WEIL-PETERSSON TEICHMÜLLER SPACE III: DEPENDENCE OF RIEMANN MAPPINGS FOR WEIL-PETERSSON CURVES

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ABSTRACT. The primary purpose of the paper is to study how a Riemann mapping depends on the corresponding Jordan curve. We are mainly concerned with those Jordan curves in the Weil-Petersson class, namely, the corresponding Riemann mappings can be quasiconformally extended to the whole plane with Beltrami coefficients being square integrable under the Poincaré metric. We endow the space of all normalized Weil-Petersson curves with a new real Hilbert manifold structure and show that it is topologically equivalent to the standard complex Hilbert manifold structure.

1 BACKGROUND

We begin with the well-known Riemann mapping theorem: Let Ω be the left domain bounded by a Jordan curve Γ passing through the point at infinity in the extended complex plane $\hat{\mathbb{C}}$. Then there exists a univalent analytic function f which maps the upper half plane $\mathbb{U} \doteq \{z = x+iy : y > 0\}$ conformally onto Ω with $f(\infty) = \infty$. There also exists a univalent analytic function g which maps the lower half plane $\mathbb{U}^* \doteq \{z = x+iy : y < 0\}$ conformally onto the right domain Ω^* bounded by Γ with $g(\infty) = \infty$. Both f and g are uniquely determined up to an affine mapping $z \mapsto az + b$ with $a > 0$, $b \in \mathbb{R}$, the real line. f and g determine an increasing homeomorphism $h : \mathbb{R} \to \mathbb{R}$ by $h = f^{-1} \circ g$, which is called a conformal sewing mapping of the curve Γ. h is uniquely determined up to two affine mappings. The primary purpose of this paper is to study how the mappings f, g and h depend on a Jordan curve Γ. As explained by Semmes [Se4], this is a good example of a problem in nonlinear Fourier analysis.

An interesting result due to Coifman-Meyer [CM] says that a Riemann mapping f depends on Γ real-analytically when Γ is a chord-arc curve passing through the point at infinity (see [Wu] for an analogous result for bounded chord-arc curves). Recall that a

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Jordan curve Γ passing ∞ is a chord-arc (or Lavrentiev) curve with constant $k \geq 0$ if it is locally rectifiable and

$$|s_1 - s_2| \leq (1+k)|z(s_1) - z(s_2)| \tag{1.1}$$

for all $s_1 \in \mathbb{R}$ and $s_2 \in \mathbb{R}$, where $z(s)$ is a parametrization of Γ by the arc-length $s \in \mathbb{R}$ (see [La], [Po2]). To make this precise, let Γ be a chord-arc curve passing through 0 and ∞, and $z(s)$ be the (unique) arc-length parametrization of Γ with $z(0) = 0$. David [Da] showed that there exists some function b in $\mathrm{BMO}_{\mathbb{R}}$ (or more precisely, $\mathrm{BMO}_{\mathbb{R}}/\mathbb{R}$), the space of all real-valued functions of bounded mean oscillation on the real line (see [FS], [Gar], [Po2], [Zh] and section 3 below), such that $z'(s) = e^{ib(s)}$, and these BMO functions $b's$ form an open subset $\mathcal{L}$ of $\mathrm{BMO}_{\mathbb{R}}/\mathbb{R}$. A Riemann mapping $f : \mathbb{U} \to \Omega$ induces an increasing homeomorphism $h_1 : \mathbb{R} \to \mathbb{R}$ by $f \circ h_1 = z$. A classical result of Lavrentiev [La] implies that h_1 is locally absolutely continuous so that h_1' belongs to the class of weights A^∞ introduced by Muckenhoupt (see [CF], [Gar]), in particular, $\log h_1'$ is a BMO function. The precise statement of the result of Coifman-Meyer [CM] is: The correspondence $b \mapsto \log h_1'$ induces a well-defined real-analytic map from $\mathcal{L}$ into $\mathrm{BMO}_{\mathbb{R}}/\mathbb{R}$. A different approach to this result was given later by Semmes [Se3] (see also [Se1]).

To see how a Riemann mapping f itself, not just the induced mapping h_1 by $f \circ h_1 = z$, depends on the curve Γ when it is a chord-arc curve, we recall a result of Pommerenke [Po1] (see also [Zi]), which says that $\log f'$ belongs to BMOA, the space of analytic functions in $\mathbb{U}$ of bounded mean oscillation (see [FS], [Gar]). From

$$f \circ h_1 = z \Rightarrow (f' \circ h_1)h_1' = z' = e^{ib} \Rightarrow \log(f' \circ h_1) + \log h_1' = ib \Rightarrow \log f' = (ib - \log h_1') \circ h_1^{-1},$$

it is not clear how the mapping f depends on the curve Γ (or on the function b), although $\log h_1'$ depends real analytically on b. For example, we do not know whether $\log f'$ depends continuously on b. It is also not clear how a conformal sewing mapping h depends on the curve Γ (or on the function b) when it is a chord-arc curve. Actually, Katznelson-Nag-Sullivan [KNS] asked whether $\log(h^{-1})'$ depends continuously on b for a chord-arc curve Γ. Anyhow, by means of some results in our paper [SW] (see also [AZ]), we conclude that, under some normalized conditions, $\log f'$ depends continuously on $\log(h^{-1})'$ and $\log(h^{-1})'$ depends continuously on $\log f'$, which implies that the continuous dependence of $\log f'$ on Γ (or b) would imply the continuous dependence of $\log(h^{-1})'$ on Γ (or b), and vice versa[1].

Since it is not clear how a Riemann mapping f (or g) depends on the curve Γ even when it is a chord-arc curve, it is desirable to find a subclass of chord-arc curves on which a Riemann mapping does depend continuously. In this paper, we will consider an important sub-class of chord-arc curves, which we call them Weil-Petersson curves. This class of Jordan curves and its Teichmüller space has been much investigated in recent years (see [Cu], [Fi], [GGPPR], [GR], [Sh], [ST], [STW], [TT] and section 2 below).

[1] We conjecture that in general neither $\log f'$ nor $\log(h^{-1})'$ depends continuously on Γ (or b).

We say a Jordan curve Γ passing through ∞ is a Weil-Petersson curve if a Riemann mapping g, which maps the lower half plane $\mathbb{U}^*$ conformally onto the right domain Ω^* bounded by Γ with $g(\infty)=\infty$, has a quasiconformal extension to the whole plane whose Beltrami coefficient is square integrable in the Poincaré metric. An open problem is to give a geometric characterization of a Weil-Petersson curve without using a Riemann mapping g or its quasiconformal extensions. An analogous problem was proposed by Takhtajan-Teo [TT] for bounded Weil-Petersson curves and a partial answer was given by Gallardo-Gutiérrez, González, Pérez-González, Pommerenke and Rättyä [GGPPR] in this case. We will show that for the arc-length parametrization $z(s)$, $z(0)=0$, of a Weil-Petersson curve Γ passing through the points 0 and ∞, there exists some function b in the real Sobolev class $H^{\frac{1}{2}}_{\mathbb{R}}/\mathbb{R}$ such that $z'(s)=e^{ib(s)}$, and these $H^{\frac{1}{2}}$ functions $b's$ form an open subset of $H^{\frac{1}{2}}_{\mathbb{R}}/\mathbb{R}$. Then we will show that for those Weil-Petersson curves Γ with $z(0)=0$, $z(1)>0$, $z(\infty)=\infty$, an appropriately chosen Riemann mapping f (or g) and the corresponding conformal sewing mapping h depend continuously on Γ (and vice versa). The precise statements of these results will be given in section 2. As far as we know, this is the first result on continuous dependence of the Riemann mappings and conformal sewing mappings on non-smooth Jordan curves.

In the paper, C, C_1, C_2 $\cdots$ will denote universal constants that might change from one line to another, while $C(\cdot)$, $C_1(\cdot)$, $C_2(\cdot)$ $\cdots$ will denote constants that depend only on the elements put in the brackets. The notation $A \lesssim B$ ($A \gtrsim B$) means that there is a positive constant C independent of A and B such that $A \leq CB$ ($A \geq CB$). The notation $A \asymp B$ means both $A \lesssim B$ and $A \gtrsim B$.

2 Introduction and statement of results

In this section, we will give some basic definitions and results on the universal Teichmüller space and the Weil-Petersson Teichmüller space (see the books [Ah], [GL], [Le], [Na] and the papers [Sh], [TT] for more details). We will also state the main results of this paper.

2.1 Various models of the universal Teichmüller space Let $M(\mathbb{U})$ denote the open unit ball of the Banach space $L^\infty(\mathbb{U})$ of essentially bounded measurable functions on the upper half plane $\mathbb{U}$ in the complex plane $\mathbb{C}$. For $\mu \in M(\mathbb{U})$, let f^μ be the unique quasiconformal mapping from $\mathbb{U}$ onto itself which has complex dilatation μ and fixes the points 0, 1 and ∞, and f_μ be the unique quasiconformal mapping on the extended plane $\hat{\mathbb{C}}$ which has complex dilatation μ in $\mathbb{U}$, is conformal in the lower half plane $\mathbb{U}^*$ and fixes the points 0, 1 and ∞. We say two elements μ and ν in $M(\mathbb{U})$ are equivalent, denoted by $\mu \sim \nu$, if $f^\mu|_{\mathbb{R}} = f^\nu|_{\mathbb{R}}$, or equivalently, $f_\mu|_{\mathbb{U}^*} = f_\nu|_{\mathbb{U}^*}$. Then $T = M(\mathbb{U})/_\sim$ is the Bers model of the universal Teichmüller space. We let Φ denote the natural projection from $M(\mathbb{U})$ onto T so that $\Phi(\mu)$ is the equivalence class $[\mu]$. $[0]$ is called the base point of T.

An increasing homeomorphism h from the real line $\mathbb{R}$ onto itself is said to be quasisymmetric if there exists a (least) positive constant $C(h)$, called the quasisymmetric constant of h, such that $|h(I_1)| \leq C(h)|h(I_2)|$ for all pairs of adjacent intervals I_1 and I_2 on $\mathbb{R}$ with the same length $|I_1| = |I_2|$. Beurling-Ahlfors [BA] proved that an increasing homeomorphism h from the real line $\mathbb{R}$ onto itself is quasisymmetric if and only if there exists some quasiconformal homeomorphism of $\mathbb{U}$ onto itself which has boundary values h. A Jordan curve Γ passing through ∞ is said to be a quasicircle if it is the image of the extended real line $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ under a quasiconformal mapping on the whole plane. It is easy to see that a Jordan curve Γ passing through ∞ is a quasicircle if and only if a Riemann mapping f (or g) can be extended to a quasiconformal mapping on the whole plane. Therefore, the universal Teichmüller space T can also be defined as:

- The set of all quasisymmetric homeomorphisms of the real line onto itself with 0, 1 and ∞ fixed ($[\mu] \mapsto f^{\mu}|_{\mathbb{R}}$).
- The set of all conformal mappings on the lower half plane $\mathbb{U}^*$ which can be quasiconformally extended to the whole plane with the points 0, 1 and ∞ fixed ($[\mu] \mapsto f_{\mu}|_{\mathbb{U}^*}$).
- The set of all quasicircles through the points 0, 1 and ∞ ($[\mu] \mapsto f_{\mu}(\hat{\mathbb{R}})$).

