Minimal varieties in riemannian manifolds

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0. Introduction

Our object in this paper is twofold. First, we give a basic exposition of immersed minimal varieties in a riemannian manifold. The principal result of this general investigation is the derivation of the linear elliptic second order equation satisfied by the second fundamental form of any minimal variety in any ambient manifold (cf. Theorem 4.2.1).

Second, we apply these general results in a more detailed study of minimal varieties in the sphere and in euclidean space. This study includes an estimation of a lower bound for the index and the nullity of a non-totally geodesic closed minimal variety immersed in S^n; a theorem which generalizes to arbitrary co-dimensions the theorem of De Giorgi [8] concerning the image of the Gauss map of a closed co-dimension 1 variety in S^n; a theorem which estimates an upper bound for the minimum value taken by the scalar curvature of any closed non-totally geodesic minimal variety in S^n; a theorem which calculates an explicit neighborhood of the standard metric on S^p isolating it in the space of metric obtained from non-totally geodesic minimal immersions in S^n; a theorem generalizing the result of Almgren [9] in which we show that the cone over any co-dimension 1 non-totally geodesic closed minimal variety in S^n is unstable with respect to its boundary for $n \leqq 6$, and an example of a cone over such a variety in S^7 which is at least locally stable. The consequences of this last result are an extension through dimension 7 of interior regularity of solutions to the co-dimension 1 Plateau problem, and an extension through dimension 8 of the Bernstein conjecture.

We wish to express our indebtedness to F. J. Almgren who acquainted us with several of the outstanding problems in this field, and with whom we have had a number of useful and highly informative conversations.

1. Riemannian vector bundles

1.1. *Definitions.* Let M denote a p-dim riemannian manifold with or without boundary. We will always take M to be oriented. $T(M)$ will denote its tangent bundle, and $T(M)_m$ its tangent space at $m \in M$. For $x, y \in T(M)_m$, $\langle x, y \rangle$ will denote their inner product. This inner product extends to an inner product on all tensors of any given type, and we shall use the same $\langle\ ,\ \rangle$ to denote this extension. The riemannian connection on $T(M)$ extends naturally to a connection on the vector bundle of tensors of any given type. This connection preserves the above mentioned inner product.

If Z is a tensor field on M, and $x \in T(M)_m$, we shall use $\nabla_x Z$ to denote the covariant derivative of Z in the x direction. $\nabla_x Z$ is a tensor at m of the same type as Z. Because the connection preserves inner products we have

$$\nabla_x\langle Z, W\rangle = \langle \nabla_x Z, W\rangle + \langle Z, \nabla_x W\rangle\ , \tag{1.1.1}$$

where Z, W are tensor fields of the same type.

If X, Y are vector fields on M, we have the first structural equation

$$\nabla_X Y - \nabla_Y X = [X, Y]\ . \tag{1.1.2}$$

For $x, y \in T(M)_m$, $R_{x,y}$ will denote the corresponding curvature transformation. $R_{x,y}: T(M)_m \to T(M)_m$ and is a skew symmetric linear transformation. $R_{x,y}$ extends naturally to a skew symmetric endomorphism of all the associated tensor spaces. The skew symmetry is with respect to the above mentioned inner product on these spaces. Let $x, y \in T(M)_m$, and let z be a tensor at m. Choose X and Y, vector fields on M, and Z a tensor field on M, which extend

x, y, and z, respectively. We then have the usual second structural equation

$$R_{x,y}z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \,. \tag{1.1.3}$$

Definition. Let $V(M)$ be a vector bundle over M. Suppose the fibres of $V(M)$ have a euclidean inner product, and suppose there is a linear connection on $V(M)$ which preserves this inner product. Then $V(M)$ is called a *riemannian vector bundle* over M.

We shall use the same $\langle\, ,\rangle$ to denote inner products on the fibres of $V(M)$. If ψ is a cross-section in $V(M)$, and $x \in T(M)_m$, we shall use the same $\nabla_x\psi$ to denote the co-variant derivative of ψ in the x direction. $\nabla_x\psi \in V(M)_m$. Since the connection preserves inner products, we have

$$\nabla_x\langle\psi, \varphi\rangle = \langle\nabla_x\psi, \varphi\rangle + \langle\psi, \nabla_x\varphi\rangle \tag{1.1.4}$$

where ψ, φ are cross-sections in $V(M)$.

The connections and inner products on $V(M)$ and $T(M)$ naturally extend to a connection and an inner product on the tensor product of all associated vector bundles. In particular, it defines a connection and an invariant inner product on $H(T, V) = \operatorname{Hom}\big(T(M), V(M)\big)$. In fact, if $r, s \in H(T, V)_m$, we set

$$\langle r, s\rangle = \textstyle\sum_{i=1}^{p} \langle r(e_i), s(e_i)\rangle \,, \tag{1.1.5}$$

where $\{e_i\}$ is a frame in $T(M)_m$. If $\mathcal{H}$ is a cross-section in $H(T, V)$, we set

$$\nabla_x(\mathcal{H})(y) = \nabla_x\big(\mathcal{H}(Y)\big) - \mathcal{H}(\nabla_x Y) \,, \tag{1.1.6}$$

where $x, y \in T(M)_m$, and Y is an arbitrary vector field extending y. This connection and inner product are easily seen to be well defined, and it is also easy to check that the inner product is invariant.

1.2. *The Laplace operator.* If ψ is a cross-section in $V(M)$, it gives rise to $\nabla\psi$, a cross-section in $H(T, V)$, by setting

$$\nabla\psi(x) = \nabla_x\psi \,. \tag{1.2.1}$$

Using the connection on $H(T, V)$, given $x, y \in T(M)_m$, we define $\nabla_{x,y}\psi \in V(M)$.

$$\nabla_{x,y}\psi = \nabla_x(\nabla\psi)(y) \,. \tag{1.2.2}$$

Clearly, the map $(x, y) \rightarrow \nabla_{x,y}\psi$ is a bilinear form on $T(M)_m$ with values in $V(M)_m$.

Definition. Let ψ be a cross-section in $V(M)$. We define $\nabla^2\psi$ to be a new cross-section in $V(M)$ by setting $\nabla^2\psi(m) =$ trace of the bilinear form $(x, y) \rightarrow \nabla_{x,y}\psi$.

PROPOSITION 1.2.1. *Let $e_1, \cdots, e_p$ be a frame in $T(M)_m$. Extend them to vector fields $E_1, \cdots, E_p$ such that $\langle E_i, E_j\rangle = \delta_{ij}$ and $\nabla_{E_i}E_j(m) = 0$. If ψ is a cross-section in $V(M)$, we have*

$$\nabla^2\psi(m) = \textstyle\sum_{i=1}^{p} \nabla_{E_i}\nabla_{E_i}\psi(m) \,. \tag{1.2.3}$$

PROOF. By the definition of trace, and by (1.2.2), we see that

$$\begin{aligned}\nabla^2\psi(m) &= \textstyle\sum_{i=1}^{p} \nabla_{e_i,e_i}\psi = \sum_{i=1}^{p} \nabla_{e_i}(\nabla\psi)(e_i) \\ &= \textstyle\sum_{i=1}^{p} \nabla_{E_i}\big(\nabla\psi(E_i)\big)(m) - \sum_{i=1}^{p} \nabla\psi(\nabla_{E_i}E_i)(m) \\ &= \textstyle\sum_{i=1}^{p} \nabla_{E_i}\nabla_{E_i}\psi(m) \,. \qquad \text{q.e.d.}\end{aligned}$$

PROPOSITION 1.2.2. *∇^2 is a differential operator on the cross-sections of $V(M)$. If M is compact and closed, or if M is compact with boundary, and ψ, φ vanish on ∂M, we have*

$$\int_M \langle\nabla^2\psi, \varphi\rangle = \int_M \langle\psi, \nabla^2\varphi\rangle = -\int_M \langle\nabla\psi, \nabla\varphi\rangle \,. \tag{1.2.4}$$

Thus, ∇^2 is a negative, semi-definite, self adjoint differential operator.

PROOF. Using (1.2.3) and (1.1.4),

$$\begin{aligned}\langle\nabla^2\psi, \varphi\rangle &= \textstyle\sum_{i=1}^{p} \langle\nabla_{E_i}\nabla_{E_i}\psi, \varphi\rangle \\ &= \textstyle\sum_{i=1}^{p} \nabla_{E_i}\langle\nabla_{E_i}\psi, \varphi\rangle - \sum_{i=1} \langle\nabla_{e_i}\psi, \nabla_{e_i}\varphi\rangle \,.\end{aligned}$$

From (1.1.5), we see that the second term is $-\langle\nabla\psi, \nabla\varphi\rangle$. Let θ be a real valued differential 1-form on M defined by

$$\theta(x) = \langle\nabla_x\psi, \varphi\rangle \,.$$

The first term is then $\delta\theta$, and we thus have

$$\langle\nabla^2\psi, \varphi\rangle = \delta\theta - \langle\nabla\psi, \nabla\varphi\rangle \,.$$

By Stokes' theorem,

$$\int_M \langle\nabla^2\psi, \varphi\rangle = -\int_M \langle\nabla\psi, \nabla\varphi\rangle + \int_{\partial M} \theta^* \,.$$

But, by our assumption, $\theta^* \mid \partial M = 0$. q.e.d.

PROPOSITION 1.2.3. *Let L be an arbitrary C^∞ cross-section in* Hom$\big(V(M), V(M)\big)$. *Suppose L is symmetric at each point. Then $\nabla^2 + L$ is strongly elliptic and has uniqueness in the Cauchy problem at each point m;* i.e., *if ψ satisfies $\nabla^2\psi + L(\psi) = 0$, and $\psi \mid U \equiv 0$ where U is an open set, then $\psi \equiv 0$ everywhere.*

PROOF. Let $x_1, \cdots, x_p$ be a normal coordinate system in a neighborhood of m. Let $v_1, \cdots, v_d$ be a frame in $V(M)_m$. Extend these to local cross-sections, $V_1, \cdots, V_d$, by parallel translation along geodesic rays emanating from m. If ψ is any local cross-section at m, it may be written as

$$\psi = u_1 V_1 + \cdots + u_d V_d \,,$$

where the $u_i \in C^\infty(M)$. Set $u = (u_1, \cdots, u_d)$. At m we clearly have

$$\nabla^2\psi = \Big[\sum_{j=1}^{p} \frac{\partial^2 u_1}{\partial x_j^2}\Big] V_1 + \cdots + \Big[\sum_{j=1}^{p} \frac{\partial^2 u_d}{\partial x_j^2}\Big] V_d .$$

Thus, if we identify the space of local cross-sections with the space of d-tuples of functions $u = (u_1, \cdots, u_d)$ at m, the system looks like

$$\sum_{i=1}^{n} \frac{\partial^2 u_1}{\partial x_i^2} = L_1(u_1, \cdots, u_d)$$
$$\vdots$$
$$\sum_{i=1}^{n} \frac{\partial^2 u_d}{\partial x_i^2} = L_d(u_1, \cdots, u_d) ,$$

where the L_j are defined by:

$$L\big(\sum u_k V_k\big) = -\sum_j L_j(u_1, \cdots, u_k) V_k .$$

It is trivial to check that the complex characteristics at m of the above linear system are all distinct. Therefore, since m was arbitrarily chosen, they are distinct everywhere. One may thus apply the main theorem of [1]. q.e.d.

2. Immersed submanifolds

2.1. *Connections in the tangent and normal bundles*. Let M be a p-dim manifold, with or without boundary, and let $\bar{M}$ be an n-dimensional riemannian manifold. Suppose $f\colon M \to \bar{M}$ is a C^∞ immersion.

Let $T(M)$ and $N(M)$ denote the tangent and normal bundles of M. The connection and metric on $\bar{M}$ lead to connections and invariant inner products on $T(M)$ and $N(M)$. That on $T(M)$ is, of course, the unique riemannian connection induced by the inherited metric. We will define these connections explicitly.

Let Y be a vector field on M. Restrict Y to a neighborhood of $m \in M$ small enough to be mapped diffeomorphically into $\bar{M}$. Let us identify Y with $df(Y)$, a vector field in $\bar{M}$ defined along the image of M. Let $x \in T(M)_m$. We set

$$\nabla_x Y = (\bar{\nabla}_x Y)^T \tag{2.1.1}$$

where $\bar{\nabla}$ is the riemannian connection in $T(\bar{M})$, and where $(\ )^T$ denotes projection into $T(M)_m$.

PROPOSITION 2.1.1. *∇ is the unique riemannian connection on $T(M)$ with respect to the metric on M inherited from the immersion.*

PROOF. Let Y and Z be vector fields on M. Let $x \in T(M)_m$.

$$\begin{aligned}\langle \nabla_x Y, Z\rangle + \langle Y, \nabla_x Z\rangle &= \langle (\bar{\nabla}_x Y)^T, Z\rangle + \langle Y, (\bar{\nabla}_x Z)^T\rangle \\ &= \langle \bar{\nabla}_x Y, Z\rangle + \langle Y, \bar{\nabla}_x Z\rangle \\ &= \bar{\nabla}_x\langle Y, Z\rangle = \nabla_x\langle Y, Z\rangle .\end{aligned}$$

Thus, ∇ preserves inner products on $T(M)$. To show that ∇ has torsion 0,

$$\begin{aligned}\nabla_Y Z - \nabla_Z Y - [Y, Z] &= (\overline{\nabla}_Y Z)^T - (\overline{\nabla}_Z Y)^T - [Y, Z] \\ &= (\overline{\nabla}_Y Z)^T - (\overline{\nabla}_Z Y)^T - [Y, Z]^T \\ &= \big(\overline{\nabla}_Y Z - \overline{\nabla}_Z Y - [Y, Z]\big)^T = 0 .\end{aligned}$$

q.e.d.

The connection in $N(M)$ is defined similarly. Let V be a cross-section in $N(M)$. Restrict V to a neighborhood of $m \in M$ which is mapped diffeomorphically into M. We may now regard V as a vector field in $\bar{M}$ defined along the image of M. Let $x \in T(M)_m$. We set

$$\nabla_x V = (\overline{\nabla}_x V)^N \tag{2.1.2}$$

where $(\ \ )^N$ denotes projection into $N(M)_m$.

PROPOSITION 2.1.2. *∇ is a connection on $N(M)$ which preserves inner products.*

PROOF. Same as that of Proposition 2.1.1. q.e.d.

2.2. *The second fundamental form.* We define two vector bundles over M associated to $T(M)$ and $N(M)$. Let $S(M)$ be the bundle whose fibre at each point is the space of symmetric linear transformations of $T(M)_m \to T(M)_m$. Let $H(M) = \operatorname{Hom}\big(N(M), S(M)\big)$. If $L \in H(M)_m$ and $w \in N(M)_m$, we will let $L^w\colon T(M)_m \to T(M)_m$ denote the associated symmetric linear transformation. The second fundamental form A is a cross-section in $H(M)$ which we shall define below.

Definition. Let $w \in N(M)_m$. Extend w to an arbitrary vector field W in $\bar{M}$, such that W is normal to $f(M)$ in a neighborhood of $f(m)$. We define $A^w\colon T(M)_m \to T(M)_m$ by

$$A^w(x) = -(\nabla_x W)^T . \tag{2.2.1}$$

A is called the *second fundamental form* of the immersion.

PROPOSITION 2.2.1. *A is well defined by* (2.2.1) *and is a C^∞ cross-section in $H(M)$.*

PROOF. Let W' be another local normal field in $\bar{M}$ which extends w. Let $y \in T(M)_m$, and let Y denote any vector field in $\bar{M}$ which is tangent to M and which extends y. We then have

$$\begin{aligned}&\langle -(\nabla_x W)^T, y\rangle - \langle -(\nabla_x W')^T, y\rangle \\ &\quad = \langle \nabla_x W' - \nabla_x W, y\rangle = \langle \nabla_x (W' - W), y\rangle \\ &\quad = \langle \nabla_x (W' - W), Y\rangle = \nabla_x \langle W' - W, Y\rangle - \langle W' - W, \nabla_x Y\rangle .\end{aligned}$$

But both of these terms are 0, and thus A^w is well defined. To show that A^w is symmetric, extend both x, y to X and Y, vector fields on $\bar{M}$ tangent to M.

We then have

$$\begin{aligned}\langle A^w(x), y\rangle - \langle A^w(y), x\rangle \\ &= -\langle \overline{\nabla}_x W, Y\rangle + \langle \overline{\nabla}_y W, X\rangle \\ &= -\overline{\nabla}_x\langle W, Y\rangle + \overline{\nabla}_y\langle W, X\rangle + \langle W, \overline{\nabla}_x Y\rangle - \langle W, \overline{\nabla}_y X\rangle \\ &= 0 + \langle W, [X, Y]\rangle = 0 .\end{aligned}$$

Clearly A is linear in all variables and therefore does define a cross-section in $H(M)$. A calculation of A in local coordinates would show that $A \in C^\infty$.

q.e.d.

