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WEIL-PETERSSON TEICHMÜLLER SPACE

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Abstract. The paper presents some recent results on the Weil-Petersson geometry theory of the universal Teichmüller space, a topic which is important in Teichmüller theory and has wide applications to various areas such as mathematical physics, differential equation and computer vision. (1) It is shown that a sense-preserving homeomorphism h on the unit circle belongs to the Weil-Petersson class, namely, h can be extended to a quasiconformal mapping to the unit disk whose Beltrami coefficient is square integrable in the Poincaré metric if and only if h is absolutely continuous and $\log h'$ belongs to the Sobolev class $H^{\frac{1}{2}}$. This solves an open problem posed by Takhtajan-Teo in 2006 and investigated later by Figalli, Gay-Balmaz-Marsden-Ratiu and others. The intrinsic characterization (1) of the Weil-Petersson class has the following applications which are also explored in this paper: (2) It is proved that there exists a quasisymmetric homeomorphism of the Weil-Petersson class which belongs neither to the Sobolev class $H^{\frac{3}{2}}$ nor to the Lipschitz class Λ^1, which was conjectured very recently by Gay-Balmaz-Ratiu when studying the classical Euler-Poincaré equation in the new setting that the involved sense-preserving homeomorphisms on the unit circle belong to the Weil-Petersson class. (3) It is proved that the flows of the $H^{\frac{3}{2}}$ vector fields on the unit circle are contained in the Weil-Petersson class, which was also conjectured by Gay-Balmaz-Ratiu in their above mentioned research. (4) A new metric is introduced on the Weil-Petersson Teichmüller space. It is shown to be topologically equivalent to the Weil-Petersson metric.

1. Introduction and statement of the main results. We begin with some basic definitions and notations. Let $\Delta = \{z : |z| < 1\}$ denote the unit disk in the extended complex plane $\hat{\mathbb{C}}$. $\Delta^* = \hat{\mathbb{C}} - \overline{\Delta}$ is the exterior of Δ, $S^1 = \partial\Delta = \partial\Delta^*$ is the unit circle, and $\mathbb{R}$ is the real line. For any function $f = f(z)$ defined on the unit circle S^1, we always denote by $\hat{f}$ the function defined on the real line $\mathbb{R}$ by $\hat{f}(\theta) = f(e^{i\theta})$.

Let $\mathrm{Hom}^+(S^1)$ denote the set of all sense-preserving homeomorphisms of S^1 onto itself. A homeomorphism $h \in \mathrm{Hom}^+(S^1)$ is said to be quasisymmetric if

$$h_{QS} \doteq \sup\Big\{\max\big(q_h(\theta,t), q_h^{-1}(\theta,t)\big) : \theta \in \mathbb{R},\ t > 0\Big\} < +\infty, \tag{1.1}$$

where

$$q_h(\theta,t) \doteq \left|\frac{\hat{h}(\theta+t) - \hat{h}(\theta)}{\hat{h}(\theta) - \hat{h}(\theta-t)}\right|. \tag{1.2}$$

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Beurling-Ahlfors [BA] proved that $h \in \mathrm{Hom}^+(S^1)$ is quasisymmetric if and only if there exists some quasiconformal homeomorphism of Δ onto itself which has boundary values h. Later Douady-Earle [DE] gave a quasiconformal extension of h to the unit disk which is conformally invariant.

The universal Teichmüller space T is a universal parameter space for all hyperbolic Riemann surfaces and can be defined as the right coset space $T = \mathrm{QS}(S^1)/\mathrm{Möb}(S^1)$, where $\mathrm{QS}(S^1)$ denotes the group of all quasisymmetric homeomorphisms of the unit circle, and $\mathrm{Möb}(S^1)$ the subgroup of Möbius transformations of the unit disk. The universal Teichmüller space T plays a significant role in Teichmüller theory, and it is also a fundamental object in mathematics and in mathematical physics. On the other hand, several subclasses of quasisymmetric homeomorphisms and their Teichmüller spaces were introduced and studied for various purposes in the literature. We refer to the books [Ah, FM, Ga, GL, Hu, IT, Le, Na, Po2] and the papers [AZ, Cu, GS, FH, FHS1, FHS2, HS, SW, TT2, TWS, WS] for an introduction to the subject and more details. In this paper, we are mainly concerned with the so-called Weil-Petersson Teichmüller space.

It is well known that the universal Teichmüller space T has a natural complex Banach manifold structure under which the hyperbolic Kobayashi metric is the classical Teichmüller metric (see [Ga, Le, Na, Ro]), and the tangent space to T was identified by Reimann [Re] and later by Gardiner-Sullivan [GS]. Let Λ^* denote the Zygmund space in the usual sense (see [Zy]), which consists of all continuous functions u on the unit circle such that

$$u_{\Lambda^*} \doteq \sup\left\{\frac{|\hat{u}(\theta+t)-2\hat{u}(\theta)+\hat{u}(\theta-t)|}{t} : \theta \in \mathbb{R},\ t>0\right\} < +\infty. \tag{1.3}$$

Then the tangent space to T at the identity map is the set of all functions $u \in \Lambda^*$ which in addition satisfy the normalized conditions

$$\Re\bar{\eta}u(\eta) = 0, \quad \eta \in S^1 \tag{1.4}$$

and

$$u(1) = u(-1) = u(i) = 0. \tag{1.5}$$

More generally, Reimann [Re] proved, given a continuous vector field $u(t,\cdot) \in C^0([0,M],\Lambda^*)$ with the normalized condition (1.4), that the flow maps $h(t,\zeta)$ of the differential equation

$$\begin{cases} \dfrac{dh}{dt} = u(t,h) \\ h(0,\zeta) = \zeta \end{cases} \tag{1.6}$$

are quasisymmetric homeomorphisms, namely, $h(t,\cdot) \in \mathrm{QS}(S^1)$ for each fixed $t \in [0, M]$.

It is also known that the Kobayashi-Teichmüller metric on any Teichmüller space is only induced from a Finsler structure (see [Ob]) and is not a Riemannian metric in general. On the other hand, there does exist a Riemannian metric on a finite dimensional Teichmüller space, the Weil-Petersson metric, which has attracted a good bit of attention (see [Hu, IT, Mi, TT2]). In order to extend the definition of the Weil-Petersson metric to the universal Teichmüller space, Nag-Verjovsky [NV] introduced a formal formula for the Weil-Petersson metric, which converges only at those vectors on the unit circle that belong to the Sobolev space $H^{\frac{3}{2}}$, however. To overcome this difficulty, Takhtajan-Teo [TT2] endowed the universal Teichmüller space with a new complex Hilbert manifold structure, under which the Weil-Petersson metric is a convergent Riemannian metric. But, under this new complex Hilbert manifold structure, the universal Teichmüller space T is not connected and has uncountably many connected components. Nowadays, the component containing the identity map is usually called the Weil-Petersson (universal) Teichmüller space, which is denoted by T_0 in this paper. Takhtajan-Teo [TT2] proved that, under the Weil-Petersson metric, T_0 is precisely the completion of $\mathrm{Diff}_+(S^1)/\mathrm{M\ddot{o}b}(S^1)$, the space of all normalized C^∞ diffeomorphisms on the unit circle. Recall that the complex Fréchet manifold $\mathrm{Diff}_+(S^1)/\mathrm{M\ddot{o}b}(S^1)$ plays an important role in one of the approaches to non-perturbative bosonic closed string field theory based on Kähler geometry (see [BR1, BR2]), and also has an interpretation as a coadjoint orbit of the Bott-Virasoro group (see [Ki, KY]).

We say a quasi-symmetric homeomorphism h belongs to the Weil-Petersson class, which is denoted by $\mathrm{WP}(S^1)$, if it represents a point in T_0. Then $T_0 = \mathrm{WP}(S^1)/\mathrm{M\ddot{o}b}(S^1)$. It is known that a quasi-symmetric homeomorphism h belongs to $\mathrm{WP}(S^1)$ if and only if h has a quasiconformal extension f to the unit disk whose Beltrami coefficient μ satisfies the property that $\iint_\Delta |\mu(z)|^2(1-|z|^2)^{-2}dxdy < \infty$ (see [Cu, TT2]). Due to their importance and wide applications to various areas such as mathematical physics (see [BR1, BR2, Ki, KY, RSW1, RSW2, RSW3, RSW4]), differential equation and computer vision (see [GMR, GR, Ku]), the Weil-Petersson class and its Teichmüller space T_0 have been much investigated in recent years (see [Fi, GMR, GR, HS, Ku, TT1, TT2, Wu]). Recently, motivated by conformal field theory, Radnell-Schippers-Staubach [RSW1, RSW2, RSW3, RSW4] have a programm to extend the Weil-Petersson theory of the universal Teichmüller space to the case of Teichmüller spaces of bordered Riemann surfaces. Yanagishita (see [Ya1, Ya2] and also [MY]) has even dealt with the Weil-Petersson Teichmüller spaces of general Riemann surfaces with a mild geometric condition. However, it is still an open problem how to characterize intrinsically the elements in $\mathrm{WP}(S^1)$ without using quasiconformal extensions. This problem was proposed by Takhtajan-Teo in 2006 (see [TT2, p. 68]) and was investigated later by Figalli [Fi], Gay-Balmaz-Marsden-Ratiu [GMR, GR] and some others. In this paper, we will study this problem and prove the following result, which gives an intrinsic characterization of a quasisymmetric homeomorphism in the Weil-Petersson class. Recall that, for a function f defined on a set Γ, f' denotes the derivative of f, namely, for $z \in \Gamma$,

$$f'(z) \doteq \lim_{\Gamma \ni \zeta \to z} \frac{f(\zeta) - f(z)}{\zeta - z} \tag{1.7}$$

provided the limit exists, while $f'(z) \doteq 0$ otherwise.

THEOREM 1.1. *A sense-preserving homeomorphism h on the unit circle belongs to the Weil-Petersson class* $\mathrm{WP}(S^1)$ *if and only if h is absolutely continuous (with respect to the arc-length measure) and* $\log h'$ *belongs to the Sobolev class* $H^{\frac{1}{2}}$.

Theorem 1.1 has several applications which we proceed to explore. It is known that T_0 is modeled on the Sobolev space $H^{\frac{3}{2}}$, namely, the tangent space to T_0 at the identity consists of precisely the $H^{\frac{3}{2}}$ vector fields on the unit circle with the normalized conditions (1.2) and (1.3) (see [NV, TT2]). Recall that when $s > \frac{3}{2}$ the group $\mathrm{Diff}^s_+(S^1)$ of all orientation preserving H^s diffeomorphisms of the unit circle and its model space H^s have the same Sobolev H^s regularity. An important question is whether the same result holds in the critical case $s = \frac{3}{2}$, namely, whether an element in $\mathrm{WP}(S^1)$ also has $H^{\frac{3}{2}}$-regularity (see [Fi, GMR, GR]). In fact, based on the results by Figalli [Fi], Gay-Balmaz-Marsden-Ratiu [GMR, GR] were able to prove that each homeomorphism in $\mathrm{WP}(S^1)$ belongs to $H^{\frac{3}{2}-\epsilon}$ for each $\epsilon > 0$. However, we shall prove that the $H^{\frac{3}{2}}$-regularity may fail for a quasisymmetric homeomorphism in the Weil-Petersson class, which was conjectured very recently by Gay-Balmaz-Ratiu during their study of the Euler-Weil-Petersson equation (see [GR, Conjecture (2), p. 760]).

THEOREM 1.2. *There exists a quasisymmetric homeomorphism in* $\mathrm{WP}(S^1)$ *which belongs neither to the Sobolev class $H^{\frac{3}{2}}$ nor to the Lipschitz class Λ^1.*

We will also deal with the flows of $H^{\frac{3}{2}}$ vector fields on the unit circle. It is easy to see that the Weil-Petersson class $\mathrm{WP}(S^1)$ can be generated by the flows of the $H^{\frac{3}{2}}$ vector fields on the unit circle (see [GMR, GR]). However, it is still an open problem whether or not the flows of the $H^{\frac{3}{2}}$ vector fields are contained in $\mathrm{WP}(S^1)$, although it is hoped to be so (see [Fi]). Actually, in the recent paper [GR] by Gay-Balmaz-Ratiu, the authors conjectured that the flows of the $H^{\frac{3}{2}}$ vector fields are contained in $\mathrm{WP}(S^1)$ (see [GR, p. 760, Conjecture], and also Conjecture 9.2 below). The following result provides an affirmative answer to this problem. A more precise statement will be given in Theorem 7.3 below.

THEOREM 1.3. *The flows of the $H^{\frac{3}{2}}$ vector fields on the unit circle are always contained in* $\mathrm{WP}(S^1)$.

