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# Universal Teichmüller space and BMO[☆]

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**Abstract**

We first give some new characterizations on BMOA–Teichmüller space and various characterizations on VMOA–Teichmüller space as well. In particular, we prove that a quasisymmetric conformal welding $h$ corresponds to an asymptotically smooth curve in the sense of Pommerenke (1978) [32] precisely when $h$ is absolutely continuous with $\log h' \in$ VMO. We then show that these BMO–Teichmüller spaces have natural complex structures.


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*Keywords:* Universal Teichmüller space; BMOA; VMOA; Carleson measure

## 1. Introduction

Let $\Delta = \{z : |z| < 1\}$ denote the unit disk in the extended complex plane $\hat{\mathbb{C}}$. $\Delta^* = \hat{\mathbb{C}} - \overline{\Delta}$ is the exterior of $\Delta$, and $S^1 = \partial\Delta = \partial\Delta^*$ is the unit circle. $C, C_1, C_2 \cdots$ will denote universal constants that might change from one line to another, while $C(\cdot), C_1(\cdot), C_2(\cdot) \cdots$ will denote constants that depend only on the elements put in the brackets.

The Bers model $T$ of the universal Teichmüller space can be characterized in the following ways (see [1,29,30]).

($T_1$) $T$ is the set of all conformal mappings (up to a Möbius transformation) on $\Delta$ which can be extended to a quasiconformal mapping in the whole complex plane $\mathbb{C}$.

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($T_2$) $T$ is the set of all quasicircles (up to a Möbius transformation) in the extended complex plane $\hat{\mathbb{C}}$. Here a closed Jordan curve $\Gamma$ in the extended complex plane $\hat{\mathbb{C}}$ is a quasicircle if there exists a constant $C(\Gamma) > 0$ such that the diameter $(\widetilde{\zeta z}) \le C(\Gamma)|\zeta - z|$ for the smaller subarc $\widetilde{\zeta z}$ of $\Gamma$ joining any two finite points $z$ and $\zeta$ of $\Gamma$. Let $f$ be a conformal mapping on the unit disk $\Delta$. Then $f$ satisfies condition $T_1$ if and only if $\Gamma = f(S^1)$ is a quasicircle.

($T_3$) $T$ is the set of all quasisymmetric homeomorphisms (up to a Möbius transformation of $\Delta$) of the unit circle $S^1$. Here a sense preserving self-homeomorphism $h$ of the unit circle $S^1$ is quasisymmetric if there exists a constant $C(h) > 0$ such that $|h(I^*)| \le C(h)|h(I)|$ for any interval $I \subset S^1$ with $|I| \le \pi$, where $I^*$ is the interval with same center as $I$ but with double length and $|\cdot|$ denotes the Lebesgue measure. Let $f$ be a conformal mapping on the unit disk $\Delta$, and $g$ be a conformal mapping from $\Delta^*$ onto $\hat{\mathbb{C}} - \overline{f(\Delta)}$. Then $f$ satisfies condition $T_1$ if and only if $h = f^{-1} \circ g$ is quasisymmetric. $h$ is called the conformal welding corresponding to $f$.

It is well known that a quasisymmetric homeomorphism need not be absolutely continuous, and a quasicircle need not be locally rectifiable (see [10,11,37]). An important problem of long time has been to determine when a quasisymmetric homeomorphism is absolutely continuous, when a quasicircle is locally rectifiable, and much work has been done in this direction (see [2,5, 12,13,17,20,22,38]). Carleson [13] initiated such an investigation, giving a sufficient condition on the dilatation of a quasiconformal self-mapping of the unit disk to have an absolutely continuous boundary value. A much satisfactory answer is given by the following theorem. Recall that a positive measure $\lambda$ defined in a simply connected domain $\Omega$ is called a Carleson measure if

$$\|\lambda\|_c^2 = \sup\left\{\frac{\lambda(\Omega \cap D(z,r))}{r} : z \in \partial\Omega, 0 < r < \text{diameter}(\partial\Omega)\right\} < \infty, \tag{1.1}$$

where $D(z,r)$ is the disk with center $z$ and radius $r$. A Carleson measure $\lambda$ is called a vanishing Carleson measure if $\lim_{r\to 0} \lambda(\Omega \cap D(z,r))/r = 0$ uniformly for $z \in \partial\Omega$. We denote by $CM(\Omega)$ and $CM_0(\Omega)$ the set of all Carleson measures and vanishing Carleson measures on $\Omega$, respectively.

**Theorem A.** *Let $f$ be a conformal mapping on $\Delta$ and $h = f^{-1} \circ g$ be the corresponding quasisymmetric conformal welding. Then the following statements are equivalent.*

($B_1$) *$f$ can be extended to a quasiconformal mapping in the whole plane whose complex dilatation $\mu$ induces a Carleson measure $|\mu(z)|^2/(|z|^2 - 1) \in CM(\Delta^*)$.*

($B_2$) $\log f'$ *belongs to* $\mathrm{BMOA}(\Delta)$*, the space of analytic functions in $\Delta$ of bounded mean oscillation (see* [23,24,33,43]*), or equivalently, $N_f = f''/f'$ induces a Carleson measure $|N_f(z)|^2(1 - |z|^2) \in CM(\Delta)$.*

($B_3$) *$S_f = N_f' - N_f^2/2$ induces a Carleson measure $|S_f(z)|^2(1 - |z|^2)^3 \in CM(\Delta)$.*

($B_4$) *$\Gamma = f(S^1)$ is a quasicircle satisfying the Bishop–Jones condition (see* [12]*).*

($B_5$) *$h$ is strongly quasisymmetric, namely, for each $\epsilon > 0$ there is a $\delta > 0$ such that*

$$|E| \le \delta|I| \Rightarrow |h(E)| \le \epsilon|h(I)|$$

*whenever $I \subset S^1$ is an interval and $E \subset I$ a measurable subset.*

$B_5 \Rightarrow B_1$ was proved by Fefferman–Kenig–Pipher [22] by means of a Beurling–Ahlfors [10] type extension. Conversely, $B_1 \Rightarrow B_5$ is implied by a theorem in [22], and was reproved by Astala–Zinsmeister [5] and Bishop–Jones [12]. ($B_1 \Rightarrow B_3 \Rightarrow B_2$ and $B_4 \Rightarrow B_5$ were proved by Astala–Zinsmeister [5], while $B_2 \Leftrightarrow B_4$ was proved by Bishop–Jones [12], who also gave a new proof of $B_3 \Rightarrow B_2$. A direct proof of $B_1 \Rightarrow B_2$ without using the Schwarzian derivative was given later by Dynkin [20].) This was proved formerly by Semmes [38] when the Carleson norm of $|\mu(z)|^2/(1-|z|^2)$ is small. Recently, Cui–Zinsmeister [16] have proved that the Douady–Earle [19] extension of a strongly quasisymmetric homeomorphism also satisfies $B_1$ using $B_5 \Rightarrow B_1$.

In this paper, we shall continue to study the BMO theory of the universal Teichmüller space, because of its great importance in the application to harmonic analysis (see [17,22,27,38]) and also of its own interest. We first give two more characterizations of this BMOA–Teichmüller space (see Theorem 3.1) and various characterizations of the VMOA–Teichmüller space (see Theorem 4.1), the set of all conformal mappings $f$ on $\Delta$ which can be extended to a quasiconformal mapping in the whole complex plane $\mathbb{C}$ such that $\log f'$ belongs to VMOA($\Delta$), the space of analytic functions in $\Delta$ of vanishing mean oscillation (see [24,33,43]). Then we show that these BMO–Teichmüller spaces have natural complex structures. As an application, we prove that $h$ is the quasisymmetric conformal welding corresponding to $f$ with $\log f' \in$ VMOA($\Delta$) precisely when $h$ is absolutely continuous with $\log h'$ belongs to VMO($S^1$), the space of integrable functions on $S^1$ of vanishing mean oscillation (see [24,33,35,43]). We hope that this complex analytic theory could find applications to some other problems in complex analysis, and also real analysis.

## 2. Kernel functions and corresponding operators

In this section, as a sequel to [26], we shall continue to discuss some kernel functions induced by a quasisymmetric homeomorphism or by the corresponding conformal mapping. These kernel functions will be used to give some new characterizations for a strongly quasisymmetric homeomorphism. The results in this section also have independent interests of their own.

First we recall the Hilbert space $\mathcal{A}^2$ of all holomorphic functions $\phi$ in $\Delta$ with the inner product and norm

$$\langle \phi, \psi \rangle = \frac{1}{\pi} \iint_{\Delta} \phi(z)\overline{\psi(z)}dxdy, \quad \|\phi\| = \left( \frac{1}{\pi} \iint_{\Delta} |\phi(z)|^2 dxdy \right)^{\frac{1}{2}} < \infty. \tag{2.1}$$

Then, $J\phi(z) = \overline{\phi(\bar{z})}$ determines an isometric isomorphism of $\mathcal{A}^2$ onto itself.

### 2.1. Kernel functions induced by a quasisymmetric homeomorphism

For a quasisymmetric homeomorphism $h$, two kernel functions were introduced in a previous paper [26] by Hu and the first author. They are

$$\phi_h(\zeta, z) = \frac{1}{2\pi i} \int_{S^1} \frac{h(w)}{(1-\zeta w)^2(1-zh(w))} dw, \quad (\zeta, z) \in \Delta \times \Delta, \tag{2.2}$$

$$\psi_h(\zeta, z) = \frac{1}{2\pi i} \int_{S^1} \frac{h(w)}{(\zeta - w)^2(1-zh(w))} dw, \quad (\zeta, z) \in \Delta \times \Delta. \tag{2.3}$$

Clearly, both $\phi_h$ and $\psi_h$ are holomorphic functions. Note that the function $\phi_h$ already appeared in Cui [15]. It was used in [26] to characterize when a quasisymmetric homeomorphism is symmetric or even belongs to the Weil–Petersson class. Here it will be used to study the BMO theory of the universal Teichmüller space.