It will sometimes be more convenient to regard the second fundamental form as a symmetric bilinear form on $T(M)_m$ with values in $N(M)_m$. Let x, $y \in T(M)_m$. We define $B(x, y) \in N(M)_m$ by

$$\langle B(x, y), w\rangle = \langle A^w(x), y\rangle . \tag{2.2.2}$$

PROPOSITION 2.2.2. *Let x, $y \in T(M)_m$. Extend y to a vector field Y in $\bar{M}$ which is tangent to $f(M)$. Then*

$$B(x, y) = (\overline{\nabla}_x Y)^N . \tag{2.2.3}$$

PROOF. From (2.2.2) and (2.2.1) we see

$$\begin{aligned}\langle B(x, y), w\rangle &= \langle A^w(x), y\rangle = -\langle \overline{\nabla}_x W, y\rangle \\ &= -\langle \overline{\nabla}_x W, Y\rangle = -\overline{\nabla}_x\langle W, Y\rangle + \langle W, \overline{\nabla}_x Y\rangle \\ &= 0 + \langle \overline{\nabla}_x Y, w\rangle .\end{aligned}$$

q.e.d.

A gives rise to three important cross-sections in bundles associated to $T(M)$ and $N(M)$. The first of these is the mean curvature.

Definition. Since B is a vector valued bilinear form on each $T(M)_m$ taking values in $N(M)_m$, we may define its trace with respect to the inner product on $T(M)_m$. This will be a cross-section in $N(M)$, and we will denote it by K. K is called the *mean curvature* of the immersed M. Let $e_1, \cdots, e_p$ be a frame in $T(M)_m$. Then

$$K = \textstyle\sum_{i=1}^{p} B(e_i, e_i) . \tag{2.2.4}$$

Since A is a cross-section in Hom $(N(M), S(M))$, and since each of these bundles has a euclidean inner product, we may construct ${}^tA$, the transpose of A. ${}^tA$ is a cross-section in Hom $(S(M), N(M))$. I.e., if $s \in S(M)_n$ and $v \in N(H)_n$, $\langle {}^tA(s), v\rangle = \langle A^v, s\rangle$. We then set

$$\widetilde{A} = {}^tA \circ A . \tag{2.2.5}$$

We now observe that, if $v, w \in N(M)_m$ and $s \in S(M)_m$, $[A^v, [A^w, s]] \in S(M)_m$, i.e., the map $(v, w) \to \operatorname{ad} A^v \operatorname{ad} A^w$ is a bilinear form on $N(M)_m$ with values in Hom $(S(M)_m, S(M)_n)$. We define $\underset{\sim}{A}$ to be the trace of this form; i.e., if $v_1, \cdots,$

v_{n-p} is a frame in $N(M)_m$,

$$\underset{\sim}{A} = \sum_{i=1}^{n-p} \operatorname{ad} A^{v_i} \operatorname{ad} A^{v_i} . \tag{2.2.6}$$

PROPOSITION 2.2.3. *$\tilde{A}$ is a C^∞ cross-section in* Hom $(N(M), N(M))$, *and $\underset{\sim}{A}$ is a C^∞ cross-section in* Hom $(S(M), S(M))$. *At each point m, $\tilde{A}$ and $\underset{\sim}{A}$ are symmetric, positive semi-definite operators.*

PROOF. C^∞ is easily checked. Let $v, w \in N(M)_m$. By (2.2.5) we have $\langle \tilde{A}(v), w\rangle = \langle {}^tA(A^v), w\rangle = \langle A^v, A^w\rangle$. Thus $\tilde{A}$ is symmetric, and

$$\langle \tilde{A}(v), v\rangle = \| A^v \|^2 . \tag{2.2.7}$$

Let $s_1, s_2 \in S(M)_m$. Then by (2.2.6)

$$\begin{aligned} \langle \underset{\sim}{A}(s_1), s_2\rangle &= \textstyle\sum_{i=1}^{n-p} \langle [A^{v_i}, [A^{v_i}, s_1]], s_2\rangle \\ &= \textstyle\sum_{i=1}^{n-p} \langle [A^{v_i}, s_1], [A^{v_i}, s_2]\rangle . \end{aligned} \tag{2.2.8}$$

Thus $\underset{\sim}{A}$ is symmetric and positive semi-definite.

2.3. *Curvature in $T(M)$ and $N(M)$.* Using the second fundamental form and the curvature form of $\bar{M}$, we can easily compute the curvature forms in $T(M)$ and $N(M)$.

For $x, y \in T(M)_m$, we define $Q^N_{x,y}$: $N(M)_m \longrightarrow N(M)_m$ by

$$\langle Q^N_{x,y} v, w\rangle = \langle [A^v, A^w](x), y\rangle . \tag{2.3.1}$$

Clearly, $Q^N_{x,y}$ is skew symmetric in x and y, and is a skew symmetric linear transformation of $N(M)_m \longrightarrow N(M)_m$.

Let $\bar{R}$ denote the curvature tensor in $\bar{M}$. For $x, y \in T(M)_m$, we define $\bar{R}^N_{x,y}$: $N(M)_m \longrightarrow N(M)_m$ by

$$\bar{R}^N_{x,y} v = (\bar{R}_{x,y} v)^N . \tag{2.3.2}$$

Again it is clear that $\bar{R}^N_{x,y}$ is skew symmetric in x and y and is a skew symmetric linear transformation of $N(M)_m \longrightarrow N(M)_m$.

PROPOSITION 2.3.1. *Let R denote the curvature tensor for $N(M)$. Then for $x, y \in T(M)_m$,*

$$R_{x,y} = \bar{R}^N_{x,y} + Q^N_{x,y} .$$

PROOF. Let $v, w \in N(M)_m$. Extend them to local cross-sections V, W in $N(M)_m$, and extend x, y to local vector fields X, Y on M. Then by (2.1.2)

$$\begin{aligned} \langle R_{x,y} v, w\rangle &= \langle \nabla_X \nabla_Y V, W\rangle - \langle \nabla_Y \nabla_X V, W\rangle - \langle \nabla_{[X,Y]} V, W\rangle \\ &= \langle \bar{\nabla}_X (\bar{\nabla}_Y V)^N, W\rangle - \langle \bar{\nabla}_Y (\bar{\nabla}_X V)^N, W\rangle - \langle \bar{\nabla}_{[X,Y]} V, W\rangle \\ &= \langle \bar{\nabla}_X \bar{\nabla}_Y V, W\rangle - \langle \bar{\nabla}_Y \bar{\nabla}_X V, W\rangle - \langle \bar{\nabla}_{[X,Y]} V, W\rangle \\ &\qquad - \langle \bar{\nabla}_X (\bar{\nabla}_Y V)^T, W\rangle + \langle \bar{\nabla}_Y (\bar{\nabla}_X V)^T, W\rangle . \end{aligned}$$

Now, using (2.2.1), (2.2.2), and (2.2.3),

$$= \langle \bar{R}_{x,y}v, w\rangle + \langle \bar{\nabla}_X(A^V(Y)), W\rangle - \langle \bar{\nabla}_Y(A^V(X)), W\rangle$$
$$= \langle \bar{R}_{x,y}v, w\rangle + \langle B(x, A^v(y)), w\rangle - \langle B(y, A^v(x)), w\rangle$$
$$= \langle \bar{R}^N_{x,y}v, w\rangle + \langle A^w A^v(y), x\rangle - \langle A^w A^v(x), y\rangle$$
$$= \langle \bar{R}^N_{x,y}v, w\rangle + \langle [A^v, A^w](x), y\rangle \,. \qquad \text{q.e.d.}$$

For $x, y \in T(M)_m$, we define $Q^T_{x,y}$: $T(M)_m \longrightarrow T(M)_m$ by

$$\langle Q^T_{x,y}z, u\rangle = -\langle B(x, z), B(y, u)\rangle + \langle B(x, u), B(y, z)\rangle \,. \tag{2.3.3}$$

Clearly $Q^T_{x,y}$ is skew symmetric in x and y, and is a skew symmetric linear transformation of $T(M)_m \longrightarrow T(M)_m$.

We also define $\bar{R}^T_{x,y}$: $T(M)_m \longrightarrow T(M)_m$ by

$$\bar{R}^T_{x,y}z = (\bar{R}_{x,y}z)^T \,. \tag{2.3.4}$$

PROPOSITION 2.3.2. *Let R denote the curvature tensor for $T(M)$. Then*

$$R_{x,y} = \bar{R}^T_{x,y} + Q^T_{x,y} \,.$$

PROOF. The proof is a calculation similar to that of the previous proposition.

In future calculations, it will sometimes be helpful to have $Q^T_{x,y}$ defined in terms of A rather than B. In the proposition below, we make the identification between $\Lambda^2 T(M)_m$ and skew symmetric linear endomorphisms of $T(M)_m$.

PROPOSITION 2.3.3. *Let $v_1, \cdots, v_{n-p}$ be a frame in $N(M)_m$. Then*

$$Q^T_{x,y} = -\sum_{i=1}^{n-p} A^{v_i}(x) \wedge A^{v_i}(y) \,.$$

PROOF. Straightforward calculation.

2.4. *Variations. Definition.* Let $\{f_t\}$ be a 1-parameter family of immersions of $M \longrightarrow \bar{M}$ with the property that $f_0 = f$, and that the map F: $M \times [0, 1] \longrightarrow \bar{M}$, defined by $F(m, t) = f_t(m)$, be C^∞. Then $\{f_t\}$ is called a *variation* of f.

If $\{f_t\}$ is a variation of f, it induces a vector field in $\bar{M}$ defined along the image of M. We shall denote this field by E, and it is constructed as follows. Let $\partial/\partial t$ be the standard vector field in $M \times [0, 1]$. We set

$$E(m) = dF(\partial/\partial t(m, 0)) \,. \tag{2.4.1}$$

E gives rise to cross-sections E^N and E^T in $N(M)$ and $T(M)$ respectively, by orthogonally projecting E into the appropriate space. We can easily see

PROPOSITION 2.4.1. *Let $\{f_t\}$ be a variation of f. Then E^N and E^T are C^∞ cross-sections in $N(M)$ and $T(M)$ respectively.*

Since E^T is a vector field on M, and M has a volume form induced by its metric, E^T corresponds to a differential $(p-1)$-form on M. We denote this form by θ_{E^T}; i.e.,

$$\theta_{E^T}(x_1, \cdots, x_{p-1}) = \langle E^T \wedge x_1 \wedge \cdots \wedge x_{p-1}, e_1 \wedge \cdots \wedge e_p \rangle , \tag{2.4.2}$$

where $e_1, \cdots, e_p$ is any positively oriented frame in $T(M)_m$.

THEOREM 2.4.1. *Suppose M is compact. Let $\mathcal{A}(t) = p$-dim area of $f_t(M)$. Then $\mathcal{A}(t) \in C^\infty$ and*

$$\mathcal{A}'(0) = -\int_M \langle E^N, K \rangle + \int_{\partial M} \theta_{E^T} , \tag{2.4.3}$$

where the integration over M of the function $\langle E^N, K \rangle$ is understood to be with respect to the volume form corresponding to its riemannian structure.

PROOF. As this is a well known result, we shall not give its proof here. One may rather easily do the calculation in local coordinates, cf. [2]. One may also do it globally by lifting each f_t to a map $\hat{f}_t$ of M into $G^p(\bar{M})$, the Grassmann bundle of oriented p-planes, cf. [3]. In this bundle is a natural p-form ω, such that $\mathcal{A}(t) = \int_M \omega \circ d\hat{f}_t$. The calculation then proceeds easily *via* a simple formula which relates $d\omega$ to mean curvature. This global approach is particularly useful in calculating the second variation.

3. Minimal varieties

3.1. *Definitions and examples.* As in the previous section, we have $f: M \to \bar{M}$ an immersion.

Definition. M is called a minimal variety in $\bar{M}$ if $K \equiv 0$, i.e., the mean curvature of M vanishes at each point.

The simplest examples of minimal varieties are where $\dim M = 1$. In this case they are simply geodesics in $\bar{M}$. In higher dimensions every totally geodesic submanifold is a minimal variety, in fact, the condition "totally geodesic" is equivalent to the condition that the entire second fundamental form vanish identically.

The class of minimal varieties is much richer than the class of totally geodesic submanifolds. Geodesics in $\bar{M}$ are the projection down of the integral curves of a direction field in the sphere bundle. In an analogous fashion, minimal varieties in $\bar{M}$ are the projection down of the p-dim integral submanifolds of a differential ideal in $G^p(\bar{M})$, the bundle of oriented p-planes, cf. [3]. Moreover, this ideal has no $(p-1)$-dim characteristics which do not intersect the vertical. One may thus use the Cartan-Kähler theorem to prove

THEOREM 3.1.1. *Suppose $\bar{M}$ is real analytic. Let U be a neighborhood of 0 in R^{p-1}, and $g: U \to \bar{M}$ an analytic imbedding. For $u \in U$, let $dg(u)$ de-*

note the $(p-1)$-plane tangent to $g(U)$ at $g(u)$. Suppose along $g(U)$ we are given an analytic field of oriented p-planes P, such that $dg(u) \subset P(u)$. Then there exists a neighborhood of 0, $V \subseteqq U$, and an $\varepsilon > 0$, and an imbedding $f: V \times [0, \varepsilon] \rightarrow \bar{M}$ which satisfies

(1) $f(v, 0) = g(v)$;

(2) $df(v, 0) = P(v)$;

(3) $V \times [0, \varepsilon)$ *is imbedded as a minimal variety in* $\bar{M}$.

Moreover, f is unique up to analytic parametrization.

The above theorem is of no use in constructing local minimal varieties in C^∞ manifolds. Moreover, since the system of partial differential equations defining a minimal variety is elliptic, the natural problem to try to solve is the boundary value problem. Extensive research has been carried out in this area, and in the past few years a great deal of progress has been made. A complete set of references may be found in [4].

The most interesting minimal varieties, at least from the geometric point of view, are the closed minimal varieties in compact manifolds, and the complete minimal varieties in non-compact manifolds. In [5] Hsiang has demonstrated that, under the action of a closed subgroup of isometries of a compact manifold, an orbit of maximum area is a closed minimal variety. This produces a wide class of examples, including new examples of closed, co-dimension 1, minimal varieties in S^n. Of equal interest are complete minimal varieties in R^n. The well known Bernstein conjecture states that, if $f(x_1, \cdots, x_{n-1})$ is defined everywhere and its graph is a minimal variety in R^n, then f is a linear function. This conjecture has been proved for $n = 3, 4, 5$, see [6], [7], [8], and [9]. In the last section of the present paper, we show that the conjecture holds through $n = 8$. It is interesting to note, as we have mentioned in § 0, that there is a close relationship between complete minimal varieties in euclidean space and closed minimal varieties in the sphere.

As a final class of examples, we turn to Kähler manifolds. The following theorem is well known.

THEOREM 3.1.2. *Let $\bar{M}$ be a $2n$-dimensional Kähler manifold, and let M be a $2p$-dimensional immersed Kähler submanifold. Then M is a minimal variety in $\bar{M}$.*

PROOF. Let J denote the covariant constant linear transformation with $J^2 = -1$. Let $m \in M$. Since M is sub-Kähler, if $x \in T(M)_m$, then $J(x) \in T(M)_m$. We first show

$$B(x, x) + B\big(J(x), J(x)\big) = 0 . \qquad (3.1.1)$$

Extend x to a vector field X on M. Then by (2.2.3) and the invariance of J,

$$B\big(J(x), J(x)\big) = \big(\overline{\nabla}_{J(x)}J(X)\big)^N = \big(J(\overline{\nabla}_{J(x)}X)\big)^N = J\big((\overline{\nabla}_{J(x)}X)^N\big) .$$

Since B is symmetric

$$= J\big((\overline{\nabla}_x J(X))^N\big) = J^2\big(B(x, x)\big) = -B(x, x) .$$

Now, we may choose a frame in $T(M)_m, e_1, \cdots, e_p, f_1, \cdots, f_p$ where $J(e_i) = f_i$. Then from (2.2.4) and the above, the theorem follows. q.e.d.

3.2. *The first and second variations.* In this section and all the following, M will be a p-dim manifold, $\bar{M}$ an n-dim riemannian manifold, and $f: M \to \bar{M}$ an immersion of M as a minimal variety in $\bar{M}$.

THEOREM 3.2.1. *Let $\{f_t\}$ be a variation of f. Suppose that for all t, $f_t(\partial M) = f(\partial M)$. Then if M is compact and $\mathfrak{A}(t) =$ area of M under f_t, $\mathfrak{A}'(0) = 0$.*

PROOF. Follows directly from Theorem 2.4.1 and the observation that our boundary conditions imply that $\theta_{E^T} \mid \partial M = 0$. q.e.d.

The above theorem shows that compact minimal varieties are critical points of the area function. This being the case, we should be able to calculate the second variation simply in terms of E^N.