Theorem 1.1 is also hoped to be useful to the further study of the geometry and structure of T_0. As we shall see later (see Remark 5.1 below), $\mathrm{WP}(S^1)/\mathrm{Rot}(S^1)$ has a very simple model, namely, it can be identified as the real Hilbert space $H_{\mathbb{R}}^{\frac{1}{2}}/\mathbb{R}$ under the bijection $h \mapsto \log|h'|$. Here and in what follows, $\mathrm{Rot}(S^1)$ denotes the group of all rotations about the circle S^1. Based on this observation, we will introduce a new metric on T_0, which can be defined roughly as follows:

$$d(h_1, h_2) \doteq \|\log|h_1'| - \log|h_2'|\|_{H^{\frac{1}{2}}}, \quad h_1, h_2 \in T_0. \tag{1.8}$$

A precise formula will be given below during the proof of Theorem 1.4 (see (8.5) below). The advantage of this metric is that, as being a global metric, it gives directly the distances between two points in T_0. This is in contrast to the case for the Weil-Petersson metric, which is an infinitesimal Riemann metric on the tangent bundle of T_0. Anyhow, we shall prove

THEOREM 1.4. *The metric d and the Weil-Petersson metric induce the same topology on T_0.*

We end this Introduction section with the organization of the paper. In Section 2, we give some basic definitions and results on the universal Teichmüller space and the Weil-Petersson Teichmüller space. In particular, we establish the complex analytic theory of the pre-logarithmic derivative model of the Weil-Petersson Teichmüller space, which plays an important role in the proof of Theorem 1.4. As we shall see later, our results and their proofs involve much use of the theory of function spaces, in particular, of Sobolev spaces of fractional order. Therefore, in Sections 3 and 4, we recall some basic definitions on Sobolev spaces, the BMO space and establish some lemmas that will be frequently used in the proof of Theorems 1.1 and 1.2; we also deal with the pull-back operator on $H^{\frac{1}{2}}$ by a quasisymmetric homeomorphism and establish several basic results which are needed to prove Theorems 1.1 and 1.3. In Sections 5–8, we give the proofs of Theorems 1.1–1.4. In Section 9, we list several open problems related to this work. In the final Appendix section, we prove Propositions 4.1 and 4.3 stated in Section 4.

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2. Preliminary results on the Weil-Petersson Teichmüller space. In this section, we give some basic definitions and results on the Weil-Petersson Teichmüller space. The results turn out to be essential in the proof of Theorem 1.4. We follow the lines in our recent paper [SW], where the BMO theory of the universal Teichmüller space was investigated.

We begin with the standard theory of the universal Teichmüller space (see [Ah, Le, Na]). Let $M(\Delta^*)$ denote the open unit ball of the Banach space $L^\infty(\Delta^*)$ of essentially bounded measurable functions on Δ^*. For $\mu \in M(\Delta^*)$, let f_μ be the quasiconformal mapping on the extended plane $\hat{\mathbb{C}}$ with complex dilatation equal to μ in Δ^*, conformal in Δ, normalized by $f_\mu(0) = f'_\mu(0) - 1 = f''_\mu(0) = 0$. We say two elements μ and ν in $M(\Delta^*)$ are equivalent, denoted by $\mu \sim \nu$, if $f_\mu|_\Delta = f_\nu|_\Delta$. Then $T = M(\Delta^*)/\sim$ is the Bers model of the universal Teichmüller space. We let Φ denote the natural projection from $M(\Delta^*)$ onto T so that $\Phi(\mu)$ is the equivalence class $[\mu]$. $[0]$ is called the base point of T. The Teichmüller distance between two points $[\mu_1]$ and $[\mu_2]$ in T is defined as

$$\tau([\mu_1],[\mu_2]) \doteq \inf\left\{\frac{1}{2}\log\frac{1+\|\frac{\nu_1-\nu_2}{1-\overline{\nu}_1\nu_2}\|_\infty}{1-\|\frac{\nu_1-\nu_2}{1-\overline{\nu}_1\nu_2}\|_\infty} : [\nu_1]=[\mu_1],[\nu_2]=[\mu_2]\right\}. \tag{2.1}$$

Let $B_2(\Delta)$ denote the Banach space of functions ϕ holomorphic in Δ with norm

$$\|\phi\|_{B_2} \doteq \sup_{z\in\Delta}(1-|z|^2)^2|\phi(z)|. \tag{2.2}$$

Consider the map $S : M(\Delta^*) \to B_2(\Delta)$ which sends μ to the Schwarzian derivative of $f_\mu|_\Delta$. Recall that for any locally univalent function f, its Schwarzian derivative S_f is defined by

$$S_f = N'_f - \frac{1}{2}N_f^2, \quad N_f = (\log f')'. \tag{2.3}$$

It is known that S is a holomorphic split submersion onto its image, which descends down to a map $\beta : T \to B_2(\Delta)$ known as the Bers embedding. Via the Bers embedding, T carries a natural complex Banach manifold structure so that Φ is a holomorphic split submersion.

We proceed to define the Weil-Petersson Teichmüller space (For details, see [TT2] and also [Cu]). We denote by $\mathcal{L}(\Delta^*)$ the Banach space of all essentially bounded measurable functions μ with norm

$$\|\mu\|_{\mathrm{WP}} \doteq \|\mu\|_\infty + \left(\frac{1}{\pi}\iint_{\Delta^*}\frac{|\mu(z)|^2}{(|z|^2-1)^2}dxdy\right)^{\frac{1}{2}}. \tag{2.4}$$

Set $\mathcal{M}(\Delta^*) = M(\Delta^*) \cap \mathcal{L}(\Delta^*)$. Then $T_0 = \mathcal{M}(\Delta^*)/_\sim$ is one of the models of the Weil-Petersson Teichmüller space. Actually, T_0 is the base point component of the universal Teichmüller space under the complex Hilbert manifold structure introduced by Takhtajan-Teo [TT2].

We denote by $\mathcal{B}(\Delta)$ the Banach space of functions ϕ holomorphic in Δ with norm

$$\|\phi\|_{\mathcal{B}} \doteq \left(\frac{1}{\pi}\iint_\Delta |\phi(z)|^2(1-|z|^2)^2dxdy\right)^{\frac{1}{2}}. \tag{2.5}$$

Then, $\mathcal{B}(\Delta) \subset B_2(\Delta)$, and the inclusion map is continuous. Under the Bers projection $S : M(\Delta^*) \to B_2(\Delta)$, $S(\mathcal{M}(\Delta^*)) = S(M(\Delta^*)) \cap \mathcal{B}(\Delta)$ (see [Cu, TT2]). Moreover, we have

PROPOSITION 2.1. [TT2] $S : \mathcal{M}(\Delta^*) \to \mathcal{B}(\Delta)$ *is a holomorphic split submersion from* $\mathcal{M}(\Delta^*)$ *onto its image. Consequently,* T_0 *has a unique complex Hilbert manifold structure such that* $\beta : T_0 \to \mathcal{B}(\Delta)$ *is a bi-holomorphic map from* T_0 *onto a domain in* $\mathcal{B}(\Delta)$*. Under this complex Hilbert manifold structure, the natural projection* Φ *from* $\mathcal{M}(\Delta^*)$ *onto* T_0 *is a holomorphic split submersion.*

It is well known that a quasiconformal self-mapping of Δ^* induces a biholomorphic automorphism of the universal Teichmüller space (see [Le, Na]). Precisely, let $w : \Delta^* \to \Delta^*$ be a quasiconformal mapping with complex dilatation μ. Then w induces an bi-holomorphic isomorphism $R_w : M(\Delta^*) \to M(\Delta^*)$ as

$$R_w(\nu) = \left(\frac{\nu - \mu}{1 - \bar{\mu}\nu} \frac{\partial w}{\overline{\partial w}} \right) \circ w^{-1}. \tag{2.6}$$

R_w descends down a bi-holomorphic isomorphism $w^* : T \to T$ by $w^* \circ \Phi = \Phi \circ R_w$.

PROPOSITION 2.2. *Suppose* w *is quasi-isometric under the Poincaré metric with Beltrami coefficient* $\mu \in \mathcal{M}(\Delta^*)$*. Then* $w^* : T_0 \to T_0$ *is bi-holomorphic.*

Proof. Clearly, R_w maps $\mathcal{M}(\Delta^*)$ into itself, and $R_w : \mathcal{M}(\Delta^*) \to \mathcal{M}(\Delta^*)$ is bi-holomorphic. It follows from Proposition 2.1 that $w^* : T_0 \to T_0$ is biholomorphic. □

We continue to consider the pre-logarithmic derivative model of the Weil-Petersson Teichmüller space. Let $B(\Delta)$ denote the space of functions ϕ holomorphic in Δ with semi-norm

$$\|\phi\|_B \doteq \sup_{z \in \Delta} (1 - |z|^2)|\phi'(z)|, \tag{2.7}$$

and $B_0(\Delta)$ the subspace of $B(\Delta)$ which consists of those functions ϕ satisfying the condition $\lim_{|z| \to 1}(1 - |z|^2)\phi'(z) = 0$. Recall that the pre-logarithmic derivative model $\hat{T}$ of the universal Teichmüller space consists of all functions $\log f'$ (in $B(\Delta)$), where f belongs to the well-known class S_Q of all univalent analytic functions f in the unit disk Δ with the normalized condition $f(0) = f'(0) - 1 = 0$ that can be extended to a quasiconformal mapping in the whole plane (see [AG, Zhu]). Under the topology of Bloch norm (2.7), $\hat{T}$ is a disconnected open set. Precisely, $\hat{T} = \hat{T}_b \cup_{\theta \in [0, 2\pi)} \hat{T}_\theta$, where $\hat{T}_b = \{\log f' : f \in S_Q \text{ is bounded}\}$ and $\hat{T}_\theta = \{\log f' : f \in S_Q \text{ satisfies } f(e^{i\theta}) = \infty\}$, $\theta \in [0, 2\pi)$, are the all connected components of $\hat{T}$ (see [Zhu]). Each $\hat{T}_\theta$ is a copy of the Bers model T, while $\hat{T}_b$ is a fiber space over T. In fact, $\hat{T}_b$ is a model of the universal Teichmüller curve (see [Ber, Te]).

Let $\mathcal{AD}(\Delta)$ denote the space of all functions ϕ holomorphic in Δ with seminorm

$$\|\phi\|_{\mathcal{AD}} \doteq \left(\frac{1}{\pi}\iint_{\Delta}|\phi'(z)|^2dxdy\right)^{\frac{1}{2}}, \tag{2.8}$$

and $\mathcal{AD}_0(\Delta)=\{\phi\in\mathcal{AD}(\Delta):\phi(0)=0\}$. Then $\mathcal{AD}(\Delta)\subset B_0(\Delta)$, and the inclusion map is continuous. We may define $\mathcal{AD}(\Delta^*)$ similarly. Define

$$\Lambda(\phi)=\phi''-\frac{1}{2}(\phi')^2,\quad \phi\in\mathcal{AD}(\Delta). \tag{2.9}$$

Then the following basic result holds.

LEMMA 2.3. [TT2] $\Lambda:\mathcal{AD}(\Delta)\to\mathcal{B}(\Delta)$ *is holomorphic.*

We come back our situation. Fix $z_0\in\Delta^*$. For $\mu\in M(\Delta^*)$, let $g_\mu^{z_0}$ (abbreviated to be g_μ) be the quasiconformal mapping on the extended plane $\hat{\mathbb{C}}$ with complex dilatation equal to μ in Δ^*, conformal in Δ, normalized by $g_\mu(0)=g_\mu'(0)-1=0$, $g_\mu(z_0)=\infty$. Then $\mu\sim\nu$ if and only if $g_\mu|_\Delta=g_\nu|_\Delta$. Consider the map L_{z_0} on $M(\Delta^*)$ by setting $L_{z_0}(\mu)=\log g_\mu'$. Then $\cup_{z_0\in\Delta^*}L_{z_0}(M(\Delta^*))=\hat{T}_b$, and $\cup_{z_0\in\Delta^*}L_{z_0}(\mathcal{M}(\Delta^*))=\hat{T}_b\cap\mathcal{AD}_0(\Delta)$ (see [Cu, TT2]). We have the following result.

THEOREM 2.4. *For each* $z_0\in\Delta^*$, $L_{z_0}:\mathcal{M}(\Delta^*)\to\mathcal{AD}_0(\Delta)$ *is holomorphic.*

Proof. We first show $L=L_{z_0}:\mathcal{M}(\Delta^*)\to\mathcal{AD}_0(\Delta)$ is continuous. Recall that L is continuous on $M(\Delta^*)$ in the topology of Bloch norm (2.7) (see [Le]), namely,

$$\sup_{z\in\Delta}|N_{g_\nu}(z)-N_{g_\mu}(z)|(1-|z|^2)\le C(\|\mu\|_\infty)\|\nu-\mu\|_\infty,\quad \mu,\nu\in M(\Delta^*). \tag{2.10}$$

Then,

$$\begin{aligned}
&\|L(\nu)-L(\mu)\|_{\mathcal{AD}}^2\\
&=\frac{1}{\pi}\iint_\Delta|N_{g_\nu}(z)-N_{g_\mu}(z)|^2dxdy\\
&\le C_1\left(|N_{g_\nu}(0)-N_{g_\mu}(0)|^2+\iint_\Delta(1-|z|^2)^2|N'_{g_\nu}(z)-N'_{g_\mu}(z)|^2dxdy\right)\\
&\le C_2\left(\|\nu-\mu\|_\infty^2+\iint_\Delta\left(1-|z|^2\right)^2\left|\left(S_\nu(z)-S_\mu(z)\right)+\frac{1}{2}\left(N_{g_\nu}^2(z)-N_{g_\mu}^2(z)\right)\right|^2dxdy\right)\\
&\le C_3(\|\nu-\mu\|_\infty^2+\|S(\nu)-S(\mu)\|_{\mathcal{B}}^2+(\|L(\nu)\|_{\mathcal{AD}}^2+\|L(\mu)\|_{\mathcal{AD}}^2)\|\nu-\mu\|_\infty^2).
\end{aligned}$$

By the holomorphy of $S:\mathcal{M}(\Delta^*)\to\mathcal{B}(\Delta)$, we conclude that $L:\mathcal{M}(\Delta^*)\to\mathcal{AD}_0(\Delta)$ is continuous.

