CURRENTS AND FLAT CHAINS ASSOCIATED TO VARIFOLDS, WITH AN APPLICATION TO MEAN CURVATURE FLOW

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Abstract

We prove under suitable hypotheses that convergence of integral varifolds implies convergence of associated mod 2 *flat chains and subsequential convergence of associated integer-multiplicity rectifiable currents. The convergence results imply restrictions on the kinds of singularities that can occur in mean curvature flow.*

1. Introduction

Let U be an open subset of $\mathbf{R}^N$. Let $\mathcal{L}_{m\text{-rec}}(U, \mathbf{Z}^+)$ denote the space of functions on U that take values in nonnegative integers, that are locally $\mathcal{L}^1$ with respect to Hausdorff m-dimensional measure on U, and that vanish except on a countable disjoint union of m-dimensional C^1-submanifolds of U. We identify functions that agree except on a set of Hausdorff m-dimensional measure zero. Let $\mathcal{L}_{m\text{-rec}}(U, \mathbf{Z}_2)$ be the corresponding space with the nonnegative integers $\mathbf{Z}^+$ replaced by $\mathbf{Z}_2$, the integers mod 2.

The space of m-dimensional integral varifolds in U is naturally isomorphic to $\mathcal{L}_{m\text{-rec}}(U, \mathbf{Z}^+)$: given any such varifold V, the corresponding function is the density function $\Theta(V, \cdot)$ given by

$$\Theta(V, x) = \lim_{r\to 0} \frac{\mu_V(\mathbf{B}(x, r))}{\omega_m r^m},$$

where μ_V is the radon measure on U determined by V and ω_n is the volume of the unit ball in $\mathbf{R}^m$. In particular, this limit exists and is a nonnegative integer for $\mathcal{H}^m$-almost every $x \in U$.

Similarly, the space of m-dimensional rectifiable mod 2 flat chains in U is naturally isomorphic to $\mathcal{L}_{m\text{-rec}}(U, \mathbf{Z}_2)$: given any such flat chain A, the corresponding function

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is the density function $\Theta(A, \cdot)$ given by

$$\Theta(A, x) = \lim_{r \to 0} \frac{\mu_A(\mathbf{B}(x, r))}{\omega_m r^m} = \lim_{r \to 0} \frac{M(A \cap \mathbf{B}(x, r))}{\omega_m r^m},$$

where μ_A is the radon measure on U determined by A. In particular, this limit exists and is 0 or 1 for $\mathcal{H}^m$-almost every $x \in U$.

The surjective homomorphism

$$[\cdot] : \mathbf{Z}^+ \to \mathbf{Z}_2,$$
$$k \mapsto [k]$$

determines a homomorphism from $\mathcal{L}_{m\text{-rec}}(U, \mathbf{Z}^+)$ to $\mathcal{L}_{m\text{-rec}}(U, \mathbf{Z}_2)$ and thus also a homomorphism from the additive semigroup of integral varifolds in U to the additive group of rectifiable mod 2 flat chains in U. If V is such a varifold, we let $[V]$ denote the corresponding rectifiable mod 2 flat chain. Thus $[V]$ is the unique rectifiable mod 2 flat chain in U such that

$$\Theta([V], x) = [\Theta(V, x)]$$

for $\mathcal{H}^m$-almost every $x \in U$.

Although in some ways integral varifolds and rectifiable mod 2 flat chains are similar, the notions of convergence are quite different. Typically (and throughout this article), convergence of varifolds means weak convergence as radon measures on $U \times \mathrm{G}_m(\mathbf{R}^N)$ (where $\mathrm{G}_m(\mathbf{R}^N)$ is the set of m-dimensional linear subspaces of $\mathbf{R}^N$), and convergence of flat chains means convergence with respect to the flat topology (see Section 4). A sequence $V(i)$ of integral varifolds may converge even though the associated flat chains $[V(i)]$ do not converge. Similarly, the flat chains $[V(i)]$ may converge even though the varifolds $V(i)$ do not. Furthermore, the $V(i)$ and $[V(i)]$ may converge to limits V and A, respectively, with $A \neq [V]$ (see Section 2.4 for examples).

This article identifies an important situation in which convergence of integral varifolds implies convergence of the corresponding mod 2 flat chains to the expected limit. In practice, one often proves existence of convergent sequences of integral varifolds by appealing to Allard's compactness theorem (described in Section 3). Here we prove that if a sequence of integral varifolds with limit V satisfies the hypotheses of Allard's compactness theorem plus one additional hypothesis, then the corresponding mod 2 flat chains converge to $[V]$.

THEOREM 1.1
Let $V(i)$ be a sequence of m-dimensional integral varifolds in an open set U of $\mathbf{R}^N$ that converges to a limit V. Suppose that

(1) *the $V(i)$ satisfy the hypotheses of Allard's compactness theorem for integral varifolds, and*

(2) *the boundaries $\partial[V(i)]$ of the mod 2 flat chains $[V(i)]$ converge in the flat topology.*

Then the chains $[V(i)]$ converge in the flat topology to $[V]$.

I do not know whether hypothesis (2) is really necessary.

There is an analogous theorem with rectifiable currents in place of mod 2 flat chains. Suppose that A is an m-dimensional integer-multiplicity rectifiable current in U and that V is an m-dimensional integral varifold in U. Recall that A determines an integral varifold $\mathbf{v}(A)$ by forgetting orientations (see [Si, Section 27]). We say that A and V are *compatible* provided that

$$V = \mathbf{v}(A) + 2W$$

for some integral varifold W in U. Thus A and V are compatible if and only if they determine the same mod 2 rectifiable chain. Equivalently, A and V are compatible provided that

$$\Theta(V, x) - \Theta(A, x)$$

is a nonnegative, even integer for $\mathcal{H}^m$-almost every $x \in U$.

The analog of Theorem 1.1 for integer-multiplicity currents is the following theorem.

THEOREM 1.2

Let $V(i)$ and $A(i)$ be sequences of m-dimensional integral varifolds and integer-multiplicity currents, respectively, in U, such that $V(i)$ and $A(i)$ are compatible for each i. Suppose that the $V(i)$ satisfy the hypotheses of Allard's compactness theorem for integral varifolds. Suppose also that the boundaries $\partial A(i)$ converge (in the integral flat topology) to a limit current. Then there is a subsequence $i(k)$ such that the $V(i(k))$ converge to an integral varifold V, the $A(i(k))$ converge to a limit integer-multiplicity current A, and A and V are compatible.