Let $v \in N(M)_m$. Let $e_1, \cdots, e_p$ be a frame in $T(M)_p$. We define

$$\bar{R}(v) = \textstyle\sum_{i=1}^{p} (\bar{R}_{e_i,v}e_i)^N . \tag{3.2.1}$$

Equation (3.2.1) defines a linear transformation from $N(M)_n$ into itself. Note $\langle \bar{R}(v), w\rangle = \langle \bar{R}(w), v\rangle$. Clearly this definition did not depend on choice of frame. It is a sort of partial Ricci transformation.

THEOREM 3.2.2. *Let $\{f_t\}$ be a variation of f such that each $f_t(\partial M) = f(M)$. Suppose M is compact. Set $V = E^N$, the cross-section in $N(M)$ defined in Proposition* 2.4.1. *Then*

$$\mathfrak{A}''(0) = \int_M \langle \nabla v, \nabla v\rangle + \langle \bar{R}(v), v\rangle - \langle \tilde{A}(v), v\rangle . \tag{3.2.2}$$

PROOF. (3.2.2) is simply a restatement in our terminology of the similar calculation in [10]. A formula of this type may also be found in [11], however it has a sign error in the term involving the second fundamental form, and this interferes with the author's geometric applications.

Using Proposition 1.2.2, we may rewrite (3.2.2) as

$$\mathfrak{A}''(0) = \int_M \langle -\nabla^2 V + \bar{R}(V) - \tilde{A}(V), V\rangle . \tag{3.2.3}$$

Let M be compact and V, W cross-sections in $N(M)$ which vanish on ∂M. We set

(3.2.4) $$I(V, W) = \int_M \langle -\nabla^2 V + \bar{R}(V) - \tilde{A}(V), W \rangle .$$

From Proposition 1.2.3 and the standard facts about strongly elliptic differential operators, we see

PROPOSITION 3.2.1. *I is a symmetric bilinear form on the space of cross-sections in $N(M)$ which vanish on ∂M. I may be diagonalized with respect to the standard inner product, and has distinct, real eigenvalues $\{\lambda_i\}$ such that*

$$\lambda_1 < \lambda_2 < \cdots < \lambda_i < \cdots \longrightarrow +\infty .$$

Moreover, the dimension of each eigenspace is finite.

Definition. The *index* of a compact M is the sum of the dimensions of the eigenspaces corresponding to negative eigenvalues. The *nullity* of M is the dimension of the 0-eigenspace.

3.3. *Jacobi fields. Definition.* A cross-section V in $N(M)$ is called a *Jacobi field* if it satisfies

(3.3.1) $$\nabla^2 V = \bar{R}(V) - \tilde{A}(V) .$$

We may easily see

PROPOSITION 3.3.1. *Let M be compact. Then the space of Jacobi fields on M which vanish on ∂M is finite dimensional and is equal to the kernel of I. Thus, the dimension of this space is the nullity of M.*

The following theorem is the p-dimensional analogue of the usual theorem for geodesics. The proof is a straightforward, although involved, calculation, and we therefore leave it for the reader to verify.

THEOREM 3.3.1. *Let $\{f_t\}$ be a variation of M. Suppose each (M, f_t) is an immersed minimal variety. Then E^N is a Jacobi field on M.*

COROLLARY 3.3.1. *Let V be a killing vector field on $\bar{M}$. Let V^N be the cross-section in $N(M)$ obtained by projection of V. Then V^N is a Jacobi field on M.*

COROLLARY 3.3.2. *Let $\bar{M}$ be a Kähler manifold and suppose V is a conformal vector field on $\bar{M}$,* i.e., *the Lie derivative of J in the V direction vanishes. Then, if M is an immersed Kähler submanifold, V^N is a Jacobi field on M.*

In a subsequent section we will characterize all Jacobi fields on compact, closed, Kähler submanifolds.

In the case of geodesics, the converse of Theorem 3.3.1 is also true. As far as we know, it is an open question as to whether or not even a local version of this converse is true for higher dimensional minimal varieties. Namely, given a Jacobi field on M, does it always arise, at least locally, from a one-parameter family of minimal varieties? In the real analytic case, such a result could probably be obtained *via* Theorem 3.1.1.

3.4. *The Morse index theorem.* The original Morse index theorem is a formula which relates the index of a geodesic segment to its conjugate points relative to one end point. For a statement and proof see [12]. Recently, Smale [13] has substantially generalized this result to a theorem which, in a similar way, evaluates the index of any symmetric, strongly elliptic, differential operator on the cross-sections of a vector bundle. The theorem applies perfectly to the case of minimal varieties, and gives a natural generalization of the index theorem for geodesics. In [13], Smale was kind enough to credit us with this result; however, while we did have a correct statement of the theorem, we were never able to produce a correct proof. The following will be an exposition of Smale's result in our context.

Let M be compact, and suppose that $\dim \partial M = p - 1$.

Definition. Let $\{g_t\}$ be a 1-parameter family of diffeomorphisms of M into itself. Suppose $g_0 =$ identity, and the map $(m, t) \longrightarrow g_t(m)$ is C^∞. Let M_t denote the image of M under g_t. If $M_t \subsetneqq M_s$, whenever $t > s$, $\{g_t\}$ is called a *contraction* of M. $\{g_t\}$ is said to be of *ε-type* if $\mathcal{A}(M_t) < \varepsilon$ for sufficiently large t.

LEMMA 3.4.1. *Let $C_0^\infty N(M_t)$ denote the space of C^∞ cross-sections in $N(M_t)$ which vanish on $\partial(M_t)$. Then there exists an $\varepsilon > 0$ such that, for any contraction $\{g_t\}$, $\mathcal{A}(M_t) < \varepsilon$ implies that $I_t(V, V) > 0$ for any $V \in C_0^\infty N(M_t)$ where*

$$I_t(V, V) = \int_{M_t} \langle -\nabla^2 V + \bar{R}(V) - \tilde{A}(V), V \rangle .$$

PROOF. See [13].

Definition. Given a fixed contraction $\{g_t\}$, ∂M_t is called a *conjugate boundary* if there is a Jacobi field in $C_0^\infty N(M_t)$. The *order* of a conjugate boundary is defined to be the dimension of the space of such Jacobi fields.

THEOREM 3.4.1. *Let ε be chosen as in Lemma 3.4.1, and let $\{g_t\}$ be a contraction of ε-type. Then there exist only a finite number of conjugate boundaries, ∂M_{t_i}, and the index of M is the sum of the orders of these boundaries for all $t_i \neq 0$.*

PROOF. Since we have shown in Proposition 1.2.3 that $-\nabla^2 + \bar{R} - \tilde{A}$ is strongly elliptic and has uniqueness in the Cauchy problem, we may directly apply the main theorem of [13]. q.e.d.

3.5. *Jacobi fields on Kähler submanifolds.* In [17], Federer shows that a compact Kähler submanifold of a Kähler manifold is an absolute minimum of area among all homologous competitors with the same boundary. A simple consequence of this is that the index of such a minimal variety must be 0. An interesting question then is, what is the nullity? To give an example of these notions of index and nullity, we prove Theorem 3.5.1 below.

In the following, we will assume that M is a $2p$ dimensional immersed Kähler submanifold of a $2n$ dimensional Kähler manifold $\bar{M}$. The J operator gives an automorphism of each $T(M)_m$ and $N(M)_m$. A cross-section V in $N(M)_m$ will be called *holomorphic* if, for any $x \in T(M)_m$,

$$\nabla_{J(x)} V = J(\nabla_x V) .$$

THEOREM 3.5.1. *Let M be compact with* $\dim \partial M = 2p - 1$. *Then the index of M = the nullity of M = 0. Let M be compact and closed. Then the index of M = 0, and the nullity of M is equal to the dimension of the space of globally defined holomorphic cross-sections in $N(M)$.*

The proof of this theorem will follow from a series of lemmas.

LEMMA 3.5.1. *For any $v \in N(M)_m$,*

$$J \circ A^v = A^{J(v)} = -A^v \circ J . \tag{3.5.1}$$

$$\widetilde{A} \circ J = J \circ \widetilde{A} . \tag{3.5.2}$$

PROOF. Extend v to V, a local cross-section in $N(M)_m$. Let $x \in T(M)_m$. Using (2.2.1),

$$\begin{aligned} J(A^v(x)) &= -J((\bar{\nabla}_x V)^T) = -(J(\bar{\nabla}_x V))^T \\ &= -(\bar{\nabla}_x J(V))^T = A^{J(v)}(x) . \end{aligned}$$

Thus, $J \circ A^v = A^{J(v)}$. Also, for $x, y \in T(M)_m$,

$$\begin{aligned} \langle A^v(J(x)), y \rangle &= \langle A^v(y), J(x) \rangle = -\langle J(A^v(y)), x \rangle \\ &= \langle A^{J(v)}(y), x \rangle = -\langle A^{J(v)}(x), y \rangle . \end{aligned}$$

Thus $A^v \circ J = -A^{J(v)}$, and this proves (3.5.1).

To prove (3.5.2), we use (3.5.1) and (2.2.5), which defines $\widetilde{A}$. Let $v, w \in N(M)_m$.

$$\begin{aligned} \langle \widetilde{A}(J(v)), w \rangle &= \langle A^{J(v)}, A^w \rangle = \langle J \circ A^v, A^w \rangle \\ &= -\langle A^v, J \circ A^w \rangle = -\langle A^v, A^{J(w)} \rangle \\ &= -\langle \widetilde{A}(v), J(w) \rangle = \langle J(\widetilde{A}(v)), w \rangle . \end{aligned}$$ q.e.d.

Let R denote the curvature form in $N(M)$. Let $e_1, \cdots, e_p, f_1, \cdots, f_p$ be a frame in $T(M)_m$ with $f_i = J(e_i)$. We define R_J, a symmetric, linear transformation of $N(M)_m$ into itself, by

$$R_J = -J \circ \sum_{i=1}^{p} R_{e_i, f_i} .$$

Clearly R_J is well defined.

LEMMA 3.5.2. $R_J(v) = \bar{R}(v) - \tilde{A}(v)$.

PROOF. From Proposition 2.3.1, we know

$$(*) \qquad R_{e_i, f_i} = \bar{R}^N_{e_i, f_i} + Q^N_{e_i, f_i} .$$

Using the first Bianchi identity, and the fact that the curvature transformations in M commute with J, we see that, for $v \in N(M)_m$,

$$\begin{aligned} \bar{R}^N_{e_i, f_i} v &= (\bar{R}_{e_i, f_i} v)^N = (-\bar{R}_{v, e_i} f_i - \bar{R}_{f_i, v} e_i)^N \\ &= \big(\bar{R}_{e_i, v} J(e_i) + \bar{R}_{f_i, v} J(f_i)\big)^N = J(\bar{R}_{e_i, v} e_i + \bar{R}_{f_i, v} f_i)^N . \end{aligned}$$

Thus by (3.2.1) which defines $\bar{R}(v)$,

$$(**) \qquad -J \circ \sum_{i=1}^{p} \bar{R}^N_{e_i, f_i} v = \bar{R}(v) .$$

Let $v, w \in N(M)_m$. By (2.3.1) which defines Q^N, and using Lemma 3.5.1,

$$\begin{aligned} \langle Q^N_{e_i, f_i} v, w \rangle &= \langle [A^v, A^w](e_i), f_i \rangle \\ &= \langle A^w(e_i), A^v(f_i) \rangle - \langle A^w(f_i), A^v(e_i) \rangle \\ &= -\langle A^w(e_i), A^{J(v)}(e_i) \rangle - \langle A^w(f_i), A^{J(v)}(f_i) \rangle . \end{aligned}$$

Thus

$$\begin{aligned} \sum_{i=1}^{p} \langle Q^N_{e_i, f_i} v, w \rangle &= -\langle A^w, A^{J(v)} \rangle \\ &= -\big\langle \tilde{A}(J(v)), w \big\rangle = -\langle J \circ \tilde{A}(v), w \rangle . \end{aligned}$$

So we get

$$(***) \qquad -J \sum_{i=1}^{p} Q^N_{e_i, f_i} v = -\tilde{A}(v) .$$

Putting together $(*)$, $(**)$ and $(***)$ the lemma is proved. q.e.d.

From the definition of Jacobi field (3.3.1) and of the form $I(V, W)$ (3.2.4), the above lemma shows

LEMMA 3.5.3. *Let V be a cross-section in $N(M)$. Then V is a Jacobi field if and only if*

$$\nabla^2 V = R_J(V) .$$

Moreover, for any V with $V \mid \partial M = 0$,

$$I(V, V) = \int_M \langle -\nabla^2 V + R_J(V), V \rangle .$$

LEMMA 3.5.4. *Let V be a holomorphic cross-section in $N(M)$. Then V is a Jacobi field.*

PROOF. Choose $e_1, \cdots, e_p, f_1, \cdots, f_p$ a frame in $T(M)_m$ with $f_i = J(e_i)$. Extend these to local vector fields $\{E_i, F_i\}$, such that they form a frame at each point, $F_i = J(E_i)$, and $\nabla_x E_i(m) = \nabla_x F_i(m) = 0$ for all i and all $x \in T(M)$. Now, using (1.2.3) and the definition of holomorphic cross-section,

$$\begin{aligned}\nabla^2 V &= \textstyle\sum_{i=1}^{p} (\nabla_{E_i}\nabla_{E_i} V + \nabla_{F_i}\nabla_{F_i} V) \\ &= \textstyle\sum_{i=1}^{p} \big(\nabla_{E_i}\nabla_{E_i} V + J(\nabla_{F_i}\nabla_{E_i} V)\big) \\ &= \textstyle\sum_{i=1}^{p} \big(\nabla_{E_i}\nabla_{E_i} V + J(\nabla_{E_i}\nabla_{F_i} V) + J(R_{f_i,e_i} V)\big) \\ &= \textstyle\sum_{i=1}^{p} \big(\nabla_{E_i}\nabla_{E_i} V + J^2(\nabla_{E_i}\nabla_{E_i} V) - J(R_{e_i,f_i} V)\big) = R_J(V)\,. \quad \text{q.e.d.}\end{aligned}$$

PROOF OF THEOREM. If V is any cross-section in $N(M)$, it gives rise to DV, a cross-section in $\operatorname{Hom}\big(T(M), N(M)\big)$ defined by

$$DV(x) = \nabla_{J(x)} V - J(\nabla_x V)\,.$$

Clearly V is holomorphic if and only if $DV \equiv 0$. Now, using the standard Stokes' theorem technique (cf. Proposition 1.2.2), one may easily show that, for any V with $V \mid \partial M = 0$, we have

$$\int_M \langle DV, DV\rangle = 2\int_M \langle -\nabla^2 V + R_J(V), V\rangle\,.$$

Using Lemma 3.5.3, we see

$$(*) \qquad I(V, V) = \frac{1}{2}\int_M \langle DV, DV\rangle\,.$$

Case I: $\dim \partial M = 2p - 1$. The above (*) shows $I(V, V) \geqq 0$, which implies the index of $M = 0$; (*) also shows that $I(V, V) = 0$ implies V is holomorphic. But since $V \mid \partial M = 0$, we must have $V \equiv 0$. Thus the nullity of $M = 0$.

Case II: $\partial M = 0$. (*) again shows that the index of $M = 0$. Since V a Jacobi field implies $I(V, V) = 0$, (*) also shows that all Jacobi fields are holomorphic. Lemma 3.5.4 shows that holomorphic cross-sections are Jacobi fields. Thus, the nullity of M is equal to the dimension of the space of global holomorphic cross-sections. q.e.d.

3.6. *An extension of the Synge lemma.* The well known Synge lemma states that, in a manifold of strictly positive sectional curvature, any closed geodesic admitting a parallel normal vector field may be deformed to a closed curve of shorter length. Such a normal field may always be constructed, for example, if the dimension of the manifold is even. There is an easy generalization of this theorem to compact, closed minimal varieties of arbitrary dimension.

If x, y are tangent vectors at $\bar{m} \in \bar{M}$, let $k(x, y)$ denote the sectional curvature of the plane section they span. If $\|x\| = \|y\| = 1$ and $\langle x, y\rangle = 0$,

$$(3.6.1) \qquad k(x, y) = -\langle \bar{R}_{x,y} x, y\rangle\,.$$

If P is a tangent p-plane at $\bar{m} \in \bar{M}$, and $y \in P^N$, the normal space to P, we set

$$k(P, y) = \sum_{i=1}^{p} k(e_i, y) = -\sum_{i=1}^{p} \langle \bar{R}_{e_i, y} e_i, y \rangle , \tag{3.6.2}$$

where $e_1, \cdots, e_p$ is a frame spanning P. Clearly $k(P, y)$ is defined independently of the choice of frame. Note that for $p = \dim P = n - 1$, $k(P, y)$ is the Ricci curvature of y.

THEOREM 3.6.1. *Let $\bar{M}$ be a riemannian manifold having the property that, for any p-plane P and any $y \in P^N$, $k(P, y) > 0$. Then, if M is a compact, closed minimal variety immersed in $\bar{M}$ such that there exists a globally defined, parallel cross-section in $N(M)$, M is deformable to a closed manifold of smaller area.*

COROLLARY 3.6.1. *If $\bar{M}$ has positive Ricci curvature, then any co-dimension 1 closed minimal variety immersed in $\bar{M}$ is deformable to a closed manifold of smaller area.*

PROOF. The corollary follows trivially from the theorem, since the unit normal field is always parallel in $N(M)$ under the co-dimension 1 assumption.

