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# On the tangent space to the BMO-Teichmüller space ☆

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ABSTRACT

In a previous paper [31] we established the complex analytic manifold theory of the BMO-Teichmüller space. In this paper we identify the function space which is the tangent space to the BMO-Teichmüller space.



## 1. Introduction

We begin with some basic definitions and notations. Let $\Delta = \{z : |z| < 1\}$ denote the unit disk in the extended complex plane $\hat{\mathbb{C}}$. $\Delta^* = \hat{\mathbb{C}} - \overline{\Delta}$ is the exterior of $\Delta$, and $S^1 = \partial\Delta = \partial\Delta^*$ is the unit circle. For any function $f = f(\zeta)$ defined on the unit circle $S^1$, we always denote by $\hat{f}$ the function defined by $\hat{f}(\theta) = f(e^{i\theta})$. $C, C_1, C_2, \cdots$ will denote universal constants that might change from one line to another, while $C(\cdot), C_1(\cdot), C_2(\cdot), \cdots$ will denote constants that depend only on the elements put in the brackets.

The universal Teichmüller space $T$ is a universal parameter space for all Riemann surfaces and one of its models can be defined as the space of all normalized quasisymmetric homeomorphisms on the unit circle, namely, $T = \mathrm{QS}(S^1)/\mathrm{Möb}(S^1)$. Here, $\mathrm{QS}(S^1)$ denotes the group of all quasisymmetric homeomorphisms of the unit circle, and $\mathrm{Möb}(S^1)$ the subgroup of Möbius transformations of the unit disk. Recall that a sense preserving self-homeomorphism $h$ of the unit circle $S^1$ is quasisymmetric if there exists a constant $C(h) > 0$ such that

$$\frac{1}{C(h)} \le \frac{|h(I_1)|}{|h(I_2)|} \le C(h)$$

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for all pairs of adjacent arcs $I_1$ and $I_2$ on $S^1$ with the same arc-length $|I_1| = |I_2|$ ($\leq \pi$). Beurling and Ahlfors [6] proved that a sense preserving self-homeomorphism $h$ is quasisymmetric if and only if there exists some quasiconformal homeomorphism of $\Delta$ onto itself which has boundary values $h$. Later Douady and Earle [9] gave a quasiconformal extension of $h$ to the unit disk which is also conformally invariant.

It is known that the universal Teichmüller space $T$ is an infinite dimensional complex manifold modeled on a Banach space (see [22,23]), and the tangent space to $T$ was identified by Reimann [27] and later by Gardiner and Sullivan [16]. Let $\Lambda$ denote the Zygmund space in the usual sense (see [34]), which consists of all continuous functions $H$ on the real line satisfying the condition

$$\big|H(x+t) - 2H(x) + H(x-t)\big| = O(t) \tag{1.1}$$

for all real number $x$ and $t > 0$. Then the tangent space to $T$ at the identity map is the set of all functions $H$ on the unit circle which satisfy the condition $\hat{H} \in \Lambda$ and the normalized conditions

$$\Re\bar{\zeta}H(\zeta) = 0 \tag{1.2}$$

and

$$H(1) = H(-1) = H(i) = 0. \tag{1.3}$$

It is also known that the universal Teichmüller space plays a significant role in Teichmüller theory, and it is also a fundamental object in mathematics and in mathematical physics. In addition, several subclasses of quasisymmetric homeomorphisms and their Teichmüller spaces were introduced and studied for various purposes in the literature. We refer to the books [2,13–15,20,22,23,25] and the papers [4,7,16,18,31,32] for an introduction to the subject and more details. In this paper, we shall continue to study the BMO-Teichmüller space and VMO-Teichmüller space, which were introduced and investigated in our recent paper [31].

A quasisymmetric homeomorphism $h$ is said to be strongly quasisymmetric if for each $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|E| \leqslant \delta|I| \quad \Rightarrow \quad \big|h(E)\big| \leqslant \varepsilon\big|h(I)\big| \tag{1.4}$$

whenever $I \subset S^1$ is an interval and $E \subset I$ a measurable subset. In other words, $h$ is strongly quasisymmetric if and only if $h$ is absolutely continuous so that $\hat{h}'$ belongs to the class of weights $A^\infty$ introduced by Muckenhoupt (see [17]), in particular, $\log \hat{h}'$ belongs to $\mathrm{BMO}(S^1)$, the space of integrable functions on $S^1$ of bounded mean oscillation (see [12,17,25,33] and also the next section). This sub-class of quasisymmetric homeomorphisms was much investigated because of its great importance in the application to harmonic analysis (see [8,11,21,29]). Let $\mathrm{SQS}(S^1)$ denote the set of all strongly quasisymmetric homeomorphisms of the unit circle. Then $T_b = \mathrm{SQS}(S^1)/\mathrm{M\ddot{o}b}(S^1)$ is a model of the BMO-Teichmüller space. We say a quasisymmetric homeomorphism $h$ is strongly symmetric if it is absolutely continuous such that $\log \hat{h}'$ belongs to $\mathrm{VMO}(S^1)$, the space of integrable functions on $S^1$ of vanishing mean oscillation (see [17,25,28,33] and also the next section). We denote by $\mathrm{SS}(S^1)$ the set of all strongly symmetric homeomorphisms of the unit circle. Then $T_v = \mathrm{SS}(S^1)/\mathrm{M\ddot{o}b}(S^1)$ is a model of the VMO-Teichmüller space.

