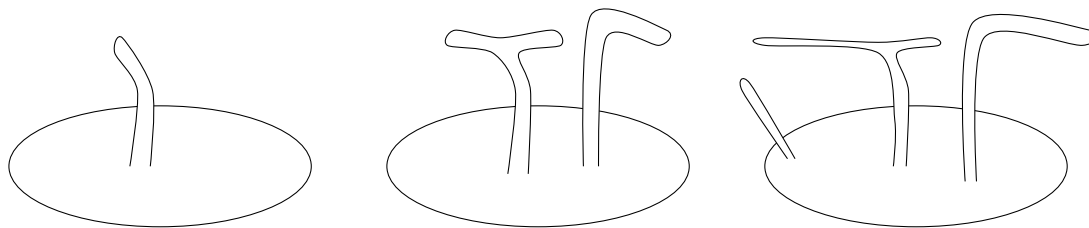
FIGURE 1. A disk with small fingers can approach the minimal area to span a disk, yet it can look very different from the (unique) minimizer, the flat disk.

This is a problem that we have to overcome in order to have some hope of solving Plateau's problem.

1.1. **History.** Briefly, we recall the history of the Plateau problem. In the 1930's Douglas and Radó gave solutions for 2-dimensional surfaces (c.f. [Dou31, Rad30]). They used the 2-dimensional hypothesis strongly, as their proof relied on conformal parametrization. Then, around 1960, De Giorgi solved the problem for general hypersurfaces (c.f. [DG61]). Soon after, Federer-Fleming solved the problem in arbitrary dimension and codimension in "Normal and Integral Currents," [FF60]. Soon after, in 1964, Fleming gave a different approach in "Flat chains over a finite coefficient groupm" [Fle66]. This is the approach we will mostly follow in these notes.

2. Introduction to Flat Chains

2.1. **Coefficient Group.** To discuss flat chains, we need to first choose a *normed abelian group* $(G, |\cdot|)$.

Definition 2.1. *A normed abelian group $(G, |\cdot|)$ is an abelian group with a function $|\cdot| : G \to \mathbb{R}$ satisfying*

(1) $|g| \geq 0$ *with equality exactly if $g = 0$.*
(2) $|g + h| \leq |g| + |h|$.
(3) $|-g| = |g|$.

From the norm, we can define a metric $d(g, h) := |g - h|$. Notice that this is translation invariant, and conversely any translation invariant metric comes from a norm. Some examples of this are

Example 2.2. *Examples of normed abelian groups include*

(1) $\mathbb{Z}$ *with the usual absolute value. This will correspond to oriented chains.*
(2) $\mathbb{Z}/2$ *with $|0| = 0, |1| = 1$. This will correspond to non-oriented chains.*
(3) $\mathbb{R}$ *with absolute value. This will be less important.*
(4) $\mathbb{R}$ *with norm $|x| := \sqrt{|x|_{usual}}$. This will come up later as a counterexample.*

We'll briefly indicate how the solution to the Plateau problem will depend on the chosen norm (of course we have not actually defined this precisely, so this will just be an intuitive picture).

Example 2.3. *Consider solving the 1-dimensional Plateau problem in the plane. Figure 2 shows an example where solutions of Plateau's problem can be different with different norms on the group. The left hand diagram exhibits the solution to Plateau's problem in this case with the group $\mathbb{Z}$ along with the usual absolute value, while the right hand diagram shows the solution with $\mathbb{Z}$ with the norm that is 1 unless the element is zero. The reason that this has less "area" is that the overlapping part is counted only once (according to the norm, $|2| = 1$, so multiplicity 2 does not affect the measured "area").*

Figure 2. Plateau's problem with different group norms.

2.2. **Polyhedral Chains.** Fix a normed abelian group G, and let $E = \mathbb{R}^N$ be Euclidean space. For $m \leq N = \dim E$, consider the abelian group generated by formal sums (with coefficients in G) of compact, convex, oriented m dimensional polyhedra[1]. Then, quotient by the equivalence relation requiring $\sigma \sim -\hat{\sigma}$, where $\hat{\sigma}$ is σ with the opposite orientation, and if σ is formed by gluing σ_1, σ_2 along a face (with the correct orientation), then $\sigma \sim \sigma_1 + \sigma_2$.

Definition 2.4. *The quotient group is called "polyhedral chains," and will be denoted $\mathscr{P}_m(E, G)$. We will often drop the G, when it is clear from context.*

It is not hard to show that for $P \in \mathscr{P}_m(E)$, there is a representation

$$P = \sum_{\text{finite}} g_i[\sigma_i],$$

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[1] We'll define these to be intersection of half planes, for example.

where the σ_i are non-overlapping. Now, by defining the *mass functional*

(2.1) $$\mathbb{M}(P) = \inf\left\{\sum |g_i|\operatorname{area})\sigma_i) : P = \sum g_i[\sigma_i]\right\} = \underbrace{\sum |g_i|\operatorname{area}(\sigma_i)}_{\sigma_i \text{ non-overlapping}},$$

this makes $\mathscr{P}_m(E, G)$ into a normed group, and gives it a topology from $d_M(P, Q) = \mathbb{M}(P - Q)$. We note that there is a *boundary map* $\partial : \mathscr{P}_m(E) \to \mathscr{P}_{m-1}(E)$, which sends each polygon to the sum of its faces with the induced orientations. It can be checked that this is compatible with the relations in $\mathscr{P}_*(E)$. Furthermore, it is not hard to check that $\partial(\partial P) = 0$.

We now remark that this topology has no chance of allowing us to solve the Plateau problem, as the following examples illustrate.

Example 2.5. *Consider two intervals a distance $\epsilon > 0$ apart, as in Figure 3. It is clear that $d_{\mathbb{M}}(P_\epsilon, P) = \mathbb{M}(P_\epsilon - P) \equiv 2$, so they cannot converge to each other*
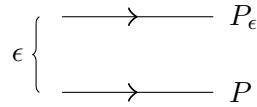

FIGURE 3. The interval P_ϵ does not converge to P in the topology induced by $\mathbb{M}$, contrary to what we would like.

More disturbingly, consider a $1 \times \epsilon$ rectangle Q and its boundary ∂Q, as in Figure 4. Clearly $\mathbb{M}(Q) = \epsilon$, so $Q \xrightarrow{\mathbb{M}} 0$, but $\mathbb{M}(\partial Q) = 2 + 2\epsilon$, which shows that ∂Q does not converge to 0 in the $\mathbb{M}$-topology.
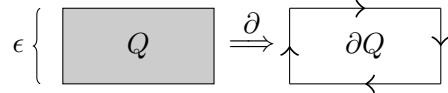
FIGURE 4. The boundary operator is not continuous in the mass norm topology, as this example illustrates.

2.3. **The Flat Norm.** To resolve the issues illustrated in Example 2.5, Whitney introduced the *flat norm*:

(2.2) $$\mathscr{F}(P) := \inf\{\mathbb{M}(P - \partial Q) + \mathbb{M}(Q) : Q \in \mathscr{P}_{m+1}(E)\} = \inf\{\mathbb{M}(S) + \mathbb{M}(T) : P = S + \partial T\}$$

To illustrate how this resolves the issues that the $\mathbb{M}$ topology had, consider the following example.

Example 2.6. *In Figure 5 we again have the two lines P, P_ϵ, a distance ϵ apart. However, in the flat chain distance, subtract the boundary of the rectangle Q from $P_\epsilon - P$, leaving only the sides of the rectangle, which have total length 2ϵ. In particular, we thus see that*

$$\mathscr{F}(P_\epsilon - P) \leq \mathbb{M}(P_\epsilon - P - \partial Q) + \mathbb{M}(Q) = 2\epsilon + \epsilon \to 0,$$

so we have that $P_\epsilon \to P$ in the $\mathscr{F}$-topology.

FIGURE 5. The interval P_ϵ converges to P in the flat norm topology.

Notice that we have not quite yet proven that $\mathscr{F}$ is a norm on $\mathscr{P}_m(E)$. Most of the properties of a norm are not hard to show, but it is not immediately obvious that $\mathscr{F}(P) = 0$ implies that $P = 0$. This true, and thus $\mathscr{F}$ is a norm on $\mathscr{P}_m(E)$, but we'll prove it later. It will be important later to notice that (given that it is actually a norm) $\mathscr{F}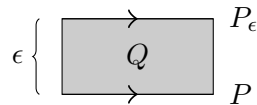$ is the largest norm on $\mathscr{P}$ such that (1) $\mathscr{F}(\cdot) \leq \mathbb{M}(\cdot)$ and (2) $\mathscr{F}(\partial(\cdot)) \leq \mathscr{F}(\cdot)$.

Now, taking K to be a compact subset of E, we define $\mathscr{P}_m(K) \subset \mathscr{P}_m(E)$ to be the m-dimensional polyhedral chains supported in K. We can thus complete this as a metric space with respect to the flat norm $\mathscr{F}(\cdot)$, denoting the completion $\mathscr{F}_m(K)$. The union of these thus gives the normed group of *flat chains*

$$\mathscr{F}_m(E) := \bigcup_{K \text{ compact}} \mathscr{F}_m(K).$$

As in any completion, the flat norm $\mathscr{F}$ continues to be a norm on $\mathscr{F}_m(E)$ in the obvious manner.

Example 2.7. *Notice that $\mathbb{M}(\cdot)$ is not continuous with respect to the $\mathscr{F}$ topology, as illustrated in Figure 6. There, we have let Q be a $1 \times \epsilon$ rectangle, and it is clear that*

$$\mathscr{F}(\partial Q) \leq \mathbb{M}(Q) = \epsilon \to 0,$$

but $\mathbb{M}(\partial Q) = 2 + 2\epsilon \nrightarrow 0$
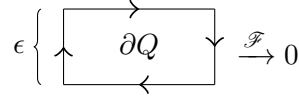
FIGURE 6. Mass is not continuous in the flat norm topology.

A slightly more sophisticated example of this is the following

Example 2.8. *The staircase paths shown in Figure 7 certainly converge to the straight line in the $\mathscr{F}$ topology, but the mass of the staircase chains is 2, while the mass of the line is $\sqrt{2}$. By modifying the idea of this example, it is not hard to give an example where the mass diverges, but the mass of the $\mathscr{F}$ limit is finite.*
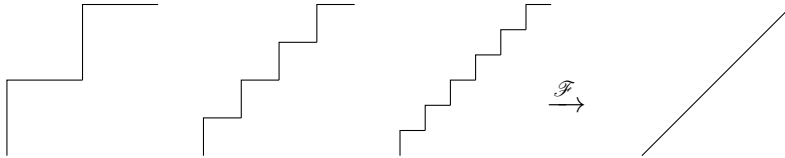
FIGURE 7. Staircase paths converging to a straight line. This is not convergence in mass, even though all of the chains have finite, nonzero mass.

Motivated by Examples 2.7 and 2.8, we define

$$(2.3) \qquad \tilde{\mathbb{M}}(A) := \liminf_{\epsilon \to 0} \{\mathbb{M}(P) : P \in \mathscr{P}(E), \mathscr{F}(A - P) < \epsilon\}$$

It turns out that

Lemma 2.9. *This new mass agrees with the old one on polygons, i.e. $\tilde{\mathbb{M}}(P) = \mathbb{M}(P)$. Equivalently, for polygons converging to another polygon in the $\mathscr{F}$ norm, $P_i \xrightarrow{\mathscr{F}} P$, we have that*

$$\mathbb{M}(P) \leq \liminf_{i \to \infty} \mathbb{M}(P_i)$$

As a result of this, in the sequel we will drop the tilde on $\tilde{\mathbb{M}}$ without any notational confusion. We will prove this later.

We make several observations concerning the flat norm.
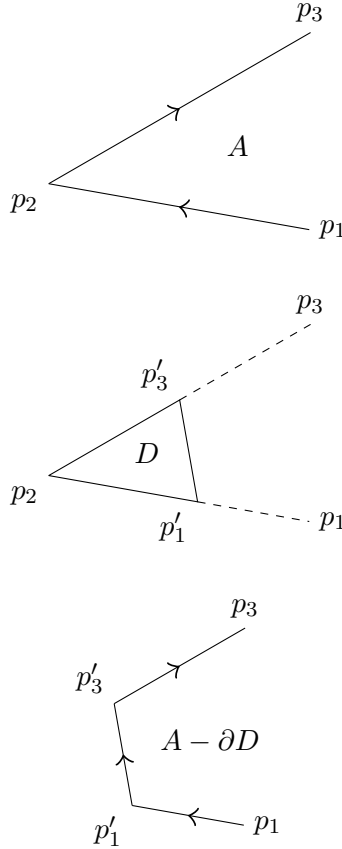
FIGURE 8. Example showing that the flat norm can be strictly smaller than the mass norm.

Example 2.10 (See [Whi57][2]). *Here we give an example to show that the flat norm may be strictly smaller than the mass norm. For three points $p_1, p_2, p_3 \in \mathbb{R}^2$, consider the 1-chain A, and 2-chain D as illustrated in Figure 8. It is not hard to see that for D small enough, $\mathbb{M}(A-\partial D)+\mathbb{M}(D) < \mathbb{M}(A)$, so we see that $\mathcal{F}(A) < \mathbb{M}(A)$.*

Example 2.11 (See [Whi57]). *Here, we give another example to show that the flat norm can be strictly smaller than the mass norm. We first describe a general construction, which we will use in demonstrating this. Given A, a polyhedral r-chain in $\mathbb{R}^r \subset \mathbb{R}^n$, and a vector $v \in \mathbb{R}^n$ which does not lie entirely in $\mathbb{R}^r$, we define $\mathscr{D}_v A := \{p + tv : p \in A, t \in [0,1]\}$ (in the sequel, we'll abuse notation and also use this notation for polyhedral r'-chains in $\mathbb{R}^r$), along with the obvious orientation. This is illustrated in Figure 9. We have defined $T_v A$ as the translation of A under v. Now, notice that*

$$\partial \mathscr{D}_v A = T_v A - A - \mathscr{D}_v \partial A$$

and it is not hard to see that $\mathbb{M}(\mathscr{D}_v A) = |v^\perp|\mathbb{M}(A)$, for $v^\perp$ the orthogonal projection away from $\mathbb{R}^r$. Furthermore, we have that

$$\mathcal{F}(T_v A - A) \leq \mathbb{M}(T_v A - A - \partial \mathscr{D}_v A) + \mathbb{M}(\mathscr{D}_v A)$$
$$= \mathbb{M}(\mathscr{D}_v \partial A) + |v^\perp|\mathbb{M}(A)$$
$$\leq |v|\left(\mathbb{M}(\partial A) + \mathbb{M}(A)\right).$$

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[2]Thanks to Nick Alikakos for suggesting this and the two subsequent examples.

Now, we specialize to the case where A is the 1 chain $[p, q]$ with $|q - p| = \alpha < 2$. Suppose that $|v| = \beta < \frac{\alpha}{2}$. Then, letting $B = T_v A - A$, using the above, we have that

$$\mathscr{F}(B) = \mathscr{F}(T_v A - A) \leq |v|(2 + \alpha) \leq 2\beta + \alpha\beta < \alpha + \alpha\beta < 2\alpha = \mathbb{M}(B),$$
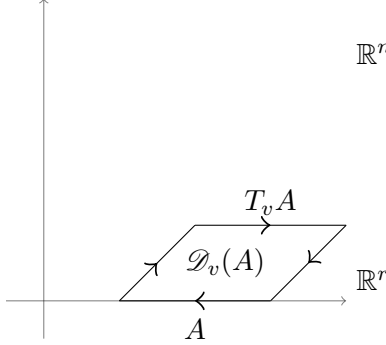
as desired.

FIGURE 9. Definition of $\mathscr{D}_v A$ for Example 2.11

Example 2.12 (See [Whi57]). *We now show that the flat norm is not additive and cannot be associated with a measure. We define the points $a_i = -2^{-i}, b_i = 2^{-i}$. We define a 1-chain $B_i = [a_i, b_i]$. Furthermore we define 0-chains $A'_i = b_i - a_i$ and $A_k = A'_0 + \cdots + A'_k$. We may compute*

$$\mathscr{F}(A'_i) = \mathscr{F}(\partial B_i) \leq \mathbb{M}(B_i) = 2^{1-i}$$

and

$$\mathscr{F}(A_m - A_n) = \mathscr{F}(A'_{n+1} + \cdots + A'_m) \leq \sum_{j=n+1}^{m} 2^{1-j}.$$

This clearly shows that A_k is a Cauchy sequence in the flat norm, so $\lim_{k \to \infty} A_k := A$ exists. Note that A is a 0-chain composed of infinitely many points but finite $\mathscr{F}$-norm.

On the other hand, for any point p, the associated 0-chain must have $\mathscr{F}(p) = 1$. Thus, we see that $\mathscr{F}$ cannot be a measure, in the sense that it is a function from (certain) subsets of $\mathbb{R}$ to $\mathbb{R}^+$, because it is not additive.

2.4. Extending maps from Polyhedral Chains to Flat Chains.

Proposition 2.13. *Suppose that $f : \mathscr{P}_*(E) \to \mathscr{P}_*(E)$ (or $\mathscr{F}_*(E)$) is a chain homomorphism (i.e. f respects the grading, $f(A + B) = f(A) + f(B)$ and $f \circ \partial = \partial \circ F$), then if f is Lipchitz with respect to $\mathbb{M}$, then it is also Lipchitz with respect to $\mathscr{F}$, and thus induces a unique extension $\mathscr{F}_*(E) \to \mathscr{F}_*(E)$.*

Proof. If $A = B + \partial C$, then because f is a chain homomorphism, $f(A) = f(B) + \partial f(C)$. Thus

$$\mathscr{F}(f(A)) \leq \mathbb{M}(f(B)) + \mathbb{M}(f(C)) \leq \mathrm{Lip}(f)(\mathbb{M}(B) + \mathbb{M}(C)).$$

Taking the infimum over B, C with $A = B + \partial C$ gives

$$\mathscr{F}(f(A)) \leq \mathrm{Lip}(f)\mathscr{F}(A). \qquad \square$$

Now, we show that we can take cones of flat chains.[3] Suppose that $0 \in K$. Then, for a polygon $\sigma \in \mathscr{P}_m(K)$, we define the cone over it to be the convex hull of σ and 0, which we denote

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[3]We will use the notation cone$(\cdot)$ for cone, as well as cone$_x(\cdot)$ for the cone centered at x, when it is important to specify the center. We remark that an alternative notation for the cone of σ centered at x is $[x] \not\Vvdash [\sigma]$, which we will not use here.

$\tilde{\sigma} \in \mathscr{P}_{m+1}(K)$. We give $\tilde{\sigma}$ the orientation so that the component of $\partial\tilde{\sigma}$ corresponding to σ has the same orientation as σ. It is not hard to check that this respects the equivalence relations on polygons, and thus extends to a homomorphism, cone $: \mathscr{P}_m \to \mathscr{P}_{m+1}$. However, notice that it is not a chain homomorphism, and in fact

$$\partial \operatorname{cone}(P) = P - \operatorname{cone}(\partial P).$$

Thus, we see that $\partial \operatorname{cone} + \operatorname{cone} \partial = \operatorname{id}$, so $\operatorname{cone}(\cdot)$ is what is known as a chain homotopy.

Proposition 2.14. *Suppose $H : \mathscr{P}_* \to \mathscr{P}_*$ is a homomorphism. Then*

(1) *$P := \partial H + H\partial$ is a chain homomorphism.*
(2) *If H, P are Lipchitz with respect to $\mathbb{M}$, then they are Lipchitz with respect to $\mathscr{F}$ (and they thus extend to maps $\mathscr{F}_* \to \mathscr{F}_*$).*

Proof. Part (1) of the claim is trivially checked by using $\partial^2 = 0$. To prove part (2), we let $A = B + \partial C$. Then, we have that

$$H(A) = H(B) + H(\partial C) = H(B) + P(C) - \partial H(C)$$

implying that

$$\mathscr{F}(H(A)) \le \mathbb{M}(H(B) + P(C)) + \mathbb{M}(H(C)) \le (\operatorname{Lip}_{\mathbb{M}}(H) + \operatorname{Lip}_{\mathbb{M}}(P))(\mathbb{M}(B) + \mathbb{M}(C)),$$

so taking the infimum over all such B, C proves the claim. $\qquad\square$

2.5. **Statement of Main Results about Flat Chains.** We now state some of the main results we will want to prove about flat chains. These will depend on the choice of $(G, |\cdot|)$, as we will discuss below.

Definition 2.15. *We call a compactness theorem the following statement: If $K \subset \mathbb{R}^N$ is a compact, convex set, and $\lambda < \infty$ is a real number, then defining*

$$\mathcal{S}_\lambda := \{A \in \mathscr{F}_m(K) : \mathbb{M}(A) + \mathbb{M}(\partial A) \le \lambda\},$$

then $\mathcal{S}_\lambda$ is compact in the flat norm topology.

We will show that

Theorem 2.16. *A compactness theorem holds if and only if $(G, |\cdot|)$ has the property that the set*

$$\{g \in G : |g| \le r\}$$

is compact for every r.

Another important notion is

Definition 2.17. *We will say that we have a rectifiability theorem if for $A \in \mathscr{F}_*$ with $\mathbb{M}(A) < \infty$, then A is rectifiable.*

Similarly, we will show that

Theorem 2.18. *A rectifiability theorem holds if and only if there does not exist a continuous, non-constant path of finite length in G.*

As an example, the normed group $(\mathbb{R}, |\cdot|_{\text{usual}})$ admits continuous, non-constant paths of finite length, but not the group $(\mathbb{R}, \sqrt{|\cdot|_{\text{usual}}})$.

2.6. More Properties of Flat Chains.

Theorem 2.19. *For $A \in \mathscr{F}_*$, we have that*

$$\mathscr{F}(A) = \inf\{\mathbb{M}(B) + \mathbb{M}(C) : A = B + \partial C\}.$$

Note that this was the definition of $\mathscr{F}(\cdot)$ if everything were polygonal chains, but we're claiming that it holds in general.

Proof. Let

$$\alpha := \inf\{\mathbb{M}(B) + \mathbb{M}(C) : A = B + \partial C\}.$$

We first claim that $\mathscr{F}(A) \geq \alpha$: Choose $A_n \in \mathscr{P}_*$ such that

$$\mathscr{F}(A - A_n) < \frac{\epsilon}{2^{n+1}}.$$

This easily implies that

$$\mathscr{F}(A_{n+1} - A_n) < \frac{\epsilon}{2^n},$$

and thus we can find polygonal chains P_n, Q_n with

$$A_{n+1} - A_n = P_n + \partial Q_n$$

and such that

$$\mathbb{M}(P_n) + \mathbb{M}(Q_n) < \frac{\epsilon}{2^n}.$$

Now, define $P, Q \in \mathscr{F}_*$

$$P := \sum_{n=1}^{\infty} P_n \qquad Q := \sum_{n=1}^{\infty} Q_n.$$

These two sums converge with respect to the $\mathbb{M}$ norm (and in fact, $\mathbb{M}(P) + \mathbb{M}(Q) < \epsilon$), and thus certainly also converge for the $\mathscr{F}$ norm. Now, we have

$$A_n - A_0 = \sum_{k=1}^{n} P_k + \partial\left(\sum_{k=1}^{n} Q_k\right),$$

and taking the limit as $n \to \infty$ (in the $\mathscr{F}$ topology ∂ is continuous) we have

$$A = A_0 + P + \partial Q.$$

Because A_0 is polyhedral, we have that there are polyhedral B_0, C_0 such that $A_0 = B_0 + \partial C_0$ and

$$\mathbb{M}(B_0) + \mathbb{M}(C_0) < \mathscr{F}(A_0) + \epsilon$$

Combining this with the bounds on the mass of P, Q, we have that because

$$A = (B_0 + P) + \partial(C_0 + Q),$$

is a candidate for the infimum defining α, we see that

$$\alpha \leq \mathbb{M}(B_0 + P) + \mathbb{M}(C_0 + Q) < 2\epsilon + \mathscr{F}(A_0) < 3\epsilon + \mathscr{F}(A).$$

Because $\epsilon > 0$ was arbitrary, we have thus shown that $\mathscr{F}(A) \geq \alpha$.

