ON F.RIESZ' FUNDAMENTAL THEOREM ON SUBHARMONIC FUNCTIONS

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1. In this paper, I shall prove simply the following F. Riesz' fundamental theorem on subharmonic functions.[1]

THEOREM 1. *Let $u(z)$ be a subharmonic function in a domain D on the z-plane, then there exists a positive mass-distribution $\mu(e)$ defined for Borel sets e in D, such that for any bounded domain $D_1 \subset D$, which is contained in D with its boundary,*

$$u(z) = v(z) - \int_{D_1} \log \frac{1}{|z-a|} \, d\mu(a) \quad (z \in D_1),$$

where $v(z)$ is harmonic in D_1 and such $\mu(e)$ is unique.

The main idea of the proof is as follows.

Let z be any point of D and a disc $\Delta(\rho, z) : |\zeta - z| < \rho$ be contained in D and put

$$L(r, z : u) = \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{i\theta}) d\theta \quad (0 < r < \rho). \tag{1}$$

Then $L(r, z : u)$ is an increasing convex function of $\log r$,[2] so that $r\,dL(r, z{:}u)/dr \geqq 0$ exists, except at most a countable number of values of r. We call such a disc $\Delta(r, z)$ a non-exceptional disc. We define the mass μ contained in a non-exceptional disc by

$$\mu(\Delta(r, z)) = \frac{r\,dL(r, z : u)}{dr} \geqq 0. \tag{2}$$

Let e be any set in D. We cover e by at most a countable number of non-exceptional discs $\Delta(r_\nu, z_\nu)$ and put

$$\mu^*(e) = \inf \sum_\nu \mu(\Delta(r_\nu, z_\nu)). \tag{3}$$

Then $\mu^*(e)$ is the Carathéodory's outer measure of e, which is an additive set function for Borel sets e. The main difficulty of the proof is prove that for a non-exceptional disc

$$\mu^*(\Delta(r, z)) = \mu(\Delta(r, z)) \quad \text{(Lemma 3)}.$$

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1) F. RIESZ, Sur les fonctions subharmoniques et leur rapport à la théorie du potentiel II, Acta Math., 54(1930). G. C. EVANS, On potentials of positive mass I, Trans. Amer. Math. Soc., 37(1938). T. RADÓ, Subharmonic functions, Berlin (1937).

2) Rado's book, p. 8.

Hence for Borel sets e, if we write $\mu(e)$ instead of $\mu^*(e)$, then (3) becomes

$$\mu(e) = \inf \sum_{\nu} \mu(\Delta(r_\nu, z_\nu)). \tag{3'}$$

If we put

$$u(z) = v(z) - \int_{D_1} \log \frac{1}{|z-a|} d\mu(a) \qquad (z \in D_1), \tag{4}$$

then we can prove that $v(z)$ is harmonic in D_1.

2. First we shall prove three lemmas.

LEMMA 1. *Let $f(x)$, $f_n(x)$ $(n = 1, 2, \cdots)$ be convex functions of x in $[a, b]$, such that $\lim_{n\to\infty} f_n(x) = f(x)$. Let x_0 $(a < x_0 < b)$ be such a point, that $f'(x_0)$, $f_n'(x_0)$ $(n = 1, 2, \cdots)$ exist, then*

$$f_n'(x_0) \to f'(x_0) \qquad (n \to \infty).$$

PROOF. Suppose that $\lim_{n\to\infty} f_n'(x_0) \neq f'(x_0)$. We may assume that $\lim_{n\to\infty} f_n'(x_0) = \alpha$ exists and $\alpha \neq f'(x_0)$, since otherwise, we take a suitable subsequence from n. If $\alpha > f'(x_0)$, then $f_n'(x_0) > \alpha_1 > f'(x_0)$ $(n \geqq n_0)$, so that since $f_n(x)$ is convex,

$$f_n(x) - f_n(x_0) \geqq \alpha_1(x - x_0) \qquad (x_0 < x < b),$$

hence for $n \to \infty$,

$$f(x) - f(x_0) \geqq \alpha_1(x - x_0),$$

so that $f'(x_0) \geqq \alpha_1$, which contradicts the hypothesis. Similarly we are lead to the contradiction, if we suppose that $\alpha < f'(x_0)$. Hence $f_n'(x_0) \to f'(x_0)$.