In our previous paper [31] we showed that both the BMO-Teichmüller space $T_b$ and the VMO-Teichmüller space $T_v$ are complex manifolds modeled on certain Banach spaces. Thus, it is reasonable to identify the function spaces which are the tangent spaces to $T_b$ and $T_v$, respectively. By right translations it is enough to restrict attention to the tangent spaces at the identity map. To state our results, we denote by $\Lambda_b$ the set of all functions $H$ on the unit circle $S^1$ such that $H$ is absolutely continuous with $\hat{H}' \in \mathrm{BMO}(S^1)$, and $\Lambda_v$ the subset of $\Lambda_b$ which consists of those functions $H$ with $\hat{H}' \in \mathrm{VMO}(S^1)$. We shall prove

**Theorem 1.1.** *The tangent space at the identity to the manifold $T_b$ is the function space consisting of all functions $H \in \Lambda_b$ with the normalized conditions (1.2) and (1.3).*

**Theorem 1.2.** *The tangent space at the identity to the manifold $T_v$ is the function space consisting of all functions $H \in \Lambda_v$ with the normalized conditions (1.2) and (1.3).*

## 2. Preliminaries

In this section, we shall review the BMO-Teichmüller theory established in our previous paper [31]. Before this, we recall some basic definitions and results on Carleson measures and BMO-functions. For primary references, see [17,22,23].

### 2.1. Carleson measure and BMO-function

A positive measure $\lambda$ defined in a simply connected domain $\Omega$ is called a Carleson measure if

$$\|\lambda\|_c^2 = \sup\left\{\frac{\lambda(\Omega \cap D(z,r))}{r} : z \in \partial\Omega, 0 < r < diameter(\partial\Omega)\right\} < \infty, \tag{2.1}$$

where $D(z,r)$ is the disk with center $z$ and radius $r$. A Carleson measure $\lambda$ is called a vanishing Carleson measure if

$$\lim_{r\to 0} \frac{\lambda(\Omega \cap D(z,r))}{r} = 0$$

uniformly for $z \in \partial\Omega$. We denote by $\mathrm{CM}(\Omega)$ and $\mathrm{CM}_0(\Omega)$ the sets of all Carleson measures and vanishing Carleson measures on $\Omega$, respectively.

We denote by $\mathrm{BMO}(S^1)$ the space of all integrable functions on $S^1$ of bounded mean oscillation. Namely, $\psi \in L^1(S^1)$ belongs to $\mathrm{BMO}(S^1)$ if

$$\|\psi\|_{\mathrm{BMO}} = \sup_I \frac{1}{|I|} \int_I |\psi - \psi_I| d\theta < \infty, \tag{2.2}$$

where $I$ is any arc on $S^1$, and

$$\psi_I = \frac{1}{|I|} \int_I \psi d\theta$$

is the average of $\psi$ over $I$. We say $\psi \in \mathrm{BMO}(S^1)$ is of vanishing mean oscillation and belongs to the class $\mathrm{VMO}(S^1)$ if

$$\lim_{|I|\to 0} \frac{1}{|I|} \int_I |\psi - \psi_I| d\theta = 0.$$

Denote by $\mathrm{H}^1$ the Hardy space in the usual sense. Namely, $\phi \in \mathrm{H}^1$ if $\phi$ is holomorphic in $\Delta$ with

$$\|\phi\|_{\mathrm{H}^1} = \sup_{0<r<1} \int_0^{2\pi} \left|\phi\left(re^{i\theta}\right)\right| d\theta < \infty. \tag{2.3}$$

We denote by BMOA(Δ) the subspace of $\mathrm{H}^1$ which consists of those $\phi$ such that $\phi|_{S^1} \in \mathrm{BMO}(S^1)$, and by VMOA(Δ) the subspace of $\mathrm{H}^1$ which consists of those $\phi$ such that $\phi|_{S^1} \in \mathrm{VMO}(S^1)$. We say $\phi \in \mathrm{BMOA}(\Delta^*)$ if $\phi(z^{-1}) \in \mathrm{BMOA}(\Delta)$, and $\phi \in \mathrm{VMOA}(\Delta^*)$ if $\phi(z^{-1}) \in \mathrm{VMOA}(\Delta)$. For more information on BMOA functions, we refer to [5,17,33].

### 2.2. Teichmüller spaces

We begin with the standard theory of the universal Teichmüller space (see [2,15,22,23]). Let $M(\Delta)$ denote the open unit ball of the Banach space $L^\infty(\Delta)$ of essentially bounded measurable functions on Δ. For $\mu \in M(\Delta)$, let $f_\mu$ be the quasiconformal mapping of Δ onto itself with complex dilatation equal to $\mu$ and keeping the points 1, −1 and $i$ fixed. We say two elements $\mu$ and $\nu$ in $M(\Delta)$ are equivalent, denoted by $\mu \sim \nu$, if $f_\mu|_{S^1} = f_\nu|_{S^1}$. Then $M(\Delta)/_\sim$ is the Bers model of the universal Teichmüller space $T$. There exists the one to one map $\Psi$ which maps $M(\Delta)/_\sim$ onto $T = \mathrm{QS}(S^1)/\mathrm{Möb}(S^1)$ by sending $[\mu]$ to $f_\mu|_{S^1}$. It is known that $T = \mathrm{QS}(S^1)/\mathrm{Möb}(S^1) = M(\Delta)/_\sim$ carries a natural complex structure so that the natural projection $\Phi$ from $M(\Delta)$ onto $T$ is a holomorphic split submersion.