Since $L : \mathcal{M}(\Delta^*) \to \mathcal{AD}_0(\Delta)$ is continuous, we conclude that L is holomorphic by the infinite dimensional holomorphy (see [Le, Na]). For completeness, we write down the standard proof. For each $z \in \Delta$, define $l_z(\phi) = \phi(z)$ for $\phi \in \mathcal{AD}_0(\Delta)$. Then, $l_z \in \mathcal{AD}_0^*(\Delta)$, that is, l_z is a continuous linear functional on the Banach space $\mathcal{AD}_0(\Delta)$. Set $A = \{l_z : z \in \Delta\}$. A is a total subset of $\mathcal{AD}_0^*(\Delta)$ in the sense that $l_z(\phi) = 0$ for all $z \in \Delta$ implies that $\phi = 0$. Now for each $z \in \Delta$, each pair $(\mu, \nu) \in \mathcal{M}(\Delta^*) \times \mathcal{L}(\Delta^*)$ and small t in the complex plane, by the well-known holomorphic dependence of quasiconformal mappings on parameters (see [Ah, Le, Na]), we conclude that $l_z(L(\mu + t\nu)) = L(\mu + t\nu)(z)$ is a holomorphic function of t. By a general result about the infinite dimensional holomorphy (see [Le, Na]), it follows that $L : \mathcal{M}(\Delta^*) \to \mathcal{AD}_0(\Delta)$ is holomorphic. □

THEOREM 2.5. *$\hat{T}_b \cap \mathcal{AD}_0(\Delta)$ is a connected open subset of $\mathcal{AD}_0(\Delta)$, and Λ is a holomorphic split submersion from $\hat{T}_b \cap \mathcal{AD}_0(\Delta)$ onto $\beta(T_0)$.*

Proof. Clearly, $\hat{T}_b \cap \mathcal{AD}_0(\Delta)$ is an open subset of $\mathcal{AD}_0(\Delta)$. We need to show that each point of $\hat{T}_b \cap \mathcal{AD}_0(\Delta)$ can be connected to 0 by a path in $\hat{T}_b \cap \mathcal{AD}_0(\Delta)$.

Let $\log f' \in \hat{T}_b \cap \mathcal{AD}_0(\Delta)$. Then f can be extended to a quasiconformal mapping in the whole plane whose Beltrami coefficient μ belongs to $\mathcal{M}(\Delta^*)$, and $z_0 = f^{-1}(\infty) \in \Delta^*$. For each $t \in [0,1]$, let $f_t \in S_Q$ be the unique mapping whose quasiconformal extension to the whole plane has Beltrami coefficient $t\mu$, and $f_t(z_0) = \infty$. Theorem 2.4 implies that $\log f_t'$, $t \in [0,1]$, is a path in $\hat{T}_b \cap \mathcal{AD}_0(\Delta)$ joining $\log f_0'$ to $\log f'$. Now, if $z_0 = \infty$, then $f_0(z) = z$, and we are done. If $z_0 \neq \infty$, then $f_0(z) = z_0 z/(z_0 - z)$, and $\log f_0'(r\cdot)$, $r \in [0,1]$, is a curve in $\hat{T}_b \cap \mathcal{AD}_0(\Delta)$ connecting 0 and $\log f_0'$.

Clearly, Lemma 2.3 implies that Λ is holomorphic on $\hat{T}_b \cap \mathcal{AD}_0(\Delta)$. Choose $z_0 \in \Delta^*$. Since $S = \Lambda \circ L_{z_0}$, we conclude that Λ is a holomorphic split submersion from $\hat{T}_b \cap \mathcal{AD}_0(\Delta)$ onto $\beta(T_0)$ since $L_{z_0} : \mathcal{M}(\Delta^*) \to \hat{T}_b \cap \mathcal{AD}_0(\Delta)$ is holomorphic, and $S : \mathcal{M}(\Delta^*) \to \beta(T_0)$ is a holomorphic split submersion. □

3. Some lemmas. In this section, we give some lemmas needed to prove Theorems 1.1 and 1.2. First we recall some basic definitions and results on Sobolev spaces, the harmonic Dirichlet space and the BMO space that will be frequently used in the rest of the paper (see [Gar, RS, Tr]).

For any $s > 0$, the Sobolev space H^s consists of all integrable functions $u \in L^1(S^1)$ on the unit circle with semi-norm

$$\|u\|_{H^s} \doteq \left(\sum_{n=-\infty}^{+\infty} |n|^{2s} |a_n(u)|^2 \right)^{\frac{1}{2}}, \tag{3.1}$$

where, as usual, $a_n(u)$ is the n-th Fourier coefficient of u, namely,

$$a_n(u) = \frac{1}{2\pi} \int_0^{2\pi} \hat{u}(\theta) e^{-in\theta} d\theta. \tag{3.2}$$

In this paper, the two cases we are concerned are $s = \frac{3}{2}$ and $s = \frac{1}{2}$. Recall that $u \in H^{\frac{3}{2}}$ if and only if u is absolutely continuous with $u' \in H^{\frac{1}{2}}$. It is also known that an integrable function u on the unit circle belongs to $H^{\frac{1}{2}}$ if and only if

$$\int_0^{2\pi}\int_0^{2\pi} \frac{|\hat{u}(s)-\hat{u}(t)|^2}{|\sin((s-t)/2)|^2} dsdt < +\infty. \tag{3.3}$$

We need another description of the space $H^{\frac{1}{2}}$. Let $\mathcal{D}(\Delta)$ denote the space of all harmonic functions u in the unit disk Δ with semi-norm

$$\|u\|_{\mathcal{D}} \doteq \left(\frac{1}{\pi}\iint_{\Delta}(|\partial_z u|^2 + |\partial_{\bar{z}} u|^2)dxdy\right)^{\frac{1}{2}}. \tag{3.4}$$

Then, $\mathcal{D}(\Delta) = \mathcal{AD}(\Delta) \oplus \overline{\mathcal{AD}(\Delta)}$, or precisely, for each $u \in \mathcal{D}(\Delta)$, there exists a unique pair of holomorphic functions ϕ and ψ in $\mathcal{AD}(\Delta)$ with $\phi(0) - u(0) = \psi(0) = 0$ such that $u = \phi + \overline{\psi}$. Here it is a convenient place to introduce two basic operators on the Dirichlet space $\mathcal{D}(\Delta)$. They are P^+ and P^-, defined respectively by $P^+u = \phi$ and $P^-u = \overline{\psi(\bar{z})}$ for $u = \phi + \overline{\psi}$. It is well known that each function $u \in \mathcal{D}(\Delta)$ has boundary values almost everywhere on the unit circle, and the boundary function, still denoted by u, belongs to $H^{\frac{1}{2}}$, and conversely each function in $H^{\frac{1}{2}}$ is obtained in this way (see [Zy]). In fact, the usual Poisson integral operator P establishes a one-to-one map from $H^{\frac{1}{2}}$ onto $\mathcal{D}(\Delta)$ with $\|Pu\|_{\mathcal{D}} = \|u\|_{H^{\frac{1}{2}}}$.

Let I_0 be a connected (closed) arc on the unit circle S^1. An integrable function $u \in L^1(I_0)$ is said to have bounded mean oscillation if

$$\|u\|_{\mathrm{BMO}(I_0)} \doteq \sup \frac{1}{|I|}\int_I |u(z) - u_I||dz| < +\infty, \tag{3.5}$$

where the supremum is taken over all sub-intervals I of I_0, while u_I is the average of u on the interval I, namely,

$$u_I = \frac{1}{|I|}\int_I u(z)|dz|. \tag{3.6}$$

In particular, $u_{S^1} = a_0(u)$. If u also satisfies the condition

$$\lim_{|I|\to 0}\frac{1}{|I|}\int_I |u(z) - u_I||dz| = 0, \tag{3.7}$$

we say u has vanishing mean oscillation. These functions are denoted by $\mathrm{BMO}(I_0)$ and $\mathrm{VMO}(I_0)$, respectively. In the following, we are mostly concerned with the case $I_0 = S^1$. Then it is well known that $H^{\frac{1}{2}} \subset \mathrm{VMO}(S^1)$, and the inclusion map is continuous (see [Zh]).

We need some basic results on BMO functions. By the well-known theorem of John-Nirenberg for BMO functions (see [Gar]), there exist two universal positive constants C_1 and C_2 such that for any $\mathrm{BMO}(I_0)$ function u, any subinterval I of I_0 and any $\lambda > 0$, it holds that

$$\frac{|\{z \in I : |u(z) - u_I| \geq \lambda\}|}{|I|} \leq C_1 \exp\left(\frac{-C_2\lambda}{\|u\|_{\mathrm{BMO}(I_0)}}\right). \tag{3.8}$$

For any $p \geq 1$, by Chebychev's inequality, we have

$$\begin{aligned}\frac{1}{|I|}\int_I \left(e^{|u(z)-u_I|} - 1\right)^p |dz| &= \frac{1}{|I|}\int_0^\infty |\{z \in I : |u(z) - u_I| \geq \lambda\}|\, d\left((e^\lambda - 1)^p\right) \\ &\leq pC_1 \int_0^\infty (e^\lambda - 1)^{p-1} e^\lambda \exp\left(\frac{-C_2\lambda}{\|u\|_{\mathrm{BMO}(I_0)}}\right) d\lambda.\end{aligned}$$

When $p\|u\|_{\mathrm{BMO}(I_0)} < C_2$, we obtain

$$\frac{1}{|I|}\int_I \left(e^{|u(z)-u_I|} - 1\right)^p |dz| \leq \frac{pC_1\|u\|_{\mathrm{BMO}(I_0)}}{C_2 - p\|u\|_{\mathrm{BMO}(I_0)}}. \tag{3.9}$$

We will repeatedly use the following basic result:

LEMMA 3.1. *Let $u \in \mathrm{BMO}(I_0)$ and $p \geq 1$. Then $e^u \in L^p(I_0)$ when $p\|u\|_{\mathrm{BMO}(I_0)}$ is small. In particular, if $u \in \mathrm{VMO}(I_0)$, then $e^u \in L^p(I_0)$ for any real number $p \geq 1$.*

Proof. When $p\|u\|_{\mathrm{BMO}(I_0)} < C_2$, it follows from (3.9) that

$$\frac{1}{|I_0|}\left\|e^{u-u_{I_0}} - 1\right\|_p^p = \frac{1}{|I_0|}\int_{I_0} |e^{u(z)-u_{I_0}} - 1|^p |dz| \leq \frac{pC_1\|u\|_{\mathrm{BMO}(I_0)}}{C_2 - p\|u\|_{\mathrm{BMO}(I_0)}}. \tag{3.10}$$

Consequently,

$$\|e^u\|_p \leq e^{\|u\|_1}(\|e^{u-u_{I_0}} - 1\|_p + |I_0|^{\frac{1}{p}}) < +\infty.$$

Now suppose $u \in \mathrm{VMO}(I_0)$, and $p \geq 1$ is any real number. By (3.7), for any sufficiently small subinterval I of I_0, u has small BMO norm on I so that $e^u \in L^p(I)$. Decompose I_0 as the union of finitely many small subintervals I_j so that $e^u \in L^p(I_j)$, we conclude that $e^u \in L^p(I_0)$ as required. □

LEMMA 3.2. *Let $u \in \mathrm{VMO}(S^1)$ and $u_n \in \mathrm{BMO}(S^1)$ on the unit circle. Suppose that $\|u_n - u\|_{\mathrm{BMO}(S^1)} \to 0$ and $a_0(u_n - u) \to 0$ when $n \to \infty$, then for any $p \geq 1$, we have $\|e^{u_n} - e^u\|_p \to 0$ as $n \to \infty$.*

Proof. By (3.10),

$$\left\|e^{(u_n-u)-a_0(u_n-u)} - 1\right\|_{2p}^{2p} \leq \frac{2pC_1\|u_n - u\|_{\mathrm{BMO}(S^1)}}{C_2 - 2p\|u_n - u\|_{\mathrm{BMO}(S^1)}} \longrightarrow 0, \quad n \longrightarrow \infty.$$

On the other hand, since $u \in \mathrm{VMO}(S^1)$, Lemma 3.1 implies that $e^u \in L^{2p}(S^1)$. Consequently,

$$\begin{aligned}\|e^{u_n} - e^u\|_p &\le \left\|e^{u_n - u} - 1\right\|_{2p} \|e^u\|_{2p} \\ &\le \|e^u\|_{2p} \left(e^{a_0(u_n-u)} \|e^{(u_n-u)-a_0(u_n-u)} - 1\|_{2p} + \|e^{a_0(u_n-u)} - 1\|_{2p} \right),\end{aligned}$$

which implies $\|e^{u_n} - e^u\|_p \to 0$ as $n \to \infty$. □

Recall that for each sense-preserving homeomorphisms h of the unit circle onto itself, there exists some strictly increasing continuous function ϕ on the real line with $\phi(\theta + 2\pi) - \phi(\theta) \equiv 2\pi$ such that $h(e^{i\theta}) = e^{i\phi(\theta)}$. Then

$$h'(e^{i\theta}) = e^{i(\phi(\theta)-\theta)}\phi'(\theta). \tag{3.11}$$

Furthermore, h is absolutely continuous on the unit circle if and only if ϕ is absolutely continuous on the real line.