The existence of a subsequence for which the limits V and A exist follows immediately from Allard's compactness theorem for integral varifolds and from the Federer-Fleming compactness theorem for integer-multiplicity currents. What is new here is the compatibility of the limits A and V.

2. Preliminaries

2.1. Terminology

For mod 2 flat chains, see Fleming's original paper [Fl] or, for a different approach, Federer's book [Fe, Section 4.2.26]. Unfortunately (for the purposes of this article), a multiplicity [1] plane does not qualify as a mod 2 flat chain under either definition.* By contrast, a multiplicity 1 plane does qualify as an integral varifold. Thus, in order for the map $V \mapsto [V]$ (as described in Section 1) to be a homomorphism from integral varifolds to mod 2 flat chains, one must either restrict the class of varifolds or enlarge the class of flat chains.

If one prefers to restrict, then one should (throughout this particle) replace "varifold" with "compactly supported varifold" and "flat chain" with "compactly supported flat chain." (Federer's flat chains are automatically compactly supported, but Fleming's need not be.) Likewise, $\mathcal{L}_{m\text{-rec}}(U, \mathbf{Z}^+)$ and $\mathcal{L}_{m\text{-rec}}(U, \mathbf{Z}_2)$ should be replaced by the subsets consisting of compactly supported functions. In particular, the main theorem, Theorem 3.3, remains true with those replacements.

However, in this article we have chosen instead to enlarge the class of flat chains. Fortunately, only a slight modification in Fleming's definition (or Federer's) is required to produce the enlarged class of flat chains. (Flat chains so defined would, in the terminology of [Fe], be called "locally flat chains" However, although locally flat chains over the integers are briefly mentioned in [Fe, Section 4.1.24], the mod 2 versions are not.)

See Section 4 for the required modification.

When the coefficient group is the integers (with the standard metric), the "correct" class of flat chains is defined in [Si], and the rectifiability and compactness theorems are proved there.

2.2. Notation

Suppose that M is a Borel subset of a properly embedded m-dimensional C^1-submanifold of U, or of a countable union of such manifolds. If M has locally finite $\mathcal{H}^m$ measure, we let $[M]$ denote the mod 2 flat chain associated to M, and we let $\mathbf{v}(M)$ denote the integral varifold associated to M. More generally, if $f : M \to \mathbf{Z}^+$ is a function such that the extension

$$F : U \to \mathbf{Z}^+,$$

$$F(x) = \begin{cases} f(x) & \text{if } x \in M, \\ 0 & \text{if } x \in U \setminus M \end{cases}$$

*Federer's definition requires that a flat chain have compact support, and Fleming's definition requires that a flat chain have finite flat norm.

is in $\mathcal{L}_{m\text{-rec}}(U, \mathbf{Z}^+)$, then we let $\mathbf{v}(M, f)$ be the integral varifold in U corresponding to F.

2.3. Pushforwards

Suppose that V is an integral varifold in U and that $\phi : U \to W$ is a C^1-map that is proper on $U \cap \operatorname{spt}(\mu_V)$. Then the pushforward $\phi_\# V$ is also an integral varifold in W, and it satisfies

$$\Theta(\phi_\# V, y) = \sum_{\phi(x)=y} \Theta(V, x) \tag{1}$$

for $\mathcal{H}^m$-almost every $y \in W$.

Similarly, if A is a rectifiable mod 2 flat chain in U, and if $\phi : U \to W$ is locally Lipschitz and proper on $U \cap \operatorname{spt} \mu_A$, then the image chain $\phi_\# A$ satisfies

$$[\Theta(\phi_\# A, y)] = \sum_{\phi(x)=y} [\Theta(A, x)] \tag{2}$$

for $\mathcal{H}^m$-almost every $y \in W$.

Note that (2) determines $\Theta(\phi_\# A, y)$ for $\mathcal{H}^m$-almost every y since its value is 0 or 1 almost everywhere. In other words, for $\mathcal{H}^m$-almost every $y \in W$,

$$\Theta(\phi_\# A, y) = \begin{cases} 1 & \text{if } \sum_{\phi(x)=y} \Theta(A, y) \text{ is odd,} \\ 0 & \text{if the sum is even.} \end{cases} \tag{3}$$

Together (1) and (3) imply that

$$\phi_\#[V] = [\phi_\# V].$$

We need pushforwards only in the special cases where ϕ is a dilation or an affine projection.

2.4. Examples

Although they are not needed in this article, some examples illustating the differences between flat chain convergence and varifold convergence may be instructive.

First, consider a sequence of smooth, simple closed curves C_i lying in a compact region of $\mathbf{R}^2$ such that the lengths tend to infinity but the enclosed areas tend to zero. Let $V_i = \mathbf{v}(C_i)$ be the corresponding one-dimensional integral varifolds. Then the varifolds V_i do not converge, but the corresponding mod 2 flat chains $[V_i]$ converge to zero.

Next, let

$$J_n = \bigcup \left\{ \left[\frac{k}{2n}, \frac{k+1}{2n} \right] : k \text{ odd}, 0 < k < 2n \right\}, \tag{4}$$

and let

$$S_n = J_n \times \left\{ \frac{0, 1}{(2n)} \right\} \subset \mathbf{R}^2.$$

Thus S_n consists of $2n$ horizontal intervals, each of length $1/(2n)$. Let $V_n = \mathbf{v}(S_n)$ be the corresponding integral varifold. Then the V_n converge to $\mathbf{v}(I)$, where

$$I = \{(x, 0) : 0 \leq x \leq 1\}. \tag{5}$$

However, the corresponding mod 2 flat chains $[V_n]$ do not converge. To see this, suppose to the contrary that the $[V_n]$ converge to a limit chain T. Let $f, g : \mathbf{R}^2 \to \mathbf{R}$ be the projections given by $f(x, y) = x$ and $g(x, y) = x - y$. Then $f_\#[V_n] = 0$, and $g_\#[V_n] = [[0, 1]]$. Passing to the limit, we get

$$f_\# T = 0, \qquad g_\# T = [[0, 1]]. \tag{6}$$

However, T is clearly supported in I, and $f|I = g|I$, so $f_\# T = g_\# T$ (by (2)), contradicting (6). This proves that the $[V_n]$ do not converge.