In [26] we introduced two operators on $\mathcal{A}^2$ induced by $\phi_h$ and $\psi_h$, respectively. They are

$$T_h^- \psi(\zeta) = \frac{1}{\pi} \iint_\Delta \phi_h(\zeta, \bar{z}) \psi(z) dxdy, \quad \psi \in \mathcal{A}^2, \zeta \in \Delta, \tag{2.4}$$

and

$$T_h^+ \psi(\zeta) = \frac{1}{\pi} \iint_\Delta \psi_h(\zeta, \bar{z}) \psi(z) dxdy, \quad \psi \in \mathcal{A}^2, \zeta \in \Delta. \tag{2.5}$$

It was proved in [26] that both $T_h^-$ and $T_h^+$ are bounded operators from $\mathcal{A}^2$ into itself. In fact, $T_h^+$ is an isomorphism from $\mathcal{A}^2$ onto itself (see Lemma 2.3).

Now, as in [26], for each $z \in \Delta$, we define

$$\phi_h(z) = \left( \frac{1}{\pi} \iint_\Delta |\phi_h(\zeta, z)|^2 d\xi d\eta \right)^{\frac{1}{2}}. \tag{2.6}$$

Then we have the following results (for details, see [26]).

**Lemma 2.1.** *Let $\nu$ be the Beltrami coefficient of a quasiconformal extension of $h^{-1}$ to the unit disk. Then it holds that*

$$\phi_h^2(z) \le \frac{1}{\pi} \iint_\Delta \frac{|\nu(w)|^2}{1 - |\nu(w)|^2} \frac{1}{|1 - zw|^4} dudv. \tag{2.7}$$

**Lemma 2.2.** *Let $E(h)$ denote the Douady–Earle extension of $h$, and $\nu(h)$ denote the Beltrami coefficient of the inverse mapping $E^{-1}(h)$. Then there exists some constant $C(h)$ such that*

$$\frac{|\nu(h)(w)|^2}{1 - |\nu(h)(w)|^2} \le C(h) \phi_h^2(\bar{w})(1 - |w|^2)^2. \tag{2.8}$$

### *2.2. Grunsky kernel*

Let $f$ be a conformal mapping on $\Delta$. Set

$$U(f, \zeta, z) = \frac{f'(\zeta) f'(z)}{[f(\zeta) - f(z)]^2} - \frac{1}{(\zeta - z)^2}, \quad (\zeta, z) \in \Delta \times \Delta. \tag{2.9}$$

Then $S_f(z) = -6U(f, z, z)$ is the Schwarzian derivative. $f$ determines the so-called Grunsky operator on $\mathcal{A}^2$, defined as

$$G_f \psi(\zeta) = \frac{1}{\pi} \iint_\Delta U(f, \zeta, \bar{z}) \psi(z) dxdy. \tag{2.10}$$

It is known that $G_f$ is a bounded operator from $\mathcal{A}^2$ into itself with $\|G_f\| \le 1$, and $\|G_f\| < 1$ if and only if $f$ can be extended to a quasiconformal mapping to the whole plane. For details, see [8,28,36] and also [31].

For each $z \in \Delta$, set

$$\psi_z(\zeta) = \frac{1}{(1 - z\zeta)^2}. \tag{2.11}$$

Clearly, $\psi_z \in \mathcal{A}^2$, and $\|\psi_z\| = (1-|z|^2)^{-1}$. We have

$$\begin{aligned} G_f\psi_z(\zeta) &= \frac{1}{\pi}\iint_\Delta U(f,\zeta,\bar{w})\psi_z(w)dudv \\ &= \frac{1}{\pi}\int_0^1 rdr\int_{|w|=r}\frac{U(f,\zeta,w)}{(1-z\bar{w})^2}\frac{dw}{iw} \\ &= U(f,\zeta,z). \end{aligned} \tag{2.12}$$

We also have

$$\begin{aligned} \langle U(f,\cdot,z),\psi_{\bar{z}}\rangle &= \frac{1}{\pi}\iint_\Delta \frac{U(f,\zeta,z)}{(1-z\bar{\zeta})^2}d\xi d\eta \\ &= \frac{1}{\pi}\int_0^1 rdr\int_{|\zeta|=r}\frac{U(f,\zeta,z)}{(1-z\bar{\zeta})^2}\frac{d\zeta}{i\zeta} \\ &= U(f,z,z). \end{aligned} \tag{2.13}$$

Following Bazilevic [6], we define

$$U(f,z) = \left(\frac{1}{\pi}\iint_\Delta |U(f,\zeta,z)|^2 d\xi d\eta\right)^{\frac{1}{2}}. \tag{2.14}$$

Like the Schwarzian derivative $S_f(z)$, the quantity $U(f,z)$ plays an important role in the study of univalent functions (see [3,6,25,44]) and universal Teichmüller space (see [39,40]). Notice that $U(f,z) = \|G_f\psi_z\|$ by (2.12), and it follows from (2.13) that

$$|S_f(z)| = 6|U(f,z,z)| \le \frac{6U(f,z)}{1-|z|^2}, \quad z \in \Delta. \tag{2.15}$$

Now we begin to prove the main result of this section.

**Proposition 2.1.** *Let $f$ be a conformal mapping on $\Delta$ and $h = f^{-1}\circ g$ be the corresponding quasisymmetric conformal welding. Then it holds that $T_h^+\circ G_f = J\circ T_h^-\circ J$.*

**Proof.** For any $\psi \in \mathcal{A}^2$, choose $\phi$ such that $\phi' = \psi$, and $(J\phi)' = J\psi$. Then,

$$\begin{aligned} T_h^+\psi(\zeta) &= \frac{1}{\pi}\iint_\Delta \psi_h(\zeta,\bar{z})\psi(z)dxdy \\ &= \frac{1}{\pi}\iint_\Delta\left(\frac{1}{2\pi i}\int_{S^1}\frac{h(w)}{(\zeta-w)^2(1-\bar{z}h(w))}dw\right)\psi(z)dxdy \\ &= \frac{1}{2\pi^2 i}\int_{S^1}\frac{h(w)}{(\zeta-w)^2}dw\iint_\Delta\frac{\psi(z)}{1-\bar{z}h(w)}dxdy \\ &= \frac{1}{2\pi^2 i}\int_{S^1}\frac{h(w)}{(\zeta-w)^2}dw\int_0^1 rdr\int_{|z|=r}\frac{\psi(z)}{1-\bar{z}h(w)}\frac{dz}{iz} \\ &= \frac{1}{\pi i}\int_{S^1}\frac{h(w)}{(\zeta-w)^2}dw\int_0^1 r\psi(r^2h(w))dr \\ &= \frac{1}{2\pi i}\int_{S^1}\frac{\phi(h(w))}{(\zeta-w)^2}dw. \end{aligned} \tag{2.16}$$

Similarly,

$$T_h^- \psi(\zeta) = \frac{1}{2\pi i} \int_{S^1} \frac{\phi(h(w))}{(1-\zeta w)^2} dw. \tag{2.17}$$

Since

$$G_f \psi(\zeta) = \frac{1}{\pi} \iint_{\Delta} \left( \frac{f'(\zeta) f'(z)}{[f(\zeta) - f(z)]^2} - \frac{1}{(\zeta - z)^2} \right) \psi(\bar{z}) dx dy,$$

we conclude by $f \circ h = g$ and (2.16), (2.17) that

$$\begin{aligned}
&(T_h^+ \circ G_f)\psi(\zeta) \\
&= -\frac{1}{2\pi^2 i} \int_{S^1} \frac{1}{(\zeta - w)^2} dw \iint_{\Delta} \left( \frac{f'(z)}{f(h(w)) - f(z)} - \frac{1}{h(w) - z} \right) \psi(\bar{z}) dx dy \\
&= -\frac{1}{2\pi^2 i} \iint_{\Delta} \psi(\bar{z}) dx dy \int_{S^1} \frac{1}{(\zeta - w)^2} \left( \frac{f'(z)}{f(h(w)) - f(z)} - \frac{1}{h(w) - z} \right) dw \\
&= -\frac{1}{2\pi^2 i} \iint_{\Delta} \psi(\bar{z}) dx dy \int_{S^1} \frac{1}{(\zeta - w)^2} \left( \frac{f'(z)}{g(w) - f(z)} - \frac{1}{h(w) - z} \right) dw \\
&= \frac{1}{2\pi^2 i} \iint_{\Delta} \psi(\bar{z}) dx dy \int_{S^1} \frac{dw}{(\zeta - w)^2 (h(w) - z)} \\
&= \frac{1}{2\pi^2 i} \int_{S^1} \frac{dw}{(\zeta - w)^2} \iint_{\Delta} \frac{\psi(z)}{h(w) - \bar{z}} dx dy \\
&= \frac{1}{2\pi i} \int_{S^1} \frac{\phi(\overline{h(w)})}{(\zeta - w)^2} dw \\
&= (J \circ T_h^- \circ J)\psi(\zeta).
\end{aligned}$$

This completes the proof of Proposition 2.1. □

**Corollary 2.1.** *Under the assumption of Proposition 2.1, it holds that*

$$U(f, z) \le \phi_h(\bar{z}) \le \|T_h^+\| U(f, z), \quad z \in \Delta. \tag{2.18}$$

**Proof.** Recall that $U(f, z) = \|G_f \psi_z\|$. Now (2.17) implies that $\phi_h(\zeta, z) = T_h^- \psi_z(\zeta)$, so $\phi_h(z) = \|T_h^- \psi_z\|$. On the other hand, it follows from (2.11) that $J\psi_z = \psi_{\bar{z}}$. Thus, we conclude by Proposition 2.1 that $\phi_h(\bar{z}) = \|(T_h^+ \circ G_f)\psi_z\| \le \|T_h^+\| U(f, z)$. The first inequality in (2.18) follows from the following lemma. □

**Lemma 2.3.** *For any quasisymmetric homeomorphism h, it holds that*

$$\|T_h^+ \psi\|^2 = \|\psi\|^2 + \|T_h^- \psi\|^2, \quad \psi \in \mathcal{A}^2. \tag{2.19}$$