To show the other inequality, $\mathscr{F}(A) \leq \alpha$, suppose that $A = B + \partial C$. Then, we have

$$\mathscr{F}(A) \leq \mathscr{F}(B) + \mathscr{F}(\partial C) \leq \mathscr{F}(B) + \mathscr{F}(C) \leq \mathbb{M}(B) + \mathbb{M}(C).$$

and taking the infimum over allowed B, C we have shown the desired inequality.

$\square$

Theorem 2.19 gives the following property, which will be important in solving Plateau's problem:

Theorem 2.20. *Every cycle bounds a chain of finite mass.*[4] *More precisely, if $A \in \mathscr{F}_*$ with $\partial A = 0$, then there is $T \in \mathscr{F}_*$ with $\partial T = A$ and $\mathbb{M}(T) < \infty$.*

Proof. By definition, $\mathscr{F}(A) < \infty$, so there are B, C with $A = B + \partial C$ and $\mathbb{M}(B) + \mathbb{M}(C) < \infty$. However, because A is a cycle, we have that

$$0 = \partial A = \partial B + \partial^2 C = \partial B.$$

We thus have that $B = \partial(\mathrm{cone}(B))$, where $\mathrm{cone}(B)$ is the cone over B, so

$$A = \partial(\mathrm{cone}(B) + C).$$

Because $\mathbb{M}(B) < \infty$, we have shown that $\mathbb{M}(\mathrm{cone}(B)) < \infty$ (by using the fact that B is contained in some compact set K), and thus we've written A as the boundary of a chain with finite mass. $\square$

2.7. Flat n-chains in $\mathbb{R}^n$.

The case of top-dimensional chains is considerably easier to understand, as we now discuss. Take $K \subset \mathbb{R}^n$ a compact, convex subset. For $A \in \mathscr{P}_n(K; G)$, we can write

$$A = \sum g_i[\sigma_i].$$

We can furthermore assume that all of the σ_i are oriented with the induced orientation from $\mathbb{R}^n$. Consider the associated function $g_A : K \to G$

$$g_A := \sum g_i \mathbb{1}_{\sigma_i},$$

where $\mathbb{1}_{\sigma_i}$ is the indicator function of σ_i. It is not hard to see that $A \mapsto g_A$ is a homomorphism

$$\mathscr{P}_n(K; G) \to L^1(K; G),$$

and furthermore, that

$$\mathscr{F}(A) = \mathbb{M}(A) = \int |g_A| d\mathcal{L}^n = \|g_A\|_{L^1}.$$

The first equality follows from the fact that there are no nontrivial $n + 1$ chains in $\mathbb{R}^n$, so from the definition of the flat norm, it is just mass on top dimensional chains. Thus, the above map is an isometric embedding. However, it is clear that the image is dense in L^1, so taking the $\mathscr{F}$ completion, we have that this map extends to an isometry

$$\mathscr{F}_n(K; G) \xrightarrow{\cong} L^1(K; G).$$

In addition, we have an isometry

$$\mathscr{F}_n(\mathbb{R}^n; G) \xrightarrow{\cong} L^1_{\mathrm{cpt}}(\mathbb{R}^n; G).$$

Thus, we see that flat chains can be very singular objects! Note that higher codimension flat chains can be even worse, because there $\mathscr{F}$ is weaker than $\mathbb{M}$.

2.8. Intersecting flat chains with subsets of $\mathbb{R}^n$.

We first give an example to show that there may not be a reasonable notion of the intersection of a flat chain with a subset, even a very nice subset such as a half plane.

Example 2.21. *Take the coefficient group to be $(\mathbb{Z}, |\cdot|_{abs})$ so that orientations matter. Consider the flat chain Q and its boundary ∂Q as shown in Figure 10. To be precise, we assume that Q is an infinite collection of lines with finite total mass. Then, the boundary of Q is the sequence of points as shown. This is certainly a flat chain, but suppose that we were to intersect ∂Q with the left half-plane, obtaining a set of points as shown in Figure 11. It is not hard to see that this cannot be a flat chain, the reason for which is informally that it has "infinite $\mathscr{F}$ norm," because no line can connect any two of the points in $\partial Q \cap H$, due to the orientations.*

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[4]It will be important later (in the proof of Theorem 7.18) that every cycle bounds a cycle which is the $\mathbb{M}$-limit of polyhedral chains, which is obvious from the proof. In particular, it is clear that such chains are rectifiable, so we have that every chain bounds a rectifiable chain of finite mass.
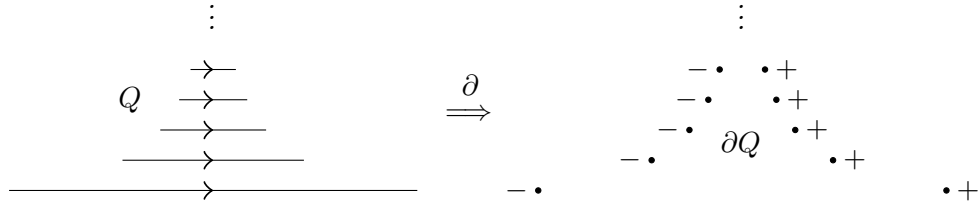
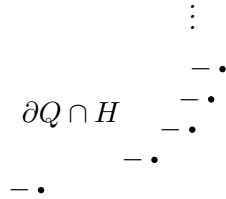FIGURE 10. The flat chain Q and its boundary, which will not intersect nicely with the left half-plane.

FIGURE 11. Attempting to intersect ∂Q with the left half-plane H does not result in a flat chain.

However, we claim that

Claim 2.22. *For A a flat chain,*

(1) *If A has finite mass, and S is any Borel set, then there is a well defined notion of "the part of A inside S."*

(2) *For general A, if $f : \mathbb{R}^n \to \mathbb{R}$ is nonconstant and linear, there is a well defined notion of "the part of A inside $\{f \leq t\}$" for a.e. t. Furthermore, this notion is continuous under convergence in the $\mathscr{F}$ topology.*

(3) *Similarly, for a.e. t, it is possible to intersect A with $\{f = t\}$.*

We'll first study how to intersect polyhedral chains with half spaces, which is not very difficult. For P a polyhedral m-chain $P = \sum g_i[\sigma_i]$ and H a half-space, let

$$P \llcorner H := \sum g_i[\sigma_i \cap H]$$

where we use the induced orientation, and throw away lower dimensional $\sigma_i \cap H$ pieces. It is not hard to see that this is well defined, and that

$$P \mapsto P \llcorner H$$

is a homomorphism. However, it is not a chain homomorphism, as it does not commute with the boundary operator. In fact, we define

(2.4) $$A \cap [\partial H] := \partial(A \llcorner H) - (\partial A) \llcorner H.$$

In many situations this gives exactly what we expect, but in general it may behave differently than simply a set-theoretic intersection. Figure 12 gives an example of the intersection of a triangle with a hyperplane in which the definition gives what we'd expect.

On the other hand, Figures 13 and 14 are examples of where the definition in (2.4) can have non-intuitive behavior. We briefly make note of two properties which will be useful in the sequel. First of all, for P a polyhedral chain and H a half plane,

$$\partial(P \cap [\partial H]) = -\partial((\partial P) \llcorner H).$$

Secondly, it is clear that

$$\mathbb{M}(P \llcorner H) \le \mathbb{M}(P).$$

We have the following theorem

Theorem 2.23. *For $f : \mathbb{R}^N \to \mathbb{R}$, linear with $|Df| = 1$, letting $H_t = \{f \le t\}$, and A be a polyhedral chain, we have that*

$$\int \mathbb{M}(A \cap [\partial H_t]) \le \mathbb{M}(A)$$
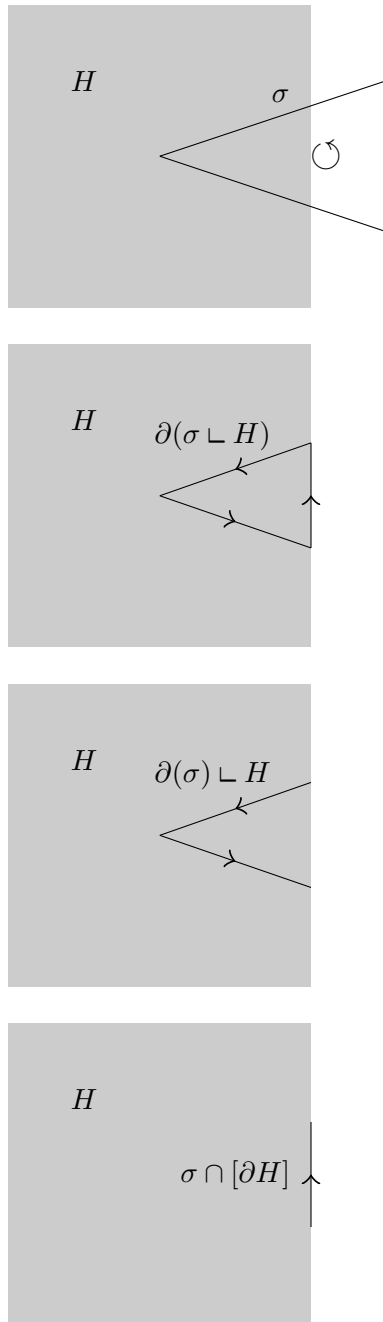
FIGURE 12. The intersection of a polygonal 2-chain with a hyperplane. In this case, things behave like we'd expect.
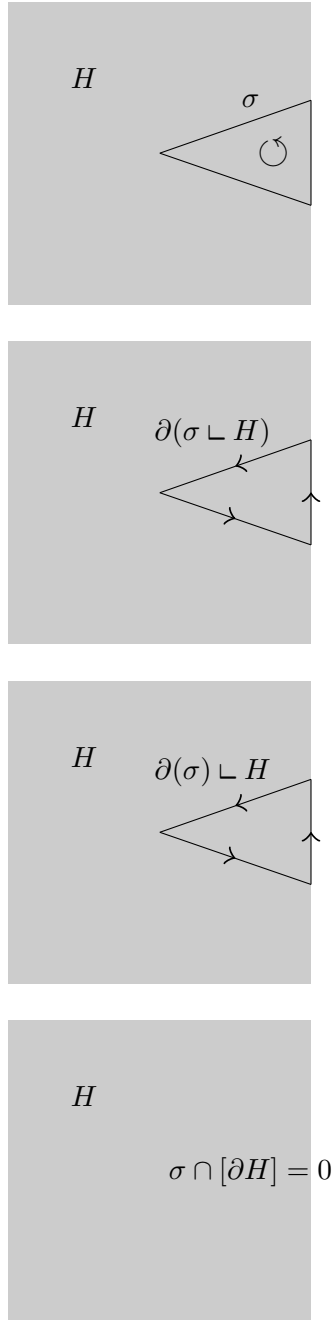
FIGURE 13. The intersection of a polygonal 2-chain with a hyperplane. Here, we get that the intersection is empty, contrary to intuition.

This is easily checked on individual polygons, because if they are perpendicular to ∂H_t, then there is equality, and in general if they are diagonal, the right inequality clearly holds.

This theorem fails if we were to replace mass with the flat norm as shown Figure 15, because if H is any half plane in between the points, we have that $\mathscr{F}(A) \leq n\epsilon$, but $\mathscr{F}(A \llcorner H) = n$ (in particular, from this example we see that $A \mapsto A \llcorner H$ is not continuous in the flat norm topology).

However, we do have

$$\partial(\sigma \llcorner H) = \partial(0) = 0$$

$$\partial(\sigma) \llcorner H$$

$$\sigma \cap [\partial H]$$
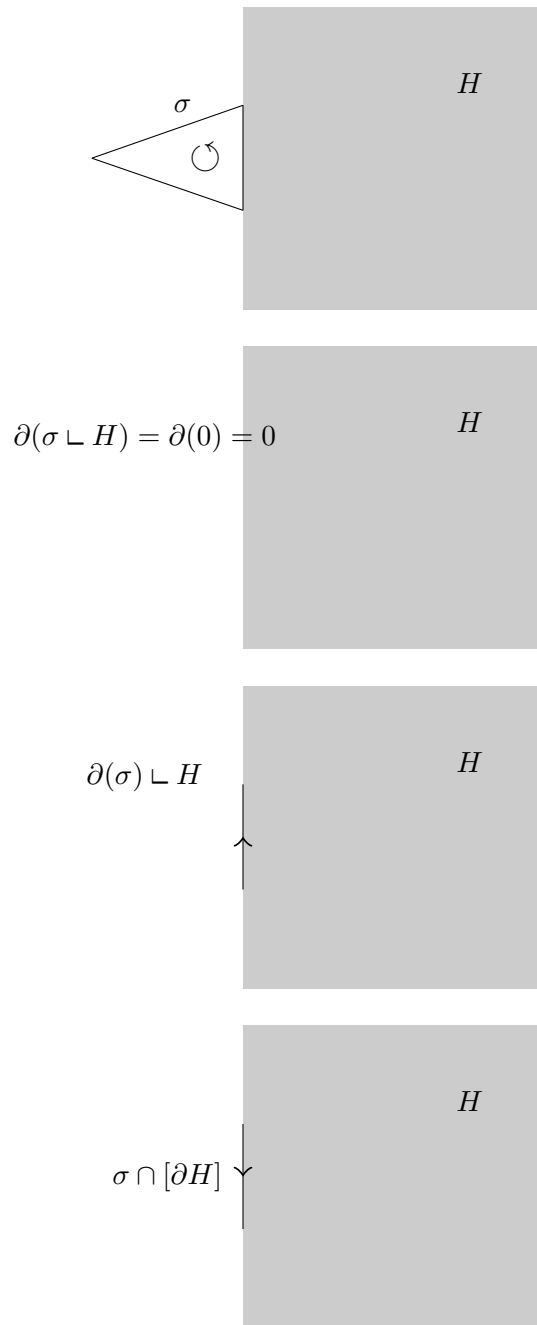
FIGURE 14. The intersection of a polygonal 2-chain with a hyperplane. Here, we get that the intersection is non-empty, unlike Figure 14. This again, runs contrary to the usual notion of intersection.

Theorem 2.24. *With the same hypothesis as Theorem 2.23, we do have that*

$$\int_a^{a+L} \mathscr{F}(A \llcorner H_t)dt \le (1+L)\mathscr{F}(A)$$

Proof. Writing $A = B + \partial C$, notice that

$$A \llcorner H_t = B \llcorner H_t + (\partial C) \llcorner H_t$$

FIGURE 15. Theorem 2.23 is false if we replace mass with the flat norm.

$$= B \llcorner H_t + \partial(C \llcorner H_t) \underbrace{-\partial(C \llcorner H_t) + \partial C \llcorner H_t}_{=-C \cap [\partial H_t]}$$

$$= B \llcorner H_t + \partial(C \llcorner H_t) - C \cap [\partial H_t]$$

Thus, we see that

$$\mathscr{F}(A \llcorner H_t) \leq \mathbb{M}(B) + \mathbb{M}(C) + \mathbb{M}(C \cap [\partial H_t])$$

and integrating, we have that

$$\int_a^{a+L} \mathscr{F}(A \llcorner H_t) dt \leq L(\mathbb{M}(B) + \mathbb{M}(C)) + \int_a^{a+L} \mathbb{M}(C \cap [\partial H_t]) dt$$

We may bound the last integrand by the previous theorem, thus obtaining

$$\int_a^{a+L} \mathscr{F}(A \llcorner H_t) dt \leq (1 + L)(\mathbb{M}(B) + \mathbb{M}(C))$$

and taking the infimum over such B, C, we can conclude as desired. $\qquad \square$

2.9. **Mass is Lower Semicontinuous on Polygons.** In Lemma 2.9, we promised a proof of the fact that for polyhedral chains $\mathbb{M}$ is lower semicontinuous, or in other words, we have equality:

$$\mathbb{M}(A) = \tilde{\mathbb{M}}(A) := \liminf_{\epsilon \to 0}\{\mathbb{M}(P) : \mathscr{F}(A - P) < \epsilon, \ P \text{ is polyhedral}\}$$

Now that we know how to intersect polygons with half spaces, we can prove this.

Proof. Case 1. A is an m-chain totally contained in a m-dimensional plane V: In this case, we'll project everything down to V and then use the simplicity of top dimensional chains. Let $\Pi^V : \mathbb{R}^N \to V$ be the projection onto V. It is trivial to see that it induces a chain homomorphism on polyhedra, which we denote $\Pi_\#^V$, and that $\mathbb{M}(\Pi_\#^V P) \leq \mathbb{M}(P)$. In particular, this implies that $\mathscr{F}(\Pi_\#^V P) \leq \mathscr{F}(\Pi_\#^V P)$.

Now, take a sequence of polygonal chains $A_i \xrightarrow{\mathscr{F}} A$. Thus, we may choose B_i, C_i polygons such that

$$A - A_i = B_i + \partial C_i \qquad \text{and} \qquad \mathbb{M}(B_i) + \mathbb{M}(C_i) \to 0.$$

Projecting to V gives

$$A - \Pi_\#^V A_i = \Pi_\#^V B_i + \Pi_\#^V \partial C_i.$$

Notice that $\Pi_\#^V \partial C_i = \partial \Pi_\#^V C_i = 0$, because $\Pi_\#^V C_i$ is a $m+1$ dimensional chain in a m-dimensional space, and is thus zero. As such,

$$\mathbb{M}(A - \Pi_\#^V A_i) = \mathbb{M}(\Pi_\#^V B_i) \leq \mathbb{M}(B_i) \to 0.$$

This shows that
$$\mathbb{M}(A) = \lim_{i \to \infty} \mathbb{M}(\Pi_{\#}^V A_i) \leq \liminf_{i \to \infty} \mathbb{M}(A_i),$$
and taking the infimum over all such sequences A_i gives
$$\mathbb{M}(A) \leq \tilde{\mathbb{M}}(A).$$

In general, we have

Case 2. The general case: We'll need

Claim 2.25. *For $f : \mathbb{R}^N \to \mathbb{R}$, letting $H_t = \{f \leq t\}$,*
$$\tilde{\mathbb{M}}(A \llcorner H_t) + \tilde{\mathbb{M}}(A \llcorner H_t^c) = \tilde{\mathbb{M}}(A)$$

where $A \llcorner H_t^c = A - A \llcorner H_t$.

Proof of Claim. That
$$\tilde{\mathbb{M}}(A \llcorner H_t) + \tilde{\mathbb{M}}(A \llcorner H_t^c) \geq \tilde{\mathbb{M}}(A)$$
is trivial by the triangle inequality, so we only need prove the other implication. To this end, let $A_i \xrightarrow{\mathscr{F}} A$ be a sequence of polygonal chains converging to A with $\mathbb{M}(A_i) \to \tilde{\mathbb{M}}(A_i)$. By extracting a subsequence, we may assume that
$$\sum \mathscr{F}(A_i - A) < \infty$$
(this is a useful trick, and in this case we'll say that A_i converges to A "quickly"). Thus, by Theorem 2.24
$$\int_a^{a+1} \mathscr{F}((A_i - A) \llcorner H_t) dt \leq 2\mathscr{F}(A_i - A).$$
By monotone convergence, we have that
$$\int_a^{a+1} \sum_{i=1}^\infty \mathscr{F}((A_i - A) \llcorner H_t) dt \leq 2 \sum_{i=1}^\infty \mathscr{F}(A_i - A) < \infty.$$

This shows that $\mathscr{F}((A_i - A) \llcorner H_t) \to 0$ for a.e. t, so $A_i \llcorner H_t \xrightarrow{\mathscr{F}} A \llcorner H_t$ for a.e. t. This shows that
$$\tilde{\mathbb{M}}(A \llcorner H_t) \leq \liminf \mathbb{M}(A_i \llcorner H_t)$$
and similarly,
$$\tilde{\mathbb{M}}(A \llcorner H_t^c) \leq \liminf \mathbb{M}(A_i \llcorner H_t^c).$$
Thus, we have that
$$\tilde{\mathbb{M}}(A \llcorner H_t) + \tilde{\mathbb{M}}(A \llcorner H_t^c) \leq \liminf[\mathbb{M}(A_i \llcorner H_t) + \mathbb{M}(A_i \llcorner H_t^c)] = \liminf \mathbb{M}(A_i) = \tilde{\mathbb{M}}(A)$$
for a.e. t. We claim that this holds for all t, which follows if we can show that both terms on the left hand side are right continuous in t, which follows from
$$\mathbb{M}(A \llcorner H_{t+\delta} - A \llcorner H_t) \to 0$$
as $\delta \searrow 0$. Thus, $A \llcorner H_{t+\delta} \xrightarrow{\mathscr{F}} A \llcorner H_t$ and
$$\tilde{\mathbb{M}}(A \llcorner H_t) \leq \liminf_{\delta \searrow 0} \mathbb{M}(A \llcorner H_{t+\delta})$$
finishing the claim. $\qquad \square$

This claim basically finishes the proof. We can induct down on the number of hyperplanes cutting out each polygon by using
$$\tilde{\mathbb{M}}(A \llcorner V) + \tilde{\mathbb{M}}(A \llcorner V^c) = \tilde{\mathbb{M}}(A)$$
and the fact that each polygon is the intersection of a finite number of half-planes. $\qquad \square$

3. THE DEFORMATION THEOREM

This section contains a discussion of what is known as the "deformation theorem" for flat chains. However, a similar theorem holds in various other contexts, and many of the ideas in this section extend to other settings as well.

We declare that in this section, everything will be polyhedral unless stated otherwise.

Definition 3.1. *If $\partial A = 0$, we define*

$$\mathscr{F}^*(A) = \inf\{\mathbb{M}(Q) : \partial Q = A\},$$

where both A and Q are polyhedral, as mentioned above.

This is closely related to the flat norm as shown by the next theorem:

Theorem 3.2. *If A is supported in a set K and $\partial A = 0$ then*

$$\mathscr{F}(A) \leq \mathscr{F}^*(A) \leq c_K \mathscr{F}(A).$$

Proof. The fist inequality follows trivially because we are taking the infimum over a smaller set on the right hand side. To prove the second inequality, write $A = B + \partial C$. Because $\partial A = 0$, we have that $\partial B = 0$, as well. We have shown that this implies that $B = \partial(\mathrm{cone}(B))$, so we see that

$$\mathscr{F}^*(A) \leq \mathbb{M}(\mathrm{cone}(B)) + \mathbb{M}(C) \leq (\mathrm{diam}\, K)\mathbb{M}(B) + \mathbb{M}(C),$$

so letting $c_K = 1 + \mathrm{diam}\, K$ and taking the infimum over all such B, C the second inequality is proven. $\square$

We will also make use of the following:

Lemma 3.3 (Simultaneous Approximation Lemma)**.** *For A a flat chain (not necessarily polyhedral, in fact the theorem will only be interesting in the case that A is not polyhedral) with*

$$\mathbb{M}(A), \mathbb{M}(\partial A) < \infty,$$

then there are polyhedral A_i such that $A_i \xrightarrow{\mathscr{F}} A$ and such that

$$\mathbb{M}(A_i) \to \mathbb{M}(A) \qquad and \qquad \mathbb{M}(\partial A_i) \to \mathbb{M}(\partial A)$$

Finally, if $\partial A = 0$, we can choose the A_i to be cycles as well, i.e. $\partial A_i = 0$.

