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Total curvature of complete surfaces in hyperbolic space ☆

Gil Solanes

Departament de Matemàtiques, Univeristat Autònoma de Barcelona, Edifici Cc. Campus de la UAB, 08193 Bellaterra, Spain

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Abstract

We prove a Gauss–Bonnet formula for the extrinsic curvature of complete surfaces in hyperbolic space under some assumptions on the asymptotic behavior. The result is given in terms of the measure of geodesics intersecting the surface non-trivially, and of a conformal invariant of the curve at infinity.

MSC: 53C65

Keywords: Integral geometry; Total curvature; Hyperbolic space; Open surfaces

1. Introduction and main results

In this paper we prove a Gauss–Bonnet formula for the total extrinsic curvature of complete surfaces in hyperbolic space. Our result is analogous to those obtained by Dillen and Kühnel in [2] for submanifolds of euclidean space, where the total curvature of a submanifold S is given in terms of the Euler characteristic $\chi(S)$, and the geometry of S at infinity (see also Dutertre's work [3] on semi-algebraic sets).

Our starting point is the following well-known equality for $S \looparrowright \mathbb{H}^3$, a compact surface with boundary immersed in hyperbolic 3-space:

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E-mail address: solanes@mat.uab.cat.

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$$\int_S K\,dS = 2\pi\chi(S) + F(S) - \int_{\partial S} k_g\,ds \tag{1}$$

being K the extrinsic curvature of S (i.e. the product of its principal curvatures), $F(S)$ the area, and k_g the geodesic curvature of ∂S in S. This formula follows from the classical (intrinsic) Gauss–Bonnet theorem, and the Gauss equation. We plan to make S expand over a complete non-compact surface, but the last two terms in (1) are likely to become infinite. To avoid an indeterminate form, we add and subtract the *area* enclosed by the curve ∂S. Such a notion was defined by Banchoff and Pohl (cf. [1] and also [12]) for any closed space curve C as

$$\mathcal{A}(C) := \frac{1}{\pi}\int_{\mathcal{L}} \lambda^2(\ell, C)\,d\ell$$

where $\mathcal{L}$ is (in our case) the space of geodesics in $\mathbb{H}^3$, $d\ell$ is the invariant measure on $\mathcal{L}$ (unique up to normalization), and $\lambda(\ell, C)$ is the linking number of C with $\ell \in \mathcal{L}$. This definition was motivated by the Crofton formula which states

$$F(S) = \frac{1}{\pi}\int_{\mathcal{L}} \#(\ell \cap S)\,d\ell, \tag{2}$$

where # stands for the cardinal. Hence, we can rewrite (1) as follows

$$\int_S K\,dS = 2\pi\chi(S) + \frac{1}{\pi}\int_{\mathcal{L}} \big(\#(\ell \cap S) - \lambda^2(\ell, \partial S)\big)\,d\ell + \mathcal{A}(\partial S) - \int_{\partial S} k_g\,ds.$$

Our main result is a similar formula for complete surfaces in $\mathbb{H}^3$ defining a smooth curve C in $\partial_\infty\mathbb{H}^3$, the ideal boundary of hyperbolic space. In that case, the last two terms of the previous equation are replaced by a conformal (or Möbius) invariant of the geometry of C in $\partial_\infty\mathbb{H}^3$. To be precise, our result applies to surfaces with *cone-like ends* in the sense defined next. A similar notion of cone-like ends for submanifolds in euclidean space appears in [2].

Definition 1.1. Let $f : S \looparrowright \mathbb{H}^3$ be an immersion of a $\mathcal{C}^2$-differentiable surface S in hyperbolic space. We say S has *cone-like ends* if

i) S is the interior of a compact surface with boundary $\overline{S}$, and taking the Poincaré half-space model of hyperbolic space, f extends to a $\mathcal{C}^2$-differentiable immersion $f : \overline{S} \looparrowright \mathbb{R}^3$,
ii) $C = f(\partial\overline{S})$ is a collection of simple closed curves contained in $\partial_\infty\mathbb{H}^3$, the boundary of the model, and
iii) $f(\overline{S})$ is orthogonal to $\partial_\infty\mathbb{H}^3$ along C.

In particular, such a surface is complete with the induced metric. We will see that surfaces with cone-like ends have finite total extrinsic curvature. There are also examples of complete non-compact surfaces with finite total extrinsic curvature which do not fulfill i) or ii) in the previous definition. Condition iii) however is necessary for the total curvature to be finite: the limit of the extrinsic curvature of S at an ideal point $x \in C$ is $\cos^2(\beta)$ where β is the angle between S and $\partial_\infty \mathbb{H}^3$ at x.

In the Klein (or projective) model, the definition reads the same, but replacing the word 'orthogonal' by 'transverse'. We will mainly work with the Poincaré half-space model. Unless otherwise stated all the metric notions (such as length, area or curvature) will refer to the hyperbolic metric.

Given a connected oriented curve $C \subset \partial_\infty \mathbb{H}^3 \equiv \mathbb{R}^2$, and a pair of distinct points $x, y \in C$, let us consider the oriented angle at x from C to the oriented circle through x that is positively tangent to C at y. This angle admits a unique continuous determination $\theta : C \times C \to \mathbb{R}$ that vanishes on the diagonal. Note that $\theta(y,x) = \theta(x,y)$ and θ is independent of the orientation of C.

We will prove the following result.

Theorem 1. *Let $S \subset \mathbb{H}^3$ be a simply connected surface of class $\mathcal{C}^2$, embedded in the Poincaré half-space model of hyperbolic space, and with a (connected) cone-like end $C \subset \partial_\infty \mathbb{H}^3$. Then, the integral over S of the extrinsic curvature K is*

$$\int_S K\, dS = \frac{1}{\pi} \int_{\mathcal{L}} \big(\#(\ell \cap S) - \lambda^2(\ell, C)\big)\, d\ell - \frac{1}{\pi} \int_{C \times C} \theta \sin\theta \frac{dx\, dy}{\|y - x\|^2} \tag{3}$$

where

- *$d\ell$ is an invariant measure on the space of geodesics $\mathcal{L}$,*
- *$\lambda^2(\ell, C)$ is 1 if the ideal endpoints of ℓ are on different components of $\partial_\infty \mathbb{H}^3 \setminus C$ and 0 otherwise, and*
- *dx, dy denote length elements on C with respect to the euclidean metric $\|\cdot\|$ on $\partial_\infty \mathbb{H}^3 \equiv \mathbb{R}^2$.*

The integrals in (3) *are absolutely convergent.*

Remark 1. The most interesting term in (3) is the last one, which we call the *ideal defect* of S. It defines a functional for plane curves which is invariant under the action of the Möbius group. In fact, the form $dx\, dy / \|y - x\|^2$, as well as $\theta(x,y)$, is invariant under Möbius transformations. Similar expressions for space curves appear often in the study of conformally invariant knot energies (cf. [5]).

The first term in the right-hand side of (3) is positive, and can be considered as a 'truncated area' of S, in view of (2). We call this term the *measure of non-trivial geodesics of S*. From Proposition 6, it will be clear that it is a natural functional of S.

The idea of the proof is roughly the following. We pull-back $d\ell$ to the space of point pairs of S. Integration gives the measure of non-trivial geodesics. Applying Stokes' theorem yields then the result. This procedure was already used by Pohl in the euclidean setting in [8], but here we use a different 'primitive' of $d\ell$. This leads to a somehow dual construction, where the total curvature instead of the area appears. This dual approach is not possible in euclidean space.

From Theorem 1 one gets easily a formula for a general surface with cone-like ends.

Corollary 2. *Let $S \looparrowright \mathbb{H}^3$ be a $\mathcal{C}^2$-immersed complete surface with cone-like ends $C_1, \ldots, C_n$, the curves C_i being simple and closed. Then*

$$\int_S K\,dS = 2\pi\big(\chi(S) - n\big) + \frac{1}{\pi}\int_{\mathcal{L}}\left(\#(\ell \cap S) - \sum_{i=1}^{n} \lambda^2(\ell, C_i)\right) d\ell$$
$$- \frac{1}{\pi}\sum_{i=1}^{n} \int_{C_i \times C_i} \theta \sin\theta \frac{dx\,dy}{\|y - x\|^2},$$

and the previous integrals are absolutely convergent.