We proceed to discuss the BMO-Teichmüller spaces. We denote by $\mathcal{L}(\Delta)$ the Banach space of all essentially bounded measurable functions $\mu$ on Δ each of which induces a Carleson measure $\lambda_\mu(z) = |\mu(z)|^2/(1-|z|^2) \in \mathrm{CM}(\Delta)$. The norm on $\mathcal{L}(\Delta)$ is defined as

$$\|\mu\|_c = \|\mu\|_\infty + \|\lambda_\mu\|_c, \tag{2.4}$$

where $\|\lambda_\mu\|_c$ is the Carleson norm of $\lambda_\mu$ defined in (2.1). $\mathcal{L}_0(\Delta)$ is the subspace of $\mathcal{L}(\Delta)$ consisting of all elements $\mu$ such that $\lambda_\mu \in \mathrm{CM}_0(\Delta)$. Similarly, we denote by $\mathcal{L}(\Delta^*)$ the Banach space of all essentially bounded measurable functions $\mu$ on $\Delta^*$ each of which induces a Carleson measure $\lambda_\mu(z) = |\mu(z)|^2/(|z|^2-1) \in \mathrm{CM}(\Delta^*)$, and by $\mathcal{L}_0(\Delta^*)$ the subspace of $\mathcal{L}(\Delta^*)$ consisting of all elements $\mu$ such that $\lambda_\mu \in \mathrm{CM}_0(\Delta^*)$. Set $\mathcal{M}(\Delta) = M(\Delta) \cap \mathcal{L}(\Delta)$, $\mathcal{M}_0(\Delta) = M(\Delta) \cap \mathcal{L}_0(\Delta)$. Then $\mathcal{M}(\Delta)/_\sim$ is a model of the BMOA Teichmüller space $T_b$, while $\mathcal{M}_0(\Delta)/_\sim$ is a model of the VMOA Teichmüller space $T_v$. We proved in [31] that both $T_b = \mathrm{SQS}(S^1)/\mathrm{Möb}(S^1) = \mathcal{M}(\Delta)/_\sim$ and $T_v = \mathrm{SS}(S^1)/\mathrm{Möb}(S^1) = \mathcal{M}_0(\Delta)/_\sim$ have natural complex structures so that the projections $\Phi : \mathcal{M}(\Delta) \to T_b$ and $\Phi : \mathcal{M}_0(\Delta) \to T_v$ are holomorphic split submersions.

## 3. Quasiconformal deformation extensions for functions in $\Lambda_b$ and $\Lambda_v$

Our discussion will be based on the theory of quasiconformal deformations. According to Ahlfors [3], a complex-valued function $F$ defined in a domain $\Omega$ is called a quasiconformal deformation (abbreviated to q.d.) if it has the generalized derivative $\bar\partial F$ such that $\bar\partial F \in L^\infty(\Omega)$. There are several reasons for being interested in quasiconformal deformations; because of their close relation with quasiconformal mappings and Teichmüller spaces (see [1,16,22,23]) and also of their own interests (see [3,26,30]). In particular, the notion of quasiconformal deformations is closely related to that of Zygmund functions. Reich and Chen [26] proved that any function $H$ on $S^1$ with $\hat{H} \in \Lambda$ has a q.d. extension to the unit disk and conversely, any continuous function $H$ on the unit circle which has a q.d. extension to the unit disk must satisfy $\hat{H} \in \Lambda$ if $H$ also satisfies the normalized condition (1.2). Later, the second-named author [30] showed that for a continuous function $H$ on the unit circle, $\hat{H} \in \Lambda$ if and only if $H$ can be extended to a quasiconformal deformation $\widetilde{H}$ of the whole plane $\mathbb{C}$ so that $\widetilde{H}(z) = O(z^2)$ as $z \to \infty$. Furthermore, it was proved that

$$E(H)(z) = \frac{|1-|z|^2|^3}{2\pi i} \int\limits_{S^1} \frac{H(\zeta)}{(1-\bar{z}\zeta)^3(\zeta - z)} d\zeta, \quad z \in \mathbb{C} \setminus S^1 \tag{3.1}$$

is a desired extension of $H$ when $\hat{H} \in \Lambda$. For details, see [26] and [19].

In this section, we are concerned with the q.d. extensions for functions in $\Lambda_b$ and $\Lambda_v$ based on the discussion from our previous paper [30]. For completeness and for the paper to be self-contained, we will repeat some discussion from [30]. We need some basic results.

**Lemma 3.1.** *(See [31].) Let $\alpha > 0$, $\beta > 0$. For a positive measure $\lambda$ on $\Delta$, set*

$$\tilde{\lambda}(z) = \iint\limits_{\Delta} \frac{(1-|z|^2)^{\alpha}(1-|w|^2)^{\beta}}{|1-\bar{z}w|^{\alpha+\beta+2}}\lambda(w)dudv. \tag{3.2}$$

*Then $\tilde{\lambda} \in \mathrm{CM}(\Delta)$ if $\lambda \in \mathrm{CM}(\Delta)$, and $\|\tilde{\lambda}\|_c \le C\|\lambda\|_c$, while $\tilde{\lambda} \in \mathrm{CM}_0(\Delta)$ if $\lambda \in \mathrm{CM}_0(\Delta)$.*

**Lemma 3.2.** *Let $F$ be analytic in $\Delta$. Then $F(z) \in \mathrm{BMOA}(\Delta)$ if and only if $zF(z) \in \mathrm{BMOA}(\Delta)$, and $F(z) \in \mathrm{VMOA}(\Delta)$ if and only if $zF(z) \in \mathrm{VMOA}(\Delta)$.*

**Proof.** Recall that an analytic function $f$ on the unit disk $\Delta$ belongs to $\mathrm{BMOA}(\Delta)$ if and only if there exists some $b \in L^\infty(S^1)$ such that

$$f(z) = \int\limits_{S^1} \frac{b(\zeta)}{\zeta - z}d\zeta.$$

Furthermore, $F \in \mathrm{VMOA}(\Delta)$ if and only if $b$ can be chosen to be continuous (see [5,33]). We simply say that $f$ corresponds to $b$.