Proof. From the definition of the mass of a flat chain, there are X_i, Y_i polyhedral such that $X_i \to A$ and $Y_i \to \partial A$ and

$$\mathbb{M}(X_i) \to \mathbb{M}(A) \qquad and \qquad \mathbb{M}(Y_i) \to \mathbb{M}(\partial A).$$

By induction on the dimension of A (the base case is trivial), we may choose the Y_i's to be cycles. Thus,

$$\underbrace{\partial X_i - Y_i}_{\text{cycle}} \xrightarrow{\mathscr{F}} \partial A - \partial A = 0.$$

This implies that $\mathscr{F}(\partial X_i - Y_i) \to 0$, and by the previous theorem $\mathscr{F}^*(\partial X_i - Y_i) \to 0$. By definition, $\partial X_i - Y_i = \partial Q_i$ for some polyhedral Q_i with $\mathbb{M}(Q_i) \to 0$. Thus, we have that $\partial(X_i - Q_i) = Y_i$, and in particular $A_i := X_i - Q_i \xrightarrow{\mathscr{F}} A$.

We check that the A_i satisfy the desired properties. Notice that

$$\mathbb{M}(A_i) \leq \mathbb{M}(X_i) + \mathbb{M}(Q_i) \to \mathbb{M}(A)$$

because $\mathbb{M}(Q_i) \to 0$ and $\mathbb{M}(X_i) \to \mathbb{M}(A)$, by our choices above. Also,

$$\mathbb{M}(\partial A_i) = \mathbb{M}(Y_i) \to \mathbb{M}(A).$$

Finally, if $\partial A = 0$, we could have taken $Y_i = 0$ in the above, so we've also proven this case. $\square$

3.1. **Intuitive Description.** Here we'll give an intuitive picture of the deformation theorem, which we will make precise in the next section. The general idea is to partition $\mathbb{R}^N$ into cubes of size $\epsilon > 0$, i.e. cubes with vertices in $\epsilon\mathbb{Z}^N$. Given a m-polygon (or in general a flat chain), we'll "deform" it so that it lies on the m-skeleton of the cubes. Roughly speaking, we'll preform a radial deformation from the center of each cube, pushing the chain out to the boundary. An example of this is given in Figure 16.
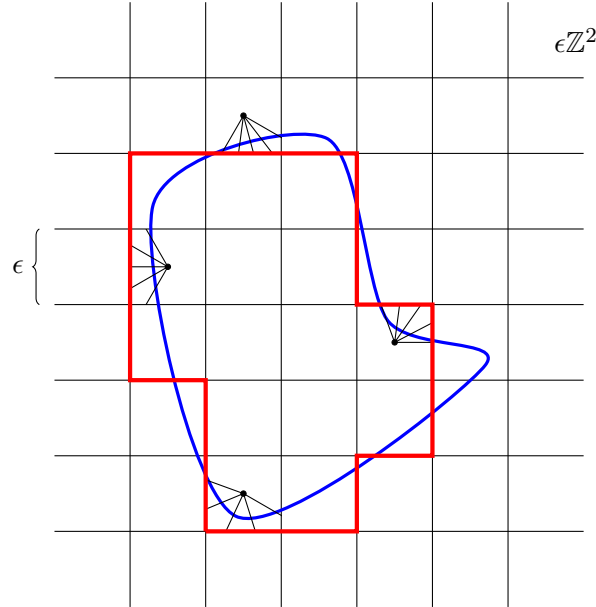
FIGURE 16. An intuitive diagram of the deformation process. The blue curve misses the middles of the squares, so we can preform a radial deformation, which is indicated by the radial lines in some of the squares, thus obtaining the thick red chain, which is contained in the 1-skeleton of the squares.

In the case that the curve has boundary, we would like the boundary to end up in the 0 skeleton (in general for a k-chain, we'd like the the deformed chain to lie in the k-skeleton and the boundary to be deformed to the $(k-1)$-skeleton). To accomplish this, we first radially deform the curve to the 1-skeleton, and then radially deform the resulting curve on the 1-skeleton on the 1-faces of the cube. This is illustrated in Figure 17

We've been a bit dishonest here, because it is certainly possible that the initial chain intersects the centers of the cubes, or in one of the preceding deformation on the lower dimensional faces, the curve might intersect the centers of the face. Then, the radial deformation at this point would not be well defined.

What we will do to solve this, is instead of using the exact center of the cubes, we let $X(\epsilon) = \epsilon\mathbb{Z}^N$ be the original cubes, and define, for $v \in \mathbb{R}^N$ with $0 < v_i < \epsilon$, $Y_v(\epsilon) := X(\epsilon) + v$. The case we have described before is with $v = (\epsilon/2, \epsilon/2, \ldots, \epsilon/2)$, but now we are allowing general such v. We will say that a polygon in $\mathbb{R}^N$ which does not intersect the $N - k - 1$ skeleton of $Y_v(\epsilon)$ is in "general position" (with respect to $Y_\epsilon(v)$). It is easy to see that this agrees with what we have said above, and in this case the successive radial deformations are well defined.

More formally, given $v \in \epsilon(0, 1)^N$, we will define two maps

$$P_v^\epsilon : \left\{ \begin{array}{c} \text{polygonal } k\text{-chains in general position} \\ \text{with respect to } Y_v(\epsilon) \end{array} \right\} \to \left\{ \begin{array}{c} \text{polygonal } k\text{-chains which are sums of} \\ k\text{-cubes of } X(\epsilon) \end{array} \right\}$$
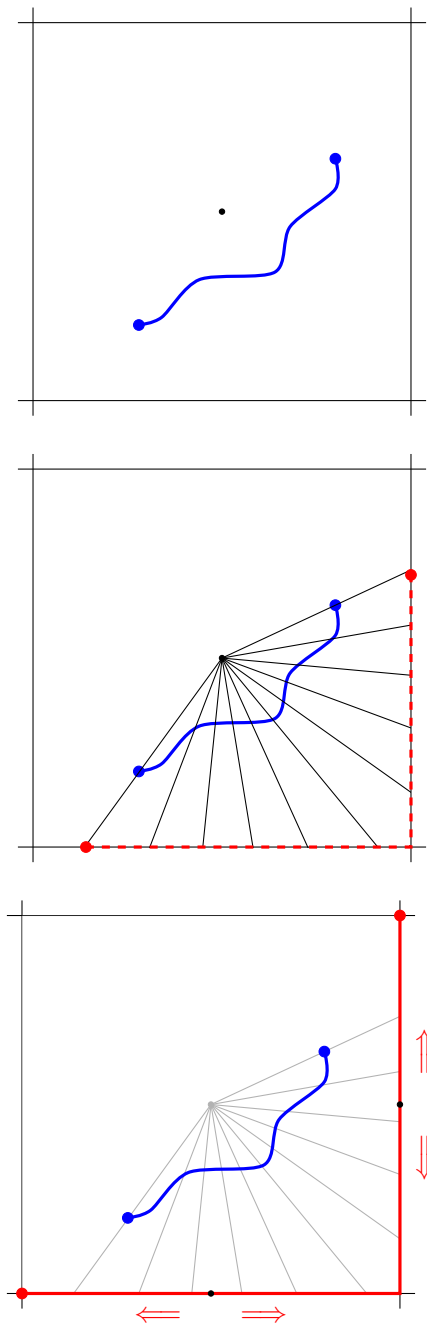
FIGURE 17. Deforming a curve (in blue) with boundary to the cube. In the first step, the curve, along with its boundary, is radially deformed to the 1-skeleton of the cube (drawn as a dashed red line). Then, in the second step, a radial deformation is preformed on each boundary, resulting in the red line.

$$H_v^\epsilon : \left\{ \begin{array}{c} \text{polygonal } k\text{-chains in general position} \\ \text{with respect to } Y_v(\epsilon) \end{array} \right\} \to \{\text{polygonal } k+1\text{-chains}\}$$

Here, P_v^ϵ is the deformation map described above and H_v^ϵ is the "region swept out by the deformation," as in Figure 18. An example in $\mathbb{R}^3$ is given in Figure 19, where the fact that $H_v^\epsilon(A)$ is formed from the regions swept out by each radial deformation is illustrated.
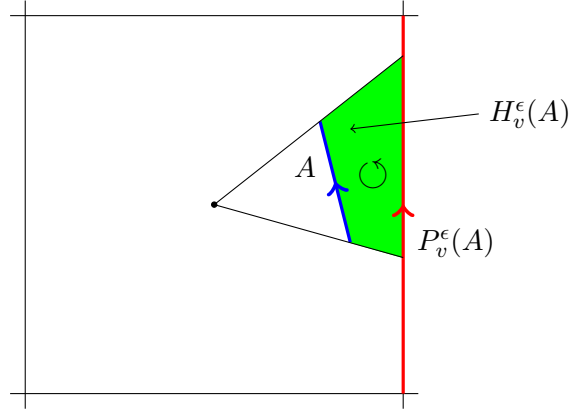
FIGURE 18.   The general picture of $H_v^\epsilon(A)$ and $P_v^\epsilon(A)$ for A a 1-polygon with boundary. We will return to this case in Figure 20.
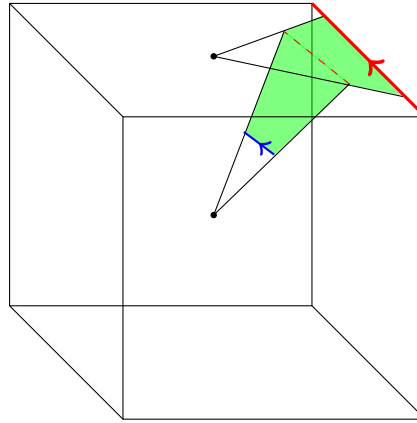
FIGURE 19. In the case where A has codimension bigger than 1, $H_v^\epsilon(A)$ is formed of "all of the regions swept out by the successive radial deformations," as shown here. As in Figure 18, A is the blue line, $P_v^\epsilon(A)$ is the red line, and $H_v^\epsilon(A)$ is the green region.

We claim that for A in general position with respect to $Y_v(\epsilon)$,

$$(3.1) \qquad P_v^\epsilon(A) - A = \partial(H_v^\epsilon(A)) + H_v^\epsilon(\partial A).$$

We give a simple picture proof of this in Figure 20 in the case of the a 1-polygon, as in Figure 18.

However, even in the case that A is in general position with respect to $Y_v(\epsilon)$, the processes P_v^ϵ and H_v^ϵ are very discontinuous. For example, a small square around the center will be made into almost all of the region by H_v^ϵ. However, we claim that by averaging over the possible v, we will solve this problem (and in fact solve the other problem of having to assume A is in general position, as the v with A not in general position will be a set of measure zero, and thus can be neglected in the integral defining the average). In this case, we have that

Claim 3.4. *For A a k-chain, writing a bar over a quantity to denote the average value taken over $v \in \epsilon(0,1)^N$, we have that*

$$\overline{\mathbb{M}(P_v^\epsilon(A))} \le c\mathbb{M}(A)$$

and

$$\overline{\mathbb{M}(H_v^\epsilon(A))} \le \epsilon c\mathbb{M}(A)$$

where c_N is a constant depending only on the dimension.
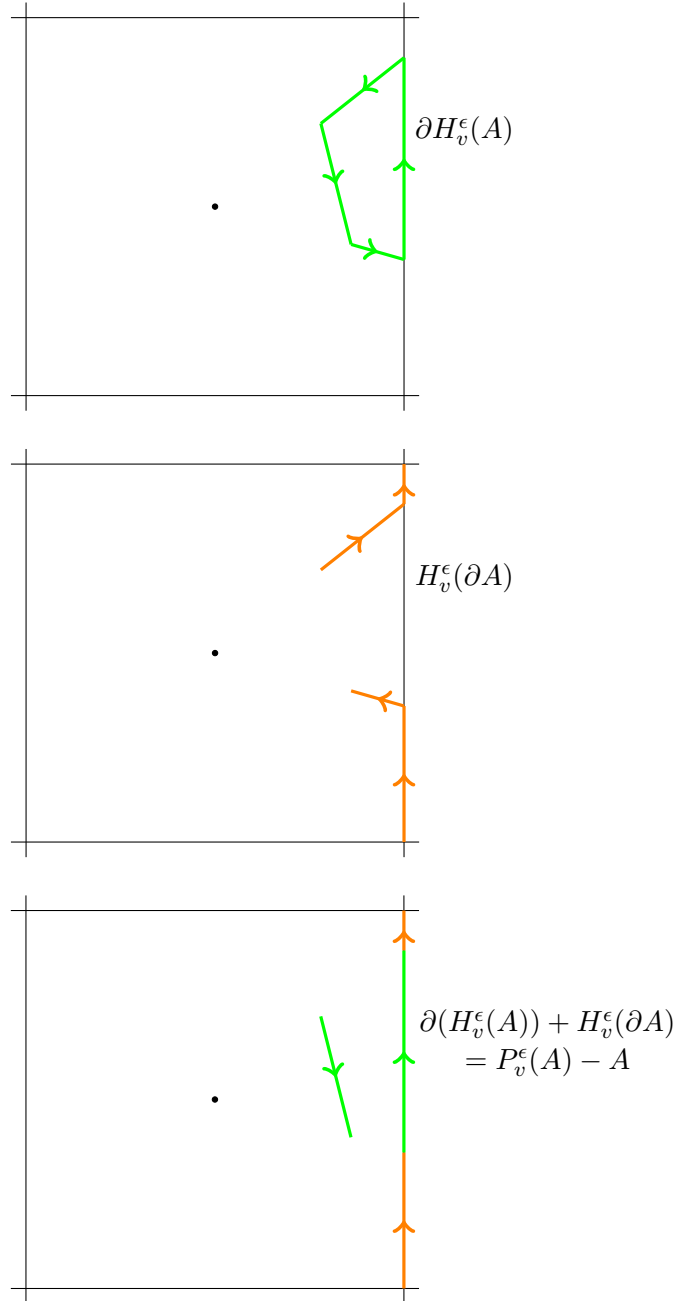
FIGURE 20. A picture proof of the fact that H is a chain homotopy between P and the identity, as in (3.1). The original chain A is the same as in Figure 18.

3.2. The Deformation Theorem for Polyhedral Chains.

As described in the previous section, we have

Theorem 3.5 (Deformation Theorem). *Fix an $\epsilon > 0$. Then for $v \in (0, \epsilon)^N$, we have maps*

$$H = H_v^\epsilon : \mathscr{P}_m^{gen(v)}(\mathbb{R}^N; G) \to \mathscr{P}_{m+1}(\mathbb{R}^N; G)$$

$$P = P_v^\epsilon : \mathscr{P}_m^{gen(v)}(\mathbb{R}^N; G) \to \mathscr{P}_m(\mathbb{R}^N; G)$$

as described before. Here the $\mathrm{gen}(v)$ *superscript denotes that the maps are only defined for polyhedral* m*-chains in general position with respect to* v*. Furthermore, the maps have the following properties, for* A, B m*-chains in general position with respect to* v*:*

(1) $H(A + B) = H(A) + H(B)$.
(2) $P(A + B) = P(A) + P(B)$.
(3) $P(A) - A = \partial H(A) + H(\partial A)$.
(4) $P(\partial A) = \partial P(A)$.
(5) $\mathrm{spt}(H(A)), \mathrm{spt}(P(A)) \subset \sqrt{N}\epsilon$*-neighborhood of* $\mathrm{spt}(A)$.
(6) $P(A) = \sum g_i[\sigma_i]$ *where the* σ_i *are* m*-cubes in* $X(\epsilon)$.
(7) $\overline{\mathbb{M}(P(A))} \leq c\mathbb{M}(A)$.
(8) $\overline{\mathbb{M}(H(A))} \leq c\epsilon\mathbb{M}(A)$.
(9) $\overline{\mathscr{F}(P(A))} \leq c\mathscr{F}(A)$
(10) $\overline{\mathscr{F}(H(A))} \leq c(1 + \epsilon)\mathscr{F}(A)$
(11) $\overline{\mathscr{F}(P(A) - A)} \leq c\epsilon(\mathbb{M}(A) + \mathbb{M}(\partial A))$

Proof. We have already defined H, and we then take (3) as the definition of P. It is not hard to check that for a fixed v

(3.2) $$\mathbb{M}(P(A)) = \epsilon^m \cdot |\#\{A \cap Y_\epsilon(v)^{N-m}\}|$$

where we take into consideration weights and orientations when counting the number of points in the intersection. Using this, it is possible to prove (7). To show (8), we reference Figure 21, and claim that for small A (it is enough to do it for such A because we can always divide A into small pieces and prove it for each such piece), we have that the volume of $H(A)$ in the figure is approximately

$$\frac{1}{m + 1} \frac{\epsilon^{m+1}}{\mathrm{dist}(x, Y^{N-m-1})^m}.$$

To see this, first notice that in Figure 21, the small white triangle formed by the center of the square and A has area

$$\frac{1}{m + 1}\mathbb{M}(A)\,\mathrm{dist}(x, Y^{N-m-1}),$$

so that the area of $H(A)$ bounded above by this, rescaled by $\epsilon/\mathrm{dist}(x, Y^{N-m-1})$ to the appropriate power (in this case m as we are considering m-area).

FIGURE 21. A diagram for the setup for the proof of (8).

More precisely, this shows that for small A we have that

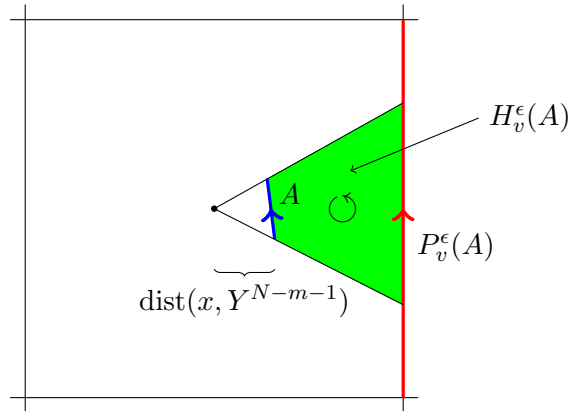$$\mathbb{M}(H(A)) \leq \int_{\mathrm{spt}(A)} \left(\frac{\epsilon}{\mathrm{dist}(x, Y_v(\epsilon)^{N-m-1})} \right)^m \epsilon |g(x)| d\mathcal{H}^m(x)$$

where $g(x)$ is the density of A, i.e. a G valued function consisting of the sum of characteristic functions of the components of A multiplied by their weights. Integrating over v (and using Fubini's theorem to interchange the x and v integrals) thus gives

$$\int_{(0,\epsilon)^N} \mathbb{M}(H(A)) d\mathcal{L}^N(v) \leq \int_{\mathrm{spt}(A)} \epsilon^{m+1} |g(x)| \underbrace{\int_{(0,\epsilon)^N} \left(\frac{1}{\mathrm{dist}(x, Y_v(\epsilon)^{N-m-1})} \right)^m d\mathcal{L}^N(v)}_{\leq c\epsilon^{N-m}} d\mathcal{H}^m(x).$$

To prove the asserted inequality, first notice that if we had originally taken $\epsilon = 1$,

$$\int_{(0,1)^N} \left(\frac{1}{\mathrm{dist}(x, Y_v(1)^{N-m-1})} \right)^m d\mathcal{L}^N(v) \leq c < \infty$$

for some constant c independent of x. This is because the distance term is roughly linear in $|v - x|$, and we may assume $m < N$ (because H of a N chain is a $N + 1$ chain which is empty), so the integrand is integrable, and it is not hard to see that it can be bounded independently of x. How, we must determine what happens when we rescale everything down to the ϵ scale. We get a factor of ϵ^N from rescaling the measure, and the dist term scales as ϵ, as argued before. Thus, we have an overall scale of ϵ^{N-m}, as asserted above. This combines to show (8), as desired.

Now, the rest of the properties follow similarly to things we have done before. For example, we may follow the proof of Proposition 2.14, to show (10), for example (the proposition itself, does not include the averaging we are considering here, but the method of proof is basically identical) but one must be a bit careful of the factors of ϵ, which is why there is a $1 + \epsilon$ term present. $\qquad \square$

Now that we have shown the deformation theorem for polyhedral chains, we claim that it holds in some sense for flat chains.

3.3. The Deformation Theorem for Flat Chains. We will show that we have "maps"

$$H_v^\epsilon : \mathscr{F}_m(\mathbb{R}^N; G) \to \mathscr{F}_{m+1}(\mathbb{R}^N; G)$$
$$P_v^\epsilon : \mathscr{F}_m(\mathbb{R}^N; G) \to \mathscr{P}_m(\mathbb{R}^N; G)$$

in the sense that for any flat chain $A \in \mathscr{F}_m$, the two maps are defined for a.e. $v \in (0, \epsilon)^N$. In particular, we will easily recover the properties of the deformation theorem for polyhedral chains, Theorem 3.5, once we show that the above maps (and really just H, as P is defined in terms of H) are well defined.

Theorem 3.6. *We have an extension of H_v^ϵ on polygons in polyhedral chains to a "map" in the sense described above.*

Proof. Choose polyhedral chains $A_i \in \mathscr{P}_m$ such that $A_i \xrightarrow{\mathscr{F}} A$ quickly (i.e. $\sum \mathscr{F}(A_i - A) < \infty$) and $\mathbb{M}(A_i) \to \mathbb{M}(A)$. This implies that $\sum \mathscr{F}(A_i - A_{i+1}) < \infty$.

Now, writing a line over quantities which are averaged over v, we have that

$$\overline{\mathscr{F}(H_v(A_i) - H_v(A_{i+1}))} = \overline{\mathscr{F}(H(A_i - A_{i+1}))} \leq c(1 + \epsilon) \mathscr{F}(A_i - A_{i+1}).$$

Summing over i, and using the quick convergence of the A_i gives

$$\sum \overline{\mathscr{F}(H_v(A_i) - H_v(A_{i+1}))} < \infty.$$

We may interchange the sum and average (i.e. integral) because each term is positive, thus obtaining

$$\overline{\sum \mathscr{F}(H_v(A_i) - H_v(A_{i+1}))} < \infty.$$

This implies that for a.e. v,
$$\sum \mathscr{F}(H_v(A_i) - H_v(A_{i+1})) < \infty,$$
so for such a v, $H_v(A_i)$ is Cauchy, so we may define $H_v(A) = \lim_{i \to \infty} H_v(A_i)$. To see that this is independent of the sequence A_i chosen for a.e. v, for any other such sequence A_i' we can let C_i be the sequence $A_1, A_1', A_2, A_2', \ldots$ and applying the above argument to C_i, we see that the definition of H_v is independent of A_i for a.e. v. $\qquad\square$

3.4. **Isoperimetric Inequality.**
Federer and Fleming used the deformation theorem to give a very short proof of the isoperimetric inequality, as we now describe:

Theorem 3.7 (Isoperimetric Inequality). *Suppose that $A \in \mathscr{F}_m(\mathbb{R}^n; G)$ has $\partial A = 0$, and furthermore that G has the property that for $g \in G - \{0\}$, $|g| \geq 1$. Then, there is a flat chain B with $\partial B = A$ and*
$$\mathbb{M}(B) \leq c\mathbb{M}(A)^{\frac{m+1}{m}}$$

Note that one may recover the usual isoperimetric inequality for A a hypersurface in $\mathbb{R}^n$ from this more general version, because in this case there is a unique B with $\partial B = A$.