Proof. Take a compact set $K \subset \mathbb{H}^3$ with $\mathcal{C}^2$ boundary ∂K transverse to S, and such that $S \setminus K = S_1 \cup \cdots \cup S_n$, where each S_i is an embedded topological cylinder over C_i. Applying (1) and (2) to $R = S \cap K$ yields

$$\int_R K\,dR = 2\pi\chi(R) - \int_{\partial R} k_g(s)\,ds + \frac{1}{\pi}\int_{\mathcal{L}} \#(\ell \cap R)\,d\ell \tag{4}$$

where k_g is the geodesic curvature in R.

Let R_i be a compact surface with boundary such that $T_i = R_i \cup S_i$ is a complete embedded simply connected surface. Combining again (1) and (2), gives

$$\int_{R_i} K\,dR_i = 2\pi - \int_{\partial R_i} k_g(s)\,ds + \frac{1}{\pi}\int_{\mathcal{L}} \#(\ell \cap R_i)\,d\ell. \tag{5}$$

Applying Theorem 1 to each T_i, and comparing with (5) yields

$$\int_{S_i} K\,dS_i = -2\pi + \frac{1}{\pi}\int_{\mathcal{L}} \big(\#(\ell \cap S_i) - \lambda^2(\ell, C_i)\big)\,d\ell$$
$$- \frac{1}{\pi}\int_{C_i \times C_i} \theta \sin\theta \frac{dx\,dy}{\|y - x\|^2} + \int_{\partial R_i} k_g(s)\,ds. \tag{6}$$

Addition of (4) and (6) finishes the proof. □

1.1. The ideal defect

The last term in (3), which we call the *ideal defect*, can also be described as an integral in the space of point pairs of $\partial_\infty \mathbb{H}^3 \equiv \mathbb{R}^2$, with respect to the Möbius invariant measure on this space.

Proposition 3. *Let $\Omega \subset \mathbb{R}^2$ be a compact domain bounded by a simple closed curve C of class $\mathcal{C}^2$. Then*

$$\int_{C \times C} \theta \sin\theta \frac{dx\,dy}{\|y - x\|^2} = 4 \int_{NT(\Omega)} \frac{dz\,dw}{\|z - w\|^4}$$

where $NT(\Omega) \subset \Omega \times \Omega$ is the set of point pairs (z, w) such that any circle $\xi \subset \mathbb{R}^2$ containing z and w intersects $\mathbb{R}^2 \setminus \Omega$ (i.e. $z, w \in \xi \Rightarrow \xi \not\subset \Omega$).

Proof. Let $Q \subset \mathbb{H}^3$ be the convex hull of $\Omega^c = \partial_\infty \mathbb{H}^3 \setminus \Omega$; i.e. Q is the minimal convex set containing Ω^c. Using the Klein model, Q can be seen as the euclidean convex hull of Ω^c. Let us consider the boundary $S = \partial Q \subset \mathbb{H}^3$, which is a surface of class $\mathcal{C}^1$. Next we construct a sequence of convex sets $Q_n \subset \mathbb{H}^3$ such that: $Q_n \supset Q_{n+1}$, $Q = \bigcap_{n=1}^\infty Q_n$, and $S_n = \partial Q_n$ is a $\mathcal{C}^2$ surface with cone-like end C. First, let $X \in \mathfrak{X}(\mathbb{R}^3)$ be a vector field in the Klein model such that X vanishes only at C, and $X|_\Omega$ points to the interior of the model. Then, for small $t > 0$, the flow φ_t brings Ω to a surface $\varphi_t(\Omega)$ with a cone-like end on C, and bounding a convex domain D. On the other hand, let Q be approximated by a decreasing sequence $Q'_n \subset \mathbb{R}^3$ of euclidean convex sets with boundary of class $\mathcal{C}^2$ (cf. [11]). Then, smoothening the corners of $D \cap Q'_n$ yields the desired sequence.

By Theorem 1

$$\int_{S_n} K\, dS_n = \frac{1}{\pi}\int_{\mathcal{L}} \big(\#(\ell \cap S_n) - \lambda^2(\ell, C)\big)\, d\ell - \frac{1}{\pi}\int_{C\times C} \theta \sin\theta \frac{dx\, dy}{\|y-x\|^2}.$$

Using, for instance, the arguments in [6], one can show

$$\lim_n \int_{S_n} K\, dS_n = 0.$$

On the other hand, by monotone convergence,

$$\lim_n \int_{\mathcal{L}} \big(\#(\ell \cap S_n) - \lambda^2(\ell, C)\big)\, d\ell = \int_{\mathcal{L}} \big(\#(\ell \cap S) - \lambda^2(\ell, C)\big)\, d\ell.$$

Hence,

$$\int_{C\times C} \theta \sin\theta \frac{dx\, dy}{\|y-x\|^2} = \int_{\mathcal{L}} \big(\#(\ell \cap S) - \lambda^2(\ell, C)\big)\, d\ell.$$

The right-hand side above is the measure of geodesics intersecting Q but not Ω. We determine each geodesic $\ell \in \mathcal{L}$ by its ideal endpoints (z, w). This allows to express $d\ell$ as in (16). Finally, we just need to note that a geodesic ℓ intersects the convex hull Q if and only if every geodesic 2-plane containing ℓ intersects Ω. □

1.2. Integral of the inverse of the chord

Next we express the ideal defect in an alternative way which is not invariant, but still interesting. Let $C \subset \partial_\infty \mathbb{H}^3$ be a $\mathcal{C}^2$-differentiable simple closed curve, and consider $S = C \times (0, \infty) \subset \mathbb{H}^3$. We may think of S as a surface with one end by closing the top end at infinity with an infinitesimally small surface. Then, the total curvature of S equals 2π, and Theorem 1 applied to S yields

$$
\begin{aligned}
2\pi + \frac{1}{\pi}\int_{C\times C} \theta \sin\theta \frac{dx\,dy}{\|y-x\|^2} &= \frac{2}{\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \bigl(\#(\overline{zw}\cap C) - \lambda^2(z,w;C)\bigr)\frac{dz\,dw}{\|w-z\|^4} \\
&= \frac{2}{\pi}\int_{A(2,1)} \sum_{x,y\in L\cap C} \frac{(-1)^{\#(\overline{xy}\cap C)}}{\|y-x\|} dL \qquad (7)
\end{aligned}
$$

where $\overline{zw}$ denotes the line segment joining $z, w \in \mathbb{R}^2$, and dL is the invariant measure on the space $A(2,1)$ of (unoriented) lines of $\mathbb{R}^2$, normalized as in [10]. The first equality uses (16). The second equality is Proposition 10.

As a consequence, the integral in (7) is invariant under Möbius transformations, which was *a priori* not obvious. In fact, if C bounds a convex domain Ω, then (7) is

$$
\frac{4}{\pi}\int_{A(2,1)} \frac{1}{\sigma(L\cap\Omega)} dL \qquad (8)
$$

where $\sigma(L\cap\Omega)$ is the chord length. The previous functional (8) is one of the so-called Franklin invariants of convex sets, defined by Santaló in [9] as a generalization of a functional introduced by Franklin with motivations from stereology (cf. [4]). These functionals had the nice property of being invariant by dilatations. For instance, the integral (8) could in principle be used to estimate, by means of line sections, the number of particles in a plane region, if these particles have the same shape but possibly different size.

An immediate consequence of our results is that (8) is in fact invariant under the Möbius group. An interesting question is to determine which of the Franklin functionals enjoy this bigger invariance. Besides, it was conjectured that the Franklin invariants are minimal for balls (cf. [4] and [9]). This was shown by Franklin among ellipsoids while Santaló obtained some general non-sharp inequalities. As a consequence of our results, we can prove this conjecture in the planar case.

Corollary 4. *For a convex set $\Omega \subset \mathbb{R}^2$ we have*

$$
\int_{A(2,1)} \frac{1}{\sigma(L\cap\Omega)} dL \geqslant \frac{\pi^2}{2} \qquad (9)
$$

where σ is the length of the chord, and $A(2,1)$ is the space of lines. Equality holds in (9) if and only if Ω is a round disk. Moreover, the left-hand side of (9) is invariant by Möbius transformations (keeping Ω convex).