It is known that the universal Teichmüller space T is an infinite dimensional complex Banach manifold. To make this precise, we first recall some important Banach spaces. Let D be an arbitrary simply connected domain in the extended complex plane $\hat{\mathbb{C}}$ which is conformally equivalent to the upper half plane. The hyperbolic metric $\lambda_D(z)|dz|$ (with curvature constantly equal to -1) in D is defined by

$$\lambda_D(f(z))|f'(z)| = \frac{1}{y}, \quad z = x + iy \in \mathbb{U}, \tag{2.1}$$

where $f : \mathbb{U} \to D$ is any conformal mapping. Let $B_2(D)$ denote the Banach space of functions ϕ holomorphic in D with norm

$$\|\phi\|_{B_2(D)} \doteq \sup_{z \in D} |\phi(z)|\lambda_D^{-2}(z), \tag{2.2}$$

and $\mathcal{B}(D)$ the Banach space of functions ϕ holomorphic in D with finite norm

$$\|\phi\|_{\mathcal{B}(D)} \doteq \left(\frac{1}{\pi} \iint_D |\phi(z)|^2 \lambda_D^{-2} dxdy \right)^{\frac{1}{2}}. \tag{2.3}$$

Then, $\mathcal{B}(D) \subset B_2(D)$, and the inclusion map is continuous (see [Zh]).

Now we consider the map $S : M(\mathbb{U}) \to B_2(\mathbb{U}^*)$ which sends μ to the Schwarzian derivative of $f_{\mu}|_{\mathbb{U}^*}$. Recall that for any locally univalent function f, its Schwarzian derivative S_f is defined by

$$S_f \doteq N_f' - \frac{1}{2}N_f^2, \quad N_f \doteq (\log f')'. \tag{2.4}$$

S is a holomorphic split submersion onto its image, which descends down to a map $\beta : T \to B_2(\mathbb{U}^*)$ known as the Bers embedding. Via the Bers embedding, T carries a natural complex Banach manifold structure so that Φ is a holomorphic split submersion.

Besides the Schwarzian derivative model, the universal Teichmüller space has another important model, the pre-logarithmic derivative model. In the unit disk case, the pre-logarithmic derivative model of the universal Teichmüller spaces was much investigated (see [AGe], [Po2], [Zhu]). Here we consider the upper half plane case. Let $B_1(D)$ denote the Bloch space of functions ϕ holomorphic in a simply connected domain D with semi-norm

$$\|\phi\|_{B_1(D)} \doteq \sup_{z\in D} |\phi'(z)|\lambda_D^{-1}(z), \tag{2.5}$$

and $\mathcal{D}(D)$ denote the Dirichlet space of functions ϕ holomorphic in D with semi-norm

$$\|\phi\|_{\mathcal{D}(D)} \doteq \left(\frac{1}{\pi}\iint_D |\phi'(z)|^2 dxdy\right)^{\frac{1}{2}}. \tag{2.6}$$

It is known that $\mathcal{D}(D) \subset B_1(D)$, and the inclusion map is continuous (see [Zh]). It is also known that, for each holomorphic function ϕ on D, $\phi'' \in \mathcal{B}(D)$ if $\phi \in \mathcal{D}(D)$, and $\phi'' \in B_2(D)$ if $\phi \in B_1(D)$. The converse is also true, with some normalized conditions at ∞ whenever D is not a bounded domain (see [ST], [STW]).

Now Koebe distortion theorem implies that $\log(f_\mu|_{\mathbb{U}^*})' \in B_1(\mathbb{U}^*)$ for $\mu \in M(\mathbb{U})$. Furthermore, the map L induced by the correspondence $\mu \mapsto \log(f_\mu|_{\mathbb{U}^*})'$ is a continuous map from $M(\mathbb{U})$ into $B_1(\mathbb{U}^*)$ (see [Le]). Actually, $L : M(\mathbb{U}) \to B_1(\mathbb{U}^*)$ is even holomorphic (see [Ha]).

2.2 Various models of the Weil-Petersson Teichmüller space Now we define the Weil-Petersson Teichmüller space. We denote by $\mathcal{L}^\infty(D)$ the Banach space of all essentially bounded measurable functions μ on a simply connected domain D with norm

$$\|\mu\|_{\mathrm{WP}} \doteq \|\mu\|_\infty + \left(\frac{1}{\pi}\iint_D |\mu(z)|^2 \lambda_D^2(z) dxdy\right)^{\frac{1}{2}}. \tag{2.7}$$

Set $\mathcal{M}(\mathbb{U}) = M(\mathbb{U}) \cap \mathcal{L}^\infty(\mathbb{U})$. Then $\mathcal{T} = \mathcal{M}(\mathbb{U})/_\sim$ is known as the Weil-Petersson Teichmüller space. Actually, $\mathcal{T}$ is the base point component of the universal Teichmüller space under the complex Hilbert manifold structure introduced by Takhtajan-Teo [TT]. Under the Bers projection $S : M(\mathbb{U}) \to B_2(\mathbb{U}^*)$, $S(\mathcal{M}(\mathbb{U})) = S(M(\mathbb{U})) \cap \mathcal{B}(\mathbb{U}^*)$ (see [Cu], [TT]). More precisely, $S : \mathcal{M}(\mathbb{U}) \to \mathcal{B}(\mathbb{U}^*)$ is a holomorphic split submersion onto its image, which induces a natural complex Hilbert manifold on $\mathcal{T}$ so that $\Phi : \mathcal{M}(\mathbb{U}) \to \mathcal{T}$ is a holomorphic split submersion. Very recently, we proved that under the pre-logarithmic derivative projection $L : M(\mathbb{U}) \to B_1(\mathbb{U}^*)$, $L(\mathcal{M}(\mathbb{U})) = L(M(\mathbb{U})) \cap \mathcal{D}(\mathbb{U}^*)$, and $L : \mathcal{M}(\mathbb{U}) \to \mathcal{D}(\mathbb{U}^*)$ is holomorphic (see [STW]).

We proceed to introduce the quasisymmetric homeomorphism model of the Weil-Petersson Teichmüller space. For simplicity, we say a quasiconformal mapping on a simply connected domain D is of the Weil-Petersson class if its Beltrami coefficient is in $\mathcal{L}^\infty(D)$. A quasiconformal mapping f on the whole plane is said to belong to the Weil-Petersson class (with respect to the real line) if both $f|_{\mathbb{U}}$ and $f|_{\mathbb{U}^*}$ are of the Weil-Petersson class. A sense preserving homeomorphism h of the real line $\mathbb{R}$ onto itself is said to belong the Weil-Petersson class if it can be extended a Weil-Petersson quasiconformal mapping to the upper half plane $\mathbb{U}$. We denote by $\mathrm{WP}(\mathbb{R})$ the class of all Weil-Petersson homeomorphisms on $\mathbb{R}$, and by $\mathrm{WP}_0(\mathbb{R})$ be those with the points 0, 1 and ∞ fixed. Then we have the following result.

Proposition 2.1. *An increasing homeomorphism h from the real line $\mathbb{R}$ onto itself belongs to the Weil-Petersson class if and only if h is locally absolutely continuous with* $\log h' \in H^{\frac{1}{2}}(\mathbb{R})$.

Recall that the Sobolev class $H^{\frac{1}{2}}$ ($H^{\frac{1}{2}}_{\mathbb{R}}$) on the real line $\mathbb{R}$ is the collection of all locally integrable (real) functions u with

$$\|u\|^2_{H^{\frac{1}{2}}} \doteq \frac{1}{4\pi^2}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|u(s)-u(t)|^2}{(s-t)^2} dsdt < +\infty. \tag{2.8}$$

Proposition 2.1 was proved in our papers [ST], [STW]. An analogous result on the unit circle was first proved by the first author [Sh], which solves a problem proposed by Takhtajan-Teo in 2006 (see page 68 in [TT] and also [Fi], [GR]). The following result (see [ST], [STW]) says that the normalized Weil-Petersson class $\mathrm{WP}_0(\mathbb{R})$, the quasisymmetric homeomorphism model of the Weil-Petersson Teichmüller space $\mathcal{T}$, can be endowed with a real Hilbert manifold structure from $H^{\frac{1}{2}}_{\mathbb{R}}/\mathbb{R}$ by the correspondence $h \mapsto \log h'$, which is real analytically equivalent to the standard complex Hilbert manifold structure on $\mathcal{T}$ given by Takhtajan-Teo [TT]. Both Propositions 2.1 and 2.2 will play an important role in our later discussion.

Proposition 2.2. *The correspondence $h \mapsto \log h'$ induces a real analytic map Ψ from the normalized Weil-Petersson class $\mathrm{WP}_0(\mathbb{R})(= \mathcal{T})$ onto the real Sobolev space $H^{\frac{1}{2}}_{\mathbb{R}}/\mathbb{R}$ whose inverse Ψ^{-1} is also real analytic.*

Before we state the main results, we summarize that the Weil-Petersson Teichmüller space $\mathcal{T}$ can also be defined in the following ways:
- The set $\mathrm{WP}_0(\mathbb{R})$ of all normalized Weil-Petersson homeomorphisms with 0, 1, ∞ fixed ($[\mu] \mapsto f^{\mu}|_{\mathbb{R}}$).
- The set of all conformal mappings g on the upper half plane $\mathbb{U}^*$ which can be extended to Weil-Petersson quasiconformal mappings on the whole plane and satisfies the following normalized conditions ($[\mu] \mapsto g_{\mu}|_{\mathbb{U}^*}$):

$$g(0) = 0,\ g(\infty) = \infty,\ g(1) > 0,\ \int_0^1 |g'(t)|dt = 1. \tag{2.9}$$

- The set of all normalized Weil-Petersson curves on the whole plane ($[\mu] \mapsto g_{\mu}(\hat{\mathbb{R}})$).

For later purposes, here we have used some normalized conditions different from the universal Teichmüller space case. A Weil-Petersson curve Γ is called normalized if it passes through 0 and ∞ and the arclength parametrization $z = z(s)$ of Γ with $z(0) = 0$ satisfies $z(1) > 0$. For $\mu \in \mathcal{M}(\mathbb{U})$, g_{μ} is the unique quasiconformal mapping on the extended plane $\hat{\mathbb{C}}$ which has complex dilatation μ in $\mathbb{U}$, is conformal in $\mathbb{U}^*$ and with the normalized conditions (2.9). It is easy to see that μ and ν in $\mathcal{M}(\mathbb{U})$ are equivalent if and only if $g_{\mu}|_{\mathbb{U}^*} = g_{\nu}|_{\mathbb{U}^*}$. Actually, it holds that $g_{\mu} = g_{\mu}(1)f_{\mu}$.