To prove the theorem, let V be a parallel normal cross-section in $N(M)$, and choose a variation $\{f_t\}$ such that $E^N = V$. (This may be done, for example, *via* the exponential map.) Looking at the second variation formula (3.2.2) we see that

$$\mathcal{A}''(0) = \int_M \langle \bar{R}(V), V \rangle - \langle \tilde{A}(V), V \rangle . \tag{$*$}$$

But, if $P(m)$ denotes the tangent p-plane to $f(M)$ at $f(m)$, (3.6.2) shows that

$$\langle \bar{R}(V), V \rangle = -k\big(P(m), V(n)\big) .$$

Since by (2.2.7), $\langle \tilde{A}(V), V \rangle = \| A^V \|^2$, we see that both terms in $(*)$ are negative, and so $\mathcal{A}''(0) < 0$. Since $\mathcal{A}'(0) = 0$, we see that area is decreasing. q.e.d.

4. The fundamental elliptic equation

4.1. *The first order system.* We have defined the second fundamental form A to be a cross-section in $H(M) = \mathrm{Hom}\big(N(M), S(M)\big)$. Since this bundle is a riemannian vector bundle, we may use its connection to make calculations involving derivatives of A. In the event that M is a minimal variety, such calculations show that A satisfies an elliptic first order system and an elliptic second order system. In both cases the coefficients depend only on the curvature tensor of the ambient manifold and on A itself. When the ambient manifold is special, e.g., the sphere or euclidean space, the equations take a particularly nice form and enable one to make geometric conclusions about the immersed variety.

As in the previous section, M will denote a p-dim manifold, $\bar{M}$ an n-dim riemannian manifold, and f an immersion of M into $\bar{M}$ as a minimal variety.

THEOREM 4.1.1. *Let $B(,)$ denote the second fundamental form of M, when that object is regarded as a symmetric bilinear form on $T(M)_m$ with values in $N(M)_m$. Then*

$$\nabla_x(B)(y, z) - \nabla_y(B)(x, z) = (\bar{R}_{x,y}z)^N \qquad \forall x, y, z \in T(M)_m \tag{4.1.1}$$

$$\textstyle\sum_{i=1}^p \nabla_{e_i}(B)(e_i, z) = \sum_{i=1}^p (\bar{R}_{e_i,z}e_i)^N \qquad \forall z \in T(M)_m\ . \tag{4.1.2}$$

PROOF. We first prove (4.1.1), which holds for any submanifold of M whether or not it is a minimal variety.

Extend x, y, z to X, Y, Z, local vector fields on M such that all the covariant constant at m with respect to ∇. Then, using the standard facts about covariant differentiation,

$$\begin{aligned}\nabla_x(B)(y, z) &= \nabla_x\big(B(Y, Z)\big) = \nabla_x(\bar{\nabla}_Y Z)^N \\ &= \big(\bar{\nabla}_x(\bar{\nabla}_Y Z)^N\big)^N = (\bar{\nabla}_x\bar{\nabla}_Y Z)^N - \big(\bar{\nabla}_x(\bar{\nabla}_Y Z)^T\big)^N \\ &= (\bar{\nabla}_x\bar{\nabla}_Y Z)^N - B(x, \nabla_y Z) = (\bar{\nabla}_x\bar{\nabla}_Y Z)^N\end{aligned}$$

since $\nabla_y Z = 0$. Interchanging x and y, we see that

$$\begin{aligned}\nabla_x(B)(y, z) - \nabla_y(B)(x, z) &= (\bar{\nabla}_x\bar{\nabla}_Y Z - \bar{\nabla}_y\bar{\nabla}_X Z)^N \\ &= (\bar{R}_{x,y}z)^N + (\bar{\nabla}_{[X,Y]}Z)^N = (\bar{R}_{x,y}z)^N\end{aligned}$$

since $[X, Y] = 0$.

To prove (4.1.2) let $e_1, \cdots, e_p$ be a frame in $T(M)_m$. Then, using (4.1.1), and the symmetry of B,

$$\begin{aligned}\textstyle\sum_{i=1}^p \nabla_{e_i}(B)(e_i, z) &= \textstyle\sum_{i=1}^p \nabla_{e_i}(B)(z, e_i) \\ &= \textstyle\sum_{i=1}^p \big(\nabla_z(B)(e_i, e_i) + (\bar{R}_{e_i,z}e_i)^N\big)\ .\end{aligned}$$

But since M is a minimal variety, $\operatorname{tr} B = 0$, which implies $\operatorname{tr} \nabla_z(B) = 0$. The theorem now follows. q.e.d.

4.2. *The second order system.* We now define several new cross-sections in $H(M)$. Let $e_1, \cdots, e_p$ be a frame in $T(M)_m$. For $x, y \in T(M)_m$ and $w \in N(M)_m$, set

$$\langle \bar{R}'^w(x), y\rangle = \textstyle\sum_{i=1}^p \big(\langle \bar{\nabla}_x(\bar{R})_{e_i,y}e_i, w\rangle + \langle \bar{\nabla}_{e_i}(\bar{R})_{e_i,x}y, w\rangle\big)\ . \tag{4.2.1}$$

Clearly $\bar{R}'$ is defined independently of the choice of frame and is linear in w, x, and y. To see that it is symmetric in x and y, use the second Bianchi identity on the first piece and the first Bianchi identity on the second piece.

We also set

$$(4.2.2)\qquad \langle \bar{R}(A)^w(x), y\rangle = \sum_{i=1}^{p} \left\{ \begin{array}{l} 2\langle \bar{R}_{e_i,y}B(x, e_i), w\rangle + 2\langle \bar{R}_{e_i,x}B(y, e_i), w\rangle \\ -\langle A^w(x), \bar{R}_{e_i,y}e_i\rangle - \langle A^w(y), \bar{R}_{e_i,x}e_i\rangle \\ +\langle \bar{R}_{e_i,B(x,y)}e_i, w\rangle - 2\langle A^w(e_i), \bar{R}_{e_i,x}y\rangle \end{array} \right\} .$$

Again, $\bar{R}(A)$ is defined independently of the choice of frame. To see that it is symmetric in x and y, we observe that interchanging these two variables interchanges term one with term two, and term three with term four. Term five is obviously symmetric in x and y. So is term six, and this may be seen by using the first Bianchi identity and the fact that A^w is symmetric and $\bar{R}^T_{x,y}$ is anti-symmetric.

We have thus seen

PROPOSITION 4.2.1. *$\bar{R}'$ and $\bar{R}(A)$ are C^∞ cross-sections in $H(M)$. $\bar{R}'$ depends only on the covariant differential of $\bar{R}$ in the ambient $\bar{M}$. $\bar{R}(A)$ is linear in $\bar{R}$ and A.*

From Proposition 2.2.3, we see that $A \circ \tilde{A}$ and $\underset{\sim}{A} \circ A$ are also cross-sections in $H(M)$.

THEOREM 4.2.1. *Let A be the second fundamental form of a minimal variety. Then A satisfies*

$$\nabla^2 A = -A \circ \tilde{A} - \underset{\sim}{A} \circ A + \bar{R}(A) + \bar{R}' .$$

PROOF. Let $e_1, \cdots, e_p$ denote a frame in $T(M)_m$, and let $E_1, \cdots, E_p$ be local, orthonormal vector fields on M which extend $e_1, \cdots, e_p$, and which are covariant constant with respect to ∇ at m. Let $x, y \in T(M)_m$, and let X, Y be local extensions which are also covariant constant with respect to ∇. Since the subsequent calculation is rather lengthy, we break it up into a series of lemmas. The curvature form for all bundles associated with $T(M)$ and $N(M)$ will always be denoted by R.

LEMMA a.

$$\nabla^2(B)(x, y) = \sum_{i=1}^{p} \big(R_{e_i,x}(B)(e_i, y) + \big(\bar{\nabla}_X(\bar{R}_{E_i,Y}E_i)^N\big)^N + \big(\bar{\nabla}_{E_i}(\bar{R}_{E_i,X}Y)^N\big)^N\big) .$$

PROOF. Using (4.1.1) and (4.1.2), we have

$$\begin{aligned} \nabla^2(B)(x, y) &= \sum_{i=1}^{p} \nabla_{E_i}\nabla_{E_i}(B)(x, y) = \sum_{i=1}^{p} \nabla_{E_i}\big(\nabla_{E_i}(B)(X, Y)\big) \\ &= \sum_{i=1}^{p} \big(\nabla_{E_i}\big(\nabla_X(B)(E_i, Y)\big) + \nabla_{E_i}(\bar{R}_{E_i,X}Y)^N\big) \\ &= \sum_{i=1}^{p} \big(\nabla_{E_i}\nabla_X(B)(e_i, y) + \nabla_{E_i}(\bar{R}_{E_i,X}Y)^N\big) \\ &= \sum_{i=1}^{p} \big(R_{e_i,x}(B)(e_i, y) + \nabla_X\nabla_{E_i}(B)(e_i, y) + \nabla_{E_i}(\bar{R}_{E_i,X}Y)^N\big) \\ &= \sum_{i=1}^{p} \big(R_{e_i,x}(B)(e_i, y) + \nabla_X\big(\nabla_{E_i}(B)(E_i, Y)\big) + \nabla_{E_i}(\bar{R}_{E_i,X}Y)^N\big) \\ &= \sum_{i=1}^{p} \big(R_{e_i,x}(B)(e_i, y) + \nabla_X(\bar{R}_{E_i,Y}E_i)^N + \nabla_{E_i}(\bar{R}_{E_i,X}Y)^N\big) \\ &= \sum_{i=1}^{p} \big(R_{e_i,x}(B)(e_i, y) + \big(\bar{\nabla}_X(\bar{R}_{E_i,Y}E_i)^N\big)^N + \big(\bar{\nabla}_{E_i}(\bar{R}_{E_i,X}Y)^N\big)^N\big) . \end{aligned}$$

q.e.d.

We now examine separately each of the three terms in Lemma a.

LEMMA b.

$$\sum_{i=1}^{p}\big(\overline{\nabla}_X(\bar{R}_{E_i,Y}E_i)^N\big)^N = \sum_{i=1}^{p}\big((\overline{\nabla}_x(\bar{R})_{e_i,y}e_i)^N + (\bar{R}_{B(x,e_i),y}e_i)^N + (\bar{R}_{e_i,B(x,y)}e_i)^N + \big(\bar{R}_{e_i,y}B(x, e_i)\big)^N - B\big(x, (\bar{R}_{e_i,y}e_i)^T\big)\big) .$$

PROOF.

$$\begin{aligned}\sum_{i=1}^{p}&\big(\overline{\nabla}_X(\bar{R}_{e_i,Y}E_i)^N\big)^N \\ &= \sum_{i=1}^{p}\big((\overline{\nabla}_X(\bar{R}_{E_i,Y}E_i))^N - (\overline{\nabla}_X(\bar{R}_{E_i,Y}E_i)^T)^N\big) \\ &= \sum_{i=1}^{p}\big((\overline{\nabla}_x(\bar{R})_{e_i,y}e_i)^N + (\bar{R}_{\overline{\nabla}_X E_i,Y}E_i)^N + (\bar{R}_{E_i,\overline{\nabla}_X Y}E_i)^N \\ &\qquad + (\bar{R}_{E_i,Y}\nabla_X E_i)^N - B\big(x, (\bar{R}_{e_i,y}e_i)^T\big)\big) .\end{aligned}$$

Since at n, $\nabla_X E_i = \nabla_X Y = 0$, we see that at m, $\overline{\nabla}_X E_i = B(x, e_i)$ and $\overline{\nabla}_X Y = B(x, y)$. Plugging these into the last line of the calculation proves the lemma. q.e.d.

LEMMA c.

$$\begin{aligned}\sum_{i=1}^{p}&\big(\overline{\nabla}_{E_i}(\bar{R}_{E_i,X}Y)^N\big)^N \\ &= \sum_{i=1}^{p}\big((\overline{\nabla}_{e_i}(\bar{R})_{e_i,y}e_i)^N + (\bar{R}_{e_i,B(e_i,x)}y)^N \\ &\qquad + \big(\bar{R}_{e_i,x}B(e_i, y)\big)^N - B\big(e_i, (\bar{R}_{e_i,x}y)^T\big)\big) .\end{aligned}$$

PROOF. Similar to that of the preceding lemma. One term disappears since

$$\sum_{i=1}^{p} B(e_i, e_i) = 0 . \qquad \text{q.e.d.}$$

LEMMA d.

$$\begin{aligned}\sum_{i=1}^{p}&R_{e_i,x}(B)(e_i, y) \\ &= \sum_{i=1}^{p}\big((\bar{R}_{e_i,x}B(e_i, y))^N - B\big((\bar{R}_{e_i,x}e_i)^T, y\big) \\ &\qquad - B\big(e_i, (\bar{R}_{e_i,x}y)^T\big) + Q^N_{e_i,x}B(e_i, y) \\ &\qquad - B(Q^T_{e_i,x}e_i, y) - B(e_i, Q^T_{e_i,x}y)\big) .\end{aligned}$$

PROOF.

$$\begin{aligned}\sum_{i=1}^{p}&R_{e_i,x}(B)(e_i, y) \\ &= \sum_{i=1}^{p}\big(R_{e_i,x}\big(B(e_i, y)\big) - B(R_{e_i,x}e_i, y) - B(e_i, R_{e_i,x}y)\big) .\end{aligned}$$

The lemma now follows from Propositions 2.3.1 and 2.3.2 which express R in terms of $\bar{R}$, and Q^N, and Q^T. q.e.d.

Plugging the formulas proved in Lemmas b—d into the formula in Lemma a, and using the definitions of $\bar{R}(A)$ and $\bar{R}'(A)$, we have proved

LEMMA e.

$$\langle \nabla^2(B)(x, y), w\rangle = \langle \bar{R}(A)^w(x), y\rangle + \langle \bar{R}'^w(x), y\rangle + \sum_{i=1}^p \big(\langle Q^N_{e_i,x}B(e_i, y), w\rangle - \langle B(Q^T_{e_i,x}e_i, y), w\rangle - \langle B(e_i, Q^T_{e_i,x}y), w\rangle\big) .$$

We now examine the terms involving Q.

LEMMA f. *Let $v_1, \cdots, v_{n-p}$ be a frame in $N(M)_m$. Then*

$$\sum_{i=1}^p \langle Q^N_{e_i,x}B(e_i, y), w\rangle = \sum_{j=1}^{n-p} \langle A^{v_j} \circ [A^w, A^{v_j}](x), y\rangle .$$

PROOF.

$$\begin{aligned}\sum_{i=1}^p \langle Q^N_{e_i,x}B(e_i, y), w\rangle &= -\sum_{i=1}^p \langle Q^N_{e_i,x}w, B(e_i, y)\rangle \\ &= -\sum_{i=1}^p \sum_{j=1}^{n-p} \langle Q^N_{e_i,x}w, v_j\rangle\langle B(e_i, y), v_j\rangle \\ &= \sum_{j=1}^{n-p} \sum_{i=1}^p \langle [A^w, A^{v_j}](x), e_i\rangle\langle A^{v_j}(y), e_i\rangle \\ &= \sum_{j=1}^{n-p} \langle [A^w, A^{v_j}](x), A^{v_j}(y)\rangle \\ &= \sum_{j=1}^{n-p} \langle A^{v_j} \circ [A^w, A^{v_j}](x), y\rangle .\end{aligned}$$

where we have used (2.3.1) which defines Q^N. q.e.d.

LEMMA g.

$$-\sum_{i=1}^p \langle B(Q^T_{e_i,x}e_i, y), w\rangle = -\sum_{j=1}^{n-p} \langle A^w A^{v_j} A^{v_j}(x), y\rangle .$$

PROOF. We recall Proposition 2.3.3 which defines Q^T.

$$\begin{aligned}&-\sum_{i=1}^p \langle B(Q^T_{e_i,x}e_i, y), w\rangle \\ &= -\sum_{i=1}^p \langle A^w(y), Q^T_{e_i,x}e_i\rangle \\ &= \sum_{i=1}^p \sum_{j=1}^{n-p} \langle A^w(y), \big(A^{v_j}(e_i) \wedge A^{v_j}(x)\big)(e_i)\rangle \\ &= \sum_{i=1}^p \sum_{j=1}^{n-p} \langle A^w(y), \langle A^{v_j}(e_i), e_i\rangle A^{v_j}(x) - \langle A^{v_j}(x), e_i\rangle A^{v_j}(e_i)\rangle \\ &= -\sum_{j=1}^{n-p} \sum_{i=1}^p \langle A^w(y), A^{v_j}(e_i)\rangle\langle A^{v_j}(x), e_i\rangle \\ &= -\sum_{j=1}^{n-p} \langle A^{v_j} A^w(y), A^{v_j}(x)\rangle = -\sum_{j=1}^{n-p} \langle A^w A^{v_j} A^{v_j}(x), y\rangle .\end{aligned}$$

q.e.d.