Proof. By (7) we have

$$
\frac{4}{\pi}\int_{A(2,1)} \frac{1}{\sigma(L\cap\Omega)} dL = 2\pi + \frac{1}{\pi}\int_{C\times C} \theta\sin\theta \frac{dx\,dy}{\|y-x\|^2} \geqslant 2\pi,
$$

and the equality occurs if and only if $\theta \equiv 0$. Indeed, since C is convex it is easy to see that $-\pi < \theta < \pi$. □

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2. The space of geodesics

Let $\mathcal{F} = \{(x; g_1, g_2, g_3)\}$ be the bundle of positive orthonormal frames of $\mathbb{H}^3$; i.e., each $(g_i)_{i=1,2,3}$ is a positive orthonormal basis of $T_x\mathbb{H}^3$. We consider on $\mathcal{F}$ the dual and connection forms

$$\omega_i = \langle dx, g_i\rangle, \qquad \omega_{ij} = \langle \nabla g_i, g_j\rangle,$$

where $\langle\,,\rangle$ denotes the (hyperbolic) metric in $\mathbb{H}^3$, and ∇ is the corresponding riemannian connection. The structure equations read

$$d\omega_i = \omega_j \wedge \omega_{ji}, \qquad d\omega_{ij} = \omega_i \wedge \omega_j + \omega_{ik} \wedge \omega_{kj}. \tag{10}$$

Let $\mathcal{L}^+$ be the space of oriented geodesics of $\mathbb{H}^3$. Clearly $\mathcal{L}^+$ is a double cover of $\mathcal{L}$. Consider $\pi_1 : \mathcal{F} \to \mathcal{L}^+$ given by $\pi_1(x; g_1, g_2, g_3) = \ell$ with $x \in \ell$, and $g_1 \in T_x\ell$ pointing in the positive direction. The space $\mathcal{L}^+$ can be endowed with a differentiable structure such that π_1 is a smooth submersion. Moreover, $\mathcal{L}^+$ admits a volume form $d\ell$ invariant under isometries of $\mathbb{H}^3$, which is unique up to normalization, and characterized by (cf. [10])

$$\pi_1^*(d\ell) = \omega_2 \wedge \omega_{12} \wedge \omega_3 \wedge \omega_{13}. \tag{11}$$

Similarly, one can consider $\mathcal{L}_2$, the space of (unoriented) totally geodesic surfaces (geodesic planes) of $\mathbb{H}^3$. We will use the space of *flags*

$$\mathcal{L}_{1,2} = \{(\ell, \wp) \in \mathcal{L}^+ \times \mathcal{L}_2 \mid \ell \subset \wp\},$$

and the canonical projection $\pi : \mathcal{L}_{1,2} \to \mathcal{L}^+$ which makes $\mathcal{L}_{1,2}$ a principal $\mathbb{S}^1$-bundle over $\mathcal{L}^+$. Let us project $\pi_{1,2} : \mathcal{F} \to \mathcal{L}_{1,2}$ so that $\pi_{1,2}(x; g_i) = (\ell, \wp)$ with $\wp \supset \ell = \pi_1(x; g_i)$ and $g_3 \perp T_x\wp$. Then $\omega_{23} = \pi_{1,2}^*\varphi$ for a certain form $\varphi \in \Omega^1(\mathcal{L}_{1,2})$, which is an invariant global angular form (or connection) of the bundle π.

Proposition 5. *There exists a unique* 2*-form* $\alpha \in \Omega^2(\mathcal{L}^+)$ *such that*

$$\pi^*(\alpha) = d\varphi \in \Omega^2(\mathcal{L}_{1,2}),$$

where φ *is the global angular form of* π*. Moreover* $\alpha \wedge \alpha = 2\,d\ell$*, so that* α *is an invariant symplectic form on* $\mathcal{L}^+$*.*

Proof. Assuming α exists, structure equations (10) give

$$\pi_1^*(\alpha) = d\omega_{23} = \omega_2 \wedge \omega_3 - \omega_{12} \wedge \omega_{13}, \tag{12}$$

whence

$$\pi_1^*(\alpha \wedge \alpha) = -2\omega_2 \wedge \omega_3 \wedge \omega_{12} \wedge \omega_{13} = 2\pi_1^*(d\ell).$$

Therefore $\alpha \wedge \alpha = 2\,d\ell$ (as $d\pi_1$ is exhaustive).

Let $X \in \mathfrak{X}(\mathcal{L}_{1,2})$ be the tangent vector field along the fibers of π such that $\varphi(X) = 1$. By (12), for any $\tilde{X} \in \mathfrak{X}(\mathcal{F})$ such that $d\pi_{1,2}\tilde{X} = X$,

$$\pi_{1,2}^*(i_X\, d\varphi) = i_{\tilde{X}}\, d\omega_{23} = 0,$$

whence $i_X\, d\varphi = 0$. Then $L_X\varphi = 0$, and

$$L_X\, d\varphi = dL_X\,\varphi = 0.$$

Hence, $d\varphi$ is constant along the fibers of π, and null on their tangent vectors, which shows the existence of α. The uniqueness follows from the injectivity of π^*. □

It follows from the previous proposition that

$$d\big(\pi^*\alpha \wedge \varphi\big) = 2 \cdot \pi^*(d\ell). \tag{13}$$

This will be used in Section 4 to prove Theorem 1 by means of Stokes' theorem.

Remark 2. The forms φ, α are in some sense dual to the forms ω_1, dI used in [8]. In fact, many of the subsequent constructions are parallel to those of [8]. However, choosing α leads us to results involving the total curvature, while dI made the area appear. This choice could not be done in the euclidean setting since there $\alpha \wedge \alpha$ vanishes.

The following notation will be used throughout the paper:

$$A \ltimes B := \big\{(x, y) \in A \times B \bigm| x \neq y\big\}.$$

In the Poincaré model, by considering the ideal endpoints z, w of each geodesic ℓ, one identifies (a full-measure subset of) $\mathcal{L}^+$ with $\mathbb{R}^2 \ltimes \mathbb{R}^2$. Then, an elementary computation with moving frames (cf. (17)) gives the following expression for α at a point $(z, w) \equiv \ell \in \mathcal{L}^+$:

$$\alpha = \frac{2}{\|w - z\|^2}(dz_1 \wedge dw_2 + dz_2 \wedge dw_1) \tag{14}$$

where the coordinate system of $\mathbb{R}^2$ has been chosen in such a way that $z_2 = w_2 = 0$ and $w_1 = -z_1$. In particular, if $z = z(x)$ and $w = w(y)$ are curves parametrized by arc-length, then

$$\alpha = 2\sin\theta(x, y)\frac{dx \wedge dy}{\|w(y) - z(x)\|^2} \tag{15}$$

where $\theta(x, y)$ is the oriented angle between the two oriented circles through $z(x), w(y)$, tangent to $z'(x)$ and $w'(y)$ respectively.

Using (14) we can also obtain an expression for the measure of geodesics. Indeed,

$$d\ell = \frac{1}{2}\alpha \wedge \alpha = 4\frac{dz \wedge dw}{\|w - z\|^4} \tag{16}$$

where dz, dw denote the area elements of the ideal endpoints z, w in $\mathbb{R}^2$.

Remark 3. The following complex valued two form in $\mathbb{C} \ltimes \mathbb{C}$ was introduced by Langevin and O'Hara in [5] under the name *infinitesimal cross-ratio*

$$\omega_{cr} = \frac{d(z_1 + iz_2) \wedge d(w_1 + iw_2)}{(w - z)^2}, \quad (z, w) \in \mathbb{C} \ltimes \mathbb{C}.$$

This form ω_{cr} is invariant under the diagonal action of the Möbius group $Sl(2, \mathbb{C})$. Using this fact, one checks easily that $-\alpha/2$ coincides with $\mathfrak{Im}(\omega_{cr})$, the imaginary part of the infinitesimal cross-ratio.

We end the section by showing that the measure of non-trivial geodesics is a natural quantity. This fact was already noticed in the euclidean setting by Pohl (cf. [8], Eq. (6.5)).