LEMMA 2. *Let $f(x)$ be a convex function in (a, b) and x_0, x_ν $(\nu = 1, 2, \cdots)$ be points in (a, b), such that $x_\nu \to x_0$ and $f'(x_0)$, $f'(x_\nu)$ $(\nu = 1, 2, \cdots)$ exist, then*

$$f'(x_\nu) \to f'(x_0) \quad (\nu \to \infty).$$

PROOF. Suppose that $x_1 > x_2 > \cdots > x_\nu \to x_0$, then $f'(x_\nu)$ decreases with ν, so that $\lim_{\nu\to\infty} f'(x_\nu) = \alpha \geqq f'(x_0)$. If $\alpha > f'(x_0)$, then $f'(x_\nu) > \alpha_1 > f'(x_0)$ $(\nu \geqq \nu_0)$, so that

$$f(x) - f(x_\nu) \geqq \alpha_1(x - x_\nu) \quad (x_\nu < x < b),$$

hence for $\nu \to \infty$,

$$f(x) - f(x_0) \geqq \alpha_1(x - x_0) \quad (x_0 < x < b),$$

so that $f'(x_0) \geqq \alpha_1$, which contradicts the hypothesis. Similarly we can prove, if $x_1 < x_2 < \cdots < x_\nu \to x_0$.

LEMMA 3. *In* (3), *for a non-exceptional disc,*

$$\mu^*(\Delta(r, z)) = \mu(\Delta(r, z)).$$

PROOF. Let z be any point of D. For a sufficiently small ρ, we put

(5) $$u_\rho^{(1)}(z) = A_\rho(z, u) = \frac{1}{\pi\rho^2}\int_0^\rho\int_0^{2\pi} u(z + re^{i\theta})r\,dr\,d\theta,$$

(6) $$u_\rho^{(2)}(z) = A_\rho(z, u_\rho^{(1)}),\quad u_\rho^{(3)}(z) = A_\rho(z, u_\rho^{(2)}).$$

Then $u_\rho^{(3)}(z)$ is subharmonic and has continuous partial derivatives of the second order and

(7) $$u(z) \leqq u_\rho^{(3)}(z),\quad \lim_{\rho\to 0} u_\rho^{(3)}(z) = u(z),$$

(8) $$u_{\rho_1}^{(3)}(z) \leqq u_{\rho_2}^{(3)}(z)\quad (\rho_1 < \rho_2)$$[3].

We put

(9) $$L(r, z : u) = \frac{1}{2\pi}\int_0^{2\pi} u(z + re^{i\theta})d\theta,$$

$$L(r, z : u_\rho^{(3)}) = \frac{1}{2\pi}\int_0^{2\pi} u_\rho^{(3)}(z + re^{i\theta})d\theta.$$

Then $L(r, z : u)$, $L(r, z : u_\rho^{(3)})$ are increasing convex functions of $\log r$ and by (7), (8),

$$L(r, z : u_\rho^{(3)}) \to L(r, z : u)\quad (\rho\to 0)$$

by decreasing, so that by Lemma 1, for a non-exceptional r,

(10) $$\frac{r\,dL(r, z : u_\rho^{(3)})}{dr} \to \frac{r\,dL(r, z : u)}{dr}\quad (\rho\to 0).$$

Since $u_\rho^{(3)}(z)$ is subharmonic, its Laplacian $\Delta u \geqq 0$, so that for any domain $\Delta \subset D$, whose boundary Γ consists of a finite number of analytic curves, we define the mass μ_ρ contained in Δ by

(11) $$\mu_\rho(\Delta) = \frac{1}{2\pi}\iint_\Delta \Delta u\,dx\,dy = \frac{1}{2\pi}\int_\Gamma \frac{\partial u_\rho^{(3)}}{\partial \nu}ds,$$

where ν is the outer normal of Γ. Then

(12) $$\mu_\rho(\Delta) \leqq M = M(\Delta),$$

where M is a constant independent of ρ.[4]

$\mu_\rho(\Delta)$ is a finitely additive function of a domain. Let e be any set in D. We cover e by at most a countable number of non-exceptional discs $\Delta(r_\nu, z_\nu)$ and put

(13) $$\mu_\rho^*(e) = \inf \sum_\nu \mu_\rho(\Delta(r_\nu, z_\nu)),$$

then $\mu_\rho^*(e)$ is the Carathéodory's outer measure of e, which is an additive set

3) Rado's book, p. 11.
4) Rado's book, p. 12.

functions for Borel sets e. We can prove easily $\mu_\rho^*(\Delta(r,z)) = \mu_\rho(\Delta(r,z))$ for a non-exceptional disc.