Now suppose $F \in \mathrm{BMOA}(\Delta)$ so that there exists some $b \in L^\infty(S^1)$ such that

$$F(z) = \int\limits_{S^1} \frac{b(\zeta)}{\zeta - z}d\zeta.$$

Then,

$$zF(z) = \int\limits_{S^1} \frac{zb(\zeta)}{\zeta - z}d\zeta = \int\limits_{S^1} \frac{\zeta b(\zeta)}{\zeta - z}d\zeta - \int\limits_{S^1} b(\zeta)d\zeta.$$

Thus $zF(z)$ corresponds to $\zeta b(\zeta)$ and belongs to $\mathrm{BMOA}(\Delta)$.

Conversely, if $zF(z) \in \mathrm{BMOA}(\Delta)$ so that there exists some $b \in L^\infty(S^1)$ such that

$$zF(z) = \int\limits_{S^1} \frac{b(\zeta)}{\zeta - z}d\zeta.$$

Noting that $b$ must satisfy the condition

$$\int\limits_{S^1} \frac{b(\zeta)}{\zeta}d\zeta = 0,$$

we conclude that

$$zF(z) = \int\limits_{S^1} \frac{b(\zeta)}{\zeta - z}d\zeta - \int\limits_{S^1} \frac{b(\zeta)}{\zeta}d\zeta = \int\limits_{S^1} \frac{z\bar{\zeta}b(\zeta)}{\zeta - z}d\zeta,$$

which implies that

$$F(z)=\int\limits_{S^1}\frac{\bar{\zeta}b(\zeta)}{\zeta-z}d\zeta.$$

Thus $F(z)$ corresponds to $\bar{\zeta}b(\zeta)$ and belongs to $\mathrm{BMOA}(\Delta)$.

The second statement can be proved by the same way. □

We now prove

**Proposition 3.1.** *Let $F$ be analytic in $\Delta$. Then the following statements are equivalent*:

(1) *$F$ is continuous in $\Delta\cup S^1$ with $F|_{S^1}\in\Lambda_b$;*

(2) *$F'\in\mathrm{BMOA}(\Delta)$;*

(3) *$|F'''(z)|^2(1-|z|^2)^3\in\mathrm{CM}(\Delta)$;*

(4) *$F$ can be extended to a quasiconformal deformation $\widetilde{F}$ to the whole plane so that $\bar{\partial}\widetilde{F}\in\mathcal{L}(\Delta^*)$, and $\widetilde{F}(z)=O(z^2)$ as $z\to\infty$.*

**Proof.** We first point out (see [10]) that for an analytic function $F$ on the unit disk $\Delta$, $F$ is continuous in $\Delta\cup S^1$ such that $F|_{S^1}$ is absolutely continuous in $S^1$ if and only if $F'\in H^1$, and

$$\hat{F}'(\theta)=ie^{i\theta}F'\bigl(e^{i\theta}\bigr). \tag{3.3}$$

Now suppose that $F$ is analytic in $\Delta$ and is continuous in $\Delta\cup S^1$ with $F|_{S^1}\in\Lambda_b$. Then $F'\in\mathrm{H}^1$, which implies that $zF'(z)\in\mathrm{H}^1$. Thus, $zF'(z)\in\mathrm{BMOA}(\Delta)$, which implies by Lemma 3.2 that $F'\in\mathrm{BMOA}(\Delta)$. Conversely, suppose that $F'\in\mathrm{BMOA}(\Delta)$. Then $F$ is continuous in $\Delta\cup S^1$ such that $F|_{S^1}$ is absolutely continuous. On the other hand, Lemma 3.2 implies that $zF'(z)\in\mathrm{BMOA}(\Delta)$, which implies by (3.3) that $\hat{F}'\in\mathrm{BMO}(S^1)$. Thus $F|_{S^1}\in\Lambda_b$. This completes the proof of (1) ⇔ (2).

(2) ⇔ (3) is a well-known result (see [17,33]).

To prove (3) ⇒ (4), we assume that $|F'''(z)|^2(1-|z|^2)^3\in\mathrm{CM}(\Delta)$. We conclude by Lemma 4.1 in [31] that

$$\sup_{z\in\Delta}\bigl|F'''(z)\bigr|\bigl(1-|z|^2\bigr)^2<\infty. \tag{3.4}$$

Set

$$\widetilde{F}(z)=F\bigl(\bar{z}^{-1}\bigr)-\bigl(\bar{z}^{-1}-z\bigr)F'\bigl(\bar{z}^{-1}\bigr)+1/2\bigl(\bar{z}^{-1}-z\bigr)^2F''\bigl(\bar{z}^{-1}\bigr),\quad z\in\Delta^*\setminus\{\infty\}.$$

Clearly, $\widetilde{F}(z)=O(z^2)$ as $z\to\infty$. A direct computation shows that

$$\bar{\partial}\widetilde{F}(z)=-1/2\bigl(\bar{z}^{-1}-z\bigr)^2F'''\bigl(\bar{z}^{-1}\bigr)\bar{z}^{-2},$$

which implies that

$$\bigl|\bar{\partial}\widetilde{F}(z)\bigr|=1/2\bigl|F'''\bigl(\bar{z}^{-1}\bigr)\bigr|\bigl(1-\bigl|z^{-1}\bigr|^2\bigr)^2.$$

We conclude by (3.4) that $\bar{\partial}\widetilde{F}\in L^\infty(\Delta^*)$ so that $\widetilde{F}$ is a quasiconformal deformation.

Set $\lambda(z)=|\bar{\partial}\widetilde{F}(z)|^2/(|z|^2-1)$. Then

$$\lambda\left(\frac{1}{z}\right)\frac{1}{|z|^2}=\frac{1}{4}\bigl|F'''(\bar{z})\bigr|^2\bigl(1-|z|^2\bigr)^3.$$

Thus,

$$\lambda\left(\frac{1}{z}\right)\frac{1}{|z|^2} \in \mathrm{CM}(\Delta),$$

which implies that $\lambda(z) \in \mathrm{CM}(\Delta^*)$ and (4) follows.