Note that in the statement of Theorem 1.1, the quasi-symmetry of the homeomorphism h is not assumed. The following result gives a sufficient condition for an absolutely continuous sense-preserving homeomorphism to be quasisymmetric, which will be used in the proof of Theorem 1.1.

LEMMA 3.3. *Let h be an absolutely continuous sense-preserving homeomorphism on the unit circle such that* $\log h' \in \mathrm{VMO}(S^1)$. *Then h is a quasisymmetric homeomorphism.*

Proof. Partyka (see [Pa1, Theorem 3.4.7]) asserted that h is actually a symmetric homeomorphism in the sense of Gardiner-Sullivan [GS], namely, for any pair of adjacent subintervals I_1 and I_2 in S^1 with $|I_1| = |I_2|$, it holds that

$$\frac{|h(I_1)|}{|h(I_2)|} = 1 + o(1), \quad |I_1| = |I_2| \to 0+. \tag{3.12}$$

A detailed proof of this fact was given in [Pa2]. Here we give a fast proof for completeness.

Set $v = \log|h'|$ for simplicity. Then $v \in \mathrm{VMO}(S^1)$. For any small subinterval I in S^1 such that the BMO-norm of v on I is small, we conclude by (3.9) (with $p = 1$) that

(3.13)
$$\int_I e^{|v(z)-v_I|}|dz| \le |I|\left(1 + \frac{C_1\|v\|_{\mathrm{BMO}(I)}}{C_2 - \|v\|_{\mathrm{BMO}(I)}}\right) = |I|(1+o(1)), \quad |I| \longrightarrow 0.$$

Noting that

$$|h(I)| = \int_I |h'(z)||dz| = \int_I e^{v(z)}|dz| = e^{v_I}\int_I e^{v(z)-v_I}|dz|,$$

we obtain from (3.13) that, as $|I| \to 0$,

$$|h(I)| \le e^{v_I} \int_I e^{|v(z)-v_I|}|dz| \le |I| e^{v_I}(1+o(1)),$$
$$|h(I)| \ge e^{v_I} \int_I e^{-|v(z)-v_I|}|dz| \ge \frac{|I|^2 e^{v_I}}{\int_I e^{|v(z)-v_I|}|dz|} \ge |I| e^{v_I}(1+o(1)),$$

and so

$$|h(I)| = |I| e^{v_I}(1+o(1)), \quad |I| \longrightarrow 0. \tag{3.14}$$

Now let I_1 and I_2 be two adjacent subintervals in $[0, 2\pi]$ with $|I_1| = |I_2| = l$ being small such that the BMO-norm of v on $I_1 \cup I_2$ is small. It holds that (see [Gar, (1.3) in Chapter VI])

$$|v_{I_1} - v_{I_2}| = 2|v_{I_1} - v_{I_1 \cup I_2}| \le 4\|v\|_{\mathrm{BMO}(I_1 \cup I_2)} = o(1), \quad l \longrightarrow 0+. \tag{3.15}$$

Then (3.12) follows from (3.14)–(3.15) immediately. □

LEMMA 3.4. *Let h be an absolutely continuous sense-preserving homeomorphism on the unit circle. Then $\log h' \in H^{\frac{1}{2}}$ if and only if $\log|h'| \in H^{\frac{1}{2}}$.*

Proof. Let $h(e^{i\theta}) = e^{i\phi(\theta)}$ as before. Without loss of generality, we assume that $h(1) = 1$ so that $\phi(0) = 0$, $\phi(2\pi) = 2\pi$. Then $|h'(e^{i\theta})| = \phi'(\theta)$, and

$$\log h'(e^{i\theta}) = \log|h'(e^{i\theta})| + i(\phi(\theta) - \theta). \tag{3.16}$$

It is clear that $\log|h'| \in H^{\frac{1}{2}}$ if $\log h' \in H^{\frac{1}{2}}$.

Conversely, we suppose that $\log|h'| \in H^{\frac{1}{2}}$. Set $u = \Im \log h'$ so that $\hat{u}(\theta) = \phi(\theta) - \theta$. We will show that $u \in H^1$, which implies that $\log h' \in H^{\frac{1}{2}}$. In fact, the n-th ($n \ne 0$) Fourier coefficient of u is

$$a_n = \frac{1}{2\pi} \int_0^{2\pi} \hat{u}(\theta) e^{-in\theta} d\theta = \frac{1}{2\pi} \int_0^{2\pi} (\phi(\theta) - \theta) e^{-in\theta} d\theta$$
$$= \frac{1}{2n\pi i} \int_0^{2\pi} (\phi'(\theta) - 1) e^{-in\theta} d\theta = \frac{1}{2n\pi i} \int_0^{2\pi} (|h'(e^{i\theta})| - 1) e^{-in\theta} d\theta.$$

Thus, by Parseval's equality, we conclude by Lemma 3.1 that

$$\sum_{n \ne 0} n^2 |a_n|^2 = \frac{1}{4\pi^2} \sum_{n \ne 0} \left| \int_0^{2\pi} \big(|h'(e^{i\theta})| - 1\big) e^{-in\theta} d\theta \right|^2 = \||h'| - 1\|_2^2 < +\infty.$$

This completes the proof. □

4. Pull-back operator revisited. In this section, we deal with the pull-back operator on the Sobolev space $H^{\frac{1}{2}}$ (and also on the Dirichlet space $\mathcal{D}(\Delta)$) by a quasisymmetric homeomorphism. The results will be used in the following sections to prove Theorems 1.1 and 1.3 and have independent interests of their own.

Let h be a quasisymmetric homeomorphism. Then h induces a pull-back operator by

$$P_h u = u \circ h, \quad u \in H^{\frac{1}{2}}. \tag{4.1}$$

P_h is a bounded isomorphism from $H^{\frac{1}{2}}$ onto itself with $P_h^{-1} = P_{h^{-1}}$. By the well-known quasi-invariance of Dirichlet integral under quasiconformal mappings, we have

$$\|P_h\| \doteq \sup\{\|P_h u\|_{H^{\frac{1}{2}}} : \|u\|_{H^{\frac{1}{2}}} = 1\} \leq e^{\tau(0,h)}. \tag{4.2}$$

This operator has played an important role in the study of Teichmüller theory (see [HS, NS, Pa1, SW, TT2]). As stated in the introduction, the universal Teichmüller space has a quasisymmetric homeomorphism model, namely, $T = \mathrm{QS}(S^1)/\mathrm{M\ddot{o}b}(S^1)$. Nag-Sullivan [NS] proved that the universal Teichmüller space T can be embedded in the universal Siegel period matrix space by means of the operator P_h (see also [TT2]). Notice that P_h (or more precisely, $P \circ P_h$, the composition of P_h with the Poisson integral operator P) is also a bounded isomorphism from $\mathcal{D}(\Delta)$ onto itself, and $P_h^{-1} = P_{h^{-1}}$.

We will need the following result.

PROPOSITION 4.1. *Let h and h_0 be quasisymmetric homeomorphisms which keep the points 1, -1 and i fixed. Then for each fixed $u \in H^{\frac{1}{2}}$, $\|P_h u - P_{h_0} u\|_{H^{\frac{1}{2}}} \to 0$ when $\tau(h, h_0) \to 0$.*

As far as the author know, Proposition 4.1 is not available in the literature. We will prove it in the final Appendix section. A natural question to ask is:

QUESTION 4.2. *Under the assumption of Proposition 4.1, is it true that $\|P_h - P_{h_0}\| \to 0$ when $\tau(h, h_0) \to 0$?*

We proceed to investigate the pull back operator P_h induced by a quasisymmetric homeomorphism. When restricted to $\mathcal{AD}(\Delta)$, P_h (more precisely, $P \circ P_h$) is a bounded operator from $\mathcal{AD}(\Delta)$ into $\mathcal{D}(\Delta)$. So we may define two further operators $P_h^+ = P^+ \circ P_h$ and $P_h^- = P^- \circ P_h$. Both P_h^+ and P_h^- are bounded operators from $\mathcal{AD}(\Delta)$ into itself. For completeness, we recall that P_h^- is a compact operator if and only if h is symmetric, while P_h^- is a Hilbert-Schmidt operator if and only if h belongs to the Weil-Petersson class $\mathrm{WP}(S^1)$ (see [HS]). We will not use this result in this paper.

The following result will play an important role in the proof of Theorem 1.1.

PROPOSITION 4.3. *P_h^+ is a bounded isomorphism from $\mathcal{AD}(\Delta)$ onto itself. Moreover, it holds that*

$$\|P_h^+\phi\|_{\mathcal{AD}}^2 = \|\phi\|_{\mathcal{AD}}^2 + \|P_h^-\phi\|_{\mathcal{AD}}^2, \quad \phi \in \mathcal{AD}(\Delta). \tag{4.3}$$

Proposition 4.3 may be a known result, but, to the best of the author's knowledge, a proof does not appear in the literature. We will give the proof in the final Appendix section.

We now establish a technical result used to prove Theorem 1.1. We consider the harmonic conjugation operator H in the usual sense. Precisely, for a real valued integrable function u on the unit circle, there exists a unique harmonic function v on the unit disk with $v(0) = 0$ such that $Pu + iv$ is analytic. Then $Hu = v|_{S^1}$. When u is complex valued, set $Hu = H\Re u + iH\Im u$. Then, $\overline{Hu} = H\overline{u}$, and $H\phi = -i(\phi - \phi(0))$ when ϕ is holomorphic. We have the following basic result:

LEMMA 4.4. *For each $\phi \in \mathcal{AD}(\Delta)$, it holds that*

$$\big(HP_h + P_hH\big)\phi = -i\big(2P_h^+\phi - P_h^+\phi(0) - \phi(0)\big).$$

Proof. The proof goes as follows:

$$\begin{aligned}\big(HP_h + P_hH\big)\phi(z) &= H\big(P_h^+\phi(z) + P_h^-\phi(\bar{z})\big) - iP_h\big(\phi(z) - \phi(0)\big)\\ &= -i\big(P_h^+\phi(z) - P_h^+\phi(0)\big) + iP_h^-\phi(\bar{z})\\ &\quad - i\big(P_h^+\phi(z) + P_h^-\phi(\bar{z}) - \phi(0)\big)\\ &= -i\big(2P_h^+\phi(z) - P_h^+\phi(0) - \phi(0)\big). \qquad \square\end{aligned}$$

COROLLARY 4.5. *Let $v \in H^{\frac{1}{2}}$ be real valued. Then there exists some $u \in H^{\frac{1}{2}}$ such that $\|(HP_h + P_hH)u - v\|_{H^{\frac{1}{2}}} = 0$. Furthermore, $2\|u\|_{H^{\frac{1}{2}}} \leq \|v\|_{H^{\frac{1}{2}}}$.*

Proof. Set $\psi = i(v + iHv)/2$. Then $P\psi \in \mathcal{AD}(\Delta)$. By Proposition 4.3, there exists $\phi \in \mathcal{AD}(\Delta)$ such that $P_h^+\phi = P\psi$. Letting $u = \Re\phi$, we obtain by Lemma 4.4 that

$$\begin{aligned}\big(HP_h + P_hH\big)u = \Re\big(HP_h + P_hH\big)\phi &= \Im\big(2P_h^+\phi - \big(P_h^+\phi(0) - \phi(0)\big)\big)\\ &= v - \Im\big(P_h^+\phi(0) - \phi(0)\big).\end{aligned}$$

Consequently, $\|(HP_h + P_hH)u - v\|_{H^{\frac{1}{2}}} = 0$, and by (4.3),

$$4\|u\|_{H^{\frac{1}{2}}}^2 = 2\|\phi\|_{\mathcal{AD}}^2 \leq 2\|P\psi\|_{\mathcal{AD}}^2 = \|v\|_{H^{\frac{1}{2}}}^2. \qquad \square$$

5. Proof of Theorem 1.1. In this section, we will give the proof of Theorem 1.1. We first recall the normalized decomposition of a quasisymmetric homeomorphism. For any quasisymmetric homeomorphism h, there exists a unique pair of conformal mappings $f \in S_Q$ and g on Δ and Δ^*, respectively, such that

$f(0) = f'(0) - 1 = 0$, $g(\infty) = \infty$, $h = f^{-1} \circ g$ on S^1. We call this a normalized decomposition of h. Conversely, for each $f \in S_Q$ which maps the unit disk onto a bounded Jordan domain, there exists a quasisymmetric h with the normalized decomposition $h = f^{-1} \circ g$. It is clear that h is uniquely determined if $h(1) = 1$, and in this case we say h is the normalized conformal sewing mapping of f.

Proof of "only if" part. Suppose $h \in \mathrm{WP}(S^1)$. Consider the above normalized decomposition $h = f^{-1} \circ g$. Then, $\log f' \in \mathcal{AD}(\Delta)$, $\log g' \in \mathcal{AD}(\Delta^*)$. For details, see [TT2] and also [Cu]. Then, h is absolutely continuous on S^1, and from $f \circ h = g$ we obtain $(f' \circ h)h' = g'$. Thus,

$$\log h' = \log g' - \log f' \circ h = \log g' - P_h \log f'. \tag{5.1}$$

Consequently, $\log h' \in H^{\frac{1}{2}}$. □

Proof of "if" part. The proof of this direction is more difficult. Suppose h is an absolutely continuous homeomorphism on the unit circle such that $\log h' \in H^{\frac{1}{2}}$. Lemma 3.3 implies that h is a quasisymmetric homeomorphism so that Corollary 4.5 may be used. Without loss of generality, we assume $h(1) = 1$. Then $h(e^{i\theta}) = e^{i\phi(\theta)}$, where ϕ is a strictly increasing and absolutely continuous function on the real line $\mathbb{R}$ such that $\phi(0) = 0$, $\phi(\theta + 2\pi) - \phi(\theta) \equiv 2\pi$.