2.3 Statement of main results As stated in section 1, an open problem is to give a geometric characterization of a Weil-Petersson curve without using a Riemann mapping

f (or g) or its quasiconformal extensions. A basic geometric notion to a locally rectifiable curve is an arc-length parametrization. Therefore, a natural question is to characterize an arclength parametrization of a Weil-Petersson curve. Our first result goes in this direction.

Theorem 2.1. *Let Γ be a normalized Weil-Petersson curve and $z = z(s)$ be the arc-length parametrization of Γ with $z(0) = 0$. Then there exists some function b in the real Sobolev class $H^{\frac{1}{2}}_{\mathbb{R}}/\mathbb{R}$ such that $z'(s) = e^{ib(s)}$. Moreover, the set $\hat{\mathcal{T}}$ of these $H^{\frac{1}{2}}$ functions $b's$ is an open subset of $H^{\frac{1}{2}}_{\mathbb{R}}/\mathbb{R}$.*

Actually, we have the following geometric characterization of a Weil-Petersson curve by means of the arc-length parametrization under the geometric assumption of chord-arc property.

Theorem 2.2. *Let Γ be a locally rectifiable Jordan curve passing through ∞ and $z = z(s)$ be an arc-length parametrization of Γ. Then Γ is a Weil-Petersson curve if and only if Γ is a chord-arc curve and there exists some function b in the real Sobolev class $H^{\frac{1}{2}}_{\mathbb{R}}/\mathbb{R}$ such that $z'(s) = e^{ib(s)}$. In other words, $\hat{\mathcal{T}} = \mathcal{L} \cap H^{\frac{1}{2}}_{\mathbb{R}}/\mathbb{R}$.*

Theorem 2.1 implies that the set of all normalized Weil-Petersson curves, a model of the Weil-Petersson Teichmüller space $\mathcal{T}$, can be endowed with a real Hilbert manifold structure in a geometric manner by the correspondence $\Gamma \mapsto b$. We will show that this new real Hilbert manifold structure is topologically equivalent to the standard complex Hilbert manifold structure given by Takhtajan-Teo [TT]. To be precise, we introduce some notations. For a normalized Weil-Petersson curve Γ with arc-length parametrization $z(s)$, $z(0) = 0$, we denote by f_Γ the unique Riemann mapping which takes $\mathbb{U}$ onto the left domain Ω bounded Γ with the normalized conditions (2.9), that is, $f_\Gamma(s) = z(s)$ for $s = 0, 1, \infty$. Similarly, we denote by g_Γ the unique Riemann mapping which takes $\mathbb{U}^*$ onto the right domain Ω^* bounded Γ such that $g_\Gamma(s) = z(s)$ for $s = 0, 1, \infty$. Denote by h_Γ the unique conformal sewing mapping determined by f_Γ and g_Γ, that is, $h_\Gamma = f_\Gamma^{-1} \circ g_\Gamma$. For $b \in \hat{\mathcal{T}}$, we may assume without loss of generality that $\int_0^1 e^{ib(t)}dt > 0$, and then denote by Γ_b the unique normalized Weil-Petersson curve whose arc-length parametrization z_b with $z_b(0) = 0$ satisfies $z_b' = e^{ib}$, namely,

$$z_b(s) = \int_0^s e^{ib(t)}dt. \tag{2.10}$$

Finally, for $b \in \hat{\mathcal{T}}$, we set $f_b = f_{\Gamma_b}$, $g_b = g_{\Gamma_b}$, and $h_b = h_{\Gamma_b}$. Then we have

Theorem 2.3. *The correspondence $b \mapsto h_b$ induces a homeomorphism from $\hat{\mathcal{T}}$ onto the normalized Weil-Petersson class $\mathrm{WP}_0(\mathbb{R})$ $(= \mathcal{T})$, or equivalently, the correspondence $b \mapsto \log h_b'$ induces a homeomorphism from $\hat{\mathcal{T}}$ onto the real Sobolev space $H^{\frac{1}{2}}_{\mathbb{R}}/\mathbb{R}$.*

Theorem 2.3 can also be restated as

Theorem 2.4. *The correspondence $b \mapsto g_b$ induces a homeomorphism from $\hat{\mathcal{T}}$ onto (the conformal mapping model of) the Weil-Petersson Teichmüller space $\mathcal{T}$. Equivalently, the correspondence $b \mapsto \log g_b'$ induces a homeomorphism from $\hat{\mathcal{T}}$ onto its image in $\mathcal{D}(\mathbb{U}^*)/\mathbb{C}$, or equivalently, the correspondence $b \mapsto \log f_b'$ induces a homeomorphism from $\hat{\mathcal{T}}$ onto its image in $\mathcal{D}(\mathbb{U})/\mathbb{C}$.*

Since the Weil-Petersson Teichmüller space $\mathcal{T}$ is contractible, we obtain

Corollary 2.1. *The arc-length parametrization space $\hat{\mathcal{T}}$ of the normalized Weil-Petersson curves is contractible.*

Remark. It is not known whether the arc-length parametrization space $\mathcal{L}$ of the normalized chord-arc curves is contractible. Actually, it is even not known whether $\mathcal{L}$ is connected. This is known to be a difficult open problem (see [AGo], [AZ], [CM], [Se1]).

3 BMO functions revisited

In order to prove Theorems 2.3-4, we need a construction concerning quasiconformal extensions of strongly quasisymmetric homeomorphisms introduced by Semmes [Se2-2], which relies heavily on BMO estimates. In this section we recall some basic definitions and results on BMO functions (see [Gar]).

A locally integrable function $u \in L^1_{loc}(\mathbb{R})$ is said to have bounded mean oscillation and belongs to the space BMO if

$$\|u\|_{\mathrm{BMO}} \doteq \sup \frac{1}{|I|}\int_I |u(t)-u_I|dt < +\infty, \tag{3.1}$$

where the supremum is taken over all finite sub-intervals I of $\mathbb{R}$, while u_I is the average of u on the interval I, namely,

$$u_I = \frac{1}{|I|}\int_I u(t)dt. \tag{3.2}$$

If u also satisfies the condition

$$\lim_{|I|\to 0} \frac{1}{|I|}\int_I |u(t)-u_I|dt = 0,$$

we say u has vanishing mean oscillation and belongs to the space VMO. It is well known that $H^{\frac{1}{2}} \subset \mathrm{VMO}$, and the inclusion map is continuous (see [Zh]). In the following, we denote by $\mathrm{BMO}_{\mathbb{R}}$ the set of all real-valued BMO functions.

We need some basic results on BMO functions. By the well-known theorem of John-Nirenberg for BMO functions (see [Gar]), there exist two universal positive constants C_1 and C_2 such that for any BMO function u, any subinterval I of $\mathbb{R}$ and any $\lambda > 0$, it holds that

$$\frac{|\{t \in I : |u(t)-u_I| \geq \lambda\}|}{|I|} \leq C_1 \exp\left(\frac{-C_2\lambda}{\|u\|_{\mathrm{BMO}}}\right). \tag{3.3}$$

By Chebychev's inequality, we obtain that for u with $\|u\|_{\mathrm{BMO}} < C_2$,

$$\begin{aligned}\frac{1}{|I|}\int_I (e^{|u-u_I|}-1)dt &= \frac{1}{|I|}\int_0^\infty |\{t\in I: |u-u_I|\geq \lambda\}|\, d(e^\lambda - 1)\\ &\leq C_1 \int_0^\infty e^\lambda \exp\left(\frac{-C_2\lambda}{\|u\|_{\mathrm{BMO}}}\right) d\lambda\\ &\leq \frac{C_1\|u\|_{\mathrm{BMO}}}{C_2-\|u\|_{\mathrm{BMO}}}.\end{aligned} \tag{3.4}$$

Similarly, for any $p\geq 1$ we have

$$\frac{1}{|I|}\int_I |u-u_I|^p dt \lesssim C(p)\|u\|^p_{\mathrm{BMO}}. \tag{3.5}$$

Lemma 3.1. *Let ϕ be a C^∞ function on the real line which is supported on $[-1,1]$ and satisfies $\int_{\mathbb{R}}\phi(x)dx = 1$. Set $\phi_y(x) = |y|^{-1}\phi(|y|^{-1}x)$ for $y\neq 0$, and consider the convolution*

$$\phi_y * w(x) = \int_{\mathbb{R}} \phi_y(x-t)w(t)dt. \tag{3.6}$$

Suppose $v\in L^\infty(\mathbb{R})$ and $|\phi_y * v|\geq \epsilon_0$ for some $\epsilon_0>0$. Then for

$$R_y(u)(x) = \frac{\phi_y*(vu)(x)}{\phi_y*v(x)}$$

it holds that

$$|R_y(e^u)| \asymp |e^{R_y(u)}| \tag{3.7}$$

when $\|u\|_{\mathrm{BMO}}$ is small.

Proof. Lemma 3.1 appeared implicitly in [Se3] though not stated in this form. For completeness and for convenience of later use, we write down the detailed proof here (see [Se3], [ST]). Actually, besides Lemma 3.1 itself, both of the estimates (3.8) and (3.9) below will be frequently used in section 6.