LEMMA h.

$$-\sum_{i=1}^p \langle B(e_i, Q^T_{e_i,x}y), w\rangle = -\langle A^{\tilde{A}(w)}(x), y\rangle + \sum_{j=1}^{n-p} \langle A^{v_j} A^w A^{v_j}(x), y\rangle .$$

PROOF.

$$\begin{aligned}&-\sum_{i=1}^p \langle B(e_i, Q^T_{e_i,x}y), w\rangle \\ &= -\sum_{i=1}^p \langle A^w(e_i), Q^T_{e_i,x}y\rangle \\ &= \sum_{i=1}^p \sum_{j=1}^{n-p} \langle A^w(e_i), \big(A^{v_j}(e_i) \wedge A^{v_j}(x)\big)(y)\rangle \\ &= \sum_{i=1}^p \sum_{j=1}^{n-p} \langle A^w(e_i), \langle A^{v_j}(e_i), y\rangle A^{v_j}(x) - \langle A^{v_j}(x), y\rangle A^{v_j}(e_i)\rangle \\ &= \sum_{i=1}^p \sum_{j=1}^{n-p} \big(\langle A^w(e_i), A^{v_j}(x)\rangle\langle A^{v_j}(e_i), y\rangle - \langle A^w(e_i), A^{v_j}(e_i)\rangle\langle A^{v_j}(x), y\rangle\big) \\ &= \sum_{j=1}^{n-p} \big(\langle A^w A^{v_j}(x), A^{v_j}(y)\rangle - \langle \tilde{A}(w), v_j\rangle\langle A^{v_j}(x), y\rangle\big) \\ &= -\langle A^{\tilde{A}(w)}(x), y\rangle + \sum_{j=1}^{n-p} \langle A^{v_j} A^w A^{v_j}(x), y\rangle .\end{aligned}$$

q.e.d.

PROOF OF THEOREM. Plugging the formulas proved in Lemmas f, g, and h back into the formula of Lemma e yields

$$\langle\nabla^2(B)(x, y), w\rangle = \langle\bar{R}(A)^w(x), y\rangle + \langle\bar{R}'^w(x), y\rangle \\ - \langle A^{\tilde{A}(w)}(x), y\rangle - \langle(\underset{\sim}{A}(A^w))(x), y\rangle .$$

Clearly $\langle\nabla^2(B)(x, y), w\rangle = \langle\nabla^2(A)^w(x), y\rangle$, and thus the theorem is proved. q.e.d.

5. Closed minimal varieties in spheres

5.1. *Index and nullity.* The simplest example of a closed, p-dim, minimal variety in S^n is S^p with the usual totally geodesic imbedding. In order to compare other minimal varieties to S^p, we observe

PROPOSITION 5.1.1. *When S^p is regarded as a minimal variety in S^n, its index is $n - p$ and its nullity is $(p + 1)(n - p)$.*

PROOF. Since the normal bundle $N(S^p)$ is trivial, we may choose $V_1, \cdots, V_{n-p}$, to be covariant constant cross-sections in $N(S^p)$ such that, at each $m \in S^p$, $\{V_i(m)\}$ is a frame in $N(S^p)_m$. We also note that, for *any* p-dim minimal variety in S^n, (3.2.1) gives

$$\bar{R}(v) = -pv . \tag{5.1.1}$$

Thus, since in this case $A \equiv 0$, (3.2.4) becomes

$$I(V, W) = \int_{S^p}\langle -\nabla^2 V - pV, W\rangle . \tag{5.1.2}$$

Therefore the eigen spaces of I, are exactly those of $-\nabla^2$, and if λ is an eigenvalue of $-\nabla^2$, the corresponding eigenvalue of I is $\lambda - p$. Any cross-section V in $N(S^p)$ is of the form $V = \sum_{i=1}^p g_i V_i$ where the $\{g_i\}$ are functions on S^p. Since the $\{V_i\}$ are covariant constant, $-\nabla^2 V = \sum_{i=1}^{n-p} - \nabla^2(g_i) V_i$. Thus, $-\nabla^2 V = \lambda V$ if and only if

$$(-\nabla^2(g_1), \cdots, -\nabla^2(g_{n-p})) = (\lambda g_1, \cdots, \lambda g_{n-p}) .$$

Therefore the λ-eigenspace of $-\nabla^2$ acting on cross-sections in $N(S^p)$ consists exactly of $(n - p)$-tuple of λ-eigenvectors of $-\nabla^2$ acting on functions on S^p. Now, on functions, $-\nabla^2$ has a 1-dim 0 eigenspace consisting of the constants, and it has a $(p - 1)$ dim eigenspace corresponding to the eigenvalue p. This space consists of the restriction to S^p of linear functionals on R^{p+1}. The other eigenvalues of $-\nabla^2$ on functions are all strictly greater than p. Thus, on cross-sections in $N(S^p)$, $-\nabla^2$ has an $(n - p)$ dim 0-eigenspace, and $(p + 1)(n - p)$ dimensional p-eigenspace, and all other eigenvalues are greater than p. Thus, I has an $(n - p)$ dim eigenspace corresponding to $-p$, and a $(p + 1)(n - p)$ dim 0-eigenspace, and all other eigenvalues are positive. q.e.d.

THEOREM 5.1.1. *Let M be a compact, closed p-dim minimal variety immersed in S^n. Then the index of M is greater than or equal to $(n-p)$, and equality holds only when M is S^p. The nullity of M is greater than or equal to $(p+1)(n-p)$, and equality holds only when M is S^p.*

PROOF. Let ξ denote the $(n+1)$ dim vector space of vector fields on S^n which are tangential projections onto S^n of parallel vector field in R^{n+1}.

LEMMA 5.1.1. *Let $Z \in \xi$. Then given any $m \in S^n$, there is a λ such that, for any $x \in T(S^n)_m$,*

$$\overline{\nabla}_x Z = \lambda_x \ , \tag{5.1.3}$$

where $\overline{\nabla}$ denotes covariant differentiation in S^n.

PROOF. Let $\overline{\overline{\nabla}}$ denote covariant differentiation in R^{n+1}. $Z = W^T$ where W is a parallel field in R^{n+1}. Thus

$$\overline{\nabla}_x Z = (\overline{\overline{\nabla}}_x Z)^T = (\overline{\overline{\nabla}}_x W^T)^T = -(\overline{\overline{\nabla}}_x W^N)^T = \bar{A}^{W^N}(x) \ ,$$

where $\bar{A}$ denotes the second fundamental form of S^n in R^{n+1}. But $\bar{A}^{W^N} = \lambda I$, where I denotes the identity transformation. q.e.d.

Since $Z \in \xi$ is a vector field on S^n, restricting it to M, and projecting into normal and tangential components, gives cross-sections Z^N and Z^T in $N(M)$ and $T(M)$ respectively.

LEMMA 5.1.2. *Let $Z \in \xi$. Then when Z^N and Z^T are regarded as cross-sections in $T(M)$ and $N(M)$, they satisfy*

$$\nabla_x Z^N = -B(x, Z^T) \tag{5.1.4}$$

$$\nabla_x Z^T = A^{Z^N}(x) + \lambda x \ , \tag{5.1.5}$$

where $x \in T(M)_m$, and λ is independent of x.

PROOF. Using (5.1.3), we see

$$\begin{aligned}\nabla_x Z^N &= (\overline{\nabla}_x Z^N)^N = (\overline{\nabla}_x Z - \overline{\nabla}_x Z^T)^N \\ &= (\lambda x - \overline{\nabla}_x Z^T)^N = 0 - B(x, Z^T) \ .\end{aligned}$$

Again by (5.1.3),

$$\nabla_x Z^T = (\overline{\nabla}_x Z^T)^T = (\overline{\nabla}_x Z - \overline{\nabla}_x Z^N)^T = \lambda x + A^{Z^N}(x) \ . \qquad \text{q.e.d.}$$

LEMMA 5.1.3. *Let $Z \in \xi$. Then, when Z^N is regarded as a cross-section in $N(M)$, it satisfies*

$$\nabla^2(Z^N) = -\widetilde{A}(Z^N) \ . \tag{5.1.6}$$

PROOF. Let $e_1, \cdots, e_p$ be a frame in $T(M)_m$, and let $E_1, \cdots, E_p$ be extensions to orthonormal vector fields in a neighborhood of m such that $\nabla_{e_i} E_j = 0$. Then, by (5.1.4),

$$\nabla^2(Z^N) = \sum_{i=1}^p \nabla_{E_i}\nabla_{E_i}(Z^N) = -\sum_{i=1}^p \nabla_{E_i}\big(B(E_i, Z^T)\big)$$
$$= -\sum_{i=1}^p \big(\nabla_{e_i}(B)(e_i, Z^T) + B(\nabla_{e_i}E_i, Z^T) + B(e_i, \nabla_{e_i}Z^T)\big) .$$

Using Theorem 4.1.1, and the fact that the ambient $\bar{M}$ is S^n, we see that the first term is 0. The second term vanishes since $\nabla_{e_i}E_i = 0$. Thus

$$\nabla^2(Z^N) = -\sum_{i=1}^p B(e_i, \nabla_{e_i}Z^T) .$$

Now, using (5.1.5),

$$\nabla^2(Z^N) = -\sum_{i=1}^p \big(\lambda B(e_i, e_i) + B(e_i, A^{Z^N}(e_i))\big) .$$

The first term vanishes since M is a minimal variety. Slight algebraic manipulation shows that the second term is exactly $-\tilde{A}(Z^N)$. q.e.d.

LEMMA 5.1.4. *Let $Z \in \xi$. Then, when Z^N is regarded as a cross-section in $N(M)$, we have*

$$I(Z^N, Z^N) = -p\int_M \| Z^N \|^2 .$$

PROOF. Since (5.1.1) holds for any p-dim minimal variety in S_n, (3.2.1) gives

$$I(Z^N, Z^N) = \int_M \langle -\nabla^2(Z^N) - pZ^N - \tilde{A}(Z^N), Z^N\rangle .$$

Lemma 5.1.3 now gives the desired conclusion. q.e.d.

Let ξ^N denote the vector space of cross-sections in $N(M)$ consisting of the elements Z^N where $Z \in \xi$. We have proved

LEMMA 5.1.5. *The index form, $I(,)$, when restricted to the finite dimensional vector space ξ^N, is negative definite.*

LEMMA 5.1.6. Dim $\xi^N \geqq n - p$. Dim $\xi^N = n - p$ *if and only if M is diffeomorphic to S^p, and imbedded in the standard way as a totally geodesic submanifold.*

PROOF. At each $m \in M$, ξ spans the entire tangent space $T(S^n)_m$. Thus, at each m, ξ^N spans $N(M)_m$. Therefore dim $\xi^N \geqq (n - p)$.

Suppose dim $\xi^N = (n - p)$. Let η be the kernel of the homomorphism of $\xi \to \xi^N$. If $Z \in \eta$, $Z^T = Z$ at every point of M. Now for some $m \in M$, let β_m be the kernel of the homomorphism $\xi \to N(M)_m$ defined by $Z \to (Z^N)(m)$, Clearly $\eta \subseteqq \beta_m$. On the other hand, dim $\beta_m = n + 1 - (n - p)$, and our assumption that dim $\xi^N = (n - p)$ implies that dim $\tau = n + 1 - (n - p)$. Thus $\eta = \beta_m$. Since $Z \to Z^T(m)$ maps β_m onto $T(M)_m$, it also maps η onto $T(M)_m$. We have therefore shown that dim $\xi^N = (n - p)$ implies

(*) Given $z \in T(M)_m$, there exists $Z \in \xi$ such that $Z(m) = z$, and Z is everywhere tangent to M.

Using (*) and (5.1.4) we see that, for $x, z \in T(M)_m$,

$$B(x, z) = -\nabla_x Z^N = 0 .$$

Thus $B \equiv 0$, and so M is totally geodesic. The only such immersed submanifold of S^n is S^p. q.e.d.

Lemmas 5.1.5 and 5.1.6 prove the first half of the theorem.

Let Ω denote the vector space of Killing vector fields on S^n. If $W \in \Omega$, W^N defines a cross-section in $N(M)$ by normal projection. From Corollary 3.3.1, we see

LEMMA 5.1.7. *For $W \in \Omega$, W^N is a Jacobi field on M.*

The above lemma could actually be proved directly, in this case, *via* calculations similar to those used in proving Lemma 5.1.3. However, we shall omit this alternative.

Let Ω^N denote the finite dimensional space of cross-sections in $N(M)$ consisting of the Jacobi fields W^N.

LEMMA 5.1.8. *For fixed $m \in M$, let $v \in N(M)_m$, and $h \in \operatorname{Hom}(T(M)_m, N(M)_m)$. Then $\exists V \in \Omega^N$ such that*

$$V(m) = v \qquad \textit{and} \qquad (\nabla_x V)(m) = h(x) .$$

PROOF. Let g be some skew symmetric endomorphism of $T(S^n)_m$ such that $g \mid T(M)_m = h$. By standard facts about the killing vector fields on S^n, there exists a unique $W \in \Omega$ such that

$$(*) \qquad W(m) = v \qquad \text{and} \qquad (\overline{\nabla}_x W)(m) = g(x) \qquad \forall x \in T(S^n)_m .$$

Let us set $V = W^N$. Clearly $V(m) = v$. Now, using (*),

$$\begin{aligned}\nabla_x V = \nabla_x W^N = (\overline{\nabla}_x W^N)^N &= (\overline{\nabla}_x W)^N - (\overline{\nabla}_x W^T)^N \\ &= g(x) - B\big(x, W^T(m)\big) = h(x) ,\end{aligned}$$

since $W^T(m) = 0$. q.e.d.

LEMMA 5.1.9. $\operatorname{Dim} \Omega^N \geqq (p+1)(n-p)$. $\operatorname{Dim} \Omega^N = (p+1)(n-p)$ *if and only if M is diffeomorphic to S^p, and imbedded in the standard way as a totally geodesic submanifold.*

PROOF. For fixed $m \in M$, we define

$$\begin{aligned}&\varphi_m \colon \Omega^N \longrightarrow N(M)_m \oplus \operatorname{Hom}\big(T(M)_m, N(M)_m\big) \\ &\varphi_m(V) \;=\; \big(V(m), h\big) \qquad\qquad \text{where} \quad h(x) = \nabla_x V .\end{aligned}$$

Clearly φ_m is a linear transformation, and Lemma 5.1.8 shows that φ_n is onto. Thus, $\dim \Omega^N \geqq (n-p) + p(n-p) = (p+1)(n-p)$.

Suppose $\dim \Omega^N = (p+1)(n-p)$. This means that φ_n is an isomorphism. Thus, if $W \in \Omega$, such that $W^N(m) = 0$ and $\nabla_x W^N(m) = 0$ for all $x \in T(M)_m$, then $W^N \equiv 0$, i.e., W is everywhere tangent to M.

Let G_m be the subgroup of the full isometry group of S^n consisting of elements g which satisfy

$$g(m) = m\ , \qquad dg\big(T(M)_m\big) = T(M)_m\ , \qquad dg \mid N(M)_m = I\ .$$

The elements dg are exactly the full orthogonal group of $T(M)_m$. Clearly, the killing vector fields corresponding to G_m are elements $W \in \Omega$ satisfying

$$(*) \qquad \begin{aligned} &W(m) = 0, \quad \overline{\nabla}_x W \in T(M)_m && \text{for } x \in T(M)_m \\ &\overline{\nabla}_v W = 0 && \text{for } v \in N(M)_m\ . \end{aligned}$$

However, if W satisfies $(*)$ we see that $W^N(m) = 0$, and

$$\begin{aligned} \nabla_x W^N = (\overline{\nabla}_x W^N)^N &= (\overline{\nabla}_x W)^N - (\overline{\nabla}_x W^T)^N \\ &= 0 - B(x, W^T) = 0\ . \end{aligned}$$

Thus, by the assumption $\dim \Omega^N = (p + 1)(n - p)$, we see that W is everywhere tangent to M. Therefore G_m acts on M, mapping it into itself. Since the orthogonal transformations $dg(m)$ are transitive on the unit vectors in $T(M)_m$, and hold the normal space fixed, we may conclude that $B(e_i, e_i) = B(e_j, e_j)$ for e_i, e_j distinct unit vectors in $T(M)_m$. But this implies $B \equiv 0$. Since m was arbitrary, M is totally geodesic. q.e.d.

Lemma 5.1.7 and 5.1.9 now prove the second half of the theorem.

5.2. *An extrinsic rigidity theorem.* In [8] it is proved that any non-parametric, cone shaped, minimal variety in R^{n+1} is a hyperplane. This is equivalent to a statement about the image of the Gauss map of a closed, co-dimension 1, minimal variety in S^n. Below we shall give a short proof of this theorem, and then go on to obtain a similar result for minimal varieties in S^n of arbitrary co-dimension.