(4) ⇒ (3) Let $\widetilde{F}$ be a q.d. extension of $F$ to the whole plane so that $\bar{\partial}\widetilde{F} \in \mathcal{L}(\Delta^*)$, and $\widetilde{F}(z) = O(z^2)$ as $z \to \infty$. Set $\bar{\partial}\widetilde{F} = \mu$. For any $z \in \Delta$, noting that $\widetilde{F}(z) = O(z^2)$ as $z \to \infty$, we conclude by Cauchy and Green formulas that

$$F'''(z) = \frac{3}{\pi i}\int_{S^1} \frac{F(\zeta)}{(\zeta - z)^4}d\zeta = \frac{3}{\pi i}\int_{S^1} \frac{\widetilde{F}(\zeta)}{(\zeta - z)^4}d\zeta = -\frac{6}{\pi}\iint_{\Delta^*} \frac{\mu(\zeta)}{(\zeta - z)^4}d\xi d\eta.$$

Set $\lambda(z) = |\mu(z^{-1})|^2/(1 - |z|^2)$ so that $\lambda \in \mathrm{CM}(\Delta)$. By Hölder's inequality we have

$$\begin{aligned}\left|F'''(z)\right|^2 &\le \frac{36}{\pi^2}\iint_{\Delta^*} \frac{|\mu(\zeta)|^2}{|\zeta - z|^4}d\zeta d\eta \iint_{\Delta^*} \frac{1}{|\zeta - z|^4}d\xi d\eta \\ &= \frac{36}{\pi(1 - |z|^2)^2}\iint_{\Delta} \frac{(1 - |\zeta|^2)}{|1 - \zeta z|^4}\lambda(\zeta)d\xi d\eta.\end{aligned}$$

Therefore,

$$\left|F'''(z)\right|^2\left(1 - |z|^2\right)^3 \le \frac{36}{\pi}\iint_{\Delta} \frac{(1 - |\zeta|^2)(1 - |z|^2)}{|1 - \zeta z|^4}\lambda(\zeta)d\xi d\eta.$$

By means of Lemma 3.1 with $\alpha = \beta = 1$ we conclude that $|F'''(z)|^2(1 - |z|^2)^3 \in \mathrm{CM}(\Delta)$. □

Examining the proof of Proposition 3.1, we have

**Proposition 3.2.** *Let $F$ be analytic in $\Delta$. Then the following statements are equivalent*:

(1) *$F$ is continuous in $\Delta \cup S^1$ with $F|_{S^1} \in \Lambda_v$*;

(2) $F' \in \mathrm{VMOA}(\Delta)$;

(3) $|F'''(z)|^2(1 - |z|^2)^3 \in \mathrm{CM}_0(\Delta)$;

(4) *$F$ can be extended to a quasiconformal deformation $\widetilde{F}$ to the whole plane so that $\bar{\partial}\widetilde{F} \in \mathcal{L}_0(\Delta^*)$, and $\widetilde{F}(z) = O(z^2)$ as $z \to \infty$.*

**Proposition 3.3.** *Let $G$ be analytic in $\Delta^*$. Then the following statements are all equivalent*:

(1) *$G$ is continuous in $\Delta^* \cup S^1$ with $G|_{S^1} \in \Lambda_b$*;

(2) $G' \in \mathrm{BMOA}(\Delta^*)$;

(3) $|G'''(z)|^2(|z|^2 - 1)^3 \in \mathrm{CM}(\Delta^*)$;

(4) *$G$ can be extended to a quasiconformal deformation $\widetilde{G}$ to the whole plane so that $\bar{\partial}\widetilde{G} \in \mathcal{L}(\Delta)$.*

**Proof.** Set

$$F(z) = G\left(z^{-1}\right)z^2, \quad z \in \Delta.$$

Then $G$ is analytic in $\Delta$ with

$$\begin{aligned}F'(z) &= -G'\left(z^{-1}\right), \\ F'''(z) &= -G'''\left(z^{-1}\right)z^{-4}.\end{aligned}$$

It is easy to see that $F$ satisfies one of the conditions in Proposition 3.1 precisely when $G$ satisfies the corresponding condition in Proposition 3.3. Thus, Proposition 3.3 follows directly from Proposition 3.1. □

By the same reasoning as in the proof of Proposition 3.3, we conclude by Proposition 3.2 the following

**Proposition 3.4.** *Let $G$ be analytic in $\Delta^*$. Then the following statements are all equivalent*:

(1) *$G$ is continuous in $\Delta^* \cup S^1$ with $G|_{S^1} \in \Lambda_v$*;
(2) $G' \in \mathrm{VMOA}(\Delta^*)$;
(3) $|G'''(z)|^2(|z|^2-1)^3 \in \mathrm{CM}_0(\Delta^*)$;
(4) *$G$ can be extended to a quasiconformal deformation $\widetilde{G}$ to the whole plane so that $\bar{\partial}\widetilde{G} \in \mathcal{L}_0(\Delta)$.*

We proceed to discuss the q.d. extensions for functions in $\Lambda_b$ and $\Lambda_v$. For a continuous function $H$ on the unit circle, we consider the Cauchy integral

$$C(H)(z) = \frac{1}{2\pi i}\int_{S^1} \frac{H(\zeta)}{\zeta - z} d\zeta, \quad z \in \Delta \cup \Delta^*.$$

More precisely, we always set $F(z) = C(H)(z)$ for $z \in \Delta$, $G(z) = C(H)(z)$ for $z \in \Delta^*$ in the rest of this section. Then $F$ and $G$ are holomorphic in $\Delta$ and $\Delta^*$, respectively. We also let $J$ denote the harmonic conjugation operator in the usual sense (see [10,17]), namely, $J(H)$ is the following Cauchy principle value integral

$$J(H)(z) = -\frac{1}{\pi}\int_{S^1} \frac{H(\zeta)}{\zeta - z} d\zeta, \quad z \in S^1. \tag{3.5}$$

It is well known that $J$ preserves the Zygmund space $\Lambda$ and the BMO spaces $\mathrm{BMO}(S^1)$ and $\mathrm{VMO}(S^1)$ (see [17]). We now show that $J$ also preserves the spaces $\Lambda_b$ and $\Lambda_v$.