For $x\in\mathbb{R}$ and $y>0$, consider $I=[x-y,x+y]$ so that

$$u_I = \frac{1}{2y}\int_{x-y}^{x+y} u(t)dt.$$

Since $\int_{\mathbb{R}}\phi(x)dx=1$, which implies that $\int_{\mathbb{R}}\phi_y(x)dx = 1$, and $R_y(1)\equiv 1$, we obtain

$$\begin{aligned}|R_y(u)(x)-u_I| = |R_y(u-u_I)(x)| &\leq \frac{1}{\epsilon_0}|\phi_y*(v(u-u_I))(x)|\\ &\leq \frac{C(\phi)\|v\|_\infty}{\epsilon_0}\frac{1}{|I|}\int_I |u(t)-u_I|dt \lesssim \|u\|_{\mathrm{BMO}}.\end{aligned} \tag{3.8}$$

Since $|e^z-1|\le|ze^z|\le|z|e^{|z|}$, we have

$$\begin{aligned}&\frac{1}{|I|}\int_I|e^{u(t)-R_y(u)(x)}-1|dt\\&\le\frac{1}{|I|}\int_I|e^{u(t)-R_y(u)(x)}||u(t)-R_y(u)(x)|dt\\&\le\frac{|e^{u_I-R_y(u)(x)}|}{|I|}\int_I|e^{u(t)-u_I}|(|u(t)-u_I|+|u_I-R_y(u)(x)|dt.\end{aligned}$$

Using Hölder inequality, we conclude from (3.4), (3.5) and (3.8) that

$$\frac{1}{|I|}\int_I|e^{u(t)-R_y(u)(x)}-1|dt\lesssim\|u\|_{\mathrm{BMO}}\tag{3.9}$$

when $\|u\|_{\mathrm{BMO}}$ is small. Noting that

$$R_y(e^u)(x)-e^{R_y(u)(x)}=e^{R_y(u)(x)}R_y(e^{u-R_y(u)(x)}-1)(x),$$

we obtain

$$\begin{aligned}|R_y(e^u)(x)-e^{R_y(u)(x)}|&=|e^{R_y(u)(x)}||R_y(e^{u-R_y(u)(x)}-1)(x)|\\&\lesssim\frac{|e^{R_y(u)(x)}|}{|I|}\int_I|e^{u(t)-R_y(u)(x)}-1|dt,\end{aligned}$$

which implies by (3.9) the required relation (3.7). □

4 Proof of Theorems 2.1 and 2.2

In this section, we will give simple proof of Theorems 2.1 and 2.2. However, to prove Theorems 2.3 and 2.4, we need a concrete approach to the openness of $\hat{\mathcal{T}}$, which will be given in section 6.

Let Γ be a locally rectifiable Jordan curve passing through ∞ and $z=z(s)$ be an arc-length parametrization of Γ. Let f map the upper half plane $\mathbb{U}$ conformally onto the left domain Ω bounded by Γ with $f(\infty)=\infty$. Set $h_1:\mathbb{R}\to\mathbb{R}$ by $f\circ h_1=z$ as before. Then we have

Theorem 4.1. *Under the above notations, the following statements are equivalent*:
(1) Γ *is a Weil-Petersson curve*;
(2) $h_1\in\mathrm{WP}(\mathbb{R})$;
(3) $\arg z'\circ h_1^{-1}\in H^{\frac12}_{\mathbb{R}}$.

Proof. From $f\circ h_1=z$ we obtain $f'=(z'\circ h_1^{-1})(h_1^{-1})'$, which implies that

$$\Re\log f'=\log(h_1^{-1})',\ \Im\log f'=\arg z'\circ h_1^{-1}.\tag{4.1}$$

Now Γ is a Weil-Petersson curve if and only if

$$\log f'\in\mathcal{D}(\mathbb{U})\Leftrightarrow\Re\log f'\in H^{\frac12}_{\mathbb{R}}\Leftrightarrow\Im\log f'\in H^{\frac12}_{\mathbb{R}}.$$

This completes the proof by (4.1) and Proposition 2.1. □

To prove Theorem 2.2, we need the following well-known result (see [BA], [NS]).

Proposition 4.1. *Let h be a sense-preserving homeomorphism from $\mathbb{R}$ onto itself. Then the pull-back operator P_h defined by $P_h(u) = u \circ h$ is a bounded operator from $H^{\frac{1}{2}}$ into itself if and only if h is quasisymmetric.*

Proof of Theorem 2.2 Let Γ be a locally rectifiable Jordan curve passing through ∞ and $z = z(s)$ be an arc-length parametrization of Γ. If Γ is a Weil-Petersson curve, then it is a chord-arc curve. We conclude by David's result (see [Da]) that there exists a real-valued BMO function $b \in \mathrm{BMO}_{\mathbb{R}}/\mathbb{R}$ such that $z'(s) = e^{ib(s)}$. Now Theorem 4.1 implies that $b \circ h_1^{-1} = \arg z' \circ h_1^{-1} \in H^{\frac{1}{2}}/\mathbb{R}$ and $h_1 \in \mathrm{WP}(\mathbb{R})$, which implies by Proposition 4.1 that $b \in H^{\frac{1}{2}}_{\mathbb{R}}/\mathbb{R}$. Conversely, suppose Γ is a chord-arc curve and there exists some function b in the real Sobolev class $H^{\frac{1}{2}}_{\mathbb{R}}/\mathbb{R}$ such that $z'(s) = e^{ib(s)}$. Then, as stated in section 1, h_1' belongs to the class of weights A^∞ introduced by Muckenhoupt (see [CF], [Gar]). Thus h_1 is strongly quasisymmetric in the sense of Semmes [Se3] (see also [AZ], [SW] and section 6 below) and consequently quasisymmetric. We conclude by Proposition 4.1 again that $\arg z' \circ h_1^{-1} = b \circ h_1^{-1} \in H^{\frac{1}{2}}_{\mathbb{R}}$, which implies that Γ is a Weil-Petersson curve by Theorem 4.1. $\square$

Proof of Theorem 2.1 Theorem 2.2 says that $\hat{\mathcal{T}} = \mathcal{L} \cap H^{\frac{1}{2}}_{\mathbb{R}}/\mathbb{R}$. Since $\mathcal{L}$ is open in $\mathrm{BMO}_{\mathbb{R}}/\mathbb{R}$ (see [Da]), we conclude by a standard discussion that $\hat{\mathcal{T}}$ is open in $H^{\frac{1}{2}}_{\mathbb{R}}/\mathbb{R}$ by the continuity of the inclusion $H^{\frac{1}{2}}_{\mathbb{R}}/\mathbb{R} \hookrightarrow \mathrm{BMO}_{\mathbb{R}}/\mathbb{R}$. $\square$

5 On Weil-petersson quasiconformal mappings

In this section, we give some preliminary results on Weil-Petersson quasiconformal mappings, which will be frequently used in the rest of the paper. They also generalize Proposition 2.1 from the real line case to the setting of Weil-Petersson curves.

Proposition 5.1. *Let F be a Weil-Petersson class quasiconformal mapping on the whole plane with $F(\infty) = \infty$. Then $\Gamma = F(\hat{\mathbb{R}})$ is a Weil-Petersson curve. Furthermore, $h = F|_{\mathbb{R}}$ is locally absolutely continuous such that $\log h' \in H^{\frac{1}{2}}$.*

Proof. Let g be a Riemann mapping which takes the lower half plane $\mathbb{U}^*$ to the right domain Ω^* bounded by Γ. Noting that $g^{-1} \circ F$ is a Weil-Petersson quasiconformal mapping of the lower half plane $\mathbb{U}^*$ onto itself, we conclude by Proposition 2.1 that $\tilde{h} = (g^{-1} \circ F)|_{\mathbb{R}}$ belongs to the Weil-Petersson class $\mathrm{WP}(\mathbb{R})$, which implies by Proposition 2.1 again that the inverse mapping $\tilde{h}^{-1} \in \mathrm{WP}(\mathbb{R})$ can be extended to a Weil-Petersson quasiconformal mapping H of the upper half plane $\mathbb{U}$ onto itself. It is known that H can be so chosen that it is bi-Lipschitz under the Poincaré metric $\lambda_{\mathbb{U}}(z)|dz|$ (see [Cu], [ST]). It is easy to see that $\tilde{g} = F \circ H$ is a Weil-Petersson quasiconformal extension of g to the whole plane, which implies by definition that Γ is a Weil-Petersson curve. Similarly, a Riemann mapping f which takes the upper half plane $\mathbb{U}$ to the left domain Ω bounded by Γ can also be extended to a Weil-Petersson quasiconformal mapping on the whole plane.

Now since Γ is a Weil-Petersson curve, we conclude that $\log g' \in \mathcal{D}(\mathbb{U}^*)$. From $h = F|_{\mathbb{R}} = g \circ \tilde{h}$ we obtain that h is locally absolutely continuous, and

$$\log h' = \log g' \circ \tilde{h} + \log \tilde{h}' = P_{\tilde{h}}(\log g') + \log \tilde{h}',$$

which implies by Propositions 2.1 and 4.1 that $\log h' \in H^{\frac{1}{2}}$ as required. □

The next result gives the converse to Proposition 5.1.

Proposition 5.2. *Let h be a sense-preserving homeomorphism from the real line onto a Weil-Petersson curve Γ such that h is locally absolutely continuous with $\log|h'| \in H^{\frac{1}{2}}$. Then h can be extended to a Weil-Petersson quasiconformal mapping on the whole plane.*

Proof. Let $z = z(s)$ be an arc-length parametrization of Γ so that $|z'| = 1$, and f be a Riemann mapping from the upper half plane $\mathbb{U}$ onto the left domain Ω bounded by Γ so that $\log f' \in \mathcal{D}(\mathbb{U})$. Consider two increasing homeomorphisms h_1 and h_2 of the real line $\mathbb{R}$ onto itself by $z \circ h_1 = f$ and $z \circ h_2 = h$, respectively. Noting that $|z'| = 1$, we obtain $h_1' = |f'|$, $h_2' = |h'|$, which implies that $\log h_1' = \Re \log f' \in H^{\frac{1}{2}}_{\mathbb{R}}$, $\log h_2' = \log|h'| \in H^{\frac{1}{2}}_{\mathbb{R}}$. We conclude by Proposition 2.1 that both h_1 and h_2 are in the Weil-Petersson class $\mathrm{WP}(\mathbb{R})$. Consequently, $h_1^{-1} \circ h_2$ also belongs to the Weil-Petersson class $\mathrm{WP}(\mathbb{R})$ and can be extended to a Weil-Petersson quasiconformal mapping H to the upper half plane $\mathbb{U}$ onto itself. Then $f \circ H$ is a Weil-Petersson quasiconformal extension of $f \circ h_1^{-1} \circ h_2 = h$ from the upper half plane $\mathbb{U}$ onto Ω. By the same way, we can extend h to a Weil-Petersson quasiconformal extension of from the lower half plane $\mathbb{U}^*$ onto the right domain Ω^* bounded by Γ. □

Proposition 5.3. *Let $z = z(s)$ be an arc-length parametrization of a Weil-Petersson curve Γ. Then z can be extended to a Weil-Petersson quasiconformal mapping on the whole plane which is bi-Lipschitz under the Euclidian metric.*

Proof. In Proposition 5.2, replacing h with the arc-length parametrization z, we conclude that z can be extended to a Weil-Petersson quasiconformal mapping on the whole plane. In this case, h_2 is the identity map. Semmes [Se3] showed that the extension mapping $f \circ H$ is actually bi-Lipschitz under the Euclidian metric. □

6 More on the openness of $\hat{\mathcal{T}}$

In section 4, we proved that $\hat{\mathcal{T}}$ is an open subset of $H^{\frac{1}{2}}_{\mathbb{R}}/\mathbb{R}$ by means of the openness of $\mathcal{L}$. However, to prove Theorems 2.3 and 2.4, we need more information about the the openness of $\hat{\mathcal{T}}$ in $H^{\frac{1}{2}}_{\mathbb{R}}/\mathbb{R}$. For any $b \in \hat{\mathcal{T}}$, we will show that there exists some neighbourhood $U(b)$ in $H^{\frac{1}{2}}/\mathbb{C}$ such that for each $u \in U(b)$ the induced mapping γ_u defined by (6.1) below can be extended to a Weil-Petersson quasiconformal on the whole plane whose Beltrami coefficient depends holomorphically on u (see Proposition 6.2 and Lemma 7.2 below), a fact which will play an essential role in the proof of Theorems 2.3 and 2.4. In particular, γ_u maps the real line $\hat{\mathbb{R}}$ onto a Weil-Ptersson curve and $b \in \hat{\mathcal{T}}$ is an interior point.