**Proof.** By means of an approximation process, we may assume that $\psi$ is smooth on $\overline{\Delta}$. Choose $\phi$ such that $\phi' = \psi$ as before. Let $P(u)$ denote the Poisson integral of an integrable function $u$ on the unit circle $S^1$. Then,

$$P(\phi \circ h)(z) = \frac{1}{2\pi i} \int_{S^1} \mathrm{Re} \frac{w + z}{w - z} \frac{\phi(h(w))}{w} dw. \tag{2.20}$$

By means of (2.16), and (2.17) we have

$$\partial P(\phi \circ h) = T_h^+ \psi, \qquad \overline{\partial} P(\phi \circ h) = \overline{J \circ T_h^- \psi}. \tag{2.21}$$

On the other hand, we have the following computation by Green's formula, as observed in [34]:

$$\iint_{\Delta} |\phi'(w)|^2 dudv = \frac{1}{2i} \int_{S^1} \bar{\phi} d\phi = \frac{1}{2i} \int_{S^1} \overline{\phi \circ h} d(\phi \circ h) = \frac{1}{2i} \int_{S^1} \overline{P(\phi \circ h)} dP(\phi \circ h)$$
$$= \iint_{\Delta} (|\partial P(\phi \circ h)|^2 - |\overline{\partial} P(\phi \circ h)|^2) dxdy.$$

Then, (2.19) follows immediately. □

The following estimates follow from Lemma 2.1, (2.15) and Corollary 2.1 immediately.

**Corollary 2.2.** *Let $h = f^{-1} \circ g$ be the corresponding quasisymmetric conformal welding of the conformal mapping $f$ on $\Delta$, and $\nu$ be the Beltrami coefficient of a quasiconformal extension of $h^{-1}$ to the unit disk. Then it holds that*

$$\frac{(1-|z|^2)^2}{36} |S_f(z)|^2 \le U^2(f, z) \le \phi_h^2(\bar{z}) \le \frac{1}{\pi} \iint_{\Delta} \frac{|\nu(w)|^2}{1 - |\nu(w)|^2} \frac{1}{|1 - \bar{z}w|^4} dudv. \tag{2.22}$$

## 3. BMOA–Teichmüller space

We will give two characterizations for BMOA–Teichmüller space by means of the kernels introduced in Section 2. Precisely, we will prove the following

**Theorem 3.1.** *Let $f$ be a conformal mapping on $\Delta$ and $h = f^{-1} \circ g$ be the corresponding quasisymmetric conformal welding. Then each of the following two statements is equivalent to those in Theorem A.*

($B_6$) *$\phi_h$ induces a Carleson measure $\phi_h^2(\bar{z})(1 - |z|^2) \in CM(\Delta)$.*
($B_7$) *$U(f, \cdot)$ induces a Carleson measure $U^2(f, z)(1 - |z|^2) \in CM(\Delta)$.*

**Proof.** We first point out that $B_6$ was asserted in [26]. Corollary 2.1 implies that $B_6 \Leftrightarrow B_7$. (2.15) implies that $B_7 \Rightarrow B_3$. We will show that $B_1 \Rightarrow B_6$ (note that the converse follows from (2.8)).

Suppose $f$ can be extended to a quasiconformal mapping to the whole plane whose complex dilatation $\mu$ induces a Carleson measure $|\mu(z)|^2/(|z|^2 - 1) \in CM(\Delta^*)$. Then $|\mu(z^{-1})|^2/(1 - |z|^2) \in CM(\Delta)$. Noting that $h = f^{-1} \circ g$, we conclude that $h^{-1}$ may be extended to a quasiconformal mapping to $\Delta^*$ with the same complex dilatation $\mu$. By reflection, $h^{-1}$ may be extended to a quasiconformal mapping to $\Delta$ whose complex dilatation $\nu$ satisfies $|\nu(z)| = |\mu(z^{-1})|$ so that $|\nu(z)|^2/(1 - |z|^2) \in CM(\Delta)$. It follows from Lemma 2.1 and the following lemma (with $\alpha = \beta = 1$) that $\phi_h^2(\bar{z})(1 - |z|^2) \in CM(\Delta)$. This completes the proof of Theorem 3.1. □

**Lemma 3.1.** *Let $\alpha > 0$, $\beta > 0$. For a positive measure $\lambda$ on $\Delta$, set*

$$\tilde{\lambda}(z) = \iint_{\Delta} \frac{(1 - |z|^2)^{\alpha} (1 - |w|^2)^{\beta}}{|1 - \bar{z}w|^{\alpha+\beta+2}} \lambda(w) dudv. \tag{3.1}$$

*Then $\tilde{\lambda} \in CM(\Delta)$ if $\lambda \in CM(\Delta)$, and $\|\tilde{\lambda}\|_c \le C\|\lambda\|_c$, while $\tilde{\lambda} \in CM_0(\Delta)$ if $\lambda \in CM_0(\Delta)$.*

**Proof.** When $\alpha = \beta = 1$, the first statement was proved by Cui–Zinsmeister [16] by definition. We will give a new proof by means of a characterization of Carleson measure (see [24,43]). Our approach also gives the proof of the second statement simultaneously.

Recall that for a positive measure $\lambda$ on $\Delta$, $\lambda \in CM(\Delta)$ if and only if

$$\sup_{\zeta\in\Delta}\iint_{\Delta}\frac{1-|\zeta|^2}{|1-\bar{\zeta}z|^2}\lambda(z)dxdy < \infty, \tag{3.2}$$

while $\lambda \in CM_0(\Delta)$ if and only if

$$\lim_{|\zeta|\to 1^-}\iint_{\Delta}\frac{1-|\zeta|^2}{|1-\bar{\zeta}z|^2}\lambda(z)dxdy = 0. \tag{3.3}$$

Furthermore, there exist two universal positive constants $C_1, C_2$ such that

$$C_1\|\lambda\|_c^2 \le \sup_{\zeta\in\Delta}\iint_{\Delta}\frac{1-|\zeta|^2}{|1-\bar{\zeta}z|^2}\lambda(z)dxdy \le C_2\|\lambda\|_c^2. \tag{3.4}$$

On the other hand, by Lemma 1 in Zhao [42], there exists some universal constant $C_3 > 0$ such that for any $\zeta, w \in \Delta$,

$$\iint_{\Delta}\frac{(1-|z|^2)^{\alpha}}{|1-\bar{z}w|^{\alpha+\beta+2}|1-\bar{\zeta}z|^2}dxdy \le \frac{C_3}{(1-|w|^2)^{\beta}|1-\bar{\zeta}w|^2}. \tag{3.5}$$

It follows from (3.1) and (3.5) that

$$\begin{aligned}\iint_{\Delta}\frac{1-|\zeta|^2}{1-\bar{\zeta}z|^2}\tilde{\lambda}(z)dxdy &= \iint_{\Delta}\frac{1-|\zeta|^2}{|1-\bar{\zeta}z|^2}dxdy\iint_{\Delta}\frac{(1-|z|^2)^{\alpha}(1-|w|^2)^{\beta}}{|1-\bar{z}w|^{\alpha+\beta+2}}\lambda(w)dudv\\ &= \iint_{\Delta}\frac{1-|\zeta|^2}{(1-|w|^2)^{-\beta}}\lambda(w)dudv\\ &\quad\times\iint_{\Delta}\frac{(1-|z|^2)^{\alpha}}{|1-\bar{z}w|^{\alpha+\beta+2}|1-\bar{\zeta}z|^2}dxdy\\ &\le C_3\iint_{\Delta}\frac{1-|\zeta|^2}{|1-\bar{\zeta}w|^2}\lambda(w)dudv.\end{aligned}$$

Consequently, $\tilde{\lambda} \in CM(\Delta)$ if $\lambda \in CM(\Delta)$, and $\|\tilde{\lambda}\|_c \le C\|\lambda\|_c$, while $\tilde{\lambda} \in CM_0(\Delta)$ if $\lambda \in CM_0(\Delta)$. □

**Remark 3.1.** If $h = f^{-1} \circ g$ is the quasisymmetric conformal welding corresponding to $f$, then $h^{-1} = g^{-1} \circ f$ is the quasisymmetric conformal welding corresponding to $j \circ g \circ j$, where $j(z) = \bar{z}^{-1}$ is the standard reflection of the unit circle. By a result of Coifman–Fefferman [14], a quasisymmetric homeomorphism $h$ is strongly quasisymmetric if and only if $h^{-1}$ is strongly quasisymmetric. We conclude that in Theorems A and 3.1 the corresponding statements hold for $g$ on $\Delta^*$.

## 4. VMOA–Teichmüller space

In this section, we shall give various characterizations on VMOA–Teichmüller space. We first note the following basic result.

**Lemma 4.1.** *Let $\phi$ be analytic in $\Delta$, $n \in \mathbb{N}$, $\alpha > 0$. Then the following statements hold.*

(1) *If* $|\phi(z)|^n(1-|z|^2)^\alpha \in CM(\Delta)$, *then* $\sup_{z\in\Delta} |\phi(z)|^n(1-|z|^2)^{\alpha+1} < \infty$.
(2) *If* $|\phi(z)|^n(1-|z|^2)^\alpha \in CM_0(\Delta)$, *then* $\lim_{|z|\to 1^-} |\phi(z)|^n(1-|z|^2)^{\alpha+1} = 0$.