Proposition 6. *Let $S \subset \mathbb{H}^3$ be an embedded surface with cone-like ends $C \subset \partial_\infty \mathbb{H}^3$. Let $\Phi : S \ltimes S \to \mathcal{L}^+$ be such that $\Phi(x, y)$ is the oriented geodesic going first through x and then through y. Then*

$$\int_{S \ltimes S} \Phi^*(d\ell) = \frac{1}{2}\int_{\mathcal{L}^+} \big(\#(\ell \cap S) - \lambda^2(\ell, C)\big)\, d\ell.$$

Proof. By the coarea formula

$$\int_{S \ltimes S} \Phi^*(d\ell) = \int_{\mathcal{L}^+} \mu(\ell)\, d\ell$$

where

$$\mu(\ell) = \sum_{(x,y)\in\Phi^{-1}(\ell)} -\epsilon(x)\epsilon(y)$$

being $\epsilon(u)$ the sign at u of the algebraic intersection $\ell \cdot S$. Now, let p (resp. q) be the number of points of $\ell \cap S$ with $\epsilon = 1$ (resp. $\epsilon = -1$), so that

$$\#(\ell \cap S) = p + q, \qquad \lambda(\ell, C) = \ell \cdot S = p - q.$$

Then $\Phi^{-1}(\ell)$ contains $(p(p-1) + q(q-1))/2$ pairs (x, y) with $\epsilon(x) = \epsilon(y)$, and pq elements with $\epsilon(x) = -\epsilon(y)$. Therefore $2\mu(\ell) = 2pq - p(p-1) - q(q-1) = \#(\ell \cap S) - \lambda^2(\ell, C)$. $\square$

3. Convergence results

Next we establish the convergence of the integrals appearing in Theorem 1. In the whole section, $S \subset \mathbb{H}^3$ will denote a complete surface with a connected cone-like end $C \subset \partial_\infty \mathbb{H}^3$. Here $\mathbb{H}^3$ denotes the Poincaré half-space model. For $h > 0$, we set $S_h = \{x \in S \mid x_3 \geqslant h\}$ which is a compact surface with boundary $C_h = \partial S_h = \{x \in S \mid x_3 = h\}$.

Proposition 7. *If K denotes the extrinsic curvature of S, and dS is the area element, then*

$$\int_S K\, dS$$

is absolutely convergent.

Proof. Let us consider the global orthonormal frame $e_i(x) = x_3 \partial/\partial x_i$ $(i = 1, 2, 3)$, defined for all $x \in \mathbb{H}^3$. The connection forms $\theta_{ij} = \langle \nabla e_i, e_j \rangle$ are then given by

$$\theta_{i3} = \frac{dx_i}{x_3}, \qquad \theta_{ij} = 0 \quad \text{for } i, j \neq 3. \tag{17}$$

Let us fix now $y \in S$. After a change of coordinates, we can assume $e_2(y) \in T_y S$. Let v_1, v_2, v_3 be a frame locally defined on S (around y) so that $v_2(y) = e_2(y)$, and $v_1(x), v_2(x) \in T_x S$. Then $v_i(x) = a_{ij}(x) e_j(x)$ for an orthogonal matrix $(a_{ij}(x)) \in O(3)$. In particular $v_1(y) = \cos\alpha e_1 + \sin\alpha e_3$, and $v_3(y) = -\sin\alpha e_1 + \cos\alpha e_3$ for some $\alpha \in [0, 2\pi)$. Then $(\omega_{ij})_y = \langle \nabla v_i, v_j \rangle_y$ are given by

$$\begin{aligned}(\omega_{12})_y &= \langle \nabla(a_{1i} e_i), e_2 \rangle_y = da_{12} + a_{12}(y)\langle \nabla e_1, e_2\rangle + a_{32}(y)\langle \nabla e_3, e_2\rangle \\ &= da_{12} + \cos\alpha\theta_{12} + \sin\alpha\theta_{32} \overset{(17)}{=} da_{12} - \sin\alpha \frac{dx_2}{y_3},\end{aligned} \tag{18}$$

$$\begin{aligned}(\omega_{13})_y &= \langle \nabla(a_{1i} e_i), -\sin\alpha e_1 + \cos\alpha e_3 \rangle_y \\ &= -\sin\alpha\big(da_{11} + a_{1i}(y)\theta_{i1}\big) + \cos\alpha\big(da_{13} + a_{1i}(y)\theta_{i3}\big) \\ &= -\sin\alpha da_{11} + \cos\alpha da_{13} + \frac{dx_1}{y_3},\end{aligned}$$

and similarly

$$(\omega_{23})_y = -\sin\alpha\, da_{21} + \cos\alpha\, da_{23} + \cos\alpha \frac{dx_2}{y_3}. \tag{19}$$

In particular

$$\omega_{13}(v_1) = -\sin\alpha\, da_{11}(v_1) + \cos\alpha\, da_{13}(v_1) + \frac{dx_1(v_1)}{y_3} = O(y_3) + \frac{dx_1(v_1)}{y_3}.$$

But $dx_1(v_1/y_3) = \cos\alpha = O(y_3)$. Indeed, $\cos\alpha = \langle e_1, v_1\rangle$ is a $\mathcal{C}^1$ function on $\bar{S} = S \cup C$ and vanishes at C. One checks similarly that $\omega_{i3}(v_j) = O(y_3)$ for $i, j = 1, 2$. We have thus that

$$K(y) = \det\bigl(\omega_{i3}(v_j) \bigm| i, j = 1, 2\bigr) = \omega_{13}(v_1)\omega_{23}(v_2) - \omega_{13}(v_2)\omega_{23}(v_1) = O\bigl(y_3^2\bigr).$$

The result follows since $y_3^2\,dS$ is the euclidean area element of S in the model. □

The following proposition is a first step towards the existence of formula (3).

Proposition 8. *Let $S, R \subset \mathbb{H}^3$ be two surfaces with the same cone-like end $\partial_\infty S = \partial_\infty R \subset \partial_\infty \mathbb{H}^3$. Then*

$$\int_S K\,dS - \int_R K\,dR = 2\pi\bigl(\chi(S) - \chi(R)\bigr) + \lim_{h\to 0}\frac{1}{\pi}\int_{\mathcal{L}}\bigl(\#(\ell\cap S_h) - \#(\ell\cap R_h)\bigr)\,d\ell.$$

Proof. From (1) and (2) one gets

$$\int_{S_h} K\,dS_h = 2\pi\chi(S_h) + \frac{1}{\pi}\int_{\mathcal{L}}\#(\ell\cap S_h)\,d\ell - \int_{\partial S_h} k_g(s)\,ds$$

and similarly for R_h. We must show that

$$\int_{\partial S_h} k_g(s)\,ds - \int_{\partial R_h} k_g(s)\,ds$$

tends to zero as $h \to 0$. By Eq. (18) we have

$$k_g = -\omega_{12}(v_2) = -da_{12}(v_2) + \sin\alpha.$$

In the previous proof we learned that $\cos\alpha = O(h)$, and thus $\sin\alpha = 1 + O(h^2)$. Besides, in the choice of the local frame v_1, v_2, v_3 one could further assume that v_1 is everywhere orthogonal to e_2. Hence $a_{12} = \langle v_1, e_2\rangle \equiv 0$, so $k_g = \sin\alpha = 1 + O(h^2)$, and

$$\int_{\partial S_h}\bigl(k_g(s) - 1\bigr)\,ds = \int_{\partial S_h} O\bigl(h^2\bigr)\,ds = O(h),$$

and similarly for ∂R_h. Thus, it suffices to show that the difference of (hyperbolic) lengths of ∂S_h and ∂R_h tends to zero as $h \to 0$. This follows from the fact that $\partial_\infty S$ is an euclidean geodesic of both S and R, and geodesics are extremals of the length. Indeed, the euclidean lengths of ∂S_h and ∂R_h differ both from the length of $\partial_\infty S$ with an order $O(h^2)$. Hence, their respective hyperbolic lengths have a difference of order $O(h)$. □

Next we study the convergence of the measure of non-trivial geodesics.

Lemma 9. *If $\lambda(\ell, C_h)$ denotes the linking number (defined up to sign) of a geodesic ℓ with the curve C_h, then*

$$\lim_{h\to 0}\int_{\mathcal{L}} \big(\#(\ell\cap S_h)-\lambda^2(\ell, C_h)\big)\,d\ell=\int_{\mathcal{L}}\big(\#(\ell\cap S)-\lambda^2(\ell, C)\big)\,d\ell \tag{20}$$

where $\lambda(\ell, C)$ is the limit of $\lambda(\ell, C_h)$ when $h\to 0$.

Proof. Let $\ell\in\mathcal{L}$ be transverse to S, which happens for almost every ℓ. Then $\#(\ell\cap S_h)$ is an increasing function of h. For h small enough, C_h is connected, and thus $\lambda^2(\ell, C_h)\leqslant 1$. Therefore $\#(\ell\cap S_h)-\lambda^2(\ell, C_h)$ is an increasing function of h. Then (20) follows by monotone convergence. □

We will see below, that the limit in (20) is finite. For the moment, we show this fact for the infinite cylinder over C.