We first quickly review some results in our paper [ST], where we explored such an approach at the base point $0 \in \hat{\mathcal{T}}$. We begin with a basic result of Coifman-Meyer [CM]. For $u \in \mathrm{BMO}/\mathbb{C}$ on the real line, without loss of generality we assume $\int_0^1 e^{iu(t)}dt > 0$ and set

$$\gamma_u(x) = \int_0^x e^{iu(t)}dt, \quad x \in \mathbb{R}. \tag{6.1}$$

Coifman-Meyer [CM] showed that γ_u is a strongly quasisymmetric homeomorphism from the extended real line $\hat{\mathbb{R}}$ onto a chord-arc curve $\Gamma_u = \gamma_u(\hat{\mathbb{R}})$ when $\|u\|_{\mathrm{BMO}}$ is small. Here a sense preserving homeomorphism h on $\hat{\mathbb{R}}$ is said to be strongly quasisymmetric if it is locally absolutely continuous so that $|h'| \in A^\infty$ and it maps $\hat{\mathbb{R}}$ onto a chord-arc curve passing through the point at infinity (see [Se3]). Later, Semmes [Se3] showed that, when $\|u\|_{\mathrm{BMO}}$ is small, γ_u can be extended a quasiconformal mapping to the whole plane whose Beltrami coefficient satisfies certain Carleson measure condition. To be precise, let φ and ψ be two C^∞ real-valued function on the real line supported on $[-1, 1]$ such that φ is even, ψ is odd and $\int_{\mathbb{R}} \varphi(x)dx = 1$, $\int_{\mathbb{R}} \psi(x)xdx = 1$. Define

$$\rho(x, y) = \rho_u(x, y) = \varphi_y * \gamma_u(x) - i(\mathrm{sgn} y)\psi_y * \gamma_u(x), \quad z = x + iy \in \mathbb{U} \cup \mathbb{U}^*, \tag{6.2}$$

and $\rho(x, 0) = \gamma_u(x)$ for $x \in \mathbb{R}$. Then ρ is a quasiconformal mapping on the whole plane whose Beltrami coefficient satisfies certain Carleson measure condition when $\|u\|_{\mathrm{BMO}}$ is small. We proved in [ST] that ρ_u is in the Weil-Petersson class when $u \in H^{\frac{1}{2}}/\mathbb{C}$ is small.

Proposition 6.1. *There exists some universal constant $\delta > 0$ such that, for any $u \in U(0, \delta) \doteq \{u \in H^{\frac{1}{2}}/\mathbb{C} : \|u\|_{H^{\frac{1}{2}}} < \delta\}$, the mapping $\rho = \rho_u$ defined by (6.2) is a Weil-Petersson quasiconformal on the whole plane whose Beltrami coefficient μ satisfies $\|\mu|_{\mathbb{U}}\|_{\mathrm{WP}} \lesssim \|u\|_{H^{\frac{1}{2}}}$ and $\|\mu|_{\mathbb{U}^*}\|_{\mathrm{WP}} \lesssim \|u\|_{H^{\frac{1}{2}}}$.*

By Proposition 5.1, we conclude that $\Gamma_u = \gamma_u(\hat{\mathbb{R}})$ is a Weil-Petersson curve when $\|u\|_{H^{\frac{1}{2}}}$ is small. Moreover, when $u \in H^{\frac{1}{2}}_{\mathbb{R}}/\mathbb{R}$, γ_u is the normalized arc-parametrization z_u of the normalized Weil-Petersson curve Γ_u. Consequently, when $u \in H^{\frac{1}{2}}_{\mathbb{R}}/\mathbb{R}$ is small, $u \in \hat{\mathcal{T}}$, which implies that 0 is an interior point of $\hat{\mathcal{T}}$.

During the proof of Proposition 6.1, we established the following result, which will be frequently used later.

Lemma 6.1. *Let $\mu \in L^\infty(\mathbb{U})$ and $u \in H^{\frac{1}{2}}$ satisfy the following condition*

$$|\mu(x + iy)|^2 \lesssim \frac{1}{y} \int_{-y}^{y} |u(t + x) - u(x)|^2 dt. \tag{6.3}$$

Then $\mu \in \mathcal{L}^\infty(\mathbb{U})$.

Proof. We reproduce the proof from [ST]. It goes as follows:

$$
\begin{aligned}
\iint_{\mathbb{U}} \frac{|\mu(z)|^2}{y^2} dxdy &\lesssim \iint_{\mathbb{U}} \int_{-y}^{y} \frac{|u(t+x)-u(x)|^2}{y^3} dtdxdy \\
&= \int_{-\infty}^{+\infty} dx \int_0^{+\infty} \frac{dy}{y^3} \int_{-y}^{y} |u(t+x)-u(x)|^2 dt \\
&= \int_{-\infty}^{+\infty} dx \int_0^{+\infty} \frac{dy}{y^3} \int_0^{y} (|u(x+t)-u(x)|^2 + |u(x-t)-u(x)|^2) dt \\
&= \int_{-\infty}^{+\infty} dx \int_0^{+\infty} (|u(x+t)-u(x)|^2 + |u(x-t)-u(x)|^2) dt \int_t^{+\infty} \frac{dy}{y^3} \qquad (6.4) \\
&= \int_{-\infty}^{+\infty} dx \int_0^{+\infty} \frac{|u(x+t)-u(x)|^2 + |u(x-t)-u(x)|^2}{2t^2} dt \\
&= \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} \frac{|u(x+t)-u(x)|^2}{2t^2} dt \asymp \|u\|^2_{H^{\frac{1}{2}}}. \quad \square
\end{aligned}
$$

We also recall the following result from [ST], which will be used in the proof of Theorem 2.3 in the next section.

Proposition 6.2. *For $u \in U(0,\delta)$, let $\Lambda(u)$ denote the Beltrami coefficient on the upper half plane $\mathbb{U}$ for the quasiconformal mapping ρ_u defined by (6.2). Then $\Lambda : U(0,\delta) \to \mathcal{M}(\mathbb{U})$ is holomorphic.*

In the rest of this section, we will extend the above approach to a general point of $\hat{\mathcal{T}}$. Let $b \in \hat{\mathcal{T}}$ be a non-zero element and Γ_b be the normalized Weil-Petersson curve whose normalized arc-length parametrization z_b satisfies $z_b' = e^{ib}$. By Proposition 5.3, there exists a Weil-Petersson quasiconformal mapping τ on the whole plane which is bi-Lipschitz under the Euclidian metric and satisfies $\tau(x) = z_b(x)$ for $x \in \mathbb{R}$. Now for any $u \in H^{\frac{1}{2}}/\mathbb{C}$, by adding some constant we may assume that $\int_0^1 e^{i(b(t)+u(t))} dt > 0$ and set

$$
\omega_u(x) = \int_0^x e^{i(b(t)+u(t))} dt, \quad x \in \mathbb{R}. \tag{6.5}
$$

We will show that $\Gamma_{b+u} = \gamma_{b+u}(\hat{\mathbb{R}}) = \omega_u(\hat{\mathbb{R}})$ is a Weil-Petersson curve when $u \in H^{\frac{1}{2}}/\mathbb{C}$ is small. In particular, this implies that $b+u \in \hat{\mathcal{T}}$ when $u \in H^{\frac{1}{2}}_{\mathbb{R}}/\mathbb{R}$ is small, that is, $b \in \hat{\mathcal{T}}$ is an interior point. By Proposition 5.1, it is sufficient to show that ω_u can be extended to a Weil-Petersson quasiconformal mapping on the whole plane when $u \in H^{\frac{1}{2}}/\mathbb{C}$ is small.

We use Semmes' construction (see [Se3]). Let φ be a C^∞ real-valued even function on the real line supported on $[-1,1]$ such that $\int_{\mathbb{R}} \varphi(x)dx = 1$. On setting $\varphi_y(x) = |y|^{-1}\varphi(|y|^{-1}x)$ for $y \neq 0$, we also want that $|\varphi_y * z_b'| \geq \epsilon_0$ for some $\epsilon_0 > 0$. As pointed out by Semmes [Se3], this can be done as soon as Γ_b is a chord-arc curve, especially a Weil-Petersson curve. As in Lemma 3.1, we consider

$$R_y(w)(x) = \frac{\varphi_y * (z_b' w)(x)}{\varphi_y * z_b'(x)}. \tag{6.6}$$

Now we are ready to define

$$\rho(z) = \rho_u(z) = \varphi_y * \omega_u(x) + R_y(e^u)(x)\{\tau(z) - \varphi_y * \tau(x)\},\ z = x + iy \in \mathbb{U} \cup \mathbb{U}^*, \tag{6.7}$$

and $\rho(x) = \omega_u(x)$ for $x \in \mathbb{R}$. Semmes [Se3] showed that ρ is quasiconformal on the whole plane when $\|u\|_{\mathrm{BMO}}$ is small. We will show that ρ is in the Weil-Petersson class on the whole plane when $\|u\|_{H^{\frac{1}{2}}}$ is small. We only consider the upper half-plane case. The lower half-plane case can be treated similarly.

We proceed to estimate the derivatives of ρ. From (6.7) we have

$$\begin{aligned} \bar\partial\rho(z) =& R_y(e^u)(x)\bar\partial\tau(z) + \bar\partial(R_y(e^u)(x))\{\tau(z) - \varphi_y * \tau(x)\} \\ &+ \bar\partial(\varphi_y * \omega_u(x)) - R_y(e^u)(x)\bar\partial(\varphi_y * \tau(x)). \end{aligned} \tag{6.8}$$

For $x \in \mathbb{R}$ and $y > 0$, consider $I = [x - y, x + y]$ as before so that

$$u_I = \frac{1}{2y}\int_{x-y}^{x+y} u(t)dt.$$

Since $\int_{\mathbb{R}} \varphi(x)dx = 1$, which implies that $\int_{\mathbb{R}} \varphi_y(x)dx = 1$, we obtain

$$|\tau(z) - \varphi_y * \tau(x)| = |\varphi_y * (\tau - \tau(z))(x)| \lesssim \frac{1}{|I|}\int_I |\tau(t) - \tau(z)|dt \lesssim |z - t| \lesssim y, \tag{6.9}$$

since τ is bi-Lipschitz under the Euclidean metric.