**Proof.** The proof is standard. Set $\lambda(z) = |\phi(z)|^n(1-|z|^2)^\alpha$. Denote by $D(\zeta, r)$ the ball of center $\zeta$ with radius $r$ as before. For any $z \in \Delta$, it is easy to see that $D(z, (1-|z|)/2) \subset \Delta \cap D(z/|z|, 3(1-|z|)/2)$. Thus,

$$\iint_{D(z,(1-|z|)/2)} |\phi(\zeta)|^n(1-|\zeta|^2)^\alpha d\xi d\eta \le \lambda(\Delta \cap D(z/|z|, 3(1-|z|)/2)).$$

On the other hand, for $\zeta \in D(z, (1-|z|)/2)$, $1-|\zeta|^2 \ge (3+|z|)(1-|z|)/4$, so

$$\begin{aligned}\iint_{D(z,(1-|z|)/2)} & |\phi(\zeta)|^n(1-|\zeta|^2)^\alpha d\xi d\eta \\ &\ge \frac{(3+|z|)^\alpha(1-|z|)^\alpha}{4^\alpha} \iint_{D(z,(1-|z|)/2)} |\phi(\zeta)|^n d\xi d\eta \\ &\ge \pi \frac{(3+|z|)^\alpha(1-|z|)^\alpha}{4^\alpha} \frac{(1-|z|)^2}{4} |\phi(z)|^n.\end{aligned}$$

Consequently, there exists some constant $C(\alpha)$ such that

$$|\phi(z)|^n(1-|z|^2)^{\alpha+1} \le C(\alpha) \frac{\lambda(\Delta \cap D(z/|z|, 3(1-|z|)/2))}{1-|z|},$$

from which we obtain the required results immediately. $\square$

**Theorem 4.1.** *Let $f$ be a bounded conformal mapping on $\Delta$ and $h = f^{-1} \circ g$ be the corresponding quasisymmetric conformal welding. Then the following statements are equivalent.*

($V_1$) *$f$ can be extended to a quasiconformal mapping to the whole plane whose complex dilatation $\mu$ induces a vanishing Carleson measure* $|\mu(z)|^2/(|z|^2-1) \in CM_0(\Delta^*)$.
($V_2$) $\log f'$ *belongs to* $\mathrm{VMOA}(\Delta)$, *or equivalently,* $|N_f(z)|^2(1-|z|^2) \in CM_0(\Delta)$.
($V_3$) $|S_f(z)|^2(1-|z|^2)^3 \in CM_0(\Delta)$.
($V_4$) *$\Gamma = f(S^1)$ is asymptotically smooth in the sense of Pommerenke* [32], *namely,* $\lim_{|\zeta-z|\to 0} |\widetilde{\zeta z}|/|\zeta - z| = 1$ *for any two points $z$ and $\zeta$ of $\Gamma$.*
($V_5$) *$h$ is absolutely continuous, and* $\log h' \in \mathrm{VMO}(S^1)$.
($V_6$) $\phi_h^2(\bar{z})(1-|z|^2) \in CM_0(\Delta)$.
($V_7$) $U^2(f,z)(1-|z|^2) \in CM_0(\Delta)$.

**Proof.** $V_1 \Leftrightarrow V_4$ was proved by Pommerenke [32]. By the same reasoning as $B_1 \Rightarrow B_6$ in Section 3, we find that $V_1 \Rightarrow V_6$ follows from Lemmas 2.1 and 3.1. $V_6 \Leftrightarrow V_7$ follows from Corollary 2.1. $V_7 \Rightarrow V_3$ follows from (2.15). It remains to prove $V_3 \Rightarrow V_2 \Rightarrow V_1$ and $V_2 \Leftrightarrow V_5$. It should be pointed out that $V_1 \Rightarrow V_2$ was implicitly proved by Dynkin [20].

If $\log f'$ belongs to $\mathrm{VMOA}(\Delta)$, then $\log f'$ belongs to the little Bloch space, namely, $|N_f(z)|(1-|z|^2) \to 0$ as $|z| \to 1^-$ (see [24,33,43]). This also follows from the fact that $|N_f(z)|^2(1-|z|^2) \in CM_0(\Delta)$ along with Lemma 4.1 (with $n = 2, \alpha = 1$). By a result of Becker–Pommerenke [7], $f$ can be extended to a quasiconformal mapping in the whole plane whose complex dilatation $\mu$ satisfies $|\mu(z)| = |N_f(\bar{z}^{-1})|(1-|z|^{-2})|z|^{-1}$ as $|z| \to 1^+$. Since

$|N_f(z)|^2(1-|z|^2) \in CM_0(\Delta)$, we conclude that $|\mu(z)|^2/(|z|^2-1) \in CM_0(\Delta^*)$. This proves $V_2 \Rightarrow V_1$.

Now we prove $V_3 \Rightarrow V_2$. For any $\zeta \in \Delta$, set

$$\gamma_\zeta(z) = \frac{z-\zeta}{1-\bar{\zeta}z}, \quad z \in \Delta. \tag{4.1}$$

Then,

$$\gamma'_\zeta(z) = \frac{1-|\zeta|^2}{(1-\bar{\zeta}z)^2}, \qquad N_{\gamma_\zeta} = \frac{\gamma''_\zeta(z)}{\gamma'_\zeta(z)} = \frac{2\bar{\zeta}}{1-\bar{\zeta}z}. \tag{4.2}$$

Set $f_\zeta = f \circ \gamma_\zeta^{-1}$ so that $f = f_\zeta \circ \gamma_\zeta$. Then

$$N_f = N_{f_\zeta \circ \gamma_\zeta} = (N_{f_\zeta} \circ \gamma_\zeta)\gamma'_\zeta + N_{\gamma_\zeta} = (N_{f_\zeta} \circ \gamma_\zeta - N_{\gamma_\zeta^{-1}} \circ \gamma_\zeta)\gamma'_\zeta. \tag{4.3}$$

Now suppose $|S_f(z)|^2(1-|z|^2)^3 \in CM_0(\Delta)$. In order to prove $|N_f(z)|^2(1-|z|^2) \in CM_0(\Delta)$, by (3.3) it is sufficient to show that

$$\lim_{|\zeta|\to 1^-} \iint_\Delta |\gamma'_\zeta(z)|(1-|z|^2)|N_f(z)|^2 dxdy = 0. \tag{4.4}$$

By Lemma 4.1 (with $n = 2, \alpha = 3$), $|S_f(z)|(1-|z|^2)^2 \to 0$ as $|z| \to 1^-$. By another result of Becker–Pommerenke [7], $|N_f(z)|(1-|z|^2) \to 0$ as $|z| \to 1^-$. Thus, we have

$$\begin{aligned}
&\lim_{|\zeta|\to 1^-} \iint_\Delta |\gamma'_\zeta(z)|(1-|z|^2)|N_f(z)|^2 dxdy \\
&= \lim_{|\zeta|\to 1^-} \iint_\Delta (1-|\gamma_\zeta|^2)|(N_{f_\zeta} \circ \gamma_\zeta - N_{\gamma_\zeta^{-1}} \circ \gamma_\zeta)\gamma'_\zeta|^2 dxdy \\
&= \lim_{|\zeta|\to 1^-} \iint_\Delta (1-|w|^2)|N_{f_\zeta}(w) - N_{\gamma_\zeta^{-1}}(w)|^2 dudv \\
&\le C_1 \lim_{|\zeta|\to 1^-} \left( |N_{f_\zeta}(0) - N_{\gamma_\zeta^{-1}}(0)|^2 + \iint_\Delta (1-|w|^2)^3 |N'_{f_\zeta}(w) - N'_{\gamma_\zeta^{-1}}(w)|^2 dudv \right) \\
&= C_1 \lim_{|\zeta|\to 1^-} \left( |N_f(\zeta)|^2(1-|\zeta|^2)^2 + \iint_\Delta (1-|w|^2)^3 |S_{f_\zeta}(w) \right. \\
&\quad \left. + \frac{1}{2}(N^2_{f_\zeta}(w) - N^2_{\gamma_\zeta^{-1}}(w))|^2 dudv \right) \\
&\le C_2 \lim_{|\zeta|\to 1^-} \iint_\Delta (1-|z|^2)^3 |\gamma'_\zeta(z)|(|S_f(z)|^2 + |N_f(z) - 2N_{\gamma_\zeta}(z)|^2 |N_f(z)|^2) dxdy \\
&\le C_3 \lim_{|\zeta|\to 1^-} \iint_\Delta (1-|z|^2)^3 |\gamma'_\zeta(z)|(|N_f(z)|^2 + |N_{\gamma_\zeta}(z)|^2)|N_f(z)|^2 dxdy.
\end{aligned}$$

It remains to show that

$$\lim_{|\zeta|\to 1^-} \iint_\Delta (1-|z|^2)^3 |\gamma'_\zeta(z)|(|N_f(z)|^2 + |N_{\gamma_\zeta}(z)|^2)|N_f(z)|^2 dxdy = 0. \tag{4.5}$$

For any $\epsilon > 0$, choose some $0 < r < 1$ such that $|N_f(z)|(1-|z|^2) < \epsilon$ as $r < |z| < 1$. Then, by (3.4) we have

$$\begin{aligned}
&\iint_{r<|z|<1} (1-|z|^2)^3 |\gamma_\zeta'(z)|(|N_f(z)|^2 + |N_{\gamma_\zeta}(z)|^2)|N_f(z)|^2 dxdy \\
&\le \epsilon^2 \iint_\Delta (1-|z|^2)|\gamma_\zeta'(z)|(|N_f(z)|^2 + |N_{\gamma_\zeta}(z)|^2) dxdy \\
&\le \epsilon^2 \left( C_4 \| \, |N_f(z)|^2(1-|z|^2)\|_c^2 + 4|\zeta|^2(1-|\zeta|^2) \iint_\Delta \frac{1-|z|^2}{|1-\bar{\zeta}z|^4} dxdy \right) \\
&\le \epsilon^2 (C_4 \| \, |N_f(z)|^2(1-|z|^2)\|_c^2 + C_5).
\end{aligned} \tag{4.6}$$

On the other hand,

$$\begin{aligned}
&\iint_{|z|<r} (1-|z|^2)^3 |\gamma_\zeta'(z)|(|N_f(z)|^2 + |N_{\gamma_\zeta}(z)|^2)|N_f(z)|^2 dxdy \\
&\le 6^4 \iint_{|z|<r} \frac{1-|\zeta|^2}{(1-|z|^2)|1-\bar{\zeta}z|^2} dxdy \\
&\le \frac{6^4 \pi}{(1-r^2)(1-r)^2}(1-|\zeta|^2).
\end{aligned} \tag{4.7}$$

Now (4.5) follows from (4.6) and (4.7).