Suppose M has co-dimension 1 in S^n. Having chosen an orientation let $N(m)$ denote the unit normal vector at m. $N(m)$ is parallel to a unit vector in R^{n+1} based at the origin, and thus defines a point on S^n which we denote by m^*. Let M^* denote the image of M under the mapping $m \longrightarrow m^*$. The following is equivalent to the theorem of [8].

THEOREM 5.2.1. *Suppose M is a closed minimal variety of co-dimension 1. Then either M^* is a single point, in which case $M = S^{n-1}$, or M^* lies in no open hemisphere of S^n.*

PROOF. Let $w \in S^n$. By parallel translation, w defines a unique parallel vector field W in R^{n+1}. *Via* tangential projection onto S^n, W corresponds to a vector field $Z \in \xi$ (see Lemma 5.1.1). Clearly

$$\langle m^*, w\rangle = \langle N(m), Z\rangle\ .$$

Thus, it is sufficient to prove

(*) Let $Z \in \xi$. Then $\langle N(m), Z\rangle > 0$ everywhere implies that $M = S^{n-1}$.

To prove (*), we set $F(m) = \langle N(m), Z\rangle$. Then F is a real valued function on M. Now, $\langle N(m), Z\rangle = \langle N(m), Z^N\rangle$, and since $N(m)$ is covariant constant, we see

$$\nabla^2 F = \langle N(m), \nabla^2(Z^N)\rangle = -\langle N(m), \tilde{A}(Z^N)\rangle$$

by Lemma 5.1.3. However, since we are in co-dimension 1, the definition of $\tilde{A}$ shows that $\tilde{A}(Z^N) = \| A \|^2 Z^N$. Thus

$$\nabla^2 F = -\| A \|^2 F .$$

Since M is closed, Stokes' theorem gives

$$\int_M \| A \|^2 F = 0 .$$

But, if $F > 0$ everywhere, $\| A \|^2 = 0$ everywhere, and this implies $A \equiv 0$, which implies M is totally geodesic. q.e.d.

Let G^{n-p}_{n+1} denote the Grassmann manifold of oriented $(n-p)$-planes in R^{n+1}. The elements of G^{n-p}_{n+1} may be identified with the decomposable elements of unit norm in the Grassmann algebra $\Lambda^{n-p}R^{n+1}$. If $g_1, g_2 \in G^{n-p}_{n+1}$, we let $\langle g_1, g_2\rangle$ denote their inner product under this identification.

Definition. Let $-1 < \delta < 1$. If $g_1 \in G^{n-p}_{n+1}$, we define the *δ-ball about* g_1 to equal the set of $g_2 \in G^{n-p}_{n+1}$ such that $\langle g_1, g_2\rangle > \delta$.

Clearly the δ-ball about g_1 is an open neighborhood of g_1. If $n - p = 1$, $G^{n-p}_{n+1} = S^{n+1}$, and the 0-ball about g_1 is the open hemisphere with g_1 as center.

Let M be a p-dimensional submanifold of S^{n+1}. Having chosen an orientation for M, let $N(m)$ denote the oriented $(n - p)$-dimensional normal space at m. $N(m)$ is parallel to an oriented $(n - p)$-plane through the origin, and thus defines a point in G^{n-p}_{n+1}, which we denote by m^*. Let M^* denote the image of M under the mapping $m \longrightarrow m^*$.

THEOREM 5.2.2. *There exists a constant δ depending only on p and n such that, for any closed, p-dim immersed minimal variety M, either M is S^p, in which case M^* is a single point; or M^* lies in no δ-ball in G^{n-p}_{n+1}.*

PROOF. Let $w \in G^{n-p}_{n+1}$. w may be identified with $w_1 \wedge \cdots \wedge w_{n-p}$, where each w_j is a vector in R^{n+1}, and where $\| w_1 \wedge \cdots \wedge w_{n-p} \| = 1$. By parallel translation, w defines a unique parallel field $W = W_1 \wedge \cdots \wedge W_{n-p}$ in all of R^{n+1}. *Via* tangential projection onto S^n, W corresponds to $Z = Z_1 \wedge \cdots \wedge Z_{n-p}$, a field on S^n, where each $Z_i \in \xi$. We clearly have

$$\langle m^*, w\rangle = \langle N(m), Z\rangle ,$$

where the right hand side is the inner product of two tensors in $T(S^n)_m$. We also note that $\| Z \| \leqq 1$ at each point. Clearly

$$\langle N(m), Z \rangle = \langle N(m), Z_1^N \wedge \cdots \wedge Z_{n-p}^N \rangle .$$

The theorem may now be restated.

(*) There exists a $\delta < 1$ such that, for any closed p-dim minimal variety M, and any $Z_1, \cdots, Z_{n-p} \in \xi$ with $\| Z_1 \wedge \cdots \wedge Z_{n-p} \| \leqq 1$, everywhere, $F(m) = \langle N(m), Z_1^N \wedge \cdots \wedge Z_{m-p}^N \rangle < \delta$ at some m, unless M is S^p.

Now, $N(m)$ and $Z_1^N \wedge \cdots \wedge Z_{n-p}^N$ are both cross-sections in $\Lambda^{n-p}N(M)$. Moreover, $N(m)$ is parallel. Thus, for $x \in T(M)_m$

$$\nabla_x F = \sum_{j=1}^{n-p} \langle N(m), Z_1^N \wedge \cdots \wedge \nabla_x Z_j^N \wedge \cdots \wedge Z_{n-p}^N \rangle ,$$

and therefore

$$\begin{aligned} \nabla^2 F = & \sum_{j=1}^{n-p} \langle N(m), Z_1^N \wedge \cdots \wedge \nabla^2(Z_j^N) \wedge \cdots \wedge Z_{n-p}^N \rangle \\ & + 2 \sum_{j<k} (-1)^{j+k+1} \Big\langle N(m), \big[\textstyle\sum_{i=1}^n \nabla_{e_i} Z_j^N \wedge \nabla_{e_i} Z_k^N\big] \\ & \qquad \wedge Z_1^N \wedge \cdots \wedge \hat{Z}_i^N \wedge \cdots \wedge \hat{Z}_j^N \wedge \cdots \wedge Z_{n-p}^N \Big\rangle . \end{aligned}$$

Using Propositions 5.1.2 and 5.1.3, we see

$$\begin{aligned} \nabla^2 F = & \overbrace{-\sum_{j=1}^{n-p} \langle N(m), Z_1^N \wedge \cdots \wedge \tilde{A}(Z_j^N) \wedge \cdots \wedge Z_{n-p}^N \rangle}^{①} \\ & \overbrace{+ 2\sum_{j<k} (-1)^{j+j+1} \Big\langle N(m), \big[\textstyle\sum_{i=1}^n B(e_i, Z_j^T) \wedge B(e_i, Z_k^T)\big]}^{②} \\ & \qquad \wedge Z_1^N \wedge \cdots \wedge \hat{Z}_i^N \wedge \cdots \wedge \hat{Z}_j^N \wedge \cdots \wedge Z_{n-p}^N \Big\rangle . \end{aligned}$$

The first piece of the above is exactly $-\operatorname{tr} \tilde{A} \langle N(m), Z_1^N \wedge \cdots \wedge Z_{n-p}^N \rangle$. It is easily seen from the definition of $\tilde{A}$ that $\operatorname{tr} \tilde{A} = \| A \|^2$. Thus

$$① = - \| A \|^2 F .$$

Let us now suppose that $F(m) > 0$. Thus, $Z_1^N \wedge \cdots \wedge Z_{n-p}^N(m) \neq 0$. Since $Z_1 \wedge \cdots \wedge Z_{n-p}$ is invariant under any unimodular transformation, we may assume that the following conditions hold at m:

$$\begin{aligned} \| Z_j \| &= \lambda & &\text{all } j = 1, \cdots, n-p ; \\ \langle Z_j, Z_k \rangle &= 0 & &\text{all } j \neq k ; \\ \langle Z_j^N, Z_k^N \rangle &= 0 & &\text{all } j \neq k ; \\ \langle Z_j^T, Z_k^T \rangle &= 0 & &\text{all } j \neq k . \end{aligned}$$

The last condition is implied by the previous two conditions. Using these we see

$$\begin{aligned} ② &= \Big[2 \sum_{j<k} \frac{1}{\| Z_j^N \|^2 \| Z_k^N \|^2} \sum_{i=1}^p \langle B(e_j, Z_j^T) \wedge B(e_i, Z_k^T), Z_j^N \wedge Z_k^N \rangle \Big] F \\ &= \Big[\sum_{j,k=1}^{n-p} \frac{1}{\| Z_j^N \|^2 \| Z_k^N \|^2} \langle [A^{Z_j^N}, A^{Z_k^N}](Z_j^T), Z_k^T \rangle \Big] F . \end{aligned}$$

Letting φ denote the coefficient of F in ②, our equation at m now reads:

$$\nabla^2 F = (\varphi - \|A\|^2)F \, . \tag{5.2.1}$$

We wish to estimate $|\varphi|$. Clearly

$$|\varphi| \leqq \sum_{j,k=1}^{n-p} \frac{1}{\|Z_j^N\|^2 \|Z_k^N\|^2} |\langle [A^{Z_j^N}, A^{Z_k^N}](Z_j^T), Z_k^T \rangle| \, .$$

For $j = 1, \cdots, n - p$ set $v_j = (1/\|Z_j^N\|)Z_j^N$. Then $v_1, \cdots, v_{n-p}$ is a frame in $N(M)_m$. Also, for $j = 1, \cdots, n - p$, set

$$e_j = \frac{1}{\|Z_j^T\|} Z_j^T \qquad \text{if} \quad \|Z_j^T\| \neq 0$$

$$e_j = 0 \qquad \text{if} \quad \|Z_j^T\| = 0 \, .$$

$$\begin{aligned} \therefore \quad |\varphi| &\leqq \sum_{j,k=1}^{n-p} \frac{\|Z_j^T\| \, \|Z_k^T\|}{\|Z_j^N\| \, \|Z_k^N\|} |\langle [A^{v_j}, A^{v_k}](e_j), e_k \rangle| \\ &\leqq \sup_{j=1,\dots,n-p} \left\{ \frac{\|Z_j^T\|^2}{\|Z_j^N\|^2} \right\} \cdot \sum_{j,k=1}^{n-p} \big(|\langle A^{v_j}(e_k), A^{v_k}(e_j) \rangle| \\ &\qquad + |\langle A^{v_j}(e_j), A^{v_k}(e_k) \rangle| \big) \, . \end{aligned}$$

Examining these terms, and using the fact that the $\{e_j\}$ are a partial frame in $T(M)_m$, we have

$$\begin{aligned} \sum_{j,k=1}^{n-p} |\langle A^{v_j}(e_k), A^{v_k}(e_j) \rangle| &\leqq \tfrac{1}{2} \sum_{j,k=1}^{n-p} \big(\|A^{v_j}(e_k)\|^2 + \|A^{v_k}(e_j)\|^2 \big) \\ &= \sum_{j,k=1}^{n-p} \|A^{v_j}(e_j)\|^2 \leqq \|A\|^2 \, . \end{aligned}$$

We also have

$$\begin{aligned} \sum_{j,k=1}^{n-p} |\langle A^{v_j}(e_j), A^{v_k}(e_k) \rangle| &\leqq \tfrac{1}{2} \sum_{j,k=1}^{n-p} \big(\|A^{v_j}(e_j)\|^2 + \|A^{v_k}(e_k)\|^2 \big) \\ &= (n - p) \sum_j \|A^{v_j}(e_j)\|^2 \leqq (n - p) \|A\|^2 \, . \end{aligned}$$

We have therefore proved

$$|\varphi| \leqq \sup_{j=1,\dots,n-p} \frac{\|Z_j^T\|^2}{\|Z_j^N\|^2} (n - p + 1) \|A\|^2 \, ,$$

Since

$$\|Z_j^N\|^2 + \|Z_j^T\|^2 = \|Z_j\|^2 = \lambda^2 \leqq 1 \, ,$$

we see

$$\|Z_j^T\|^2 \leqq 1 - \|Z_j^N\|^2 \, ,$$

and therefore

$$\frac{\|Z_j^T\|^2}{\|Z_j^N\|^2} \leqq \frac{1}{\|Z_j^N\|^2} - 1 \, .$$

Since $F^2(m) = \|Z_1^N\|^2 \cdots \|Z_{n-p}^N\|^2$, and each of the factors is smaller than 1, we see that $\| Z_j^N \|^2 \geqq F^2(m)$. Thus

$$\frac{1}{\| Z_j^N \|^2} - 1 \leqq \frac{1}{F^2(m)} - 1 .$$

Therefore, at m we have

$$| \varphi | \leqq (n - p + 1)\Big(\frac{1}{F^2(m)} - 1\Big) \| A \|^2 .$$

Up to this point, we have used only the assumption that $F(m) > 0$. Plugging our estimate for $| \varphi |$ into (5.2.1), we have proved

$$\frac{\nabla^2 F}{F} \leqq \Big[(n - p + 1)\Big(\frac{1}{F^2} - 1\Big) - 1\Big] \| A \|^2 ,$$

where the above holds at all m where $F(m) > 0$.

Now, suppose that, at all $m \in M$, F satisfies the inequality

$$F(m) > \Big(\frac{n - p + 1}{n - p + 2}\Big)^{1/2} . \tag{5.2.2}$$

Then certainly $F > 0$ everywhere, and inserting this inequality into the previous one, yields

$$\frac{\nabla^2 F}{F} \leqq -\varepsilon \| A \|^2 \qquad \text{everywhere} ,$$

where $\varepsilon > 0$ is a constant. Thus

$$\nabla^2 F \leqq -\varepsilon \| A \|^2 F ,$$

and so by Stokes' theorem,

$$\int_M \| A \|^2 F \leqq 0 .$$

Since F satisfies (5.2.2), we must have $\| A \|^2 = 0$ everywhere, and so M is totally geodesic. We have thus proved (*) with

$$\delta = \Big(\frac{n - p + 1}{n - p + 2}\Big)^{1/2} . \qquad \text{q.e.d.}$$

It would be nice if one could prove Theorem 5.2.2 with $\delta = 0$, as it is in the co-dimension 1 case. If this were true, it would imply that the $(p + 1)$-plane in R^{n+1}, which is spanned by $T(M)_m$ and the radial vector m has non-trivial intersection with every fixed $(n - p)$-plane in R^{n+1}. In any event, the result is strong enough to prove a rigidity theorem.

THEOREM 5.2.3. *Let $f: S^p \to S^n$ be the usual totally geodesic imbedding. Then there is a C' neighborhood of f in the space of C^∞ immersions of $S^p \to S^n$ such that no f' in this neighborhood is a minimal immersion, unless $f' =$*

$G \circ f$ where $G \in O^{n+1}(R^{n+1})$.

PROOF. Since $(S^p)^*$ is a single point in G^{n-p}_{n+1}, we can obviously choose a C' neighborhood of f such that, if $M = S^p$ together with f' in this neighborhood, then $M^* \subseteqq \delta$-ball in G^{n-p}_{n+1}. Thus M is not minimal unless M is totally geodesic, in which case $f' = G \circ f$ for some orthogonal G. q.e.d.

5.3. *The fundamental equation and an intrinsic rigidity theorem.* In this section we examine the fundamental elliptic equation in the special case where $\bar{M} = S^n$. It has several interesting consequences, one of which bears on the internal geometry of M.

THEOREM 5.3.1. *Let M be a p-dim minimal variety immersed in S^n. Then the second fundamental form satisfies*

$$\nabla^2 A = pA - A \circ \tilde{A} - \underset{\sim}{A} \circ A . \tag{5.3.1}$$

If M is of co-dimension 1, the equation becomes

$$\nabla^2 A = pA - \| A \|^2 A . \tag{5.3.2}$$

PROOF. Since the ambient manifold is S^n, formula (4.2.1) shows $\bar{R}' \equiv 0$. Also, for any $t_1, t_2, t_3 \in T(S^n)_m$, we have

$$\bar{R}_{t_1, t_2} t_3 = -\langle t_1, t_3 \rangle t_2 + \langle t_2, t_3 \rangle t_1 .$$

Using this formula in (4.2.2) shows

$$\bar{R}(A) = pA .$$

Plugging these facts into Theorem 4.2 yields formula (5.3.1). To prove (5.3.2) we observe that, from the definitions of $\tilde{A}$ and $\underset{\sim}{A}$, (2.25) and (2.26), the assumption of co-dimension 1 shows that

$$\underset{\sim}{A} \circ A = 0 \qquad \text{and} \qquad A \circ \tilde{A} = \| A \|^2 A.$$

Thus (5.3.2) follows from (5.3.1). q.e.d.