For a final example, let

$$Q_n = J_n \times \left[0, \left(\frac{1}{n^2}\right)\right],$$

where J_n is given by (4). Thus Q_n is the union of n closed rectangles, each with base $1/(2n)$ and height $1/n^2$. Let V_n be the one-dimensional varifold associated to the set-theoretic boundary of Q_n: $V_n = \mathbf{v}(\partial Q_n)$. Then the V_n converge to $V = \mathbf{v}(I)$, where I is given by (5), but the flat chains $[V_n]$ converge to zero since the area of Q_n tends to zero. Thus the varifolds V_n converge to V and the chains $[V_n]$ converge to zero, but $[V] \neq 0$.

3. Proofs of the main results

Let $V(i)$ be a sequence of m-dimensional varifolds in an open subset U of $\mathbf{R}^N$. If the $V(i)$ converge to a varifold V, then of course

$$\limsup \mu_{V(i)} W < \infty \quad \text{for all } W \subset\subset U. \tag{7}$$

Conversely, if (7) holds, then the $V(i)$ have a convergent subsequence (by the compactness theorem for radon measures).

Definition 3.1

Suppose that $V(i)$, $i = 1, 2, 3, \ldots$, and V are m-dimensional varifolds in an open subset U of $\mathbf{R}^N$. In this article, we say that $V(i)$ converges with *locally bounded first* *variation* to V provided that $V(i) \to V$ as varifolds and

$$\limsup_{i\to\infty} \|\delta V(i)\|(W) < \infty \tag{8}$$

for every $W \subset\subset U$.

To understand the definition, the reader may find it helpful to recall that if V is the mutiplicity 1 varifold associated to a smooth, embedded manifold with boundary M, then

$$\|\delta V\|(W) = \mathcal{H}^{m-1}(W \cap \partial M) + \int_{M\cap W} |H(x)|\, d\mathcal{H}^m x,$$

where $H(x)$ is the mean curvature vector of M at x. Thus, for a sequence $V(i)$ of such integral varifolds, condition (8) means that the areas of the boundaries and the L^1-norms of the mean curvature are uniformly bounded on compact subsets of U (see [A] or [Si, Section 39] for the general definition of $\|\delta V\|$).

The following closure theorem of Allard is one of the key results in the theory of varifolds (see [A, Theorem 6.4] or [Si, Section 42.8]).

THEOREM 3.2
If $V(i)$ is a sequence of integral varifolds that converges with locally bounded first variation to V, then V is also an integral varifold.

Here we prove the following theorem.

THEOREM 3.3
Suppose that $V(i)$ is a sequence of integral varifolds that converges with locally bounded first variation to an integral varifold V. If the boundaries $\partial[V(i)]$ converge (as mod 2 flat chains) to a limit chain Γ, then

$$[V(i)] \to [V],$$

and therefore $\partial[V] = \Gamma$.

The last assertion ($\partial[V] = \Gamma$) follows because the boundary operator is continuous with respect to flat convergence.

The result is already interesting in the case where $\partial[V(i)] = 0$ for all i.

Proof
Since V is rectifiable, there is a countable union $\bigcup \mathcal{M}$ of m-dimensional C^1 embedded manifolds such that

$$\mu_V\Big(U \setminus \bigcup \mathcal{M}\Big) = 0$$

Without loss of generality, we may assume that the manifolds in $\mathcal{M}$ are disjoint.

By the compactness theorem for flat chains of locally finite mass (see Theorem 4), a subsequence of the $[V(i)]$ converges to such a flat chain A. (Here and throughout the proof, "flat chain" means "mod 2 flat chain.") Using the rectifiability theorem (see Theorem 4), we can conclude that A is rectifiable.

We remark that one may prove rectifiability of A directly (without invoking Theorem 4). One sees this as follows. By the lower-semicontinuity of mass with respect to flat convergence, the inequality

$$\mu_{[V(i)]} \le \mu_{V(i)}$$

implies that

$$\mu_A \le \mu_V \tag{9}$$

and therefore that

$$\mu_A\Big(\mathbf{R}^N \setminus \bigcup \mathcal{M}\Big) \le \mu_V\Big(\mathbf{R}^N \setminus \bigcup \mathcal{M}\Big) = 0. \tag{10}$$

Hence A is rectifiable.

To show that $A = [V]$, it suffices by (9) to show that

$$\Theta(\mu_V, x) - \Theta(\mu_A, x) \quad \text{is an even integer} \tag{11}$$

for μ_V-almost every $x \in U$. By (10), it suffices to show that (11) holds for μ_V-almost every $x \in \cup\mathcal{M}$.

For $\mathcal{H}^m$-almost $x \in \bigcup \mathcal{M}$ (and therefore in particular for μ_V-almost every $x \in \bigcup \mathcal{M}$), we have

$$\begin{aligned} \eta_{x,\lambda\#} V &\to \Theta(V, x)\mathbf{v}(P), \\ \eta_{x,\lambda\#} A &\to \Theta(A, x)[P] \end{aligned} \tag{12}$$

as $\lambda \to 0$, where P is the tangent plane at x to the unique $M \in \mathcal{M}$ that contains x. Here $\eta_{x,\lambda} : \mathbf{R}^N \to \mathbf{R}^N$ is translation by $-x$ followed by dilation by $1/\lambda$:

$$\eta_{x,\lambda}(y) = \frac{1}{\lambda}(y - x).$$

The proof of [Si, Lemma 42.9] shows that μ_V-almost every x has an additional property, namely,

$$\liminf_i \|\delta V(i)\| \mathbf{B}(x, r) \leq cr^m \quad \text{for all } r \in (0, 1), \tag{13}$$

where $c = c(x) < \infty$.

We complete the proof by showing that if x has properties (12) and (13), then $\Theta(V, x)$ and $\Theta(A, x)$ differ by an even integer.