To prove $V_2 \Rightarrow V_5$, we consider the pull-back operator $P_h(u) = u \circ h$ induced by a quasisymmetric function $h$. Jones [27] proved that $P_h$ is a bounded operator from $\mathrm{BMO}(S^1)$, the space of integrable functions on $S^1$ of bounded mean oscillation (see [24,33,43]), into itself if and only if $h$ is strongly quasisymmetric. In this case, $P_h$ maps $\mathrm{VMO}(S^1)$ into itself, as observed in [2]. In fact, $\mathrm{VMO}(S^1)$ is the closed subspace of $\mathrm{BMO}(S^1)$ which is precisely the closure of the space of all continuous functions in $S^1$ under the BMO norm (see [24,35,43]). Since $P_h$ maps a continuous function $u$ to a continuous function $u \circ h$, we conclude that $P_h$ maps $\mathrm{VMO}(S^1)$ into itself.

Now suppose $\log f' \in \mathrm{VMOA}(\Delta)$. Then, $\Gamma = f(S^1)$ is asymptotically smooth. Without loss of generality, we may assume that $\Gamma = f(S^1)$ does not pass through 0. Then $j(\Gamma)$ is also asymptotically smooth. As pointed out in Remark 3.1, $h^{-1} = g^{-1} \circ f$ is the quasisymmetric conformal welding corresponding to $j \circ g \circ j$. We conclude that $\log g' \in \mathrm{VMOA}(\Delta^*)$. Clearly, $h = f^{-1} \circ g$ is strongly quasisymmetric (by Theorem A) and thus absolutely continuous. Noting that $h' = g'/(f' \circ h)$, we see that $\log h' = \log g' - \log f' \circ h \in \mathrm{VMO}(S^1)$.

The proof of $V_5 \Rightarrow V_2$ will be given at the end of the paper. □

**Remark 4.1.** Theorem 4.1 does not hold if $f$ is an unbounded mapping. For example, let $f(z) = z/(1 - e^{i\theta}z)$ so that $h(z) = z$, but $\log f'(z) = -2\log(1 - e^{i\theta}z)$ does not belong to $\mathrm{VMOA}(\Delta)$.

**Remark 4.2.** $V_2 \Rightarrow V_5$ generalizes a related result by Anderson–Becker–Lesley [2], who proved that if a quasisymmetric homeomorphism $h$ can be extended to a quasiconformal mapping to $\Delta$ whose complex dilatation $\mu$ satisfies

$$\int_0^1 \left( \operatorname*{ess\,sup}_{1-t \le |z| < 1} |\mu(z)| \right)^2 \frac{dt}{t} < \infty, \tag{4.8}$$

then $h$ is absolutely continuous, and $\log h'$ belongs to $\mathrm{VMO}(S^1)$. It is easy to see that (4.8) implies that $|\mu(z)|^2/(1-|z|^2) \in CM_0(\Delta)$. Notice that the condition (4.8) first appeared in Carleson's paper [13] and later in many other papers (see [20]).

## 5. On Schwarzian derivative model

In the rest of the paper, we will discuss the complex analytic theory of BMO–Teichmüller spaces, and then complete the proof of ($V_5 \Rightarrow V_2$ of) Theorem 4.1. In this section, we will define BMO–Teichmüller spaces and discuss their Schwarzian derivative models.

We begin with the standard theory of the universal Teichmüller space (see [1,29,30]). Let $M(\Delta^*)$ denote the open unit ball of the Banach space $L^\infty(\Delta^*)$ of essentially bounded measurable functions on $\Delta^*$. For $\mu \in M(\Delta^*)$, let $f_\mu$ be the quasiconformal mapping on the extended plane $\hat{\mathbb{C}}$ with complex dilatation equal to $\mu$ in $\Delta^*$, conformal in $\Delta$, normalized by $f_\mu(0) = f'_\mu(0) - 1 = f''_\mu(0) = 0$. We say two elements $\mu$ and $\nu$ in $M(\Delta^*)$ are equivalent, denoted by $\mu \sim \nu$, if $f_\mu|_\Delta = f_\nu|_\Delta$. Then $T = M(\Delta^*)/_\sim$ is the Bers model of the universal Teichmüller space. We let $\Phi$ denote the natural projection from $M(\Delta^*)$ onto $T$ so that $\Phi(\mu)$ is the equivalence class $[\mu]$. $[0]$ is called the base point of $T$.

Let $B(\Delta)$ denote the Banach space of functions $\phi$ holomorphic in $\Delta$ with norm

$$\|\phi\|_B = \sup_{z\in\Delta}(1-|z|^2)^2|\phi(z)|. \tag{5.1}$$

$B_0(\Delta)$ is the subspace of $B(\Delta)$ consisting of all functions $\phi$ such that $(1-|z|^2)^2|\phi(z)| \to 0$ as $|z| \to 1^-$. Consider the map $S : M(\Delta^*) \to B(\Delta)$ which sends $\mu$ to the Schwarzian derivative of $f_\mu|_\Delta$. It is known that $S$ is a holomorphic split submersion onto its image, which descends down to a map $\beta : T \to B(\Delta)$ known as the Bers embedding. Via the Bers embedding, $T$ carries a natural complex structure so that $\Phi$ is a holomorphic split submersion.

Now we begin to discuss the BMO–Teichmüller spaces. We denote by $\mathcal{L}(\Delta^*)$ the Banach space of all essentially bounded measurable functions $\mu$ on $\Delta^*$ each of which induces a Carleson measure $\lambda_\mu(z) = |\mu(z)|^2/(|z|^2-1) \in CM(\Delta^*)$. The norm on $\mathcal{L}(\Delta^*)$ is defined as

$$\|\mu\|_c = \|\mu\|_\infty + \|\lambda_\mu\|_c, \tag{5.2}$$

where $\|\lambda_\mu\|_c$ is the Carleson norm of $\lambda_\mu$ defined in (1.1). $\mathcal{L}_0(\Delta^*)$ is the subspace of $\mathcal{L}(\Delta^*)$ consisting of all elements $\mu$ such that $\lambda_\mu \in CM_0(\Delta^*)$. Set $\mathcal{M}(\Delta^*) = M(\Delta^*) \cap \mathcal{L}(\Delta^*)$, $\mathcal{M}_0(\Delta^*) = M(\Delta^*) \cap \mathcal{L}_0(\Delta^*)$. We call $T_b = \mathcal{M}(\Delta^*)/_\sim$ the BMOA–Teichmüller space, and $T_v = \mathcal{M}_0(\Delta^*)/_\sim$ the VMOA–Teichmüller space.

We denote by $\mathcal{B}(\Delta)$ the Banach space of functions $\phi$ holomorphic in $\Delta$ each of which induces a Carleson measure $\lambda_\phi(z) = |\phi(z)|^2(1-|z|^2)^3 \in CM(\Delta)$. The norm on $\mathcal{B}(\Delta)$ is

$$\|\phi\|_{\mathcal{B}} = \|\lambda_\phi\|_c. \tag{5.3}$$

Lemma 4.1 implies that $\mathcal{B}(\Delta) \subset B(\Delta)$, and the inclusion map is continuous. We denote by $\mathcal{B}_0(\Delta)$ the subspace of $\mathcal{B}(\Delta)$ consisting of all functions $\phi$ such that $\lambda_\phi \in CM_0(\Delta)$. Then $\mathcal{B}_0(\Delta) \subset B_0(\Delta)$.

We proceed to consider the Bers projection $S : M(\Delta^*) \to B(\Delta)$. Theorem A implies that $S(\mathcal{M}(\Delta^*)) = S(M(\Delta^*)) \cap \mathcal{B}(\Delta)$, so $S(\mathcal{M}(\Delta^*))$ is an open subset of $\mathcal{B}(\Delta)$ by the openness of $S(M(\Delta^*))$ in $B(\Delta)$. Similarly, $S(\mathcal{M}_0(\Delta^*)) = S(M(\Delta^*)) \cap \mathcal{B}_0(\Delta)$, so $S(\mathcal{M}_0(\Delta^*))$ is an open subset of $\mathcal{B}_0(\Delta)$. Now we prove the following

**Theorem 5.1.** *$S : \mathcal{M}(\Delta^*) \to \mathcal{B}(\Delta)$ is a holomorphic split submersion from $\mathcal{M}(\Delta^*)$ onto its image. Consequently, $T_b$ has a unique complex structure such that $\beta : T_b \to \mathcal{B}(\Delta)$ is a biholomorphic map from $T_b$ onto a domain in $\mathcal{B}(\Delta)$. Under this complex structure, the natural projection $\Phi$ from $\mathcal{M}(\Delta^*)$ onto $T_b$ is a holomorphic split submersion.*

**Proof.** It is sufficient to show that $S : \mathcal{M}(\Delta^*) \to \mathcal{B}(\Delta)$ is a holomorphic split submersion from $\mathcal{M}(\Delta^*)$ onto its image. We first show that $S : \mathcal{M}(\Delta^*) \to \mathcal{B}(\Delta)$ is continuous. We borrow some discussion from Astala–Zinsmeister [5]. By an integral representation of the Schwarzian derivative by means of the representation theorem of quasiconformal mappings, Astala–Zinsmeister [5] proved that for any two elements $\mu$ and $\nu$ in $M(\Delta^*)$, there exists some constant $C_1(\|\mu\|_\infty)$ such that

$$|S(\nu)(z) - S(\mu)(z)|^2 \le \frac{C_1(\|\mu\|_\infty)}{(1-|z|^2)^2} \times \iint_{\Delta^*} \frac{|\nu(\zeta)-\mu(\zeta)|^2 + \|\nu-\mu\|_\infty^2 |\mu(\zeta)|^2}{|\zeta - z|^4} d\xi d\eta. \tag{5.4}$$

By Lemma 2.1 we conclude that there exists some constant $C_2(\|\mu\|_\infty)$ such that

$$\begin{aligned}\|S(\nu) - S(\mu)\|_{\mathcal{B}}^2 &\le C_2(\|\mu\|_\infty)(\|\lambda_\nu - \lambda_\mu\|_c^2 + \|\nu-\mu\|_\infty^2 \|\lambda_\mu\|_c^2) \\ &\le C_2(\|\mu\|_\infty)(1 + \|\mu\|_c^2)\|\nu - \mu\|_c^2.\end{aligned}$$

Consequently, $S : \mathcal{M}(\Delta^*) \to \mathcal{B}(\Delta)$ is continuous.