**Theorem 3.1.** *The harmonic conjugation operator $J$ keeps the spaces $\Lambda_b$ and $\Lambda_v$ invariant.*

**Proof.** Suppose $H \in \Lambda_b$ so that $H$ is absolutely continuous on $S^1$ with $\hat{H}' \in \mathrm{BMO}(S^1)$. For $F(z) = C(H)(z)$,

$$F'(z) = \frac{1}{2\pi i}\int_{S^1} \frac{H(\zeta)}{(\zeta - z)^2} d\zeta = \frac{1}{2\pi i}\int_{S^1} \frac{H'(\zeta)}{\zeta - z} d\zeta.$$

Then

$$\begin{aligned} zF'(z) &= \frac{1}{4\pi i}\int_{S^1} \frac{\zeta + z}{\zeta - z} H'(\zeta) d\zeta - \frac{1}{4\pi i}\int_{S^1} H'(\zeta) d\zeta \\ &= \frac{1}{4\pi i}\int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \hat{H}'(\theta) d\theta - \frac{1}{4\pi i}\int_0^{2\pi} \hat{H}'(\theta) d\theta. \end{aligned}$$

Thus $zF'(z)$ is analytic in $\Delta$ with boundary values $(\hat{H}' + iJ(\hat{H}'))/2$ up to a constant. Since $\hat{H}' \in \mathrm{BMO}(S^1)$, and $J$ preserves $\mathrm{BMO}(S^1)$, we conclude that $zF'(z) \in \mathrm{BMOA}(\Delta)$, which implies $F' \in \mathrm{BMOA}(\Delta)$ by Lemma 3.2. Thus, $F$ is continuous in $\Delta \cup S^1$ and absolutely continuous in $S^1$. Noting that $F = (H + iJ(H))/2$ on $S^1$, we conclude that $J(H)$ is absolutely continuous in $S^1$, and $\hat{H}'(\theta) + i\widehat{J(H)}'(\theta) = 2\hat{F}'(\theta) = 2ie^{i\theta}F'(e^{i\theta}) \in \mathrm{BMO}(S^1)$, which implies that $\widehat{J(H)}' \in \mathrm{BMO}(S^1)$. Thus, $J(H) \in \Lambda_b$ as required.

Replacing $\mathrm{BMO}(S^1)$ by $\mathrm{VMO}(S^1)$ in the proof, we find out that $J$ also keeps $\Lambda_v$ invariant. □

Now we can prove the main results in this section.

**Theorem 3.2.** *Let $H$ be continuous on the unit circle. Then the following statements are equivalent*:

(1) $H \in \Lambda_b$;

(2) $|F'''(z)|^2(1-|z|^2)^3 \in \mathrm{CM}(\Delta)$, *and* $|G'''(z)|^2(|z|^2-1)^3 \in \mathrm{CM}(\Delta^*)$;

(3) *$F$ and $G$ have q.d. extensions $\widetilde{F}$ and $\widetilde{G}$ respectively to the whole plane so that* $\bar\partial\widetilde{F} \in \mathcal{L}(\Delta^*)$, $\bar\partial\widetilde{G} \in \mathcal{L}(\Delta)$, *and* $\widetilde{F}(z) = O(z^2)$ *as* $z \to \infty$;

(4) *$H$ can be extended to a quasiconformal deformation $\widetilde{H}$ to the whole plane so that* $\bar\partial\widetilde{H}|_\Delta \in \mathcal{L}(\Delta)$, $\bar\partial\widetilde{H}|_{\Delta^*} \in \mathcal{L}(\Delta^*)$, *and* $\widetilde{H}(z) = O(z^2)$ *as* $z \to \infty$.

**Proof.** Noting that $F = (H + iJ(H))/2$, $G = (-H + iJ(H))/2$ on $S^1$, we conclude that (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) by means of Theorem 3.1 and Propositions 3.1 and 3.3.

(3) $\Rightarrow$ (4) Suppose $F$ and $G$ have q.d. extensions $\widetilde{F}$ and $\widetilde{G}$ respectively to the whole plane so that $\bar\partial\widetilde{F} \in \mathcal{L}(\Delta^*)$, $\bar\partial\widetilde{G} \in \mathcal{L}(\Delta)$, and $\widetilde{F}(z) = O(z^2)$ as $z \to \infty$. Define $\widetilde{H}$ by $\widetilde{H}(z) = F(z) - \widetilde{G}(z)$ on $\Delta \cup S^1$, and $\widetilde{H}(z) = \widetilde{F}(z) - G(z)$ on $\Delta^* \cup S^1 \setminus \{\infty\}$. Then $\widetilde{H}$ is the desired q.d. extension of $H$ to the whole plane.