We first assume $\|\log h'\|_{H^{\frac{1}{2}}}$ is small. By Corollary 4.5, there exists some $u \in H^{\frac{1}{2}}$ and a real constant c_1 such that

$$(HP_h + P_h H)u = -H \log|h'| - \Im \log h' + c_1, \tag{5.2}$$

and $2\|u\|_{H^{\frac{1}{2}}} \leq \|H \log|h'| + \Im \log h'\|_{H^{\frac{1}{2}}}$ is small. Then there exists a locally univalent analytic function f on the unit disk with $f(0) = f'(0) - 1 = 0$ such that for some constant c_2,

$$\log f'(z) = P(u + iHu)(z) + c_2. \tag{5.3}$$

Since $\|\log f'\|_{\mathcal{AD}} = \|u + iHu\|_{H^{\frac{1}{2}}}$ is small, by the continuity of the inclusion of $\mathcal{AD}(\Delta)$ into $B(\Delta)$, $\|\log f'\|_B$ is also small. It is well known that f is univalent in Δ and can be extended to a quasiconformal mapping in the whole plane (see [Be] and also [AG]). Consequently, $\log f' \in \hat{T}_b \cap \mathcal{AD}_0(\Delta)$.

Now we set $v = P_h u + \log|h'|$. Then $\|v\|_{H^{\frac{1}{2}}}$ is small. In fact, when $\|\log h'\|_{H^{\frac{1}{2}}}$ is small, $\|\log h'\|_{\mathrm{BMO}(S^1)}$ is also small by the continuity of the inclusion $H^{\frac{1}{2}}$ into $\mathrm{VMO}(S^1) \subset \mathrm{BMO}(S^1)$. Then h can be extended to a quasiconformal mapping in the unit disk whose Beltrami coefficient μ has small norm $\|\mu\|_\infty$ (see [AZ, Be]), which in turn implies by (4.2) that $\|P_h u\|_{H^{\frac{1}{2}}}$ is small and so $\|v\|_{H^{\frac{1}{2}}}$ is also small. By the same reasoning as above, there exists a quasiconformal mapping g on the whole plane with $g(\infty)=\infty$ such that g is conformal in Δ^* with $\log g' \in \mathcal{AD}(\Delta^*)$ and

$$\begin{aligned}\log g' &= v - iHv + \big(c_2 + ic_1\big) \\ &= P_h u + \log|h'| - iHP_h u - iH\log|h'| + c_2 + ic_1.\end{aligned} \tag{5.4}$$

Now it follows from (5.2)–(5.4) that

$$\begin{aligned}P_h \log f' - \log g' &= \big(P_h u + iP_h Hu + c_2\big) \\ &\quad - \big(P_h u + \log|h'| - iHP_h u - iH\log|h'| + c_2 + ic_1\big) \\ &= i\big(P_h Hu + HP_h u\big) - \log|h'| + iH\log|h'| - ic_1 \\ &= -i\big(H\log|h'| + \Im \log h'\big) - \log|h'| + iH\log|h'| \\ &= -\log h'.\end{aligned}$$

Consequently, adding some constant to g if necessary, it holds that $g = f\circ h$. Since $\log f' \in \hat{T}_b \cap \mathcal{AD}_0(\Delta)$, we conclude that h belongs to the Weil-Petersson class under the assumption that $\|\log h'\|_{H^{\frac{1}{2}}}$ is small. It should be pointed out that the above reasoning was inspired by David [Da] in an other setting of BMO theory of the universal Teichmüller space.

When $\|\log h'\|_{H^{\frac{1}{2}}}$ is not necessarily small, we use an approximation process. Since $\log h' \in H^{\frac{1}{2}}$, there exists a sequence (u_n) of real valued (real) analytic functions such that $\|u_n - \log|h'|\|_{H^{\frac{1}{2}}} \to 0$ as $n\to\infty$. Replacing u_n by $u_n - a_0(u_n) + a_0(\log|h'|)$ if necessary, we may assume that $a_0(u_n) = a_0(\log|h'|)$. Define $h_n(e^{i\theta}) = e^{i\phi_n(\theta)}$ by

$$\phi_n(\theta) = \frac{2\pi}{\int_0^{2\pi} e^{\hat{u_n}(t)}dt}\int_0^{\theta} e^{\hat{u_n}(t)}dt, \quad \theta\in\mathbb{R}. \tag{5.5}$$

Then, $h_n \in \mathrm{WP}(S^1)$ since ϕ_n is a real analytic diffeomorphism.

We first show that $\|\log h_n' - \log h'\|_{H^{\frac{1}{2}}} \to 0$ as $n\to\infty$. By our construction, $\|\log|h_n'| - \log|h'|\|_{H^{\frac{1}{2}}} \to 0$ as $n\to\infty$. We need to show that $\|\Im\log h_n' - \Im\log h'\|_{H^{\frac{1}{2}}} \to 0$ as $n\to\infty$. For simplicity, we set $\lambda_n = \Im\log h_n' - \Im\log h'$ so that $\hat{\lambda_n} = \phi_n - \phi$. Recall that $H^{\frac{1}{2}} \subset \mathrm{VMO}(S^1)$, and the inclusion map is continuous. Noting that

$$|a_0(e^{u_n}) - 1| = \frac{1}{2\pi}\left|\int_0^{2\pi}\big(e^{\hat{u_n}(t)} - e^{\log\phi'(t)}\big)dt\right| \le \|e^{u_n} - e^{\log|h'|}\|_1,$$

we conclude by Lemma 3.2 that $a_0(e^{u_n}) \to 1$ as $n\to\infty$. Now (5.5) implies that $\log|h_n'| = u_n - \log a_0(e^{u_n})$, which implies $a_0(\log|h_n'|) = a_0(u_n) - \log a_0(e^{u_n}) \to a_0(\log|h'|)$ as $n\to\infty$. By Lemma 3.2 again, we conclude that, for any $p\ge 1$, $\||h_n'| - |h'|\|_p \to 0$ as $n\to\infty$. Now since the $m-$th $(m\ne 0)$ Fourier coefficient of

λ_n is

$$\begin{aligned} a_m &= \frac{1}{2\pi}\int_0^{2\pi} \hat{\lambda_n}(\theta)e^{-im\theta}d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} (\phi_n(\theta)-\phi(\theta))e^{-im\theta}d\theta \\ &= \frac{1}{2m\pi i}\int_0^{2\pi} (\phi_n'(\theta)-\phi'(\theta))e^{-im\theta}d\theta \\ &= \frac{1}{2m\pi i}\int_0^{2\pi} (|h_n'|(e^{i\theta})-|h'|(e^{i\theta}))e^{-im\theta}d\theta, \end{aligned}$$

we conclude by Parseval's equality that

$$\begin{aligned} \|\lambda_n\|_{H^1} &= \sum_{m\neq 0} m^2|a_m|^2 = \frac{1}{4\pi^2}\sum_{m\neq 0}\left|\int_0^{2\pi}\big(|h_n'|\big(e^{i\theta}\big)-|h'|\big(e^{i\theta}\big)\big)e^{-im\theta}d\theta\right|^2 \\ &= \big\||h_n'|-|h'|\big\|_2^2, \end{aligned}$$

which implies $\|\lambda_n\|_{H^{\frac{1}{2}}} \leq \|\lambda_n\|_{H^1} \to 0$ as $n\to\infty$. Thus, $\|\log h_n' - \log h'\|_{H^{\frac{1}{2}}} \to 0$ as $n\to\infty$.

Now we consider $\tilde{h}_n = h_n \circ h^{-1}$. Then $\tilde{h}_n$ is absolutely continuous. Noting that

$$\log \tilde{h}_n' = \big(\log h_n' - \log h'\big)\circ h^{-1} = P_h^{-1}\big(\log h_n' - \log h'\big),$$

we find that $\|\log \tilde{h}_n'\|_{H^{\frac{1}{2}}} \to 0$ as $n\to\infty$. By what we have proved in the small norm case, $\tilde{h}_n \in \mathrm{WP}(S^1)$. Since $\mathrm{WP}(S^1)$ is a group (see [Cu, TT2]), we conclude that $h\in\mathrm{WP}(S^1)$. Now the proof of Theorem 1.1 is completed. □

Remark 5.1. By means of Theorem 1.1, we can give a new model of the Weil-Petersson Teichmüller space. More precisely, let $H_{\mathbb{R}}^{\frac{1}{2}}$ denote the subspace of all real-valued functions in $H^{\frac{1}{2}}$. By Theorem 1.1, $\log|h'|\in H_{\mathbb{R}}^{\frac{1}{2}}$ for $h\in\mathrm{WP}(S^1)$. Conversely, suppose $u\in H_{\mathbb{R}}^{\frac{1}{2}}$. Adding to a constant if necessary, we may assume that $\int_0^{2\pi} e^{\hat{u}(t)}dt = 2\pi$. Set $h(e^{i\theta}) = e^{i\phi(\theta)}$ by

$$\phi(\theta) = \int_0^{\theta} e^{\hat{u}(t)}dt, \quad \theta\in\mathbb{R}. \tag{5.6}$$

Then h is an absolutely continuous sense-preserving homeomorphism of the unit circle with $\log|h'| = u$. By Lemma 3.4 and Theorem 1.1, we get $h\in\mathrm{WP}(S^1)$. Consequently, the correspondence $h\mapsto \log|h'|$ establishes a one-to-one map from $\mathrm{WP}(S^1)/\mathrm{Rot}(S^1)$ onto $H_{\mathbb{R}}^{\frac{1}{2}}/\mathbb{R}$. By means of the $H^{\frac{1}{2}}$ metric, a metric can be assigned to $\mathrm{WP}(S^1)/\mathrm{Rot}(S^1)$. This will be done in Section 8.

6. A counterexample: Proof of Theorem 1.2. Combining Theorem 1.1 with the following result gives the proof of Theorem 1.2.

THEOREM 6.1. *Fix* $\alpha > 1$. *Define* $h(e^{i\theta}) = e^{i\varphi(\theta)}$

$$\varphi(\theta) = c_\alpha \int_0^\theta \left(\left(\log\alpha - \log\sin\frac{t}{2} \right)^2 + \frac{(\pi - t)^2}{4} \right) dt, \quad \theta \in [0, 2\pi], \tag{6.1}$$

where $c_\alpha > 0$ *is a constant so that* $\varphi(2\pi) = 2\pi$. *Then* h *is a sense-preserving homeomorphism which is absolutely continuous such that* $\log h' \in H^{\frac{1}{2}}$, *but* h *is neither in* $H^{\frac{3}{2}}$ *nor Lipschitz.*

Proof. We first point out that φ can be extended to the whole real line $\mathbb{R}$ by means of $\varphi(\theta + 2\pi) - \varphi(\theta) \equiv 2\pi$. Consider

$$g(z) = \log\log\frac{2\alpha}{1-z}. \tag{6.2}$$

g is holomorphic in Δ, and except for $e^{i\theta} = 1$, $\lim_{z\to e^{i\theta}} g(z)$ exists and equals

$$g(e^{i\theta}) = \log\left(\log\alpha - \log\sin\frac{\theta}{2} + i\frac{\pi - \theta}{2} \right).$$

We first show that $g \in \mathcal{AD}(\Delta)$. Noting that

$$g'(z) = \frac{1}{(1-z)\log\frac{2\alpha}{1-z}},$$

it is sufficient to show that

$$\iint_{\{|z-1|<1\}} |g'(z)|^2 dxdy < +\infty.$$

This can be be done as follows:

$$\begin{aligned}
\iint_{\{|z-1|<1\}} |g'(z)|^2 dxdy &= \int_{\{|w|<1\}} \frac{1}{|w\log\frac{2\alpha}{w}|^2} dudv \\
&= \int_0^1 \rho d\rho \int_0^{2\pi} \frac{d\theta}{\rho^2(\log^2\frac{2\alpha}{\rho} + \theta^2)} \\
&= \int_0^1 \frac{1}{\rho\log\frac{2\alpha}{\rho}} \arctan\frac{2\pi}{\log\frac{2\alpha}{\rho}} d\rho \\
&= \int_{\log 2\alpha}^{+\infty} \frac{\arctan\frac{2\pi}{x}}{x} dx \\
&< 2\pi \int_{\log 2\alpha}^{+\infty} \frac{1}{x^2} dx = \frac{2\pi}{\log 2\alpha}.
\end{aligned}$$

Thus, $g \in H^{\frac{1}{2}}$, which implies that $\Re g \in H^{\frac{1}{2}}$. By Lemma 3.1, we obtain that $\exp(2\Re g) \in L^1(S^1)$. Noting that

$$\Re g(e^{i\theta}) = \log\left|\log\alpha - \log\sin\frac{\theta}{2} + i\frac{\pi-\theta}{2}\right| = \frac{1}{2}\log\left(\left(\log\alpha - \log\sin\frac{\theta}{2}\right)^2 + \frac{(\pi-\theta)^2}{4}\right)$$

when $\theta \in (0,2\pi)$, we conclude that our function φ defined in (6.1) is well defined, strictly increasing and absolutely continuous with

$$\varphi'(\theta) = c_\alpha\left(\left(\log\alpha - \log\sin\frac{\theta}{2}\right)^2 + \frac{(\pi-\theta)^2}{4}\right), \quad \theta \in (0,2\pi). \tag{6.3}$$

Thus, h is an absolutely continuous sense-preserving homeomorphism of the unit circle onto itself. Since $\|\varphi'\|_\infty = \infty$, h is not Lipschitz. On the other hand, since $\log|h'| = \log c_\alpha + 2\Re g \in H^{\frac{1}{2}}$, we conclude by Lemma 3.4 that $\log h' \in H^{\frac{1}{2}}$.