Now set

$$\psi(x) = \frac{1}{2}((1 - ix)\phi(x))' = \frac{1}{2}(\phi'(x) - i(\phi(x) + x\phi'(x))).$$

Clearly, ψ is a C^∞ function on the real line which is supported on $[-1, 1]$ and satisfy $\int_{\mathbb{R}} \psi(x)dx = 0$. A direct computation yields that

$$y\bar\partial(\varphi_y * w(x)) = \psi_y * w(x).$$

Since $|\varphi_y * z_b'| \geq \epsilon_0$, we have

$$\begin{aligned} &|y\bar\partial(R_y(e^u)(x))| \\ &= \left|\frac{y\bar\partial(\varphi_y * ({z_b}' e^u)(x)) - yR_y(e^u)(x)\bar\partial(\varphi_y * z_b'(x))}{\varphi_y * z_b'(x)}\right| \\ &= \left|\frac{\psi_y * (z_b' e^u)(x) - R_y(e^u)(x)\psi_y * z_b'(x)}{\varphi_y * z_b'(x)}\right| \\ &\lesssim |\psi_y * (z_b'(e^u - e^{u_I}))(x)| + |\psi_y * z_b'(x)||e^{u_I} - R_y(e^u)(x)|. \end{aligned} \tag{6.10}$$

For the first term in (6.10),

$$\begin{aligned}|\psi_y*(z_b'(e^u-e^{u_I}))(x)| &\lesssim \frac{1}{|I|}\int_I |z_b'(t)(e^{u(t)}-e^{u_I})|dt\\ &\lesssim \frac{1}{|I|}\int_I |e^{u(t)-u_I}-1||e^{u_I}|dt\\ &\lesssim \frac{1}{|I|}\int_I |u(t)-u_I||e^{u(t)}|dt. \end{aligned} \tag{6.11}$$

Then, using Hölder inequality we obtain from Lemma 3.1 and (3.4), (3.5), (3.8) that

$$\begin{aligned}\frac{|\psi_y*(z_b'(e^u-e^{u_I}))(x)|}{|R_y(e^u)(x)|} &\lesssim \frac{1}{|I|}\int_I |u(t)-u_I||e^{u(t)-R_y(u)(x)}|dt\\ &\lesssim \frac{1}{|I|}\int_I |u(t)-u_I|e^{|u(t)-u_I|+|R_y(u)(x)-u_I|}dt\\ &\lesssim \|u\|_{\mathrm{BMO}} \end{aligned} \tag{6.12}$$

when $\|u\|_{\mathrm{BMO}}$ is small.

For the second term in (6.10),

$$|\psi_y*z_b'(x)||e^{u_I}-R_y(e^u)(x)| \lesssim |R_y(e^u-e^{u_I})(x)| \lesssim |\varphi_y*(z_b'(e^u-e^{u_I}))(x)|.$$

Repeat the reasoning in (6.11) and (6.12), we have

$$\frac{|\psi_y*z_b'(x)||e^{u_I}-R_y(e^u)(x)|}{|R_y(e^u)(x)|} \lesssim \|u\|_{\mathrm{BMO}} \tag{6.13}$$

when $\|u\|_{\mathrm{BMO}}$ is small. So by (6.9), (6.10), (6.12) and (6.13) we have

$$|\bar\partial(R_y(e^u)(x))\{\tau(z)-\varphi_y*\tau(x)\}| \lesssim y|\bar\partial(R_y(e^u)(x))| \lesssim \|u\|_{\mathrm{BMO}}|R_y(e^u)(x)| \tag{6.14}$$

when $\|u\|_{\mathrm{BMO}}$ is small.

Next, we will prove $|y\bar\partial(R_y(e^u)(x))||R_y(e^u)(x)|^{-1}$ is square integrable in the Poincaré metric when $\|u\|_{H^{\frac12}}$ and consequently $\|u\|_{\mathrm{BMO}}$ is small. Similar to (6.10),

$$\begin{aligned}&|y\bar\partial(R_y(e^u)(x))|\\ &= \left|\frac{\psi_y*(z_b'e^u)(x)-R_y(e^u)(x)\psi_y*z_b'(x)}{\varphi_y*z_b'(x)}\right|\\ &\lesssim |\psi_y*(z_b'(e^u-e^{u(x)}))(x)| + |\psi_y*z_b'(x)||e^{u(x)}-R_y(e^u)(x)|. \end{aligned} \tag{6.15}$$

For the first part of (6.15), similar to (6.11)

$$|\psi_y*(z_b'(e^u-e^{u(x)}))(x)| \lesssim \frac{1}{|I|}\int_I |u(t)-u(x)||e^{u(t)}|dt. \tag{6.16}$$

Then, by lemma 3.1,

$$\frac{|\psi_y * (z_b'(e^u - e^{u(x)}))(x)|}{|R_y(e^u)(x)|} \lesssim \frac{1}{|I|}\int_I |u(t)-u(x)||e^{u(t)-R_y(u)(x)}|dt. \tag{6.17}$$

By Hölder inequality and (3.9), we conclude

$$\begin{aligned}\frac{|\psi_y * (z_b'(e^u - e^{u(x)}))(x)|^2}{|R_y(e^u)(x)|^2} &\lesssim \frac{1}{|I|^2}\int_I |u(t)-u(x)|^2dt\int_I |e^{u(t)-R_y(u)(x)}|^2dt\\ &\lesssim \frac{1}{|I|}\int_I |u(t)-u(x)|^2dt\\ &\lesssim \frac{1}{y}\int_{-y}^{y} |u(t+x)-u(x)|^2dt. \end{aligned}\tag{6.18}$$

Consequently, by Lemma 6.1 and (6.4),

$$\iint_{\mathbb{U}} \frac{|\psi_y * (z_b'(e^u - e^{u(x)}))(x)|^2}{|R_y(e^u)(x)|^2}\frac{1}{|y|^2}dxdy \lesssim ||u||^2_{H^{\frac{1}{2}}}. \tag{6.19}$$

For the second part of (6.15),

$$|\psi_y * z_b'||e^{u(x)} - R_y(e^u)(x)| \lesssim |R_y(e^u - e^{u(x)})(x)| \lesssim |\varphi_y * (z_b'(e^u - e^{u(x)}))(x)|.$$

Doing the same as (6.16)-(6.19), we can obtain that

$$\iint_{\mathbb{U}} \frac{|\varphi_y * (z_b'(e^u - e^{u(x)}))(x)|^2}{|R_y(e^u)(x)|^2}\frac{1}{|y|^2}dxdy \lesssim \|u\|^2_{H^{\frac{1}{2}}}. \tag{6.20}$$

Therefore, by (6.15), (6.19) and (6.20) we obtain that

$$\iint_{\mathbb{U}} \frac{|\bar\partial(R_y(e^u)(x))|^2}{|R_y(e^u)(x)|^2}dxdy \lesssim \|u\|^2_{H^{\frac{1}{2}}} \tag{6.21}$$

when $\|u\|_{H^{\frac{1}{2}}}$ is small.

Now, we consider the third and forth parts of (6.8). According to Semmes' discussion,

$$\frac{\partial}{\partial x}(\varphi_y * \omega_u(x)) - R_y(e^u)(x)\frac{\partial}{\partial x}(\varphi_y * \tau(x)) = 0, \tag{6.22}$$

while

$$\frac{\partial}{\partial y}(\varphi_y * \omega_u(x)) - R_y(e^u)(x)\frac{\partial}{\partial y}(\varphi_y * \tau(x)) = \alpha_y * (z_b' e^u)(x) - R_y(e^u)(x)\alpha_y * z_b'(x), \tag{6.23}$$

where α is a C^∞ function on the real line which is supported on [-1,1] and satisfies $\int_{\mathbb{R}} \alpha(x)dx = 0$. Noting that

$$|\alpha_y*(z_b'e^u)(x)-R_y(e^u)(x)\alpha_y*z_b'(x)| \leq |\alpha_y*(z_b'(e^u-e^{u_I}))(x)|+|\alpha_y*(z_b'(R_y(e^u)(x)-e^{u_I}))(x)|,$$

we do the same as (6.11), (6.12) and obtain

$$|\alpha_y * (z_b'e^u)(x) - R_y(e^u)(x)\alpha_y * z_b'(x)| \lesssim |R_y(e^u)(x)|\|u\|_{\mathrm{BMO}} \tag{6.24}$$

when $\|u\|_{\mathrm{BMO}}$ is small. Similar to (6.16)-(6.19), we can prove that

$$\iint_{\mathbb{U}} \frac{|\alpha_y * (z_b'e^u)(x) - R_y(e^u)(x)\alpha_y * z_b'(x)|^2}{|R_y(e^u)(x)|^2}\frac{1}{y^2}dxdy \lesssim \|u\|^2_{H^{\frac{1}{2}}} \tag{6.25}$$

when $\|u\|_{H^{\frac{1}{2}}}$ is small.

Summarizing the above, we have

$$|\bar\partial\rho(z) - R_y(e^u)(x)\bar\partial\tau(z)| \lesssim |R_y(e^u)(x)|||u||_{H^{\frac{1}{2}}}, \tag{6.26}$$

and

$$\iint_{\mathbb{U}} \frac{|\bar\partial\rho(z) - R_y(e^u)(x)\bar\partial\tau(z)|^2}{|R_y(e^u)(x)|^2}\frac{1}{y^2}dxdy \lesssim \|u\|_{H^{\frac{1}{2}}} \tag{6.27}$$

if $\|u\|_{H^{\frac{1}{2}}}$ is small enough.