(4) $\Rightarrow$ (3) Suppose $H$ can be extended to a quasiconformal deformation $\widetilde{H}$ to the whole plane so that $\bar\partial\widetilde{H}|_\Delta \in \mathcal{L}(\Delta)$, $\bar\partial\widetilde{H}|_{\Delta^*} \in \mathcal{L}(\Delta^*)$, and $\widetilde{H}(z) = O(z^2)$ as $z \to \infty$. Denote $\bar\partial\widetilde{H} = \mu$. Set

$$\widetilde{G}(z) = \frac{1}{\pi}\iint_\Delta \frac{\mu(\zeta)}{\zeta - z}\,d\xi d\eta,$$

and $\widetilde{F}(z) = \widetilde{H}(z) + \widetilde{G}(z)$. Clearly, both $\widetilde{F}$ and $\widetilde{G}$ are quasiconformal deformations on the whole plane, $\bar\partial\widetilde{F}|_{\Delta^*} \in \mathcal{L}(\Delta^*)$, $\bar\partial\widetilde{G}|_\Delta \in \mathcal{L}(\Delta)$, and $\widetilde{F}(z) = O(z^2)$ as $z \to \infty$. For any $z \in \Delta^*$, Green's formula implies that

$$\widetilde{G}(z) = \frac{1}{\pi}\iint_\Delta \frac{\mu(\zeta)}{\zeta - z}\,d\xi d\eta = \frac{1}{2\pi i}\int_{S^1} \frac{H(\zeta)d\zeta}{\zeta - z} = G(z).$$

Thus $\widetilde{G}$ is a q.d. extension of $G$. On the other hand, when $z \in \Delta$, Pompeiu's formula implies that

$$\widetilde{F}(z) = \widetilde{H}(z) + \widetilde{G}(z) = \widetilde{H}(z) + \frac{1}{\pi}\iint_\Delta \frac{\mu(\zeta)}{\zeta - z}\,d\xi d\eta = \frac{1}{2\pi i}\int_{S^1} \frac{H(\zeta)d\zeta}{\zeta - z} = F(z).$$

Thus $\widetilde{F}$ is a q.d. extension of $F$.

Examining the proof of Theorem 3.2, we have

**Theorem 3.3.** *Let $H$ be continuous on the unit circle. Then the following statements are equivalent*:

(1) $H \in \Lambda_v$;

(2) $|F'''(z)|^2(1-|z|^2)^3 \in \mathrm{CM}_0(\Delta)$, *and* $|G'''(z)|^2(|z|^2-1)^3 \in \mathrm{CM}_0(\Delta^*)$;

(3) *$F$ and $G$ have q.d. extensions $\widetilde{F}$ and $\widetilde{G}$ respectively to the whole plane so that* $\bar\partial\widetilde{F} \in \mathcal{L}_0(\Delta^*)$, $\bar\partial\widetilde{G} \in \mathcal{L}_0(\Delta)$, *and* $\widetilde{F}(z) = O(z^2)$ *as* $z \to \infty$;

(4) *$H$ can be extended to a quasiconformal deformation $\widetilde{H}$ to the whole plane so that* $\bar\partial\widetilde{H}|_\Delta \in \mathcal{L}_0(\Delta)$, $\bar\partial\widetilde{H}|_{\Delta^*} \in \mathcal{L}_0(\Delta^*)$, *and* $\widetilde{H}(z) = O(z^2)$ *as* $z \to \infty$.

**Remark.** Recall that

$$E(H)(z) = \frac{|1-|z|^2|^3}{2\pi i}\int_{S^1} \frac{H(\zeta)}{(1-\bar z\zeta)^3(\zeta - z)}\,d\zeta, \quad z \in \mathbb{C} \setminus S^1,$$

defines a q.d. extension of a continuous function $H$ on $S^1$ with $\hat{H} \in \Lambda$. When $H \in \Lambda_b$, $E(H)$ is an extension of $H$ satisfying condition (4) in Theorem 3.2. In fact, by differentiating (3.1) with respect to $\bar{z}$, we obtain

$$\bar{\partial} E(H)(z) = \frac{3}{2\pi i}\big(\chi_\Delta(z) - \chi_{\Delta^*}(z)\big)\big(1-|z|^2\big)^2 \int\limits_{S^1} \frac{H(\zeta)}{(1-\bar{z}\zeta)^4} d\zeta, \quad z \in \Delta \cup \Delta^* \setminus \{\infty\},$$

where $\chi$ denotes the characteristic function of a set. Since $H \in \Lambda_b$, it follows from Theorem 3.2 that $H$ can be extended to a quasiconformal deformation $\widetilde{H}$ to the whole plane so that $\bar{\partial}\widetilde{H}|_\Delta \in \mathcal{L}(\Delta)$, $\bar{\partial}\widetilde{H}|_{\Delta^*} \in \mathcal{L}(\Delta^*)$, and $\widetilde{H}(z) = O(z^2)$ as $z \to \infty$. Thus, by Green's formula and Hölder's inequality, we obtain

$$\frac{|\bar{\partial} E(H)(z)|^2}{1-|z|^2} \leqslant \frac{9}{\pi} \iint\limits_{\Delta} \frac{|\bar{\partial}\widetilde{H}(w)|^2}{1-|w|^2} \frac{(1-|z|^2)(1-|w|^2)}{|1-\bar{z}w|^4} dudv, \quad z \in \Delta.$$

Consequently, Lemma 3.1 (with $\alpha = \beta = 1$) implies $\bar{\partial} E(H)|_\Delta \in \mathcal{L}(\Delta)$. In the same way, $\bar{\partial} E(H)|_{\Delta^*} \in \mathcal{L}(\Delta^*)$. Similarly, when $H \in \Lambda_v$, $E(H)$ is an extension of $H$ satisfying condition (4) in Theorem 3.3.

When $H$ satisfies the normalized condition (1.2), we can obtain some stronger results, which will be used to prove Theorems 1.1 and 1.2 in the next section.