For each fixed λ,

$$\begin{aligned} \eta_{x,\lambda\#} V(i) &\to \eta_{x,\lambda\#} V, \\ \eta_{x,\lambda\#} A(i) &\to \eta_{x,\lambda\#} A. \end{aligned} \tag{14}$$

Thus a standard diagonal argument (applied to (12) and (14)) shows that there is a sequence $N(i) \to \infty$ such that if $n(i) \geq N(i)$ for all i and if

$$\tilde{V}(i) = \eta_{x,1/i} V\big(n(i)\big),$$

then

$$\begin{aligned} \tilde{V}(i) &\to \Theta(V, x)\mathbf{v}(P), \\ [\tilde{V}(i)] &\to \Theta(A, x)[P]. \end{aligned} \tag{15}$$

Choose $r_i \to 0$ so that

$$R_i = i r_i \to \infty.$$
$$i^{m-1} r_i^m \to 0$$

(e.g., one can let $r_i = i^{-\alpha}$ where $(m-1)/m < \alpha < 1$.) By (13), we can choose $n(i) \geq N(i)$ so that

$$\big\|\delta V(n(i))\big\| \mathbf{B}(0, r_i) < cr_i^m.$$

Thus

$$\begin{aligned} \big\|\delta \tilde{V}(i)\big\| \mathbf{B}(0, R_i) &= i^{m-1} \big\|\delta V(n(i))\big\| \mathbf{B}(0, r_i) \\ &< c i^{m-1} r_i^m \end{aligned}$$

which tends to 0 as $i \to \infty$ by choice of r_i. Consequently,

$$\|\delta \tilde{V}(i)\| \to 0 \tag{16}$$

as radon measure.

Thus we are done if we can show that (15) and (16) imply that $\Theta(V, x) - \Theta(A, x)$ is an even integer. That is, we have reduced the theorem to the special case described in the following lemma.

LEMMA 3.4
Suppose that
(i) *a sequence $V(i)$ of integral varifolds converges to the varifold $V = n\mathbf{v}(P)$, where n is a nonnegative integer and P is an m-dimensional linear subspace of $\mathbf{R}^N$;*
(ii) *the radon measures $\|\delta V(i)\|$ converge to zero;*
(iii) *the associated mod 2 flat chains $[V(i)]$ converge to $A = a[P]$, where $a \in \mathbf{Z}_2$.*
Then $a = [n]$.

Proof
We may assume that $P = \mathbf{R}^m \times (0)^{N-m} \subset \mathbf{R}^N$. Let

$$\pi : \mathbf{R}^N \cong \mathbf{R}^m \times \mathbf{R}^{N-m} \to \mathbf{R}^m \tag{17}$$

be the orthogonal projection map.

Hypothesis (iii) implies that for almost every $R > 0$,

$$[V(i)] \llcorner \mathbf{B}^N(0, R) \to a[P] \cap \mathbf{B}^N(0, R) = a[P \cap \mathbf{B}^N(0, R)]. \tag{18}$$

We can assume that this is the case for $R = 1$. (Otherwise, dilate by $1/R$.) We write $\mathbf{B}$ for $\mathbf{B}^N(0, R) = \mathbf{B}^N(0, 1)$.

Let $W(i) = V(i) \llcorner \mathbf{B}$. By (18),

$$[\pi_\# W(i)] = \pi_\# [W(i)] \to a[\mathbf{B}^m], \tag{19}$$

where $\mathbf{B}^m = \mathbf{B}^m(0, 1)$. Also,

$$W(i) \to V \llcorner \mathbf{B} = n\mathbf{v}(P \cap \mathbf{B}),$$

and therefore

$$\pi_\# W(i) \to n\mathbf{v}(\mathbf{B}^m). \tag{20}$$

Note that

$$\pi_\# W(i) = \mathbf{v}(\mathbf{B}^m, \theta_i), \tag{21}$$

where

$$\theta_i(x) = \sum_{y \in \mathbf{B} \cap \pi^{-1} x} \Theta\big(W(i), y\big).$$

From hypotheses (i) and (ii), it follows that

$$\mathcal{L}^m Q_i \to 0, \tag{22}$$

where

$$Q_i = \{x \in \mathbf{B}^m : \theta_i(x) \neq n\}. \tag{23}$$

(This is a very nontrivial fact. Indeed, it is a key part of the proof given in [Si] of the closure theorem for integral varifolds. See Remark 3.5 for a more detailed discussion.)

Now

$$[\pi_\# W(i)] = \big[\{x \in \mathbf{B}^m : \theta_i(x) \text{ is odd}\}\big].$$

Thus

$$[\pi_\# W(i)] - [n\mathbf{v}(\mathbf{B}^m)] = \big[\{x \in \mathbf{B}^m : \theta_i(x) - n \text{ is odd}\}\big],$$

and so (by (22) and (23))

$$M\big([\pi_\# W(i)] - [n\mathbf{v}(\mathbf{B}^m)]\big) \le \mathcal{L}^m(Q_i) \to 0.$$

Consequently,

$$[\pi_\# W(i)] \to [n\mathbf{v}(\mathbf{B}^m)].$$

This together with (19) implies that $a[\mathbf{B}^m] = [n\mathbf{v}(\mathbf{B}^m)]$ and thus that $a = [n]$. □

This completes the proof of Theorem 3.3. □

Remark 3.5

Here we elaborate on statement (22) of the proof above, because it may not be immediately apparent to one who reads [Si] that the lemma we cite, [Si, Lemma 42.9], does actually justify that step. Note that

$$\int_{\mathbf{B}^m} \theta_i \to n\mathcal{L}^m(\mathbf{B}^m) \tag{24}$$

by (20) and (21). Let $\epsilon > 0$. Write

$$\theta_i(x) = F_{i,\epsilon}(x) + G_{i,\epsilon}(x), \tag{25}$$

where

$$F_{i,\epsilon}(x) = \sum \big\{\Theta(W(i), y) : y \in \mathbf{B} \cap \pi^{-1}(x),\ |y| < \epsilon\big\}$$

and

$$G_{i,\epsilon}(x) = \sum \left\{ \Theta(W(i), y) : y \in \mathbf{B} \cap \pi^{-1}(x),\ |y| \geq \epsilon \right\}.$$

Now

$$\int G_{i,\epsilon} \to 0 \tag{26}$$

since $W(i) \to n\mathbf{v}(P)$. This together with (24) implies that

$$\int F_{i,\epsilon} \to n\mathcal{L}^m(\mathbf{B}^m). \tag{27}$$

According to [Si, Lemma 42.9],

$$\limsup_{i\to\infty} \int_{\mathbf{B}^m} (F_{i,\epsilon} - n)^+ \, d\mathcal{L}^m \leq \omega(\epsilon) \tag{28}$$

for some function $\omega(\cdot)$ such that $\omega(\epsilon) \to 0$ as $\epsilon \to 0$.