Proof. We know that
$$\overline{\mathbb{M}(H_\epsilon^v(A))} \leq c\epsilon\mathbb{M}(A)$$
$$\overline{\mathbb{M}(P_\epsilon^v(A))} \leq c\mathbb{M}(A)$$
so adding these gives
$$\frac{\overline{\mathbb{M}(H_\epsilon^v(A))}}{\epsilon} + \mathbb{M}(P_\epsilon^v(A)) \leq c\mathbb{M}(A)$$
(with a different constant c). We choose $\epsilon = (2c\mathbb{M}(A))^{1/m}$, for reasons that will be clear in a moment. Now, because it is true on average, we must have a v so that
$$(3.3) \qquad \frac{\mathbb{M}(H_\epsilon^v(A))}{\epsilon} + \mathbb{M}(P_\epsilon^v(A)) \leq c\mathbb{M}(A).$$
However, notice that if $P(A)$ is nonzero, then $P(A)$ must contain at least one entire m-face of $X(\epsilon)$, so $\mathbb{M}(P(A)) \geq |g|\epsilon^m$ where g is some nonzero element which would be the coefficient of this face. However, the choice of ϵ along with the bound $|g| \geq 1$ shows that this cannot happen, so $P(A) = 0$. This, combined with $\partial A = 0$, implies that
$$A = \partial(\underbrace{-H(A)}_{:=B}).$$
We thus have that by (3.3),
$$\mathbb{M}(B) = \mathbb{M}(H(A)) \leq c\epsilon\mathbb{M}(A) = c(2c\mathbb{M}(A))^{1/m}\mathbb{M}(A).$$
$$\qquad\square$$

A few remarks about this are in order. First of all, this argument does not give the optimal constant, which was established by Almgren by other GMT methods. We do remark that in the version we have stated the theorem, the assumptions on G are necessary, because if for an arbitrary group G, we have the isoperimetric inequality, then for $A = g[S_R]$, the circle of radius R weighted by $g \in G$, there is a unique B with $\partial B = A$, namely $B = g[B_R]$. Thus, if there was some c such that the isoperimetric inequality held, then
$$\mathbb{M}(A) = \pi R^2|g| \leq c\mathbb{M}(B)^2 = c(2\pi R|g|)^2,$$
which implies that
$$|g| \geq \frac{1}{4\pi c}.$$

In spite of this example, Almgren was able to extend the isoperimetric inequality to general groups in the following form

$$\mathbb{M}(B) \leq c(\operatorname{size} A)^{1/m}\mathbb{M}(A)$$

where $\operatorname{size} A$ is the mass of A without the multiplicities, i.e. in the norm which takes 1 on all elements besides the identity. We will prove this below.

Finally, we remark that the isoperimetric inequality shows that for area minimizing B (among other chains with the same boundary),

$$\mathbb{M}(B) \leq c\mathbb{M}(\partial B)^{\frac{m+1}{m}},$$

and as remarked above, Almgren was able to establish the optimal constant c. The same isoperimetric inequality can be shown if we only assume that B is a minimal surface (in the sense of stationary varifolds), but it is not known what the optimal constant is in this case.

We now prove Almgren's isoperimetric inequality for general coefficient groups. To do so, we define the size of a polyhedral chain $\sum g_i[\sigma_i]$ to be

$$\operatorname{size}\left(\sum g_i[\sigma_i]\right) := \sum \operatorname{area}(\sigma_i)$$

where we assume that the σ_i are pairwise disjoint. For a general flat chain, we define the size by

$$\operatorname{size}(A) = \inf\left\{\liminf_{i\to\infty} \operatorname{size}(A_i) : A_i \in \mathscr{P}_m, A_i \to A\right\}.$$

Theorem 3.8 (Almgren's General Isoperimetric Inequality)**.** *If A is an m chain with $\partial A = 0$, then there is a $m+1$ chain B with $\partial B = A$ and*

$$(3.4) \qquad\qquad \mathbb{M}(B) \leq c(\operatorname{size} A)^{1/m}\mathbb{M}(A)$$

Proof. By repeating the arguments for mass in the proof of the deformation theorem, it is not hard to show that

$$(3.5) \qquad\qquad \overline{\operatorname{size}(P_v^\epsilon(A))} \leq c\operatorname{size}(A).$$

We already know that

$$(3.6) \qquad\qquad \overline{\mathbb{M}(H_v^\epsilon(A))} \leq c\epsilon\mathbb{N}(A),$$

where $\mathbb{N}(A) = \mathbb{M}(A) + \mathbb{M}(\partial A)$. Note that $\mathbb{N}(A) = \mathbb{M}(A)$ because $\partial A = 0$.

Let $\epsilon^m = 10c\operatorname{size}(A)$ and choose v such that

$$\operatorname{size}(P(A)) \leq 2c\operatorname{size}(A)$$

and

$$\mathbb{M}(H(A)) \leq 2c\epsilon\mathbb{M}(A),$$

which we may do so, by the above. If $P(A) \neq 0$, then

$$\operatorname{size}(P(A)) \geq \epsilon^m = 10c\operatorname{size}(A),$$

contradicting our choice of v. Thus, $A = \partial H(-B)$, and

$$\mathbb{M}(H(B)) \leq 2c\epsilon\mathbb{M}(A) = \tilde{c}\operatorname{size}(A)^{1/m}\mathbb{M}(A). \qquad\qquad \square$$

3.5. **Compactness Theorem.** The compactness theorem is also a consequence of the deformation theorem. Recall that the compactness theorem is

Theorem 3.9. *For K a compact subset of $\mathbb{R}^N$, if $\lambda < \infty$ then*

$$\{A \in \mathscr{F}_m(K; G) : \mathbb{N}(A) \leq \lambda\}$$

is compact if (and obviously only if) balls are compact in G, i.e.

$$\{g \in G : |g| \leq R\}$$

is compact for all $R > 0$.

Proof. Recall the following elementary fact about metric spaces: If Z is a complete metric space, and there is a subset $S \subset Z$ such that for all $\epsilon > 0$, there is a compact set $K_\epsilon \subset Z$ such that S is in the ϵ-neighborhood of K_ϵ, i.e.

$$S \subset \{z \in Z : \mathrm{dist}(z, K_\epsilon) < \epsilon\},$$

then the closure $\overline{S}$ is compact.

We have that for a given $\epsilon > 0$, the properties of the deformation theorem give (the third line follows from the first two)

$$\overline{\mathbb{M}(P(A))} \leq c\mathbb{M}(A),$$
$$\overline{\mathbb{M}(P(\partial A))} \leq c\mathbb{M}(\partial A),$$
$$\overline{\mathbb{N}(P(A))} \leq c\mathbb{N}(A),$$
$$\overline{\mathscr{F}(A - P(A))} \leq c\epsilon\mathbb{N}(A).$$

Thus, for each A with $\mathbb{N}(A) \leq \lambda$, there is $P = P(A)$ supported in $X(\epsilon)$ such that

$$\mathbb{N}(P) \leq c\mathbb{N}(A) \leq c\lambda,$$

and

$$\mathscr{F}(A - P) \leq c\epsilon\lambda.$$

However, notice that such P's are G-linear combinations of m-faces of $X(\epsilon)$ contained in a fixed compact set K (actually they will extend outside of K a little bit, but this can be fixed by making K a bit larger), and by the assumption that $\mathbb{N}(P) \leq c\lambda$, each coefficient of each face is bounded by $c\lambda$, and because there are a finite number of possible faces, this is a compact set of flat chains. Thus, using our characterization of compactness in the beginning of the proof, we have shown the desired conclusion. $\qquad\square$

3.6. **Further Consequences of the Deformation Theorem.** One might wonder how good of an approximation $P(A)$ is to A. We know that on average, it is a good approximation, i.e.

$$\overline{\mathscr{F}(A - P(A))} \leq c\epsilon\mathbb{N}(A),$$

but what about a pointwise bound? We prove the following theorem:

Theorem 3.10. *Given an m-chain A, there exists $\epsilon_i \to 0$ such that $P_{\epsilon_i}(A) \to A$ for a.e. v.*

Proof. Choose A_i polyhedral converging to A quickly (i.e. $\sum \mathscr{F}(A - A_i) < \infty$). We have that

$$\overline{\mathscr{F}(A - P(A))} \leq \mathscr{F}(A - A_i) + \overline{\mathscr{F}(A_i - P(A_i))} + \overline{\mathscr{F}(P(A_i) - P(A))}$$
$$\leq \mathscr{F}(A - A_i) + c\epsilon\mathbb{N}(A_i) + c\epsilon\mathscr{F}(A_i - A).$$

Choosing $\epsilon_i > 0$ so that $c\epsilon_i\mathbb{N}(A_i) \leq \mathscr{F}(A - A_i)$, we have that

$$\overline{\mathscr{F}(A - P_{\epsilon_i}(A))} \leq \tilde{c}\mathscr{F}(A - A_i).$$

Summing over i and switching the integral (average) with the sum, by $A_i \to A$ quickly, we have that

$$\overline{\sum \mathscr{F}(A - P_{\epsilon_i}(A))} = \sum \overline{\mathscr{F}(A - P_{\epsilon_i}(A))} < \infty,$$

which thus shows that $\mathscr{F}(A - P_{\epsilon_i}(A)) \to 0$ for a.e. v. $\square$

4. Support of a Flat Chain

It will be useful to have a notion of support of a flat chain. We will give two equivalent definitions. For the first one, we define seminorms $\mathbb{M}_W$ for $W \subset \mathbb{R}^N$ open: for A polyhedral, we define

$$(4.1) \qquad \mathbb{M}_W(A) = \sum |g_i| \operatorname{area}(\sigma_i \cap W),$$

where $A = \sum g_i[\sigma_i]$ is a representation of A by disjoint polygons. As with regular mass, it is possible to show that $\mathbb{M}_W$ is lower semicontinuous (with respect to the $\mathscr{F}$ topology), and thus we may extend it to a lower semicontinuous function on flat chains by

$$(4.2) \qquad \mathbb{M}_W(A) = \inf\{\liminf_{i \to \infty} \mathbb{M}_W(A_i) : A_i \in \mathscr{P}_m, \ A_i \to A\}.$$

With these seminorms, we can give our first definition of support of a flat chain:

Definition 4.1. *For a flat chain A, we define the (complement of the) support by*

$$(4.3) \qquad (\operatorname{supp} A)^c = \{x \in \mathbb{R}^N : \text{there exists an open set } W \ni x, \text{ with } \mathbb{M}_W(A) = 0\}.$$

Alternatively, we define

Definition 4.2. *For a closed set K, we say that a flat chain A is supported in K if there is a sequence A_i of polyhedral chains converging to A and such that*

$$\bigcap_{i=1}^{\infty} \overline{\bigcup_{i>n} A_i} \subset K.$$

The following theorem will show that these two notions are compatible:

Theorem 4.3. *For a flat chain A, there is a smallest (closed) set K such that A is supported in K. Furthermore, this K is exactly equal to $\operatorname{supp} A$, as defined in Definition 4.1, and there exists a sequence of polyhedral chains A_i with $A_i \to A$, $\mathbb{M}(A_i) \to \mathbb{M}(A)$ and*

$$\bigcap_{i=1}^{\infty} \overline{\bigcup_{i>n} A_i} = K.$$

Proof. By a diagonal argument, it is easy to see that a nested sequence of such sets have a smallest element, and thus by Zorn's lemma, there is in fact a smallest element. Denote it by K. If $x \notin K$, then by examining a sequence of polygonal chains converging to A, missing x, it is not hard to see that $x \notin \operatorname{supp} A$.

Conversely, suppose that $x \notin \operatorname{supp} A$, i.e. there is an open set $W \ni x$ with $\mathbb{M}_W(A) = 0$. Take a sequence of polyhedral chains A_i converging quickly to A. By assumption there is a sequence of polyhedral chains B_i converging to A (extracting a subsequence, we assume they converge quickly as well) with $\mathbb{M}_W(B_i) \to 0$. In particular, we have that

$$\sum \mathscr{F}(A_i - B_i) < \infty.$$

Let $Q(x, r)$ denote the cube with center x and inradius r. We claim that for a.e. $r > 0$, we have that

$$\sum \mathscr{F}((A_i - B_i) \llcorner Q(x, r)) < \infty.$$

This follows from the general theory of intersecting flat chains with half spaces (as in Theorem 2.24), just applied multiple times for each coordinate axes to intersect to the cube. We may thus choose such an r with $Q(x, r) \subset W$. Furthermore, by assumption

$$\mathbb{M}(B_i \llcorner Q) \leq \mathbb{M}_W(B_i) \to 0,$$

so $B_i \llcorner Q \to 0$. This shows that $A_i \llcorner Q \to 0$ as well, so we have that

$$\lim(\underbrace{A_i - A_i \llcorner Q}_{:=A_i'}) = \lim A_i = A.$$

which shows that $K = \operatorname{supp} A$.

Finally, it remains to show that we can choose the A_i so that the masses converge as well. To show this, by the same argument, we can find K', a closed set, minimal among sets such that there is A_i with $A_i \to A$, $\mathbb{M}(A_i) \to \mathbb{M}(A)$ and

$$\bigcap_{i=1}^{\infty} \overline{\bigcup_{i>n} A_i} = K'.$$

If $K' \neq K = \operatorname{supp} A$, then there is some $x \in K' \backslash K$, Because K is closed, we can find a cube $Q(x, r)$ disjoint from K such that $A_i \llcorner Q \xrightarrow{\mathbb{M}} 0$ (this uses the fact we have just proven, which is that $K = \operatorname{supp} A$). Then, defining $A_i' = A_i - A_i \llcorner Q$, we have that

$$\lim \mathbb{M}(A_i') \leq \lim \mathbb{M}(A_i) = \mathbb{M}(A)$$

and by lower semicontinuity of mass, we thus have that $\mathbb{M}(A_i') \to \mathbb{M}(A)$. This contradicts the minimality of K', because we have now shown that $K' \backslash Q$ has the same properties and is strictly smaller. Thus we must have had $K' = K$ to begin with, finishing the proof. $\qquad \square$

Notice that this theorem shows that the support of a nonempty chain is nonempty, because there thus exist (nonempty) A_i converging to A

$$\bigcap_{i=1}^{\infty} \overline{\bigcup_{i>n} A_i} = \operatorname{supp} A.$$

and each set in the intersection are compact sets, all contained in a fixed compact set, so certainly have nonempty intersection.

5. Intersecting Flat Chains

In this section, we define a notion of the intersection of two flat chains. To begin with, for $A \in \mathscr{P}_m(\mathbb{R}^N; G), B \in \mathscr{P}_n(\mathbb{R}^N; \mathbb{Z})$ (where $\mathbb{Z}$ has its standard absolute value norm), if $A = \sum g_i[\sigma_i]$ and $B = \sum n_j[\gamma_j]$, then we'd like to define

$$A \cap B = \sum g_i n_j [\sigma_i \cap \gamma_j].$$

First of all, even just working with polygons, to make sense of their intersection σ_i and γ_j have to be in general position with respect to each other. In particular, we require that they intersect transversely. In this case, the intersection will have

$$\dim[\sigma_i \cap \gamma_j] = m + n - N.$$

In this section, we will not be careful about the orientation of the intersections. The intersections are oriented by the transverse assumption, but we must make a global choice as to how we choose the orientations. For simplicity, we ignore this point. The reader could replace $\mathbb{Z}$ with $\mathbb{Z}/2\mathbb{Z}$ and G by a $\mathbb{Z}/2\mathbb{Z}$ module if they desire.

Notice that if we let τ_v be affine translation by v, then $A \cap (\tau_v)_{\#} B$ is defined for a.e. $v \in \mathbb{R}^N$. In fact, as in Theorem 2.23,

Claim 5.1. *For polyhedral chains A, B*

$$\int_{v \in \mathbb{R}^N} \mathbb{M}(A \cap (\tau_v)_\# B) d\mathscr{L}_v^N \leq \mathbb{M}(A)\mathbb{M}(B).$$

To show this, it is enough to check on individual polygons, where Theorem 2.23 can be applied inductively. This result extends reasonably to flat chains as we now show. It is not hard to see that for A, B in general position

$$\partial(A \cap B) = \partial A \cap B \pm A \cap \partial B$$

where the sign is determined by the orientation, which will not be important to us here. Writing $A = P + \partial Q$, we thus have that for a.e. v

$$A \cap (\tau_v)_\# B = P \cap (\tau_v)_\# B + \partial Q \cap (\tau_v)_\# B$$
$$= P \cap (\tau_v)_\# B + \partial(Q \cap (\tau_v)_\# B) \pm Q \cap (\tau_v)_\# \partial B.$$

Thus, taking the flat norm and integrating in v gives

$$\int \mathscr{F}(A \cap (\tau_v)_\# B) d\mathscr{L}_v^N \leq \mathbb{M}(P)\mathbb{M}(B) + \mathbb{M}(Q)\mathbb{M}(B) + \mathbb{M}(Q)\mathbb{M}(\partial B)$$
$$\leq (\mathbb{M}(P) + \mathbb{M}(Q))\mathbb{N}(B).$$

Taking the infimum over such P, Q gives:

Claim 5.2. *For A, B polyhedral,*

$$\int \mathscr{F}(A \cap (\tau_v)_\# B) d\mathscr{L}_v^N \leq \mathscr{F}(A)\mathbb{N}(B).$$

For any flat chain A, approximating it (quickly) by polyhedral A_i and applying this claim shows that $A_i \cap (\tau_v)_\# B$ is Cauchy, and we denote the limit by $A \cap (\tau_v)_\# B$ (it is not hard to show that the limit is unique), by the previous claim. Now, for a flat chain A with $\mathbb{N}(A) < \infty$ and a polyhedral chain, we can repeat the above argument to show that

$$\int \mathscr{F}(A \cap (\tau_v)_\# B) d\mathscr{L}_v^N \leq \mathbb{N}(A)\mathscr{F}(B),$$

which then in turn shows

Theorem 5.3. *For flat chains A, B with $\mathbb{N}(A) < \infty$, approximating by polyhedral chains gives a well defined notion of $A \cap (\tau_v)_\# B$ for a.e. v. Furthermore, we continue to have the bound*

$$\int \mathscr{F}(A \cap (\tau_v)_\# B) d\mathscr{L}_v^N \leq \mathbb{N}(A)\mathscr{F}(B).$$

5.1. **The Deformation Theorem via Intersections.** Now that we have a notion of intersection, we can add an additional conclusion to the deformation theorem. For $A \in \mathscr{F}_m$,

$$(5.1) \qquad\qquad P_v^\epsilon(A) = \sum_{\sigma \in X^m(\epsilon)} \overline{\partial}(A \cap \tilde{\sigma}_v)[\sigma]$$

for a.e. v. Here, we are summing of m-faces in the grid $X(\epsilon)$, $\overline{\partial} : \mathscr{F}_0 \to G$ is the map which takes a flat 0-chain to the sum of its coefficients, $\tilde{\sigma}_v$ is the "dual" $N - m$ face in $Y_v(\epsilon)$ which intersects σ (there is a unique such face). Notice that in the case that A is polyhedral, this is clear from our discussion of the deformation theorem, so what remains to show is that it also holds for flat chains. To do so, take A_i polyhedral chains converging to A rapidly. Then, the identity holds for each i, and from the previous section it is clear that $A_i \cap \tilde{\sigma}_v \to A \cap \tilde{\sigma}_v$ for a.e. v. Because the boundary operator is continuous, we may take the limit for (5.1) applied to A_i for a.e. v, obtaining (5.1) in general.

As a consequence of this, we can now show

Theorem 5.4. *If $A \in \mathscr{F}_m$ has support $\operatorname{supp} A$ with the property that it has $\mathscr{H}^m$-measure zero projections onto each m-coordinate plane, then $A = 0$.*

Proof. By the hypothesis combined with the description of $P_v^\epsilon(A)$ with intersections in (5.1), it is clear that $P_v^\epsilon(A) = 0$ for a.e. v. But, by Theorem 3.10, we may choose $\epsilon_i \to 0$ such that $P_v^{\epsilon_i}(A) \to A$, which shows that $A = 0$. $\qquad\qquad\square$

6. Flat Chains with Finite Mass

We will omit/sketch many of the proofs in this section, because they are mostly similar to things which we have done before, and because we'd like to move on from just the fundamentals of the theory. Most of the proofs can be found in Fleming's 1966 paper *Flat chains over a finite coefficient group*, see [Fle66].

6.1. **Radon Measures Associated to Finite Mass Flat Chains.** We define

(6.1) $$\mathscr{M}_m(\mathbb{R}^n; G) := \{A \in \mathscr{F}_m(\mathbb{R}^N; G) : \mathbb{M}(A) < \infty\}.$$

We can associate a Radon measure to such chains. If $A = \sum g_i[\sigma_i]$ is polyhedral and $S \subset \mathbb{R}^N$ is Borel, we can define

$$\mu_A(S) = \sum |g_i| \mathscr{H}^m(\sigma_i \cap S).$$

In general, we have

Theorem 6.1. *For any $A \in \mathscr{M}_m$, there is a unique Radon measure μ_A such that*
  (1) *For $W \subset \mathbb{R}^N$ an open set,*
$$\mu_A(W) = \mathbb{M}_W(A).$$
  (2) *If $A_i \to A$ and $\mathbb{M}(A_i) \to \mathbb{M}(A)$ then $\mu_{A_i} \to \mu_A$ weakly.*

We'll skip the proof, but simply remark that for a fixed $A \in \mathscr{M}_m$, a sequence of polygonal A_i as in the second part of the theorem has a subsequence whose associated measures converge weakly, by Banach-Alaoglu. We can then take the limit measure as our definition of μ_A, provided we show that it is well defined.

Similarly, for finite mass chains, we have the notion of "the part inside a Borel set"

Theorem 6.2. *For every $A \in \mathscr{M}_m(\mathbb{R}^N; G)$ and Borel set S, there is $A \llcorner S \in \mathscr{M}_m$, and this operation has the following properties:*
  (1) *For S_i Borel,*
$$A \llcorner \left(\bigcup_{i=1}^\infty S_i \right) = \sum_{i=1}^\infty A \llcorner S_i$$
  (2) *The map $A \mapsto A \llcorner S$ is a homomorphism.*
  (3) $\mathbb{M}(A \llcorner S) = \mu_A(S)$.

Recall that we cannot do this for infinite mass chains, as discussed in Example 2.21.

6.2. **Images of Flat Chains Under Maps Between Spaces.** Let $F : \mathbb{R}^N \to \mathbb{R}^M$ be a piecewise linear map. This means that we can partition the domain into polygonal regions so that F is linear on each region, and F is globally continuous. Clearly, there is a chain homomorphism

$$F_\# : \mathscr{P}_m(\mathbb{R}^N; G) \to \mathscr{P}_m(\mathbb{R}^M; G),$$

wich is (locally) Lipschitz, and thus extends to a map

$$F_\# : \mathscr{F}_m(\mathbb{R}^N; G) \to \mathscr{F}_m(\mathbb{R}^M; G).$$

Now, if we have a Lipchitz map $F : \mathbb{R}^N \to \mathbb{R}^M$, we'd like to approximate it by piecewise linear maps so that we can use the previous observation to define $F_\#$. Partitioning $\mathbb{R}^N$ into a cubes with

side lengths ϵ, we define a piecewise linear function $G : \mathbb{R}^N \to \mathbb{R}^M$ as follows. What we will do is inductively define G as a piecewise linear function on the k-skeleton of the partition. First, on the 0-vertices let $G(v) = F(v)$. Then, to extend from the $(k-1)$-skeleton, if q is the center of a k-face, let $G(q) = F(q)$, and then to extend to the whole k-face, let G radially interpolate between the center face and the corresponding radial point on the boundary of the k-face. We have illustrated the case of $N = 2$, and extending from the 1-skeleton to the 2-skeleton in Figure 22.
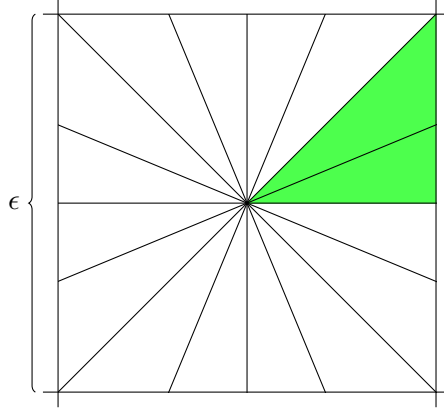

FIGURE 22. Extending the function G piecewise linearly from the 1-skeleton to the 2-skeleton when $N = 2$. We have already linearly interpolated on each edge of the square, from the center to each vertex. Then, we let $G(\text{center}) = F(\text{center})$, and linearly interpolate in the radial direction as is illustrated here. It is clear that G is linear in the region shaded green (as well as the 8 other congruent regions covering the square).