For Borel sets e, we write $\mu_\rho(e)$ instead of $\mu_\rho^*(e)$. In virtue of (12), we can find $\rho_\nu \to 0$, such that μ_{ρ_ν} converges to an additive set function μ, in the sense that if the boundary of a Borel set e does not contain μ-mass, then $\mu_{\rho_\nu}(e) \to \mu(e)$. [5]

We shall prove that if $\Delta(r,z)$ is a non-exceptional disc, then its boundary does not contain μ-mass.

Now by (10), for a non-exceptional r,

$$\mu_\rho(\Delta(r,z)) = \frac{1}{2\pi}\int_\Gamma \frac{\partial u_\rho^{(3)}}{\partial \nu}\,ds = \frac{r\,dL(r,z:u_\rho^{(3)})}{dr} \to \frac{r\,dL(r,z:u)}{dr} \quad (\rho \to 0). \tag{14}$$

By Lemma 2, for any $\varepsilon > 0$, we can choose non-exceptional r_1, r_2 $(r_1 < r < r_2)$, such that

$$0 \leqq \frac{r_2\,dL(r_2,z:u)}{dr_2} - \frac{r_1\,dL(r_1,z:u)}{dr_1} < \varepsilon.$$

Hence by (14),

$$0 \leqq \mu_\rho(\Delta(r_2,z)) - \mu_\rho(\Delta(r_1,z)) < 2\varepsilon,$$

if $\rho < \rho_0$ (ε). From this we see that the boundary of $\Delta(r,z)$ does not contain μ-mass, so that $\mu_\rho(\Delta(r,z)) \to \mu(\Delta(r,z))$, hence by (14),

$$\mu(\Delta(r,z)) = \frac{r\,dL(r,z:u)}{dr}. \tag{15}$$

Let e be any set in D. We cover e by at most a countable number of non-exceptional discs $\Delta(r_\nu, z_\nu)$ and put

$$\mu^*(e) = \inf \sum_\nu \mu(\Delta(r_\nu, z_\nu)).$$

Then for Borel sets e, $\mu^*(e) = \mu(e)$, [6] so that

$$\mu(e) = \inf \sum_\nu \mu(\Delta(r_\nu, z_\nu)), \tag{16}$$

especially for a non-exceptional disc,

$$\mu(\Delta(r,z)) = \inf \sum_\nu \mu(\Delta(r_\nu, z_\nu)).$$

Hence the lemma is proved.

REMARK. Since $\mu(e)$ is defined by (15), (16), we see that $\mu(e)$ is independent of the choice of $\rho_\nu \to 0$, so that $\mu_\rho(e) \to \mu(e)$ $(\rho \to 0)$.

5) O. FROSTMAN, Potential d'équlibre et capacité des ensembles, Lund (1935).

6) E. HOPF, Ergodentheorie, Berlin (1935), p. 3.

3. Now we shall prove the theorem. Let $\mu(e)$ be defined by (2), (3'). Let $D_1 \subset D$ be any bounded domain, which is contained in D with its boundary. Since D_1 is covered by a finite number of non-exceptional discs, $\mu(D_1) < \infty$.

We put for $z \in D_1$,

$$w(z) = -\int_{D_1} \log \frac{1}{|z-a|} d\mu(a), \tag{17}$$

$$u(z) = v(z) + w(z). \tag{18}$$

We shall prove that $v(z)$ is harmonic in D_1.