(Note that there is a mistake in the statement of [Si, Lemma 42.9]; instead of (28), it asserts the weaker inequality

$$\limsup_{i\to\infty} \mathcal{L}^m \left\{ x \in \mathbf{B}^m : F_{i,\epsilon}(x) > n \right\} \leq \omega(\epsilon).$$

However, the proof of [Si, Lemma 42.9] establishes the stronger statement (28). Indeed, the stronger statement is essential in the proof of Allard's integrality theorem (see [Si, Section 42.8]). In particular, the stronger statement is used in line (8) of that proof.)

From (27) and (28), we see that

$$\limsup_{i\to\infty} \int_{\mathbf{B}^m} |F_{i,\epsilon} - n| \, d\mathcal{L}^m \leq 2\,\omega(\epsilon).$$

This together with (26) and (25) implies that

$$\limsup_{i\to\infty} \int_{\mathbf{B}^m} |\theta_i - n| \, d\mathcal{L}^m \leq 2\,\omega(\epsilon). \tag{29}$$

Letting $\epsilon \to 0$ gives

$$\limsup_{i\to\infty} \int_{\mathbf{B}^m} |\theta_i - n| \, d\mathcal{L}^m = 0,$$

and thus (since θ_i is integer valued)

$$\lim_{i\to\infty} \mathcal{L}^m \{ x \in \mathbf{B}^m : \theta_i(x) \neq n \} = 0.$$

THEOREM 3.6

Suppose that $V(i)$ is a sequence of integral varifolds that converge with locally bounded first variation to an integral varifold V. Suppose that $A(i)$ is a sequence of integer-multiplicity rectifiable currents such that $V(i)$ and $A(i)$ are compatible. If the boundaries $\partial A(i)$ converge (in the integral flat topology) to a limit integral flat chain Γ, then there is a subsequence $i(k)$ such that the $A(i(k))$ converge to an integer-multiplicity rectifiable current A. Furthermore, V and A must then be compatible, and ∂A must equal Γ.

The proof is exactly analogous to the proof of Theorem 3.3. Alternatively, one can argue as follows. The existence of a subsequence $A(i(k))$ that converges to an integer-multiplicity rectifiable current A follows from the compactness theorem for such currents (see Theorems 4 and 4). The "furthermore" statement then follows immediately from Theorem 3.3, together with the observation that an integral varifold and an integer-multiplicity rectifiable current are compatible if and only if they determine the same mod 2 rectifiable flat chain.

4. Application to mean curvature flow

Here we show how the results of this article rule out certain kinds of singularities in mean curvature flows. In another article, we will use similar arguments to prove, under mild hypotheses, boundary regularity at all times for hypersurfaces moving by mean curvature.

On both theoretical and experimental grounds, grain boundaries in certain annealing metals are believed to move by mean curvature flow (see [B, Appendix A]). In such metals, one typically sees triple junctions where three smooth surfaces come together at equal angles along a smooth curve. Of course one also sees such triple junctions in soap films, which are equilibrium solutions to mean curvature flow.

Consider the following question. Can an initially smooth surface evolve under mean curvature flow so as later to develop triple junction type singularities? More generally, can such a surface have as a blow-up flow (i.e., a limit of parabolic blow-ups) a static configuration of k half-planes (counting multiplicity) meeting along a common edge? Using Theorem 3.3, we can (for a suitable formulation of mean curvature flow) prove that the answer is "no" if k is odd.

Suppose that $\mathcal{M}$ is a Brakke flow, X_i is a sequence of spacetime points converging to $X = (x, t)$ with $t > 0$, and λ_i is a sequence of numbers tending to infinity. Translate $\mathcal{M}$ in spacetime by $-X_i$ and then dilate parabolically by λ_i to get a flow $\mathcal{M}_i$. A *blow-up flow* of $\mathcal{M}$ is any Brakke flow that can be obtained as a subsequential limit of such a sequence.

Let $I \subset \mathbf{R}$ be an interval, typically either $[0, \infty)$ or all of $\mathbf{R}$. Recall that a Brakke flow $t \in I \mapsto V(t)$ of varifolds is called an *integral Brakke flow* provided that $V(t)$ is an integral varifold for almost all $t \in I$ (see [B, Section 3] or [I, Section 6] for the definition of Brakke flow).

Definition 4.1
Let $t \mapsto V(t)$, $t \in I$, be an integral Brakke flow in $U \subset \mathbf{R}^N$. We say that $V(\cdot)$ is *cyclic mod* 2 (or *cyclic* for short) provided that $\partial[V(t)] = 0$ for almost every $t \in I$.

More generally, suppose W is an open subset of U and J is a subinterval of I. We say that the Brakke flow $V(\cdot)$ is *cyclic mod* 2 *in* $W \times J$ if for almost all $t \in J$, $[V(t)]$ has no boundary in W.

We have the following.

THEOREM 4.2
Suppose $t \mapsto V_i(t)$ is a sequence of integral Brakke flows that converge as Brakke flows to an integral Brakke flow $t \mapsto V(t)$. If the flows $V_i(\cdot)$ are cyclic mod 2, then so is the flow $V(\cdot)$. If the flows $V_i(\cdot)$ are cyclic mod 2 in $W \times J$, then so is the flow $V(\cdot)$.

Here convergence as Brakke flows means that for almost all t,

$$\mu_{V_i(t)} \to \mu_{V(t)}, \tag{30}$$

and

$$\text{there is a subsequence } i(k) \text{ (depending on } t) \text{ such that } V_{i(k)}(t) \to V(t). \tag{31}$$

(This definition may seem peculiar, but this is precisely the convergence that occurs in Ilmanen's compactness theorem for integral Brakke flows [I, Section 7].)

Theorem 4.2 follows immediately from Theorem 3.3 and the following lemma (which is implicit in [I] but is not actually stated there).