To prove that $S : \mathcal{M}(\Delta^*) \to \mathcal{B}(\Delta)$ is a holomorphic map, we use a general result about the infinite dimensional holomorphy (see [29,30]). It says that a continuous map $f$ from a domain $U$ in a complex Banach space $X$ into another complex Banach space $Y$ is holomorphic if for each pair $(u, x)$ in $U \times X$ and each element $y^*$ from a total subset $Y_0^*$ of the dual space $Y^*$, $y^*(f(u+tx))$ is a holomorphic function in a neighborhood of zero in the complex plane. Here a subset $Y_0^*$ of $Y^*$ is total if $y^*(y) = 0$ for all $y^* \in Y_0^*$ implies that $y = 0$.

Now for each $z \in \Delta$, define $l_z(\phi) = \phi(z)$ for $\phi \in \mathcal{B}(\Delta)$. Lemma 4.1 says that $|\phi(z)|(1-|z|^2)^2 \le C\|\phi\|_{\mathcal{B}}$, which implies that $\|l_z\| \le C(1-|z|^2)^{-2}$. Thus, $l_z \in \mathcal{B}^*(\Delta)$. Set $A = \{l_z : z \in \Delta\}$. Clearly, $A$ is a total subset of $\mathcal{B}^*(\Delta)$. Now for each $z \in \Delta$, each pair $(\mu, \nu) \in \mathcal{M}(\Delta^*) \times \mathcal{L}(\Delta^*)$ and small $t$ in the complex plane, by the well known holomorphic dependence of quasiconformal mappings on parameters (see [1,29,30]), we conclude that $l_z(S(\mu+t\nu)) = S(\mu+t\nu)(z)$ is a holomorphic function of $t$. Consequently, $S : \mathcal{M}(\Delta^*) \to \mathcal{B}(\Delta)$ is holomorphic.

Finally, we prove $S : \mathcal{M}(\Delta^*) \to \mathcal{B}(\Delta)$ is a split submersion onto its image, or equivalently, $S : \mathcal{M}(\Delta^*) \to \mathcal{B}(\Delta)$ has local holomorphic sections. Fix $\phi = S(\mu)$ and set $D = f_\mu(\Delta)$, $D^* = f_\mu(\Delta^*)$. Denote by $\rho_D$ the hyperbolic metric in $D$, that is, $\rho_D(f_\mu(z))|f_\mu'(z)| = (1-|z|^2)^{-1}$ for $z \in \Delta$. Consider $U_\epsilon(\phi) = \{\psi \in \mathcal{B}(\Delta) : \|\psi - \phi\|_{\mathcal{B}} < \epsilon\}$ for $\epsilon > 0$. Then for each $\psi \in U_\epsilon(\phi)$ there exists a unique locally univalent function $f_\psi$ in $\Delta$ with $f_\psi(0) = f_\psi'(0) - 1 = f_\psi''(0) = 0$ such that $S_{f_\psi} = \psi$. Set $g_\psi = f_\psi \circ f_\mu^{-1}$. Then $S_{g_\psi} = ((\psi - \phi) \circ f_\mu^{-1})(f_\mu^{-1})'^2$, and $\sup_{z\in D} \rho_D^{-2}(z)|S_{g_\psi}(z)| = \|\psi - \phi\|_B$. Since the inclusion map $i : \mathcal{B}(\Delta) \to B(\Delta)$ is continuous, $\|\psi - \phi\|_B < C\epsilon$ for $\psi \in U_\epsilon(\phi)$. When $\epsilon$ is small, Ahlfors [1] (see also Earle–Nag [21]) proved that $g_\psi$ is univalent and can be extended to a quasiconformal mapping in the whole plane whose complex dilatation $\mu_\psi$ has the form

$$\mu_\psi(z) = \frac{S_{g_\psi}(r(z))(r(z)-z)^2\bar{\partial} r(z)}{2 + S_{g_\psi}(r(z))(r(z)-z)^2\partial r(z)}, \quad z \in D^* \tag{5.5}$$

where $r : D^* \to D$ is a quasiconformal reflection which satisfies

$$C_3^{-1}(\|\mu\|_\infty) \le |r(z)-z|^2\rho_D(r(z))|\bar{\partial} r(z)| \le C_3(\|\mu\|_\infty), \quad z \in D^*. \tag{5.6}$$

It should be pointed out that we may choose $r = f_\mu \circ j \circ f_\mu^{-1}$ if we choose $\mu$ appropriately (see [1,21,5,16,22] for details). Then,

$$|\mu_\psi(z)| \le C_4(\|\mu\|_\infty)|S_{g_\psi}(r(z))|\rho_D^{-2}(r(z)), \quad z \in D^*. \tag{5.7}$$

Consequently, $f_\psi = g_\psi \circ f_\mu$ is univalent in $\Delta$ and has a quasiconformal extension to the whole plane whose complex dilatation $\nu_\psi$ is

$$\nu_\psi = \frac{\mu + (\mu_\psi \circ f_\mu)\tau}{1 + \overline{\mu}(\mu_\psi \circ f_\mu)\tau}, \quad \tau = \frac{\overline{\partial f_\mu}}{\partial f_\mu}. \tag{5.8}$$

Now, it follows from (5.7) that

$$\begin{aligned} |\mu_\psi(f_\mu(z))| &\le C_4(\|\mu\|_\infty)|S_{g_\psi}(r(f_\mu(z)))|\rho_D^{-2}(r(f_\mu(z))) \\ &= C_4(\|\mu\|_\infty)|S_{g_\psi}(f_\mu(j(z)))|\rho_D^{-2}(f_\mu(j(z))) \\ &= C_4(\|\mu\|_\infty)|\psi(j(z)) - \phi(j(z))|(1-|j(z)|^2)^2, \end{aligned}$$

which implies that $\|\lambda_{\mu_\psi \circ f_\mu}\|_c \le C_5(\|\mu\|_\infty)\|\psi - \phi\|_{\mathcal{B}}$. Thus, $\mu_\psi \circ f_\mu \in \mathcal{M}(\Delta^*)$, and we conclude by (5.8) that $\nu_\psi \in \mathcal{M}(\Delta^*)$. On the other hand, from (5.5) and (5.8) it is easy to see that $\nu_\psi$ depends holomorphically on $\psi$. Since $S(\nu_\psi) = \psi$, we conclude that $\nu : U_\epsilon(\phi) \to \mathcal{B}(\Delta)$ is a local holomorphic section to $S : \mathcal{M}(\Delta^*) \to \mathcal{B}(\Delta)$. This completes the proof of Theorem 5.1. □

Examining the proof of Theorem 5.1, we may obtain the following

**Theorem 5.2.** *$S : \mathcal{M}_0(\Delta^*) \to \mathcal{B}_0(\Delta)$ is a holomorphic split submersion from $\mathcal{M}_0(\Delta^*)$ onto its image. Consequently, $T_v$ has a unique complex structure such that $\beta : T_v \to \mathcal{B}_0(\Delta)$ is a biholomorphic map from $T_v$ onto a domain in $\mathcal{B}_0(\Delta)$. Under this complex structure, the natural projection $\Phi$ from $\mathcal{M}_0(\Delta^*)$ onto $T_v$ is a holomorphic split submersion.*

**Remark 5.1.** Let $w : \Delta^* \to \Delta^*$ be a quasiconformal mapping with complex dilatation $\mu$. Then $w$ induces a biholomorphic isomorphism $R_w : M(\Delta^*) \to M(\Delta^*)$ as

$$R_w(\nu) = \left(\frac{\nu - \mu}{1 - \bar{\mu}\nu}\frac{\partial w}{\overline{\partial w}}\right) \circ w^{-1}. \tag{5.9}$$

$R_w$ descends down a biholomorphic isomorphism $w^* : T \to T$ by $w^* \circ \Phi = \Phi \circ R_w$.

Suppose $w$ is quasi-isometric under the Poincaré such that $\mu \in \mathcal{M}(\Delta^*)$. Examining the proof of Lemma 10 in [16], we find that $R_w(\nu) \in \mathcal{M}(\Delta^*)$ with $\|R_w(\nu)\|_c \le C_1(\|\mu\|_\infty)\|\nu - \mu\|_c$ whenever $\nu \in \mathcal{M}(\Delta^*)$. It is easy to see that $R_w : \mathcal{M}(\Delta^*) \to \mathcal{M}(\Delta^*)$ is biholomorphic. It follows from Theorem 5.1 that $w^* : T_b \to T_b$ is biholomorphic.

## 6. On pre-logarithmic derivative model

Recall that the pre-logarithmic derivative model $\hat{\mathcal{T}}$ of the universal Teichmüller space consists of all functions $\log f'$, where $f$ belongs to the well known class $S_Q$ of all univalent analytic functions $f$ in the unit disk $\Delta$ with the normalized condition $f(0) = f'(0) - 1 = 0$ that can be extended to a quasiconformal mapping in the whole plane (see [4,45]). Under the topology of Bloch norm (see [33,43]), $\hat{\mathcal{T}}$ is a disconnected open set. Precisely, $\hat{\mathcal{T}} = \mathcal{T} \cup_{\theta\in[0,2\pi)} \mathcal{T}_\theta$, where $\mathcal{T} = \{\log f' : f \in S_Q \text{ is bounded}\}$ and $\mathcal{T}_\theta = \{\log f' : f \in S_Q \text{ satisfies} f(e^{i\theta}) = \infty\}, \theta \in [0, 2\pi)$, are the all connected components of $\hat{\mathcal{T}}$ (see [45]). Each $\mathcal{T}_\theta$ is a copy of the Bers model $T$, while $\mathcal{T}$ is a fiber space over $T$. In fact, $\mathcal{T}$ is a model of the universal Teichmüller curve (see [9,41]).