For another derivative of ρ,

$$\begin{aligned}\partial\rho(z) =& R_y(e^u)(x)\partial\tau(z) + \partial(R_y(e^u)(x))\{\tau(z) - \varphi_y * \tau(x)\}\\ &+ \partial(\varphi_y * \omega_u(x)) - R_y(e^u)(x)\partial(\varphi_y * \tau(x)).\end{aligned} \tag{6.28}$$

Similarly, we can prove that

$$|\partial\rho(z) - R_y(e^u)(x)\partial\tau(z)| \lesssim |R_y(e^u)(x)|||u||_{H^{\frac{1}{2}}}, \tag{6.29}$$

and

$$\iint_{\mathbb{U}} \frac{|\partial\rho(z) - R_y(e^u)(x)\partial\tau(z)|^2}{|R_y(e^u)(x)|^2}\frac{1}{y^2}dxdy \lesssim \|u\|_{H^{\frac{1}{2}}} \tag{6.30}$$

if $\|u\|_{H^{\frac{1}{2}}}$ is small enough. Since τ is bi-Lipschitz under the Euclidean metric, which implies that $|\partial\tau| \asymp 1$, we obtain from (6.26-6.30) that

$$\left\|\frac{\bar\partial\rho}{\partial\rho} - \frac{\bar\partial\tau}{\partial\tau}\right\|_\infty \lesssim \|u\|_{H^{\frac{1}{2}}}, \tag{6.31}$$

and

$$\iint_{\mathbb{U}} \left| \frac{\bar{\partial}\rho}{\partial\rho}(z) - \frac{\bar{\partial}\tau}{\partial\tau}(z) \right|^2 \frac{1}{y^2} dxdy \lesssim \|u\|_{H^{\frac{1}{2}}} \tag{6.32}$$

when $\|u\|_{H^{\frac{1}{2}}}$ is small. In particular, ρ is in the Weil-Petersson class on the upper half plane when $\|u\|_{H^{\frac{1}{2}}}$ is small. $\square$

Remark. Let $\hat{\mathcal{T}}_e$ denote the set of all $u \in H^{\frac{1}{2}}/\mathbb{C}$ such that the function γ_u defined by (6.1), that is,

$$\gamma_u(x) = \int_0^x e^{iu(t)} dt, \quad x \in \mathbb{R},$$

is a homeomorphism from the real line $\mathbb{R}$ onto a Weil-Petersson curve. Clearly, the arc-length parametrization space $\hat{\mathcal{T}}$ of the normalized Weil-Petersson curves comprises precisely the real-valued functions in $\hat{\mathcal{T}}_e$, that is, $\hat{\mathcal{T}} = \hat{\mathcal{T}}_e \cap H^{\frac{1}{2}}_{\mathbb{R}}/\mathbb{R}$. Our discussion not only shows that $\hat{\mathcal{T}}$ is an open set in $H^{\frac{1}{2}}_{\mathbb{R}}/\mathbb{R}$, namely, each point $b \in \hat{\mathcal{T}}$ is an interior point of $\hat{\mathcal{T}}$, but also shows that each point $b \in \hat{\mathcal{T}}$ is actually an interior point of $\hat{\mathcal{T}}_e$. Actually, $\hat{\mathcal{T}}_e$ is an open subset of $H^{\frac{1}{2}}/\mathbb{C}$. We will come back this in the last section.

7 Proof of Theorems 2.3 and 2.4

For $b \in \hat{\mathcal{T}}$, we assume without loss of generality that $\int_0^1 e^{ib(t)} dt > 0$, and set as before

$$z_b(s) = \int_0^s e^{ib(t)} dt. \tag{7.1}$$

We also use the notations $\Gamma_b = z_b(\hat{\mathbb{R}})$, f_b, g_b, and h_b introduced in section 2. Now we consider $h_1 = f_b^{-1} \circ z_b$ and $h_2 = g_b^{-1} \circ z_b$. Then we have the following result.

Lemma 7.1. *Both h_1 and h_2 depend on b real-analytically. Precisely, the correspondence $b \mapsto h_1$ induces a real-analytic map from $\hat{\mathcal{T}}$ into the normalized Weil-Petersson class $\mathrm{WP}_0(\mathbb{R})\,(= \mathcal{T})$, and so does the correspondence $b \mapsto h_2$.*

To prove Lemma 7.1, we need

Lemma 7.2. *For each non-zero $b \in \hat{\mathcal{T}}$, let $\delta > 0$ be small enough so that for each $v \in U(b, \delta) \doteq \{v \in H^{\frac{1}{2}}/\mathbb{C} : \|v - b\|_{H^{\frac{1}{2}}} < \delta\}$, the mapping ρ_{v-b} defined by (6.7) is a Weil-Petersson quasiconformal mapping on the whole plane and satisfies the inequalities (6.31) and (6.32). For $v \in U(b, \delta)$, let $\Lambda(v)$ denote the Beltrami coefficient for the quasiconformal mapping ρ_{v-b} on the upper half plane. Then $\Lambda : U(b, \delta) \to \mathcal{M}(\mathbb{U})$ is holomorphic.*

Proof. The proof is almost the same as the one of Proposition 6.2 given in [ST]. By (6.31) and (6.32), Λ is bounded in $U(b, \delta)$. So it is sufficient to show that, for each fixed pair of (u,v) with $u \in U(b,\delta)$, $v \in H^{\frac{1}{2}}/\mathbb{C}$, $\tilde{\Lambda}(t) \doteq \Lambda(u+tv)$ is holomorphic in a small neighbourhood of $t=0$ in the complex plane. To do so, choose

$$0 < \epsilon < \frac{\delta - \|u-b\|_{H^{\frac{1}{2}}}}{2\|v\|_{H^{\frac{1}{2}}}}$$

so that $u+tv \in U(b,\delta)$ when $|t| \le 2\epsilon$. We conclude by (6.8) and (6.28) that $\tilde{\Lambda}(t)(z)$ is holomorphic in $|t| \le 2\epsilon$ for fixed $z \in \mathbb{U}$. For $|t_0| < \epsilon$, $|t| < \epsilon$, Cauchy formula yields that

$$\begin{aligned}\left|\frac{\tilde{\Lambda}(t)(z) - \tilde{\Lambda}(t_0)(z)}{t-t_0} - \frac{d}{dt}|_{t=t_0}\tilde{\Lambda}(t)(z)\right| &= \frac{|t-t_0|}{2\pi}\left|\int_{|\zeta|=2\epsilon}\frac{\tilde{\Lambda}(\zeta)(z)}{(\zeta-t)(\zeta-t_0)^2}d\zeta\right| \\ &\le \frac{|t-t_0|}{2\pi\epsilon^3}\int_{|\zeta|=2\epsilon}|\tilde{\Lambda}(\zeta)(z)||d\zeta|.\end{aligned}$$

Thus, by (6.31),

$$\left\|\frac{\tilde{\Lambda}(t) - \tilde{\Lambda}(t_0)}{t-t_0} - \frac{d}{dt}|_{t=t_0}\tilde{\Lambda}(t)\right\|_\infty \le \frac{|t-t_0|}{2\pi\epsilon^3}\int_{|\zeta|=2\epsilon}\|\tilde{\Lambda}(\zeta)\|_\infty|d\zeta| \le C(u,v)|t-t_0|,$$

and by (6.32),

$$\begin{aligned}&\iint_{\mathbb{U}}\frac{1}{y^2}\left|\frac{\tilde{\Lambda}(t)(z) - \tilde{\Lambda}(t_0)(z)}{t-t_0} - \frac{d}{dt}|_{t=t_0}\tilde{\Lambda}(t)(z)\right|^2 dxdy \\ &\le \frac{|t-t_0|^2}{4\pi^2\epsilon^6}\iint_{\mathbb{U}}\frac{1}{y^2}\left(\int_{|\zeta|=2\epsilon}|\tilde{\Lambda}(\zeta)(z)||d\zeta|\right)^2 dxdy \\ &\le \frac{|t-t_0|^2}{\pi\epsilon^5}\iint_{\mathbb{U}}\int_{|\zeta|=2\epsilon}\frac{|\tilde{\Lambda}(\zeta)(z)|^2}{y^2}|d\zeta|dxdy \\ &= \frac{|t-t_0|^2}{\pi\epsilon^5}\int_{|\zeta|=2\epsilon}\iint_{\mathbb{U}}\frac{|\tilde{\Lambda}(\zeta)(z)|^2}{y^2}dxdy|d\zeta| \\ &\lesssim C(u,v)|t-t_0|^2.\end{aligned}$$

Consequently, the limit

$$\lim_{t\to t_0}\frac{\tilde{\Lambda}(t) - \tilde{\Lambda}(t_0)}{t-t_0} = \frac{d}{dt}|_{t=t_0}\tilde{\Lambda}(t)$$

exists in $\mathcal{M}(\mathbb{U})$ and $\Lambda : U(b,\delta) \to \mathcal{M}(\mathbb{U})$ is holomorphic. $\square$

Proof of Lemma 7.1 For each $b \in \hat{\mathcal{T}}$, we consider the neighborhood $U(b,\delta)$ in Proposition 6.1 or Lemma 7.2. For each $v \in U(b,\delta)$, we denote as above by $\Lambda(v)$ the Beltrami coefficient on the upper half plane $\mathbb{U}$ of the quasiconformal mapping ρ_{v-b} defined by (6.2) or (6.7). Then $\Lambda : U(b,\delta) \to \mathcal{M}(\mathbb{U})$ is holomorphic by Proposition 6.2 or Lemma 7.2, which implies that Λ is real-analytic from $U_{\mathbb{R}}(b,\delta)$, the real-valued functions in $U(b,\delta)$, into $\mathcal{M}(\mathbb{U})$. On the other hand, when $v \in U_{\mathbb{R}}(b,\delta)$, $f_v \circ h_1 = z_v = \omega_{v-b} = \rho_{v-b}|_{\mathbb{R}}$, which implies that $h_1 = (f_v^{-1} \circ \rho_{v-b})|_{\mathbb{R}}$, or equivalently, $h_1 = f^{\Lambda(v)}|_{\mathbb{R}}$. Consequently, the correspondence $b \mapsto h_1$ induces a real-analytic map from $\hat{\mathcal{T}}$ into the normalized Weil-Petersson class $\mathrm{WP}_0(\mathbb{R})\,(=\mathcal{T})$. By the same way, we can prove that the correspondence $b \mapsto h_2$ also induces a real-analytic map from $\hat{\mathcal{T}}$ into the normalized Weil-Petersson class $\mathrm{WP}_0(\mathbb{R})\,(=\mathcal{T})$. □

Proof of Theorems 2.3 and 2.4 For each $b \in \hat{\mathcal{T}}$, we have $h_b = h_1 \circ h_2^{-1}$. Since the Weil-Petersson Teichmüller space $\mathrm{WP}_0(\mathbb{R})$ is a topological group, we conclude by Lemma 7.1 that h_b depends continuously on b, or equivalently, g_b depends continuously on b.

