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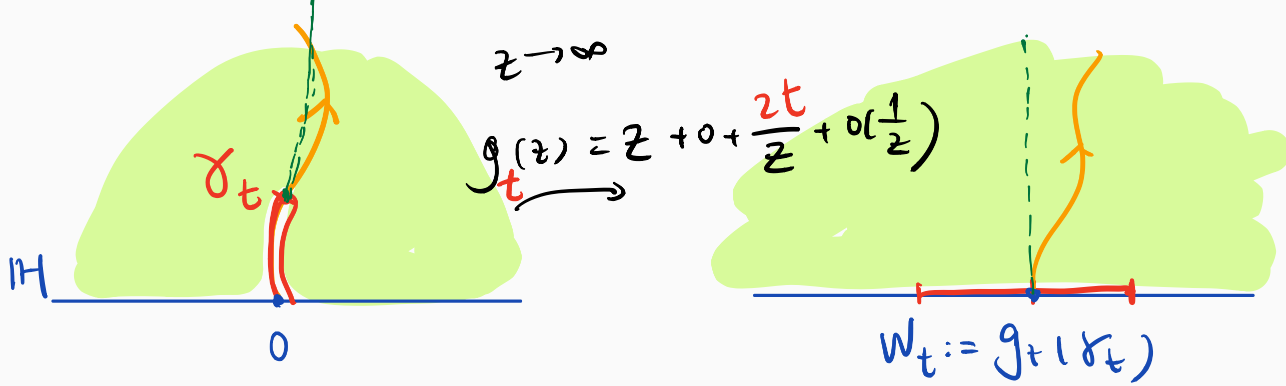
- What is Loewner energy I^L
- $I^L(\gamma) < \infty \iff \gamma$ is WP
- $I^L(\gamma) < \infty$ but with a spiral.

Chordal Loewner chains

Let γ be a simple chord in $(\mathbb{H}, 0, \infty)$.



- γ is **capacity-parametrized** by $[0, \infty)$.
- $W : \mathbb{R}_+ \to \mathbb{R}$ is called the **driving function** of γ.
- $W_0 = 0$, W is continuous.
- The curve γ can be recovered from W using Loewner's differential equation: $\partial_t g_t(z) = 2/(g_t(z) - W_t)$, $g_0(z) = z$.
- We say that γ is the **chordal Loewner curve** driven by W.

Introduced by *[Loewner '23 Math. Ann.]*.

Additivity

- The driving function of $g_t(\gamma_{[t,\infty)})$ is $s \mapsto W_{t+s}$.

Scaling

- The driving function of $c\gamma$, $c>0$

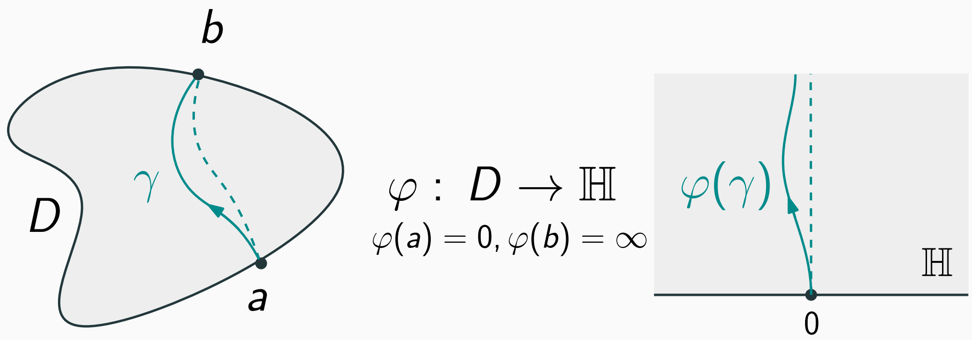
  is $t \mapsto cW_{c^{-2}t}$

- When $W_t = \sqrt{\kappa} B$ $\kappa \geq 0$

  $\gamma = SLE_\kappa$ random fractal non-self crossing curve.