We come back to our situation. Under the topology of the BMOA norm, $\hat{\mathcal{T}} \cap \text{BMOA}(\Delta)$ is a disconnected open subset of $\text{BMOA}(0) = \{\phi \in \text{BMOA}(\Delta), \phi(0) = 0\}$. Precisely, $\mathcal{T} \cap \text{BMOA}(\Delta)$ and $\mathcal{T}_\theta \cap \text{BMOA}(\Delta), \theta \in [0, 2\pi)$, are the all connected components of $\hat{\mathcal{T}} \cap \text{BMOA}(\Delta)$ (see [5]). We will show that each $\mathcal{T}_\theta \cap \text{BMOA}(\Delta)$ is biholomorphic to the BMOA–Teichmüller space $T_b$, while $\mathcal{T} \cap \text{BMOA}(\Delta)$ is a holomorphic fiber space over $T_b$.

We recall some basic results on BMOA (see [23,24,33,42]). For any $\phi \in \text{BMOA}(0)$, we set $\chi_\phi(z) = |\phi'(z)|^2(1 - |z|^2)$. Then $\chi_\phi \in CM(\Delta)$, and BMOA(0) is a Banach space with norm

$$\|\phi\|_b = \|\chi_\phi\|_c. \tag{6.1}$$

For any $\phi \in \text{BMOA}(0)$, it is well known that $\phi'' \in \mathcal{B}(\Delta), (\phi')^2 \in \mathcal{B}(\Delta)$, and $\|\phi''\|_{\mathcal{B}} \le C_1\|\phi\|_b, \|(\phi')^2\|_{\mathcal{B}} \le C_2\|\phi\|_b^3$. More generally,

$$\|(\psi')^2 - (\phi')^2\|_{\mathcal{B}} \le 2C_2(\|\psi\|_b^2 + \|\phi\|_b^2)\|\psi - \phi\|_b. \tag{6.2}$$

Set

$$\Lambda(\phi) = \phi'' - \frac{1}{2}(\phi')^2, \quad \phi \in \text{BMOA}(0). \tag{6.3}$$

Then $\Lambda(\phi) \in \mathcal{B}(\Delta)$, and $\Lambda : \text{BMOA}(0) \to \mathcal{B}(\Delta)$ is continuous.

**Lemma 6.1.** $\Lambda : \text{BMOA}(0) \to \mathcal{B}(\Delta)$ *is holomorphic.*

**Proof.** Since $\Lambda$ is continuous, it needs to show that for any $\phi, \psi$ in BMOA(0), the Frechet derivative

$$d_\phi\Lambda(\psi) = \lim_{t\to 0} \frac{\Lambda(\phi + t\psi) - \Lambda(\phi)}{t} = \psi'' + \phi'\psi'$$

exists in the norm $\|\cdot\|_{\mathcal{B}}$. This can be done as follows:

$$\lim_{t\to 0} \left\|\frac{\Lambda(\phi + t\psi) - \Lambda(\phi)}{t} - (\psi'' + \phi'\psi')\right\|_{\mathcal{B}} = \lim_{t\to 0} \frac{|t|}{2}\|(\psi')^2\|_{\mathcal{B}} = 0. \quad \square$$

Fix $z_0 \in \overline{\Delta^*}$. For $\mu \in M(\Delta^*)$, let $g_\mu$ be the quasiconformal mapping on the extended plane $\hat{\mathbb{C}}$ with complex dilatation equal to $\mu$ in $\Delta^*$, conformal in $\Delta$, normalized by $g_\mu(0) = g'_\mu(0) - 1 = 0, g_\mu(z_0) = \infty$. Consider the map $L_{z_0}$ on $M(\Delta^*)$ by setting $L_{z_0}(\mu) = \log g'_\mu$. Then Theorem A implies that $L_{e^{i\theta}}(\mathcal{M}(\Delta^*)) = \mathcal{T}_\theta \cap \text{BMOA}(\Delta)$, and $\cup_{z_0\in\Delta^*} L_{z_0}(\mathcal{M}(\Delta^*)) = \mathcal{T} \cap \text{BMOA}(\Delta)$, Theorem 4.1 implies that $\cup_{z_0\in\Delta^*} L_{z_0}(\mathcal{M}_0(\Delta^*)) = \mathcal{T} \cap \text{VMOA}(\Delta)$. We have the following result.

**Theorem 6.1.** *For each* $z_0 \in \overline{\Delta^*}$, $L_{z_0} : \mathcal{M}(\Delta^*) \to \mathrm{BMOA}(0)$ *is holomorphic.*

**Proof.** It is sufficient to show $L = L_{z_0} : \mathcal{M}(\Delta^*) \to \mathrm{BMOA}(0)$ is continuous. On proving this, we can prove $L : \mathcal{M}(\Delta^*) \to \mathrm{BMOA}(0)$ is holomorphic by the same reasoning as the proof of the holomorphy of $S : \mathcal{M}(\Delta^*) \to \mathcal{B}(\Delta)$. Recall that $L$ is continuous on $M(\Delta^*)$ in the topology of the Bloch norm (see [29]), namely,

$$\sup_{z\in\Delta} |N_{g_\nu}(z) - N_{g_\mu}(z)|(1-|z|^2) \le C(\|\mu\|_\infty)\|\nu-\mu\|_\infty, \quad \mu,\ \nu \in M(\Delta^*). \tag{6.4}$$

For each $\zeta \in \Delta$, set $\gamma_\zeta$ as in (4.1), and set $g_{\mu,\zeta} = g_\mu \circ \gamma_\zeta^{-1}$. Then, it follows from (3.4) and (6.4) that

$$\begin{aligned}
&\iint_\Delta |\gamma_\zeta'(z)|(1-|z|^2)|N_{g_\nu}(z) - N_{g_\mu}(z)|^2 dxdy \\
&= \iint_\Delta (1-|\gamma_\zeta|^2)|(N_{g_{\nu,\zeta}}\circ\gamma_\zeta - N_{g_{\mu,\zeta}}\circ\gamma_\zeta)\gamma_\zeta'|^2 dxdy \\
&= \iint_\Delta (1-|w|^2)|N_{g_{\nu,\zeta}}(w) - N_{g_{\mu,\zeta}}(w)|^2 dudv \\
&\le C_1\left(|N_{g_{\nu,\zeta}}(0) - N_{g_{\mu,\zeta}}(0)|^2 + \iint_\Delta (1-|w|^2)^3|N'_{g_{\nu,\zeta}}(w) - N'_{g_{\mu,\zeta}}(w)|^2 dudv\right) \\
&= C_1|N_{g_\nu}(\zeta) - N_{g_\mu}(\zeta)|^2(1-|\zeta|^2)^2 \\
&\quad + C_1\left(\iint_\Delta (1-|w|^2)^3|(S_{g_{\nu,\zeta}}(w) - S_{g_{\mu,\zeta}}(w)) + \frac{1}{2}(N^2_{g_{\nu,\zeta}}(w) - N^2_{g_{\mu,\zeta}}(w))|^2 dudv\right) \\
&\le C_1(\|\mu\|_\infty)\|\nu-\mu\|_\infty \\
&\quad + C_2\iint_\Delta (1-|z|^2)^3|\gamma_\zeta'(z)|(|S(\nu)(z) - S(\mu)(z)|^2 + |N^2_{g_\nu}(z) - N^2_{g_\mu}(z)|^2)dxdy \\
&\le C_2(\|\mu\|_\infty)(\|\nu-\mu\|_\infty + \|S(\nu)-S(\mu)\|^2_{\mathcal{B}} + (\|L(\nu)\|_b^2 + \|L(\mu)\|_b^2)\|\nu-\mu\|^2_\infty).
\end{aligned}$$

Consequently, it follows from (3.4) again that

$$\begin{aligned}
\|L(\nu) - L(\mu)\|_b^2 \le\ & C_2(\|\mu\|_\infty)(\|\nu-\mu\|_\infty + \|S(\nu)-S(\mu)\|^2_{\mathcal{B}} \\
&+ (\|L(\nu)\|_b^2 + \|L(\mu)\|_b^2)\|\nu-\mu\|^2_\infty).
\end{aligned}$$

By the continuity of $S : \mathcal{M}(\Delta^*) \to \mathcal{B}(\Delta)$, we conclude that $L : \mathcal{M}(\Delta^*) \to \mathrm{BMOA}(0)$ is continuous. □

**Theorem 6.2.** *For each* $\theta \in [0, 2\pi)$, $\Lambda$ *is a biholomorphic isomorphism from* $\mathcal{T}_\theta \cap \mathrm{BMOA}(\Delta)$ *onto* $\beta(T_b)$.