It is easy to see that the correspondence $b \mapsto h_b$ induces a one-to-one map from $\hat{\mathcal{T}}$ onto the normalized Weil-Petersson class $\mathrm{WP}_0(\mathbb{R})\,(=\mathcal{T})$, or equivalently, the correspondence $b \mapsto \log h_b'$ induces a one-to-one map from $\hat{\mathcal{T}}$ onto the real Sobolev space $H_{\mathbb{R}}^{\frac{1}{2}}/\mathbb{R}$. Thus, for each $h \in \mathrm{WP}_0(\mathbb{R})$ there exists unique $b \in \hat{\mathcal{T}}$ such that $h = h_b = f_b^{-1} \circ g_b$. Suppose $h_{b_n} \to h_b$ in $\mathrm{WP}_0(\mathbb{R})$, or equivalently, $\|\log h_{b_n}' - \log h_b'\|_{H^{\frac{1}{2}}} \to 0$. We need to show that $\|b_n - b\|_{H^{\frac{1}{2}}} \to 0$. Writing $h_{b_n} = f_{b_n}^{-1} \circ g_{b_n}$, we have $\|\log g_{b_n}' - \log g_b'\|_{\mathcal{D}(\mathbb{U}^*)} \to 0$. Let z_b and z_{b_n} denote the arc-length parametrization of the normalized Weil-Petersson curve Γ_b and Γ_{b_n}, respectively. Set as above that $h_2 = g_b^{-1} \circ z_b$, and $h_{2n} = g_{b_n}^{-1} \circ z_{b_n}$. Then $(h_2^{-1})' = |g_b'|^{-1}$, and $(h_{2n}^{-1})' = |g_{b_n}'|^{-1}$. Noting that

$$\log(h_{2n}^{-1})' - \log(h_2^{-1})' = -\Re(\log g_{b_n}' - \log g_b'),$$

we conclude that $\|\log(h_{2n}^{-1})' - \log(h_2^{-1})'\|_{H^{\frac{1}{2}}} \to 0$, that is, $h_{2n}^{-1} \to h_2^{-1}$ in $\mathrm{WP}_0(\mathbb{R})$, or equivalently, $h_{2n} \to h_2$ in $\mathrm{WP}_0(\mathbb{R})$. On the other hand, from

$$z_b = g_b \circ h_2 \Rightarrow z_b' = (g_b' \circ h_2)h_2' \Rightarrow ib = \log z_b' = \log(g_b' \circ h_2) + \log h_2'$$

we obtain $b = \Im \log(g_b' \circ h_2)$. Similarly, $b_n = \Im \log(g_{b_n}' \circ h_{2n})$. So we have

$$b_n - b = \Im((\log g_{b_n}') \circ h_{2n} - (\log g_b') \circ h_2).$$

Noting that $\|\log g_{b_n}' - \log g_b'\|_{\mathcal{D}(\mathbb{U}^*)} \to 0$, and $h_{2n} \to h_2$ in $\mathrm{WP}_0(\mathbb{R})$, we conclude by the following result (see Corollary 4.2 in [HWS] and also also Lemma 7.2 in [Sh]) that $\|b_n - b\|_{H^{\frac{1}{2}}} \to 0$ as required.

Proposition 7.3. *Let h_t, $t \in [0, t_0]$, be quasisymmetric homeomorphisms on the real line which keep the points 0 and 1 fixed. Suppose $u_t : [0, t_0] \to H^{\frac{1}{2}}$ and $h_t : [0, t_0] \to T$ are continuous. Then $P_{h_t} u_t : [0, t_0] \to H^{\frac{1}{2}}$ is continuous.*

To complete the proof, we need to show that the correspondence $b \mapsto \log f_b'$ induces a homeomorphism from $\hat{\mathcal{T}}$ onto its image in $\mathcal{D}(\mathbb{U})/\mathbb{C}$. This can be obtained by means of the following facts: Let $J(z) = \bar{z}$ denote the standard reflection with respect to the real line. Then for each $b \in \hat{\mathcal{T}}$, we have $z_{-b} = Jz_b$, $\Gamma_{-b} = J(\Gamma_b)$, $f_{-b} = Jg_bJ$, $g_{-b} = Jf_bJ$, $h_{-b} = h_b^{-1}$. □

8 Generalized Weil-Petersson homeomorphisms

A sense-preserving homeomorphism h on the real line $\mathbb{R}$ is called a generalized Weil-Petersson homeomorphism if h is locally absolutely continuous with $h(\infty) = \infty$ and $\log h' \in H^{\frac{1}{2}}$, or equivalently, $\log |h'| \in H^{\frac{1}{2}}$ by Propositions 5.1 and 5.2, and $h(\hat{\mathbb{R}})$ is a Weil-Petersson curve. By Propositions 5.1 and 5.2 again, a sense-preserving homeomorphism h on the real line $\mathbb{R}$ is a generalized Weil-Petersson homeomorphism if and only if h can be extended to a Weil-Petersson quasiconformal mapping on the whole plane with ∞ fixed. There are several ways to parameterize the class $\mathrm{WP}(\mathbb{C})$ of all generalized Weil-Petersson homeomorphisms on the real line. We denote by $\mathrm{WP}_0(\mathbb{C})$ the subset of all $h \in \mathrm{WP}(\mathbb{C})$ with the normalized conditions (2.9), that is,

$$h(0) = 0,\ h(\infty) = \infty,\ h(1) > 0,\ \int_0^1 |h'(t)|dt = 1. \tag{8.1}$$

We also let $\mathrm{Aff}(\mathbb{C})$ denote the set of all affine mappings $z \mapsto az + b$, $a \neq 0$.

Proposition 8.1. *The mapping Ψ_1 defined by $\Psi_1(h) = \log h'$ is a one-to-one map from $\mathcal{T}_e \doteq \mathrm{WP}(\mathbb{C})/\mathrm{Aff}(\mathbb{C})$ into $H^{\frac{1}{2}}/\mathbb{C}$. The image $\Psi_1(\mathcal{T}_e)$ is an open subset of $H^{\frac{1}{2}}/\mathbb{C}$.*

Proof. Clearly, $\Psi_1(h) = \log h'$ determines a one-to-one map Ψ_1 from $\mathcal{T}_e$ into $H^{\frac{1}{2}}/\mathbb{C}$. We need to show that $\log h_0'$ is an interior point of $\Psi_1(\mathcal{T}_e)$ for each $h_0 \in \mathcal{T}_e$. Let $w \in H^{\frac{1}{2}}/\mathbb{C}$ be given with small norm $\|w\|_{H^{\frac{1}{2}}}$. We need to find $h \in \mathcal{T}_e$ with $\log h' = \log h_0' + w$.

Without loss of generality, we may assume that h_0 satisfies the normalized condition (8.1) so that $h_0 \in \mathrm{WP}_0(\mathbb{C})$. Consider the normalized Weil-Petersson curve $\Gamma_0 = h_0(\hat{\mathbb{R}})$ with the normalized arc-length parametrization $z = z_{\Gamma_0}$. Then there exists $b \in \hat{\mathcal{T}}$ such that $z' = e^{ib}$. Consider the increasing homeomorphism g_0 on the real line determined by $z \circ g_0 = h_0$ and set

$$\tilde{z}(x) = \int_0^x e^{i(b(t) - i(w \circ g_0^{-1})(t))} dt. \tag{8.2}$$

Since $\|w\|_{H^{\frac{1}{2}}}$ is small, and $g_0 \in \mathrm{WP}_0(\mathbb{R})$ since $\log g_0' = \log |h_0'| \in H^{\frac{1}{2}}$, we obtain from Proposition 4.1 that $\|w \circ g_0^{-1}\|_{H^{\frac{1}{2}}}$ is also small. We conclude by the reasoning in section 6 that the equation (8.2) represents a Weil-Petersson curve Γ. Set $h = \tilde{z} \circ g_0$ so that h maps $\hat{\mathbb{R}}$ onto Γ. Then h is locally absolutely continuous with

$$h' = (\tilde{z}' \circ g_0) g_0' = (z' \circ g_0) g_0' e^w = h_0' e^w,$$

which implies that $\log h' = \log h_0' + w$. Consequently, $h \in \mathcal{T}_e$ is the required mapping. □

Proposition 8.2. *There is a one-to-one map from $\mathcal{T}_e$ onto $\mathrm{WP}_0(\mathbb{R}) \times \hat{\mathcal{T}}$.*

Proof. From the proof of Proposition 8.1, each $h \in \mathrm{WP}(\mathbb{C})$ induces a $g \in \mathrm{WP}(\mathbb{R})$ and a $b \in \hat{\mathcal{T}}$ such that $h = z \circ g$ maps $\hat{\mathbb{R}}$ onto a Weil-Petersson curve Γ whose parametrization

$z(s)$ by the arc-length $s \in \mathbb{R}$ satisfies $z'(s) = e^{ib(s)}$. This induces a one-to-one map Ψ_2 from $\mathcal{T}_e$ onto $\mathrm{WP}_0(\mathbb{R}) \times \hat{\mathcal{T}}$ by letting $\Psi_2(h) = (g, b)$. Actually, replacing h by $\tilde{h}$ defined as

$$\tilde{h}(x) = \frac{|h(1) - h(0)|}{h(1) - h(0)} \frac{h(x) - h(0)}{\int_0^1 |h'(t)|dt}$$

if necessary, we may assume that each $h \in \mathcal{T}_e$ satisfies the normalized condition (8.1) so that $h \in \mathrm{WP}_0(\mathbb{C})$. Then the corresponding function $g \in \mathrm{WP}(\mathbb{R})$ satisfies the normalized condition $g(0) = 0$, $g(1)$=1 so that $g \in \mathrm{WP}_0(\mathbb{R})$. □

By means of Propositions 8.1 and 8.2, the generalized Weil-Petersson Teichmüller space $\mathcal{T}_e$ can be endowed with two manifold structures. The following result says that they are topologically equivalent.

Proposition 8.3. *The mapping* $\hat{\Psi} \doteq \Psi_1 \circ \Psi_2^{-1}$ *is a homeomorphism from* $\mathrm{WP}_0(\mathbb{R}) \times \hat{\mathcal{T}}$ *onto its image domain* $\Psi_1(\mathcal{T}_e)$ *in* $H^{\frac{1}{2}}/\mathbb{C}$.

Proof. For $(g, b) \in \mathrm{WP}_0(\mathbb{R}) \times \hat{\mathcal{T}}$, $\hat{\Psi}(g, b) = \log h'$, where $h = z \circ g$ with $z = z_b$ is the normalized arc-length parametrization of the normalized Weil-Petersson curve Γ_b. Thus,

$$\log h' = \log(z' \circ g) + \log g' = ib \circ g + \log g'. \tag{8.3}$$

We conclude by Proposition 7.3 that $\hat{\Psi}$ is continuous from $\mathrm{WP}_0(\mathbb{R}) \times \hat{\mathcal{T}}$ into $H^{\frac{1}{2}}/\mathbb{C}$. Conversely, from (8.3) we obtain $\log g' = \Re \log h'$, and $b = \Im(\log h') \circ g^{-1}$, which implies by Proposition 7.3 again that $\hat{\Psi}^{-1}$ is continuous on $\Psi_1(\mathcal{T}_e)$. □

Remark. The correspondence $u \mapsto -iu$ induces an involution from $\hat{\mathcal{T}}_e$ onto the open subset $\Psi_1(\mathcal{T}_e)$ in $H^{\frac{1}{2}}/\mathbb{C}$, which is contractible by Proposition 8.3. Consequently, $\hat{\mathcal{T}}_e$ is a contractible domain in $H^{\frac{1}{2}}/\mathbb{C}$.

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