We choose ρ_0 so small that for any $z \in D_1$, a disc $|\zeta - z| < \rho_0$ is contained in D and put

$$L(r, z : w) = \frac{1}{2\pi}\int_0^{2\pi} w(z + re^{i\theta})d\theta \quad (0 < r < \rho_0, z \in D_1). \tag{19}$$

Suppose that $z = 0$ belongs to D_1, so that

$$L(r, 0 : w) = \frac{1}{2\pi}\int_0^{2\pi} w(re^{i\theta})\, d\theta \quad (0 < r < \rho_0). \tag{20}$$

Let $R = \sup_{z \in D_1} |z|$ and put

$$\Omega(r) = \int_{|a|<r} d\mu(a) \quad (0 < r < R), \tag{21}$$

then

$$\Omega(r) = \mu\,(\Delta(r, 0)) \quad (0 < r < \rho_0).$$

Since

$$\int_0^{2\pi} \log |re^{i\theta} - a|\, d\theta = 2\pi \operatorname{Max}\,(\log r,\ \log |a|),$$

we have

$$L(r, 0 : w) = \frac{1}{2\pi}\int_{D_1} d\mu(a)\int_0^{2\pi} \log |re^{i\theta} - a| d\theta = \int_{D_1} \operatorname{Max}\,(\log r,\ \log |a|) d\mu(a)$$

$$= \int_0^R \operatorname{Max}(\log r, \log t)\, d\Omega(t) = \Omega(R)\log R - \int_r^R \frac{\Omega(t)}{t} dt,$$

or

$$L(r, 0 : w) = \mu(D_1)\log R - \int_r^R \frac{\Omega(t)}{t} dt. \tag{22}$$

Since $\Omega(r)$ is continuous, except at most a countable number of values of

r, we have for a non-exceptional r,

$$\frac{r\,dL(r,0:w)}{dr}=\Omega(r)=\mu(\Delta(r,0))\quad(0<r<\rho_0).$$

Hence except at most a countable number of values of r, we have from (2),

$$\frac{r\,dL(r,0:w)}{dr}=\frac{r\,dL(r,0:u)}{dr}=\mu(\Delta(r,0)).\tag{23}$$

Since being convex functions of $\log r$, $L(r,0:w)$, $L(r,0:u)$ are absolutely continuous functions of r in any closed interval contained in $(0,\rho_0)$ and their derivatives coincide almost everywhere,

$$L(r,0:v)=L(r,0:u)-L(r,0:w)=\text{const.}\quad(0<r<\rho_0).$$

Similarly for any $z\in D_1$,

(24) $L(r,z:v)=L(r,z:u)-L(r,z:w)=\text{const.}=a(z)\quad(0<r<\rho_0)$.

Let

$$A_\rho(z:u)=\frac{1}{\pi\rho^2}\int_0^\rho\int_0^{2\pi}u(z+re^{i\theta})\,r\,dr\,d\theta\quad(0<r<\rho_0),$$

then

$$A_\rho(z:u)=\frac{2}{\rho^2}\int_0^\rho L(r,z:u)\,r\,dr,$$

so that by (24),

$$A_\rho(z,v)=A_\rho(z:u)-A_\rho(z:w)=a(z)\quad(0<\rho<\rho_0).\tag{25}$$

Since $u(z),w(z)$ are subharmonic, $\lim\limits_{\rho\to0}A_\rho(z:u)=u(z)$, $\lim\limits_{\rho\to0}A_\rho(z:w)=w(z)$, so that

$$\lim_{\rho\to0}A_\rho(z:v)=u(z)-w(z)=v(z),$$

hence by (25),

$$v(z)=A_\rho(z:v)\quad(0<\rho<\rho_0).\tag{26}$$

Since $A_\rho(z:v)$ and hence $v(z)$ is a continuous function of z, we see from (26), that $v(z)$ is harmonic in D_1.

Next we shall prove the uniqueness of μ. Suppose that

$$u(z)=v_1(z)-\int_{D_1}\log\frac{1}{|z-a|}\,d\mu_1(a)=v_2(z)-\int_{D_1}\log\frac{1}{|z-a|}\,d\mu_2(a).\tag{27}$$

If we put $v_1(z)-v_2(z)=v(z)$, $\mu_2-\mu_1=\mu$, then

$$v(z)=-\int_{D_1}\log\frac{1}{|z-a|}\,d\mu(a)\tag{28}$$

is harmonic in D_1. We put

$$\Omega_i(r) = \int\limits_{|a|<r} d\mu_i(a) \quad (i = 1, 2),$$

$$\Omega(r) = \int\limits_{|a|<r} d\mu(a) = \Omega_2(r) - \Omega_1(r) \quad (0 < r < \rho_0).$$