It is not hard to see that by shrinking ϵ in the above description, for $F : \mathbb{R}^N \to \mathbb{R}^M$, we can find $F_i : \mathbb{R}^N \to \mathbb{R}^M$ such that $F_i \to F$ in the C^0 (uniform) topology, and such that

$$\text{Lip}(F_i) \leq c_{M,N} \text{Lip}(F).$$

for some constant $c_{M,N}$ only depending on the dimension of the domain and range of F. Given this, it is possible to show

Theorem 6.3. *If A and F, F_i are as above, then $\lim_{i\to\infty}(F_i)_\# A$ exists. We thus may define $F_\# A$ by its limit (note that uniqueness is obvious from the first part of the theorem, as we do not need to pass to a subsequence.*

Without any assumptions on the mass, we have no control over the rate of convergence. However, if we do assume control on the $\mathbb{N}$ norm of A, we can show

Lemma 6.4. *For A with $\mathbb{N}(A) < \infty$ and $f, g : \mathbb{R}^N \to \mathbb{R}^M$ piecewise linear with $\text{Lip}(f), \text{Lip}(g) \leq \lambda < \infty$, then*

$$\mathscr{F}(f_\# A - g_\# A) \leq c_{N,M} C_\lambda \|f - g\|_{C^0} \mathbb{N}(A)$$

where $C_\lambda = \max\{\lambda^m, \lambda^{m+1}\}$.

In particular, this proves Theorem 6.3 if $\mathbb{N}(A) < \infty$, which is actually the first step in the proof. We do not finish the rest of the proof, but we will prove Lemma 6.4:

Proof of Lemma 6.4. We prove this by working in one extra dimension, so that we may interpolate between f and g. Consider $A \times [0, \delta]$ as a $(m+1)$-chain in $\mathbb{R}^{N+1}$ for some $\delta > 0$ which we fix later. Choose a piecewise linear homotopy $H : \mathbb{R}^N \times [0, \delta] \to \mathbb{R}^M$ between f and g. Note that interpolating linearly does not result in a piecewise linear function, because $(1-t)f(x) + tg(x)$ is

quadratic in t, x. However, we can construct H by partitioning $\mathbb{R}^N \times [0, \delta]$ into cubes of size δ, and then using the same linear interpolation argument on each skeleton of the cube that we used to approximate Lipschitz functions by piecewise linear ones. Now, notice that we can easily check that

$$
\begin{aligned}
\partial H_\#(A \times [0, \delta]) &= H_\#(\partial(A \times [0, \delta])) \\
&= H_\#(A \times \partial[0, \delta] + (-1)^m \partial A \times [0, \delta]) \\
&= H_\#(A \times [\delta] - A \times [0] + (-1)^m \partial A \times [0, \delta]) \\
&= g_\# A - f_\# A + (-1)^m H_\#(\partial A \times [0, \delta]).
\end{aligned}
$$

Thus, we have that

$$
g_\# A - f_\# A = \partial H_\#(A \times [0, \delta]) - (-1)^m H_\#(\partial A \times [0, \delta]),
$$

which implies that

$$
\begin{aligned}
\mathscr{F}(g_\# A - f_\# A) &\leq \mathbb{M}(H_\#(\partial A \times [0, \delta])) + \mathbb{M}(H_\#(A \times [0, \delta])) \\
&\leq (\operatorname{Lip} H)^m \mathbb{M}(\partial A \times [0, \delta]) + (\operatorname{Lip} H)^{m+1} \mathbb{M}(A \times [0, \delta]) \\
&= \delta(\operatorname{Lip} H)^m \mathbb{M}(\partial A) + \delta(\operatorname{Lip} H)^{m+1} \mathbb{M}(A) \\
&\leq \delta \max\{(\operatorname{Lip} H)^m, \operatorname{Lip}(H)^{m+1}\} \mathbb{N}(A).
\end{aligned}
$$

Choosing $\delta = \|f - g\|_{C^0}$, it is possible to check that for this choice of δ

$$
\operatorname{Lip} H \leq C \max\{\operatorname{Lip}(f), \operatorname{Lip}(g)\} \leq C\lambda,
$$

and inserting this into the above inequality finishes the proof. $\qquad \square$

We note that the image of a flat chain under a C^1 map has the following properties:

Theorem 6.5. *For $F : \mathbb{R}^N \to \mathbb{R}^M$ a Lipchitz map, we have that*

$$
\mathbb{M}(F_\# A) \leq (\operatorname{Lip}(F))^m \mathbb{M}(A).
$$

If we further assume that A is polyhedral, then we have

$$
\mathbb{M}(F_\# A) \leq \int \operatorname{Jac}_m(DF|_{\operatorname{supp} A}) d\mu_A(x)
$$

with equality if F is 1-1.

Recall that we know that for top dimensional chains, the flat norm is equivalent to the mass norm. The following is a generalization of this to chains supported in submanifolds of $\mathbb{R}^N$:

Theorem 6.6. *If M is an oriented C^1 m-dimensional submanifold of $\mathbb{R}^N$ and A is a flat chain supported in M, then $\mathbb{M}(A) < \infty$. In fact, there is an isometry, associating such a flat chain to a function $g_A : M \to G$*

$$
\mathscr{M}_m(M; G) \to L^1(M; G)
$$

where the space on the left is the space of flat chains with finite mass supported in M and the space on the right are L^1, G-valued functions on M.

7. Rectifiablity

In this section, we discuss the notion of rectifiability, in particular its relationship to flat chains. To begin with, we have:

Theorem 7.1. *Let $S \subset \mathbb{R}^N$. The following are equivalent*

(1) *There are sets Z and M_i such that $\mathcal{H}^m(Z) = 0$ and the M_i are embedded C^1 m-dimensional submanifolds of $\mathbb{R}^N$ such that*

$$S \subset Z \cup \left(\bigcup_{i=1}^{\infty} M_i \right)$$

(2) *The same statement as before, except we only assume that the M_i are the images of $\mathbb{R}^m$ under Lipschitz maps $F_i : \mathbb{R}^m \to \mathbb{R}^N$.*

Definition 7.2. *A set S satisfying the (equivalent) hypothesis of the previous theorem is called "countably $(\mathcal{H}^m, m)$-rectifiable" (which we will often shorten to "m-rectifiable").*

Proof of Theorem 7.1. Clearly (1) implies (2), so all that remains to show is the converse. With this goal, suppose that

$$S \subset Z \cup \left(\bigcup_{i=1}^{\infty} F_i(\mathbb{R}^m) \right)$$

for Lipschitz F_i. Recall the following two theorems (the proofs may be found in [Sim83, Theorems 5.2, 5.3]).

Theorem 7.3 (Rademacher). *If $F^m \to \mathbb{R}^N$ is Lipschitz, then it is differentiable almost everywhere.*

Theorem 7.4 (Consequence of Rademacher and Whitney's Extension Theorem). *If $F : \mathbb{R}^m \to \mathbb{R}^N$ is Lipschitz and $\epsilon > 0$ is fixed, then there is a C^1 $G : \mathbb{R}^m \to \mathbb{R}^N$ such that*

$$\mathrm{Lip}(G) \leq c_m \mathrm{Lip}(F)$$

and

$$L^m\{x : G(x) \neq F(x)\} < \epsilon.$$

Applying the second theorem gives that

$$F_i(\mathbb{R}^m) \subset \bigcup_{j=1}^{\infty} F_{ij}(\mathbb{R}^m) \cup Z_i$$

where the F_{ij} are C^1 and Z_i has measure zero. Thus, without loss of generality, we may assume that the original F_i were C^1. Now, to assure that the images are embedded C^1-submanifolds, let

$$X_i = \{x \in \mathbb{R}^m : \mathrm{rank}\, DF_i(x) < m\} = \{x \in \mathbb{R}^m : \mathrm{Jac}_m\, DF_i = 0\}$$

By the area formula (c.f. [Sim83, Formula 12.4]), $\mathcal{H}^m(F_i(X_i)) = 0$. Thus, we may write the $F_i(\mathbb{R}^m)$ as the countable union of immersed C^1-submanifolds along with a set of measure zero. Because immersions are locally embeddings, we may further decompose them to obtain the desired conclusion. $\qquad\square$

7.1. **Tangent Planes to Rectifiable Sets.** If S is m-rectifiable and Borel, then as before, we have that

$$S \subset Z \cup \left(\bigcup_{i=1}^{\infty} M_i \right)$$

where the M_i are embedded C^1-manifolds. If $x \in M_i$, then we'd like to define

$$\mathrm{Tan}(S, x) := T_x M_i,$$

but this is not necessarily well defined if x is in the pairwise intersection of any two M_i. In particular, it is not well defined on

$$\bigcup_{ij}\{x \in M_i \cap M_j T_x M_i \neq T_x M_j\},$$

and by a transversality type argument it is possible to show that each of these sets is contained in a C^1 (m-1)-submanifold, and thus has measure zero. This thus shows that $\mathrm{Tan}(S, x)$ exists for a.e. $x \in S$.

Another (equivalent) way to define tangent planes is via densities. We assume that $\mathcal{H}^m(S) < \infty$ (or even that this is true locally). We define

Definition 7.5. *We define the m-density (when the limit exists) of S at x to be*

$$\Theta^m(S, x) = \lim_{r \searrow 0} \frac{\mathcal{H}^m(S \cap B(x, r)}{\omega_n r^n}.$$

In general, we can define the upper and lower densities by

$$\Theta^{m*}(S, x) = \limsup_{r \searrow 0} \frac{\mathcal{H}^m(S \cap B(x, r))}{\omega_n r^n}$$

$$\Theta^m{}_*(S, x) = \liminf_{r \searrow 0} \frac{\mathcal{H}^m(S \cap B(x, r))}{\omega_n r^n}.$$

It is possible to prove that

Theorem 7.6. *If S is m-rectifiable, Borel, and has $\mathcal{H}^m(S) < \infty$, then for $\mathcal{H}^m$-a.e. $x \in \mathbb{R}^N$, either*

   (1) $x \notin S$, $\Theta^m(x, S) = 0$, or
   (2) $x \in S$ and there is a C^1 submanifold $M \ni x$ such that

$$\Theta^m(S \Delta M, x) = 0,$$

   where $S \Delta M$ is the symmetric difference of the two sets.

We will call a point x satisfying condition (2) of the previous theorem an "almost regular point" of S (this is potentially nonstandard notation). With this, we have

Definition 7.7. *If $x \in S$ is almost regular, then define*

$$\mathrm{Tan}(S, x) = T_x M$$

for M as in (2) of the previous theorem. It is not hard to see that this is independent of the choice of M.

We briefly remark that in spite of these nice properties, rectifiable sets can still be quite pathological. For example,

Example 7.8. *Let x_i be a dense subset of $B_1(0) \subset \mathbb{R}^3$, and choose r_i such that $\sum r_i^2 < \infty$ and $B_{r_i}(x_i) \subset B_1(0)$. Then, the set*

$$S = \bigcup_{i=1}^{\infty} \partial B_{r_i}(x_i) \subset B_1(0)$$

is a 2-rectifiable. Note that it is dense in $B_1(0)$, but also has tangent planes almost everywhere.

7.2. **Rectifiable Flat Chains.**

Definition 7.9. *Suppose that $A \in \mathcal{M}_m(\mathbb{R}^N; G)$ is a flat chain of finite mass. We will say that A is m-rectifiable if there is a m-rectifiable set $S \subset \mathbb{R}^N$ such that $\mu_A(S^c) = 0$, or equivalently $A \llcorner S = A$.*

We remark that this is *not* a property of $\mathrm{supp}\, A$, as shown by Example 7.8, which is a rectifiable 2-chain, but has support the entire ball.

Theorem 7.10. *Suppose that $A \subset \mathcal{M}_m(\mathbb{R}^N; G)$ is rectifiable. Then, for $\epsilon > 0$, there is $F : \mathbb{R}^N \to \mathbb{R}^N$ a C^1-diffeomorphism such that*

$$\|F - \mathrm{Id}_{\mathbb{R}^N}\|_{C^1} < \epsilon$$

and a polyhedral chain P such that

$$\mathbb{M}(A - F_\# P) < \epsilon,$$

or equivalently

$$\mathbb{M}((F^{-1})_\# A - P) < \epsilon.$$

Proof. By assumption, there is rectifiable S such that $\mu_A(S^c) = 0$. If Z is the $\mathcal{H}^m$-measure zero set in the definition of S, then $A \llcorner Z = 0$, by Theorem 5.4, which says that if the support of a flat chain projects onto the coordinate axes with measure zero, then the flat chain was zero to begin with. As such, we may drop any measure zero part of S, and write

$$S \subset \bigcup_{i=1}^{\infty} M_i.$$

Furthermore, we may assume that the M_i are disjoint, by writing $M_i' = M_i - \overline{\cup_{j=1}^{i-1} M_j}$ (we may be dropping a set of measure zero, but we can disregard this set, by the previous argument).

Now, because we have assumed that the M_i are disjoint,

$$\mu_A(S) = \sum_{i=1}^{\infty} \mu_A(S \cap M_i).$$

Thus, we may choose n such that

$$\mu_A \left(\bigcup_{i > n} M_i \right) < \epsilon.$$

In other words,

$$\mathbb{M} \left(A - A \llcorner \bigcup_{i=1}^{n} M_i \right) < \epsilon.$$

Furthermore, for $i = 1, \ldots, n$, (up to a set of measure zero) we may cover each M_i by countably many disjoint closed balls B_k, and in particular by this and the previous equation, we have k such that

$$\mathbb{M} \left(A - A \llcorner \bigcup_{j=1}^{k} B_j \right) < 2\epsilon.$$

We could have chosen the balls B_j so that $M_i \cap B_j$, if it is nonempty, is a graph over a tangent plane, so there is a diffeomorphism $F : \mathbb{R}^N \to \mathbb{R}^N$ which "flattens each (nonempty) $M_i \cap B_j$" and leaves the exterior of the balls fixed. If we chose the balls small enough, F is ϵ close to the identity map, as required in the statement of the theorem. Now, letting $A' = A \llcorner \bigcup_{j=1}^{k} B_j$, by construction, $F_\# A'$ is supported in the disjoint union of m-disks. The piece of A' in each m-disk is supported in an m-plane, and because the $\mathscr{F}$ norm and the $\mathbb{M}$ norm are equal for m-chains supported in an m-plane, we can approximate $F_\# A'$ well by a polygonal chain, which finishes the proof. $\qquad\square$

7.3. Rectifiability Theorems.
In this section we will examine criteria for flat chains to be rectifiable. To begin with, we have

Theorem 7.11. *For $A \in \mathscr{M}_m(\mathbb{R}^N; G)$ a flat chain, then A is rectifiable if and only if for a.e. $(N - m)$ dimensional plane, slicing A by the plane results in a rectifiable 0-chain.*

Notice that a zero chain T with $\mathbb{M}(T) < \infty$ is rectifiable if and only if there is a countable set S of points such that $\mu_T(S^c) = 0$. let $S = \{p_1, p_2, \ldots\}$. We thus have

$$(7.1) \qquad T = T \llcorner S = \sum_{i=1}^{\infty} T \llcorner p_i = \sum_{i=1}^{\infty} g_i[p_i],$$

for g_i with $\sum |g_i| < \infty$.

Note 7.12. *Notice that the analogue of Theorem 7.11 for sets instead of flat chains is not true. What we can prove is that if $S \subset \mathbb{R}^N$ is Borel and $\mathcal{H}^m(S) < \infty$, then if $\Pi_V : \mathbb{R}^N \to V$ is projection onto an m-plane, V, then it is easy to show (with no rectifiability assumptions)*

$$\int_{x \in V} \mathcal{H}^0(S \cap (\Pi_V)^{-1}(x)) d\mathcal{H}^m(x) \leq \mathcal{H}^m(S).$$

Furthermore, if S is rectifiable, we have equality once we average over V appropriately:

(7.2)
$$\overline{\int_{x \in V} \mathcal{H}^0(S \cap (\Pi_V)^{-1}(x)) d\mathcal{H}^m(x)} = \mathcal{H}^m(S).$$

However, there are examples with $0 < \mathcal{H}^m(S) < \infty$ such that $\mathcal{H}^m(\Pi_V(S)) = 0$ for all V. In particular, this satisfies the slicing criterion of Theorem 7.11, cannot be rectifiable, because (7.2) certainly fails to hold.

Now, we state the full rectifiability theorem for flat chains of finite mass

Theorem 7.13. *Every chain $A \in \mathscr{M}_m(\mathbb{R}^N; G)$ is rectifiable if and only if no two distinct elements of G can be connected by a path of finite length.*

Before proving this, we give an example of a non-rectifiable chain, because at first glance it is hard to imagine such a situation (in fact, our example will immediately generalize to show the "only if" direction in the above theorem).

Suppose that $\gamma : [0, 1] \to G$ is a non-constant finite length path parametrized by arc length (for any group where there is a non-constant path, we may reparametrize by arc length, and scaling the norm of G will allow us to have a path defined on $[0, 1]$, so this is not a restrictive assumption). Then, we may define

$$T_n := \sum_{k=1}^{2^n} \left(\gamma\left(\frac{k}{2^n}\right) - \gamma\left(\frac{k-1}{2^n}\right) \right) \left[\frac{k}{2^n}\right]$$

It is not hard to show that the T_n form a $\mathscr{F}$-Cauchy sequence, converging to some chain T. Furthermore, by our choice of γ, $\mu_T((a, b)) = |b - a|$, which shows that T cannot be supported on a finite number of points, so cannot be rectifiable.

Theorem 7.14. *We have a homeomorphism*

$$\mathscr{M}_0(\mathbb{R}^N; G) \to \left\{ \begin{array}{c} \textit{compactly supported Borel measures in } \mathbb{R}^N \textit{ taking} \\ \textit{values in } G \textit{ (with finite total mass)} \end{array} \right\},$$

given by $T \mapsto \psi_T(S) := \overline{\partial}(T \llcorner S)$. Recall that we have defined $\overline{\partial} : \mathscr{F}_0(\mathbb{R}^N; G) \to G$ to be the "sum of coefficients" 0-boundary map, not the trivial 0-boundary map.

Proof. It is easy to see that this is a well defined, continuous map, so we will show that it is invertible. Suppose that ψ is a compactly supported Borel measure (with finite mass) with values in G. Let $X(\epsilon_n)$ be a grid of cubes with side lengths $\epsilon_n = 2^{-n}$. We now define

$$T_n = \sum_{Q \text{ a cube of } X(\epsilon)} \psi(Q)[\text{center of } Q].$$

Now, it is not hard to see that $T_n \to T$ for some finite mass 0-chain T, and that $\psi_T = \psi$. $\qquad \square$

Now we will work towards the proof of the Rectifiability Theorem, Theorem 7.13. The following result, due to Fleming, will be useful. For $A \in \mathscr{M}_m(\mathbb{R}^n; G)$, recall that we have defined a measure μ_A, which satisfies for $\overline{B}(x, r)$ the closed ball of radius r around x (we are using closed balls because we'll apply a covering lemma in the proof, but it really does not change anything),

$$\mu_A(\overline{B}(x, r)) = \mathbb{M}(A \llcorner \overline{B}(x, r)).$$

We now define a second measure ν_A by

$$\nu_B(\overline{B}(x,r)) = \inf(\mathbb{M}(T) : \partial T = \partial(A \llcorner \overline{B}(x,r)))\}.$$

In particular, we see that $\nu_A(\overline{B}(x,r)) \leq \mu_A(\overline{B}(x,r))$, with inequality if and only if $A \llcorner \overline{B}(x,r)$ is minimizing among chains with the same boundary. We have the following theorem which sort of says that A is almost everywhere area minimizing for very small balls. More precisely

Theorem 7.15 (Fleming). *For any coefficient group G, for all $A \in \mathscr{M}_m(\mathbb{R}^N; G)$,*

$$\lambda(x) := \lim_{r \to 0} \frac{\nu_A(\overline{B}(x,r))}{\mu_A(\overline{B}(x,r))} = 1$$

for μ_A-a.e. x (included in the statement is that the limit exists almost everywhere).

Proof. Since the ratio is always less than or equal to 1, by definition of ν_A, we define

$$L(x) := \liminf_{r \to 0} \frac{\nu_A(\overline{B}(x,r))}{\mu_A(\overline{B}(x,r))}$$

and suppose that $L(x) < \eta < 1$ on some set S. We'd like to show that $\mu_A(S) = 0$, which we now proceed to do. For $x \in S$, there are arbitrarily small r such that

$$(7.3) \qquad\qquad\qquad \frac{\nu_A(\overline{B}(x,r))}{\mu_A(\overline{B}(x,r))} < \eta.$$

By Besicovitch Covering Theorem (c.f. the discussion in [Sim83, Lemma 4.6], which references [Fed69] for the proof), we can choose disjoint balls satisfying the inequality in (7.3), $\overline{B}_i := \overline{B}(x_i, r_i)$ with $r_i < \epsilon$, such that they cover μ_A-a.e. of S.

For each $\overline{B}_i$, there is a chain T_i such that $\partial T_i = \partial(A \llcorner \overline{B}_i)$ and

$$\mathbb{M}(T_i) < \eta \mathbb{M}(A \llcorner \overline{B}_i).$$

Define a new chain

$$A' = A + \sum_i (T_i - A \llcorner \overline{B}_i) = A + \sum_i \partial(\mathrm{cone}_{x_i}(T_i - A \llcorner \overline{B}_i)).$$

Here, cone_{x_i} is the cone with center x_i. Thus, we have that

$$\mathscr{F}(A - A') \leq \sum_{i=1}^{\infty} \mathbb{M}(\mathrm{cone}_{x_i}(T_i - A \llcorner \overline{B}_i))$$

$$\leq \sum_{i=1}^{\infty} \frac{r_i}{m+1} \mathbb{M}(T_i - A \llcorner \overline{B}_i)$$

$$\leq C\epsilon(1+\eta) \sum_{i=1}^{\infty} \mathbb{M}(A \llcorner \overline{B}_i)$$

$$\leq C\epsilon(1+\eta)\mathbb{M}(A).$$

In the last line, we used that the $\overline{B}_i$'s are disjoint. On the other hand,

$$\mathbb{M}(A') = \mathbb{M}(A' \llcorner (\cup \overline{B}_i)^c) + \sum_{i=1}^{\infty} \mathbb{M}(A' \llcorner \overline{B}_i)$$

$$= \mathbb{M}(A \llcorner (\cup \overline{B}_i)^c) + \sum_{i=1}^{\infty} \mathbb{M}(T_i)$$

$$\leq \mathbb{M}(A \llcorner (\cup \overline{B}_i)^c) + \eta \sum_{i=1}^{\infty} \mathbb{M}(A \llcorner \overline{B}_i)$$

$$= \mathbb{M}(A) - \sum_{i=1}^{\infty} \mathbb{M}(A \llcorner \overline{B}_i) + \eta \sum_{i=1}^{\infty} \mathbb{M}(A \llcorner \overline{B}_i)$$

$$= \mathbb{M}(A) - (1 - \eta) \sum_{i=1}^{\infty} \mathbb{M}(A \llcorner \overline{B}_i)$$

$$\leq \mathbb{M}(A) - (1 - \eta)\mu_A(\cup \overline{B}_i)$$

$$\leq \mathbb{M}(A) - (1 - \eta)\mu_A(S).$$

Now, by taking $\epsilon \to 0$, where ϵ is the upper bound on the balls used to define A', we have a sequence of A' which clearly converge to A in the flat norm. Taking the limit in the above inequality, because mass is is lower semicontinuous we get

$$\mathbb{M}(A) \leq \lim \mathbb{M}(A') \leq \mathbb{M}(A) - (1 - \eta)\mu_A(S).$$

Because $\eta < 1$, we thus have that $\mu_A(S) = 0$, as desired. $\qquad \square$

Now, for simplicity, we assume that the coefficient group G satisfies $|g| \geq 1$ for $g \neq 0$. This certainly implies the hypothesis of the rectafiability theorem, so we will prove the rectifiability theorem with these restricted hypothesis. To begin with, we have

Theorem 7.16. *With the hypothesis that nonzero g have $|g| \geq 1$, there is a universal constant $c_{N,m} > 0$ such that for $A \in \mathscr{M}_m(\mathbb{R}^N; G)$, we have*

$$\liminf_{r \to 0} \frac{\mu_A(\overline{B}(x,r))}{\omega_m r^m} \geq c_{N,m} > 0,$$

for μ_A-a.e. $x \notin \operatorname{supp} \partial A$.