LEMMA 4.3
Suppose that $t \in I \mapsto V_i(t)$ is a sequence of Brakke flows in $U \subset \mathbf{R}^N$ that converges to a Brakke flow $t \mapsto V(t)$. Then, for almost every $t \in I$, there is a subsequence $i(k)$ such that $V_{i(k)}(t)$ converges with locally bounded first variation to $V(t)$. Indeed, we can choose the subsequence so that $\delta V_{i(k)}$ is absolutely continuous with respect to $\mu_{V_{i(k)}}$ and so that

$$\sup_{i(k)} \int_{x \in W} \left| H(V_{i(k)}(t), x) \right|^2 d\mu_{V_{i(k)}(t)} x < \infty$$

for every $W \subset\subset U$, where $H(V_{i(k)}(t), \cdot)$ is the generalized mean curvature of $V_{i(k)}(t)$.

Proof
For simplicity, let us assume that $I = [0, \infty)$.

Recall that for almost all t, the varifold $V_i(t)$ has bounded first variation, and the singular part of the first variation measure is zero. Thus (for such t)

$$\|\delta V_i(t)\|(W) = \int_W |H_{i,t}|\, d\mu_{i,t} \le \Big(\int_W |H_{i,t}|^2\, d\mu_{i,t} \Big)^{1/2} \big(\mu_{i,t}(W)\big)^{1/2}, \tag{32}$$

where $H_{i,t}$ is the generalized mean curvature of $V_i(t)$ and $\mu_{i,t} = \mu_{V_i(t)}$.

Consider first the case where the varifolds $V_i(t)$ are all supported in some compact set. Then the initial total masses $\mathrm{M}(V_i(0)) = \mu_{V_i(0)}(\mathbf{R}^N)$ are bounded above by some $C < \infty$. Since mass decreases under mean curvature flow, the same bound holds for all $t > 0$. By definition of Brakke flow,

$$\overline{D}_t\, \mathrm{M}\big(V_i(t)\big) \le - \int \big|H(V_i(t), \cdot)\big|^2\, d\mu_{V_i(t)},$$

so

$$\int_{t\in I} \int \big|H(V_i(t), \cdot)\big|^2\, d\mu_{V_i(t)}\, dt \le C. \tag{33}$$

Thus by Fatou's theorem,

$$\int_{t\in I} \Big(\liminf_i \int \big|H(V_i(t), \cdot)\big|^2\, d\mu_{V_i(t)} \Big)\, dt \le C.$$

In particular,

$$\liminf_i \int \big|H(V_i(t), \cdot)\big|^2\, d\mu_{V_i(t)} < \infty$$

for almost every t. For each such t, there is a subsequence $i(k)$ such that

$$\sup_k \int \big|H(V_{i(k)}(t), \cdot)\big|^2\, d\mu_{V_{i(k)}(t)} < \infty.$$

This together with (32) implies that the $V_{i(k)}(t)$ converge with locally bounded first variation to $V(t)$ (in the sense of Definition 3.1).

The general case (noncompactly supported varifolds) is essentially the same, except that instead of (33) one uses the local bound

$$\sup_i \int_{t\in J} \int_{x\in W} \big|H(V_i(t), x)\big|^2\, d\mu_{V_i(t)}\, dt < \infty$$

together with the mass bound

$$\sup_i \sup_{t \in J} \mu_{V_i(t)}(W) < \infty,$$

both of which hold for all intervals $J \subset\subset I$ and open subsets $W \subset\subset U$ (see [E, Proposition 4.9]). □

Remark 4.4
The lemma and Allard's closure theorem (Theorem 3.2) imply that a limit of integral Brakke flows is also integral.

COROLLARY 4.5
Suppose that k is an odd integer. A static configuration of k half-planes (counting multiplicity) meeting along a common edge cannot occur as a blow-up flow to an integral Brakke flow that is cyclic mod 2.

Proof
Let V be the varifold corresponding to k half-planes (counting multiplicity) meeting along an edge E. If the static flow $t \mapsto V$ is a limit flow to an integral Brakke flow that is cyclic mod 2, then this static flow is also cyclic mod 2, and thus $\partial[V] = 0$. But $\partial[V]$ is the common edge E with multiplicity $[k]$, so k must then be even. □

The following theorem shows that for rather arbitrary initial surfaces, there exist nontrivial integral Brakke flows that are cyclic mod 2.

THEOREM 4.6
Let A_0 be any compactly supported rectifiable mod 2 cycle in $\mathbf{R}^N$ (e.g., A_0 could be the mod 2 rectifiable flat chain associated to a C^1 compact, embedded submanifold). Then there are an integral Brakke flow $t \in [0, \infty) \mapsto V(t)$ and a one-parameter family $t \in [0, \infty) \mapsto A(t)$ of rectifiable mod 2 flat chains with the following properties:
(1) $A(0) = A_0$ *and* $\mu_{V(0)} = \mu_{A(0)}$*;*
(2) $\partial A(t) = 0$ *for all* t*;*
(3) $t \mapsto A(t)$ *is continuous with respect to the flat topology;*
(4) $\mu_{A(t)} \leq \mu_{V(t)}$ *for all* t*;*
(5) $A(t) = [V(t)]$ *for almost every* t*.*

In particular, the flow is cyclic mod 2, and thus triple (or, more generally, odd-multiplicity) junctions cannot occur in $V(\cdot)$ by Corollary 4.5.

In assertion (5), note that since $V(\cdot)$ is an integral Brakke flow, $V(t)$ is an integral varifold for almost all t, and thus $[V(t)]$ is well defined for almost all t.

Proof
Except for assertion (5), this theorem was proved by Ilmanen [I, Sections 8.1, 8.3]. He used integer-multiplicity currents rather than mod 2 flat chains, but his proof works equally well in either context. (The $A(t)$ here is the slice T_t in Ilmanen's notation.) The flat continuity (3) is not stated there, but it follows immediately from [I, Section 8.3].

Roughly speaking, Ilmanen constructs $V(\cdot)$ and $A(\cdot)$ as limits of nice examples $V_i(\cdot)$ and $A_i(\cdot)$ for which

$$\mu_i(t) = \mu_{A(i)}(t)$$

for all t.

Now his $A_i(t)$ are not quite cycles. However, $\partial A_i(t)$ moves by translation, and it moves very fast if i is large. In particular, if $U \subset\subset \mathbf{R}^N$ and $I \subset\subset (0, \infty)$, then for sufficiently large i and for all $t \in I$, $\partial A_i(t)$ lies outside U.