It remains to show that h is not in $H^{\frac{3}{2}}$, or equivalently, h' is not in $H^{\frac{1}{2}}$. By means of (3.3), it is sufficient to show that $|h'|$ is not in $H^{\frac{1}{2}}$. To do so, we consider the function

$$f(z) = \log(1-z) \tag{6.4}$$

which is analytic in the unit disk. Then, except for $e^{i\theta} = 1$, $\lim_{z\to e^{i\theta}} f(z)$ exists and is equal to

$$f(e^{i\theta}) = \log(1-e^{i\theta}) = \log 2 + \log\sin\frac{\theta}{2} - i\frac{\pi-\theta}{2}. \tag{6.5}$$

It is easy to see that f does not belong to $\mathcal{AD}(\Delta)$, which implies that $\Re f$ is not in $H^{\frac{1}{2}}$. By (3.3) we have

$$\int_0^\pi\int_0^\pi \frac{|\log\sin s - \log\sin t|^2}{|\sin(s-t)|^2}\,ds\,dt = +\infty. \tag{6.6}$$

Fix $0 < \epsilon < \pi/4$, and set $I_\epsilon = [\pi/2-\epsilon, \pi/2+\epsilon]$, $(I_\epsilon \times I_\epsilon)^c = [0,\pi]\times[0,\pi] - I_\epsilon \times I_\epsilon$. Noting that $\log(1+x) < x$ when $x > 0$, we find that

$$|\log x - \log y| \le \frac{|x-y|}{\min(x,y)}, \quad x > 0,\ y > 0.$$

On the other hand, since $\sin x \ge (2/\pi)x$ when $0 < x < \pi/2$, we conclude that

$$\frac{|\log\sin s - \log\sin t|^2}{|\sin(s-t)|^2} \le \frac{\pi^2}{4}\frac{|\sin s - \sin t|^2}{|s-t|^2\min(\sin^2 s, \sin^2 t)} \le \frac{\pi^2}{4\cos^2\epsilon} \le \frac{\pi^2}{2}, \quad s,t \in I_\epsilon.$$

Thus,

$$\int_{I_\epsilon}\int_{I_\epsilon}\frac{|\log\sin s-\log\sin t|^2}{|\sin(s-t)|^2}dsdt<+\infty.$$

It follows from (6.6) that

$$\iint_{(I_\epsilon\times I_\epsilon)^c}\frac{|\log\sin s-\log\sin t|^2}{|\sin(s-t)|^2}dsdt=+\infty.$$

Noting that $\log\sin s<\log\cos\epsilon<0$ when $s\in I_\epsilon^c$, we conclude from the above equality that

$$\begin{aligned}&\iint_{(I_\epsilon\times I_\epsilon)^c}\frac{|(\log\alpha-\log\sin s)^2-(\log\alpha-\log\sin t)^2|^2}{|\sin(s-t)|^2}dsdt\\&\quad\geq\log^2(\alpha^2\cos\epsilon)\iint_{(I_\epsilon\times I_\epsilon)^c}\frac{|\log\sin s-\log\sin t|^2}{|\sin(s-t)|^2}dsdt=+\infty,\end{aligned}$$

which implies that

$$\int_0^\pi\int_0^\pi\frac{|(\log\alpha-\log\sin s)^2-(\log\alpha-\log\sin t)^2|^2}{|\sin(s-t)|^2}dsdt=+\infty. \tag{6.7}$$

On the other hand, consider the function u on the unit circle defined by $u(e^{i\theta})=(\pi-\theta)^2$, $\theta\in[0,2\pi]$. Then, $u\in H^{\frac{1}{2}}$. Actually, a direct computation will show that the n-th ($n\neq 0$) Fourier coefficient of u is

$$a_n=\frac{1}{2\pi}\int_0^{2\pi}\hat{u}(\theta)e^{-in\theta}d\theta=\frac{1}{2\pi}\int_0^{2\pi}(\pi-\theta)^2e^{-in\theta}d\theta=\frac{2}{n^2}.$$

Combining this with (3.3) and (6.7), we conclude that $|h'|$ is not in $H^{\frac{1}{2}}$. This completes the proof of Theorem 6.1. □

7. Proof of Theorem 1.3.

We first prove two general results.

LEMMA 7.1. *Given a continuous vector field* $u(t,\cdot)\in C^0([0,M],\Lambda^*)$ *with the normalized conditions* (1.4) *and* (1.5), *the flow maps* $h(t,\zeta)$ *of the differential equation*

$$\begin{cases}\frac{dh}{dt}=u(t,h)\\ h(0,\zeta)=\zeta\end{cases} \tag{7.1}$$

are quasisymmetric homeomorphisms, and $h(t,\cdot):[0,M]\to T$ *is continuous.*

Proof. As stated in Section 1, Reimann [Re] proved that, for each fixed $t\in[0,M]$, $h(t,\cdot)$ is a quasisymmetric homeomorphism. In fact, Agard-Kelingos (see [AK, Theorems 1 and 2]) already proved that $h(t,\cdot):[0,M]\to T$ is continuous under the assumption that $u(t,\cdot)$ can be extended to a so-called quasiconformal deformation $U(t,\cdot)$ to the unit disk with $\overline{\partial}U(t,\cdot)\in C^0([0,M],L^\infty(\Delta))$, which was proved to be true by Gardiner-Sullivan (see [GS, Section 8]) and Reich-Chen (see [RC, Theorem 2.2]) independently. A detailed proof of Lemma 7.1 can be found in our paper [HWS]. □

LEMMA 7.2. *Let h_t, $t\in[0,M]$ be quasisymmetric homeomorphisms which keep the points* 1, -1 *and i fixed. Suppose $u_t:[0,M]\to H^{\frac{1}{2}}$ and $h_t:[0,M]\to T$ are continuous. Then $P_{h_t}u_t:[0,M]\to H^{\frac{1}{2}}$ is continuous.*

Proof. Fix $t_0\in[0,M]$. By (4.2) we have

$$\begin{aligned}\|P_{h_t}u_t-P_{h_{t_0}}u_{t_0}\|_{H^{\frac{1}{2}}}&\le\|P_{h_t}u_t-P_{h_t}u_{t_0}\|_{H^{\frac{1}{2}}}+\|P_{h_t}u_{t_0}-P_{h_{t_0}}u_{t_0}\|_{H^{\frac{1}{2}}}\\&\le e^{\tau(0,h_t)}\|u_t-u_{t_0}\|_{H^{\frac{1}{2}}}+\|P_{h_t}u_{t_0}-P_{h_{t_0}}u_{t_0}\|_{H^{\frac{1}{2}}}.\end{aligned}$$

We conclude that $P_{h_t}u_t:[0,M]\to H^{\frac{1}{2}}$ is continuous by Proposition 4.1 and the continuity of u_t and h_t. □

Now we begin to prove Theorem 1.3. It is contained in

THEOREM 7.3. *Given a continuous vector field $u(t,\cdot)\in C^0([0,M],H^{\frac{3}{2}})$ with the normalized condition* (1.4)*, the flow maps $h(t,\cdot)$ of the differential equation* (7.1) *belong to the Weil-Petersson class, namely, $h(t,\cdot)\in \mathrm{WP}(S^1)$ for each fixed $t\in[0,M]$; Furthermore, the mapping $t\mapsto\log h'(t,\cdot)$ from $[0,M]$ into $H^{\frac{1}{2}}$ is continuously differentiable such that*

$$\frac{d}{dt}\log h'(t,\cdot)=u'(t,h(t,\cdot)). \tag{7.2}$$

Proof. Without loss of the generality, we assume that the vector field $u(t,\cdot)$ also satisfies the normalized condition (1.5) so that the the flow maps $h(t,\cdot)$ keep the points 1, -1 and i fixed. We first point out that by Figalli's result (see [Fi]), for each fixed $t\in[0,M]$, $h(t,\cdot)$ is absolutely continuous. As done by Figalli [Fi], differentiating both sides of the equation

$$\frac{d}{dt}h(t,\zeta)=u\big(t,h(t,\zeta)\big) \tag{7.3}$$

with respect to ζ yields

$$\frac{d}{dt}h'(t,\zeta)=u'\big(t,h(t,\zeta)\big)h'(t,\zeta),$$

that is,

$$\frac{d}{dt}\log h'(t,\zeta)=u'\big(t,h(t,\zeta)\big).$$

Noting that $h(0,\zeta)=\zeta$, we obtain

$$\log h'(t,\zeta) = \int_0^t u'\big(s, h(s,\zeta)\big)ds. \tag{7.4}$$

Recalling that the inclusion of $H^{\frac{3}{2}}$ into Λ^* is continuous, we conclude that $u'(t,h(t,\cdot)) : [0,M] \to H^{\frac{1}{2}}$ is continuous by Lemmas 7.1 and 7.2. Now Theorem 7.3 follows from the following Lemma 7.4 immediately. □

LEMMA 7.4. *Suppose* $u(t,\cdot) : [0,M] \to H^{\frac{1}{2}}$ *is continuous, and*

$$U(t,\zeta) = \int_0^t u(s,\zeta)ds, \quad \zeta \in S^1. \tag{7.5}$$

Then for each fixed $t \in [0,M]$, $U(t,\cdot) \in H^{\frac{1}{2}}$, *and* $U(t,\cdot) : [0,M] \to H^{\frac{1}{2}}$ *is continuously differentiable with*

$$\frac{d}{dt}U(t,\cdot) = u(t,\cdot). \tag{7.6}$$

Proof. For simplicity, we set $U(t,\cdot) = U_t$, $u(t,\cdot) = u_t$. By definition we have

$$\begin{aligned} a_n(U_t) &= \frac{1}{2\pi}\int_0^{2\pi}\left(\int_0^t u_s(e^{i\theta})ds\right)e^{-in\theta}d\theta \\ &= \int_0^t \left(\frac{1}{2\pi}\int_0^{2\pi} u_s(e^{i\theta})e^{-in\theta}d\theta\right)ds \\ &= \int_0^t a_n(u_s)ds. \end{aligned}$$

Then,

$$\begin{aligned} |a_n(U_t)|^2 &\le t\int_0^t |a_n(u_s)|^2 ds, \\ \|U_t\|^2_{H^{\frac{1}{2}}} &= \sum_{n=-\infty}^{+\infty} |n||a_n(U_t)|^2 \\ &\le t\sum_{n=-\infty}^{+\infty} |n|\int_0^t |a_n(u_s)|^2 ds \\ &= t\int_0^t \sum_{n=-\infty}^{+\infty} |n||a_n(u_s)|^2 ds \\ &= t\int_0^t \|u_s\|^2_{H^{\frac{1}{2}}} ds \\ &\le t^2 \max_{s\in[0,t]} \|u_s\|^2_{H^{\frac{1}{2}}}. \end{aligned}$$

Consequently, for each fixed $t \in [0, M]$, $U(t, \cdot) \in H^{\frac{1}{2}}$.

It remains to show (7.6). Fix $t_0 \in [0, M]$. Noting that

$$U_{t_0+t}(\zeta) - U_{t_0}(\zeta) - tu_{t_0}(\zeta) = \int_{t_0}^{t_0+t} (u_s(\zeta) - u_{t_0}(\zeta))ds,$$

we conclude by the reasoning as above that

$$\|U_{t_0+t} - U_{t_0} - tu_{t_0}\|_{H^{\frac{1}{2}}} \leq |t| \max_{|s-t_0|\leq|t|} \|u_s - u_{t_0}\|_{H^{\frac{1}{2}}},$$

which implies that

$$\lim_{t\to 0} \left\| \frac{U_{t_0+t} - U_{t_0}}{t} - u_{t_0} \right\|_{H^{\frac{1}{2}}} \leq \lim_{t\to 0} \left(\max_{|s-t_0|\leq|t|} \|u_s - u_{t_0}\|_{H^{\frac{1}{2}}} \right) = 0,$$

that is, U_t is differentiable at t_0, and (7.6) holds. □

Remark 7.5. Here it is an appropriate place to relate a result of Figalli [Fi]. In an attempt to study the regularity of the elements in WP(S^1), Figalli [Fi] investigated the smoothness of the flows of the $H^{\frac{3}{2}}$ vector fields and showed that there exists some $H^{\frac{3}{2}}$ vector field whose flow is neither Lipschitz nor $H^{\frac{3}{2}}$. Now our Theorem 7.3 says that the the flow maps of the $H^{\frac{3}{2}}$ vector field in Figalli's example must also belong to WP(S^1), which in turn implies (Theorem 1.2) that there exists some quasisymmetric homeomorphism which is in WP(S^1) but is neither $H^{\frac{3}{2}}$ nor Lipschitz. Our proof of Theorem 1.2 relies neither on Theorem 7.3 nor on Figalli's result. Moreover, it gives an explicit expression of a quasisymmetric homeomorphism of the Weil-Petersson class being neither $H^{\frac{3}{2}}$ nor Lipschitz.