The chordal Loewner energy (W. [1])

$D \subset \mathbb{C}$ a simply connected domain, a, b are two boundary points of D.



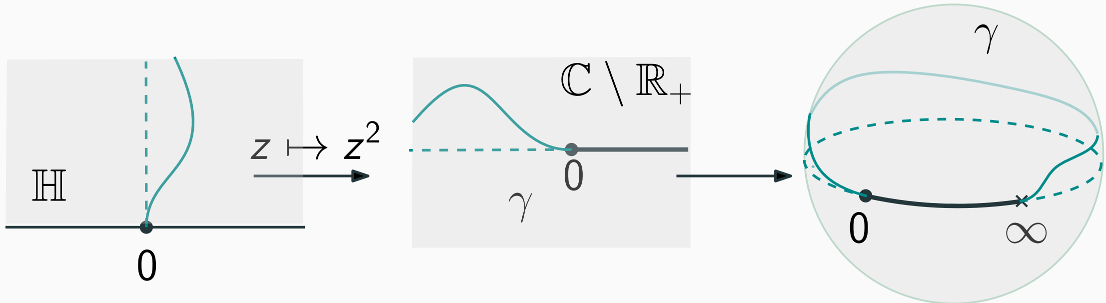
We define the **Loewner energy of a simple chord γ in** (D, a, b) to be

$$I_{D,a,b}(\gamma) := I_{\mathbb{H},0,\infty}(\varphi(\gamma)) := I(W) := \frac{1}{2}\int_0^\infty W'(t)^2 dt$$

$$= \sup_{0=t_0<t_1<\cdots<t_n} \frac{1}{2}\sum_{i=1}^{n} \frac{(W(t_i) - W(t_{i-1}))^2}{t_i - t_{i-1}}$$

where W is the driving function of $\varphi(\gamma)$.

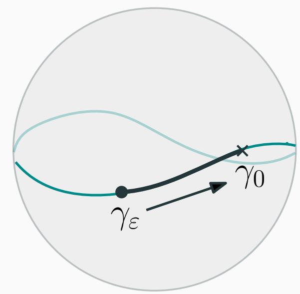
Loewner loop energy (Rohde, W. [2])



$$I^L(\gamma \cup \mathbb{R}_+, \infty) := I_{\mathbb{C}\setminus\mathbb{R}_+,0,\infty}(\gamma).$$

More generally, we define the **Loewner energy of a simple loop** $\gamma : [0,1] \mapsto \hat{\mathbb{C}}$ rooted at $\gamma_0 = \gamma_1$ to be

$$I^L(\gamma, \cancel{\gamma_0}) := \lim_{\varepsilon \to 0} I_{\hat{\mathbb{C}}\setminus\gamma[0,\varepsilon],\gamma_\varepsilon,\gamma_0}(\gamma[\varepsilon,1]).$$



Remarks: $I^L(\gamma, \gamma_0) = 0$ if and only if γ is a circle.

If $\varphi \in \mathrm{PSL}(2,\mathbb{C})$, then $I^L(\varphi(\gamma), \varphi(\gamma_0)) = I^L(\gamma, \gamma_0)$.

I. Dirichlet energy of log-derivatives of conformal maps

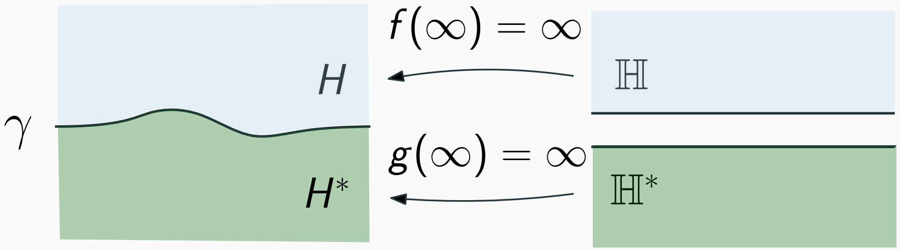
For $D \subset \mathbb{C}$, we write

$$\mathcal{D}_D(\varphi) := \frac{1}{\pi} \int_D |\nabla \varphi(z)|^2 \, dz^2.$$