Proposition 10. *Let $C\subset\partial_\infty\mathbb{H}^3$ be a simple closed curve, and let $R=C\times(0,\infty)\subset\mathbb{H}^3$. Then the following integrals converge and coincide*

$$\int_{\mathcal{L}}\big(\#(\ell\cap R)-\lambda^2(\ell, C)\big)\,d\ell=\int_{A(2,1)}\sum_{x,y\in L\cap C}\frac{(-1)^{\#(\overline{xy}\cap C)}}{\|y-x\|}\,dL<\infty$$

where dL is an invariant measure in the space $A(2,1)$ of lines in $\mathbb{R}^2$.

Proof. After a vertical projection onto $\partial_\infty\mathbb{H}^3$, each geodesic ℓ is mapped to a segment $\overline{zw}$, and R projects onto C. From the proof of Proposition 6 we know

$$\#(\ell\cap R)-\lambda^2(\ell, C)=-\sum_{x,y\in\overline{zw}\cap C}\epsilon(x)\epsilon(y)$$

where $\epsilon(u)$ is the sign at u of the algebraic intersection $\overline{zw}\cdot C$. The equality of the integrals follows from (16), together with (cf. [10], Eq. (4.2))

$$dz\,dw=\|t-s\|\,ds\,dt\,dL$$

where s, t are arc-length parameters of z, w along L. In order to check the convergence, we use the following expression of the measure of lines in $\mathbb{R}^2$ (cf. [8])

$$dL=|\sin\beta_x\sin\beta_y|\frac{dx\,dy}{\|y-x\|}$$

where x, y are intersection points with C, and β_x, β_y are the oriented angles between L and C at x, y respectively. Then the integral over $A(2,1)$ above becomes

$$-\int_{C\times C}\sin\beta_x\sin\beta_y\frac{dx\,dy}{\|y-x\|^2}.$$

This integral converges since $\beta_x, \beta_y=O(\|y-x\|)$ as one can easily prove. □

Lemma 11. *Let $S, R \subset \mathbb{H}^3$ be two surfaces with the same cone-like end $\partial_\infty S = \partial_\infty R \subset \partial_\infty \mathbb{H}^3$. Then the following integrals are uniformly bounded for all $h > 0$*

$$\int_{\mathcal{L}} \big(\lambda^2(\ell, \partial S_h) - \lambda^2(\ell, \partial R_h)\big)\, d\ell.$$

Proof. Let T_h be the region of $\{x \in \mathbb{H}^3 \mid x_3 = h\}$ bounded by ∂S_h and ∂R_h. If a geodesic ℓ is disjoint from T_h, then $\lambda^2(\ell, \partial S_h) = \lambda^2(\ell, \partial T_h)$. Hence, the integral above is bounded by the measure of geodesics intersecting T_h. By the Crofton formula (2), this measure is proportional to the area of T_h. Since S and R are tangent at infinity, the euclidean area of T_h has order $O(h^2)$. Therefore, its hyperbolic area is uniformly bounded. □

Proposition 12. *The measure of non-trivial geodesics*

$$\int_{\mathcal{L}} \big(\#(\ell \cap S) - \lambda^2(\ell, C)\big)\, d\ell$$

is absolutely convergent.

Proof. Clearly

$$\begin{aligned}\int_{\mathcal{L}} \big(\#(\ell \cap S_h) - \lambda^2(\ell, \partial S_h)\big)\, d\ell = {} & \int_{\mathcal{L}} \big(\#(\ell \cap S_h) - \#(\ell \cap R_h)\big)\, d\ell \\ & + \int_{\mathcal{L}} \big(\#(\ell \cap R_h) - \lambda^2(\ell, \partial R_h)\big)\, d\ell \\ & + \int_{\mathcal{L}} \big(\lambda^2(\ell, \partial R_h) - \lambda^2(\ell, \partial S_h)\big)\, d\ell.\end{aligned}$$

The last three integrals are uniformly bounded by Propositions 8 and 10 and Lemma 11 respectively. Thus, by monotonicity, the following limit

$$\lim_{h \to 0} \int_{\mathcal{L}} \big(\#(\ell \cap S_h) - \lambda^2(\ell, \partial S_h)\big)\, d\ell,$$

exists and is finite. Since $\#(\ell \cap S) - \lambda^2(\ell, C)$ is positive, Lemma 9 shows the absolute convergence of the integral. □

Corollary 13. *Let $S \subset \mathbb{H}^3$ be a surface with a cone-like end $C \subset \partial_\infty \mathbb{H}^3$. Then*

$$\int_S K\, dS = 2\pi \chi(S) + \frac{1}{\pi} \int_{\mathcal{L}} \big(\#(\ell \cap S) - \lambda^2(\ell, C)\big)\, d\ell - \delta(C)$$

where $\delta(C)$ depends only on the ideal curve C. All the integrals above are absolutely convergent.

Proof. The convergence has been established in Propositions 7 and 12. The result follows then from Proposition 8. □

Remark 4. We have assumed C to be connected for simplicity. If C is a collection of *disjoint* simple closed curves, each of them arbitrarily oriented, the previous results hold without change. The key fact for the convergence is that $\lambda^2(\cdot, C) \leqslant 1$ outside a compact subset of $\mathcal{L}$. As for $\delta(C)$, it depends in this case on the orientations of C, as well as the relative positions of the several components.

Remark 5. In order to get explicit expressions of $\delta(C)$, it is enough to find, for each curve $C \subset \partial_\infty \mathbb{H}^3$, a surface S with cone-like ends on C for which the total curvature and the measure of non-trivial geodesics can be computed. In fact, this is what we did in Subsections 1.1 and 1.2.

However, in order to get the expression of $\delta(C)$ that appears in Theorem 1, we will need to follow a different strategy.

4. Proof of Theorem 1

4.1. *The space of chords*

Given a $\mathcal{C}^2$-differentiable manifold S (without boundary), the *space of chords* of S is a $\mathcal{C}^1$-differentiable manifold M_S with boundary, introduced by Whitney in [13], and described in detail in [8]. This space is the blow-up of $S \times S$ along the diagonal. In particular, the interior of M_S is $S \ltimes S$, and the boundary is the sphere bundle of oriented tangent directions of S

$$\partial M_S = T^+S := \big(TS \setminus \{(x, \vec{0}) \mid x \in S\}\big)/\mathbb{R}^+.$$

The reader is referred to [8] for details on the differentiable structure of M_S. The following property describes this structure quite well: given a regular injective $\mathcal{C}^2$-differentiable curve $x : [0, 1) \to S$, the curve $c : (0, 1) \to S \ltimes S$ defined by $c(t) = (x(0), x(t))$ extends to a $\mathcal{C}^1$-differentiable curve $c : [0, 1) \to M_S$ which meets ∂M_S transversely at $c(0) = [x'(0)] \in T^+S$. Another basic property is the following: the natural projections $p_1, p_2 : S \ltimes S \to S$ extend naturally to differentiable submersions $p_1, p_2 : M_S \to S$.

Let now S be a manifold with boundary. The space M_S of chords of S is constructed as follows. We consider a manifold without boundary $\tilde{S}$ extending S. Let $p_1, p_2 : M_{\tilde{S}} \to \tilde{S}$ be the submersions mentioned above. The space of chords of S is then defined as $M_S = p_1^{-1}(S) \cap p_2^{-1}(S) \subset M_{\tilde{S}}$ (i.e. M_S contains the chords of $\tilde{S}$ with both ends in S). This space is a topological manifold with boundary, but this boundary is not smooth. Indeed, the interior of M_S is $S \ltimes S$, and the boundary is $\partial M_S = T^+S \cup (S \ltimes \partial S) \cup (\partial S \ltimes S)$. The faces T^+S, $S \ltimes \partial S$, and $\partial S \ltimes S$ are pairwise transverse outside $T^+\partial S$ (in fact, $\partial S \ltimes S$ and $S \ltimes \partial S$ are tangent at points of $T^+\partial S$). Hence, $M_S \setminus T^+\partial S$ is a manifold with corners in the usual sense (cf. for instance [7]).

4.2. *Bundles and sections*

In this subsection we use the Klein model of hyperbolic space. Hence $\mathbb{H}^3$ is the interior of the closed unit ball $\mathbb{B}^3$ in $\mathbb{R}^3$. Let $\Psi : \mathbb{B}^3 \ltimes \mathbb{B}^3 \to \mathcal{L}^+$ be such that (x, y) is mapped to the geodesic line going first through x and then through y. This map extends naturally to $\Psi : M_{\mathbb{B}^3} \setminus T^+\mathbb{S}^2 \to \mathcal{L}^+$. This extension is smooth by the results of [8].