8. Proof of Theorem 1.4. Recall that the universal Teichmüller space has a quasisymmetric homeomorphism model, namely, $T = \mathrm{QS}(S^1)/\mathrm{M\ddot{o}b}(S^1)$. Now $\mathcal{T} = \mathrm{QS}(S^1)/\mathrm{Rot}(S^1)$ is a fiber space over T and in fact is a model of the universal Teichmüller curve (see [Ber, Te, TT2]). Each point in $\mathcal{T}$ can be considered as a quasisymmetric homeomorphism which keeps 1 fixed. There exists a one-to-one map Ψ from $\mathcal{T}$ onto $\hat{T}_b$ (another model of the universal Teichmüller curve) which sends h to $\log f'$ under the normalized decomposition $h^{-1} = f^{-1} \circ g$. Via Ψ, $\mathcal{T}$ is endowed with a standard complex Banach manifold structure such that $\Psi : \mathcal{T} \to \hat{T}_b$ is a bi-holomorphic isomorphism (see [TT2] for more details).

Now we consider the Weil-Petersson class. Set $\mathcal{T}_0 = \mathrm{WP}(S^1)/\mathrm{Rot}(S^1)$. Then Ψ establishes a bijective map between $\mathcal{T}_0$ and $\hat{T}_b \cap \mathcal{AD}_0(\Delta)$. As stated in Remark 5.1, a natural metric assigned to $\mathcal{T}_0$ is the following $H^{\frac{1}{2}}$ metric:

$$d(h_1, h_2) = \| \log |h_2'| - \log |h_1'| \|_{H^{\frac{1}{2}}}, \quad h_1, h_2 \in \mathcal{T}_0. \tag{8.1}$$

Examining the last step in the proof of Theorem 1.1, we see that the metric is topologically equivalent to the following metric:

$$d'(h_1,h_2) = \|\log h_2' - \log h_1'\|_{H^{\frac{1}{2}}}, \quad h_1,h_2 \in \mathcal{T}_0. \tag{8.2}$$

Then we have the following result.

THEOREM 8.1. *$\Psi : (\mathcal{T}_0, d) \to \hat{T}_b \cap \mathcal{AD}_0(\Delta)$ is a homeomorphism.*

Proof. We first recall the fact that $\|\log h'\|_{H^{\frac{1}{2}}}$ is small if and only if $\|\log(h^{-1})'\|_{H^{\frac{1}{2}}}$ is small. Examining the proof of Theorem 1.1, we find out that $\|\Psi(h)\|_{\mathcal{AD}}$ is small if $\|\log h'\|_{H^{\frac{1}{2}}}$ is small. Thus, Ψ is continuous at the base point id. Conversely, suppose $\log f' \in \hat{T}_b \cap \mathcal{AD}_0(\Delta)$ has small norm. Let $h^{-1} = f^{-1} \circ g$ be the normalized conformal sewing mapping of f. We need to show that $\|\log h'\|_{H^{\frac{1}{2}}}$ is small, or equivalently, $\|\log(h^{-1})'\|_{H^{\frac{1}{2}}}$ is small.

Since $S_f = \Lambda(\log f')$ has small norm (2.5), by means of the well-known Ahlfors-Weil section (see [AW]), f can be extended to a quasiconformal mapping in the whole plane whose complex dilatation μ has the form

$$\mu(z) = -\frac{1}{2}(|z|^2-1)^2 S_f(\bar{z}^{-1})\bar{z}^{-4}, \quad z \in \Delta^*. \tag{8.3}$$

Thus, $\mu \in \mathcal{M}(\Delta^*)$ with small norm $\|\mu\|_{\mathrm{WP}}$. By means of Lemma 1.5 in [TT2], we have $f_\mu(\infty) = \infty$.

We first consider the special case that $f = f_\mu|_\Delta$. Let w_μ be the unique quasiconformal mapping of Δ^* onto itself with Beltrami coefficient μ and keeping the points 1 and ∞ fixed. Extending w_μ to the unit disk by symmetry, we obtain a quasiconformal mapping w_μ in the whole plane with $w_\mu(0) = 0$. Then $g = f_\mu \circ w_\mu^{-1}|_{\Delta^*}$, and $h = w_\mu|_{S^1}$. Now Lemma 2.5 in [TT2] implies that the Beltrami coefficient ν of w_μ^{-1} has small norm $\|\nu\|_{\mathrm{WP}}$. On the other hand, it is easy to see that $h = g^{-1} \circ f$ is the quasisymmetric conformal sewing mapping corresponding to $rj \circ g \circ j$, where $j(z) = \bar{z}^{-1}$ is the standard reflection of the unit circle, and r is a constant such that $r(j \circ g \circ j)'(0) = 1$. Now $rj \circ g \circ j = rj \circ f_\mu \circ w_\mu^{-1} \circ j|_\Delta$ has the quasiconformal extension $rj \circ f_\mu \circ w_\mu^{-1} \circ j|_{\Delta^*}$ which keeps the point at infinity fixed, we conclude that $\log(rj \circ g \circ j)'$ has small norm in $\mathcal{AD}_0(\Delta)$ since the Beltrami coefficient ν of w_μ^{-1} has small norm $\|\nu\|_{WP}$. Thus, $\log g'$ has small norm in $\mathcal{AD}(\Delta^*)$. It follows from (5.1) that $\|\log(h^{-1})'\|_{H^{\frac{1}{2}}}$ is small.

In the general case, since f and f_μ have the same complex dilatation μ, we conclude by the normalized conditions $f(0) = f_\mu(0) = 0$, $f'(0) = f_\mu'(0) = 1$ and $f_\mu''(0) = 0$ that $f = \gamma_1 \circ f_\mu$, where $\gamma_1(z) = \frac{z}{1-\lambda z}$ with $\lambda = f''(0)/2$. Since $\log f' \in \hat{T}_b \cap \mathcal{AD}_0(\Delta)$ has small norm, we conclude that $\lambda = f''(0)/2$ is small (see [Te]). To find the normalized conformal sewing map of f_μ, we set $z_1 = g^{-1}(-\frac{1}{\lambda})$,

$z_2 = -\frac{1-z_1}{z_1(1-\overline{z_1})}$, and

$$\gamma_2(z) = \frac{1-\overline{z_2}}{1-z_2}\frac{z-z_2}{1-\overline{z_2}z}.$$

A direct computation yields that $\gamma_2(1) = 1$, $\gamma_2(\infty) = z_1$. Noting that $g(\infty) = \infty$, we conclude that z_1 tends to infinity and z_2 is small when λ is small. Consider $\hat{g} = \gamma_1^{-1} \circ g \circ \gamma_2$. Then $\hat{g}$ is a conformal mapping from Δ^* onto $f_\mu(\Delta^*)$, and $\hat{g}(1) = f_\mu(1)$, $\hat{g}(\infty) = \infty$. Consequently, the normalized conformal sewing map of f_μ is

$$\hat{h}^{-1} = f_\mu^{-1} \circ \hat{g} = f_\mu^{-1} \circ \gamma_1^{-1} \circ g \circ \gamma_2 = f^{-1} \circ g \circ \gamma_2 = h^{-1} \circ \gamma_2,$$

which implies that $h = \gamma_2 \circ \hat{h}$. By what we have proved in the first (special) case, we conclude that $\|\log \hat{h}'\|_{H^{\frac{1}{2}}}$ is small when $\|\log f'\|_{\mathcal{AD}}$ is small. On the other hand, when $\|\log f'\|_{\mathcal{AD}}$ is small, z_2 is small, which implies that

$$\|\log \gamma_2'\|_{\mathcal{AD}}^2 = \frac{1}{\pi}\iint_\Delta \frac{4|z_2|^2}{|1-\overline{z_2}z|^2}dxdy = 4\log\frac{1}{1-|z_2|^2}$$

is also small. Therefore, we conclude by $\log h' = \log \gamma_2' \circ \hat{h} + \log \hat{h}'$ that $\|\log h'\|_{H^{\frac{1}{2}}}$ is small when $\|\log f'\|_{\mathcal{AD}}$ is small. This completes the proof that Ψ^{-1} is continuous at the base point 0.

We now handle the general case by changing a general point to the base point. We only sketch the standard procedure by using the so-called allowable mappings (see [Ber, Na, TT2] for more details). Let $h \in \mathcal{T}_0$ be fixed. Consider the map R_h defined by $R_h(k) = k \circ h^{-1}$. Then R_h is a bijective map from $\mathcal{T}_0$ onto itself. Noting that

$$d'(R_h(k_1), R_h(h_2)) = \|(\log k_2' - \log k_1') \circ h^{-1}\|_{H^{\frac{1}{2}}}, \tag{8.4}$$

we conclude that R_h is a quasi-isometric map from $\mathcal{T}_0$ onto itself under the d'-metric. Now let w be a quasiconformal extension of h to Δ^* such that w is quasi-isometric under the Poincaré metric with Beltrami coefficient $\mu \in \mathcal{M}(\Delta^*)$. The existence of such a quasiconformal extension is guaranteed by means of the well-known Douady-Earle [DE] extension of a quasisymmetric homeomorphism (see [Cu]). As stated in Proposition 2.2, R_w induces a bi-holomorphic isomorphism w^* from T_0 onto itself with $w^* \circ \Phi = \Phi \circ R_w$. In fact, it is known that R_w also induces a bi-holomorphic isomorphism $\tilde{w^*}$ from $\hat{T}_b \cap \mathcal{AD}_0(\Delta)$ onto itself which is related to R_h by $\tilde{w^*} \circ \Psi = \Psi \circ R_h$. By using the allowable mappings $\tilde{w^*}$ and R_h, we conclude that both $\Psi : (\mathcal{T}_0, d) \to \hat{T}_b \cap \mathcal{AD}_0(\Delta)$ and its inverse are continuous at a general point h (or $\Psi(h)$). □

Proof of Theorem 1.4. Let $\tilde{\Lambda}$ denote the natural projection from $\mathcal{T}_0 = \mathrm{WP}(S^1)/\mathrm{Rot}(S^1)$ onto $T_0 = \mathrm{WP}(S^1)/\mathrm{Möb}(S^1)$. The metric d on $\mathcal{T}_0$ descends down to a metric on T_0, still denoted by d, as follows:

$$d(h_1,h_2)=\inf\{d(\tilde{h}_1,\tilde{h}_2):\tilde{\Lambda}(\tilde{h}_1)=h_1,\tilde{\Lambda}(\tilde{h}_2)=h_2\},\quad h_1,h_2\in T_0. \tag{8.5}$$

By Theorem 8.1, Ψ establishes a homeomorphism from $(\mathcal{T}_0,d)$ onto $\hat{T}_b\cap\mathcal{AD}_0(\Delta)$. On the other hand, Theorem 2.5 says that $\Lambda:\hat{T}_b\cap\mathcal{AD}_0(\Delta)\to\beta(T_0)$ is a holomorphic split submersion. This already implies that $\Psi:(\mathcal{T}_0,d)\to\hat{T}_b\cap\mathcal{AD}_0(\Delta)$ induces a homeomorphism from $\tilde{\Psi}:(T_0,d)$ onto $\beta(T_0)$, which implies that the metric d and the Weil-Petersson metric induce the same topology on $T_0=\mathrm{WP}(S^1)/\mathrm{Möb}(S^1)$. □

9. Open problems. It is known that $\mathcal{T}_0=\mathrm{WP}(S^1)/\mathrm{Rot}(S^1)$ inherits a standard complex Hilbert manifold structure from $\mathcal{AD}_0(\Delta)$ by the bijection $\Psi:\mathcal{T}_0\to\hat{T}_b\cap\mathcal{AD}_0(\Delta)$ (see [TT2]). Meanwhile, $H^{\frac{1}{2}}_{\mathbb{R}}/\mathbb{R}$ provides $\mathcal{T}_0$ with a real Hilbert manifold structure by the correspondence $h\mapsto\log|h'|$ (see Remark 5.1). Now Theorem 8.1 says that theses two Hilbert manifold structures induce the same topology on $\mathcal{T}_0$. It is not clear whether these two manifold structures are compatible with each other. We believe this is the case and propose the following:

CONJECTURE 9.1. *Under the normalized decomposition $h^{-1}=f^{-1}\circ g$, both the bijective map* $\log f'\mapsto\log|h'|$ *and its inverse are real analytic.*

In the recent paper [GR], Gay-Balmaz-Ratiu made the following:

CONJECTURE 9.2. [GR] *Given a continuous vector field $u(t,\cdot)\in C^0([0,M],H^{\frac{3}{2}})$ with the normalized condition* (1.4)*, the flow maps $h(t,\zeta)$ of the differential equation*

$$\begin{cases}\dfrac{dh}{dt}=u(t,h)\\ h(0,\zeta)=\zeta\end{cases} \tag{9.1}$$

belong to the Weil-Petersson class, namely, $h(t,\cdot)\in\mathrm{WP}(S^1)$ for each fixed $t\in[0,M]$; Furthermore, the mapping $t\mapsto h(t,\cdot)$ from $[0,M]$ into $\mathrm{WP}(S^1)$ is continuously differentiable under the standard Hilbert manifold structure introduced by Takhtajan-Teo [TT2].