Thus (exactly as in the proof of Theorem 4.2, or by Remark 4.4 and Theorem 3.3), we deduce (for almost every $t \in I$) that $A(t) \llcorner U = [V(t)] \llcorner U$ and that $\partial [V(t)]$ lies outside U.

Since U is arbitrary, this gives (5). □

Remark 4.7
The description just given is a slightly simplified account of Ilmanen's proof. Actually he does not quite get the pair $(V(\cdot), A(\cdot))$ as limits of nice examples. Rather, he gets a pair of flows $(\mu^*(\cdot), A^*(\cdot))$ of one higher dimension as such a limit. The argument given above shows that $(\mu^*(\cdot), A^*(\cdot))$ has the property corresponding to property (5) in Theorem 4.6 (and Ilmanen in his proof shows that it has properties (1)–(4)). Now the pair $(\mu^*(\cdot), A^*(\cdot))$ is translation invariant in one spatial direction. By slicing, Ilmanen gets the desired pair $(\mu(\cdot), A(\cdot))$. Translational invariance implies (in a straightforward way) that properties (1)–(5) for $(\mu(\cdot), A(\cdot))$ are equivalent to the corresponding properties for $(\mu^*(\cdot), A^*(\cdot))$.

Theorem 4.6 has an analog for integer-multiplicity currents in place of mod 2 flat chains.

THEOREM 4.8
Let A_0 be any compactly supported integer-multiplicity cycle (i.e., integer-multiplicity current with $\partial A_0 = 0$.) Then there are an integral Brakke flow $t \in [0, \infty) \mapsto V(t)$

and a one-parameter family $t \in [0, \infty) \mapsto A(t)$ of integer-multiplicity currents with the following properties:

(1) $A(0) = A_0$ *and* $\mu_{V(0)} = \mu_{A(0)}$*;*

(2) $\partial A(t) = 0$ *for all* t*;*

(3) $t \mapsto A(t)$ *is continuous with respect to the flat topology;*

(4) $\mu_{A(t)} \le \mu_{V(t)}$ *for all* t*;*

(5) $A(t)$ *and* $V(t)$ *are compatible for almost every* t.

We omit the proof since it is almost identical to the proof of the mod 2 case, Theorem 4.6.

Note that if an integer-multiplicity current A is compatible with an integral varifold V, then $[V]$ is the flat chain mod 2 corresponding to A. It follows that the Brakke flow $V(\cdot)$ in Theorem 4.8 is cyclic mod 2. In particular, triple (or more generally odd-multiplicity) junctions cannot occur in $V(\cdot)$ by Corollary 4.5.

Ruling out even-multiplicity junctions is more subtle. In particular, limits of smooth Brakke flows can have quadruple junctions. For example, recall that Sherk constructed a complete, embedded, singly periodic minimal surface in $\mathbf{R}^3$, that is, away from the z-axis, asymptotic to the union of the planes $x = 0$ and $y = 0$. We may regard that surface as an equilibrium solution to mean curvature flow. Now dilate by $1/n$, and let $n \to \infty$. The limit surface is a pair of orthogonal planes and thus has a quadruple junction.

Appendix. Flat chains

Let G be a metric abelian coefficient group, that is, an abelian group with a translation-invariant metric $d(\cdot, \cdot)$. The norm $|g|$ of a group element g is defined to be its distance from zero. The groups relevant for this article are $\mathbf{Z}_2$ and $\mathbf{Z}$, both with the standard metrics. If U is an open subset of $\mathbf{R}^N$, let $\mathcal{F}_c(U; G)$ be the space of flat chains with coefficients in G and with compact support in U, as defined in [Fl]. We let $\mathcal{F}_{m,c}(U; G)$ denote the space of m-dimensional chains in $\mathcal{F}_c(U; G)$.

If W is an open subset of $\mathbf{R}^N$ and $A \in \mathcal{F}_c(\mathbf{R}^N; G)$, we let $\mathrm{M}_W(A)$ be the minimum of

$$\liminf \mu_{A(i)}(W) \tag{34}$$

among all sequences of compactly supported, finite-mass flat chains $A(i)$ such that $A(i)$ converges in the flat topology to A. By lower semicontinuity of mass, $\mathrm{M}_W(A) = \mu_A(W)$ for any chain A of finite mass.

We define the flat seminorm $\mathcal{F}_W$ by

$$\mathcal{F}_W(A) = \inf \left\{ \mathrm{M}_W(A - \partial Q) + \mathrm{M}_W(Q) \right\},$$

where the infimum is over all $Q \in \mathcal{F}_c(\mathbf{R}^N; G)$.

Let U be an open subset of $\mathbf{R}^N$. Choose a countable collection $\mathcal{W}$ of nested open sets whose union is U and each of whose closures is a compact subset of U. We define the space $\mathcal{F}_m(U;G)$ of flat m-chains in U with coefficients in G to be the completion of $\mathcal{F}_{m,c}(U;G)$ with respect to the seminorms $\mathcal{F}_W$ for $W \in \mathcal{W}$. (It is straightforward to show that the resulting space is independent of the choice of $\mathcal{W}$.)

By continuity, the seminorms $\mathcal{F}_W$ extend to all of $\mathcal{F}_m(U;G)$. We also define the mass seminorms M_W on all of $\mathcal{F}_m(U;G)$ exactly as above (34).

Convergence of flat chains means flat convergence, that is, convergence with respect to the seminorms $\mathcal{F}_W$ for all open $W \subset\subset U$ or, equivalently, for all $W \in \mathcal{W}$ for a collection $\mathcal{W}$ of nested open sets as above.

We define the support of a flat chain $A \in \mathcal{F}_m(U;G)$ as follows: $x \notin \operatorname{spt} A$ if and only if there are a sequence $A_i \in \mathcal{F}_{m,c}(U;G)$ and a ball $\mathbf{B}(x,r)$ such that $A_i \to A$ and such that $\operatorname{spt} A_i$ is disjoint from $\mathbf{B}(x,r)$ for every i.

In the proof of the main results, Theorems 3.3 and 3.6, we used the following version of the compactness theorem for flat chains. It is valid for any coefficient group G in which all sets of the form $\{g \in G : |g| \le r\})$ are compact. In particular, it is valid for the integers with the usual norm and for the integers mod 2.