**Proof.** By Lemma 6.1, $\Lambda$ is holomorphic on $\mathcal{T}_\theta \cap \mathrm{BMOA}(\Delta)$. When restricted on $\mathcal{T}_\theta \cap \mathrm{BMOA}(\Delta)$, $\Lambda$ sends $\log f'$ to $S_f$. By normalization, $\Lambda$ is one-to-one on $\mathcal{T}_\theta \cap \mathrm{BMOA}(\Delta)$. Noting that $S = \Lambda \circ L_{e^{i\theta}}$, we conclude that $\Lambda : \mathcal{T}_\theta \cap \mathrm{BMOA}(\Delta) \to \beta(T_b)$ is surjective, and $\Lambda^{-1} : \beta(T_b) \to \mathcal{T}_\theta \cap \mathrm{BMOA}(\Delta)$ is holomorphic since $L_{e^{i\theta}} : \mathcal{M}(\Delta^*) \to \mathcal{T}_\theta \cap \mathrm{BMOA}(\Delta)$ is holomorphic, and $S : \mathcal{M}(\Delta^*) \to \beta(T_b)$ is a holomorphic split submersion. □

Similarly, we can prove the following

**Theorem 6.3.** *$\Lambda$ is a holomorphic split submersion from $\mathcal{T} \cap \mathrm{BMOA}(\Delta)$ onto $\beta(T_b)$.*

**Proof.** Lemma 6.1 implies that $\Lambda$ is holomorphic on $\mathcal{T} \cap \mathrm{BMOA}(\Delta)$. Fix $z_0 \in \Delta^*$. Since $S = \Lambda \circ L_{z_0}$, we conclude that $\Lambda : \mathcal{T} \cap \mathrm{BMOA}(\Delta) \to \beta(T_b)$ is a holomorphic split submersion since $L_{z_0} : \mathcal{M}(\Delta^*) \to \mathcal{T} \cap \mathrm{BMOA}(\Delta)$ is holomorphic, and $S : \mathcal{M}(\Delta^*) \to \beta(T_b)$ is a holomorphic split submersion. □

An analogous result holds for the VMOA–Teichmüller space. We say $\phi \in \mathrm{VMOA}(0)$ if $\phi \in \mathrm{VMOA}(\Delta)$ and $\phi(0) = 0$. First note the following

**Theorem 6.4.** *For each $z_0 \in \Delta^*$, $L_{z_0} : \mathcal{M}_0(\Delta^*) \to \mathrm{VMOA}(0)$ is holomorphic.*

**Theorem 6.5.** *$\mathcal{T} \cap \mathrm{VMOA}(\Delta)$ is a connected open subset of $\mathrm{VMOA}(0)$.*

**Proof.** Clearly, $\mathcal{T} \cap \mathrm{VMOA}(\Delta)$ is an open subset of VMOA(0). It remains to show that each point of $\mathcal{T} \cap \mathrm{VMOA}(\Delta)$ can be connected to 0 by a path in $\mathcal{T} \cap \mathrm{VMOA}(\Delta)$.

Let $\log f' \in \mathcal{T} \cap \mathrm{VMOA}(\Delta)$. Theorem 4.1 implies that $f$ can be extended to a quasiconformal mapping in the whole plane whose Beltrami coefficient $\mu$ belongs to $\mathcal{M}_0(\Delta^*)$, and $z_0 = f^{-1}(\infty) \in \Delta^*$. For each $t \in [0, 1]$, let $f_t \in S_Q$ be the unique mapping whose quasiconformal extension to the whole plane has Beltrami coefficient $t\mu$, and $f_t(z_0) = \infty$. Theorem 6.4 implies that $\log f_t'$, $t \in [0, 1]$, is a path in $\mathcal{T} \cap \mathrm{VMOA}(\Delta)$ joining $\log f_0'$ to $\log f'$. Now, if $z_0 = \infty$, then $f_0(z) = z$, and we are done. If $z_0 \neq \infty$, then $f_0(z) = z_0 z/(z_0 - z)$, and $\log f_0'(r\cdot)$, $r \in [0, 1]$, is a curve in $\mathcal{T} \cap \mathrm{VMOA}(\Delta)$ connecting 0 and $\log f_0'$. This completes the proof of Theorem 6.5. □

**Theorem 6.6.** *$\Lambda$ is a holomorphic split submersion from $\mathcal{T} \cap \mathrm{VMOA}(\Delta)$ onto $\beta(T_v)$.*

**Proof.** The proof of Lemma 6.1 implies that $\Lambda : \mathrm{VMOA}(0) \to \mathcal{B}_0(\Delta)$ is holomorphic. Thus, $\Lambda$ is holomorphic on $\mathcal{T} \cap \mathrm{VMOA}(\Delta)$. Choose $z_0 \in \Delta^*$. Since $S = \Lambda \circ L_{z_0}$, we conclude that $\Lambda : \mathcal{T} \cap \mathrm{VMOA}(\Delta) \to \beta(T_v)$ is a holomorphic split submersion since $L_{z_0} : \mathcal{M}_0(\Delta^*) \to \mathcal{T} \cap \mathrm{VMOA}(\Delta)$ is holomorphic, and $S : \mathcal{M}_0(\Delta^*) \to \beta(T_v)$ is a holomorphic split submersion. □

## 7. On the quasisymmetric homeomorphism model

Recall that the universal Teichmüller space $T$ has another model (see [1,29,30]). Precisely, let $QS$ be the set of all normalized quasisymmetric homeomorphisms of the unit circle keeping 1, $i$ and $-i$ fixed. Then there exists a homeomorphism $\Psi$ between $T$ and $QS$ (with Teichmüller metric) which takes a point $\Phi(\mu)$ to the normalized quasisymmetric conformal welding corresponding to $f_\mu$.

Let *SQS* be the set of all normalized strongly quasisymmetric homeomorphisms on the unit circle. Theorem A implies that $\Psi$ establishes a bijective map between $T_b$ and *SQS*. Recall that each $h \in SQS$ is absolutely continuous, and $\log h' \in \mathrm{BMO}(S^1)$. A natural metric assigned to *SQS* is the following BMO metric:

$$d(h_1, h_2) = \| \log h_2' - \log h_1' \|_{\mathrm{BMO}}, \quad h_1,\ h_2 \in SQS. \tag{7.1}$$

Let $h \in SQS$ be given. Consider the map $R_h$ defined by $R_h(k) = k \circ h^{-1}$. Then $R_h$ is a bijective map from *SQS* onto itself. Noting that

$$d(R_h(k_1), R_h(h_2)) = \|(\log k_2' - \log k_1') \circ h^{-1}\|,$$

we conclude by Jones' result [27] that $R_h$ is a quasi-isometric map from $SQS$ onto itself under the BMO topology.

Let $w_h$ be a quasiconformal extension of $h$ to $\Delta^*$ such that $w_h$ is quasi-isometric under the Poincaré with Beltrami coefficient $\mu_h \in \mathcal{M}(\Delta^*)$. As stated in the Introduction, the existence of such an extension was first proved by Fefferman–Kenig–Pipher [22] by means of a Beurling–Ahlfors [10] type extension (see also [16] or Lemmas 2.1 and 2.2). We remarked in Section 5 that $w_h$ induces a biholomorphic isomorphism $w_h^*$ from $T_b$ onto itself. It should be pointed out that $w_h^*$ depends only on $h$, not on the extension $w_h$. In fact, it is related to $R_h$ by $\Psi \circ w_h^* = \Psi \circ R_h$.

**Theorem 7.1.** *$\Psi : T_b \to SQS$ is a homeomorphism. Consequently, $SQS$ possesses a complex structure such that $\Psi : T_b \to SQS$ is a biholomorphic isomorphism.*

**Proof.** Recall the holomorphic map $L_\infty : \mathcal{M}(\Delta^*) \to \mathcal{T} \cap \mathrm{BMOA}(\Delta)$ defined in the last section. It is known that $\|\log h'\|_{\mathrm{BMO}}$ is small if and only if $\|L_\infty(\mu_h)\|_b$ is small (see [5,18,38]). By Theorems 5.1 and 6.3, we conclude that both $\Psi : T_b \to SQS$ and its inverse are continuous at the base point $id = \Psi([0])$. By using the translations $w_h^*$ and $R_h$, we conclude that both $\Psi : T_b \to SQS$ and its inverse are continuous at a general point $h = \Psi([\mu_h])$. □

## 8. Completing the proof of Theorem 4.1

Let $h$ be the quasisymmetric conformal welding corresponding to a bounded conformal mapping $f$ on the unit disk. Suppose that $h$ is absolutely continuous, and $\log h' \in \mathrm{VMO}(S^1)$. Then, by Theorem 2 in [35], $h$ must be strongly quasisymmetric. We need to prove $\log f' \in \mathrm{VMOA}(\Delta)$.

We use an approximation process. Without loss of generality, we may assume $h$ keeps the point 1 fixed. Under the arclength parameterization of the unit circle, $h$ may be regarded as a continuous and strictly increasing function on $[0, 2\pi]$ with $h(0) = 0, h(2\pi) = 2\pi$. Recall that $\mathrm{VMO}(S^1)$ is the closed subspace of $\mathrm{BMO}(S^1)$ which is the closure of the space of all continuous functions in $S^1$ under the BMO norm (see [24,35,43]). So there exists a sequence of (real) analytic functions $\psi_n$ on $[0, 2\pi]$ such that $\|\psi_n - \log h'\|_{\mathrm{BMO}} \to 0$ as $n \to \infty$. Set

$$h_n(\theta) = 2\pi \frac{\int_0^\theta e^{\psi_n(t)} dt}{\int_0^{2\pi} e^{\psi_n(t)} dt}. \tag{8.1}$$

Then $h_n$ is an analytic and strictly increasing function from $[0, 2\pi]$ onto itself, and $\|\log h_n' - \log h'\|_{\mathrm{BMO}} \to 0$ as $n \to \infty$. Now, by Theorem 7.1, each $h_n$ may be represented as the conformal welding of a bounded conformal mapping $f_n$ on the unit disk such that $\|\log f_n' - \log f'\|_b \to 0$ as $n \to \infty$. Since $h_n$ is analytic, it is obvious that $\log f_n' \in \mathrm{VMOA}(\Delta)$. Thus, $\log f' \in \mathrm{VMOA}(\Delta)$ due to the closedness of $\mathrm{VMOA}(\Delta)$ in $\mathrm{BMOA}(\Delta)$. This completes the proof of Theorem 4.1.

We say a quasisymmetric homeomorphism $h$ is strongly symmetric if it is absolutely continuous with $\log h' \in \mathrm{VMO}(S^1)$. Let $SS$ be the set of all normalized strongly symmetric homeomorphisms on the unit circle. It is a metric space under the BMO norm (7.1). Theorem 4.1 implies that $\Psi$ establishes a bijective map between $T_v$ and $SS$. By Theorem 7.1, we have the following

**Theorem 8.1.** *$\Psi : T_v \to SS$ is a homeomorphism. Consequently, $SS$ possesses a complex structure such that $\Psi : T_v \to SS$ is a biholomorphic isomorphism.*

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