Theorem (W. [3])

If γ passes through ∞, we have the identity

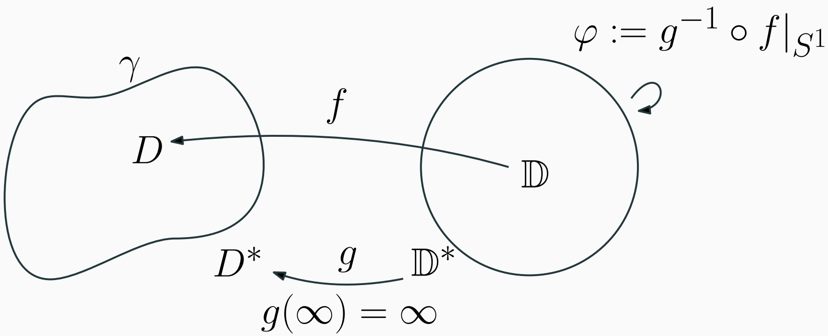
$$I^L(\gamma, \infty) = \mathcal{D}_{\mathbb{H}}(\log |f'|) + \mathcal{D}_{\mathbb{H}^*}(\log |g'|).$$



The identity is related to SLE/GFF couplings but the proof is purely analytic.

Further connection to SLE/GFF couplings is studied in *[Viklund, W. 4]*.

Universal Liouville action



Theorem [Takhtajan & Teo '06 Memoir AMS]

The **universal Liouville action S_1** : $T_0(1) \to \mathbb{R}$,

$$S_1(\overset{\gamma}{\cancel{[\varphi]}}) := \int_{\mathbb{D}} \left| \frac{f''}{f'}(z) \right|^2 \mathrm{d}z^2 + \int_{\mathbb{D}^*} \left| \frac{g''}{g'}(z) \right|^2 \mathrm{d}z^2 + 4\pi \log \left| \frac{f'(0)}{g'(\infty)} \right|$$

is a **Kähler potential** for the Weil-Petersson metric.

$$\mathrm{WP} \longleftrightarrow I^L(\gamma) < \infty$$

"Equivalent descriptions of Loewner energy"

Theorem (W. [3])

A bounded simple loop γ has finite Loewner energy if and only if γ is WP. *Moreover,*

$$I^L(\gamma) = \mathbf{S}_1([\varphi])/\pi.$$

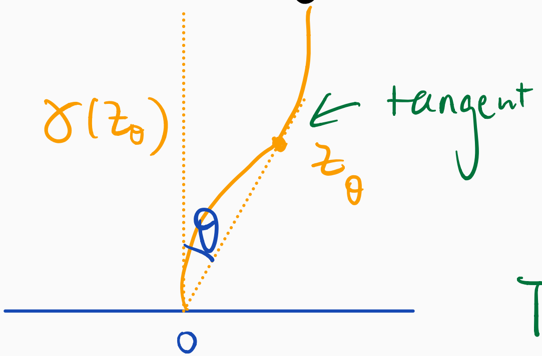
growth model definition

static

A spiral

Lemma:

The minimal energy needed to hit a point of $\arg = \frac{\pi}{2} - \theta$ is $-8 \log \cos \theta$.



$$\sim -8 \log\left(1 - \frac{\theta^2}{2}\right)$$

$$\sim 4\theta^2 \quad \text{as } \theta \to 0$$

Take $\theta_n = \frac{\varepsilon}{n}$ $\rightsquigarrow$ $\sum \theta_n = \infty$

but $\sum 4\theta_n^2 = 4\varepsilon^2 \sum \frac{1}{n^2} < \infty$