Suppose that $z = 0$ belongs to D_1, then from (28), we have as before

$$(29) \quad v(0) = \frac{1}{2\pi}\int_0^{2\pi} v(re^{i\theta})\,d\theta = \mu(D_1)\log R - \int_r^R \frac{\Omega(t)}{t}\,dt \quad (0 < r < \rho_0).$$

Since $\Omega(r)$ is continuous, except at most a countable number of values of r, we have for a non-exceptional r, by differentiating the both sides of (29), $\Omega(r) = 0$, $\Omega_1(r) = \Omega_2(r)$, or

$$\mu_1(\Delta(r, 0)) = \mu_2(\Delta(r, 0)).$$

Similarly

$$(30) \qquad \mu_1(\Delta(r, z)) = \mu_2(\Delta(r, z)) \quad (z \in D_1)$$

except at most a countable number of values of r.

Let e be any set in D_1. We cover e by at most a countable number of non-exceptional discs $\Delta(r_\nu, z_\nu)$ and put

$$\mu_i^*(e) = \inf \sum_\nu \mu_i(\Delta(r_\nu, z_\nu)) \quad (i = 1, 2).$$

Then for Borel sets e, $\mu_i^*(e) = \mu_i(e)$ $(i = 1, 2)$, so that by (30), $\mu_1(e) = \mu_2(e)$ for Borel sets e, which proves the uniqueness of μ.

REMARK. From (22), (23), we see that $rdL(r, 0:u)/dr$ exists, when and only when there is no mass on $|z| = r$ and $r\,dL(r, 0:u)/dr$ is equal to the mass contained in $|z| < r$ and $(r\,dL(r, 0:u)/dr)_+ - (r\,dL(r, 0:u)/dr)_-$ is equal to the mass contained on $|z| = r$.

4. In the above, we assumed that D is a schlicht domain, but if D is a domain on a Riemann surface, we can prove similarly the following theorem.

THEOREM 2. *Let $u(z)$ be a subharmonic function in a domain D, then there exists a positive mass-distribution $\mu(e)$ in D, such that for any compact domain $D_1 \subset D$, which is contained in D with its boundary,*

$$(31) \qquad u(z) = v(z) - \int\limits_{D_1} g(z, a)\,d\mu(a) \quad (z \in D_1)$$

where $v(z)$ is harmonic in D_1 and $g(z, a)$ is the Green's function of D with a as its pole, and such $\mu(e)$ is unique.

If $\int\limits_D g(z, a)\,d\mu(a) \not\equiv \infty$, then $-\int\limits_D g(z, a)\,d\mu(a)$ is subharmonic in D and we can take $D_1 = D$ in (31), such that

$$u(z) = v(z) - \int_D g(z, a)\, d\mu(a) \quad (z \in D),$$

where $v(z)$ is harmonic in D.

Since $g(z, a) > 0$, $u(z) < v(z)$ in D. Hence there exists a harmonic majorant of $u(z)$ in D. Conversely, if there exists a harmonic function $h(z)$ in D, such that

$$u(z) \leqq h(z) \qquad \text{in } D, \tag{32}$$

then we shall prove that $\int_D g(z, a)\, d\mu(a) \not\equiv \infty$.

Let $u(z_0) \neq -\infty$. We approximate D by a sequence of compact domains $D_1 \subset D_2 \subset \cdots \subset D_k \to D$, such that $z_0 \in D_1$, $\bar{D}_k \subset D_{k+1}$ and the boundary Γ_k of D_k consists of a finite number of analytic Jordan curves and let $g_k(z, z_0)$ be the Green's function of D_k, with z_0 as its pole. By (31),

$$u(z) = v_k(z) - \int_{D_k} g(z, a)\, d\mu(a) \quad (z \in D_k), \tag{33}$$

where $v_k(z)$ is harmonic in D_k.