In fact, the $x \notin \operatorname{supp} \partial A$ assumption is not strictly necessary, but will be easier for us to prove. Notice that, for example, a "fat cantor set" is 1-rectifiable, but its boundary is certainly not.

Proof. Fix $x \notin \operatorname{supp} \partial A$ so that

$$\lim_{r \to 0} \frac{\nu_A(\overline{B}(x,r))}{\mu_A(\overline{B}(x,r))} = 1.$$

By Theorem 7.15, this holds for μ_A almost every x. Thus, for this x, there is $\delta > 0$ so that

(7.4)
$$\frac{\nu_A(\overline{B}(x,r))}{\mu_A(\overline{B}(x,r))} > \frac{1}{2}$$

for $r < \delta$. Shrinking δ if necessary, we can also arrange $\delta < d(x, \operatorname{supp} \partial A)$. Let $S_r = A \cap \partial \overline{B}(x,r) = \partial(A \llcorner \overline{B}(x,r))$. By the isoperimetric inequality, Theorem 3.7 (by assumption that nonzero g have $|g| \geq 1$, the hypothesis are satisfied), we have that there is a chain T_r with $\partial T_r = S_r$ and

$$\mathbb{M}(T_r) \leq c\mathbb{M}(S_r)^{\frac{m}{m-1}}.$$

By definition of ν_A, this implies that

$$\nu_A(\overline{B}(x,r)) \leq \mathbb{M}(T_r) \leq c\mathbb{M}(S_r)^{\frac{m}{m-1}}.$$

Define $u(r) = \mathbb{M}(A \llcorner \overline{B}(x,r))$. We certainly have

$$u(R) = \mathbb{M}(A \llcorner \overline{B}(x,R) \geq \int_0^R \mathbb{M}(S_r) dr.$$

We may differentiate this, giving (to be precise, we mean in either the a.e. sense, or the distributional sense, either of which holds)

$$u'(r) \geq \mathbb{M}(S_r).$$

This, combined with the isoperimetric inequality above, as well as (7.4) gives that

$$u'(r) \geq \mathbb{M}(S_r) \geq \left[c^{-1}\nu_A(\overline{B}(x,r))\right]^{1-\frac{1}{m}} \geq \left[2c^{-1}\mu_A(\overline{B}(x,r))\right]^{1-\frac{1}{m}} = \left[2c^{-1}u(r)\right]^{1-\frac{1}{m}}.$$

By integration, this readily implies that

$$u(r) \geq \tilde{c}r^m,$$

for a constant $\tilde{c}$ only depending on dimension, which finishes the proof. $\square$

Now, we recall the the following result concerning upper densities

Theorem 7.17 ([Sim83, Theorem 3.2 (2)]). *If μ is a Radon measure on $\mathbb{R}^N$ and $S \subset \mathbb{R}^N$ is Borel, if*

$$\limsup_{r \to 0} \frac{\mu(\overline{B}(x,r))}{\omega_n r^n} \geq c$$

for $x \in S$, then we have

$$\mu(S) \geq c2^{-m}\mathcal{H}^m(S).$$

This will allow us to prove the first step towards rectifiability:

Theorem 7.18. *If G satisfies $|g| \geq 0$ for $g \neq 0$, then for $A \in \mathscr{M}_m(\mathbb{R}^N; G)$, there is a Borel set $S \subset \mathbb{R}^N$ such that $\mathcal{H}^m(S) < \infty$ and $\mu_A(S^c) = 0$.*

Proof. First, assume that $\partial A = 0$. Because there is no boundary, Theorem 7.16 implies that

$$\liminf_{r \to 0} \frac{\mu_A(\overline{B}(x,r)}{\omega_m r^m} \geq c_{N,m} > 0.$$

for μ_A-a.e. x. Let S be the set of x such that it holds. Because it holds μ_A-a.e., clearly $\mu_A(S^c) = 0$, so it remains to show that $\mathcal{H}^m(S) < \infty$. However, this follows immediately from Theorem 7.17, because $\mu_A(S) < \infty$, as A has finite mass.

Now, for the general case, notice that if the result is true for A_1 and A_2, then it is certainly true for $A_1 + A_2$. Furthermore, we have shown in Theorem 2.20 that cycles bound chains of finite mass, and examining the proof shows that in fact every cycle bounds a rectifiable chain of finite mass. Thus $\partial A = \partial R$ for R rectifiable. We thus have that $A = (A - R) + R$. Because $A - R$ has trivial boundary, we may apply the first part of the proof to establish the theorem for it. Furthermore, R is easily seen to satisfy the desired conclusion, because it is rectifiable. In particular, R is supported in $\cup M_i$ for a countable number of C^1 manifolds. As such, we have a density $g : \cup M_i \to G$ such that

$$\infty > \mathbb{M}(A) = \int_{\cup M_i} |g|(x)\mathcal{H}^m(x) \geq \mathcal{H}^m\left(\underbrace{\{x \in \cup M_i : g \neq 0\}}_{:=S}\right).$$

Taking S as just defined clearly satisfies the desired conclusion. $\square$

Now, we will need the following theorem, which is discussed in [Sim83, Theorem 13.2]. A proof can be found in [Fed69].

Theorem 7.19 (Federer-Besicovitch Structure Theorem). *Suppose that $S \subset \mathbb{R}^N$ is a Borel set with $\mathcal{H}^m(S) < \infty$. Then,*

$$S = R \cup U$$

where R is a rectifiable set and U is a "purely unrectifiable set," i.e. $\mathcal{H}^m(U \cap M) = 0$ for any C^1, m-dimensional manifold M. Furthermore, this property is equivalent to

$$\mathcal{H}^m(\Pi_V(U)) = 0$$

for a.e. m-plane V.

Before applying this theorem to the rectifiablility theorem, we give an example of a purely unrectifiable set.

Example 7.20. *We give an example of a purely unrectifiable set T in Figure 23. To construct T, start with a (solid) triangle and remove a (regular) inscribed hexagon (which must have one third of the side length). Repeat this process inductively, and then let T be the intersection of all of these sets. Now, notice that it is easy to show that $\mathcal{H}^1(T) \leq 1$. Establishing a lower bound is usually more difficult, but as illustrated in Figure 24, T projects down to the x-axis to be the solid interval $[0, 1]$, and because projection can only decrease the measure, we see that $\mathcal{H}^1(T) = 1$. On the other hand, projecting onto the y-axis (and any of the other two lines which are perpendicular to a side of the triangle) gives the (standard) middle thirds Cantor set.*

This does not quite imply purely unrectifiability, as we do not yet know that T projects to have measure zero for a.e. axis, but it is not hard to see that if T contains a 1-rectifiable subset of positive measure, that subset can only project to have measure zero in at most one direction (if so, up to a set of measure zero it clearly must be contained in lines perpendicular to the axis). Thus, we see that T is purely unrectifiable, and in fact we thus know that it projects to have measure zero in almost every direction.
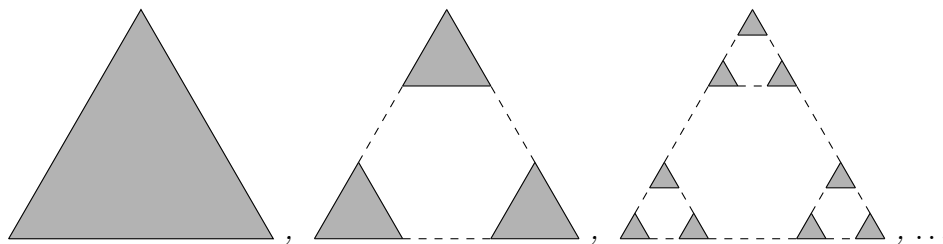

FIGURE 23. Constructing a totally unrectifiable set T as described in Example 7.20. At each step we remove a hexagon from the middle of each (filled) triangle, as pictured. We then define T to be the intersection of each of the successive iteration.

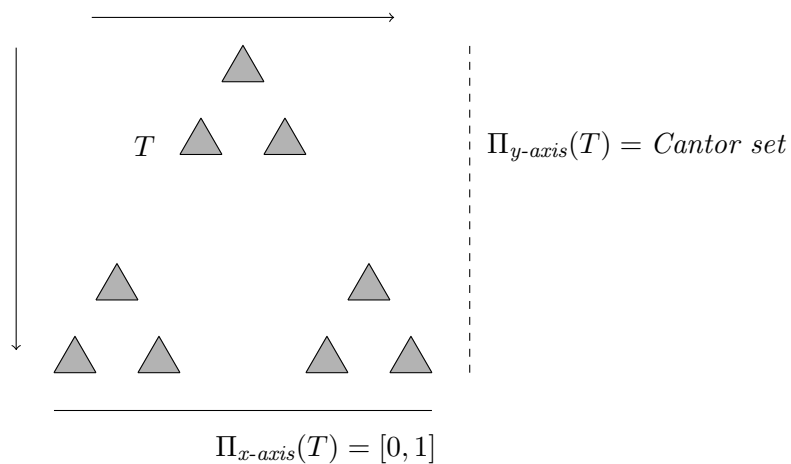FIGURE 24. The set T constructed in Figure 23 projects down to the x-axis to be the unit interval, but projects to the y-axis to be the Cantor set.

We now prove the rectifiability theorem. For simplicity, we take slightly stronger hypothesis than the general theorem.

Theorem 7.21. *Under the hypothesis $|g| \geq 1$ for nonzero G, all $A \in \mathscr{M}_m(\mathbb{R}^N; G)$ are rectifiable.*

Proof. By Theorem 7.18, there is a Borel set S with $\mathcal{H}^m(S) < \infty$ and $\mu_A(S^c) = 0$. By the structure theorem, $S = R \cup U$ for R rectifiable and U purely unrectifiable. Thus, $A = A \llcorner S + A \llcorner U$. We will be finished if we can show that $A \llcorner U = 0$. Rotating the coordinate frames, the structure theorem implies that $\mathcal{H}^m(\Pi_V(U)) = 0$ for V the coordinate m-planes. Now, recall that

$$\mu_A(U) = \sup_{\text{compact } K \subset U} \mu_A(K).$$

Each such K also has $\mathcal{H}^m(\Pi_V(K)) = 0$, and thus $A \llcorner K$ is supported in a set with $\mathcal{H}^m$-measure zero projections onto the m-coordinate planes. Theorem 5.4 now implies that $A \llcorner K = 0$, so $\mu_A(K) = 0$ for all compact K in U. Thus $\mu_A(U) = 0$, as desired. $\square$

Remark 7.22. *We briefly remark that if $S \subset \mathbb{R}^N$ is a m-rectifiable set inside some compact set K, then we may give S the structure of a flat chain, as follows. For simplicity, we take $G = \mathbb{Z}/2\mathbb{Z}$, with the obvious norm (in general, there is some ambiguity of how to associate a flat chain to a rectifiable set, because we are free to choose the multiplicity for each piece). Recall that in Section 2.7, we showed that m-dimensional chains (over G) in $\mathbb{R}^m$ are just $L^1_{cpt}(\mathbb{R}^n; G)$. However, it is easy to extend this to m-dimensional submanifolds.*

In particular, for the set S as above, S is contained (up to a set of measure zero, which we may disregard in this context) inside a countable number of m-dimensional C^1 manifolds, and for each one of these, it is not hard to construct a flat chain (over $\mathbb{Z}/2\mathbb{Z}$) whose "underlying set" is $S \cap M_i$, and then adding these up gives the desired flat chain, because S has finite $\mathcal{H}^m$-measure.

8. Mass Minimizing Chains

In the following section we will assume that G satisfies the hypothesis of the compactness theorem, unless we state otherwise. We now show that if $\partial \Gamma = 0$ for Γ a $(m-1)$-dimensional flat chain, then there exists a least mass A spanning Γ, i.e. $\partial A = \Gamma$. Notice that the compactness theorem guarantees a convergent sequence A_i bounding Γ whose mass approaches the infimum of the mass of chains spanning Γ, but because mass can jump in the limit, it is not a priori clear that we can construct mass minimizers, which we now do.

Theorem 8.1. *Suppose that A_i are mass minimizing (recall that this means that the A_i have minimal mass among all chains with the same boundary) and $A_i \to A$. Then, A is mass minimizing and $\mathbb{M}(A_i) \to \mathbb{M}(A)$.*

Proof. Suppose that $\partial A' = \partial A$ (so A' is a competitor with respect to A being mass minimizing). Because $A_i \to A$, there are Q_i, R_i such that

$$A_i - A = R_i + \partial Q_i \quad \text{and} \quad \mathbb{M}(R_i) + \mathbb{M}(Q_i) \to 0.$$

Applying the boundary operator gives

$$\begin{aligned} \partial A_i &= \partial A + \partial R_i \\ &= \partial A' + \partial R_i \\ &= \partial(A' + R_i), \end{aligned}$$

by the assumption on A'. Thus, by the assumption that the A_i are minimizing

$$\mathbb{M}(A_i) \leq \mathbb{M}(A' + R_i).$$

By the triangle inequality, and the fact that $\mathbb{M}(R_i) \to 0$, we see that

$$\limsup_{i \to \infty} \mathbb{M}(A_i) \leq \mathbb{M}(A').$$

However, by lower semicontinuity of mass,

$$\mathbb{M}(A) \leq \liminf_{i \to \infty} \mathbb{M}(A_i).$$

Combining these two (using the obvious inequality $\liminf \mathbb{M}(A_i) \leq \limsup \mathbb{M}(A_i)$, we see that $\mathbb{M}(A) \leq \mathbb{M}(A')$ implying that A is mass minimizing because. Furthermore, taking $A' = A$ gives that the last two inequalities must be equalities, and thus

$$\mathbb{M}(A) = \lim_{i \to \infty} \mathbb{M}(A_i),$$

finishing the proof. $\hfill\square$

8.1. **The Monotonicity Formula.** The following theorem is of fundamental importance in the study of mass-minimizing chains.

Theorem 8.2 (The Monotonicity Formula)**.** *Suppose that A is an m-dimensional mass minimizing chain, and that $x \notin \operatorname{supp} \partial A$. Then, defining the mass ratio*

$$\Theta(A, x, r) := \frac{\mathbb{M}(A \llcorner B(x, r))}{\omega_m r^m} = \frac{\mu_A(B(x, r))}{\omega_m r^m}$$

(where ω_m is the volume of an m-ball of radius 1), then Θ is an increasing function of r for $0 < r < d(x, \operatorname{supp} \partial A)$.

Proof. Fix such an $x \notin \operatorname{supp} \partial A$. Defining

$$u(r) := \mathbb{M}(A \llcorner B(x, r)),$$

we have the trivial inequality

(8.1) $$u(r) \geq \int_0^r \mathbb{M}(A \cap \partial B(x, t)) dt$$

(which has nothing to do with the mass-minimizing properties of A, as it is just the radial analogue of Theorem 2.23). Thus (in the a.e. as well as the distributional sense)

$$u'(r) \geq \mathbb{M}(A \cap \partial B(x, t)).$$

By the assumption that A is mass minimizing, for a.e. r, we have

$$u(r) = \mathbb{M}(A \llcorner B(x, r)) \leq \mathbb{M}(\operatorname{cone}_x(\partial(A \llcorner B(x, r))))$$
$$\leq \frac{r}{m} \mathbb{M}(\partial(A \llcorner B(x, r)))$$
$$= \frac{r}{m} \mathbb{M}(A \cap \partial B_r)$$
$$\leq \frac{r}{m} u'.$$

In the second to last line we used that $r < d(x, \operatorname{supp} \partial A)$. Thus, rearranging this gives that (in the a.e. or distributional sense)

$$\left(\frac{u}{r^m}\right)' \geq 0,$$

which exactly proves that Θ is increasing. $\hfill\square$

Corollary 8.3. *The density exists for every $x \notin \operatorname{supp} \partial A$, i.e.*

$$\Theta(A, x) = \lim_{r \to 0} \Theta(A, x, r)$$

exists.

Furthermore, by analyzing the equality case of (8.1) and in the subsequent argument, we also have

Corollary 8.4. *For $0 < r < d(x, \operatorname{supp} \partial A)$,*

$$\Theta(A, x) < \Theta(A, x, r)$$

unless $A \llcorner B(x, r) = \operatorname{cone}_x(\partial(A \llcorner B(x, r)))$.

The monotonicity formula allows us to prove

Theorem 8.5. *The function $\Theta(A, x)$ is upper semicontinuous as a function of A and x (for $x \notin \operatorname{supp} \partial A$). More precisely, if A_i are mass-minimizing with $A_i \to A$, and $x_i \to x$ with $d(x, \operatorname{supp} \partial A_i) \geq \delta > 0$, then*

$$\Theta(A, x) \geq \limsup_{i \to \infty} \Theta(A_i, x_i).$$

Proof. Translating A_i by x_i and A by x, we may assume, without loss of generality, that in fact $x_i = x = 0$. For a.e. $0 < r \leq \delta$ we have that

$$A_i \llcorner B(0, r) \to A \llcorner (0, r) \quad \text{and} \quad \mathbb{M}(A_i \llcorner B(0, r)) \to \mathbb{M}(A \llcorner B(0, r)).$$

As such, we have that

$$\Theta(A, 0, r) = \lim_{i \to \infty} \Theta(A_i, 0, r)$$

for a.e. such r. By monotonicity

$$\Theta(A_i, 0, r) \geq \Theta(A_i, 0).$$

Taking the limit in i thus gives

$$\Theta(A, 0, r) \geq \limsup_{i \to \infty} \Theta(A_i, 0),$$

and then taking the limit as $r \to 0$, we thus have

$$\Theta(A, 0) \geq \limsup_{i \to \infty} \Theta(A_i, 0). \qquad \square$$

8.2. **Tangent Cones to Mass-Minimizing Chains.** We now show that mass-minimizing chains have tangent cones in the following sense

Theorem 8.6. *Let A be a mass-minimizing chain with $0 \notin \operatorname{supp} \partial A$. For a sequence of positive real numbers $\lambda_i \to \infty$, we may find a subsequence $\lambda_{i'} \to \infty$ such that*

$$(\lambda_{i'})_\# A \xrightarrow{\mathcal{F}_{loc}} C$$

where C is a mass minimizing cone. Here, we have written $(\lambda)_\# A$ for the rescaling of A by λ. The convergence above is in the local $\mathcal{F}$ topology (i.e. $\mathcal{F}$ convergence inside any ball). By a mass-minimizing cone (or any locally flat chain) we mean that the part of it inside any ball is mass-minimizing.

Proof. We must show the existence of a subsequencial limit by the compactness theorem. Notice that for every $r, \lambda > 0$

$$\Theta((\lambda)_\# A, 0, r) = \Theta\left(A, 0, \frac{r}{\lambda}\right).$$

Thus, for each $r > 0$,

(8.2) $$\lim_{\lambda \to \infty} \Theta((\lambda)_\# A, 0, r) = \lim_{\lambda \to \infty} \Theta\left(A, 0, \frac{r}{\lambda}\right) = \Theta(A, 0).$$

In particular, for every $B(0, r)$, we have that

$$\limsup_{i \to \infty} \mathbb{M}_{B(0,r)}((\lambda_i)_\# A) < \infty.$$

Furthermore, for large enough i (depending on r)

$$\mathbb{M}_{B(0,r)}((\lambda_i)_\# \partial A) = 0.$$

Thus, by the compactness theorem and a diagonal argument, we may find such a subsequence $\lambda_{i'} \to \infty$ with

$$(\lambda_{i'})_\# A \xrightarrow{\mathcal{F}_{loc}} C,$$

for some chain C (which certainly has $\partial C = 0$). By Theorem 8.1, C is mass-minimizing. Furthermore, for a.e. $r > 0$, we have that

$$((\lambda_{i'})_\# A) \llcorner B(0, r) \to C \llcorner B(0, r),$$

so by the C-mass minimizing, Theorem 8.1 applied again gives

$$\mathbb{M}_{B(0,r))}(C) = \mathbb{M}(C \llcorner B(0, r)) = \lim_{i \to \infty} \mathbb{M}(((\lambda_{i'})_\# A) \llcorner B(0, r)).$$

However, by (8.2), this gives

$$\Theta(C, 0, r) = \Theta(A, 0).$$

We may thus conclude that C is a cone, by Corollary 8.4. $\hspace{1cm}\square$

Remark 8.7. *We remark that Theorem 8.6 makes no claims as to the uniqueness of the tangent cones. In fact, it is a long standing open problem to establish uniqueness for minimizing chains. In Figure 25 we give an example of a (non-minimal) 1-chain in $\mathbb{R}^2$ which has tangent cones at the origin pointing in every direction.*

8.3. **Mass-Minimizing Cones.** Thanks to Theorem 8.6, we are thus motivated to study mass-minimizing cones, because they naturally arise as the tangent cones to mass-minimizing chains.[5] Suppose that A is any locally flat chain which is mass-minimizing with $\partial A = 0$. It will be useful in the study of mass-minimizing cones to discuss the limit as $r \to \infty$ of the mass ratio $\Theta(A, x, r)$. Notice that because $B(x, r) \subset B(0, |x| + r)$

$$\mathbb{M}(A \llcorner B(x, r)) \leq \mathbb{M}(A \llcorner B(0, |x| + r)).$$

This implies that

$$\begin{aligned}
\Theta(A, x, r) &= \frac{\mathbb{M}(A \llcorner B(x, r))}{\omega_m r^m} \\
&\leq \left(\frac{|x| + r}{r}\right)^m \frac{\mathbb{M}(A \llcorner B(0, |x| + r))}{\omega_m (|x| + r)^m} \\
&= \left(\frac{|x| + r}{r}\right)^m \Theta(A, 0, |x| + r).
\end{aligned}$$

By the monotonicity formula, the limit as $r \to \infty$ of both sides of this inequality exist (of course it could possibly be infinity—an example of this is given in Figure 26).

Thus, we have that

$$\Theta(A, x, \infty) \leq \Theta(A, 0, \infty)$$

with the obvious interpretation if one side is infinite. We could repeat this argument with 0 and x switched, thus obtaining the opposite inequality. Thus, we see that

$$\Theta(A, x, \infty) = \Theta(A, 0, \infty)$$

for all x. Thus, we are motivated to define

Definition 8.8. *For A a locally flat chain which is (locally) mass-minimizing with $\partial A = 0$, we define*

$$\Theta(A, \infty) = \lim_{r \to \infty} \Theta(A, x, r)$$

The above computation shows that this definition is independent of the choice of x.