LEMMA 5.3.1. *The second fundamental form A of any p-dim variety in any manifold always satisfies*

$$\langle A \circ \tilde{A} + \underset{\sim}{A} \circ A, A \rangle \leqq q \| A \|^4$$

where

$$q = 2 - \frac{1}{n-p} .$$

PROOF. Using Proposition 2.2.3, we may choose a frame $v_1, \cdots, v_{n-p}$ in $N(M)_m$ such that

$$\tilde{A}(v_i) = \lambda_i^2 v_i \qquad \text{and} \qquad \| A \|^2 = \textstyle\sum_{i=1}^{n-p} \lambda_i^2 .$$

Thus

$$\langle A \circ \tilde{A}, A\rangle = \sum_{i=1}^{n-p} \langle A \circ \tilde{A}(v_i), A^{v_i}\rangle$$
$$= \sum_{i=1}^{n-p} \lambda_i^2 \langle A^{v_i}, A^{v_i}\rangle = \sum_{i=1}^{n-p} \lambda_i^2 \langle \tilde{A}(v_i), v_i\rangle = \sum_{i=1}^{n-p} \lambda_i^4 .$$

We also see from Proposition 2.2.3

$$\langle \underset{\sim}{A} \circ A, A\rangle = \sum_{i,j=1}^{n-p} \| [A^{v_i}, A^{v_j}] \|^2 \leqq 2 \sum_{i \neq j}^{n-p} \| A^{v_i} \|^2 \| A^{v_j} \|^2 = 2 \sum_{i \neq j}^{n-p} \lambda_i^2 \lambda_j^2 .$$

Thus combining the two estimates,

$$\langle \tilde{A} \circ A + \underset{\sim}{A} \circ A, A\rangle \leqq \sum_{i=1}^{n-p} \lambda_i^4 + 2 \sum_{i \neq j} \lambda_i^2 \lambda_j^2$$
$$= 2 \left[\sum_{i=1}^{n-p} \lambda_i^2\right]^2 - \sum_{i=1}^{n-p} \lambda_i^4$$
$$\leqq \left[2 - \frac{1}{n-p}\right]\left[\sum_{i=1}^{n-p} \lambda_i^2\right]^2 = q \| A \|^4 . \qquad \text{q.e.d.}$$

THEOREM 5.3.2. *Let M be a closed, p-*dim *minimal variety immersed in S^n. Then its second fundamental form A satisfies the inequality*

$$\int_M \left(\| A \|^2 - \frac{p}{q}\right) \| A \|^2 \geqq 0 . \tag{5.3.3}$$

PROOF. Using (5.3.1) and the fact that ∇^2 is negative semi-definite, we see

$$0 \leqq - \int_M \langle \nabla^2 A, A\rangle = \int_M - p \| A \|^2 + \langle A \circ \tilde{A}, A\rangle + \langle \underset{\sim}{A} \circ A, A\rangle .$$

Now, using Lemma 5.3 we see

$$0 \leqq \int_M - p \| A \|^2 + q \| A \|^4 = \int_M q \| A \|^2 \left(\| A \|^2 - \frac{p}{q}\right) . \qquad \text{q.e.d.}$$

Remark 5.3.1. The lower bound for this estimate is, of course, achieved when $M = S^n$. However, there are non-trivial examples when the lower bound is achieved. A class of these is the following. Let $S^n(r)$ denote the n-sphere of radius r in R^{n+1}. Then

$$S^n\left(\frac{\sqrt{2}}{2}\right) \times S^n\left(\frac{\sqrt{2}}{2}\right)$$

sits naturally in S^{2n+1} as a $2n$-dim minimal variety. An easy calculation shows $\| A \|^2 \equiv 2n$.

COROLLARY 5.3.2. *Let M be a closed, p-*dim *minimal variety immersed in S^n. Then either M is the totally geodesic S^p, or $\| A \|^2 \equiv p/q$, or at some $m \in M$, $\| A \|^2 (m) > p/q$.*

PROOF. Suppose $\| A \|^2 \leqq p/q$ everywhere. Then (5.3.3) shows

$$\int_M \left(\| A \|^2 - \frac{p}{q}\right) \| A \|^2 = 0 .$$

Thus $\nabla^2 A \equiv 0$, which implies that A is covariant constant, which implies that

$\| A \|^2$ is a constant. Thus either $\| A \|^2 \equiv 0$, or $\| A \|^2 \equiv p/q$. In the first case M is S^p. Therefore, except for these two possibilities, $\| A \|^2 (m) > p/q$ somewhere. q.e.d.

Although the above theorem appears to be an extrinsic comparison of M with S^p, it is in fact an intrinsic comparison, as we show below.

Definition. If M is a riemannian manifold and $m \in M$, we let $k(m)$ denote the *scalar curvature* of M at m. We choose to define $k(m)$ to be the *average* of all the sectional curvatures at m; i.e.,

$$k(m) = \frac{-1}{p(p-1)} \sum_{i,j=1}^{p} \langle R_{e_i,e_j} e_i, e_j \rangle .$$

PROPOSITION 5.3.1. *Let M be a p-*dim *minimal variety immersed in S^n. Then*

$$k(m) = 1 - \frac{1}{p(p-1)} \| A \|^2 . \tag{5.3.4}$$

PROOF. Using Propositions 2.3.2 and 2.3.3, and the fact that the ambient manifold is S^n, we see

$$\begin{aligned} \langle R_{e_i,e_j} e_i, e_j \rangle = & -\langle e_i \wedge e_j, e_i \wedge e_j \rangle \\ & - \sum_{k=1}^{n-p} \langle A^{v_k}(e_i) \wedge A^{v_k}(e_j), e_i \wedge e_j \rangle . \end{aligned}$$

Thus

$$\begin{aligned} k(m) = 1 + \frac{1}{p(p-1)} \sum_{k=1}^{n-p} \sum_{i,j=1}^{p} \big(& \langle A^{v_k}(e_i), e_i \rangle \langle A^{v_k}(e_j), e_j \rangle \\ & - \langle A^{v_k}(e_i), e_j \rangle \langle A^{v_k}(e_j), e_i \rangle \big) . \end{aligned}$$

But since each A^{v_k} is symmetric, and has trace 0, we get

$$\begin{aligned} k(m) &= 1 - \frac{1}{p(p-1)} \sum_{k=1}^{n-p} \sum_{i,j=1}^{p} \langle A^{v_k}(e_i), e_j \rangle^2 \\ &= 1 - \frac{1}{p(p-1)} \| A \|^2 \end{aligned}$$ q.e.d.

THEOREM 5.3.3. *Let M be a closed, p-*dim *minimal variety immersed in S^n. Then its scalar curvature k satisfies the inequalities*

$$k(m) \leqq 1 \qquad \textit{everywhere},$$

and

$$\int_M (1 - k(m))\Big(1 - \frac{1}{q(p-1)} - k(m)\Big) \geqq 0 .$$

PROOF. Follows directly from (5.3.3) and (5.3.4). q.e.d.

COROLLARY 5.3.3. *Let M be a closed, p-*dim *minimal variety immersed* *in S^n. Then either M is S^p, in which case $k(m) \equiv 1$, or $k(m) \equiv 1-1/(q(p-1))$, or at some $m \in M$, $k(m) < 1 - 1/(q(p-1))$.*

PROOF. Follows directly from (5.3.4) and Corollary (5.3.2).

An immediate consequence of the above corollary is the following rigidity theorem.

THEOREM 5.3.4. *Let g denote the standard metric on S^p. Then there is a C^2 neighborhood of g in the space of non-equivalent riemannian metric such that S^p, together with any g' in this neighborhood, cannot be isometrically immersed in S^n as a minimal variety.*

PROOF. Let $U = \{g' \mid g' \nsim g$ and $k'(m) > 1 - 1/(q(p-1))$ at all $m \in S^p\}$, where k' denotes the scalar curvature associated to g'. An isometric minimal immersion of any such (S^p, g') would imply, by Corollary 5.3.3, that the image was totally geodesic. But this would mean that $g' \sim g$. q.e.d.

5.4. *Holomorphic quadratic differentials.* In the case that $p = 2$ and $n = 3$, the above Theorem 5.3.4 may be strengthened. In fact, it is shown in [9] that S^2, together with *no* riemannian structure, may be isometrically immersed in S^3, unless that structure is equivalent to the standard one. The proof follows from the observation that the second fundamental form, in this case a real-valued bilinear form, is the real part of a holomorphic quadratic differential with respect to the conformal structure induced by the inherited metric, and the fact that the Riemann-Roch theorem implies that any such on S^2 must be zero.

If we remain in co-dimension 1, the second fundamental form is still a real-valued bilinear form on M. The first order conditions that make it the real part of a holomorphic quadratic differential when $p = 2$ are simply the fundamental first order systems (4.1.1) and (4.1.2) applied to $\bar{M} = S^n$. That is, we have already shown

THEOREM 5.4.1. *Let M be a co-dimension 1 minimal variety in S^n. Then the second fundamental form B is a symmetric real-valued bilinear form which satisfies*

$$(5.4.1) \qquad \operatorname{tr} B = 0 \quad \text{and} \quad \nabla_x(B)(y, z) - \nabla_y(B)(x, z) = 0 \qquad \forall x, y, z .$$

When $\dim M = 2$, B satisfies (5.4.1) if and only if the form $Q(x) = B(x, x) - iB(x, J(x))$ is a holomorphic quadratic differential (J being the usual 90° rotation). In higher dimensions, and on a compact manifold, bilinear forms satisfying (5.4.1) span a finite dimensional space, however there is no obvious relation between its dimension and some topological or differential invariants of the manifold. To ask that the dimension of this space be stable under all changes in metric is probably a bit much, however one can easily show that it is stable in a C^2 neighborhood of the standard metric on S^p.

THEOREM 5.4.2. *If B satisfies* (5.4.1), *then B satisfies*

$$\nabla^2 B(x, y) = \sum_{i=1}^{p} R_{e_i,x}(B)(e_i, y) \ . \tag{5.4.2}$$

Conversely, if M is compact and B satisfies (5.4.2), *then $B = \lambda g + H$ where λ is a constant, g is the metric, and H satisfies* (5.4.1).

PROOF. The first part is essentially Lemma a of Theorem 4.2.1. The converse follows easily from Stokes's theorem.

If we set $R(B)(x, y) = \sum_{i=1}^{p} R_{e_i,x}(B)(e_i, y)$, it is easy to see that, with the standard metric on S^p, $R(B) = pB$ so long as $\operatorname{tr} B = 0$. Thus on the space of B with $\operatorname{tr} B = 0$, equation (5.4.2) becomes $\nabla^2 B = pB$. This clearly has no solutions on S^p, and thus the dimension of the space satisfying (5.4.1) is zero. Via perturbation of the standard metric we see

THEOREM 5.4.3. *Let g' be any metric on S^p such that for all symmetric bilinear forms B with $\operatorname{tr} B = 0$, $g'(R'(B), B) > 0$, where R' is the curvature associated with g', and $R'(B)$ is as defined above. Then such g' form a C^2 neighborhood of the standard metric, and with respect to any such g', the dimension of the solution space to* (5.4.1) *is zero.*

6. Minimal varieties in euclidean space

6.1. *Cone shaped varieties.* In [9] Almgren showed that the cone over any 2-dimensional minimal variety in S^3, except for the totally geodesic S^2, is unstable with respect to its boundary. This fact has important consequences which are outlined in the next section. The method he used depended on the conformal analysis of 2-dimensional manifolds and on the Gauss-Bonnet theorem. In the present section we show that this instability theorem is true for the cone over any immersed, co-dimension 1 minimal variety in S^n for $n \leqq 6$. The proof depends on the elliptic methods developed in the previous chapters. We also give an example of a minimal variety in S^7, the cone over which is locally stable in the sense that every variation holding the boundary fixed is initially increasing area.

Definition. Let M be an immersed submanifold in S^n. The *cone over M*, CM, is the mapping of $M \times [0, 1] \rightarrow R^{n+1}$ defined by $(m, t) \rightarrow tm$.

The *ε-truncated cone over M*, CM_ε, is the same mapping restricted to $M \times [\varepsilon, 1]$.

PROPOSITION 6.1.1. *Let M be a closed minimal variety in S^n. Then $CM - 0$ is a minimal variety immersed in R^{n+1}. CM_ε is a compact minimal variety immersed in R^{n+1} and $\partial CM_\varepsilon = M \cup M_\varepsilon$, where M_ε denotes the set of* *points εm for $m \in M$.*

PROOF. Let τ denote the unit vector field on CM corresponding to the coordinate t. The integral curves of τ are rays to the origin. Let $\overline{\nabla}$ denote covariant differentiation in R^{n+1}. Clearly $\overline{\nabla}_\tau \tau = 0$. Let $m \in M$ be fixed, and let $e_1, \cdots, e_p$ be a frame in $T(M)_m$. Extend them to orthonormal local vector fields on M, $E_1, \cdots, E_p$, chosen so that they are covariant constant at m with respect to the connection on M. By parallel translation in R^{n+1}, extend them up and down the rays to get vector fields on CM. For any t, $\{E_i(m, t)\}$ are orthonormal. Then, since M is minimal in S^n, an easy calculation shows

$$\sum_{i=1}^{p} \overline{\nabla}_{E_i} E_i = -\frac{p}{t}\tau \qquad \forall t \in (0, 1] . \tag{6.1.1}$$

Thus

$$[\overline{\nabla}_\tau \tau + \sum_{i=1}^{p} \overline{\nabla}_{E_i} E_i]^N = 0 ,$$

where $[\quad]^N$ denotes projection into the normal space to CM at (m, t). Thus $CM - 0$ is a minimal variety in R^{n+1}. So therefore is CM_ε, and clearly its boundary is $M \cup M_\varepsilon$. q.e.d.

LEMMA 6.1.1. *Let $A(m, t)$ denote the second fundamental form of CM or CM_ε at (m, t). Let $A(m)$ denote the second fundamental form of M at m. Then*

$$\| A(m, t) \|^2 = \frac{1}{t^2} \| A(m) \|^2 .$$

PROOF. This is simply a statement about the way in which principal curvatures behave under dilations. q.e.d.

If $F(m, t)$ is a function on CM or CM_ε, for each fixed t, let F_t be the function on M defined by $F_t(m) = F(m, t)$.

LEMMA 6.1.2. *Suppose M is a p-dim minimal variety in S^n, and $F(m, t)$ is a C^∞ function on CM or CM_ε. Then*

$$\nabla^2(F)(m, t) = \frac{1}{t^2}\nabla^2(F_t)(m) + \frac{p}{t}\frac{\partial F}{\partial t}(m, t) + \frac{\partial^2 F}{\partial t^2}(m, t) .$$

PROOF. Let $E_1, \cdots, E_p$ be defined as in Proposition 6.1.1. We first observe

$$\nabla_{E_i}(F)(m, t) = \frac{1}{t}\nabla_{E_i}(F_t)(m) .$$

The above formula holds since the E_i were defined along CM by parallel translation up and down the rays, and not as the image of the e_i under $m \longrightarrow tm$. Moreover,

$$(*) \qquad \textstyle\sum_{i=1}^{p} \nabla_{E_i}\nabla_{E_i}(F)(m, t) = \frac{1}{t^2}\sum \nabla_{E_i}\nabla_{E_i}(F_t)(m) = \frac{1}{t^2}\nabla^2(F_t)(m) .$$

Thus,

$$\begin{aligned}\nabla^2(F)(m, t) &= \frac{\partial^2 F}{\partial t^2}(m, t) + \textstyle\sum_{i=1}^{p} \nabla_{e_i, e_i}(F)(m, t) \\ &= \frac{\partial^2 F}{\partial t^2}(m, t) + \textstyle\sum_{i=1}^{p} \big(\nabla_{E_i}\nabla_{E_i}(F)(m, t) - \nabla_{\nabla_{E_i}E_i}(F)(m, t)\big) .\end{aligned}$$

Now, using the above (*) and (6.1.1), the Lemma follows. q.e.d.

LEMMA 6.1.3. *Let M be a co-dimension* 1, *closed minimal variety immersed in S^n. Let $N(m, t)$ denote the unit normal field on CM_ε. Let $F(m, t)$ be a C^∞ function on CM_ε such that $F(m, 1) = F(m, \varepsilon) = 0$ for all m. Set $V(m, t) = F(m, t)\cdot N(m, t)$. Then $V(m, t)$ is a cross-section in $N(CM_\varepsilon)$ which vanishes on ∂CM_ε, and we have*

$$I(V, V) = \int_{M\times[\varepsilon,1]}\left\langle -\nabla^2(F_t) - \|A(m)\|^2 F - tp\frac{\partial F}{\partial t} - t^2\frac{\partial^2 F}{\partial t^2}, t^{p-2}F\right\rangle$$

where the integration is carried out with respect to the product measure, and $p = n - 1 = \dim M$.

PROOF. Using the fact that the ambient manifold is euclidean space, which causes the curvature term to drop out, and the fact that we are in co-dimension 1, which makes $\tilde{A}(V) = \|A\|^2 V$, formula (3.2.4) gives

$$I(V, V) = \int_{CM_\varepsilon}\langle -\nabla^2(F)(m, t) - \|A(m, t)\|^2 F, F\rangle$$

Lemmas 6.1.2 and 6.1.1 now give

$$I(V, V) = \int_{CM_\varepsilon}\left\langle -\nabla^2(F_t) - \|A(m)\|^2 F - tp\frac{\partial F}{\partial t} - t^2\frac{\partial^2 F}{\partial t^2}, \frac{1}{t^2}F\right\rangle .$$

Clearly the volume form on CM_ε is t^p times the volume form on $M \times [\varepsilon, 1]$. The Lemma now follows. q.e.d.