Let Γ'_k be the niveau curve: $g_k(z, z_0) = \delta > 0$. Since Γ'_k consists of a finite number of analytic Jordan curves, $u(z)$ is integrable on Γ'_k[7], hence

$$\frac{1}{2\pi} \int_{\Gamma'_k} v_k(z) \frac{\partial g_k(z, z_0)}{\partial \nu} ds = v_k(z_0),$$

$$\frac{1}{2\pi} \int_{\Gamma'_k} u(z) \frac{\partial g_k(z, z_0)}{\partial \nu} ds = v_k(z_0) - \frac{1}{2\pi} \int_{\Gamma'_k} \frac{\partial g_k(z, z_0)}{\partial \nu} \left(\int_{D_k} g(z, a) d\mu(a) \right) ds,$$

where ν is the outer normal of Γ'_k. Since $u(z) \leqq h(z)$ on Γ'_k,

$$\begin{aligned} & v_k(z_0) - \frac{1}{2\pi} \int_{\Gamma'_k} \frac{\partial g_k(z, z_0)}{\partial \nu} \left(\int_{D_k} g(z, a)\, d\mu(a) \right) ds \\ & \leqq \frac{1}{2\pi} \int_{\Gamma'_k} h(z) \frac{\partial g_k(z, z_0)}{\partial \nu} ds = h(z_0). \end{aligned} \tag{34}$$

Let D'_k be the domain bounded by Γ'_k, then if $a \in D_k - \bar{D}'_k$,

$$\frac{1}{2\pi} \int_{\Gamma'_k} g(z, a) \frac{\partial g_k(z, z_0)}{\partial \nu} ds = g(z_0, a). \tag{35}$$

Since the right hand side of (35) is a continuous function of a, (35) holds if

7) Rado's book, p. 5.

a lies on Γ'_k.

If $a \in D'_k$, then since $g_k(z, a) = \delta$ on Γ'_k and $g(z, a) - g_k(z, a) + \delta$ is harmonic in D'_k,

$$
\begin{aligned}
(36) \quad \frac{1}{2\pi}\int_{\Gamma'_k} g(z, a)\, \frac{\partial g_k(z, z_0)}{\partial \nu}\, ds &= \frac{1}{2\pi}\int_{\Gamma'_k} (g(z, a) - g_k(z, a) + \delta)\, \frac{\partial g_k(z, z_0)}{\partial \nu}\, ds \\
&= g(z_0, a) - g_k(z_0, a) + \delta.
\end{aligned}
$$

Since $D_k = D'_k + (D_k - D'_k)$, we have by (34), (35),

$$v_k(z_0) - \int_{D_k} g(z_0, a)\, d\mu(a) + \int_{D'_k} g_k(z_0, a)\, d\mu(a) \leqq h(z_0) + 0(\delta),$$

so that for $\delta \to 0$,

$$v_k(z_0) - \int_{D_k} g(z_0, a)\, d\mu(a) + \int_{D_k} g_k(z_0, a)\, d\mu(a) \leqq h(z_0). \tag{37}$$

Since

$$u(z_0) = v_k(z_0) - \int_{D_k} g(z_0, a)\, d\mu(a),$$

we have from (37),

$$\int_{D_k} g_k(z_0, a) d\mu(a) \leqq h(z_0) - u(z_0) \quad (k = 1, 2, \cdots\cdot),$$

so that for $\nu = 1, 2, \cdots\cdot$

$$\int_{D_k} g_{k+\nu}(z_0, a)\, d\mu(a) \leqq \int_{D_{k+\nu}} g_{k+\nu}(z_0, a)\, d\mu(a) \leqq h(z_0) - u(z_0),$$

hence for $\nu \to \infty$,

$$\int_{D_k} g(z_0, a) \leqq h(z_0) - u(z_0),$$

and for $k \to \infty$,

$$\int_D g(z, a) d\mu(a) \leqq h(z_0) - u(z_0) < \infty. \qquad \text{q.e.d.}$$

Hence we have proved :

THEOREM 3.[8] *Let $u(z)$ be subharmonic in a domain D, then the necessary and sufficient condition that $u(z)$ can be expressed in the form:*

$$u(z) = v(z) - \int_D g(z, a)\, d\mu(a) \quad (z \in D)$$

8) F. RIESZ, l. c. 1). Rado's book, p. 45.

is that there exists a harmonic majorant of $u(z)$ *in D, where* $v(z)$ *is harmonic in D and* $g(z, a)$ *is the Green's function of D.*

It can be proved easily that $v(z)$ is the least harmonic majorant of $u(z)$.

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