**Theorem 3.4.** *Let $H$ be continuous on the unit circle which satisfies the normalized condition (1.2). Then the following statements are equivalent:*

(1) *$H \in \Lambda_b$;*

(2) *$H$ can be extended to a quasiconformal deformation $\widetilde{H}$ to the whole plane so that $\bar{\partial}\widetilde{H}|_\Delta \in \mathcal{L}(\Delta)$, $\bar{\partial}\widetilde{H}|_{\Delta^*} \in \mathcal{L}(\Delta^*)$, and $\widetilde{H}(z) = O(z^2)$ as $z \to \infty$;*

(3) *$H$ can be extended to a quasiconformal deformation $H_1$ to $\Delta$ so that $\bar{\partial} H_1 \in \mathcal{L}(\Delta)$;*

(4) *$H$ can be extended to a quasiconformal deformation $H_2$ to $\Delta^* \setminus \{\infty\}$ so that $\bar{\partial} H_2 \in \mathcal{L}(\Delta^*)$ and $H_2(z) = O(z^2)$ as $z \to \infty$.*

**Proof.** It follows from Theorem 3.2 that (1) $\Leftrightarrow$ (2). It is obvious that (2) implies both (3) and (4), and vice versa. So we only need to show that (3) $\Leftrightarrow$ (4). Let $H_1$ be a q.d. extension of $H$ to the unit disk so that $\bar{\partial} H_1 \in \mathcal{L}(\Delta)$. Set $H_2(z) = -z^2\overline{H_1(\bar{z}^{-1})}$. Noting that $|\bar{\partial} H_2(z)|^2/(|z|^2-1) = |\bar{\partial} H_1(\bar{z}^{-1})|^2/(|z|^2-1)$, we conclude that $H_2$ is a quasiconformal deformation in $\Delta^* \setminus \{\infty\}$ with $H_2(z) = O(z^2)$ as $z \to \infty$, and $\bar{\partial} H_2 \in \mathcal{L}(\Delta^*)$. Since $H(\zeta) = -\zeta^2\overline{H(\zeta)}$ by means of the assumption $\Re\bar{\zeta}H(\zeta) = 0$ on $S^1$, $H_2 = H$ on the unit circle. Thus (3) $\Rightarrow$ (4) follows. By the same reasoning, we obtain (4) $\Rightarrow$ (3). $\square$

Similarly, by Theorem 3.3 we obtain

**Theorem 3.5.** *Let $H$ be continuous on the unit circle which satisfies the normalized condition (1.2). Then the following statements are equivalent:*

(1) *$H \in \Lambda_v$;*

(2) *$H$ can be extended to a quasiconformal deformation $\widetilde{H}$ to the whole plane so that $\bar{\partial}\widetilde{H}|_\Delta \in \mathcal{L}_0(\Delta)$, $\bar{\partial}\widetilde{H}|_{\Delta^*} \in \mathcal{L}_0(\Delta^*)$, and $\widetilde{H}(z) = O(z^2)$ as $z \to \infty$;*

(3) *$H$ can be extended to a quasiconformal deformation $H_1$ to $\Delta$ so that $\bar{\partial} H_1 \in \mathcal{L}_0(\Delta)$;*

(4) *$H$ can be extended to a quasiconformal deformation $H_2$ to $\Delta^* \setminus \{\infty\}$ so that $\bar{\partial} H_2 \in \mathcal{L}_0(\Delta^*)$ and $H_2(z) = O(z^2)$ as $z \to \infty$.*

## 4. The proofs of Theorems 1.1 and 1.2

**Proof of Theorem 1.1.** Suppose we are given a curve of strongly quasisymmetric mappings $h^t(\zeta)$ ($t > 0$ is small) normalized to fix $\pm 1$ and $i$, which is the identity for $t = 0$ and differentiable with respect to $t$ for the manifold structure on $T_b$. Denote

$$h^t(\zeta) = \zeta + tH(\zeta) + o(t), \quad t \to 0.$$

Since the natural projection $\Phi : \mathcal{M}(\Delta) \to T_b$ is a holomorphic split submersion, we conclude that there is a differentiable curve of Beltrami coefficients $\nu_t \in \mathcal{M}(\Delta)$ such that $h^t$ is the restriction to the unit circle of the normalized quasiconformal mapping $f_{\nu_t}$. Now there exists some $\mu \in \mathcal{L}(\Delta)$ such that

$$\nu_t = t\mu + o(t).$$

Consequently,

$$f_{\nu_t}(z) = z + t\dot{f}[\mu](z) + o(t), \quad t \to 0.$$

Here $\dot{f}[\mu]$ satisfies the normalized conditions (1.2) and (1.3) and is uniquely determined by the condition $\bar{\partial}\dot{f}[\mu] = \mu$ (see [1,23,24]). Noting that $H = \dot{f}[\mu]|_{S^1}$, we conclude by Theorem 3.4 that $H \in \Lambda_b$ and satisfies the normalized conditions (1.2) and (1.3).

Conversely, suppose we are given a function $H \in \Lambda_b$ satisfying the normalized conditions (1.2) and (1.3). By Theorem 3.4, we deduce that $H$ can be extended to the unit disk to a quasiconformal deformation $\widetilde{H}$ with $\bar{\partial}$-derivative $\mu = \bar{\partial}\widetilde{H} \in \mathcal{L}(\Delta)$. Set $\mu_t = t\mu$ for small $t > 0$. Then

$$f_{\mu_t}(z) = z + t\dot{f}[\mu](z) + o(t), \quad t \to 0.$$

Noting that both $\dot{f}[\mu]$ and $\widetilde{H}$ satisfy the normalized conditions (1.2) and (1.3) and have the same $\bar{\partial}$-derivative $\mu$, we conclude that $\dot{f}[\mu] = \widetilde{H}$. Then,

$$f_{\mu_t}(z) = z + t\widetilde{H}(z) + o(t), \quad t \to 0.$$

Set $h^t = f_{\mu_t}|_{S^1}$. Then it holds that

$$h^t(\zeta) = \zeta + tH(\zeta) + o(t), \quad t \to 0,$$

which implies that $h^t$ is a differentiable curve in $T_b$ with the tangent vector $H$. □

**Proof of Theorem 1.2.** Replace Theorem 3.4 by Theorem 3.5 in the proof of Theorem 1.1. □

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