The above lemma suggests the definition of two differential operators:

$$\begin{aligned} &L_1\colon C^\infty(M) \longrightarrow C^\infty(M) , \qquad && L_1(f) = -\nabla^2(f) - \|A\|^2 f \\ &L_2\colon C^\infty[\varepsilon, 1] \longrightarrow C^\infty[\varepsilon, 1] , \qquad && L_2(g) = -t^2 g'' - ptg' .\end{aligned}$$

LEMMA 6.1.4. *L_1 may be diagonalized in $C^\infty(M)$ by eigenfunctions $\{f_i\}$. To each i corresponds an eigenvalue λ_i, and we have*

$$\lambda_1 \leqq \lambda_2 \leqq \cdots \leqq \lambda_i \leqq \cdots \longrightarrow \infty .$$

If $i \neq j$, then $\int_M f_i f_j = 0$. If $f \in C^\infty(M)$, then it has a unique decomposition $f = \sum_{i=1}^{\infty} a_i f_i$.

PROOF. This is essentially a restatement of Proposition 3.2.1. q.e.d.

LEMMA 6.1.5. *Let* $C_0^\infty[\varepsilon, 1]$ *denote the* C^∞ *functions on* $[\varepsilon, 1]$ *which vanish on the end points. Then* L_2 *may be diagonalized on this space by eigenfunctions* $\{g_i\}$. *To each* g_i *corresponds an eigenvalue* δ_i, *and we have* $\delta_1 < \delta_2 < \cdots < \delta_i < \cdots \to \infty$. *In fact,*

$$g_i = t^{-\frac{(p-1)}{2}} \sin\left(\frac{-i\pi}{\log \varepsilon} \log t\right)$$

$$\delta_i = \left(\frac{p-1}{2}\right)^2 + \left(\frac{i\pi}{\log \varepsilon}\right)^2 .$$

If $i \neq j$, *then* $\int_\varepsilon^1 g_i g_j t^{p-2} = 0$. *If* $g \in C_0^\infty[\varepsilon, 1]$, *then there exist unique constants* $\{a_i\}$ *such that* $g = \sum_{i=1}^\infty a_i g_i$.

PROOF. Direct calculation shows that $L_2(g_i) = \delta_i g_i$, and that these are all of the eigenvectors of L_2 in $C_0^\infty[\varepsilon, 1]$. For $g, h \in C_0^\infty[\varepsilon, 1]$, the fundamental theorem of calculus shows

$$\int_\varepsilon^1 L_2(g) \cdot h t^{p-2} = \int_\varepsilon^1 L_2(h) \cdot g t^{p-2} .$$

Thus for $i \neq j$, since $\delta_i \neq \delta_j$, g_i and g_j are orthogonal with respect to this inner product. The expansion of g follows as a consequence of orthogonality. q.e.d.

LEMMA 6.1.6. *With the hypotheses of Lemma* 6.1.3, *we may choose* $F(m, t)$ *such that* $I(V, V) < 0$ *if and only if* $\lambda_1 + \delta_1 < 0$, *where* λ_1 *and* δ_1 *are defined in the above two lemmas.*

PROOF. Since $F(m, t)$ vanishes on ∂CM_ε, Lemmas 6.1.4 and 6.1.5 show that F has a unique expansion as

$$F(m, t) = \sum_{i,j=1} a_{ij} f_i(m) g_j(t) .$$

Now, using Lemma 6.1.3,

$$\begin{aligned} I(V, V) &= \int_{M\times[\varepsilon,1]} \left\langle \sum_{i,j=1}^\infty \left(a_{ij} L_1(f_i) g_j + a_{ij} f_i L_2(g_j)\right), t^{p-2} \sum_{i,j=1}^\infty a_{ij} f_i g_j \right\rangle \\ &= \int_{M\times[\varepsilon,1]} \left\langle \sum_{i,j=1}^\infty a_{ij}[\lambda_i + \delta_j] f_i g_j, t^{p-2} \sum_{i,j=1}^\infty a_{ij} f_i g_j \right\rangle \\ &= \sum_{i,j,k,l=1}^\infty a_{ij} a_{kl} [\lambda_i + \delta_j] \int_{M\times[\varepsilon,1]} f_i f_k g_j g_l t^{p-2} . \end{aligned}$$

Using Lemma 6.1.4 and 6·1.5 again, we see

$$I(V, V) = \sum_{i,j=1}^\infty a_{ij}^2 [\lambda_i + \delta_j] \left[\int_M f_i^2\right]\left[\int_\varepsilon^1 g_j^2 t^{p-2}\right] .$$

If $I(V, V) < 0$, then some $\lambda_i + \delta_j < 0$, but since $\lambda_1 \leqq \lambda_i$ and $\delta_1 \leqq \delta_j$, this implies that $\lambda_1 + \delta_1 < 0$. On the other hand, if $\lambda_1 + \delta_1 < 0$, we may simply take $F(m, t) = f_1 \cdot g_1$ which would give $I(V, V) = \lambda_1 + \delta_1$. q.e.d.

In Lemma 6.1.5 we calculated δ_1. We now wish to examine λ_1.

LEMMA 6.1.7. *Let M be a closed, co-dimension 1 minimal variety immersed in S^n. Then $\lambda_1 \leqq -p$ unless M is the totally geodesic S^{n-1}, in which case $\lambda_1 = 0$. As before, $p = n - 1 = \dim M$.*

PROOF. If $M = S^{n-1}$, then $L_1 = -\nabla^2$, and certainly $\lambda_1 = 0$ with the other eigenvalues strictly positive. In general we have

$$(*) \qquad \lambda_1 \leqq \left[\int_M f^2\right]^{-1} \int_M L_1(f) \cdot f$$

for any $f \in C^\infty(M)$ with $f \not\equiv 0$. We will prove the theorem by choosing a one-parameter family of f's, plugging each into (*) and passing to the limit.

For any $\varepsilon > 0$, set $f_\varepsilon = (\| A \|^2 + \varepsilon)^{1/2}$. Clearly $f_\varepsilon \in C^\infty(M)$, and if M is not totally geodesic, then

$$(**) \qquad \lim_{\varepsilon \to 0} \int_M f_\varepsilon^2 = \int_M \| A \|^2 \neq 0 .$$

Let $e_1, \cdots, e_p$ be a frame in $T(M)_m$. Then

$$\nabla_{e_i} f_\varepsilon = \nabla_{e_i}(\langle A, A \rangle + \varepsilon)^{1/2} = (\langle A, A \rangle + \varepsilon)^{-1/2} \langle \nabla_{e_i} A, A \rangle .$$

Thus

$$\begin{aligned} \nabla^2 f_\varepsilon &= (\langle A, A \rangle + \varepsilon)^{-1/2} \langle \nabla^2 A, A \rangle + \sum_{i=1}^p \begin{Bmatrix} -(\langle A, A \rangle + \varepsilon)^{-3/2} \langle \nabla_{e_i} A, A \rangle^2 \\ +(\langle A, A \rangle + \varepsilon)^{-1/2} \langle \nabla_{e_i} A, \nabla_{e_i} A \rangle \end{Bmatrix} \\ &\geqq (\langle A, A \rangle + \varepsilon)^{-1/2} \langle \nabla^2 A, A \rangle + \sum_{i=1}^p \begin{Bmatrix} -(\langle A, A \rangle + \varepsilon)^{-3/2} \langle A, A \rangle \langle \nabla_{e_i} A, \nabla_{e_i} A \rangle \\ +(\langle A, A \rangle + \varepsilon)^{-1/2} \langle \nabla_{e_i} A, \nabla_{e_i} A \rangle \end{Bmatrix} \\ &\geqq (\langle A, A \rangle + \varepsilon)^{-1/2} \langle \nabla^2 A, A \rangle . \end{aligned}$$

We now use Theorem 5.3.1 which shows

$$\begin{aligned} \nabla^2 f_\varepsilon &\geqq (\langle A, A \rangle + \varepsilon)^{-1/2} \langle pA - \| A \|^2 A, A \rangle \\ &= \frac{1}{f_\varepsilon} [p \| A \|^2 - \| A \|^4] . \end{aligned}$$

Thus

$$\nabla^2 f_\varepsilon \cdot f_\varepsilon \geqq p \| A \|^2 - \| A \|^4$$

and so

$$L_1(f_\varepsilon) \cdot f_\varepsilon \leqq \| A \|^4 - p \| A \|^2 - \| A \|^2 (f_\varepsilon)^2 \leqq -p \| A \|^2 .$$

Therefore

$$\int_M L_1(f_\varepsilon) \cdot f_\varepsilon \leqq -p \int_M \| A \|^2 .$$

Thus, using (**), we have

$$\lim_{\varepsilon\to 0}\left[\int_M f_\varepsilon^2\right]^{-1}\int_M L_1(f_\varepsilon)\cdot f_\varepsilon \leqq -p\ .$$

The lemma now follows from $(*)$. q.e.d.

LEMMA 6.1.1. *Let M be a closed co-dimension 1 minimal variety in S^n. Suppose M is not the totally geodesic S^{n-1}. Then, if $n \leqq 6$, the cone CM does not minimize area with respect to its boundary.*

PROOF. Since M is not totally geodesic, Lemmas 6.1.7 and 6.1.5 show

$$\lambda_1 + \delta_1 \leqq -p + \left(\frac{p-1}{2}\right)^2 + \left(\frac{\pi}{\log\varepsilon}\right)^2 .$$

The assumption $n \leqq 6$ implies $p \leqq 5$ which implies that, for sufficiently small ε, $\lambda_1 + \delta_1 < 0$. Lemma 6.1.6 shows that, for such ε, a variation V may be chosen of the truncated cone CM_ε which holds its boundary fixed and decreases area. By extending the variation to hold fixed the set of (m, t) with $t < \varepsilon$, we get an area decreasing variation of CM. q.e.d.

This technique fails in dimensions 7 and above. In fact the following theorem gives an example of a cone over a 6-dimensional minimal variety in S^7, for which every variation holding the boundary fixed is area increasing.

THEOREM 6.1.2. *Let*

$$M = S^3\left(\frac{\sqrt{2}}{2}\right) \times S^3\left(\frac{\sqrt{2}}{2}\right)$$

considered as a minimal variety in S^7 (see Remark 5.3.1). Then every variation of CM which holds M fixed is initially increasing area.

PROOF. As was pointed out in Remark 5.3.1, $||A||^2 = 6$ everywhere. Thus $L_1(f) = -\nabla^2(f) - 6f$, and therefore $\lambda_1 = -6$. For any ε,

$$\delta_1 = \left(\frac{6-1}{2}\right)^2 + \left(\frac{\pi}{\log\varepsilon}\right)^2 \geqq 6\frac{1}{4} .$$

Therefore for any ε, $\lambda_1 + \delta_1 > 1/4$. By Lemma 6.1.6, $I(V, V) > 0$ for any variation of any CM_ε which holds the boundary fixed. Thus all the truncated cones are stable. A variation of CM need not hold the vertex fixed, however, since the area of the cone in the neighborhood of the vertex is going to zero like ε^{p+1} (in this case ε^7), it is not difficult to show that, given an area decreasing variation of CM, one can always construct, for sufficiently small ε, an area decreasing variation of CM_ε which holds its boundary fixed. This completes the proof. q.e.d.

6.2. *Plateau's problem and the Bernstein conjecture.* The results of the previous section may be applied to yield the solution of two well known problems in the theory of minimal varieties. The first of these is the co-dimension

1 Plateau problem.

Let S be a fixed $(n-2)$-dimensional compact, oriented manifold imbedded in R^n. S defines an integral $(n-2)$-dimensional current in the sense of [ ]. Let $\mathcal{H}_s$ denote the set of immersed C^∞ submanifolds H which satisfy

(a) $\dim H = n - 1$

(b) $\partial H = S$ as currents.

THEOREM 6.2.1. *Suppose $n \leqq 7$. Then there exists an $H \in \mathcal{H}_s$ having minimal $n-1$ dimensional area, and whose interior is a real analytic minimal variety in R^n.*

PROOF. This theorem follows as an immediate consequence of our Theorem 6.1.1, and of the extensive results of Federer-Fleming [14] along with those of De Giorgi [15] and Triscari [16]. The basic idea is to show that one may solve the Plateau problem in the framework of integral currents. These are then demonstrated to be regular, except on a set of measure 0. It is also shown that a singular point may be blown up to yield a set of tangent cones, and that these must be stable with respect to their boundary. One argues by induction that these cones must lie over regular co-dimension 1 minimal varieties in the sphere. It is finally shown that, if any of these cones is a disk, the current is regular at that point. Now, under the appropriate dimension assumptions, Theorem 6.1.1 shows that these cones are stable only if they are disks, and thus the current is regular everywhere in its interior.

A more detailed proof of this theorem may be found in [9], and since the proof in higher dimensions is identical, we simply refer the reader to that paper. q.e.d.

Remark 6.2.1. The statement of Theorem 6.2.1 is by no means the sharpest possible. We have stated it merely as an example of the type of theorem that is true under these dimension restrictions. For a more complete list of implications of our instability theorem for cones, the reader is referred to Theorems 1 and 2 and Corollaries 1, 2, 3, and 4 of [9], all of which go through in our dimensions.

Remark 6.2.2. The reader will note that, in Theorem 6.2.1, we could only conclude that $\partial H = S$ as currents. It would be nice if it could be shown that $\partial S = H$ as manifolds. Some progress in this direction has been made by W. K. Allard in his doctoral thesis at Brown University.

Remark 6.2.3. Our example given in Theorem 6.1.2 may be a counter-example to regularity for solutions to the co-dimension 1 Plateau problem in dimensions 8 and above. The cone over $S^3 \times S^3$ is an integral current with an isolated singularity and having $S^3 \times S^3$ as boundary. Moreover, it is a local minimum of the area function. Whether or not it is a global minimum is an open question.

As was mentioned in § 3.1, there have been a number of proofs of the Bernstein conjecture for graphs in R^3. Most of these have used the methods of complex analysis. One which does not is that of Fleming [7]. He shows that the Bernstein conjecture for graphs in R^n would follow from an interior regularity theorem for co-dimension 1 minimal integral currents in R^n. This gave a proof in R^3. De Giorgi [8] then showed that the conjecture for graphs in R^n would actually follow from a co-dimension 1 regularity theorem in R^{n-1}. This gave a proof in R^4. The new interior regularity theorem of Almgren [9] for co-dimension 1 minimal currents in R^4 proved the Bernstein conjecture in R^5. Our interior regularity theorem provides a uniform proof in R^n for $n \leqq 8$.

THEOREM 6.2.2. *Let $f(x_1, \cdots, x_{n-1})$ be a C^∞ function defined everywhere in R^{n-1}. Suppose its graph is a minimal variety in R^n. Then if $n \leqq 8$, f is a linear function.*

PROOF. Fleming's argument is essentially the following. He first shows that the graph absolutely minimizes area with respect to any compact boundary in its interior. Then he takes the intersection of the graph with the ball of radius r and contracts by $1/r$ to get a family of minimal varieties in the unit ball whose boundaries are co-dimension 1 submanifolds of S^{n-1}. He then takes the limit of these minimal varieties as $r \to \infty$, and shows it to be the cone over an integral current in S^{n-1}. The above mentioned absolute minimization property of the graph implies that the cone is a minimal integral current with respect to its boundary, and an interior regularity theorem would imply that it is therefore a disk. He finally shows that the cone is a disk only if the graph was a hyperplane. Thus, Theorem 6.2.1 makes this proof work for $n \leqq 7$. De Giorgi then pointed out that interior regularity in R^{n-1} already implied that the limiting cone was over a real analytic minimal variety in S^{n-1}. Moreover this variety has its normal vector making a non-negative inner product with the positive z axis. He could then use his theorem (see our Theorem 5.2.1) together with some extra work to show the cone was a disk. This then proves the theorem for $n \leqq 8$. q.e.d.

Remark 6.2.4. A key feature of Fleming's argument is the above observation that the graph is an absolute minimum of area for any boundary in its interior. It is possible that one could proceed from that point to prove the theorem without actually passing to the cone and using interior regularity. In fact, one could attempt to show that, for a sufficiently large boundary, one could construct a variation V vanishing on the boundary, and having $I(V, V) < 0$ unless the surface was a hyperplane. This approach seems reasonable since it is essentially what is done to prove that the limiting cone is unstable, and it has two possible advantages. First, it might work in more or all dimensions. Second, it might suggest a more intrinsic hypothesis on a complete minimal hypersurface which would guarantee its being a hyperplane. To be precise, we conjecture

Conjecture 6.2.1. Let $f: R^{n-1} \to R^n$ be an immersion as a complete minimal variety. Then either the image is a hyperplane; or, for sufficiently large r, the sphere of radius r in R^{n-1} is mapped in as a conjugate boundary (see § 3.4).

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