The first assertion in Conjecture 9.2 is true by our Theorem 7.3. Furthermore, Theorem 7.3 implies that the mapping $t\mapsto\log|h'(t,\cdot)|$ from $[0,M]$ into $H^{\frac{1}{2}}_{\mathbb{R}}$ is continuously differentiable. It is clear that if Conjecture 9.1 were true, then Conjecture 9.2 would also be true.

Based on Lemma 7.1, it is natural to propose the following:

Problem 9.3. Given a continuous vector field $u(t,\cdot)\in C^0([0,M],\Lambda^*)$ with the normalized condition (1.4) and (1.5), let $h(t,\cdot)$ be the flow maps of the differential equation (9.1). Determine whether or not the flow $h(t,\cdot):[0,M]\to T$ is continuously differentiable.

10. Appendix: proof of Propositions 4.1 and 4.3. In this section, we will prove Propositions 4.1 and 4.3 as we promised in Section 4. We restate them as follows.

PROPOSITION 10.1. *Let h and h_0 be quasisymmetric homeomorphisms which keep the points 1, -1 and i fixed. Then for each fixed $u\in H^{\frac{1}{2}}$, $\|P_h u - P_{h_0}u\|_{H^{\frac{1}{2}}}\to 0$ when $\tau(h,h_0)\to 0$.*

PROPOSITION 10.2. *P_h^+ is a bounded isomorphism from $\mathcal{AD}(\Delta)$ onto itself. Moreover, it holds that*

$$\|P_h^+\phi\|^2_{\mathcal{AD}} = \|\phi\|^2_{\mathcal{AD}} + \|P_h^-\phi\|^2_{\mathcal{AD}}, \quad \phi\in\mathcal{AD}(\Delta). \tag{10.1}$$

Here it is an appropriate place to point out that, though not stated in this form, Proposition 10.1 has appeared in the unpublished Master thesis [Li] of Q. Liu. To prove Propositions 10.1 and 10.2, we need two related operators. Let $\mathcal{A}^2(\Delta)$ denote the complex Hilbert space of all holomorphic functions ψ on the unit disk with norm

$$\|\psi\|_{\mathcal{A}^2} = \left(\frac{1}{\pi}\iint_\Delta |\psi(\zeta)|^2 d\xi d\eta\right)^{\frac{1}{2}}. \tag{10.2}$$

Then $D\phi(z)=\phi'(z)$ determines an isometric isomorphism from $\mathcal{AD}_0(\Delta)$ onto $\mathcal{A}^2(\Delta)$.

For a quasisymmetric homeomorphism h, two kernel functions were introduced in the previous paper [HS] by Hu and the author. They are

$$\phi_h(\zeta,z) = \frac{1}{2\pi i}\int_{S^1}\frac{h(w)}{(1-\zeta w)^2(1-zh(w))}dw, \quad (\zeta,z)\in\Delta\times\Delta, \tag{10.3}$$

$$\psi_h(\zeta,z) = \frac{1}{2\pi i}\int_{S^1}\frac{h(w)}{(\zeta-w)^2(1-zh(w))}dw, \quad (\zeta,z)\in\Delta\times\Delta. \tag{10.4}$$

The two kernels ϕ_h and ψ_h induce two bounded operators on $\mathcal{A}^2(\Delta)$ as follows:

$$T_h^-\psi(\zeta) = \frac{1}{\pi}\iint_\Delta \phi_h(\zeta,\bar z)\psi(z)dxdy, \quad \psi\in\mathcal{A}^2(\Delta),\ \zeta\in\Delta, \tag{10.5}$$

and

$$T_h^+\psi(\zeta) = \frac{1}{\pi}\iint_\Delta \psi_h(\zeta,\bar z)\psi(z)dxdy, \quad \psi\in\mathcal{A}^2(\Delta),\ \zeta\in\Delta. \tag{10.6}$$

Then, Theorem 3.1 in [HS] says that on $\mathcal{AD}(\Delta)$,

$$D \circ P_h^- = T_h^- \circ D, \quad D \circ P_h^+ = T_h^+ \circ D, \tag{10.7}$$

while Lemma 2.3 in [SW] says that

$$\|T_h^+\psi\|_{\mathcal{A}^2}^2 = \|\psi\|_{\mathcal{A}^2}^2 + \|T_h^-\psi\|_{\mathcal{A}^2}^2, \quad \psi \in \mathcal{A}^2(\Delta). \tag{10.8}$$

We first prove

LEMMA 10.3. *Let h be a quasisymmetric homeomorphism which keep the points* 1, -1 *and i fixed. Then*

(1) $\|P_h^-\| \to 0$ *when* $\tau(0,h) \to 0$.

(2) *For each fixed* $\phi \in \mathcal{AD}$, $\|P_h^+\phi - \phi\|_{\mathcal{AD}} \to 0$ *as* $\tau(0,h) \to 0$.

Proof. By the definition (2.1) of the Teichmüller metric, there exists a so-called extremal quasiconformal extension f of h so that its complex dilatation μ satisfies

$$\|\mu\|_\infty = \frac{e^{2\tau(0,h)} - 1}{e^{2\tau(0,h)} + 1}.$$

Thus, as $\tau(0,h) \to 0$, $\|\mu\|_\infty \to 0$. Since h keeps the points 1, -1 and i fixed, we conclude by Strebel's approximation theorem (see [St]) that $\partial f(z) \to 1$ for a.e. $z \in \Delta$, and $f(z) \to z$ locally uniformly in Δ.

(1) Proposition 3.1 in [HS] says that

$$\|T_h^-\| \le \frac{\|\mu\|_\infty}{\sqrt{1 - \|\mu\|_\infty^2}}, \tag{10.9}$$

which implies that, when $\tau(0,h) \to 0$, $\|T_h^-\| \to 0$ and consequently that $\|P_h^-\| \to 0$ by (10.7).

(2) By (10.7) we need to show that for each fixed $\psi \in \mathcal{A}^2$, $\|T_h^+\psi - \psi\|_{\mathcal{A}^2} \to 0$ as $\tau(0,h) \to 0$. Clearly,

$$\|T_h^+\psi - \psi\|_{\mathcal{A}^2}^2 = \|T_h^+\psi\|_{\mathcal{A}^2}^2 + \|\psi\|_{\mathcal{A}^2}^2 - \frac{2}{\pi}\Re \iint_\Delta T_h^+\psi(\zeta)\overline{\psi(\zeta)}\,d\xi d\eta. \tag{10.10}$$

Proposition 3.2 in [HS] says that

$$T_h^+\psi(\zeta) = \frac{1}{\pi}\iint_\Delta \frac{\partial f(w)\psi(f(w))}{(1-\zeta\bar{w})^2}\,dudv, \tag{10.11}$$

and

$$\|T_h^+\| \le \frac{1}{\sqrt{1 - \|\mu\|_\infty^2}}. \tag{10.12}$$

Then,

$$\begin{aligned}
&\iint_\Delta T_h^+\psi(\zeta)\overline{\psi(\zeta)}d\xi d\eta \\
&\quad = \frac{1}{\pi}\iint_\Delta \left(\iint_\Delta \frac{\partial f(w)\psi(f(w))}{(1-\zeta\bar{w})^2}dudv\right)\overline{\psi(\zeta)}d\xi d\eta \\
&\quad = \frac{1}{\pi}\iint_\Delta \left(\iint_\Delta \frac{\overline{\psi(\zeta)}}{(1-\zeta\bar{w})^2}d\xi d\eta\right)\partial f(w)\psi(f(w))dudv \\
&\quad = \iint_\Delta \partial f(w)\psi(f(w))\overline{\psi(w)}dudv.
\end{aligned} \tag{10.13}$$

By (10.10)–(10.13), we only need to show that

$$\lim_{\|\mu\|_\infty\to 0}\iint_\Delta \left(|\psi|^2 - \Re(\partial f\psi(f)\overline{\psi})\right) = 0. \tag{10.14}$$

Noting that

$$0 \le ||\psi|^2 - \partial f\psi(f)\overline{\psi}| + |\psi|^2 - |\partial f\psi(f)\overline{\psi}| \le 2|\psi|^2,$$

we conclude by Lesbegue's dominated convergence theorem that

$$\begin{aligned}
&\lim_{\|\mu\|_\infty\to 0}\iint_\Delta (||\psi|^2 - \partial f\psi(f)\overline{\psi}| + |\psi|^2 - |\partial f\psi(f)\overline{\psi}|) \\
&\quad = \iint_\Delta \lim_{\|\mu\|_\infty\to 0}(||\psi|^2 - \partial f\psi(f)\overline{\psi}| + |\psi|^2 - |\partial f\psi(f)\overline{\psi}|) = 0.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\left(\iint_\Delta |\partial f\psi(f)\overline{\psi}|\right)^2 &\le \iint_\Delta |\psi|^2 \iint_\Delta |\psi(f)|^2|\partial f|^2 \\
&= \iint_\Delta |\psi|^2 \iint_\Delta \frac{|\psi|^2}{1-|\mu(f^{-1})|^2} \\
&\le \frac{1}{1-\|\mu\|_\infty^2}\left(\iint_\Delta |\psi|^2\right)^2.
\end{aligned}$$

Combining these two inequalities together we obtain

$$\lim_{\|\mu\|_\infty\to 0}\iint_\Delta ||\psi|^2 - \partial f\psi(f)\overline{\psi}| = 0. \tag{10.15}$$

Now (10.14) follows from (10.15) by noting

$$||\psi|^2 - \Re(\partial f\psi(f)\overline{\psi})| \le ||\psi|^2 - \partial f\psi(f)\overline{\psi}|.$$

□

Proof of Proposition 10.1. Let $u = \phi + \overline{\psi}$ be given. Noting that

$$\begin{aligned} P_h u(z) - u(z) &= P_h \phi(z) + \overline{P_h \psi(z)} - \phi(z) - \overline{\psi(z)} \\ &= P_h^+ \phi(z) - \phi(z) + \overline{P_h^+ \psi(z)} - \overline{\psi(z)} + P_h^- \phi(\bar{z}) + \overline{P_h^- \psi(\bar{z})}, \end{aligned}$$

we conclude by Lemma 10.3 that $\|P_h u - u\|_{H^{\frac{1}{2}}} \to 0$ when $\tau(0,h) \to 0$. Consequently,

$$\|P_h u - P_{h_0} u\|_{H^{\frac{1}{2}}} = \|P_{h_0}(P_{h \circ h_0^{-1}} u - u)\|_{H^{\frac{1}{2}}} \leq \|P_{h_0}\| \|P_{h \circ h_0^{-1}} u - u\|_{H^{\frac{1}{2}}} \longrightarrow 0$$

when $\tau(0, h \circ h_0^{-1}) = \tau(h, h_0) \to 0$. □

To prove Proposition 10.2, we also need the so-called Grunsky operator. Consider the normalized decomposition $h = f^{-1} \circ g$ as before. Set

$$U(f, \zeta, z) = \frac{f'(\zeta) f'(z)}{[f(\zeta) - f(z)]^2} - \frac{1}{(\zeta - z)^2}, \quad (\zeta, z) \in \Delta \times \Delta. \tag{10.16}$$

Then $S_f(z) = -6U(f, z, z)$ is the Schwarzian derivative of f. f determines the so-called Grunsky operator on $\mathcal{A}^2(\Delta)$, defined as

$$G_f \psi(\zeta) = \frac{1}{\pi} \iint_\Delta U(f, \zeta, \bar{z}) \psi(z) dx dy. \tag{10.17}$$

It is known that G_f is a bounded operator from $\mathcal{A}^2(\Delta)$ into itself with $\|G_f\| < 1$ (see [Po1, Sh, TT2]). The following relation was proved by the author and Wei [SW]:

$$T_h^+ \circ G_f = J \circ T_h^- \circ J, \tag{10.18}$$

where J is the operator defined by $J\phi(z) = \overline{\phi(\bar{z})}$ so that $J^2 =$id, and $J \circ D = D \circ J$.

Proof of Proposition 10.2. (10.1) follows directly from (10.7) and (10.8). Now let $\psi \in \mathcal{AD}(\Delta)$ be given. Choose $\omega \in \mathcal{AD}_0(\Delta)$ so that $D\omega = -G_f J D\psi$. By (10.18) it holds that

$$J T_h^- D\psi + T_h^+ D\omega = J T_h^- D\psi - T_h^+ G_f J D\psi = 0.$$

By (10.7) we obtain

$$D(P_h^+ \omega + J P_h^- \psi) = T_h^+ D\omega + J T_h^- D\psi = 0.$$

Then,

$$P P_h(\psi + \overline{\omega}) = P_h^+ \psi + \overline{J P_h^- \psi} + \overline{P_h^+ \omega} + J P_h^- \omega = P_h^+ \psi + J P_h^- \omega + \overline{P_h^+ \omega(0)}.$$

Set $\phi = P_h^+\psi + JP_h^-\omega + \overline{P_h^+\omega(0)}$. Then $\phi \in \mathcal{AD}(\Delta)$, and $P_{h^{-1}}\phi = \psi + \overline{\omega}$. Consequently, $P_{h^{-1}}^+\phi = \psi$, and $P_{h^{-1}}^+$ is surjective. Replacing h^{-1} with h, we conclude that P_h^+ is surjective. □

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