THEOREM A.1 (Compactness theorem)
Let A_i be a sequence of flat m-chains in U such that the boundaries ∂A_i converge to a limit chain Γ and such that

$$\limsup_i \mathrm{M}_W(A_i) < \infty \tag{35}$$

for every open $W \subset\subset U$. Then A_i has a convergent subsequence.

We first prove the version for compact supports.

LEMMA A.2
If $A_i, \Gamma \in \mathcal{F}_c(\mathbf{R}^N;G)$ are supported in a fixed compact subset X of $\mathbf{R}^N$, if $\sup_i \mathrm{M}(A_i) < \infty$, and if $\mathcal{F}(\partial A_i - \Gamma) \to 0$, then there are a subsequence $A_{i(k)}$ and a chain A such that $\mathcal{F}(A_{i(k)} - A) \to 0$.

Proof
We may assume that X is convex. (Otherwise, replace it by its convex hull.) Since $\partial A_i \to \Gamma$, we have $\partial\Gamma = 0$. It follows that $\Gamma = \partial R$ for some chain R of finite mass.

By hypothesis,

$$\mathcal{F}(\partial A_i - \partial R) \to 0.$$

Thus there are chains Q_i such that

$$M(Q_i) + M(\partial Q_i + \partial A_i - \partial R) \to 0. \tag{36}$$

We may assume that R and the Q_i are supported in X. (Otherwise, map them into X by the nearest point retraction of $\mathbf{R}^N$ to X.)

Now, let

$$A_i^* = Q_i + A_i - R.$$

Note that

$$\limsup_i M(A_i^*) \le \sup_i M(A_i) + M(R) < \infty$$

since $M(Q_i) \to 0$ by (36). From (36) we also see that $M(\partial A_i^*) \to 0$, so in particular, $\sup_i M(\partial A_i^*) < \infty$.

Thus by the standard compactness theorem (see, e.g., [Fl, Corollary 7.5]), we may, by passing to a subsequence, assume that the A_i^* converge to a limit A^*. Hence

$$\begin{aligned}\mathcal{F}\big(A_i - (A^* + R)\big) &= \mathcal{F}(A_i^* - A^* - Q_i)\\ &\le \mathcal{F}(A_i^* - A^*) + \mathcal{F}(Q_i)\\ &\le \mathcal{F}(A_i^* - A^*) + M(Q_i)\\ &\to 0\end{aligned}$$

since $A_i^* \to A^*$ and $\mathcal{M}(Q_i) \to 0$. Thus the A_i converge to $A^* + R$. □

Proof of Theorem A.1

Let W be an open set whose closure is a compact subset of U. Choose an open set V whose interior contains the closure of W and whose closure is a compact subset of V. The idea of the proof is to work in a one-point compactification of V so that we can apply Lemma 5.2.

Let $u : \mathbf{R}^N \to [0, 1]$ be a smooth function that is 1 on W, that is strictly positive on V, and that vanishes on $\mathbf{R}^N \setminus V$.

Define $f : \mathbf{R}^N \to \mathbf{R}^{N+1}$ by

$$F(x) = u(x)(x, 1).$$

Note that f is Lipschitz and that f maps the complement of V to a point. (Indeed, $f(\mathbf{R}^N)$ may be regarded as a one-point compactification of V.) It follows that $M(f_\# S)$ and $\mathcal{F}(f_\# S)$ can be bounded by a constant times $\mathcal{M}_V(S)$ and $\mathcal{F}_V(S)$, respectively.

Let $A_i^* = f_\# A_i$. Then the hypotheses of Lemma 5.2 are satisfied for the A_i^*. Thus by passing to a subsequence we may assume that the A_i^* converge in the $\mathcal{F}$ metric.

By passing to a further subsequence, we may assume that

$$\sum_i \mathcal{F}(A_i^* - A_{i+1}^*) < \infty. \tag{37}$$

Let $H = H_\zeta$ be a half-space of the form $\mathbf{R}^N \times [\zeta, \infty)$. From (37) it follows that

$$\sum_i \mathcal{F}(A_i^* \llcorner H - A_{i+1}^* \llcorner H) < \infty \tag{38}$$

for almost every ζ (see [Fl, Lemma 2.1]). Fix such a $\zeta \in (0, 1)$ and the corresponding H.

The radial projection map

$$\pi : H \to \mathbf{R}^N,$$
$$\pi(x, y) = \frac{x}{y}$$

is Lipschitz, so by (38), the chains $A_i^\dagger := \pi_\#(A_i^* \llcorner H)$ are $\mathcal{F}$-convergent.

It follows that the $A_i^\dagger$ are also $\mathcal{F}_W$-convergent (since $\mathcal{F}_W \leq \mathcal{F}$).

But $\pi \circ f$ is the identity on W. Hence $A_i^\dagger$ and A_i coincide in W. (In other words, $A_i - A_i^\dagger$ is supported in W^c.)

Thus the A_i are also $\mathcal{F}_W$-convergent.

We have shown that for every open $W \subset\subset U$, there is an $\mathcal{F}_W$-convergent subsequence of the A_i. Now apply the diagonal argument to a nested sequence of such W's which exhaust U. □

COROLLARY A.3
Suppose that A_i are flat chains in U such that

$$\limsup_i \big(\mathbf{M}_W(A_i) + \mathbf{M}_W(\partial A_i)\big) < \infty$$

for every $W \subset\subset U$. Then A_i has a subsequence that converges in the flat topology.

Proof
By Theorem A.1 applied to the ∂A_i, there is a subsequence $i(k)$ for which the boundaries $\partial A_{i(k)}$ converge. Consequently, the $A_{i(k)}$ satisfy the hypotheses of Theorem A.1. □

THEOREM A.4 (Rectifiablity theorem)
Suppose that A is a flat m-chain in U with locally finite mass. Then A is rectifiable.

Of course, "A has locally finite mass" means "$\mathrm{M}_W(A) < \infty$ for every open $W \subset\subset U$."

The theorem was proved in the case $G = \mathbf{Z}$ by Federer and Fleming [FF]. The proof is also presented in [Fe] and in [Si]. Rather different proofs are given in [So] and [W1]. Fleming proved the rectifiabilty theorem for all finite coefficient groups [Fl]. For the most general result, see [W2], which gives a simple necessary and sufficient condition on the coefficient group in order for the rectifiablity theorem to hold.

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