Let now $S^\circ \subset \mathbb{H}^3$ be a simply connected surface with a cone-like end $C \subset \partial_\infty \mathbb{H}^3$. Then, the closure $S = S^\circ \cup C$ is a compact surface with boundary in $\mathbb{B}^3$, transverse to the ideal sphere $\mathbb{S}^2 = \partial_\infty \mathbb{H}^3$. Notice that we slightly modified, for simplicity, the notation used in the previous sections.

As seen in [8], the inclusion $M_S \subset M_{\mathbb{B}^3}$ is compatible with the differentiable structures. Hence, the mapping

$$\Phi : M_S \setminus T^+C \to \mathcal{L}^+$$

obtained as a restriction of Ψ is smooth. Note that this extends the mapping Φ defined in Proposition 6.

To simplify the notation we denote $B := M_S \setminus T^+C$. By Proposition 6, the measure of non-trivial geodesics can be obtained by integrating $\Phi^*(d\ell)$ on B. Our aim is to compute this integral by means of Stokes' theorem, using an *invariant* form whose differential is $d\ell$. Such a form is given by (13), but it lives in the bundle $\mathcal{L}_{1,2}$. In fact, there is no invariant form in $\mathcal{L}^+$ whose differential is $d\ell$. We are thus led to consider the pull-back by Φ of the $\mathbb{S}^1$-bundle $\pi : \mathcal{L}_{1,2} \to \mathcal{L}^+$. More precisely, we consider $E := \Phi^*(\mathcal{L}_{1,2}) = \{(z, \wp) \in B \times \mathcal{L}_2 \mid \Phi(z) \subset \wp\}$, and the following commutative diagram with the obvious mappings

$$\begin{array}{ccc} E & \xrightarrow{\Phi'} & \mathcal{L}_{1,2} \\ {\scriptstyle \Phi^*\pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ B & \xrightarrow{\Phi} & \mathcal{L}^+ \end{array} \tag{21}$$

It would be desirable to define a section of $\Phi^*\pi : E \to B$. This section should be canonically constructed in some geometric way. This can be done quite naturally, but only at the boundary ∂B; in fact only on

$$\partial B \setminus (C \ltimes C) = \left(T^+S \setminus T^+C\right) \cup \left(S^\circ \times C\right) \cup \left(C \times S^\circ\right) = \partial M_S \setminus M_C.$$

Indeed, for $z = (x, [v]) \in T^+S \setminus T^+C$ we choose the geodesic plane $\wp(z)$ spanned by T_xS. For $z = (x, y) \in C \times S^\circ$, and for $z = (y, x) \in S^\circ \times C$, we choose the plane $\wp(z)$ tangent to C at x and containing y. Note that this definition does not extend to $C \ltimes C$: the two planes through $x, y \in C$ that are tangent to C at x and y respectively, form a certain angle. In fact, this is precisely the angle θ appearing in Theorem 1.

To summarize, we have defined

$$\begin{aligned} s : \partial M_S \setminus M_C &\to E, \\ z &\mapsto \left(z, \wp(z)\right) \end{aligned} \tag{22}$$

in such a way that $T_x\wp(z) = T_xS$ if $z = (x, [v]) \in T^+S$, and $T_xC \subset T_x\wp(z)$ for $z = (x, y) \in S^\circ \times C$, or $z = (y, x) \in C \times S^\circ$.

We already noted that s has a jump discontinuity in $C \ltimes C$. To solve this, we shall complete the image of s with a family of fiber intervals interpolating the two one-sided limits of s. However, these intervals are not well-defined in the $\mathbb{S}^1$-bundle E. We are led to consider an infinite cyclic cover of E that gives an $\mathbb{R}$-bundle over B. Next we define this cover, and we show it admits a lift of s. Here we take great advantage of the assumption that S is simply connected.

Proposition 14. *The principal $\mathbb{S}^1$-bundle $\Phi^*\pi : E \to B$ is trivial. Moreover, there is a bundle isomorphism $\tau : E \to B \times \mathbb{S}^1$, such that $\tau \circ s$ lifts over the covering $q : B \times \mathbb{R} \to B \times \mathbb{S}^1$; i.e., there exists a continuous function*

$$g : \partial M_S \setminus M_C \to \mathbb{R}$$

such that $q(x, g(x)) = \tau \circ s(x)$ for every $x \in \partial M_S \setminus M_C$.

Proof. Consider an isotopy of embeddings $H : S \times [0, 1] \to \mathbb{B}^3$ such that $H_0 = id$ and $H_1(S^\circ)$ is contained in a plane $\wp \in \mathcal{L}_2$. We may construct the isotopy so that $H(C \times [0, 1]) \subset \mathbb{S}^2$. Put $\tilde{H}(x, y, t) := (H_t(x), H_t(y))$ for $(x, y) \in S \ltimes S$. Clearly $\tilde{H}$ extends continuously to $\tilde{H} : B \times [0, 1] \to M_{\mathbb{B}^3} \setminus T^+\mathbb{S}^2$. Furthermore the bundle $(\Psi \circ \tilde{H}_1)^*\pi$ clearly admits a global section $s_1 \equiv \wp$. By the covering homotopy theorem, s_1 extends to a global section $\tilde{s}$ of $(\Psi \circ \tilde{H})^*\pi$, and therefore this principal bundle is trivial. This already shows that $E = (\Psi \circ \tilde{H}_0)^*(\mathcal{L}_{1,2})$ is trivial. Let

$$\tilde{\tau} : (\Psi \circ \tilde{H})^*(\mathcal{L}_{1,2}) \to M_S \times [0, 1] \times \mathbb{S}^1$$

be the isomorphism corresponding to this global section, i.e. such that $\tilde{\tau} \circ \tilde{s}(z, t) = (z, t, 1)$. For each t, the construction above (cf. (22)) yields a section s_t of the restriction of $\Psi^*\pi$ to each $\partial M_{S_t} \setminus M_{\partial S_t}$, with $s_1 \equiv \wp$, and $s_0 = s$. Clearly these fit together to give a global section $\bar{s}$ of the restriction of $(\Psi \circ \tilde{H})^*\pi$ to $\partial M_S \setminus M_C \times [0, 1]$. From the construction of $\tilde{\tau}$ it is clear that the restriction of $\tilde{\tau} \circ \bar{s}$ to $\partial M_S \setminus M_C \times \{1\}$ lifts over q. Now the covering homotopy theorem implies that $\tilde{\tau} \circ \bar{s}$ lifts over all of $\partial M_S \setminus M_C \times [0, 1]$. Hence we may take τ to be the restriction of $\tilde{\tau}$ to $(\Psi \circ \tilde{H}_0)^*(\mathcal{L}_{1,2}) = E$. $\square$

While g cannot be continuously defined over all ∂B, we can consider the continuous extensions of g to $S \ltimes C$ and $C \ltimes S$ respectively. We denote these extensions by g_1 and g_2 respectively. This way, $\theta(x, y) = g_2(x, y) - g_1(x, y)$ in the notation of Theorem 1, for every $(x, y) \in C \ltimes C$. Let $T_1 \subset B \times \mathbb{R}$ be the graph of g over $\partial B \setminus C \ltimes C$, completed with the graphs of g_1 and g_2 over $C \ltimes C$. Now we sew in a family of vertical intervals over $C \ltimes C$ interpolating these two one-sided limits. To be precise we consider $T_2 = C \ltimes C \times [0, 1]$ together with the mapping

$$\begin{aligned} \sigma : T_2 &\to C \ltimes C \times \mathbb{R}, \\ (x, y, t) &\mapsto \big(x, y, tg_1(x, y) + (1 - t)g_2(x, y)\big). \end{aligned} \tag{23}$$

Note that σ is a smooth mapping, possibly non-regular.

In the following we will need to specify some orientations. The manifold $S \ltimes S$, and hence M_S is canonically oriented by $dS \wedge dS$. This induces an orientation on ∂M_S, and hence T_1 is naturally oriented. Finally, we choose on T_2 the orientation given by $dx \wedge dy \wedge dt$. This way, T_1 and T_2 induce opposite orientations on the graphs of g_1 and g_2.