In many cases one can show that $\Theta(A, \infty) < \infty$ for any locally mass-minimizing chain. For example

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[5]Many of the results in this section actually applies to "minimal" cones (cones which are critical points for area under compact variations), because the monotonicity formula also holds under this weaker hypothesis and this is our chief tool of studying these objects. For simplicity we restrict ourselves here to mass-minimizing cones.
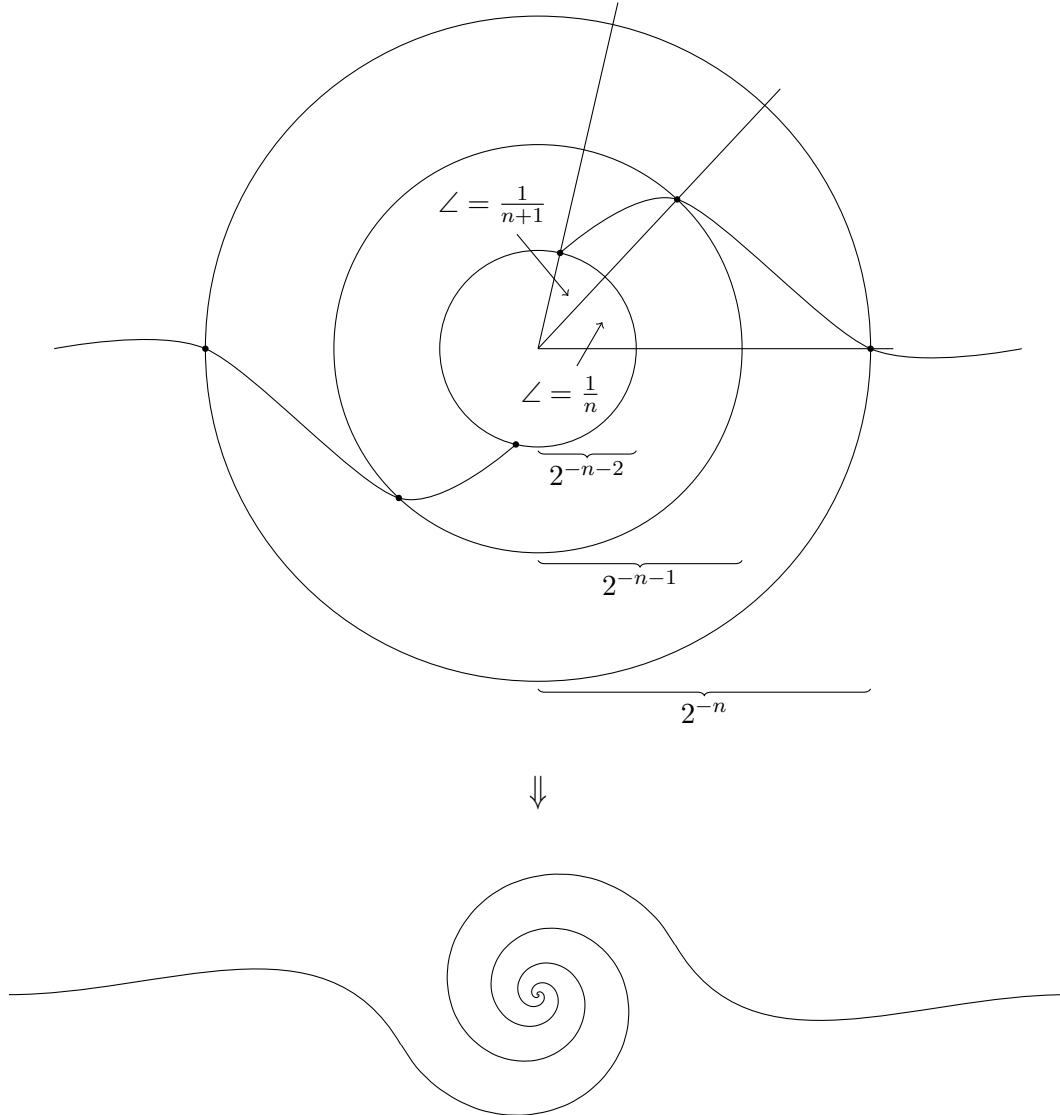
FIGURE 25. A rectifiable curve (it smooth except for at the origin) with non-unique tangent planes at the origin. We construct the spiral as illustrated on the top. By construction it certainly has finite length. However, because the harmonic series diverges, for any angle we may always find a sequence scales tending to zero so that "most" of the curve is in that direction, which thus allows us to find a tangent cone in any direction. Of course, this curve is not mass-minimizing, so it is not a counterexample to the possibility that mass-minimizing chains have unique tangent cones.

Example 8.9. *If A is a m-dimensional mass-minimizing, locally flat chain in $\mathbb{R}^{m+1}$ with the coefficient group $G = \mathbb{Z}/2\mathbb{Z}$, then for any ball B with $B \cap \operatorname{supp} \partial A$, we have that*

$$(8.3) \qquad\qquad \mathbb{M}(A \llcorner B) \leq \frac{1}{2}\mathcal{H}^m(\partial B).$$

which certainly implies that $\Theta(A, \infty) < \infty$. To show this, notice that $\partial(A \llcorner B)$ is a cycle in ∂B. Thus, $\partial(A \llcorner B) = \partial Q$ for Q (a m-chain) supported in ∂B (which is a m-sphere). Because we're working in $\mathbb{Z}/2\mathbb{Z}$, m-chains in ∂B are exactly (measurable) subsets of ∂B. Thus, Q and Q^c (the
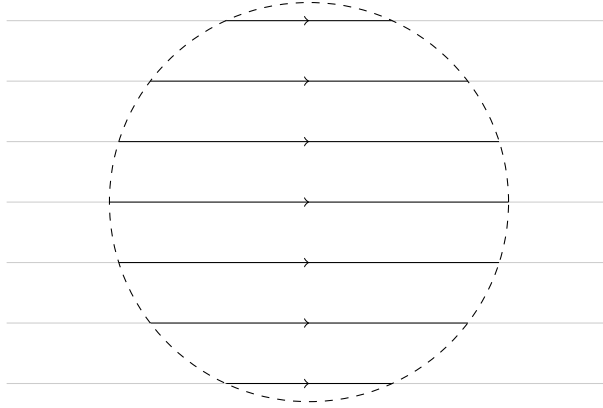
FIGURE 26. An example of a mass-minimizing locally flat chain A (for example, it is minimizing with coefficient group $(\mathbb{Z}, |\cdot|_{\text{std}})$) with $\Theta(A, 0, \infty) = \infty$.

complement in ∂B) of Q) are flat chains with $\partial Q = \partial Q^c = \partial(A \llcorner B)$. Thus, if we were to consider

$$A' := A - A \llcorner B + Q \quad and \quad A'' := A - A \llcorner B + Q^c,$$

then certainly A' and A'' are competitors for A mass-minimizing in some large ball. Thus, we see that because A', A'' must not have smaller mass in this big ball, we must have that

$$\mathbb{M}(A \llcorner B) \leq \min(\mathbb{M}(Q), \mathbb{M}(Q^c)).$$

However, because Q and Q^c are complementary subsets of ∂B, we thus see that at least one of them must have no more than $\frac{1}{2}\mathcal{H}^m(\partial B)$, establishing (8.3).

Now, we may discuss a splitting theorem for minimizing cones

Theorem 8.10. *For C a mass-minimizing cone (around 0)*

    (1) *The density $\Theta(C, x)$ attains its maximum at 0, and furthermore $\Theta(C, 0) = \Theta(C, \infty)$.*
    (2) *Letting $V := \{x : \Theta(C, x) = \Theta(C, \infty)\}$ be the set where $\Theta(C, x)$ is maximal, V is a linear subspace.*
    (3) *The cone C splits as $C = C' \times V$ where C' is a mass minimizing cone of dimension $\dim C - \dim V$ contained in $V^\perp$.*
    (4) *The cone C is translation invariant "along V."*

Proof. Notice that the monotonicity formula (in particular, Corollary 8.4) shows that

$$\Theta(C, x) \leq \Theta(C, x, r) \leq \Theta(C, \infty)$$

with equality if and only if x is a vertex of C. Because it is clear that $\Theta(C, 0) = \Theta(C, \infty)$, we see that $\Theta(C, x)$ attains its maximum at 0, proving (1).

To show (2), first notice that because C is invariant under dilations around 0, we have that for $\lambda > 0$

$$\Theta(C, x) = \Theta(C, \lambda x).$$

This implies that if $x \in V$, then so is the ray from 0 towards x. However, we could switch the role of x and 0 (because x is also a vertex of C, by Corollary 8.4), which shows that in fact the whole line through 0 and x is in V. More generally, if $x, y \in V$, we have that the whole line through them is contained in V. This shows that V is a linear subspace.

Notice that (3) and (4) are equivalent, so we will just prove the latter. For $p \in V$, note that C is invariant under dilations by λ around p, i.e.

$$x \mapsto p + \lambda(x - p).$$

However, it is also invariant under dilations around 0 by $\frac{1}{\lambda}$, i.e.

$$x \mapsto \frac{1}{\lambda} x.$$

Composing these, with $\lambda = \frac{1}{2}$ shows that C is invariant under the map

$$x \mapsto 2 \left(p + \frac{1}{2}(x - p) \right)) = x + p,$$

which is exactly the property that C is invariant under translations along V.   □

Theorem 8.11. *Suppose that we are in the situation of the previous theorem and $V \neq C$, i.e. C is not just a linear subspace. Then,*

$$\sup_{x \notin V} \Theta(C, x) < \Theta(C, \infty)$$

and in fact the supremum is attained.

Proof. For $x \in C \backslash V$, we can write $x = y + z$ for $y \in V$ and $0 \neq z \in V^{\perp}$. Translating by y, we see that $z \in C$ and $\Theta(C, x) = \Theta(C, z)$. Furthermore, because 0 is a cone point

$$\Theta(C, z) = \Theta \left(C, \frac{z}{|z|} \right).$$

Thus,

$$\sup_{x \notin V} \Theta(C, x) = \sup \{ \Theta(C, z) : z \in C \cap V^{\perp}, |z| = 1 \}.$$

Now, because $\Theta(C, z)$ is upper-semicontinuous and the set we are taking the supremum over is compact, the supremum is attained. Thus, by Theorem 8.10, the supremum must be strictly less than $\Theta(C, \infty)$.   □

Thanks to the last two theorems, we are motivated to define

Definition 8.12. *For C a minimizing cone, we denote the subspace $V = \{x : \Theta(C, x) = \Theta(C, \infty)\}$ as described above by* spine(C).

Two examples of minimizing cones.[6] (for particular coefficient groups) and their spines are given in Figure 27.

We now state (but will not prove) the following theorem, which illustrates the power of tangent cones.

Theorem 8.13. *(Almgren's Stratification Theorem) For A mass-minimizing, we define*

$$\mathscr{C}(x) = \{C : C \text{ is a tangent cone to } A \text{ at } x\},$$

for any point $x \in \operatorname{supp} A \backslash \operatorname{supp} \partial A$. Furthermore, we let $d(x) = \max\{\dim(\operatorname{spine}(C)) : C \in \mathscr{C}(x)\}$. Then

$$\{x \in \operatorname{supp}(A) \backslash \operatorname{supp} \partial A : d(x) \leq k\}$$

has Hausdorff dimension $\leq k$.

---

[6]That the tetrahedral cone on the right of Figure 27 mass minimizing for some coefficient groups is somewhat difficult to show. In fact, the first proof of this fact was that one could prove that for some coefficient groups, the cones in Figure 27 along with the plane were the only possibilities for mass-minimizing cones, and further it was possible to demonstrate mass-minimizing chains with tangent cones that could not be the plane or the join of three planes, as on the left of Figure 27.
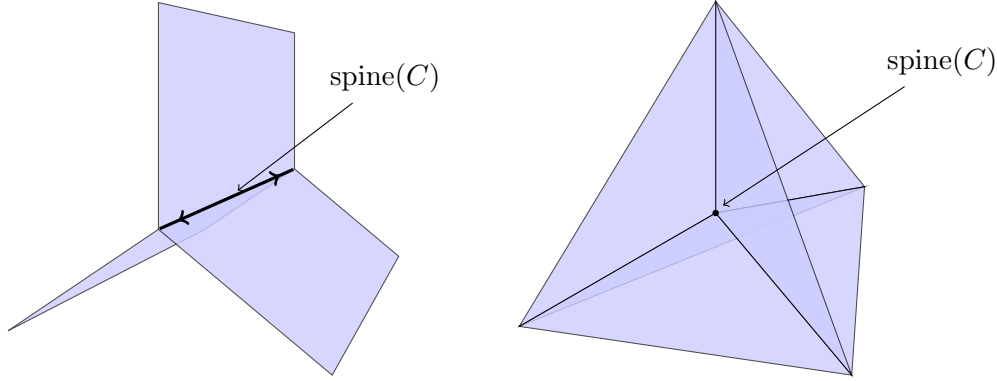
FIGURE 27. Nontrivial examples of minimal cones (we have only illustrated a finite part, but the cones actually extend radially) and their spines. The cone on the left, formed from three planes, is mass-minimizing over $(\mathbb{Z}/3\mathbb{Z}, |\cdot|_{\text{std}})$. The cone on the right, which is the cone over the 1-skeleton of a tetrahedron is minimizing over various groups, for example $\mathbb{Z}/4$ with the norm $|g| = 1$ for $g \neq 0$. To show this, consider the (locally) flat 3-chain formed by the sum of the connected components of the complement of the cone, each with a different element of $\mathbb{Z}/4\mathbb{Z}$. Then, the cone is the boundary of this chain, and it is possible to show that it is mass minimizing.

9. Interior Regularity of Minimizers

In this section, we discuss the regularity properties of mass-minimizing chains. Suppose that A is a minimizing m-chain in $\mathbb{R}^n$. We will call a point $x \in \operatorname{supp} A \setminus \operatorname{supp} \partial A$ an interior point of A.

Definition 9.1. *We will call an interior point x a regular point if x has a neighborhood U so that*

$$A \llcorner U = g[M]$$

for M, a m-dimensional properly embedded C^1-submanifold of U and $g \in G$.

Note that mass-minimizing actually implies (by elliptic regularity) that the above M will in fact be C^∞ and even real analytic. Trivially, the regular points form an open subset of the support. The following theorem is basically due (in a different form) to De Giorgi and then later independently by Reifenberg.

Theorem 9.2 (The Fundamental Regularity Theorem)**.** *Suppose that the coefficient group has the property that if $g \neq 0$, then $|g| \geq 1$. Then, there is $\epsilon > 0$ and $C < \infty$ (depending on the dimension and codimension) with the following property. Suppose that that A is a m-dimensional mass-minimizing chain so that the ball $B(0,1)$ is disjoint from $\operatorname{supp}(\partial A)$ and*

$$\mathscr{F}_{B(0,1)}(A - g[D]) < \epsilon$$

for $D = (m\text{-plane through } 0) \cap B(0,1)$ and an element $g \in G$ with $|g| = 1$. Then,

$$A \llcorner B(0, 1/2) = g[M],$$

where $M \subset \operatorname{graph}(f)$ for $f : D \to D^\perp$ with $\|f\|_{C^{1,\alpha}} \leq C$.

In particular, we have the following useful corollary about tangent planes:

Corollary 9.3. *With G as above (i.e. $|g| \geq 0$ for all nonzero g), if x is an interior point with a multiplicity g tangent plane with $|g| = 1$, then x is a regular point.*

Remark 9.4. *The hypothesis for Theorem 9.2 are not sharp. In fact, for a general group $(G, |\cdot|)$, the regularity theorem holds for $g \in G$ with $|g| = 1$ if and only if there is $\delta > 0$ so that for any $h \notin \{0, g\}$, then*

(9.1) $$|h| + |g - h| > |g| + \delta.$$

To see that this is necessary, suppose that (9.1) does not hold for g, for all $\delta > 0$. Thus, there is $h_i \neq 0$ so that

$$|h_i| + |g - h_i| \to |g|.$$

Now, it is possible to choose $\delta_i \to 0$ so that the flat chain, A, made up of two planes as illustrated in Figure 28 are mass minimizing (we must choose δ_i converging to zero slowly enough compared to $|h_i| + |g - h_i| - |g|$ so that the two lines have less mass than the other case in Figure 2). In this case, A is converging to a disk with multiplicity g. However, it is never a graph (it is two planes, not one), so the fundamental regularity theorem cannot hold in this case.
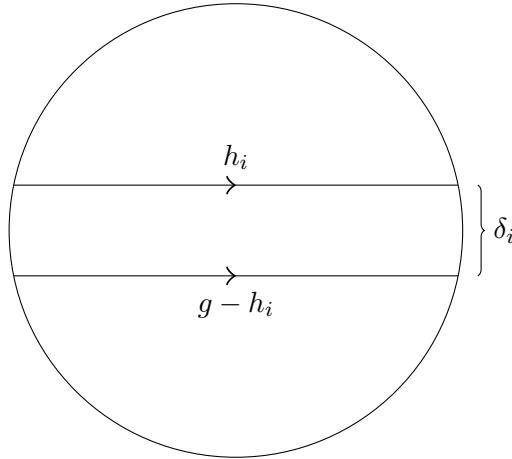

FIGURE 28. The flat chain A, as used in Remark 9.4. A is converging to a disk, but is never a graph over

In the case of hypersurfaces, the hypothesis of Corollary 9.3 can be dropped. Thus, we will state the following theorem for chains which are either "codimension 1" or "mod 2" (the latter hypothesis satisfy Theorem 9.2 and Corollary 9.3, so what is new about the following theorem is really the hypersurface statement).

Theorem 9.5. *If A is mass minimizing and either*

(1) *a codimension 1 integral current (i.e. over $(\mathbb{Z}, |\cdot|_{\text{std}})$), or*
(2) *is of any codimension, but is a chain over $\mathbb{Z}/2\mathbb{Z}$,*

then if any tangent cone at an interior point x is a plane, then x is a regular point.

Remark 9.6. *We remark that while the hypothesis of the previous theorem are certainly not sharp, some hypothesis are required. For example*

$$\left\{ (z^2, z^3) \subset \mathbb{C}^2 \simeq \mathbb{R}^4 : z \in \mathbb{Z} \right\}$$

is mass minimizing (because holomorphic varieties are calibrated) but the tangent cone at the singular point 0 is a multiplicity two plane.

9.1. **Dimension Reduction.** We will continue to assume that we are studying codimension 1 integral currents or mod 2 chains.

Theorem 9.7 (Federer's Dimension Reduction). *Suppose that A is a m-dimensional mass-minimizing chain (either a codimension 1 integral chain or a mod 2 chain) in $\mathbb{R}^{m+k}$ and x is an interior singular point. Let C be any tangent cone of x. Then, either*

(1) *C is singular and it is regular except along its spine, or*
(2) *there is a singular minimizing cone $C' \subset \mathbb{R}^{m+k}$ with $\Theta(C') < \Theta(C)$ and C' regular except for along its spine.*

Proof. By hypothesis, C is singular, so $\operatorname{spine}(C) \subset$ singular set(C). If $\operatorname{spine}(C) =$ singular set(C), we're done, so we may suppose that there is a singular point $p \in \operatorname{supp}(C) \setminus \operatorname{spine}(C)$. Let C' be a tangent cone to C at the point p. It must be that C' is a minimizing nonplanar cone with $\Theta(C') < \Theta(C)$. We claim that C' has a strictly larger spine that C. To show this, notice that translating along $\operatorname{spine}(C)$ must preserve C', but so does translation in the direction of p (and $p \notin \operatorname{spine}(C)$), so

$$\dim \operatorname{spine}(C') > \dim \operatorname{spine}(C).$$

Thus, we may repeat this process until it terminates (i.e., we cannot find such a p), in which case we are in case (1). □

Corollary 9.8. *For integral hypersurfaces, among all of the singular minimizing hypercones of dimension $\leq m$, the infimum of their density is attained by a cone which is regular except at 0. A similar theorem holds for general cones mod 2.*

Proof. By the compactness theorem, there is a convergent sequence of cones C_j tending toward a cone C with density equal to the minimum such density. If C was not singular, then we may apply the fundamental regularity theorem (Theorem 9.2) to conclude that the C_j must have been regular after some point. Thus C is a singular (mass-minimizing) hypercone achieving the minimum density among such cones.

It remains to show that we can find such a cone which is regular away from 0. By dimension reduction, we may assume that C is regular away from its spine, and as such can be written as $C' \times \mathbb{R}^p$ for some p, where C' is a minimizing hypercone regular except at its vertex, and thus clearly satisfies the conclusion of the theorem. □

Example 9.9. *In order give an example of dimension reduction (we use the coefficient group $\mathbb{Z}/3\mathbb{Z}$, which we have not included in the hypothesis of the above theorems, but the example should still be illuminating) we consider 2-dimensional mass-minimizing cones mod 3 in $\mathbb{R}^3$. It turns out that there are exactly two: the plane and the one illustrated in Figure 29.*

Theorem 9.10 (c.f. [IW10]). *If C is a singular area minimizing hypercone which is regular away from the vertex, and if C is "topologically nontrivial" (i.e. $\mathbb{R}^{m+1} \setminus C$ has a noncontractible component), then*

$$\Theta(C) > \sqrt{2}.$$

We remark that all known examples of minimizing singular cones satisfy the topologically nontrivial condition, so we have

Conjecture 9.11 (White). *This is true without the "topologically nontrivial" assumption.*

9.2. **Size of the Singular Set Over General Groups.** We first remark that for most groups, $(G, |\cdot|)$, there are 1-dimensional mass minimizing cones which are singular. We sketch the construction of such a cone. Choose $g_1, g_2, g_3 \in G$ so that
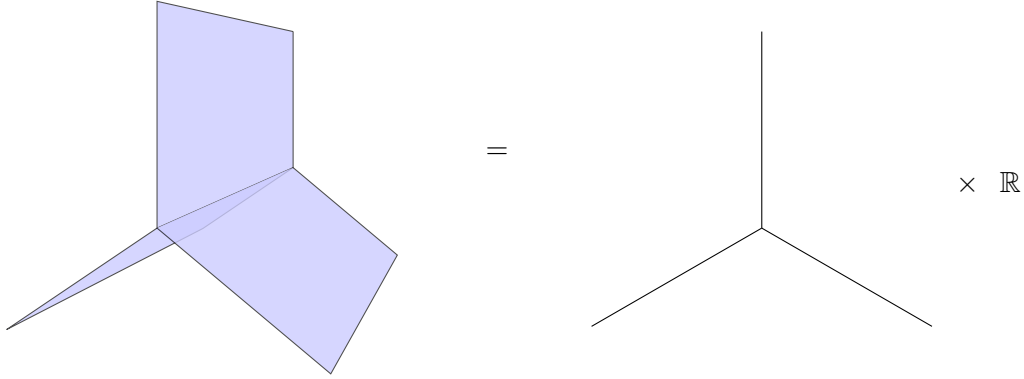
$$\sum_{i=1}^{3} g_i = 0.$$

FIGURE 29. The only singular minimizing 2-cone in $\mathbb{R}^3$ with $(\mathbb{Z}/3\mathbb{Z}, |\cdot|_{\text{std}})$ coefficients is given on the left. Notice that this cone is singular along its spine, which is one dimensional, so in fact it is the product of a minimizing singular 1-cone with $\mathbb{R}$.