4.3. Stokes' theorem

Before applying Stokes' theorem, the non-compacity of $M_S \setminus T^+C$ needs to be settled. To this end, let us consider the function $f : M_S \to [0, \infty]$ which vanishes on T^+C, and assigns to each $z \in M_S \setminus T^+C$ the euclidean distance in $\partial_\infty \mathbb{H}^3$ between the ideal endpoints of $\Phi(z)$. Here $\mathbb{H}^3$ denotes again the Poincaré model. Then

$$\Delta_\epsilon := f^{-1}\big([0, \epsilon)\big)$$

is a neighborhood of T^+C inside M_S, and $M_S \setminus \Delta_\epsilon$ is compact. By Sard's theorem, for almost every ϵ, the level set $\partial\Delta_\epsilon := f^{-1}(\epsilon)$ is smooth and transverse to ∂M_S. Therefore $M_S \setminus \Delta_\epsilon$ is a compact manifold with corners for almost every $\epsilon > 0$. We denote this manifold by $B_\epsilon := M_S \setminus \Delta_\epsilon$.

Let us consider $T_{1,\epsilon} = T_1 \setminus \Delta'_\epsilon$, being $\Delta'_\epsilon = \pi^{-1}(\Delta_\epsilon)$. Here $\pi : B \times \mathbb{R} \to B$ is the projection on the first factor. For a generic $\epsilon > 0$, Sard's theorem applied to $f \circ \pi$ ensures that $T_{1,\epsilon}$ is a compact manifold with corners. Also $T_{2,\epsilon} = T_2 \setminus \sigma^{-1}(\Delta'_\epsilon)$ is a compact manifold with corners for almost every ϵ.

Since $T_{2,\epsilon}$ can be triangulated, we may think of $(T_{2,\epsilon}, \sigma)$ as a (smooth) singular chain. Also $T_{1,\epsilon}$ can be thought of as a singular chain. Hence it makes sense to consider $T_\epsilon := T_{1,\epsilon} + T_{2,\epsilon}$ as a chain in $\partial B \times \mathbb{R} \setminus \Delta'_\epsilon$. Its boundary is a singular chain of $\partial\Delta'_\epsilon := \pi^{-1}\partial\Delta_\epsilon$, namely $\partial T_\epsilon = (T_1 \cap \partial\Delta'_\epsilon) + \sigma^{-1}(\partial\Delta'_\epsilon)$.

In the next subsection, we will construct a chain R_ϵ in $\partial\Delta'_\epsilon$ such that $\partial R_\epsilon = -\partial T_\epsilon$. This way, $T_\epsilon + R_\epsilon$ is a cycle, and hence gives an element in the homology group $H_3(B_\epsilon \times \mathbb{R})$. Since S is contractible, we have the following homotopy equivalences

$$B_\epsilon \times \mathbb{R} \simeq B_\epsilon \simeq B \simeq S \ltimes S \simeq S \times \mathbb{S}^1 \simeq \mathbb{S}^1.$$

Therefore $H_3(B_\epsilon \times \mathbb{R}) = 0$, and $T_\epsilon + R_\epsilon$ is a boundary.

By composing with $\pi : B \times \mathbb{R} \to B$ we can consider $\pi_*(T_\epsilon + R_\epsilon)$ as a cycle in $(\partial B) \setminus \Delta_\epsilon \cup \partial\Delta_\epsilon = \partial B_\epsilon$. The latter is an oriented compact manifold so $H_3(\partial B_\epsilon, \mathbb{Z}) \equiv \mathbb{Z}$, and $[\pi_*(T_\epsilon + R_\epsilon)]$ is given by some integer n. For any form $\omega \in \Omega^3(\partial B_\epsilon)$ one has

$$\int_{T_\epsilon + R_\epsilon} \pi_*\omega = n \int_{\partial B_\epsilon} \omega.$$

Note that π restricted to the interior of $T_{1,\epsilon}$ is a diffeomorphism preserving orientations. Thus, taking ω supported on the interior of $\pi(T_{1,\epsilon})$ makes clear that $n = 1$.

Now, since $H^4(B_\epsilon) = 0$, there exists some differential form $\omega \in \Omega^3(B_\epsilon)$ such that $d\omega = \Phi^*\, d\ell$. Therefore, by Stokes' theorem

$$\int_{B_\epsilon} \Phi^*\, d\ell = \int_{B_\epsilon} d\omega = \int_{\partial B_\epsilon} \omega = \int_{T_\epsilon + R_\epsilon} \pi^*\omega = \frac{1}{2} \int_{T_\epsilon + R_\epsilon} \pi^*\alpha \wedge \varphi, \tag{24}$$

since $2\pi^*\omega - \pi^*\alpha \wedge \varphi$ is closed by (13), and $T_\epsilon + R_\epsilon$ is a boundary. Here we are abusing the notation for simplicity: by α and φ we refer to $\Phi^*\alpha$ and $(\Phi' \circ \tau^{-1} \circ \pi)^*\varphi$ respectively. We will go on with this abuse, and hopefully no confusion will arise.

4.4. Total curvature and ideal defect

In this section we integrate $\pi^*\alpha \wedge \varphi$ over T_1 and T_2. We will get respectively the total curvature, and the ideal defect.

Proposition 15.

$$\lim_{\epsilon\to 0} \int_{T_1\setminus\Delta'_\epsilon} \pi^*\alpha \wedge \varphi = 2\pi \int_S K\,dS.$$

Proof. Recall that

$$T_1 = (\text{graph } g|_{T^+S}) \cup (\text{graph } g_1) \cup (\text{graph } g_2).$$

We claim that $\pi^*\alpha \wedge \varphi$ vanishes on the graphs of g_1 and g_2. Recall these functions are defined over $S \ltimes C$ and $C \ltimes S$ respectively. Indeed, let x be a local coordinate on C. Then expression (14) shows $\alpha \wedge dx = 0$. Let now $c(t)$ be the lift in the graph of g of a curve $(y(t), x) \in S \times C$ or $(x, y(t)) \in C \times S$ with x fixed. This curve corresponds to a curve $(\ell(t), \wp(t)) = \Phi' \circ q(c(t)) \in \mathcal{L}_{1,2}$. In the Poincaré model, the ideal boundaries of $\wp(t)$ are circles in $\partial_\infty \mathbb{H}^3$ tangent to C at the point x. In order to compute $\varphi(c'(t))$ we take an isometry of $\mathbb{H}^3$ sending the point $x \in C$ to infinity. This way, $\ell(t)$ become vertical lines, and the geodesic planes $\wp(t)$ are transformed into a family of parallel vertical planes. By using the expression (17) of the connection forms, it is clear that $\varphi(c'(t))$ vanishes. This shows that φ is a multiple of π^*dx (on this region of T_1), and the claim follows.

We focus now on the graph over T^+S. Given $(x, l) \in T^+S^\circ$, we take v_1, v_2, v_3 an orthonormal basis of $T_x\mathbb{H}^3$ such that $[v_1] = l$, and $v_3 \perp T_xS$. With such a moving frame, by (12)

$$\pi^*\alpha \wedge \varphi = (\omega_2 \wedge \omega_3 - \omega_{12} \wedge \omega_{13}) \wedge \omega_{23} = -\omega_{12} \wedge \omega_{13} \wedge \omega_{23} = -K(x)\omega_{12} \wedge dS.$$

By Proposition 7, this volume form has finite integral on T^1S, the euclidean unit tangent of S (in the Poincaré model). Then we may use Lebesgue's dominated convergence theorem to get

$$\begin{aligned}\lim_{\epsilon\to 0} \int_{\text{graph } g|_{T^+S}\setminus\Delta'_\epsilon} \pi^*\alpha \wedge \varphi &= \lim_{\epsilon\to 0} \int_{T^+S\setminus\Delta_\epsilon} K\,\omega_{12} \wedge dS \\ &= \int_{T^+S} K\ \omega_{12} \wedge dS = 2\pi \int_S K\,dS,\end{aligned}$$

where we used the natural orientation of T^+S, which is opposite to the one induced by M_S. □

Proposition 16.

$$\lim_{\epsilon\to 0} \int_{T_2\setminus\Delta'_\epsilon} \pi^*\alpha \wedge \varphi = 2 \int_{C\times C} \theta \sin\theta \frac{dx\,dy}{\|y-x\|^2}.$$

Proof. Recall that T_2 is mapped to the union of vertical segments in $B \times \mathbb{R}$ interpolating the one-sided limits of g along $C \ltimes C$ (cf. (23)). This segments have length θ, and φ restricted to the fibers is precisely the length element. Hence, Fubini's theorem gives

$$\lim_{\epsilon\to 0}\int_{T_2\setminus\Delta'_\epsilon} \pi^*\alpha\wedge\varphi = \lim_{\epsilon\to 0}\int_{C\times C\setminus\Delta_\epsilon} \theta\,\alpha = 2\lim_{\epsilon\to 0}\int_{C\times C\setminus\Delta_\epsilon} \theta\sin\theta\,\frac{dx\,dy}{\|y-x\|^2}$$

where we have used (15). The result follows since $\theta = O(\|y-x\|)$, which is easy to prove. □

The proof of Theorem 1 is almost finished. So far we have seen (cf. Propositions 6 and 12, Eq. (24), and Propositions 15 and 16)

$$\begin{aligned}\int_{\mathcal{L}} \big(\#(\ell\cap S) - \lambda^2(\ell, C)\big)\,d\ell &= 2\int_B \Phi^*(d\ell) = 2\lim_{\epsilon\to 0}\int_{B_\epsilon}\Phi^*(d\ell)\\ &= \lim_{\epsilon\to 0}\int_{R_\epsilon}\pi^*\alpha\wedge\varphi + 2\pi\int_S K\,dS + 2\int_{C\ltimes C}\theta\sin\theta\frac{dx\,dy}{\|y-x\|^2}.\end{aligned} \tag{25}$$

It remains only to check that the contribution of R_ϵ vanishes as $\epsilon \to 0$. This is done in the next subsection.

4.5. *Asymptotic estimations*

Next we construct a singular chain R_ϵ in $\partial\Delta'_\epsilon$ with $\partial R_\epsilon = -\partial T_\epsilon$ as promised. Let $v : \partial\Delta_\epsilon \to E$ be the section given by the vertical planes. With the same kind of arguments as in the proof of Proposition 14 one shows that $\tau\circ v$ lifts over q; i.e. there exists $h : \partial\Delta_\epsilon \to \mathbb{R}$ such that $q(x, h(x)) = \tau\circ v(x)$. Let $R_{0,\epsilon} \subset \partial\Delta'_\epsilon$ be the graph of h over $\partial\Delta_\epsilon$. In particular, both $T_1 \cap \partial\Delta'_\epsilon$ and $\partial R_{0,\epsilon}$ project by π onto $\partial B \cap \partial\Delta_\epsilon$. Next we consider the union of vertical segments joining these two graphs. More precisely, we define $R_{1,\epsilon} = (C \ltimes S \cap \partial\Delta_\epsilon) \times [0, 1]$, $R_{2,\epsilon} = (S \ltimes C \cap \partial\Delta_\epsilon) \times [0, 1]$, $R_{3,\epsilon} = (T^+S \cap \partial\Delta_\epsilon) \times [0, 1]$ together with the mappings

$$\begin{aligned}\sigma_i : R_{i,\epsilon} &\to \partial\Delta'_\epsilon,\\ (z, t) &\mapsto \big(z, tg_i(z) + (1-t)h(z)\big),\end{aligned}$$

for $i = 1, 2$. As for σ_3, we take the same definition with g in the place of g_i. We think of $\{R_{i,\epsilon}, i = 0, 1, 2, 3\}$ as singular chains in $\partial\Delta'_\epsilon$, and we define $R_\epsilon = \sum_{i=0}^3 R_{i,\epsilon}$. A careful study of the boundaries shows that $\partial R_\epsilon = -\partial T_\epsilon$.

To finish the proof of Theorem 1 we only need to establish the following.

Proposition 17.

$$\lim_{\epsilon\to 0}\int_{R_\epsilon}\pi^*\alpha\wedge\varphi = 0.$$

Proof. Here we assume that S coincides with the cylinder $C \times (0, \infty) \subset \mathbb{H}^3$ in a neighborhood of infinity. This is no loss of generality by Corollary 13, and Eq. (25).

In particular we may assume that h and g coincide over T^+S. Hence, the integral over $R_{3,\epsilon}$ vanishes. Next we concentrate on $R_{1,\epsilon}$ (the study of $R_{2,\epsilon}$ being obviously symmetric). Given $(x, y) \in C \times S \cap \partial\Delta_\epsilon$, let $\{x, z\}$ be the ideal endpoints of $\Phi(x, y)$. The euclidean distance between x and z is constant ϵ. Hence, given $x \in C$ the point z is determined by the angle γ between the straight segment $\overline{xz}$ and T_xC. This angle γ coincides with the length of the fiber interval $\pi^{-1}(x, y) \cap R_{1,\epsilon}$. By Fubini's theorem

$$\int_{R_{1,\epsilon}} \pi^*\alpha \wedge \varphi = \int_{\pi(R_{1,\epsilon})} \gamma \cdot \alpha = \int_{\pi(R_{1,\epsilon})} \gamma \cos\gamma \frac{dx\, d\gamma}{\epsilon}$$

since $\alpha = \cos\gamma\epsilon^{-1}\, dx\, d\gamma$ (cf. (14)). The previous integrals vanish when $\epsilon \to 0$ since $\gamma = O(\epsilon)$. Indeed, the chords of length smaller than ϵ make angles with C of order $O(\epsilon)$.

It remains to estimate the integral over $R_{0,\epsilon}$. Let $(x', y') \in S \ltimes S$ be a generic point in $M_S \cap \partial\Delta_\epsilon$. Let $x, y \in C$ be the vertical projections of x', y'. Let z, w be the ideal endpoints of the geodesic $\ell = \Phi(x', y')$. We choose euclidean coordinates on $\mathbb{R}^2 \equiv \partial_\infty\mathbb{H}^3$ so that $x_2 = y_2 = 0$. We can assume $z_1 < x_1 < y_1 < w_1$. Then

$$dz_2 = (t + \sigma)\frac{dx_2}{\sigma} - t\frac{dy_2}{\sigma},$$

$$dw_2 = (\sigma + t - \epsilon)\frac{dx_2}{\sigma} + (\epsilon - t)\frac{dy_2}{\sigma},$$

where $t = x_1 - z_1$, $\sigma = y_1 - x_1$, and thus $w_1 - y_1 = \epsilon - t - \sigma$. Recall that (x', y') corresponds (through $\Phi' \circ \tau^{-1} \circ \pi$) to the pair $(\ell, \wp)$ where $\wp$ is the vertical plane containing ℓ. We take an adapted orthonormal frame $(p; g_1, g_2, g_3)$ such that $p \in \ell$ projects vertically onto $\frac{1}{2}(z + w) \in \partial_\infty\mathbb{H}^3$ and $g_3 \perp \wp$. Then (17) and the equations above yield

$$\varphi = \langle\nabla g_2, g_3\rangle = \theta_{23} = \frac{1}{\epsilon}(dz_2 + dw_2) = \frac{1}{\epsilon\sigma}\big((2t + 2\sigma - \epsilon)\, dx_2 + (\epsilon - 2t)\, dy_2\big).$$

Since $\pi^*\alpha = d\varphi$ we get

$$\pi^*\alpha = \frac{2}{\epsilon\sigma}(dt \wedge dx_2 - dt \wedge dy_2) - \frac{2t - \epsilon}{\epsilon\sigma^2} d\sigma \wedge dx_2 - \frac{\epsilon - 2t}{\epsilon\sigma^2} d\sigma \wedge dy_2.$$

Hence, recalling that x, y are restricted to move along C, we get

$$\pi^*\alpha \wedge \varphi = \frac{4}{\epsilon^2\sigma} dt \wedge dx_2 \wedge dy_2 = \frac{4}{\epsilon^2} \sin\beta_x \sin\beta_y \frac{dt \wedge dx \wedge dy}{\|y - x\|},$$

where dx, dy denote arc-length elements on C, and β_x, β_y are angles between C and the segment $\overline{xy}$. Therefore

$$\int_{R_{0,\epsilon}} \pi^*\alpha \wedge \varphi = \frac{4}{\epsilon^2} \int_{x,y\in C,\ \|y-x\|\leqslant\epsilon} \sin\beta_x \sin\beta_y \big(\epsilon - \|y-x\|\big)\frac{dx\wedge dy}{\|y-x\|}$$

which goes to zero when $\epsilon \to 0$, since $\beta_x, \beta_y = O(\|y-x\|)$. $\square$

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