Depending on the size of the $|g_i|$, it is always possible to choose angles α, β, γ and form the cone in Figure 30 so that the cone is minimal, in the sense that it is a critical point for area, under deformations. Now, when the group is $(\mathbb{Z}, |\cdot|_{\text{abs}})$, it turns out that one of the angles will always be
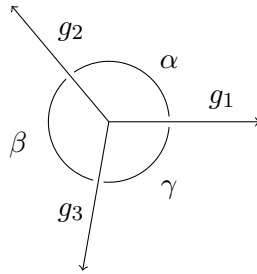

FIGURE 30. For most groups and elements g_i summing to zero, we can find angles α, β, γ so that the illustrated cone is area minimizing.

zero, so the cone will not be singular. However, if it is the case that

$$|g_1| < |g_2| + |g_3|$$
$$|g_2| < |g_1| + |g_3|$$
$$|g_3| < |g_1| + |g_2|,$$

then none of the angles will be zero. Furthermore, it is often possible to show that the cone is actually minimizing. Thus, over most groups, there are mass minimizing cones.

9.3. **Size of the Singular Set for Mod 2 Chains and Integral Hypersurfaces.** Integral hypersurfaces and mod 2 chains are two such (very important) settings where the above situation not happen, and in fact the singular set is "small." We will discuss the mod 2 case below, but everything will equally apply to integral hypersurfaces (besides of course, the results that are about higher codimension—this is discussed in the next section).

Theorem 9.12. *For a fixed codimension k, let $d(k)$ be the smallest m so that there is a m-dimensional mass minimizing mod 2 chain M in $\mathbb{R}^{m+k}$ with an interior singular point. Such an M has isolated (internal) singularities.*

Proof. Let M be such an example, and x an internal singular point. Let C be a tangent cone at x. By the basic regularity theorem (Theorem 9.2) we know that C cannot be a plane. We must have

that C is regular except at the vertex. If it has a larger singular set, then we may use the splitting theorem for cones (Theorem 8.10) to show that C splits a s the product of a (singular) cone and $\mathbb{R}^p$ for some $p > 0$, which contradicts our choice of $d(k)$. Thus, we have shown that there always exists an example of a mod 2 $d(k)$-chain in $\mathbb{R}^{d(k)+k}$ with isolated internal singuarlities, namely the cone C. However, we have claimed something stronger, namely that any such M has isolated internal singularities.

To see this, suppose that $x_i \in M$ are points converging to $0 \in M$ which is an interior singular point of M. Dilate M by $|x_i|^{-1}$ to give a chain M_i, and passing to a subsequence, we may assume that $M_i \to C$, where C is a tangent cone at 0. By the above argument, C is only singular at 0. Furthermore, notice that the rescalings of the x_i is

$$x_i' = \frac{x_i}{|x_i|}.$$

Passing to a further subsequence, we may assume that the x_i' converge to some point $x' \in C$. By the fundamental regularity theorem (Theorem 9.2), the x_i' must be regular points of M_i' for large i, because a small ball around them will be $\mathcal{F}$-close to a correspondingly small disk in the tangent plane of C at x'.

Now, suppose that the x_i' were singular points of M_i. We thus have that

$$
\begin{aligned}
1 &= \Theta(M', x') \\
&= \limsup_{i \to \infty} \Theta(M_i, x_i') \\
&\geq 1 + \rho.
\end{aligned}
$$

Here, $\rho > 0$ is some fixed constant, independent of i. The final inequality follows from Corollary 9.8, because we have assumed that the x_i' are singular points of M_i. However, this is a contradiction, which proves that x must have been isolated. $\qquad\square$

The following theorem is due to Federer, and follows from the stratification theorem (Theorem 8.13).

Theorem 9.13. *If M is a m-dimensional mass-minimizing mod 2 chain in $\mathbb{R}^{m+k}$, then the Hausdorff dimension[7] of the set*

$$\text{singular set}(M) \backslash \operatorname{spt}(\partial M)$$

is $\leq m - d(k)$, where $d(k)$ is as defined in Theorem 9.12.

It is a natural question to ask what the value of $d(k)$ is for small k. This is now well understood, and we have

Theorem 9.14. *We have that $d(1) = 7$ for mod 2 chains. This is a consequence of the following two facts*

  (1) *For hypersurfaces mod 2 of dimension < 7, there are no singularities, c.f. [Sim68].*
  (2) *There is a minimizing hypercone mod 2 in $\mathbb{R}^8$, c.f. [BDGG69].*

Idea of Proof. The proof of (1) consists of showing that any (nonplanar) singular cone with an isolated singularity is not stable (there is a one parameter family of diffeomorphisms under which the second derivative of mass is positive). To do this, Simons constructed explicit variations that achieved this goal. On the other hand, Bombieri-De Giorgi-Giusti were able to show that the cone

$$\{(x, y) \in \mathbb{R}^4 \times \mathbb{R}^4 = \mathbb{R}^8 : |x| = |y|\}$$

(and its higher dimensional variants) is mass minimizing. $\qquad\square$

On the other had for higher codimension, we have that

---

[7]We define $\dim^{\mathcal{H}}(\emptyset) = -\infty$, to make sense of the situation when $m < d(k)$.

Theorem 9.15. *For $k \geq 2$, we have that $d(k) = 2$ for mod 2 chains.*

Proof. First, we claim that $d(k) \geq 2$. Suppose that there was M a singular minimizing 1-chain. Taking the cone over a singular point, we have a singular minimizing 1-cone in $\mathbb{R}^2$. Such a cone must look like the one drawn in Figure 31, and we see that it cannot be minimizing.
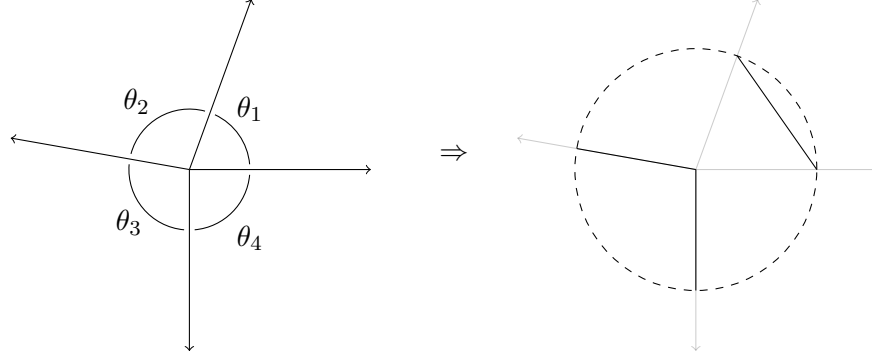

FIGURE 31. On the left is a hypothetical singular minimizing 1-cone mod 2 and on the right is a competitor which has smaller mass. Because we are working mod 2, there must be an even number of lines, so if the cone is singular, there must be at least 4. Because then, one of the angles must be $\leq \pi/2$, say θ_1, it is easy to check that the given competitor has less mass.

On the other hand, we claim that there is a codimension-2 singular minimal cone mod 2. In fact, we claim that the cone

$$\{(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2 = \mathbb{R}^4 : x = 0 \text{ or } y = 0\}$$

is minimizing. More generally, the union of any two "totally orthogonal" planes is minimizing. The following proof is due to Morgan. Let D_1, D_2 be the unit disks in the two planes. We claim that their union is mass minimizing, which is enough to prove that the cone is mass minimizing. Suppose that A is a 2-chain mod 2 with $\partial A = \partial D_1 + \partial D_2$. We will show that

$$\mathbb{M}(A) \geq \mathbb{M}(D_1) + \mathbb{M}(D_2).$$

Notice that, if we let Π_i be the projection onto the i-th plane, then

$$\begin{aligned}
\partial((\Pi_i)_\# A) &= (\Pi_i)_\#(\partial A) \\
&= (\Pi_i)_\#(\partial D_1) + (\Pi_i)_\#(\partial D_2) \\
&= \partial D_i.
\end{aligned}$$

However, $(\Pi_i)_\# A$ is a 2-chain lying entirely in the i-th plane, so it is a top chain in a plane with $\partial((\Pi_i)_\# A) = \partial D_i$, so in fact $((\Pi_i)_\# A) = D_i$. Because of this, we see that

$$\begin{aligned}
\mathbb{M}(D_1) + \mathbb{M}(D_2) &= \mathbb{M}((\Pi_1)_\# A) + \mathbb{M}((\Pi_2)_\# A) \\
&\leq \int |\operatorname{Jac} \Pi_1| d\mu_A + \int |\operatorname{Jac} \Pi_2| d\mu_A \\
&= \int (|\operatorname{Jac} \Pi_1| + |\operatorname{Jac} \Pi_2|) d\mu_A.
\end{aligned}$$

We claim that $|\operatorname{Jac} \Pi_1| + |\operatorname{Jac} \Pi_2| \leq 1$, which will finish the proof, because then we will show that

$$\mathbb{M}(D_1) + \mathbb{M}(D_2) \leq \mathbb{M}(A)$$

To prove the inequality about the Jacobians, first notice that we may assume that A is a 2-plane, because the quantity only depends on the tangent plane of A at the point (which exists a.e. in the above integral). Thus, we consider the Jacobians of the projection from a plane

$$P = \mathrm{graph}(L : \mathbb{R}^2 \to \mathbb{R}^2)$$

(if P is not a graph, then the inequality is trivial). By rotating (in both plane separately, so thus preserving the disks D_i seperately), we may thus assume that

$$L = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix},$$

so

$$P = \{(x, y, ax, by) : (x, y) \in \mathbb{R}^2\}.$$

Now, to compute the Jacobians of Π_i, we may simply compute the factor which the area of rectangle is scaled by under Π_i, because everything is linear. A rectangle on P is formed by the two vectors $v = (1, 0, a, 0)$ and $w = (0, 1, 0, b)$, and it has area $\sqrt{1 + a^2}\sqrt{1 + b^2}$. On the other hand, its projection to the 1-st plane is spanned by $v_1 = (1, 0), w_1 = (0, 1)$, with area 1. Thus

$$|\operatorname{Jac}\Pi_1| = \frac{1}{\sqrt{1 + a^2}\sqrt{1 + b^2}}.$$

On the other hand, the projection to the 2-nd plane is spanned by $v_2 = (a, 0)$ and $w_2 = (0, b)$, so it has area ab, so

$$|\operatorname{Jac}\Pi_2| = \frac{ab}{\sqrt{1 + a^2}\sqrt{1 + b^2}}.$$

Combined with the inequality $1 + ab \leq \sqrt{1 + a^2}\sqrt{1 + b^2}$, this finishes the proof. $\qquad\square$

Open Question 9.16. *Is there any other singular mass-minimizing 2-chains mod 2 in $\mathbb{R}^4$, which is not a trivial modification of this example? It is possible to show that the above cone is the only singular minimizing 2-cone in $\mathbb{R}^4$ (by studying the intersection with S^3), so the question is whether or not there is there a minimizing chain with a singularity which is modeled by the above cone, but is not exactly flat near the singularity.*

We remark that holomorphic varieties are not necessarily minimizing mod 2, contrary to the case when regarding them as integral chains.

Finally, we note that for $k > 1$, the cone (the cone over the Clifford torus)

(9.2) $$\{(x, y) \in \mathbb{R}^k \times \mathbb{R}^k : |x| = |y|\}$$

is a minimal (i.e. the a critical point of mass under any deformation) hypercone in $\mathbb{R}^{2k}$.

Open Question 9.17. *For $k > 2$, does the cone in (9.2) have least density among all minimal singular integral hypercones in $\mathbb{R}^{2k}$?*

In $k = 2$ this is known as the Willmore conjecture, and has recently been proven by Marques-Neves in [MN12].

9.4. **Higher Codimension Integral Chains.** We remark that the study of integral chains with higher codimension is a much harder subject. Almgren proved (c.f. [Alm00]) that m-dimensional mass minimizing chains (he actually worked in the context of currents) have singular sets of codimension at least 2. The proof is exceedingly long and complicated.

Of course, we know that any holomorphic k-dimensional variety in $\mathbb{C}^n$ is mass-minimizing as a $2k$-dimensional integral chain in $\mathbb{R}^{2n}$. In particular, this shows that Almgren's result is sharp, because a singular complex variety has singularities of (real) codimension 2.

Open Question 9.18. *Are there any singular mass-minimizing integral chains in $\mathbb{R}^4$ that are not complex varieties (up to change of complex structure on $\mathbb{R}^4$)?*

For example, a pair of (intersecting) 2-dimensional (real dimension) planes are maximal if and only if they are holomorphic with respect to some complex structure.

Further directions concerning this question has to with branch points (points where a minimal surface fails to be immersed). For example, if $F : D \to \mathbb{R}^4$ is area minimizing among maps with the same boundary values, then it can have branch points, as shown by

$$F(z) = (z^2, z^3) \in \mathbb{C} \times \mathbb{C} \simeq \mathbb{R}^4$$

(this is certainly area minimizing because it is holomorphic).[8] Thus, in a similar vein, we may ask

Open Question 9.19. *Are there other (non-holomorphic) examples of branch points for minimal disks in $\mathbb{R}^4$?*

10. Boundary Regularity of Minimizers

In this section we will only consider the coefficient groups $(\mathbb{Z}, |\cdot|_{\text{std}})$ and $\mathbb{Z}/2\mathbb{Z}$. The following theorem is due to Allard. We will be slightly imprecise in our statement of the theorem, so as to convey the intuition.

Theorem 10.1 (Fundamental Boundary Regularity Theorem)**.** *Let M be a minimizing (actually, the theorem holds with only the assumption that M is minimal) m-chain (over either $\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z}$, as above) so that ∂M is a smooth multiplicity 1 manifold of dimension $m-1$.*

Suppose that in some ball $B(x, \delta)$ around $x \in \text{supp}(\partial M)$, M is weakly close (i.e. in the $\mathscr{F}$ norm) to a multiplicity one half plane intersected with the ball, $D = H \cap B(x, \delta)$, and ∂M is strongly close to the boundary of the half plane (i.e. is a $C^{1,\alpha}$ graph over $\partial H \cap B(x, \delta)$ of a function with small $C^{1,\alpha}$-norm). Then, M is strongly close (in the $C^{1,\alpha}$-sense) to the half plane, in some smaller ball.

We remark that the assumption that ∂M has multiplicity 1 is automatically satisfied in the mod 2 setting. In the integral hypersurface setting, we have the following stronger result

Theorem 10.2 (Hardt-Simon)**.** *In any dimension, if M a mass-minimizing integral hypersurface with ∂M regular of class $C^{1,\alpha}$ and multiplicity 1, then there are no boundary singularities, in the sense of Figure 32.*
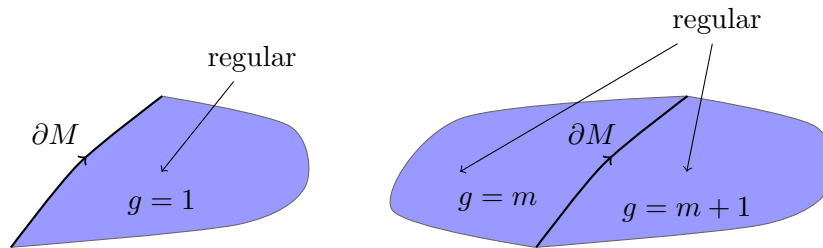

Figure 32. For M an integral hypersurface with boundary ∂M, we say that M has "no boundary singularities" if one of the two cases pictured occurs. The label "regular" means at least $C^{1,\alpha}$-smoothness, but will improve with stronger assumptions on the smoothness of ∂M.

To see that both possibilities in Figure 32 occur, we give a simple example in Figure 33. It is interesting to note that Theorem 10.2 is false mod 2. The best boundary regularity theorem in this case is

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[8]Note that if the target is $\mathbb{R}^3$ instead, there are no interior branch points. This was shown by Osserman in [Oss70] and studied further by Alt and Gulliver, c.f. [Alt72, Gul73].
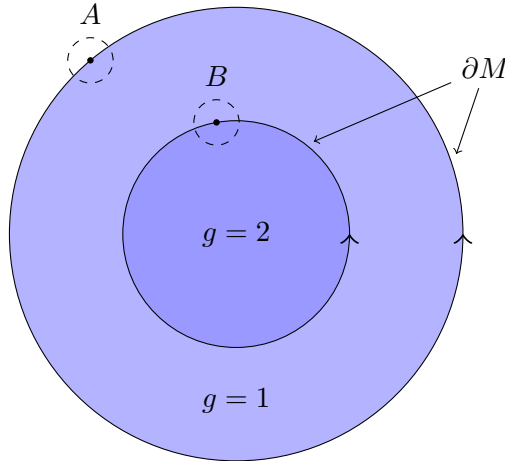
FIGURE 33. It is easy to see that the 2-chain M (with the given multiplicities) is the only chain in $\mathbb{R}^2$ which spans ∂M, and thus it must be minimizing. Furthermore, the balls marked A and B are respectively examples of the first and second kind of boundary regularity illustrated in Figure 32

Theorem 10.3. *For M a mass-minimizing hypersurface mod 2 in $\mathbb{R}^{m+1}$, if ∂M is a smooth $m-1$ submanifold, then the Hausdorff dimension of the boundary singularities is $\leq m-3$. Furthermore, if $m=3$, then the singularities are isolated.*

To see that this theorem is sharp, we give the following somewhat shocking example

Example 10.4. *First, suppose that M is a regular, compact 3-manifold with boundary. Then, it is clear that we may double M along its boundary, producing a smooth manifold $\mathrm{double}(M)$. Furthermore, it is easy to see (e.g. by considering a triangulation) that*

$$\mathcal{X}(\mathrm{double}(M)) = 2\mathcal{X}(M) - \mathcal{X}(\partial M),$$

where $\mathcal{X}(\cdot)$ is the Euler characteristic. In particular, because we may compute the Euler characteristic over $\mathbb{Z}/2\mathbb{Z}$ and apply Poincaré duality (even for non-orientable $\mathrm{double}(M)$) to show that any closed 3-manifold has zero Euler characteristic. Applying this to $\mathrm{double}(M)$ we have that

$$\mathcal{X}(M) = \frac{1}{2}\mathcal{X}(\partial M).$$

In particular, $\mathcal{X}(\partial M)$ must be even.

Now, we will apply this observation to show that Theorem 10.3 is sharp. Let Γ be a smooth, embedded real projective plane, $\mathbb{R}P^2$, in $\mathbb{R}^4$. We may solve Plateau's problem to find M a mass-minimizing chain mod 2 with $\partial M = \Gamma$. If M had no boundary singularities, then it would be a smoothly immersed submanifold, and we could thus show that $\mathcal{X}(\partial M) = \mathcal{X}(\Gamma) = 0$. However, $\mathcal{X}(\mathbb{R}P^2) = 1$, so M must have boundary singularities.

It is a natural question to ask what the cone C over boundary singularity in the previous example looks like. It can be shown that $C \cap S^3$ will be a complete, embedded, minimizing surface $\Sigma \subset S^3$ with $\partial\Sigma$ a great circle. Hardt-Simon have shown that such a surface cannot be orientable (even merely assuming minimality).

To construct a candidate for a surface $\Sigma \subset S^3$ which is minimizing and has $\partial\Sigma$ a great circle we pick a point a on the great circle in the $x_1 x_2$ plane and a point b on the great circle in the $x_3 x_4$ plane. We then form Σ as the surface swept out by the shortest geodesic between a and b as a goes around the circle and b goes around the other circle at twice the speed. This is illustrated in Figure 34. By construction it is clear that $\partial\Sigma$ is the great circle in the $x_1 x_2$ plane.
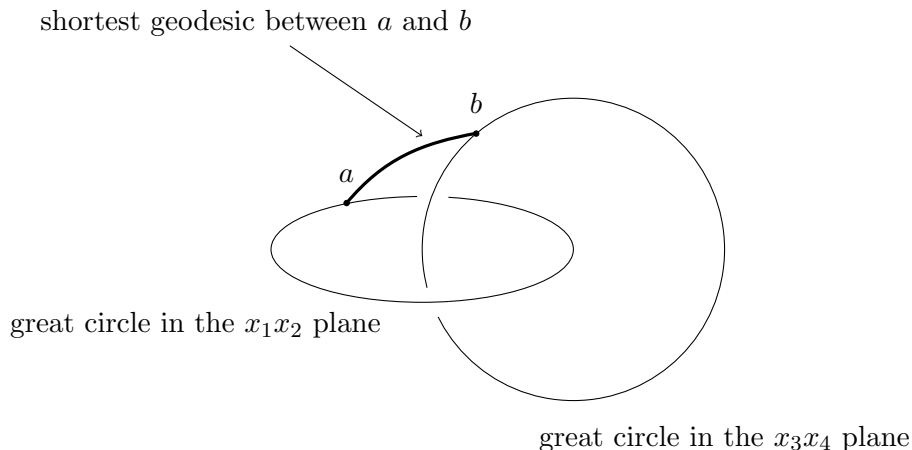
shortest geodesic between a and b

great circle in the $x_1 x_2$ plane

great circle in the $x_3 x_4$ plane

FIGURE 34. The surface Σ constructed in the text. It is a candidate for $C \cap S^3$ where C is a mass-minimizing cone arising from boundary singularities for minimizing mod 2 surfaces in $\mathbb{R}^4$, as constructed in Example 10.4.

Conjecture 10.5 (White). *The cone over Σ is minimizing mod 2.*

Rotating S^3 by $180°$ along one of the great circles yields a locally orientation reversing isometry from $\Sigma \to \Sigma$. This shows that Σ must be minimal, because if its mean curvature H is nonzero at a point, then applying this isometry gives $H = -H$, a contradiction. However, it is not yet known if Σ is minimizing.

APPENDIX A. PARTITIONING PROBLEMS

Here, we briefly describe a problem similar to Plateau's problem called the "partitioning problem" or more generally the "immiscible fluids problem." The partitioning problem is to divide a given domain Ω into a partition with prescribed volumes, as illustrated in Figure 35.
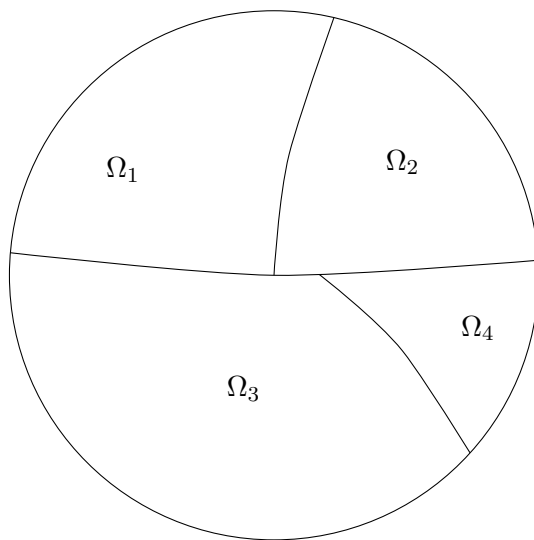

FIGURE 35. An allowable configuration for the partitioning problem.

More precisely, given elements $g_1, \ldots, g_n \in G$ with $|g_i| = 1$, we consider flat chains in Ω of the form $A = \sum g_i[\Omega_i]$, with the Ω_i disjoint subsets of Ω subject to the constraint area$(\Omega_i) = c_i$ (for

compatibility, we certainly require that $\sum c_i = \text{area}(\Omega)$).[9] Then, we'd like to minimize $\mathbb{M}_\Omega(\partial A)$ subject to the above constraint. As in Plateau's problem, compactness and lower-semicontinuity of mass combine to show the existence of a minimizer. Almgren was able to show that the minimizer of this volume constrained problem had many of the properties of a mass-minimizing chain.

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[9]More generally, the immiscible fluid problem allows $|g_i|$ to take values other than 1, which models interactions between fluids.